



DYNAMICAL SYSTEMS SEMINAR

Musical Scales and the Three Gaps Theorem

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Mini-auditorium/ MAT

Abstract. Where do the sounds Do-Re-Mi-Fa-Sol-La-Si come from? Why do the black keys on the piano repeat unevenly?

This is due to an experimental discovery by Pythagoras that says, given a note (sound frequency), which following note sounds best when played next to the first. We will justify this theory of Pythagorean harmony with principles of sound production and perception, and note that it can be seen as a well-known dynamical system: representing a note by a point on the circle, the note that sounds best with it according to Pythagoras is given by a certain irrational rotation of the circle.

A musical scale is a finite set of notes. What distinguishes one music from another is not exactly the, essentially arbitrary, frequencies of the notes on the used scale but its harmony, which is given by the gaps between consecutive notes of the scale. The gaps of a Pythagorean scale cannot all have the same lengths, since its notes come from an irrational rotation. Still, we will show that in this irregularity there is an unexpected regularity: the gaps of a scale of notes generated by an irrational rotation have at most three different lengths (Three Gaps Theorem, Steinhaus and Lon 1958).

The most used scales in Western music come from this. They are the first Pythagorean scales that are as regular as possible: with only two gap lengths. They are the 5-note scale (pentatonic), 7 notes (Do-Re-Mi-Fa-Sol-La-Si) and 12 notes (Do -...- Si with the sharps or flats). This last scale gives the notes and gaps that appear on the piano keys between two consecutive Dots: it is known as the chromatic scale and its regular irregularity gives rise to the harmonic richness of Western music.

References

- [1] Peter Shiu: *A Footnote to the Three Gaps Theorem*. Amer. Math. Monthly 125(3), 2018.
- [2] Bruno Pedroso: *Simples Música*. <http://simplesmusica.com.br/>, 2017.
- [3] Arthur H. Benade: *Horns, Strings and Harmony*. Dover Publ., Reprint, 1992.