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Multiplicity and extremal regions for existence of positive solutions for singular problems in $\mathbb{R}^N$

by

Ricardo Lima Alves

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por

Ricardo Lima Alves

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Comissão Examinadora:

Prof. Dr. Carlos Alberto Pereira dos Santos– Orientador (MAT-UnB)

Prof. Dr. José Valdo Abreu Gonçalves (MAT-UnB)

Prof. Dr. Claudianor Oliveira Alves (UFCG)

Prof. Dr. Kaye Oliveira da Silva (UFG)

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Dedicatória

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em memória, com todo amor...

Resumo

Neste trabalho, estudamos problemas elípticos semilineares com parâmetros em todo espaço $\mathbb{R}^N$ $(N \geq 3)$ envolvendo não linearidades, que podem apresentar singularidades, e potencial com sinal indefinido. Nosso objetivo principal é estabelecer a existência de regiões extremais para a existência, não-existência e multiplicidade de soluções positivas tanto para problemas envolvendo uma equação quanto para sistemas.

No caso de não linearidades singulares, nossa abordagem é baseada em um refinamento do método da Variedade Nehari que inclua pontos de inflexão da aplicação fibração gerada pelo funcional energia associado ao problema, finas estimativas e propriedades dos níveis de energia sobre componentes conexas da Variedade de Nehari e um novo teorema de supersolução. Para não linearidades não singulares, usamos o Grau Topológico de Leray-Schauder, o método de sub-supersolução e estimativas a-priori das soluções.

Palavras-chave: Singularidade; regiões extremais dos parâmetros; multiplicidade, existência e não existência de soluções; Método de Nehari para funcionais não-diferenciáveis; método de fibração, Grau de Leray-Schauder, sub-supersolução, Teorema de supersolução

Abstract

In this work, we study semilinear elliptic problems with parameters on the whole space $\mathbb{R}^N$ ($N \geq 3$) involving nonlinearities, that may present singularities, and potential with indefinite sign. Our main objective is to establish the existence of extremal regions for the existence, non-existence and multiplicity of positive solutions for both problems involving equation and systems.

In the case of singular nonlinearities, our approach is based on a refinement of the Nehari manifold method that includes inflection points of the fiber map generated by the energy functional associated to the problem, fine estimates and properties of levels of energy on connected components of the Nehari manifold, and a new supersolution theorem. For non-singular nonlinearities, we use the Leray-Schauder Topological Degree, the sub-supersolution method, and a priori estimates of the solutions.

Keywords: Singularity; extremal regions of the parameters; multiplicity, existence and non-existence of solutions; Nehari method for non-differentiable functionals; fibering method, Leray-Schauder degree, sub-supersolution, supersolution theorem

Contents

Introduction

In this thesis, we present a study on the issues related to non-existence, existence and multiplicity of positive solutions to the following class of problems

$$\begin{cases} -\Delta u + V(x)u = f(\lambda, x, u, v) \text{ in } \mathbb{R}^N, \\ -\Delta v + V(x)v = g(\lambda, x, u, v) \text{ in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} V u^2 dx < \infty, \int_{\mathbb{R}^N} V v^2 dx < \infty, \; u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{H}$$

where $N \geq 3$ and the functions f, g satisfy some technical conditions, which will be mentioned later on and may present singular behavior of one of the following types:

$(S)_1$ the function f is singular at $u = 0$, that is, $\lim_{u \to 0^+} f(\lambda, x, u, v) = +\infty$ for all fixed $(\lambda, x, v) \in (0, \infty) \times \mathbb{R}^N \times \mathbb{R}$,

$(S)_2$ the functions f and g are singular with respect to u and v at $u = 0$ and $v = 0$ respectively, that is, $\lim_{u \to 0} f(\lambda, x, u, v) = +\infty$ and $\lim_{v \to 0} g(\lambda, x, u, v) = +\infty$, for all fixed $(\lambda, x, v), (\lambda, x, u) \in (0, \infty) \times \mathbb{R}^N \times \mathbb{R}$.

Although Problems of the type (H) have been extensively studied in recent years, there are many interesting questions related to these classe of problems. However, on the whole space $\mathbb{R}^N$ there are few results about problems that can present singular behavior in nonlinearities.

According to the specificities of f and g, a refinement of Nehari manifold and the fibering method, Leray-Schauder degree and sub-supersolution techniques were employed. To use such methods some difficulties occur. For example, due to the lack of differentiability of the energy functional associated to the problem, the sets defined similarly to the classical Nehari manifold are not manifold as in the case of functionals are of class C^1. Nevertheless, we continue using the usual numeclature for these sets. In Chapter 1, to use the Nehari manifold method, the main difficulties come from the non-differentiability of the energy functional and the fact that the intersection of the boundaries of the connected components of the Nehari set is non-empty. We overcome these difficulties by exploring topological structures of that boundary to build non-empty sets whose boundaries have empty intersection and minimizing over them by controlling the energy level.

In Chapter 2, we introduced a new idea of modifying an elliptic systems in its standard form to a new elliptic systems to generalize the ideas of Chapter 1. In this new context of elliptic systems with singular nonlinearities, we will obtain a continuous curve that plays a similar role to the extremal value obtained in Chapter 1. To show the global existence of solutions, we prove a new supersolution theorem for systems with indefinite potentials and apply it to prove our main result.

In Chapter 3, in addition to the lack of compact embbedings of Sobolev spaces into Lebesgue, we have the additional difficulties of choosing the appropriate spaces to work and extending to the whole space $\mathbb{R}^N$ a sub-supersolution theorem of Cheng-Zhang [17] dealt on bounded domains. This result was very important to obtain multiplicity of radial solutions as well. We also need of new a priori estimates for some extremal curves and a new idea to obtain multiplicity of non-radial solutions claimed in the Corollary 0.0.1.

Next, we present precisely what was developed in each chapter.

In Chapter 1, we consider the scalar case of (H) with $g(\lambda, x, u, v) = 0$ for all $(\lambda, x, u, v) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^2$ and we study the singular superlinear and subcritical Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = \lambda a(x)u^{-\gamma} + b(x)u^p \text{ in } \mathbb{R}^N, \\ u > 0, \ \mathbb{R}^N, \ \displaystyle\int_{\mathbb{R}^N} V u^2 dx < \infty, \ u \in H^1(\mathbb{R}^N), \end{cases} \qquad (P_\lambda)$$

when the potential b may change its sign, $0 < a \in L^{\frac{2}{1+\gamma}}(\mathbb{R}^N)$, $b^+ \neq 0, b \in L^\infty(\mathbb{R}^N)$, $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function, $0 < \gamma < 1 < p < 2^* - 1$, $N \geq 3$ and $\lambda > 0$ is a real positive parameter.

Since the pioneering work by Fulks-Maybee [30] on singular problems, this kind of subject has drawn the attention of several researchers. They showed that if $\Omega \subset \mathbb{R}^3$ is a bounded region of the space occupied by an electrical conductor, then u satisfies the equation

$$cu_t - k\Delta u = \frac{E^2(x,t)}{u^\gamma},$$

where $u(x,t)$ denotes the temperature at the point $x \in \Omega$ and time t, $E(x,t)$ describes the local voltage drop, u^γ with $\gamma > 0$ is the electrical resistivity, c and k are the specific heat and the thermal conductivity of the conductor, respectively.

Due to the applications or mathematical purposes, the issues on multiplicity (both local and global) of solutions for elliptic problems have been largely considered in the last decades. In 1994, Ambrosetti-Brezis-Cerami in [1], by exploring the sub and super solution method and Mountain Pass Theorem, proved a global multiplicity result, i.e., there exists a $\Lambda > 0$ such that the problem

$$\begin{cases} -\Delta u = \lambda a(x)|u|^{\gamma-2}u + b(x)|u|^{p-2}u \text{ in } \Omega, \\ u > 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{Q_λ}$$

admits at least two positive solutions for $0 < \lambda < \Lambda$, at least one solution for $\lambda = \Lambda$ and no solution for $\lambda > \Lambda$, when $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $a = b = 1$, $1 < \gamma < 2 < p < 2^*$ and 2^* is the critical Sobolev exponent. Considering more general operators and hypothesis, problem (Q_λ) was generalized by Figueiredo-Gossez-Ubilla [23, 22].

Recently, a number of authors have studied problems like (Q_λ) by using only variational methods, to wit, the Nehari manifold and the fibering method of Pohozaev [53] (see [41, 58, 60, 61]). In 2018, Silva-Macedo in [58] took advantage of the C^1-regularity of the energy functional associated to problem (Q_λ) with $a = 1$ to refine the Nehari's classical arguments and show multiplicity of solutions beyond the Nehari's extremal value

$$\lambda_* = \left(\frac{2-\gamma}{p-\gamma}\right)^{\frac{2-\gamma}{p-2}} \left(\frac{p-2}{p-\gamma}\right) \inf_{0 \lneqq u \in H_0^1(\Omega), \int_\Omega b|u|^{p+1}dx > 0} \frac{(||u||^2)^{\frac{p-\gamma}{p-2}}}{\left[\int_\Omega b|u|^p dx\right]^{\frac{2-\gamma}{p-2}} \left[\int_\Omega a|u|^\gamma dx\right]},$$

as defined in Il'yasov [42].

Similar issues have been considered for singular problems of the type

$$\begin{cases} -\Delta u = \lambda a(x)u^{-\gamma} + b(x)u^p \text{ in } \Omega, \\ u > 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{R_λ}$$

where $0 < \gamma < 1 < p < 2^* - 1, \Omega \subset \mathbb{R}^N$ is a smooth bounded domain. In 2003, Haitao in [40] proved a global multiplicity result for Problem (R_λ) with $a = b = 1$ by combining sub-supersolution and variational methods. In 2008, Yijing-Shujie in [60] considered the problem (R_λ) with potentials $a, b \in C(\overline{\Omega})$ satisfying $a \geq 0, a \not\equiv 0$ and b may change sign. They proved a local multiplicity result, i.e., there exists a $\Lambda > 0$ such that the problem (R_λ) admits at least two non-negative solutions for each $\lambda \in (0, \Lambda)$. Still in this context of bounded smooth domains, we refer the reader to [52, 21, 45, 61] where different techniques, more general operators and non-linearities were considered.

On $\mathbb{R}^N$ there are a few results related with existence, multiplicity and non-existence of solutions for Problems like (R_λ). By using the sub and super solution method combined with perturbation arguments, the authors Carl-Perera [15], Gonçalves-Santos [37], Cîrstea-Rădulesco [19], Edelson [28] proved existence of $C^1(\mathbb{R}^N)$-solutions.

With respect to the variational techniques point of view, as far as we know, there is just one, to wit, Liu-Guo-Liu [46] in 2009 proved a local multiplicity result of $D^{1,2}(\mathbb{R}^N)$-solutions for the equation

$$-\Delta u = a(x)u^{-\gamma} + \lambda b(x)u^p, \ x \in \mathbb{R}^N, u > 0,$$

where $N \geq 3, \lambda > 0, 0 < \gamma < 1 < p < 2^* - 1, 0 \leq a \in L^{\frac{2^*}{2^*-(1-\gamma)}}(\mathbb{R}^N), 0 \leq b \in L^{\frac{2^*}{2^*-(1+\beta)}}(\mathbb{R}^N)$ and b may change sign. They combined a local minimization over the ball with an extension of the Mountain Pass Theorem for nonsmooth functionals (see Canino-Degiovani [13]). Due to the their techniques, it is not hard to see that their extremal value that still guarantees multiplicity of solutions is less than

$$\hat{\lambda} = \hat{C} \inf_{0 \lneqq u \in X, \int_{\mathbb{R}^N} b|u|^{p+1}dx > 0} \frac{(||u||^2)^{\frac{p+\gamma}{p-1}}}{\left[\int_{\mathbb{R}^N} b|u|^{p+1}dx\right]^{\frac{1+\gamma}{p-1}} \left[\int_{\mathbb{R}^N} a|u|^{1-\gamma}dx\right]}, \tag{1}$$

where

$$\hat{C} = (1-\gamma)\frac{(p+1)^{\frac{1+\gamma}{p-1}}}{2^{\frac{p+\gamma}{p-1}}} \left(\frac{1+\gamma}{p+\gamma}\right)^{\frac{1+\gamma}{p-1}} \left(\frac{p-1}{p+\gamma}\right),$$

because they were able to show multiplicity of solutions just in the λ-variation of the parameter λ that still produces the second solution with positive energy.

By using a new approach, we were able to prove multiplicity of solutions for Problem (P_λ) beyond $\hat{\lambda}$, that necessarily implies that all the solutions found by this method have negative energies. Besides this, we were also able to characterize a λ-behavior of the energy functional along these solutions.

To state our main results, let us assume that $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function that satisfies

$(V)_0$ $V_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0$, and one of the following conditions:

- (i) $\lim_{|x|\to\infty} V(x) = \infty$;
- (ii) $1/V \in L^1(\mathbb{R}^N)$;
- (iii) for each $M > 0$ given the $\mathcal{L}(\{x \in \mathbb{R}^N : V(x) \leq M\}) < \infty$.

Define

$$X = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\},$$

and observe that $\Phi_\lambda : X \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx - \frac{1}{p+1}\int_{\mathbb{R}^N} b(x)|u|^{p+1}dx,$$

where

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx,$$

is well-defined and continuous. One of the main difficulties to approach the problem (P_λ) is the lack of Gâteaux differentiability of the energy functional Φ_λ, which is due to the presence of the singular term.

We say that $u \in X$ is a solution of (P_λ) if

$$\int_{\mathbb{R}^N} \nabla u \nabla \psi + V(x)u\psi dx = \lambda \int_{\mathbb{R}^N} a(x)u^{-\gamma}\psi dx + \int_{\mathbb{R}^N} b(x)u^p\psi dx \text{ for all } \psi \in X.$$

Related to the structure of the functional Φ_λ, let us set (see Hirano-Saccon-Shioji [41] and Il'yasov [42])

$$\lambda_* = \left(\frac{1+\gamma}{p+\gamma}\right)^{\frac{1+\gamma}{p-1}}\left(\frac{p-1}{p+\gamma}\right) \inf_{0\lneqq u\in X, \int_{\mathbb{R}^N} b|u|^{p+1}dx>0} \frac{(\|u\|^2)^{\frac{p+\gamma}{p-1}}}{\left[\int_{\mathbb{R}^N} b|u|^{p+1}dx\right]^{\frac{1+\gamma}{p-1}}\left[\int_{\mathbb{R}^N} a|u|^{1-\gamma}dx\right]}, \tag{2}$$

which relates with $\hat{\lambda} > 0$ defined at (1) by

$$\hat{\lambda} = (1-\gamma)\frac{(p+1)^{\frac{1+\gamma}{p-1}}}{2^{\frac{p+\gamma}{p-1}}}\lambda_* < \lambda_*.$$

Our first result is

Theorem 0.0.1 *Suppose that* $0 < \gamma < 1 < p < 2^* - 1; 0 < a \in L^{\frac{2}{1+\gamma}}(\mathbb{R}^N)$, $b \in L^\infty(\mathbb{R}^N)$, $b^+ \neq 0$, $(V)_0$ *and* $[a/b]^{\frac{1}{p+\gamma}} \notin X$ *if* $b > 0$ *in* $\mathbb{R}^N$ *hold. Then there exists an* $\epsilon > 0$ *such that the problem* (P_λ) *has at least two positive solutions* $w_\lambda, u_\lambda \in X$ *for each* $0 < \lambda < \lambda_* + \epsilon$ *given. Besides this, we have:*

a) $\dfrac{d^2\Phi_\lambda}{dt^2}(tu_\lambda)\Big|_{t=1} > 0$ *and* $\dfrac{d^2\Phi_\lambda}{dt^2}(tw_\lambda)\Big|_{t=1} < 0$ *for all* $0 < \lambda < \lambda_* + \epsilon$,

b) *there exists a constant* $c > 0$ *such that* $||w_\lambda|| \geq c$ *for all* $0 < \lambda < \lambda_* + \epsilon$,

c) u_λ *is a ground state solution for all* $0 < \lambda \leq \lambda_*$, $\Phi_\lambda(u_\lambda) < 0$ *for all* $0 < \lambda < \lambda_* + \epsilon$ *and* $\lim_{\lambda \to 0} ||u_\lambda|| = 0$,

d) *the applications* $\lambda \longmapsto \Phi_\lambda(u_\lambda)$ *and* $\lambda \longmapsto \Phi_\lambda(w_\lambda)$ *are decreasing for* $0 < \lambda < \lambda_* + \epsilon$ *and are left-continuous ones for* $0 < \lambda < \lambda_*$,

e) $\Phi_\lambda(w_\lambda) > 0$ *for* $0 < \lambda < \hat{\lambda}$, $\Phi_{\hat{\lambda}}(w_{\hat{\lambda}}) = 0$ *and* $\Phi_\lambda(w_\lambda) < 0$ *for* $\hat{\lambda} < \lambda < \lambda_* + \epsilon$ *(see* $\hat{\lambda}$ *in* (1)*),*

Remark 0.0.1 *In fact, the hypothesis* $[a/b]^{\frac{1}{p+\gamma}} \notin X$ *if* $b > 0$ *in* $\mathbb{R}^N$ *is required just for* $\lambda_* \leq \lambda \leq \lambda_* + \epsilon$.

The second result gives us an estimate on how big the number $\epsilon > 0$ can be, under additional assumptions on a and b.

Theorem 0.0.2 *Suppose that the hypotheses of Theorem 0.0.1 hold. Moreover, assume that there exists a smooth bounded open set* $\Omega \subset \mathbb{R}^N$ *such that* $b > 0$ *in* Ω *and* $a \in L^\infty(\Omega)$. *Then there exists* $\lambda^* > 0$ *such that the problem* (P_λ) *has no solution at all for* $\lambda > \lambda^*$. *Moreover, we have the exact estimate*

$$0 < \lambda_* < \lambda^* = \lambda_1^{\frac{p+\gamma}{p-1}} \left(\frac{\gamma+1}{p-1}\right)^{\frac{\gamma+1}{p-1}} \left(\frac{p-1}{p+\gamma}\right)^{\frac{p+\gamma}{p-1}},$$

where $\lambda_1 := \lambda_1(\Omega) > 0$ *is given in Lemma* 1.5.1.

Some comments are in order now:

a) Theorem 0.0.1 is new in the literature by showing multiplicity of solutions with negative energies as well,

b) traditionally two solutions are found by minimizing the energy functional over connected components of the Nehari manifold which are separated in the sense that their boundaries have disjoint intersection. In this work we go further, because we find solutions in the case where such intersection is not empty even in the context of singular problems,

c) the characterization of the λ-behavior about continuity and monotonicity of the energy functional along the solutions is new as well,

d) Theorem 0.0.1 and Theorem 0.0.2 induce us to conjecture that there exists a bifurcation point $\tilde{\lambda} > 0$ with $\lambda_* + \epsilon \le \tilde{\lambda} \le \lambda^*$ for which the two solutions collapse.

The results of Chapter 1 are published in the preprint [54]. Summarizing our results in a picture we have

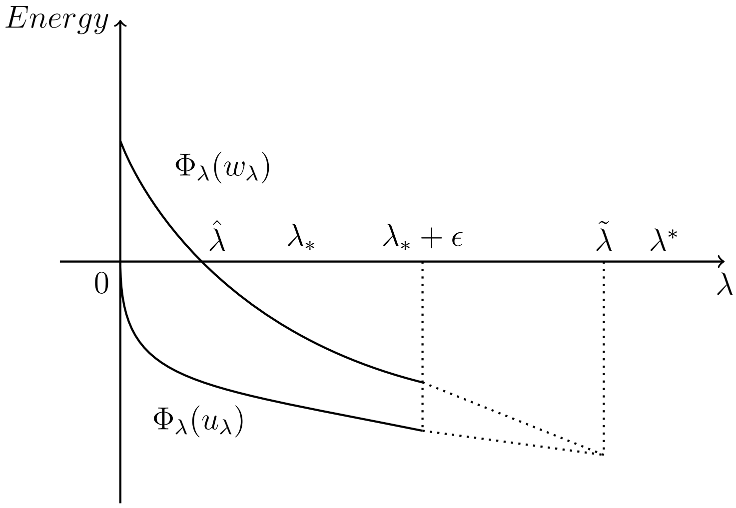


Fig. 1 Energy depending on λ

In Chapter 2, we study existence, multiplicity and non-existence of $H^1(\mathbb{R}^N)$-solutions for the following system

$$\begin{cases} -\Delta u + V(x)u = \lambda a(x)u^{-\gamma} + \dfrac{\alpha}{\alpha+\beta} b(x)u^{\alpha-1}v^{\beta} \ in\ \mathbb{R}^N, \\ -\Delta v + V(x)v = \mu c(x)v^{-\gamma} + \dfrac{\beta}{\alpha+\beta} b(x)u^{\alpha}v^{\beta-1} \ in\ \mathbb{R}^N, \\ u, v > 0,\ \mathbb{R}^N,\ \displaystyle\int_{\mathbb{R}^N} Vu^2dx + \int_{\mathbb{R}^N} Vv^2dx < \infty,\ u, v \in H^1(\mathbb{R}^N), \end{cases} \qquad (\tilde{P}_{\lambda,\mu})$$

where $0 < a, c$ in $\mathbb{R}^N$, $b^+ \not\equiv 0$, $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function; $0 < \gamma < 1 < \alpha, \beta$; $2 < \alpha + \beta < 2^*$, $N \ge 3$ and $\lambda, \mu \ge 0$ are real parameters. The potential V and the functions a, b and c satisfy some technical conditions, which will be mentioned later on.

Problems involving singular nonlinearities have been deeply studied in the last decades in the context of scalar problems (see [40, 60, 52, 21, 45, 61] again for further details). However, there are few works dealing with systems of type $(\tilde{P}_{\lambda,\mu})$ with indefinite potential even in bounded domains.

Unlike the singular case, there are a variety of works treating elliptic systems with nonsingular nonlinearities. In bounded domain, we would like to quote here, in addition to the works already mentioned above for elliptic systems, the works of Wu [64], Velin [63], Alves-de Morais filho-Souto [3], Bozhkov-Mitidieri [11], Silva-Macedo [57], Bobkov-Il'yasov [9, 10] and references therein, where the authors have used variational methods to show their main results.

In 2018, the authors Silva-Macedo in [57] considered the following system:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} + \alpha f(x)|u|^{\alpha-2}|v|^{\beta} u \ in\ \Omega, \\ -\Delta_q v = \mu |v|^{q-2} + \beta f(x)|u|^{\alpha}|v|^{\beta-2} v \ in\ \Omega, \\ \qquad\qquad (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $\lambda, \mu \in \mathbb{R}$, $1 < p, q < \infty$, $f \in L^{\infty}(\Omega)$ and f has indefinite sign, that is, f^+ and f^- are not identically zero in Ω. Also, the exponents and function f satisfy some other technical conditions (see [57]). Denote by λ_1 and μ_1 the first eigenvalue of the operators $-\Delta_p$ and $-\Delta_q$ respectively. Using the Nehari manifold method and the fibering method they proved the existence of a extremal curve $\gamma^* \subset \mathbb{R}_0^+ \times \mathbb{R}_0^+$ such that the system has at least one positive solution for $(\lambda, \mu) \in \{(\lambda, \mu) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : (\lambda_1, \mu_1) < (\lambda, \mu) \leq \gamma^*\}$, and for each $\sigma = (\lambda_0, \mu_0) \in \gamma^*$, there exists a positive real number $\epsilon_\sigma > 0$ such that the system has at least one positive solution for $(\lambda, \mu) \in [\lambda_0, \lambda_0 + \epsilon_\sigma) \times [\mu_0, \mu_0 + \epsilon_\sigma)$. This result improves the works [9, 10] and [11].

As we mentioned above, about singular elliptic systems there are few results dealing with problems of the type $(\tilde{P}_{\lambda,\mu})$. By using non-variational methods, the works of Alves-Corrêa-Gonçalves [2], Giacomoni-Schindler-Takác [33], Manouni-Perera-Shivaji [29], Gonçalves-Carvalho-Santos [35], Hai [39] and references therein, showed existence of solutions for small parameters, but they did not get multiplicity results. Using the Nehari manifold method and the fibering method of Pohozaev the authors Carvalho-Silva-Santos-Goulat [16] considered nonnegative potentials and Goyal [34] dealt with some indefinite potential to show local multiplicity results, but only minimizing the energy functional over connected components of the Nehari manifold which are separated in the sense that their boundaries have disjoint intersection. Therefore, the set of parameters where they found a solution is not the best possible, so it is possible to improve their results and this is one of the objectives of Chapter 2.

For unbounded domains we would like to quote here Marano-Marino-Moussaoui [48] and references therein, where the authors use non-variational methods to prove their results and Benrhouma [8] and references therein, where the authors used truncation arguments combined with variational methods to prove their results. Moreover, in these works they do not prove multiplicity of solutions and their potentials are nonnegative.

There are two difficulties in approaching the problem $(\tilde{P}_{\lambda,\mu})$. The first one is the same as in Chapter 1, that is, the non-differentiability of the energy functional and the fact that the intersection of the boundaries of the connected components of the Nehari set is non empty. The second one is that considering the problem with no related parameters (λ, μ), as previous works have done, a few information can be obtained about the set of parameters (λ, μ) such that $(\tilde{P}_{\lambda,\mu})$ has a solution. Thus, the main idea to overcome this difficulty is to modify problem $(\tilde{P}_{\lambda,\mu})$ to problem $(\tilde{P}_{\lambda,\theta\lambda})$ for every $\theta > 0$ fixed. With this modification we are able to solve a system similar to that considered in Chapter 1 (see (2.15)-(2.16)) and find an extremal value $\lambda_*(\theta)$, in the sense of the applicability of Nehari method. By varying $\theta > 0$ we have a continuous curve $\Gamma(\theta) = (\lambda_*(\theta), \theta\lambda_*(\theta))$ which is the boundary of a set of parameters (λ, μ) for which there is a solution for the system $(\tilde{P}_{\lambda,\mu})$, and this set is bigger than those considered by previous works. In addition, we obtain multiplicity of solution for parameters above of $\Gamma(\theta)$ but close to it. In particular, our results improve or complete the above works and generalize to the system $(\tilde{P}_{\lambda,\mu})$ the results obtained in the Chapter 1.

To state our main results, let us assume that $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function that satisfies the conditions:

$(V)_0$ $V_0 := \inf\limits_{x\in\mathbb{R}^N} V(x) > 0,$

$(V)_1$ $1/V \in L^1(\mathbb{R}^N).$

Define

$$X = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\}, \ E = X \times X,$$

and denote by $U=(u,v)$ points of E. With these, we say that $U=(u,v)\in E$ is a solution of $(\tilde{P}_{\lambda,\mu})$ if

$$\begin{aligned}&\int_{\mathbb{R}^N}[\nabla u\nabla\varphi+V(x)u\varphi]dx+\int_{\mathbb{R}^N}[\nabla v\nabla\psi+V(x)v\psi]dx\\&=\lambda\int_{\mathbb{R}^N}a(x)u^{-\gamma}\varphi dx+\mu\int_{\mathbb{R}^N}c(x)v^{-\gamma}\psi dx\\&+\frac{\alpha}{\alpha+\beta}\int_{\mathbb{R}^N}b(x)u^{\alpha-1}v^{\beta}\varphi dx+\frac{\beta}{\alpha+\beta}\int_{\mathbb{R}^N}b(x)u^{\alpha}v^{\beta-1}\psi dx\end{aligned}$$

for all $\Psi=(\varphi,\psi)\in E$.

In fact, we will prove that a solution of the problem $(\tilde{P}_{\lambda,\mu})$ must be always everywhere positive in $\mathbb{R}^N$ whenever λ,μ be positive. These kind of solutions will be named as positive solutions, while solutions (u,v) such that $uv=0$ will be called as semitrivial. These type of solutions can occurs just on the semi-axes.

About the potentials, let us assume that them satisfy:

$(A1)$ $a,c\in L^{\infty}(\mathbb{R}^N)\cap L^{\frac{2}{1+\gamma}}(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$,

$(A2)$ $b^+\not\equiv 0$ and $b\in L^{\infty}(\mathbb{R}^N)\cap L^{\frac{2^*}{2^*-\alpha-\beta}}(\mathbb{R}^N)$,

$(A3)$ $\left[\frac{a(x)}{b(x)}\right]^{\frac{1}{\alpha+\beta+\gamma-1}}\left[\frac{c(x)}{a(x)}\right]^{\frac{\beta}{(1-\gamma)(\alpha+\beta+\gamma-1)}}\notin X.$

These assumptions imply that the functionals

$$J(U)=\|u\|^2+\|v\|^2, K_{\lambda,\mu}(U)=\lambda\int_{\mathbb{R}^N}a(x)|u|^{1-\gamma}dx+\mu\int_{\mathbb{R}^N}c(x)|v|^{1-\gamma}dx$$

and

$$L(U)=\int_{\mathbb{R}^N}b(x)|u|^{\alpha}|v|^{\beta}dx$$

are well-defined and continuous on E, which lead us to infer the same to the functional $\Phi_{\lambda,\mu}:E\to\mathbb{R}$ defined by

$$\Phi_{\lambda,\mu}(U)=\frac{1}{2}J(U)-\frac{1}{1-\gamma}K_{\lambda,\mu}(U)-\frac{1}{\alpha+\beta}L(U).$$

However, this functional is not Gâteaux differentiable due to the presence of the singular terms.

Now, for every $(a,b),(c,d)\in\mathbb{R}^2$ let us denote by

$$](a,b),(c,d)]=\{(1-t)(a,b)+t(c,d):0<t\leq 1\}$$

and

$$](a,b),(c,d)[=\{1-t)(a,b)+t(c,d):0<t<1\}\,.$$

Our first result of Chapter 2 is

Theorem 0.0.3 *Suppose that $0<\gamma<1<\alpha,\beta$; $2<\alpha+\beta<2^*$; $0<a,c$ in $\mathbb{R}^N$, $(A1)-(A2)$, $(V)_0-(V)_1$ and $(A3)$ if $b>0$ in $\mathbb{R}^N$ hold. Then there exist two continuous simple arc $\Gamma_0=\{(\hat{\lambda}(\theta),\hat{\mu}(\theta)):\ \theta>0\}, \tilde{\Gamma}=\{(\lambda_*(\theta),\mu_*(\theta)):\ \theta>0\}\subset\mathbb{R}_0^+\times\mathbb{R}_0^+$, with $\Gamma_0(\theta)<\tilde{\Gamma}(\theta)$ for all $\theta>0$; $\hat{\lambda}(\theta),\lambda_*(\theta)$ non-increasing; $\hat{\mu}(\theta),\mu_*(\theta)$ non-decreasing and $\hat{\mu}(\theta)=\theta\hat{\lambda}(\theta),\mu_*(\theta)=\theta\lambda_*(\theta)$ satisfying the property: for each $\theta>0$ there exists an $\epsilon=\epsilon(\theta)>0$ such that the problem $(\tilde{P}_{\lambda,\mu})$ has at least two positive solutions $W_\lambda,U_\lambda\in E$ for each $(\lambda,\mu)\in](0,0),\tilde{\Gamma}(\theta)+(\epsilon,\theta\epsilon)[$ given. Besides this, writing $(\lambda,\mu)=(\lambda,\theta\lambda)$ we have:*

a) $\frac{d^2\Phi_{\lambda,\theta\lambda}}{dt^2}(tU_\lambda)\big|_{t=1}>0$ *and* $\frac{d^2\Phi_{\lambda,\theta\lambda}}{dt^2}(tW_\lambda)\big|_{t=1}<0$ *for all* $(\lambda,\mu)\in](0,0),\tilde{\Gamma}(\theta)+(\epsilon,\theta\epsilon)[$,

b) there exists a constant $c>0$ such that $||W_\lambda||\geq c$ for all $(\lambda,\mu)\in](0,0),\tilde{\Gamma}(\theta)+(\epsilon,\theta\epsilon)[$,

c) U_λ is a ground state solution for all $(\lambda,\mu)\in](0,0),\tilde{\Gamma}(\theta)]$, $\Phi_{\lambda,\theta\lambda}(U_\lambda)<0$ for all $(\lambda,\mu)\in](0,0),\tilde{\Gamma}(\theta)+(\epsilon,\theta\epsilon)[$ and $\lim\limits_{\lambda\to 0}||U_\lambda||=0$,

d) the applications $\lambda\longmapsto\Phi_{\lambda,\theta\lambda}(U_\lambda)$ and $\lambda\longmapsto\Phi_{\lambda,\theta\lambda}(W_\lambda)$ are decreasing for $0<\lambda<\lambda_*(\theta)+\epsilon$ and are left-continuous ones for $0<\lambda<\lambda_*(\theta)$,

e) $\Phi_{\lambda,\theta\lambda}(W_\lambda)>0$ for $(\lambda,\mu)\in](0,0),\Gamma_0(\theta)[$, $\Phi_{\Gamma_0(\theta)}(W_{\hat{\lambda}(\theta)})=0$ and $\Phi_{\lambda,\theta\lambda}(W_\lambda)<0$ for $(\lambda,\mu)\in]\Gamma_0(\theta),\tilde{\Gamma}(\theta)+(\epsilon,\theta\epsilon)[$.

Remark 0.0.2 *In fact, the hypothesis $(A3)$ is required just for $(\lambda,\mu)\in[\tilde{\Gamma}(\theta),\tilde{\Gamma}(\theta)+(\epsilon,\theta\epsilon)[$ for each $\theta>0$.*

Our second result is about extremal regions of existence of positive solutions. We have not been able to use the approach of the Theorem 0.0.3 to prove it and we need of a new argument. It is based in a new supersolution theorem and we have to keep in mind that, since the potential b may change its sign the principle of comparison cannot be used in our case, and therefore, the usual sub-supersolution theorems cannot be applied directly here. To overcome this difficulty, we proved a new supersolution theorem, to be precise the Theorem 0.0.4.

For convenience let us define supersolution for the problem $(\tilde{P}_{\lambda,\mu})$ and state the supersolution Theorem.

Definition 0.0.1 *Let* $(\lambda,\mu) > (0,0)$. *A function* $\overline{U} = (\overline{u},\overline{v}) \in E$ *is said to be a supersolution of* $(\tilde{P}_{\lambda,\mu})$ *if* $\overline{u},\overline{v} > 0$ *a.e. in* $\mathbb{R}^N$ *and*

$$\begin{aligned}&\int_{\mathbb{R}^N}[\nabla\overline{u}\nabla\varphi + V(x)\overline{u}\varphi]dx + \int_{\mathbb{R}^N}[\nabla\overline{v}\nabla\psi + V(x)\overline{v}\psi]dx\\ &\geq \lambda\int_{\mathbb{R}^N} a(x)\overline{u}^{-\gamma}\varphi dx + \mu\int_{\mathbb{R}^N} c(x)\overline{v}^{-\gamma}\psi dx\\ &+ \frac{\alpha}{\alpha+\beta}\int_{\mathbb{R}^N} b(x)\overline{u}^{\alpha-1}\overline{v}^{\beta}\varphi dx + \frac{\beta}{\alpha+\beta}\int_{\mathbb{R}^N} b(x)\overline{u}^{\alpha}\overline{v}^{\beta-1}\psi dx\end{aligned}$$

for all $\Psi = (\varphi,\psi) \in E_+$.

Theorem 0.0.4 *Suppose that* $0 < \gamma < 1 < \alpha,\beta$; $2 < \alpha+\beta < 2^*$; $0 < a, c$ *in* $\mathbb{R}^N$, $(A1)-(A2)$ *and* $(V)_0 - (V)_1$ *hold. Assume that the problem* $(\tilde{P}_{\overline{\lambda},\overline{\mu}})$ *admits a supersolution for some* $(\overline{\lambda},\overline{\mu}) > (0,0)$. *Then the problem* $(\tilde{P}_{\overline{\lambda},\overline{\mu}})$ *has at least one solution* $U_{\overline{\lambda},\overline{\mu}} = (u_{\overline{\lambda}}, v_{\overline{\mu}})$ *with* $\Phi_{\overline{\lambda},\overline{\mu}}(U_{\overline{\lambda},\overline{\mu}}) < 0$. *In particular, we have that the problem* $(\tilde{P}_{\lambda,\mu})$ *has at least one solution* $U_{\lambda,\mu}$ *satisfying* $\Phi_{\lambda,\mu}(U_{\lambda,\mu}) < 0$ *for all* $(0,0) \lneqq (\lambda,\mu) \leq (\overline{\lambda},\overline{\mu})$.

Our second result is related with extremal region of existence of positive solutions is.

Theorem 0.0.5 *Suppose that* $0 < \gamma < 1 < \alpha,\beta$; $2 < \alpha+\beta < 2^*$; $0 < a, c$ *in* $\mathbb{R}^N$, $(A1)-(A2)$, $(V)_0 - (V)_1$ *and* $(A3)$ *if* $b > 0$ *in* $\mathbb{R}^N$ *hold. Then:*

a) there exists an extended function $\Gamma^* : (0,\infty) \to \overline{\mathbb{R}}\times\overline{\mathbb{R}}$ $(\overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\})$, *with* $\Gamma^*(\theta) = (\lambda^*(\theta),\mu^*(\theta))$ *and* $\mu^*(\theta) = \theta\lambda^*(\theta)$ *such that system* $(\tilde{P}_{\lambda,\mu})$ *has at least one solution* $U_{\lambda,\mu}$ *for* $(\lambda,\mu)\in\Theta$ *and no solution for* $(\lambda,\mu)\notin\Theta$, *where*

$$\begin{aligned}\Theta = \{(\lambda,\mu) : (0,0) < (\lambda,\mu) \leq \Gamma^*(\theta),\ \theta > 0\} &\cup \{(\lambda,0) : \lambda\in[0,\infty)\}\\ &\cup\{(0,\mu) : \mu\in[0,\infty)\}\,.\end{aligned}$$

Moreover, we have $\Phi_{\lambda,\mu}(U_{\lambda,\mu}) < 0$ *if* $(\lambda,\mu)\in\Theta\setminus\{\Gamma^*(\theta):\theta>0\}$ *and* $\Phi_{\lambda,\mu}(U_{\lambda,\mu}) \leq 0$ *if* $(\lambda,\mu)\in\Gamma^*(\theta)$ *for* $\theta > 0$ *if* $\Gamma^*(\theta)\in\mathbb{R}_0^+\times\mathbb{R}_0^+$,

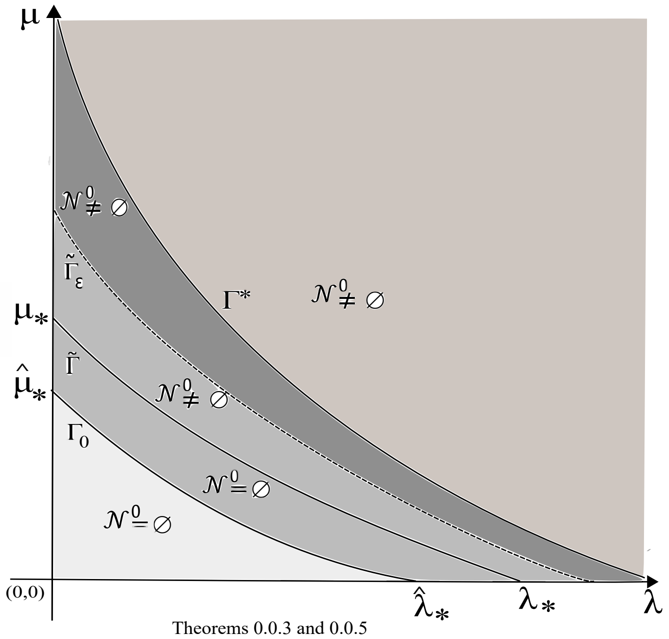
b) if in addition there exists a smooth bounded open set $\Omega\subset\mathbb{R}^N$ *such that* $b > 0$ *in* Ω, *then* $\Gamma^*\subset\mathbb{R}_0^+\times\mathbb{R}_0^+$ *and* $\Gamma^* : (0,\infty)\to\mathbb{R}_0^+\times\mathbb{R}_0^+$ *is a continuous curve, with* $0 < \lambda^*(\theta)$ *non-increasing and* $0 < \mu^*(\theta)$ *non-decreasing. In particular,*

$$\{\Gamma^*(\theta) = (\lambda^*(\theta),\mu^*(\theta)) : \theta > 0\} = \partial\Theta\cap(\mathbb{R}_0^+\times\mathbb{R}_0^+)$$

and $(\tilde{P}_{\Gamma^*(\theta)})$ *has at least one solution for all* $\theta > 0$.

To ease the interpretation of the conclusions of the above results, we draw them in the below graphics. We are writing $\tilde{\Gamma}_\epsilon(\theta) = \tilde{\Gamma}(\theta) + (\epsilon,\theta\epsilon)$.

Next, we list some of the main contributions of study of $(\tilde{P}_{\lambda,\mu})$ the literature:



Theorems 0.0.3 and 0.0.5

$\mathcal{N}^0 = \{U : \Phi_\lambda(tU)$ has critical points that are inflection points$\}$

two solutions with $\Phi_\lambda(U_\lambda) < 0$ and $\Phi_\lambda(W_\lambda) > 0$

two solutions with $\Phi_\lambda(U_\lambda) < 0$ and $\Phi_\lambda(W_\lambda) < 0$

one solution with $\Phi_\lambda(U_\lambda) < 0$

there is no positive solution

a) Theorem 0.0.3 is new in the literature by showing the existence of two curves, in one of them occurs the transition of positive to negative energy of one of the solutions (the other solution always has negative energy) and the other curve stands for the transition of the applicability of the Nehari Method. Besides this, it shows multiplicity of solutions beyond the critical curve to applicability to Nehari Method, which lead to existence of at least two solutions with negative energy. Moreover, as far as we know this result is new even when the potential is nonnegative,

b) as in the case of scalar problems, traditionally two solutions for elliptic systems are found by minimizing the energy functional over connected components of the Nehari manifold which are separated in the sense that their boundaries have disjoint intersection. In this work we go further, because we find multiplicity of solutions in the case where such intersection is not empty even in the context of singular problems,

c) the Theorem 0.0.4 is new in the literature by considering indefinite potential. The idea of its proof can be made in the context of scalar problems or bounded domain, being new in these contexts as well,

d) the Theorem 0.0.5 is new in the literature because it proves existence of the extremal region for existence of positive solutions to problems of type $(\tilde{P}_{\lambda,\mu})$. As far as we know, this result is new even when potential b is nonnegative.

In Chapter 3, we consider $V(x) = 0$ for all $x \in \mathbb{R}^N$ and we approach the multi-parameter elliptic system

$$\begin{cases} -\Delta u = \lambda w(x) f_1(u) g_1(v) \ in \ \mathbb{R}^N, \\ -\Delta v = \mu w(x) f_2(v) g_2(u) \ in \ \mathbb{R}^N, \\ u, v > 0 \ in \ \mathbb{R}^N \text{ and } u(x), v(x) \overset{|x| \to \infty}{\longrightarrow} 0 \end{cases} \tag{$P_{\lambda,\mu}$}$$

with respect to the parameters $\lambda, \mu \in \mathbb{R}^+$, where $N \geq 3$ and $\mathbb{R}^+ = [0, \infty)$. The potential w and the functions f_i, g_i $(i = 1, 2)$ satisfy some technical conditions, which will be mentioned later on.

In the last decades many authors have studied existence of solutions for elliptic systems in bounded domains, see for instance [4, 18, 17, 25, 26, 27, 50] and references therein. Cheng-Zhang in [17] studied the system

$$\begin{cases} -\Delta u = \lambda f_1(x, u) g_1(x, v) \ in \ \Omega, \\ -\Delta v = \mu f_2(x, v) g_2(x, u) \ in \ \Omega, \\ u, v > 0 \ in \ \Omega, \quad u, v \in H_0^1(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, the functions $g_i \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R}_0^+)(\mathbb{R}_0^+ = (0, \infty))$ and $f_i \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R}_0^+)(i = 1, 2)$ satisfy:

$(CZ)_1$: $f_i, g_i \in C^{\alpha(r)}(\Omega \times (-r, r), \mathbb{R}_0^+)$, for each $r > 0$ and some $\alpha(r) \in (0, 1)$,

$(CZ)_2$: g_1 and g_2 are bounded above on $\overline{\Omega} \times \mathbb{R}^+$,

$(CZ)_3$: $g_i(x, s_1) \leq g_i(x, s_2)$ for $s_1 \leq s_2$,

$(CZ)_4$: the inequality

$$\frac{\lambda_1}{\min\limits_{x \in \overline{\Omega}} g_i(x, 0)} < \liminf_{s \to \infty} \frac{\min\limits_{x \in \overline{\Omega}} f_i(x, s)}{s}$$

holds, where $\lambda_1 > 0$ is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$,

$(CZ)_5$: there exist $p_1(x), p_2(x) \in C(\overline{\Omega}, \mathbb{R}_0^+)$ and $q_1, q_2 \in (1, \frac{N}{N-2})$ such that

$$\lim_{s\to\infty} \frac{f_i(x,s)}{s^{q_i}} = p_i(x) \text{ uniformly with respect to } x \in \overline{\Omega}.$$

They proved the existence of a **bounded extremal curve** that separates $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ into two subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the system has no positive solution for $(\lambda, \mu) \in \mathcal{O}_2$, has at least two positive solutions for $(\lambda, \mu) \in \mathcal{O}_1$ and at least one positive solution for (λ, μ) in the extremal curve. We would like to point out that the idea of constructing of curve in [17] comes from of the work of Lee [47] in 2001, and the construction of curves presented by us is different from these.

In the works above mentioned the authors took advantage of the compact embeddings of Sobolev spaces into Lebesgue spaces $L^p(\Omega)$ $(1 < p < (N+2)/(N-2))$ to use the compact-operator theory on these natural functions spaces. In particular, in [17] the authors explored the boundedness from below by positive constants of the non-linearities, the positivity of the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ and combined sub-supersolution method with fixed point index on these natural settings to prove their main results.

After these works, some natural questions arise: when the problem $(P_{\lambda,\mu})$, on the whole space, has the property of global multiplicity of solutions and how different shapes the extremal curves may have. We have not found any results about these issues in literature up to now even for bounded domains. To begin to answer these questions, we have to have in mind that the lack of compact embbedings of Sobolev spaces into Lebesgue ones prevent us to build a spectral theory and compact operators associated to the problem $(P_{\lambda,\mu})$ on these natural functions spaces. Besides these, unlike to the case of bounded domains, the boundedness of the potential w from below by a positive constant may yields a first principal eigenvalue null, see for example [50].

To overcome these obstacle, we consider appropriated assumptions on w that make possible the space $D^{1,2}(\mathbb{R}^N)$ being compactly embedding into a Lebesgue space weighted by this potential. In this new context, a spectral theory becomes possible, which is essential in our approach to show non-existence of solutions to problem $(P_{\lambda,\mu})$. Among the assumptions that make possible to show the existence of a principal first eigenvalue, we should have $w \in L^1(\mathbb{R}^N)$ and this prevent us to use the blow up method to prove priori estimates for solutions of the problem $(P_{\lambda,\mu})$, because of $\liminf_{x\to\infty} w(x) = 0$.

More specifically, let us assume (see an example of such w in [5]):

$(W)_1$: $w \in C^{\alpha}_{loc}(\mathbb{R}^N, \mathbb{R}^+_0)$ for some $\alpha \in (0,1)$ and there exists $W \in C(\mathbb{R}^+_0, \mathbb{R}^+_0)$ such that $0 < w(x) \leq W(|x|)$ for all $x \in \mathbb{R}^N \setminus \{0\}$,

$(W)_2$: $\displaystyle\int_{\mathbb{R}^N} |x|^{2-N} W(|x|)dx < \infty$,

$(W)_3$: $W \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,

$(W)_4$: $\displaystyle\int_{\mathbb{R}^N} \frac{W(|y|)}{|x-y|^{N-2}} dy \leq \frac{C}{|x|^{N-2}}$ for all $x \in \mathbb{R}^N \setminus \{0\}$ and for some constant $C > 0$.

Under the hypotheses $(W)_1 - (W)_4$, it was proved in [5, 55] that the problem

$$\begin{cases} -\Delta u = \lambda w(x) u \ in \ \mathbb{R}^N, \\ u > 0 \ in \ \mathbb{R}^N, \quad u(x) \overset{|x|\to\infty}{\longrightarrow} 0 \end{cases} \tag{A}$$

has a first eigenvalue $\delta_1 > 0$ with positive eigenfunction associated $\phi_1 \in D^{1,2}(\mathbb{R}^N)$. Moreover, δ_1 is simple, isolated and any eigenfunction associated to it has a defined signal.

After this, we can fix our assumptions on the non-linearities f_i, g_i for $i \in \{1, 2\}$.

$(H)_1$: $f_i, g_i \in C^{\alpha(r)}((-r, r), \mathbb{R}^+_0)$, for each $r > 0$ and some $\alpha(r) \in (0,1)$,

$(H)_2$: $0 < \inf\limits_{s\in\mathbb{R}} g_i(s) \leq \sup\limits_{s\in\mathbb{R}} g_i(s) < \infty$,

$(H)_3$: $g_i(s_1) \leq g_i(s_2)$ for $s_1 \leq s_2$,

$(H)_4$: $\displaystyle\frac{\delta_1}{g_i(0)} < \liminf_{s\to\infty} \frac{f_i(s)}{s} \leq \infty$, where $\delta_1 > 0$ is the first eigenvalue of (A),

$(H)_5$: there exist $p_1, p_2 > 0$ and $q_1, q_2 \in (1, \frac{N}{N-2})$ such that $\displaystyle\lim_{s\to\infty} \frac{f_i(s)}{s^{q_i}} = p_i$.

To state our main results, let us set that a pair of functions $(u, v) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ is a solution of $(P_{\lambda,\mu})$ if $u, v > 0$ in $\mathbb{R}^N$; $u(x), v(x) \overset{|x|\to\infty}{\longrightarrow} 0$, and

$$\int \nabla u \nabla \phi dx = \lambda \int w(x) f_1(u) g_1(v) \phi dx \text{ and } \int \nabla v \nabla \psi dx = \mu \int w(x) f_2(v) g_2(u) \psi dx$$

for all $(\phi, \psi) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$.

First we are going to prove the existence of a **bounded extremal curve** for a global multiplicity result of radially-symmetric positive solutions for $(P_{\lambda,\mu})$, that is, solutions (u, v) of $(P_{\lambda,\mu})$ satisfying $(u(x), v(x)) = (u(|x|), v(|x|))$ for every $x \in \mathbb{R}^N$.

Theorem 0.0.6 *Assume $(W)_1-(W)_4,(H)_1-(H)_5$ for $i=1,2$ and that w is radially symmetric. Then:*

a) there exists a continuous simple arc $\tilde{\Gamma}=\{(\lambda(t),\mu(t)):\ t>0\}$, with $0<\lambda(t)$ non-increasing, $0<\mu(t)$ non-decreasing and $\mu(t)=t\lambda(t)$, connecting $(\tilde{\lambda}_*,0)$ and $(0,\tilde{\mu}_*)$, for some $\tilde{\lambda}_*,\tilde{\mu}_*>0$, that separates $\mathbb{R}_0^+\times\mathbb{R}_0^+$ into two disjoint open subsets $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ such that system $(P_{\lambda,\mu})$ has no radially symmetric positive solutions, at least one or at least two radially symmetric positive solutions according to (λ,μ) belongs to $\tilde{\Theta}_2,\tilde{\Gamma}$ or $\tilde{\Theta}_1$, respectively. Moreover, $\tilde{\Gamma}\cup[0,\tilde{\lambda}_*]\cup[0,\tilde{\mu}_*]=\partial\tilde{\Theta}_1$,

b) there exists $\tilde{\lambda}^*\geq\tilde{\lambda}_*$ and $\tilde{\mu}^*\geq\tilde{\mu}_*$ such that the system $(P_{\lambda,\mu})$ has no radially symmetric positive solution for $(\lambda,\mu)\in\left\{(\lambda,0):\lambda>\tilde{\lambda}^*\right\}\cup\left\{(0,\mu):\mu>\tilde{\mu}^*\right\}$, at least one semi-trivial radially symmetric positive solution for $(\lambda,\mu)\in\left\{(\tilde{\lambda}^*,0),(0,\tilde{\mu}^*)\right\}$ or at least two semi-trivial radially symmetric positive solutions for $(\lambda,\mu)\in\left\{(\lambda,0):\lambda<\tilde{\lambda}^*\right\}\cup\{(0,\mu):\mu<\tilde{\mu}^*\}$.

Our second result does not require w be necessarily radially symmetric, but we are not able to prove a global multiplicity result. Without the assumption of symmetry for w the region of existence of solutions given in the theorem below may be bigger than $\tilde{\Theta}_1$.

Theorem 0.0.7 *Assume that $(H)_1-(H)_4$ for $i=1,2$ and $(W)_1-(W)_4$ hold. Then:*

a) there exists a continuous simple arc Γ, with the same properties as those one in Theorem 0.0.6, which separates $\mathbb{R}_0^+\times\mathbb{R}_0^+$ into two disjoint open subsets Θ_1 and Θ_2 such that system $(P_{\lambda,\mu})$ has no positive solution and has at least one according to (λ,μ) belongs to Θ_2 and Θ_1, respectively. Moreover, $\Gamma\cup[0,\lambda_*]\cup[0,\mu_*]=\partial\Theta_1$ for some $\lambda_*,\mu_*>0$,

b) there exists $\lambda^*\geq\lambda_*$ and $\mu^*\geq\mu_*$ such that the system $(P_{\lambda,\mu})$ has no positive solutions for $(\lambda,\mu)\in\{(\lambda,0):\lambda>\lambda^*\}\cup\{(0,\mu):\mu>\mu^*\}$ and at least one for $(\lambda,\mu)\in\{(\lambda,0):\lambda<\lambda^*\}\cup\{(0,\mu):\mu<\mu^*\}$.

In the next Corollary, the solutions are not necessary radially symmetric, but the potential w is still one.

Corollary 0.0.1 *Assume that $(W)_1-(W)_4$, $(H)_1-(H)_5$ for $i=1,2$ hold and w is radially symmetric. Let $\tilde{\Theta}_1,\tilde{\Gamma},\Theta_1$ and Θ_2 as in Theorems 0.0.6 and 0.0.7. If $\Theta_1\backslash\overline{\tilde{\Theta}}_1\neq\emptyset$, then the system $(P_{\lambda,\mu})$ has no positive solution, at least one and at least two ones according to (λ,μ) in $\Theta_2,\tilde{\Gamma}$ or $\Theta_1\setminus\tilde{\Gamma}$, respectively.*

Now for $i=1,2$ let us assume:

$(H)_6$: $f_i(s_1) \leq f_i(s_2)$ for $s_1 \leq s_2$,

$(H)_7$: $0 < \lim\limits_{t\to\infty} \dfrac{g_i(t)}{t} \leq \infty$,

$(H)_8$: $\lim\limits_{t\to\infty} \dfrac{f_i(t)}{t} = 0$.

An example of functions satisfying $(H)_6 - (H)_8$ are as follows:

$$f_1(s) = \pi + arctg(s),\ g_1(t) = e^{\theta_1 t},\ f_2(t) = \pi + arctg(t),\ g_2(s) = e^{\theta_2 s} \quad \forall s, t \in \mathbb{R},$$

where $\theta_1, \theta_2 > 0$ are constant.

In our next result the **extremal curve is unbounded in both directions** λ and μ.

Theorem 0.0.8 *Assume that $(W)_1 - (W)_4$, $(H)_1, (H)_3$ and $(H)_6 - (H)_8$ for $i = 1, 2$ hold. Then there exists a continuous simple arc $\Gamma = \{(\lambda(t), \mu(t)) : t > 0\}$, with $0 < \lambda(t)$ non-increasing; $0 < \mu(t)$ non-decreasing; $\mu(t) = t\lambda(t)$; $\lim\limits_{t\to 0} \Gamma(t) = (\infty, 0)$; and $\lim\limits_{t\to\infty} \Gamma(t) = (0, \infty)$, that separates $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ into two disjoint open subsets Θ_1 and Θ_2 such that the system $(P_{\lambda,\mu})$ has no positive solution and has at least one according to (λ, μ) belongs to Θ_2 and Θ_1, respectively.*

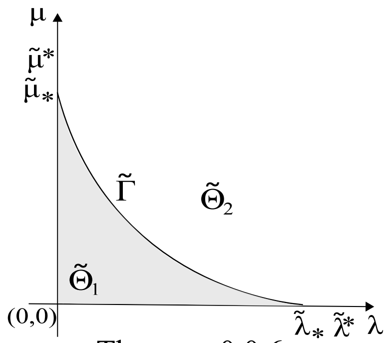
In the next theorem the **extremal curve is bounded in the direction λ and unbounded in the direction μ**.

Theorem 0.0.9 *Assume $(W)_1 - (W)_4$, $(H)_1, (H)_3, (H)_6$ for $i = 1, 2$ hold. Suppose that $(H)_2, (H)_4$ are satisfied for $i = 1$ and $(H)_7 - (H)_8$ are satisfied for $i = 2$. Then there exists a continuous simple arc $\Gamma = \{(\lambda(t), \mu(t)) : t > 0\}$, with $0 < \lambda(t)$ non-increasing; $0 < \mu(t)$ non-decreasing; $\mu(t) = t\lambda(t)$; $\lim\limits_{t\to 0} \Gamma(t) = (\lambda_*, 0)$ for some $\lambda_* > 0$; and $\lim\limits_{t\to\infty} \Gamma(t) = (0, \infty)$, that separates $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ into two disjoint open subsets Θ_1 and Θ_2 such that the system $(P_{\lambda,\mu})$ has no positive solution and has at least one according to (λ, μ) belongs to Θ_2 and Θ_1, respectively.*

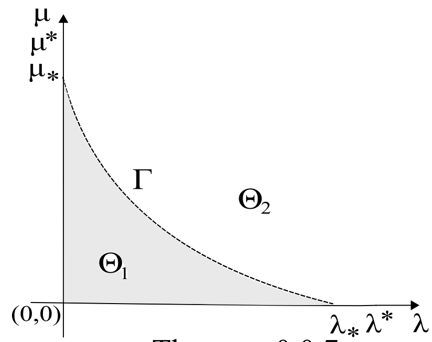
In the next theorem the **extremal curve is bounded in the direction μ and unbounded in the direction λ**.

Theorem 0.0.10 *Assume $(W)_1 - (W)_4$, $(H)_1, (H)_3, (H)_6$ for $i = 1, 2$ hold. Suppose that $(H)_7 - (H)_8$ are satisfied for $i = 1$ and $(H)_2, (H)_4$ are satisfied for $i = 2$. Then there exists a continuous simple arc $\Gamma = \{(\lambda(t), \mu(t)) : t > 0\}$, with $0 < \lambda(t)$ non-increasing; $0 < \mu(t)$ non-decreasing; $\mu(t) = t\lambda(t)$; $\lim\limits_{t\to 0} \Gamma(t) = (\infty, 0)$; and $\lim\limits_{t\to\infty} \Gamma(t) = (0, \mu_*)$ for some $\mu_* > 0$, that separates $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ into two disjoint open subsets Θ_1 and Θ_2 such that the system $(P_{\lambda,\mu})$ has no positive solution and has at least one according to (λ, μ) belongs to Θ_2 and Θ_1, respectively.*

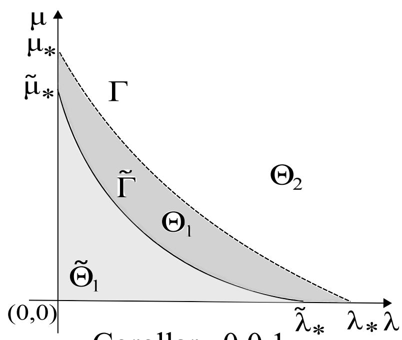
To ease the interpretation of the conclusions of the above results, we draw them in the below graphics.



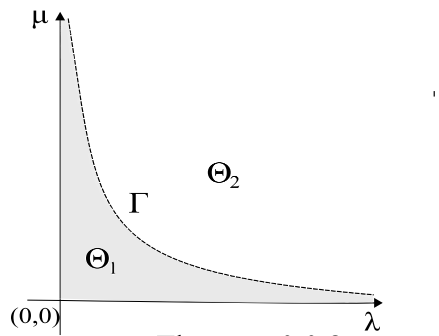
Theorem 0.0.6



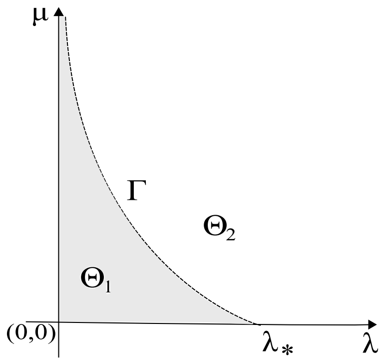
Theorem 0.0.7



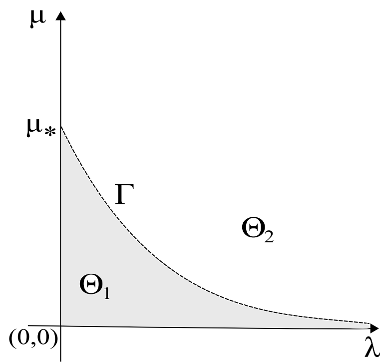
Corollary 0.0.1



Theorem 0.0.8



Theorem 0.0.9



Theorem 0.0.10

Below, let us highlight some contributions of this work to the literature:

a) Theorem 0.0.6 is new, because it presents a complete picture of the global multiplicity of radially symmetric solutions for elliptic systems with multi-parameters in the whole space,

b) Theorem 0.0.7 and Corollary 0.0.1 partially extend the main result in [17] to the whole space,

c) Theorem 3.1.1 extends to the whole space a similar result proved in [17] for bounded domains . The key point to prove Theorem 3.1.1 is that the potential

$w : \mathbb{R}^N \to (0, \infty)$ has to have appropriated properties to allow us to work with topological degree theory,

d) our approach contributes with a fine analysis to overcome the natural difficulties that problems in $\mathbb{R}^N$ bring up,

e) to the best of our knowledge, Theorem 0.0.8, 0.0.9 and 0.0.10 are new and they have not been considered in literature up to now even for bounded domains. They give a complete description of **unbounded regions of existence and nonexistence** of positive weak solutions for the problem $(P_{\lambda,\mu})$. The key point to prove them is Theorem 3.1.1 together with the representation of Riesz given in (3.1).

We would like to point out that the results of Chapter 3 have already been accepted for publication in the paper [6].

This thesis has the following structure. In Chapter 1, in the first section we study some topological structures associated to energy functional Φ_λ and apply them in the next sections. In Section 1.2, we show the multiplicity of solutions for $0 < \lambda < \lambda_*$. In Section 1.3, taking advantage of the solutions obtained in Section 1.2 and the results obtained in Section 1.1, we show multiplicity of solutions for $\lambda = \lambda_*$. In section 1.4, by controlling the energy levels we prove multiplicity of solutions for $\lambda_* < \lambda$. Finally, in the last section, we prove the Theorem 0.0.1 and Theorem 0.0.2.

In Chapter 2, we present in the first section a new concept of critical point for non-differentiable functionals and we prove abstract theorem for this class of functionals. This theorem is new in the literature and we will apply it to prove that certain minimums over the Nehari manifold are solutions of system $(\tilde{P}_{\lambda,\mu})$. In Section 2.2, we start by proving that certain minimums over the Nehari manifold are solutions of system $(\tilde{P}_{\lambda,\mu})$. Besides this, we introduce the modified problems $(\tilde{P}_{\lambda,\theta\lambda})$, for each $\theta > 0$ fixed, and study some topological structures associated to the energy functional $\Phi_{\lambda,\theta\lambda}$, which help to build the curves $\Gamma_0, \tilde{\Gamma}$ as claimed in Theorem 0.0.3. In Section 2.3, we show the multiplicity of solutions for $0 < \lambda < \lambda_*(\theta)$.

In Section 2.4, we show multiplicity of solutions for $\lambda = \lambda_*(\theta)$. In section 2.5, controlling the energy levels we prove multiplicity of solutions for $\lambda_*(\theta) < \lambda$ and we prove the theorem 0.0.3. Finally, in the last section, we prove the Theorem 0.0.4 and the Theorem 0.0.5.

In Chapter 3, in first section we introduce the spaces where we will work and we prove a sub-supersolution theorem that will be essential to prove the multiplicity of positive solutions to system $(P_{\lambda,\mu})$. This result extends to the whole space a similar result proved in [17] for bounded domains. In Section 3.2, we build the extremal curves claimed in the Theorems 0.0.6-0.0.10 and Corollary 0.0.1. In the last section we prove the Theorems 0.0.6-0.0.10 and Corollary 0.0.1.

Notation and Terminology

- c and C are possibly different positive constants which may change from line to line,
- $b^+ = \max\{b, 0\}$ is the positive part of the function b,
- $\mathbb{S} = \{u \in \mathcal{B} : ||u|| = 1\}$ is the unitary sphere, where where $(\mathcal{B}, \|\cdot\|)$ is a Banach space,
- $\langle \Phi'(u), \psi \rangle$ denotes the Gâteaux derivative of Φ at u with respect to the direction $\psi \in \mathcal{B}$,
- $|B_1(0)|$ is the volume of the unit ball in $\mathbb{R}^N$,
- if Ω is a measurable set in $\mathbb{R}^N$, we denote by $\mathcal{L}(\Omega)$ the Lebesgue measure of Ω,
- The spaces $\mathbb{R}^N$ are equipped with the Euclidean norm $\sqrt{x_1^2 + \cdots + x_N^2}$,
- $B_r(x)$ denotes the ball centered at $x \in \mathbb{R}^N$ with radius $r > 0$,
- the Banach space $\mathcal{B} \times \mathcal{B} = \{(u, v) : u, v \in X\}$ is equipped with the norm $\|(u, v)\| = \max\{\|u\|, \|v\|\}$, where $(\mathcal{B}, \|\cdot\|)$ is a Banach space as well,
- $B(u, r)$ denotes the ball centered at $u \in \mathcal{B} \times \mathcal{B}$ with radius $r > 0$,
- the notation $(a, b) > (c, d)$ means $a > c$ and $b > d$. Similarly, $(a, b) \geq (c, d)$ means $a \geq c$ and $b \geq d$ for all $(a, b), (c, d) \in \mathbb{R}^2$,
- for $(a, b), (c, d) \in \mathbb{R}^2$ denote by $](a, b), (c, d)] = \{(1 - t)(a, b) + t(c, d) : 0 < t \leq 1\}$ and $](a, b), (c, d)[= \{1 - t)(a, b) + t(c, d) : 0 < t < 1\}$,

- $\lim_{|x|\to\infty}(u(x), v(x)) = (\lim_{|x|\to\infty} u(x), \lim_{|x|\to\infty} v(x))$ for functions $u, v : \mathbb{R}^N \longrightarrow \mathbb{R}$,
- $dist(u, v) = \inf_{x\in\mathbb{R}^N} |u(x) - v(x)|$ for functions $u, v : \mathbb{R}^N \longrightarrow \mathbb{R}$,
- $[0, \tilde{\lambda}] = \left\{(\lambda, 0) : 0 \leq \lambda \leq \tilde{\lambda}\right\}$ and $[0, \tilde{\mu}] = \{(0, \mu) : 0 \leq \mu \leq \tilde{\mu}\}$ for any $\tilde{\lambda}, \tilde{\mu} > 0$,
- $deg(I - T, \mathcal{W}, 0)$ denotes the Leray-Schauder degree of $I - T$ in $\mathcal{W}$ with respect to 0, where $\mathcal{W} \subset \mathcal{B}$ is a bounded open set in a Banach space $\mathcal{B}$ and $T : \overline{\mathcal{W}} \longrightarrow \mathcal{B}$ is a compact operator.

Chapter 1

Multiplicity of solutions for singular-superlinear Schrödinger equations with indefinite-sign potential

In this chapter, we show the multiplicity and non-existence of positive solutions for the following superlinear and subcritical Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = \lambda a(x)u^{-\gamma} + b(x)u^{p} \text{ in } \mathbb{R}^N, \\ u > 0,\ \mathbb{R}^N,\ \displaystyle\int_{\mathbb{R}^N} Vu^2dx < \infty,\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_λ}$$

when the potential b may change its sign, $0 < a \in L^{\frac{2}{1+\gamma}}(\mathbb{R}^N),\ b^{+} \neq 0, b \in L^{\infty}(\mathbb{R}^N)$, $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function, $0 < \gamma < 1 < p < 2^{*} - 1,\ N \geq 3$ and $\lambda > 0$ is a real positive parameter.

To show the multiplicity of solutions we use the Nehari manifold and the fibering method of Pohozaev for non-differentiable functionals. We were motivated by Silva-Macedo [58] and would like to point out that due to the lack of Gâteaux differentiability of the energy functional Φ_λ, the ideas in [58] do not apply directly here. Thus, through of new proofs and new arguments we generalize some results of [58] to prove the Theorem 0.0.1. As we already mentioned, we intend to minimize the functional Φ_λ over the Nehari manifold when the intersection of its connected components is non empty, and we overcome these difficulties by exploring topological structures of that boundary to build non-empty sets whose boundaries have empty intersection and minimizing over them by controlling the energy level. To achieve this, we need of estimates in the projectors that are new even in the non-singular case as in [58].

This chapter follows the following structure. In the first section, we study some toplogical structures associated to the energy funcional associated to the problem (P_λ). So, we introduce the Nehari manifold associated with the problem (P_λ) and study some of its properties as well. In the Section 1.2, we show the multiplicity of solutions to problem (P_λ) to $\lambda \in (0, \lambda_*)$, where

$$\lambda_* = \left(\frac{1+\gamma}{p+\gamma}\right)^{\frac{1+\gamma}{p-1}} \left(\frac{p-1}{p+\gamma}\right) \inf_{0 \lneqq u \in X, \int_{\mathbb{R}^N} b|u|^{p+1} > 0} \frac{(||u||^2)^{\frac{p+\gamma}{p-1}}}{\left[\int_{\mathbb{R}^N} b|u|^{p+1} dx\right]^{\frac{1+\gamma}{p-1}} \left[\int_{\mathbb{R}^N} a|u|^{1-\gamma} dx\right]}.$$

In Section 1.3, using the results obtained in the sections 1.1 and 1.2, we show the multiplicity of solutions to (P_λ) when $\lambda = \lambda_*$. Here we point out an additional difficulty that we had what is to prove that the sequences of solutions u_{λ_n} and w_{λ_n} obtained in Section 1.2 converge strongly, with $\lambda_n \uparrow \lambda_*$ to functions u_{λ_*} and w_{λ_*}, respectively, which are solutions of problem (P_{λ_*}). This is due to the lack of comparison principle.

In Section 1.4, we show the multiplicity of solutions to (P_λ) when λ is bigger than λ_*, but close to it. Finally, in Section 1.5 we prove the Theorems 0.0.1 and 0.0.2. To show non-existence of solutions claimed in Theorem 0.0.2, we were motivated by Figueiredo-Gossez-Ubila [23, 22]. To prove it, we use interior regularity and an integration by parts formula given in [23, 22], that appears in it an eigenfunction associated to an eigenvalue problem in a bounded domain. To the best of our knowledge this result has not still been considered when the potential b changes its signal.

For convenience, below we recall once again all the assumptions required in the potential V throughout this chapter.

Let us assume that $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function that satisfies

$(V)_0$ $V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0$, and one of the following conditions:

- (i) $\lim_{|x| \to \infty} V(x) = \infty$;
- (ii) $1/V \in L^1(\mathbb{R}^N)$;
- (iii) for each $M > 0$ given the $\mathcal{L}(\{x \in \mathbb{R}^N : V(x) \leq M\}) < \infty$.

We also remember that

$$X = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\},$$

and $\Phi_\lambda : X \to \mathbb{R}$, defined by

$$\Phi_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx - \frac{1}{p+1}\int_{\mathbb{R}^N} b(x)|u|^{p+1}dx,$$

is the energy functional associated to the problem (P_λ), where

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx.$$

1.1 Topological structures associated to the energy functional

Throughout this chapter, let us assume the hypotheses of Theorem 0.0.1 to prove some topological properties for the functional Φ_λ. Let us endow X with the inner product

$$(u, w) = \int_{\mathbb{R}^N} \nabla u \nabla w + V(x)uw dx,$$

which turns X into a Hilbert space with induced norm given by $||u||^2 = (u, u)$. As a consequence, one deduces immediately from $(V)_0$ that X is embedded continuously into $H^1(\mathbb{R}^N)$. The below Lemma was proved in [7, 20, 51].

Lemma 1.1.1 *The subspace X is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, 2^*]$ and compactly embedded for all $q \in [2, 2^*)$.*

It follows from Lemma 1.1.1 that

Lemma 1.1.2 *If $\lambda > 0$ then Φ_λ is a continuous and weakly lower semicontinuous functional.*

Proof We prove that Φ_λ is weakly lower semicontinuous (the proof of the continuity is almost similar). Take $\{u_n\} \subset X$ such that $u_n \rightharpoonup u$. It follows from Lemma 1.1.1 that

$$u_n \to u \text{ in } L^q(\mathbb{R}^N),\ u_n \to u \text{ a.e. in } \mathbb{R}^N \text{ and } |u_n(x)| \leq g_q(x) \text{ a.e. in } \mathbb{R}^N.$$

for some $g_q \in L^q(\mathbb{R}^N)$. Since $0 < \gamma < 1$, we obtain

$$||u_n|^{1-\gamma} - |u|^{1-\gamma}|^{\frac{2}{1-\gamma}} \to 0 \text{ and } ||u_n|^{1-\gamma} - |u|^{1-\gamma}|^{\frac{2}{1-\gamma}} \leq 2^{\frac{2}{1-\gamma}} g_2^2 \in L^1(\mathbb{R}^N) \text{ a.e. in } \mathbb{R}^N.$$

From $a \in L^{2/(1+\gamma)}(\mathbb{R}^N)$, the Hölder inequality and the Lebesgue dominated convergence theorem, we conclude that

$$\Big|\int_{\mathbb{R}^N} a(x)(|u_n|^{1-\gamma} - |u|^{1-\gamma})dx\Big| \leq [\int_{\mathbb{R}^N} a^{\frac{2}{1+\gamma}}dx]^{\frac{1+\gamma}{2}}[\int_{\mathbb{R}^N} ||u_n|^{1-\gamma} - |u|^{1-\gamma}|^{\frac{2}{1-\gamma}}dx]^{\frac{1-\gamma}{2}} \to 0,$$

Again, by using Lemma 1.1.1 and $b \in L^\infty(\mathbb{R}^N)$, we have that $\int_{\mathbb{R}^N} b(x)|u_n|^{p+1}dx \to \int_{\mathbb{R}^N} b(x)|u|^{p+1}dx$ holds which completes the proof.

∎

Since we are interested in positive solutions, let us constrain Φ_λ to the cone of non-negative functions of X, that is,

$$X_+ = \{u \in X \setminus \{0\} : u \geq 0\}.$$

Define the C^∞-fiber map $\phi_{\lambda,u} : (0,\infty) \to \mathbb{R}$ by

$$\phi_{\lambda,u}(t) = \Phi_\lambda(tu) = \frac{t^2}{2}||u||^2 - \frac{t^{1-\gamma}\lambda}{1-\gamma}\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N} b(x)|u|^{p+1}dx,$$

for each $u \in X_+$ and $\lambda > 0$ given. It is clear that

$$\phi'_{\lambda,u}(t) = t||u||^2 - t^{-\gamma}\lambda\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx - t^p\int_{\mathbb{R}^N} b(x)|u|^{p+1}dx,$$

$$\phi''_{\lambda,u}(t) = ||u||^2 + \gamma t^{-\gamma-1}\lambda\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx - pt^{p-1}\int_{\mathbb{R}^N} b(x)|u|^{p+1}dx \tag{1.1}$$

and if $u \in X_+$ is a solution of (P_λ), then $u \in \mathcal{N}_\lambda$, where

$$\begin{aligned}\mathcal{N}_\lambda &\equiv \Big\{u \in X_+ : ||u||^2 - \int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx - \lambda\int_{\mathbb{R}^N} b(x)|u|^{p+1}dx = 0\Big\}\\ &= \Big\{u \in X_+ : \phi'_{\lambda,u}(1) = 0.\Big\}.\end{aligned}$$

Although $\mathcal{N}_\lambda$ does not have enough regularity, let us refer to it as the Nehari manifold associated to (P_λ) from now on. It is classical to split it in three disjoint sets

$$\begin{aligned}\mathcal{N}_\lambda^- &\equiv \Big\{u \in \mathcal{N}_\lambda : ||u||^2 + \gamma\lambda\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx - p\int_{\mathbb{R}^N} b(x)|u|^{p+1}dx < 0\Big\}\\ &= \Big\{u \in \mathcal{N}_\lambda : \phi''_{\lambda,u}(1) < 0\Big\},\\ \mathcal{N}_\lambda^+ &\equiv \Big\{u \in \mathcal{N}_\lambda : ||u||^2 + \gamma\lambda\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx - p\int_{\mathbb{R}^N} b(x)|u|^{p+1}dx > 0\Big\}\\ &= \Big\{u \in \mathcal{N}_\lambda : \phi''_{\lambda,u}(1) > 0\Big\},\\ \mathcal{N}_\lambda^0 &\equiv \Big\{u \in \mathcal{N}_\lambda : ||u||^2 + \gamma\lambda\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx - p\int_{\mathbb{R}^N} b(x)|u|^{p+1}dx = 0\Big\}\\ &= \Big\{u \in \mathcal{N}_\lambda : \phi''_{\lambda,u}(1) = 0\Big\}.\end{aligned}$$

We will study the structure of the sets $\mathcal{N}_\lambda^-, \mathcal{N}_\lambda^0, \mathcal{N}_\lambda^+$ and show existence of solutions on $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^+$. The easiest case is when $\mathcal{N}_\lambda^0 = \emptyset$. One of our main contributions to the literature of singular problems is to show existence of solutions on $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^+$ beyond the extremal value, for which $\mathcal{N}_\lambda^0$ is not empty anymore.

The next proposition is straightforward.

Proposition 1.1.1 *Let $u \in X_+$ and $\lambda > 0$. If $\int b|u|^{p+1}dx \leq 0$, then $\phi_{\lambda,u}$ has only one critical point at $t_\lambda^+(u) \in (0,\infty)$, which satisfies $\phi''_{\lambda,u}(t_\lambda^+(u)) > 0$. If $\int b|u|^{p+1}dx > 0$, then there are three possibilities:*

(I) there are only two critical points for $\phi_{\lambda,u}$. The first one is $t_\lambda^+(u)$ with $\phi''_{\lambda,u}(t_\lambda^+(u)) > 0$ and the second one is $t_\lambda^-(u)$ with $\phi''_{\lambda,u}(t_\lambda^-(u)) < 0$. Moreover, $\phi_{\lambda,u}$ is decreasing over the intervals $[0, t_\lambda^+(u)], [t_\lambda^-(u), \infty)$ and increasing over the the interval $[t_\lambda^+(u), t_\lambda^-(u)]$ (evidently $0 < t_\lambda^+(u) < t_\lambda^-(u)$),

(II) there is only one critical point $t_\lambda^0(u) > 0$ for $\phi_{\lambda,u}$, which is an inflection point. Moreover, $\phi_{\lambda,u}$ is decreasing for $t > 0$,

(III) the function $\phi_{\lambda,u}$ is decreasing for $t > 0$ and has no critical points.

Let us study the set $\mathcal{N}_\lambda^0$. One can easily see that if $u \in \mathcal{N}_\lambda^0$ then $\int_{\mathbb{R}^N} b|u|^{p+1}dx > 0$, therefore, we introduce the set

$$Z^+ \equiv \left\{ u \in X_+ : \int_{\mathbb{R}^N} b|u|^{p+1}dx > 0 \right\}.$$

Observe that Z^+ is a cone. For $u \in Z^+$ consider the system

$$\phi'_{\lambda,u}(t) = \phi''_{\lambda,u}(t) = 0,$$

that is

$$\begin{cases} t||u||^2 - t^{-\gamma}\lambda \displaystyle\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx - t^p \int_{\mathbb{R}^N} b(x)|u|^{p+1}dx = 0, \\ ||u||^2 + \gamma\lambda t^{-\gamma-1} \displaystyle\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx - pt^{p-1} \int_{\mathbb{R}^N} b(x)|u|^{p+1}dx = 0. \end{cases}$$

The system has a unique solution which is given by $(t(u), \lambda(u))$, where

$$\begin{cases} t(u) = \left(\dfrac{1+\gamma}{p+\gamma}\right)^{\frac{1}{p-1}} \left[\dfrac{||u||^2}{\int_{\mathbb{R}^N} b|u|^{p+1}dx}\right]^{\frac{1}{p-1}} \\ \lambda(u) = C(\gamma,p)\dfrac{(||u||^2)^{\frac{p+\gamma}{p-1}}}{\left[\int_{\mathbb{R}^N} b|u|^{p+1}dx\right]^{\frac{1+\gamma}{p-1}} \left[\int_{\mathbb{R}^N} a|u|^{1-\gamma}dx\right]}, \end{cases} \tag{1.2}$$

where

$$C(\gamma,p) \equiv \left(\frac{1+\gamma}{p+\gamma}\right)^{\frac{1+\gamma}{p-1}} \left(\frac{p-1}{p+\gamma}\right).$$

From the definition of $\lambda(u)$ we conclude that

Proposition 1.1.2 *Suppose that $u \in Z^+$. Then, if $\lambda \in (0, \lambda(u))$ the fiber map $\phi_{\lambda,u}$ satisfies (I) of Proposition 1.1.1, while $\phi_{\lambda(u),u}$ satisfies (II) and if $\lambda \in (\lambda(u), \infty)$ it must satisfies (III).*

Define

$$\lambda_* = \inf_{u \in Z^+} \lambda(u).$$

Lemma 1.1.3 *The function λ defined in (1.2) is continuous, 0-homogeneous and unbounded from above. Moreover, $\lambda_* > 0$ and there exists $u \in Z^+$ such that $\lambda_* = \lambda(u)$.*

Proof The continuity and 0-homogeneity are obvious. From these properties, it follows that the rest of the proof can be done by considering λ restricted to the set $Z^+ \cap \mathbb{S}$, where $\mathbb{S} = \{u \in X : \|u\| = 1\}$. To prove that λ is unbounded from above, first note that the functional $F_b : X \longrightarrow \mathbb{R}$ defined by $F_b(u) = \int_{\mathbb{R}^N} b|u|^{p+1}dx$ is continuous and therefore $F_b^{-1}((0,\infty)) \cap \mathbb{S}$ is an open set in $\mathbb{S}$. Moreover, since $F_b(tu) = t^{p+1}F_b(u)$ for $t > 0$, it follows that $F_b^{-1}((0,\infty)) \cap \mathbb{S} \neq \mathbb{S}$ and therefore there exists a sequence $\{u_n\} \subset F_b^{-1}((0,\infty)) \cap \mathbb{S}$ such that $F_b(u_n) \to 0$ in X. Consequently

$$\lim_{n\to\infty} \lambda(u_n) = \lim_{n\to\infty} \frac{C(\gamma,p)}{\left[\int_{\mathbb{R}^N} b|u_n|^{p+1}dx\right]^{\frac{1+\gamma}{p-1}} \left[\int_{\mathbb{R}^N} a|u_n|^{1-\gamma}dx\right]} = \infty,$$

which proves that λ is unbounded from above. Now observe that

$$\lambda_* = \inf_{u \in Z^+ \cap S} \lambda(u) \geq cC(\gamma,p)\|a\|_{2/(1+\gamma)}^{-1}\|b\|_\infty^{-1} > 0$$

for some $c > 0$. To end the proof, take $\{u_n\} \subset Z^+ \cap \mathbb{S}$ such that $\lambda(u_n) \to \lambda_*$. So, it follows from Lemma 1.1.1 that

$$u_n \rightharpoonup u \in X, \quad u_n \to u \text{ in } L^q(\mathbb{R}^N) \text{ for each } q \in [2, 2^*) \text{ and } u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N,$$

which lead us to infer that $u \not\equiv 0$. Otherwise, we would have

$$\lambda_* = \lim_{n\to\infty} \lambda(u_n) = \lim_{n\to\infty} \frac{C(\gamma,p)}{\left[\int_{\mathbb{R}^N} b|u_n|^{p+1}dx\right]^{\frac{1+\gamma}{p-1}} \left[\int_{\mathbb{R}^N} a|u_n|^{1-\gamma}dx\right]} = \infty,$$

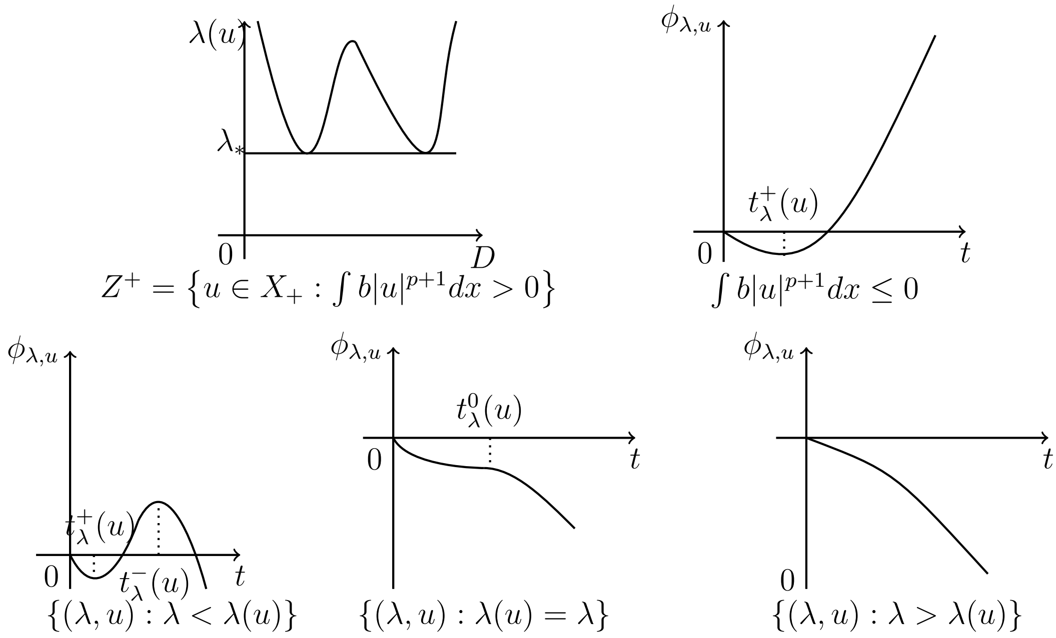
which is an absurd. Let $v = \frac{u}{||u||} \in X_+ \cap \mathbb{S}$. If $u_n \not\to u$ in X, it would follow by the weak lower semi-continuity of the norm that

$$\lambda(v) = \lambda\left(\frac{u}{\|u\|}\right) = \lambda(u) < \liminf \lambda(u_n) = \lambda_*,$$

but this is impossible. It follows that $u \in Z^+ \cap \mathbb{S}$ and $\lambda(u) = \lambda_*$. This ends the proof.

■

Proposition 1.1.1 and Lemma 1.1.3 are described on the following pictures:



From Proposition 1.1.1 and Lemma 1.1.3 we obtain

Lemma 1.1.4 *For each $\lambda > 0$ we have that $\mathcal{N}_\lambda^+, \mathcal{N}_\lambda^- \neq \emptyset$. Moreover:*

a) $\mathcal{N}_\lambda^0 = \emptyset$ *for* $0 < \lambda < \lambda_*$,

b) $\mathcal{N}_\lambda^0 \neq \emptyset$ *for* $\lambda \geq \lambda_*$.

Proof First we will to prove that $\mathcal{N}_\lambda^+, \mathcal{N}_\lambda^- \neq \emptyset$. By Lemma 1.1.3 for each $\lambda > 0$ there exists $u \in Z^+$ such that $\lambda < \lambda(u)$. Thus by Proposition 1.1.2 there exist $t_\lambda^+(u) < t_\lambda^-(u)$ such that $t_\lambda^+(u)u \in \mathcal{N}_\lambda^+$ and $t_\lambda^-(u)u \in \mathcal{N}_\lambda^-$. Hence $\mathcal{N}_\lambda^+ \neq \emptyset, \mathcal{N}_\lambda^- \neq \emptyset$.

To prove *a)* we first note that if $u \in Z^+$ then from Lemma 1.1.3 there holds $\lambda(u) \geq \lambda_*$. Hence, if $\lambda \in (0, \lambda_*)$ it follows from Proposition 1.1.2 that $u \notin \mathcal{N}_\lambda^0$. If $u \notin Z^+$, then $\int_{\mathbb{R}^N} b|u|^{p+1}dx \leq 0$ and by Proposition 1.1.1, $\phi_{\lambda,u}$ has only one critical point at $t_\lambda^+(u) \in (0,\infty)$, which satisfies $\phi''_{\lambda,u}(t_\lambda^+(u)) > 0$ which implies again that $u \notin \mathcal{N}_\lambda^0$. Therefore $\mathcal{N}_\lambda^0 = \emptyset$ for $0 < \lambda < \lambda_*$.

Now we prove b). Indeed, from the definition of $\lambda(u)$ we know that

$$t(u)u \in \mathcal{N}_{\lambda(u)}^0.$$

From Lemma 1.1.3 we know that for each $\lambda \geq \lambda^*$, there exits $u \in Z^+$ such that $\lambda(u) = \lambda$ which ends the proof.

■

Now we characterize the Nehari set $\mathcal{N}_{\lambda_*}^0$. Note that the singular term forces the non-differentiability of the function $\lambda(u)$ at some points, however, at the global minimum points we prove that it has null derivative.

Lemma 1.1.5 *There holds*

$$\mathcal{N}_{\lambda_*}^0 = \left\{ u \in \mathcal{N}_{\lambda_*} : \int_{\mathbb{R}^N} b|u|^{p+1}dx > 0, \lambda(u) = \lambda_* \right\}, \tag{1.3}$$

and

$$(u,\psi) - (p+1)\int_{\mathbb{R}^N} b(x)u^p\psi dx - (1-\gamma)\lambda_* \int_{\mathbb{R}^N} a(x)u^{-\gamma}\psi dx = 0, \ \forall \psi \in X, \tag{1.4}$$

holds for each $u \in \mathcal{N}_{\lambda_*}^0$ given.

Proof The characterization of $\mathcal{N}_{\lambda_*}^0$ is a consequence of Lemma 1.1.3. Let us prove (1.4) by splitting the proof in three steps. First, let rewrite the function $\lambda(u)$ as $\lambda(u) = C(\gamma,p)f(u)g(u)$, where

$$f(u) = \frac{1}{\int_{\mathbb{R}^N} a|u|^{1-\gamma}dx} \quad \text{and} \quad g(u) = \frac{(||u||^2)^{\frac{p+\gamma}{p-1}}}{\left[\int_{\mathbb{R}^N} b|u|^{p+1}dx\right]^{\frac{1+\gamma}{p-1}}}.$$

Step.1. $\langle f'(u),\psi\rangle$ there exists for all $\psi \in X_+$ and for all $u \in \mathcal{N}_{\lambda_*}^0$.

In fact, for such u,ψ given, it follows by continuity that $\int_{\mathbb{R}^N} b|u+t\psi|^{p+1} > 0$ for $t > 0$ small enough. Therefore $g(u+t\psi)$ is well defined for $t > 0$ small enough and $\langle g'(u),\psi\rangle$ there exists. Since, u is the minimum point for $\lambda(u)$, we have that $\lambda(u+t\psi) - \lambda(u) = \lambda(u+t\psi) - \lambda_* \geq 0$, $\forall\ t \geq 0$ enough small, that implies

$$(g(u+t\psi) - g(u))f(u+t\psi) \geq -g(u)(f(u+t\psi) - f(u)).$$

Since,

$$f(u+t\psi) - f(u) = -h(t)^{-2}\left[\int_{\mathbb{R}^N} a|u+t\psi|^{1-\gamma}dx - \int_{\mathbb{R}^N} a|u|^{1-\gamma}dx\right],$$

where

$$h(t) = \theta(t) \int_{\mathbb{R}^N} a|u + t\psi|^{1-\gamma} dx + (1 - \theta(t)) \int_{\mathbb{R}^N} a|u|^{1-\gamma} dx, \ \theta(t) \in [0, 1],$$

is a measurable function such that $h(t) \to h(0) = \int_{\mathbb{R}^N} a|u|^{1-\gamma} dx \neq 0$ with $t \to 0^+$, it follows from Fatou's lemma, that

$$\begin{aligned} \infty > \langle g'(u), \psi \rangle f(u) &\geq g(u) \left[\int_{\mathbb{R}^N} a|u|^{1-\gamma} dx \right]^{-2} \liminf_{t \to 0^+} \int_{\mathbb{R}^N} \frac{a|u + t\psi|^{1-\gamma} - a|u|^{1-\gamma}}{t} dx \\ &\geq g(u) \left[\int_{\mathbb{R}^N} a|u|^{1-\gamma} dx \right]^{-2} (1 - \gamma) \int_{\mathbb{R}^N} aG(x)\psi dx, \end{aligned}$$

where

$$G(x) = \begin{cases} u^{-\gamma}(x), & \text{if } \ u(x) \neq 0, \\ \infty, & \text{if } \ u(x) = 0. \end{cases}$$

So, by taking $\psi > 0, \ \psi \in X$ above, we obtain that $G(x) = u^{-\gamma}(x)$ for all $x \in \mathbb{R}^N$, that is, $u > 0$ in $\mathbb{R}^N$. This implies that $0 < \int_{\mathbb{R}^N} au^{-\gamma}\psi dx < \infty$ for all $\psi \in X_+$. As a consequence, we have $\langle j'(u), \psi \rangle$ there exists, where $j(u) = \int_{\mathbb{R}^N} a|u|^{1-\gamma} dx, \ \psi \in X_+$. To end the proof, we just note that $f(u) = [j(u)]^{-1}$ and hence

$$\langle f'(u), \psi \rangle = -(1 - \gamma) \left[\int_{\mathbb{R}^N} a|u|^{1-\gamma} dx \right]^{-2} \int_{\mathbb{R}^N} au^{-\gamma}\psi dx$$

holds.

Before proving (1.4), let us prove the Step 2 by assuming without loss of generality that $||u|| = 1$.

Step.2. There holds

$$2(u, \psi) - (p + 1) \int_{\mathbb{R}^N} b(x)u^p\psi dx - (1 - \gamma)\lambda_* \int_{\mathbb{R}^N} a(x)u^{-\gamma}\psi dx \geq 0, \ \forall \psi \in X_+. \quad (1.5)$$

Indeed, since $u \in X$ is minimum point of $\lambda(u)$ such that $\int b|u|^{p+1} dx > 0$, we have

$$\left\{ \frac{\left(\frac{2(p+\gamma)}{p-1}\right)(u, \psi) \left[F(u)\right]^{\frac{1+\gamma}{p-1}} - (p + 1) \left(\frac{1+\gamma}{p-1}\right) \left[F(u)\right]^{\frac{2+\gamma-p}{p-1}} \int_{\mathbb{R}^N} b(x)u^p\psi dx}{\left[F(u)\right]^{\frac{2(1+\gamma)}{p-1}}} \right\} [H(u)]^{-1} \quad (1.6)$$

$$-(1 - \gamma) \frac{[H(u)]^{-2} \left[\int_{\mathbb{R}^N} a(x)u^{-\gamma}\psi dx \right]}{\left[F(u)\right]^{\frac{1+\gamma}{p-1}}} \geq 0,$$

for all $\psi \in X_+$, where

$$F(u) = \int_{\mathbb{R}^N} b(x)u^{p+1}dx \text{ and } H(u) = \int_{\mathbb{R}^N} au^{1-\gamma}dx. \tag{1.7}$$

Once using that $u \in \mathcal{N}^0_{\lambda_*}$, we are able to infer that

$$H(u) = \int_{\mathbb{R}^N} au^{1-\gamma}dx = \frac{p-1}{\lambda_*(p+\gamma)} \text{ and } F(u) = \int_{\mathbb{R}^N} b(x)u^{p+1}dx = \frac{1+\gamma}{p+\gamma}.$$

Thus, by using these expressions in (1.6), we get (1.5) after some manipulations.

Finally, by using the characterization (1.3) and adjusting an argument from Graham-Eagle [38], we are able to show the equality (1.4).

Step.3. There holds

$$2(u,\psi) - (p+1)\int_{\mathbb{R}^N} b(x)u^p\psi dx - (1-\gamma)\lambda_*\int_{\mathbb{R}^N} a(x)u^{-\gamma}\psi dx = 0, \ \forall\ \psi \in X.$$

To do this, let us set $\Psi := (u+\epsilon\psi)^+ \in X_+$ for $\epsilon > 0$. Since (1.5) holds, it follows from splitting the whole space in $\{u+\epsilon\psi > 0\}$ and $\{u+\epsilon\psi \le 0\}$, that

$$\begin{aligned}
0 &\le 2(u,\Psi) - (p+1)\int_{\mathbb{R}^N} b(x)u^p\Psi dx - (1-\gamma)\lambda_*\int_{\mathbb{R}^N} a(x)u^{-\gamma}\Psi dx \\
&= 2||u||^2 - (p+1)\int_{\mathbb{R}^N} b(x)u^{p+1}dx - (1-\gamma)\lambda_*\int_{\mathbb{R}^N} a(x)u^{1-\gamma}dx \qquad (1.8)\\
&+\epsilon\left[\int_{\mathbb{R}^N} 2(\nabla u\nabla\psi + V(x)u\psi) - (p+1)b(x)u^p\psi - (1-\gamma)\lambda_* a(x)u^{-\gamma}\psi)dx\right] \\
&-2\int_{\{u+\epsilon\psi\le 0\}} (|\nabla u|^2 + V(x)u^2)dx + (p+1)\int_{\{u+\epsilon\psi\le 0\}} b(x)u^p(u+\epsilon\psi)dx \\
&+(1-\gamma)\lambda_*\int_{\{u+\epsilon\psi\le 0\}} au^{-\gamma}(u+\epsilon\psi)dx - 2\epsilon\int_{\{u+\epsilon\psi\le 0\}} (\nabla u\nabla\psi + V(x)u\psi)dx.
\end{aligned}$$

Now, by using $0<\gamma<1$ and again splitting $\{u+\epsilon\psi \le 0\}$ in $\{u+\epsilon\psi\le 0\}\cap\{b < 0\}$ and $\{u+\epsilon\psi\le 0\}\cap\{b\ge 0\}$, we obtain

$$\begin{aligned}
0 &\le 2(u,\psi) - (p+1)\int_{\mathbb{R}^N} b(x)u^p\psi dx - (1-\gamma)\lambda_*\int_{\mathbb{R}^N} a(x)u^{-\gamma}\psi dx \\
&\le \epsilon\left[\int_{\mathbb{R}^N} 2(\nabla u\nabla\psi + V(x)u\psi) - (p+1)b(x)u^p\psi - (1-\gamma)\lambda_* a(x)u^{-\gamma}\psi)dx\right] \qquad (1.9)\\
&- 2\epsilon\int_{\{u+\epsilon\psi\le 0\}} (\nabla u\nabla\psi + V(x)u\psi)dx + \epsilon(p+1)\int_{\{u+\epsilon\psi\le 0,\{b<0\}\}} b(x)u^p\psi dx.
\end{aligned}$$

Since the measure of the domains of integration $\{u+\epsilon\psi\le 0\}$ and $\{u+\epsilon\psi\le 0\}\cap\{b<0\}$ tends to zero as $\epsilon\to 0$, we have from (1.9) that

$$\begin{aligned}
0 &\le \int_{\mathbb{R}^N} (2(\nabla u\nabla\psi + V(x)u\psi) - (p+1)b(x)u^p\psi - (1-\gamma)\lambda_* a(x)u^{-\gamma}\psi)dx \\
&= 2(u,\psi) - (p+1)\int_{\mathbb{R}^N} b(x)u^p\psi dx - (1-\gamma)\lambda_*\int_{\mathbb{R}^N} a(x)u^{-\gamma}\psi dx
\end{aligned}$$

holds. So, the equality is a consequence of taking $-\psi$ in the above inequality. This ends the proof. ■

The following result will be very important to show multiplicity of solutions to problem (P_λ) at $\lambda = \lambda_*$ and in particular it shows that these solutions belongs to $\mathcal{N}^-_{\lambda_*}$ and $\mathcal{N}^+_{\lambda_*}$, respectively.

Corollary 1.1.1 *The problem (P_{λ_*}) has no solution $u_{\lambda_*} \in \mathcal{N}^0_{\lambda_*}$.*

Proof If there exists a solution $u_{\lambda_*} \in \mathcal{N}^0_{\lambda_*}$ for (P_{λ_*}), then it would follows from Lemma 1.1.5-(1.4) that

$$\int_{\mathbb{R}^N} [(p-1)b(x)u^p_{\lambda_*} - (1+\gamma)\lambda_* a(x)u^{-\gamma}_{\lambda_*}]\psi dx = 0, \forall \psi \in X,$$

that is,

$$(p-1)b(x)u^p_{\lambda_*}(x) = (1+\gamma)\lambda_* a(x)u^{-\gamma}_{\lambda_*}(x) \text{ a.e. in } \mathbb{R}^N.$$

Therefore we have two possibilities. If $b(x) \leq 0$ in $\Omega \subset \mathbb{R}^N$ with $\mathcal{L}(\Omega) > 0$, then $(1+\gamma)a(x)u^{-\gamma}_{\lambda_*} \leq 0$ in Ω, which is an absurd. If $b > 0$ in $\mathbb{R}^N$, then

$$u_{\lambda_*} = \left[\frac{a(x)\lambda_*(1+\gamma)}{b(x)(p-1)}\right]^{\frac{1}{p+\gamma}} \notin X,$$

which is an absurd again. ■

The following result will be essential in order to prove the existence of multiple solutions for $\lambda > \lambda_*$ as well. Due to the presence of the singular term, the arguments used for regular cases, see for instance Corollary 2 in (see [58]), does not work anymore.

Lemma 1.1.6 *The set $\mathcal{N}^0_{\lambda_*}$ is compact.*

Proof First, we note that $u \in \mathcal{N}^0_{\lambda_*}$ implies that

$$(1+\gamma)||u||^2 = (\gamma+p)\int_{\mathbb{R}^N} b(x)|u|^{p+1}dx \text{ and } (p-1)||u||^2 = \lambda_*(\gamma+p)\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx.$$

Thus, by using the Hölder's inequality and the Sobolev embeddings $X \hookrightarrow L^{p+1}(\mathbb{R}^N)$, $L^2(\mathbb{R}^N)$, we obtain

$$c \leq ||u|| \leq C \tag{1.10}$$

for some $c, C > 0$.

Set $\{u_n\} \subset \mathcal{N}^0_{\lambda_*}$. Thus we may assume that $u_n \rightharpoonup u \in X$ in X, $u_n \to u$ in $L^q(\mathbb{R}^N)$ for $q \in [2, 2^*)$ and $u \geq 0$. This, together with (1.10), imply that

$$0 < c \leq \liminf_{n\to\infty} ||u_n||^2 = \left(\frac{(\gamma+p)}{1+\gamma}\right) \lim_{n\to\infty} \int_{\mathbb{R}^N} b(x)|u_n|^{p+1}dx$$
$$= \left(\frac{(\gamma+p)}{1+\gamma}\right) \int_{\mathbb{R}^N} b(x)|u|^{p+1}dx,$$

that is, $u \not\equiv 0$.

Now, we claim that $u_n \to u$ in X. Indeed, if not, it would follow from the continuities of F and H (see (1.7)), that

$$\lambda(u) = \left(\frac{1+\gamma}{p+\gamma}\right)^{\frac{1+\gamma}{p-1}} \left(\frac{p-1}{p+\gamma}\right) \frac{(||u||^2)^{\frac{p+\gamma}{p-1}}}{\left[\int_{\mathbb{R}^N} b|u|^{p+1}dx\right]^{\frac{1+\gamma}{p-1}} \left[\int_{\mathbb{R}^N} a|u|^{1-\gamma}dx\right]} < \liminf \lambda(u_n) = \lambda_*,$$

which is an absurd, therefore, $u_n \to u$ in X and consequently $\mathcal{N}^0_{\lambda_*}$ is compact. This ends the proof. ■

Below, by taking advantage of Lemma 1.1.4, we define for each $\lambda > 0$ the non-empty set

$$\hat{\mathcal{N}_\lambda} = \left\{u \in X_+ : \int_{\mathbb{R}^N} b|u|^{p+1}dx > 0, \phi_{\lambda,u} \text{ has two critical points}\right\},$$

and the set

$$\hat{\mathcal{N}}^+_\lambda = \left\{u \in X_+ : \int_{\mathbb{R}^N} b|u|^{p+1}dx \leq 0\right\},$$

which may be empty.

Let $\overline{\hat{\mathcal{N}_\lambda} \cup \hat{\mathcal{N}}^+_\lambda}$ be the closure of $\hat{\mathcal{N}_\lambda} \cup \hat{\mathcal{N}}^+_\lambda$ with respect to the norm topology. After a few modifications in the proofs of Propositions 2.9, 2.10 and Corollary 2.11 in [58], we have

Proposition 1.1.3 *There holds:*

(i) if $\lambda_1, \lambda_2 \in (0, \lambda_*)$, then $\hat{\mathcal{N}}_{\lambda_1} = \hat{\mathcal{N}}_{\lambda_2}$,

(ii) if $u \in \hat{\mathcal{N}_\lambda} \cup \hat{\mathcal{N}}^+_\lambda$, then $tu \in \hat{\mathcal{N}_\lambda} \cup \hat{\mathcal{N}}^+_\lambda$ for all $t > 0$, that is, $\hat{\mathcal{N}_\lambda} \cup \hat{\mathcal{N}}^+_\lambda$ is a positive cone generated by the set $\mathcal{N}^+_\lambda \cup \mathcal{N}^-_\lambda$. More specifically,

$$\hat{\mathcal{N}_\lambda} \cup \hat{\mathcal{N}}^+_\lambda = \left\{tu : t > 0,\ u \in \mathcal{N}^+_\lambda \cup \mathcal{N}^-_\lambda\right\},$$

(iii) there holds

$$\overline{\hat{\mathcal{N}}_{\lambda_*} \cup \hat{\mathcal{N}}^+_{\lambda_*}} = \hat{\mathcal{N}}_{\lambda_*} \cup \hat{\mathcal{N}}^+_{\lambda_*} \cup \left\{tu : t > 0,\ u \in \mathcal{N}^0_{\lambda_*}\right\} \cup \{0\},$$

(iv) *the function* t_{λ_*} *is continuous and* $P^- : \mathbb{S} \cap \overline{\hat{\mathcal{N}}_{\lambda_*}} \to \mathcal{N}^-_{\lambda_*} \cup \mathcal{N}^0_{\lambda_*}$ *defined by* $P^-(w) = t_{\lambda_*}(w)w$ *is a homeomorphism, where*

$$t_{\lambda_*}(w) = \begin{cases} t^-_{\lambda_*}(w) & \text{if } w \in \hat{\mathcal{N}}_{\lambda_*}, \\ t^0_{\lambda_*}(w) & \text{otherwise,} \end{cases} \tag{1.11}$$

(v) *the function* s_{λ_*} *is continuous and* $P^+ : \mathbb{S} \to \mathcal{N}^+_{\lambda_*} \cup \mathcal{N}^0_{\lambda_*}$ *defined by* $P^+(u) = s_{\lambda_*}(u)u$ *is a homeomorphism, where*

$$s_{\lambda_*}(u) = \begin{cases} t^+_{\lambda_*}(u) & \text{if } u \in \hat{\mathcal{N}}_{\lambda_*} \cup \hat{\mathcal{N}}^+_{\lambda_*} \\ t^0_{\lambda_*}(u) & \text{otherwise,} \end{cases} \tag{1.12}$$

(vi) *the set* $\mathcal{N}^0_{\lambda_*} \subset \mathcal{N}_{\lambda_*}$ *has empty interior, where* $\mathcal{N}_{\lambda_*}$ *is endowed with the induced topology of the norm on* X.

As a fundamental ingredient to show multiplicity of solutions for Problem (P_λ) beyond Nehari's extremal value, we have to prove the continuity and monotonicity of the energy functional constrained on $\mathcal{N}^+_\lambda$ and $\mathcal{N}^-_\lambda$. To do these, let us define $J^+_\lambda : \hat{\mathcal{N}}_\lambda \cup \hat{\mathcal{N}}^+_\lambda \to \mathbb{R}$ and $J^-_\lambda : \hat{\mathcal{N}}_\lambda \to \mathbb{R}$ by

$$J^+_\lambda(u) = \Phi_\lambda(t^+_\lambda(u)u) \text{ and } J^-_\lambda(u) = \Phi_\lambda(t^-_\lambda(u)u) \tag{1.13}$$

and denote their infimum by

$$\tilde{J}^+_\lambda = \inf\left\{J^+_\lambda(u) : u \in \mathcal{N}^+_\lambda\right\} \text{ and } \tilde{J}^-_\lambda = \inf\left\{J^-_\lambda(u) : u \in \mathcal{N}^-_\lambda\right\},$$

respectively.

Unlikely of the non-singular case, the proof of the regularities of the functions $t^+_\lambda(u)$ and $t^-_\lambda(u)$ here are more delicated. However, by inspiring on ideas found in [41], we are able to overcome these obstacles.

Lemma 1.1.7 *Let* $u \in X_+$ *and* $I \subset \mathbb{R}$ *be an open interval such that* $t^\pm_\lambda(u)$ *are well defined for all* $\lambda \in I$. *Then:*

a) *the functions* $I \ni \lambda \to t^\pm_\lambda(u)$ *are* C^∞. *Moreover,* $I \ni \lambda \to t^-_\lambda(u)$ *is decreasing while* $I \ni \lambda \to t^+_\lambda(u)$ *is increasing.*

b) *the functions* $I \ni \lambda \to J^\pm_\lambda(u)$ *are* C^∞ *and decreasing.*

In particular, both claims hold true for $I = (0, \lambda_*)$ *and all* $u \in X_+$ *given.*

Proof Let us begin proving $a)$. To show that $I \ni \lambda \to t_\lambda^{\pm}(u)$ are C^∞, define the C^∞-function F by

$$F(\lambda, t, e, f, g) = et - \lambda f t^{-\gamma} - gt^p \ \text{ for } (\lambda, t, e, f, g) \in I \times (0, \infty) \times \mathbb{R}^3,$$

and set

$$e_1 = ||u||^2, \ f_1 = \int_{\mathbb{R}^N} a|u|^{1-\gamma} dx \text{ and } g_1 = \int_{\mathbb{R}^N} b|u|^{p+1} dx.$$

For $\lambda' \in I$, we have that

$$\frac{\partial F(\lambda', t_{\lambda'}^+(u), e_1, f_1, g_1)}{\partial t} = ||u||^2 + \gamma (t_{\lambda'}^+(u))^{-\gamma-1} \lambda' \int_{\mathbb{R}^N} a(x)|u|^{1-\gamma} dx$$
$$-p(t_{\lambda'}^+(u))^{p-1} \int_{\mathbb{R}^N} b(x)|u|^{p+1} dx > 0,$$

because $t_{\lambda'}^+(u)u \in \mathcal{N}_{\lambda'}^+$. Since

$$F(\lambda', t_{\lambda'}^+(u), e_1, f_1, g_1) = 0 \text{ and } \frac{\partial F}{\partial t}(\lambda', t_{\lambda'}^+(u), e_1, f_1, g_1) > 0,$$

it follows from the implicit function theorem that $t_\lambda^+(u) \in C^\infty((\lambda' - \epsilon, \lambda' + \epsilon), \mathbb{R})$ for some $\epsilon > 0$ and hence, by the arbitrariness of λ', we conclude that the function $I \ni \lambda \to t_\lambda^+(u)$ is C^∞. Moreover, since $F(\lambda, t_\lambda^+(u), e_1, f_1, g_1) = 0$ we also have

$$\frac{\partial F(\lambda, t_\lambda^+(u), e_1, f_1, g_1)}{\partial \lambda} + \frac{\partial F(\lambda, t_\lambda^+(u), e_1, f_1, g_1)}{\partial t} \frac{dt_\lambda^+(u)}{d\lambda} = 0,$$

that is,

$$\frac{dt_\lambda^+(u)}{d\lambda} = \frac{(t_\lambda^+(u))^{-\gamma} \int_{\mathbb{R}^N} a|u|^{1-\gamma} dx}{||u||^2 + \gamma (t_\lambda^+(u))^{-\gamma-1} \lambda \int_{\mathbb{R}^N} a(x)|u|^{1-\gamma} dx - p(t_\lambda^+(u))^{p-1} \int_{\mathbb{R}^N} b(x)|u|^{1+p} dx} > 0,$$

where the last inequality is a consequence of $t_\lambda^+(u)u \in \mathcal{N}_\lambda^+$. Therefore, the function $I \ni \lambda \to t_\lambda^+(u)$ is increasing. In a similar way, we can prove that $I \ni \lambda \to t_\lambda^-(u)$ is C^∞ and decreasing.

Now let us prove $b)$. Since $t_\lambda^+(u) > 0$ and

$$J_\lambda^+(u) = \Phi_\lambda(t_\lambda^+(u)u) = \frac{(t_\lambda^+(u))^2}{2} ||u||^2 - \frac{(t_\lambda^+(u))^{1-\gamma} \lambda}{1-\gamma} \int_{\mathbb{R}^N} a(x)|u|^{1-\gamma} dx$$
$$-\frac{(t_\lambda^+(u))^{p+1}}{p+1} \int_{\mathbb{R}^N} b(x)|u|^{p+1} dx,$$

it follows from item $a)$ the C^∞-regularity for $J_\lambda^+(u)$ with respect to λ. Besides this, we have

$$\frac{dJ_\lambda^+(u)}{d\lambda} = \phi_{\lambda,u}'(t_\lambda^+(u)) \frac{dt_\lambda^+(u)}{d\lambda} - \frac{(t_\lambda^+(u))^{1-\gamma}}{1-\gamma} \int_{\mathbb{R}^N} a(x)|u|^{1-\gamma} dx$$
$$= -\frac{(t_\lambda^+(u))^{1-\gamma}}{1-\gamma} \int_{\mathbb{R}^N} a(x)|u|^{1-\gamma} dx < 0,$$

where we used the fact that $t^+_\lambda(u)u \in \mathcal{N}^+_\lambda$ to obtain the last inequality, that is, $I \ni \lambda \to J^+_\lambda(u)$ is decreasing. Similarly, we can prove that $I \ni \lambda \to J^-_\lambda(u)$ is a continuous and decreasing function. ∎

As a consequence of the monotonicity proved above, after some adjusts on the proof of Corollary 2.15 in [58], we can prove the below Corollary.

Corollary 1.1.2 *Suppose that $u \notin \hat{\mathcal{N}}^+_{\lambda_*}$. Then*

$$\lim_{\lambda\uparrow\lambda_*} t^-_\lambda(u) = t_{\lambda_*}(u),\ \lim_{\lambda\uparrow\lambda_*} t^+_\lambda(u) = s_{\lambda_*}(u)$$

$$\lim_{\lambda\uparrow\lambda_*} J^-_\lambda(u) = \Phi_{\lambda_*}(t_{\lambda_*}(u)u),\ \lim_{\lambda\uparrow\lambda_*} J^+_\lambda(u) = \Phi_{\lambda_*}(s_{\lambda_*}(u)u),$$

where $t_{\lambda_*}(u)$ and $s_{\lambda_*}(u)$ are defined at (1.11) *and* (1.12)*, respectively.*

1.2 Multiplicity of solutions on the interval $0 < \lambda < \lambda_*$

In this section we show the existence of two solutions for problem (P_λ) when $\lambda \in (0, \lambda_*)$. Some ideas are motivated by the work of Hirano-Sacon-Shioji [41]. Like them, first we show the existence of $u_\lambda \in \mathcal{N}^+_\lambda$ and $w_\lambda \in \mathcal{N}^-_\lambda$ such that

$$\Phi_\lambda(u_\lambda) = \tilde{J}^+_\lambda,\ \ \Phi_\lambda(w_\lambda) = \tilde{J}^-_\lambda,$$

$$0 \le \int_{\mathbb{R}^N} \nabla u_\lambda \nabla\psi + V(x)u_\lambda\psi dx - \lambda\int_{\mathbb{R}^N} a(x)u_\lambda^{-\gamma}\psi dx - \int_{\mathbb{R}^N} b(x)u_\lambda^p\psi dx, \forall\psi \in X_+$$

and

$$0 \le \int_{\mathbb{R}^N} \nabla w_\lambda \nabla\psi + V(x)w_\lambda\psi dx - \lambda\int_{\mathbb{R}^N} a(x)w_\lambda^{-\gamma}\psi dx - \int_{\mathbb{R}^N} b(x)w_\lambda^p\psi dx, \forall\psi \in X_+.$$

The next step will be to adjust the arguments used to prove the Step 3 of Lemma 1.1.5 to show that the last inequalities are in fact equalities, that is, $u_\lambda \in \mathcal{N}^+_\lambda$ and $w_\lambda \in \mathcal{N}^-_\lambda$ are solutions for problem (P_λ).

To carry out this strategy, let us begin by proving the next Lemma.

Lemma 1.2.1 *Let $\lambda > 0$. Then:*

a) for all $u \in \mathcal{N}^+_\lambda$, we have that

$$||u||^2 < \frac{\lambda(\gamma+p)}{p-1}\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx \tag{1.14}$$

holds. In particular $\sup\left\{||u|| : u \in \mathcal{N}^+_\lambda\right\} < \infty$.

b) for all $w \in \mathcal{N}_\lambda^-$, we have that

$$||w||^2 < \frac{(\gamma + p)}{(1+\gamma)} \int_{\mathbb{R}^N} b|w|^{p+1} dx \tag{1.15}$$

holds and $\sup\left\{||w|| : w \in \mathcal{N}_\lambda^-, \Phi_\lambda(w) \leq M\right\} < \infty$ *for each $M > 0$ given. Moreover*

$$\inf\left\{||w|| : w \in \mathcal{N}_\lambda^-\right\} > 0.$$

Furthermore,

$$0 > \tilde{J}_\lambda^+ := \inf_{u \in \mathcal{N}_\lambda^+} \Phi_\lambda(u) > -\infty \quad \textit{and} \quad \tilde{J}_\lambda^- := \inf_{w \in \mathcal{N}_\lambda^-} \Phi_\lambda(w) > -\infty. \tag{1.16}$$

Proof Item *a)* is a consequence of $\phi''_{\lambda,u}(1) > 0$, Hölder and Sobolev embedding. The inequalities (1.15) of *b)* and $\inf\left\{||w|| : w \in \mathcal{N}_\lambda^-\right\} > 0$ are direct consequences of $\phi''_{\lambda,u}(1) < 0$, Hölder and Sobolev embedding. Now fix $M > 0$ and $w \in \mathcal{N}_\lambda^-$ such that $\Phi_\lambda(w) \leq M$. By using Hölder and Sobolev embeddings, we obtain

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)||w||^2 + \lambda\left(\frac{1}{p+1} - \frac{1}{1-\gamma}\right)C||w||^{1-\gamma} \leq \Phi_\lambda(w) \leq M,$$

where C is a positive constant. Since $0 < 1-\gamma < 2$, we have

$$\sup\left\{||w|| : w \in \mathcal{N}_\lambda^-, \Phi_\lambda(w) \leq M\right\} < \infty.$$

Now, let us prove the two first inequalities in (1.16). First, let $u_n \subset \mathcal{N}_\lambda^+$ such that $\Phi_\lambda(u_n) \to \tilde{J}_\lambda^+$. Thus, if follows from the boundedness of $\mathcal{N}_\lambda^+$ proved in *a)* that, up to a subsequence, $u_n \rightharpoonup u$ in X and hence $-\infty < \Phi_\lambda(u) \leq \liminf \Phi_\lambda(u_n) = \tilde{J}_\lambda^+$. To show the first inequality, we use (1.14) in the expression of $\Phi_\lambda(u)$ to infer that

$$\begin{aligned}
\Phi_\lambda(u) &= \left(\frac{p-1}{2(p+1)}\right)||u||^2 - \lambda\left(\frac{\gamma+p}{(p+1)(1-\gamma)}\right)\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx \\
&< \left(\frac{p-1}{2(p+1)}\right)||u||^2 - \left(\frac{(\gamma+p)(p-1)}{(p+1)(1-\gamma)(\gamma+p)}\right)||u||^2 \\
&= -\left(\frac{(1+\gamma)(p-1)}{2(1-\gamma)(p+1)}\right)||u||^2 < 0
\end{aligned}$$

holds, that is, $\tilde{J}_\lambda^+ < 0$.

In a similar way we can prove that $-\infty < \Phi_\lambda(w) \leq \liminf \Phi_\lambda(w_n) = \tilde{J}_\lambda^-$. This ends the proof. ■

Now we show that the infimum value is achieved in both Nehari manifolds $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$.

Lemma 1.2.2 *Let* $0 < \lambda < \lambda_*$. *Then there exist* $u_\lambda \in \mathcal{N}_\lambda^+$ *and* $w_\lambda \in \mathcal{N}_\lambda^-$ *such that* $\Phi_\lambda(u_\lambda) = \tilde{J}_\lambda^+$ *and* $\Phi_\lambda(w_\lambda) = \tilde{J}_\lambda^-$.

Proof First, we will show that there exists $u_\lambda \in \mathcal{N}_\lambda^+$ such that $\Phi_\lambda(u_\lambda) = \tilde{J}_\lambda^+$. Let $\{u_n\} \subset \mathcal{N}_\lambda^+$ such that $\Phi_\lambda(u_n) \to \tilde{J}_\lambda^+$. So, it follows from Lemma 1.2.1 a) that, up to a subsequence, $u_n \rightharpoonup u_\lambda$ in X and $u_\lambda \geq 0$. Suppose on the contrary that $u_\lambda = 0$, then $0 = \Phi_\lambda(u_\lambda) \leq \liminf \Phi_\lambda(u_n) = \tilde{J}_\lambda^+ < 0$, which is impossible, that is, $u_\lambda \neq 0$ and so $u_\lambda \in X_+$.

Let us prove that $u_\lambda \in \mathcal{N}_\lambda^+$. First, we claim that $\{u_n\}$ converges strongly to u_λ in X. On the contrary, we would have that $||u_\lambda|| < \liminf ||u_n||$ and thus

$$\liminf_{n\to\infty} \phi'_{\lambda,u_n}(t_\lambda^+(u_\lambda)u_n) > \phi'_{\lambda,u_\lambda}(t_\lambda^+(u_\lambda)u_\lambda) = 0,$$

which implies that $\phi'_{\lambda,u_n}(t_\lambda^+(u)u_n) > 0$ for sufficiently large n. It follows from Proposition 1.1.1 and Lemma 1.1.4 applied to the fiber map ϕ_{λ,u_n} that $1 = t_\lambda^+(u_n) < t_\lambda^+(u_\lambda)$ holds for larger n. Therefore, by coming back to the fiber map ϕ_{λ,u_λ}, we obtain from Proposition 1.1.1 again that $\Phi_\lambda(t_\lambda^+(u_\lambda)u_\lambda) < \Phi_\lambda(u_\lambda)$ and consequently

$$\tilde{J}_\lambda \leq J_\lambda^+(u) = \Phi_\lambda(t_\lambda^+(u)u) < \liminf \Phi_\lambda(u_n) = \tilde{J}_\lambda^+,$$

which is an absurd, that is, $u_n \to u$ in X and hence

$$\phi'_{\lambda,u_\lambda}(1) = \lim_{n\to\infty} \phi'_{\lambda,u_n}(1) = 0 \text{ and } \phi''_{\lambda,u_\lambda}(1) = \lim_{n\to\infty} \phi''_{\lambda,u_n}(1) \geq 0. \tag{1.17}$$

Since from Lemma 1.1.4 *b*) we have that $\mathcal{N}_\lambda^0 = \emptyset$ for $0 < \lambda < \lambda_*$, we must conclude that $u_\lambda \in \mathcal{N}_\lambda^+$ and $\Phi_\lambda(u_\lambda) = \tilde{J}_\lambda^+$.

Next, let us prove that there exists $w_\lambda \in \mathcal{N}_\lambda^-$ for which $\Phi_\lambda(w_\lambda) = \tilde{J}_\lambda^-$ holds. Let $\{w_n\} \subset \mathcal{N}_\lambda^-$ be such that $\Phi_\lambda(w_n) \to \tilde{J}_\lambda^-$. As above, we have that $w_n \rightharpoonup w_\lambda$ in X and $w_\lambda \geq 0$. Assume on the contrary that $w_\lambda = 0$ then, from Lemma 1.2.1 *b*) we obtain the absurd

$$0 < \inf\left\{||w||^2 : w \in \mathcal{N}_\lambda^-\right\} \leq \liminf_{n\to\infty} ||w_n||^2 \leq \liminf_{n\to\infty} \frac{(\gamma+p)}{(1+\gamma)} \int_{\mathbb{R}^N} b|w_n|^{p+1} dx = 0$$

where the last equality follows from the compact embedding X into $L^{p+1}(\mathbb{R}^N)$, hence $w_\lambda \neq 0$ and so $w_\lambda \in X_+$. By repeating the above arguments, we have $\int b|w_\lambda|^{p+1}dx > 0$.

We claim that $\{w_n\}$ converges strongly to w_λ in X. Suppose not. Then we may assume that $||w_n - w_\lambda|| \to \theta > 0$ and apply Brezis-Lieb lemma to infer that

$$\tilde{J}_\lambda^- = \Phi_\lambda(w_\lambda) + \frac{\theta^2}{2}, \ \phi'_{\lambda,w_\lambda}(1) + \theta^2 = 0, \text{ and } \phi''_{\lambda,w_\lambda} + \theta^2 \leq 0$$

holds. So, we would have $\phi^{'}_{\lambda,w_\lambda}(1) < 0$ and $\phi^{''}_{\lambda,w_\lambda}(1) < 0$. As a consequence of Proposition 1.1.1 and Lemma 1.1.4, there exists a $t^-_\lambda \in (0,1)$ such that $\phi^{'}_{\lambda,w_\lambda}(t^-_\lambda) = 0$, $\phi^{''}_{\lambda,w_\lambda}(t^-_\lambda) < 0$ and $t^-_\lambda w_\lambda \in \mathcal{N}^-_\lambda$.

By setting $g(t) = \phi_{\lambda,w_\lambda}(t) + \frac{\theta^2 t^2}{2}$ for $t > 0$ we conclude that $0 < t^-_\lambda < 1$, $g'(1) = 0$ and $g^{'}(t^-_\lambda) = \theta^2 t^-_\lambda > 0$, which together with Proposition 1.1.1 lead us to conclude that g is increasing on $[t^-_\lambda, 1]$. Thus, we have

$$\tilde{J}^-_\lambda = \lim \Phi_\lambda(w_n) = g(1) > g(t^-_\lambda) > \phi_{\lambda,w_\lambda}(t^-_\lambda) = \Phi_\lambda(t^-_\lambda w_\lambda) \geq \tilde{J}^-_\lambda,$$

which is a contradiction, that is $\theta = 0$ and $\{w_n\}$ converges strongly to w_λ in X. After this, we obtain that $w_\lambda \in \mathcal{N}^-_\lambda$ and $\Phi_\lambda(w_\lambda) = \tilde{J}^-_\lambda$, as done at (1.17). This ends the proof. ■

Lemma 1.2.3 *Let* $0 < \lambda < \lambda_*$. *Then there exists* $\epsilon_0 > 0$ *such that:*

a) $\Phi_\lambda(u_\lambda + \epsilon\psi) \geq \Phi_\lambda(u_\lambda)$,

b) $t^-_\lambda(w_\lambda + \epsilon\psi) \to 1$ *as* $\epsilon \downarrow 0$, *where* $t^-_\lambda(w_\lambda + \epsilon\psi)$ *is the unique positive real number, given by Proposition* 1.1.1, *satisfying* $t^-_\lambda(w_\lambda + \epsilon\psi)(w_\lambda + \epsilon\psi) \in \mathcal{N}^-_\lambda$

for each $\psi \in X_+$ *given and for each* $0 \leq \epsilon \leq \epsilon_0$.

Proof Let ψ be a function in X_+. First, let us prove $a)$. It follows from (1.1) that

$$\phi''_{\lambda,u_\lambda+\epsilon\psi}(1) = ||u_\lambda+\epsilon\psi||^2+\gamma\lambda\int_{\mathbb{R}^N} a(x)|u_\lambda+\epsilon\psi|^{1-\gamma}dx-p\int_{\mathbb{R}^N} b(x)|u_\lambda+\epsilon\psi|^{p+1}dx,\ \epsilon \geq 0,$$

which combined with the continuity of $\phi_{\lambda,u_\lambda+\epsilon\psi}(1)$ in $\epsilon \geq 0$ and the fact that $\phi^{''}_{\lambda,u_\lambda}(1) > 0$, because $u_\lambda \in \mathcal{N}^+_\lambda$, implies that there exists an $\epsilon_0 > 0$ such that $\phi''_{\lambda,u_\lambda+\epsilon\psi}(1) > 0$ for all $0 \leq \epsilon \leq \epsilon_0$.

Fix $0 \leq \epsilon \leq \epsilon_0$. Then from $\phi''_{\lambda,u_\lambda+\epsilon\psi}(1) > 0$, we obtain

$$\Phi_\lambda(u_\lambda+\epsilon\psi) = \phi_{\lambda,u_\lambda+\epsilon\psi}(1) \geq \phi_{\lambda,u_\lambda+\epsilon\psi}(t^+_\lambda(u_\lambda+\epsilon\psi)) = \Phi_\lambda(t^+_\lambda(u_\lambda+\epsilon\psi)(u_\lambda+\epsilon\psi)) \geq \Phi_\lambda(u_\lambda)$$

where the last inequality follows from Lemma 1.2.2, because $u_\lambda, t^+_\lambda(u_\lambda+\epsilon\psi)(u_\lambda+\epsilon\psi) \in \mathcal{N}^+_\lambda$.

Now we prove $b)$. By defining $F : (0,\infty)\times\mathbb{R}^3 \to \mathbb{R}$ by $F(t,e,f,g) = et - \lambda f t^{-\gamma} - g t^p$, we have that F is a C^∞ function,

$$F(1, e_1, f_1, g_1) = \phi^{'}_{\lambda,w_\lambda}(1) = 0,$$

because $w_\lambda \in \mathcal{N}_\lambda$, and

$$\frac{dF}{dt}(1, e_1, f_1, g_1) = \phi''_{\lambda, w_\lambda}(1) < 0,$$

due to the fact that $w_\lambda \in \mathcal{N}_\lambda^-$, where

$$e_1 = ||w_\lambda||^2, \;\; f_1 = \int_{\mathbb{R}^N} a|w_\lambda|^{1-\gamma} dx \text{ and } g_1 = \int_{\mathbb{R}^N} b|w_\lambda|^{p+1} dx.$$

Therefore, it follows from the implicit function theorem and from

$$F(t_\lambda^-(w_\lambda + \epsilon\psi), ||w_\lambda + \epsilon\psi||^2, \int_{\mathbb{R}^N} a(x)|w_\lambda + \epsilon\psi|^{1-\gamma} dx, \int_{\mathbb{R}^N} b(x)|w_\lambda + \epsilon\psi|^{p+1} dx) = 0,$$

thanks to Proposition 1.1.1, that

$$t(||w_\lambda + \epsilon\psi||^2, \int_{\mathbb{R}^N} a(x)|w_\lambda + \epsilon\psi|^{1-\gamma} dx, \int_{\mathbb{R}^N} b(x)|w_\lambda + \epsilon\psi|^{p+1} dx) = t_\lambda^-(w_\lambda + \epsilon\psi)$$

for $\epsilon > 0$ small enough, where $t : B \to A$ is a C^∞-function where A and B are open neighborhoods of 1 and (e_1, f_1, g_1), respectively. The continuity of t implies the claim. This finishes the proof. ■

Lemma 1.2.3 implies

Lemma 1.2.4 *Let* $0 < \lambda < \lambda_*$. *Then for each* $\psi \in X_+$ *given, there hold* $au_\lambda^{-\gamma}\psi, aw_\lambda^{-\gamma}\psi \in L^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \nabla u_\lambda \nabla\psi + V(x)u_\lambda\psi dx - \int_{\mathbb{R}^N} (\lambda a(x)u_\lambda^{-\delta}\psi dx + b(x)u_\lambda^p\psi dx) \geq 0 \tag{1.18}$$

and

$$\int_{\mathbb{R}^N} \nabla u_\lambda \nabla\psi + V(x)u_\lambda\psi dx - \int_{\mathbb{R}^N} (\lambda a(x)u_\lambda^{-\delta}\psi dx + b(x)u_\lambda^p\psi dx) \geq 0 \tag{1.19}$$

In particular, $u_\lambda, w_\lambda > 0$ *almost everywhere in* $\mathbb{R}^N$.

Proof Let $\psi \in X_+$. First, let us prove the inequality (1.18). After some manipulations, we obtain from Lemma 1.2.3 item a), that

$$\frac{||u_\lambda + \epsilon\psi||^2 - ||u_\lambda||^2}{2\epsilon} - \int_{\mathbb{R}^N} \frac{b|u_\lambda + \epsilon\psi|^{p+1} - b|u_\lambda|^{p+1}}{(p+1)\epsilon} dx \geq \lambda \int_{\mathbb{R}^N} \frac{a|u_\lambda + \epsilon\psi|^{1-\gamma} - a|u_\lambda|^{1-\gamma}}{(1-\gamma)\epsilon} dx$$

holds for sufficiently small $\epsilon > 0$.

By using similar arguments as in the proof of Lemma 1.1.5, we obtain from the last inequality that $u_\lambda > 0$ in $\mathbb{R}^N$, $au_\lambda^{-\gamma}\psi \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \nabla u_\lambda \nabla \psi + V(x)u_\lambda \psi dx - \int_{\mathbb{R}^N} (\lambda a(x)u_\lambda^{-\delta}\psi dx + b(x)u_\lambda^p \psi dx) \geq 0.$$

To prove (1.19), we note that

$$\Phi_\lambda(t_\lambda^-(w_\lambda+\epsilon\psi)(w_\lambda+\epsilon\psi)) \geq \Phi_\lambda(w_\lambda) = \phi_{\lambda,w_\lambda}(1) \geq \phi_{\lambda,w_\lambda}(t_\lambda^-(w_\lambda+\epsilon\psi)) = \Phi_\lambda(t_\lambda^-(w_\lambda+\epsilon\psi)w_\lambda),$$

where the first inequality follows from Lemma 1.2.2 and the second inequality comes from Proposition 1.1.1.

After some manipulations, we obtain from the above inequality that

$$t_\lambda^-(w_\lambda+\epsilon\psi)^2 \frac{||w_\lambda+\epsilon\psi||^2 - ||w_\lambda||^2}{2\epsilon} - t_\lambda^-(w_\lambda+\epsilon\psi)^{p+1} \int_{\mathbb{R}^N} \frac{b|w_\lambda+\epsilon\psi|^{p+1} - b|w_\lambda|^{p+1}}{(p+1)\epsilon} dx$$
$$\geq t_\lambda^-(w_\lambda+\epsilon\psi)^{1-\gamma}\lambda \int_{\mathbb{R}^N} \frac{a|w_\lambda+\epsilon\psi|^{1-\gamma} - a|w_\lambda|^{1-\gamma}}{(1-\gamma)\epsilon} dx$$

holds for $\epsilon > 0$ small enough.

So, by applying Lemma 1.2.3 item *b*), we obtain $w_\lambda > 0$ in $\mathbb{R}^N$, $aw_\lambda^{-\gamma}\psi \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \nabla w_\lambda \nabla \psi + V(x)w_\lambda \psi dx - \int_{\mathbb{R}^N} (\lambda a(x)w_\lambda^{-\delta}\psi dx + b(x)w_\lambda^p \psi dx) \geq 0$$

holds. This completes the proof. ■

Proposition 1.2.1 *Let* $0 < \lambda < \lambda_*$. *Then* $u_\lambda \in \mathcal{N}_\lambda^+$ *and* $w_\lambda \in \mathcal{N}_\lambda^-$ *are solutions of Problem* (P_λ).

Proof First we will show that u_λ is a solution for (P_λ). To this end, let $\psi \in X$ and define $\Psi_\epsilon = (u_\lambda + \epsilon\psi)^+ \in X_+$ for each $\epsilon > 0$ given. Therefore, it follows from Lemma 1.2.4 that the inequality (1.18) holds true with Ψ_ϵ in the place of ψ.

Now, by adapting the proof of Step 3 of Lemma 1.1.5 with

$$||u_\lambda||^2 - \lambda \int_{\mathbb{R}^N} a(x)|u_\lambda|^{1-\gamma}dx - \int_{\mathbb{R}^N} b(x)|u_\lambda|^{p+1}dx = 0 \ \ \text{(because } u_\lambda \in \mathcal{N}_\lambda)$$

in the place of (1.8), we are able to show that u_λ is a solution for Problem (P_λ). In a similar way, w_λ will be a solution for (P_λ) as well.

■

1.3 Multiplicity of solutions for $\lambda = \lambda_*$

In this section we prove the existence of at least two solutions for Problem (P_{λ_*}) by using the multiplicity result given in Proposition 1.2.1 for $0 < \lambda < \lambda_*$ and performing a limit process. The next proposition is a consequence of the monotonicities and regularities of the functions $t_\lambda^+(u), t_\lambda^-(u)$, J_λ^+ and J_λ^- given by Lemma 1.1.7.

Proposition 1.3.1 *There holds:*

a) the functions $(0, \lambda_*] \ni \lambda \to \tilde{J}_\lambda^\pm$ *are decreasing and left-continuous for* $\lambda \in (0, \lambda_*)$,

b) $\lim\limits_{\lambda \uparrow \lambda_*} \tilde{J}_\lambda^\pm = \tilde{J}_{\lambda_*}^\pm$.

Proposition 1.3.2 *The problem* (P_{λ_*}) *admits at least two solutions* $w_{\lambda_*} \in \mathcal{N}_{\lambda_*}^-$ *and* $u_{\lambda_*} \in \mathcal{N}_{\lambda_*}^+$.

Proof First, let us show that there exists a solution $w_{\lambda_*} \in \mathcal{N}_{\lambda_*}^-$ for (P_{λ_*}). Let $\{\lambda_n\} \subset (0, \lambda_*)$ be such that $\lambda_n \uparrow \lambda_*$ and $\{w_{\lambda_n}\} \subset \mathcal{N}_{\lambda_n}^-$ as in Proposition 1.2.1. Suppose on the contrary that $||w_{\lambda_n}|| \to \infty$, hence after applying the Hölder inequality, Sobolev embedding and the fact that $w_{\lambda_n} \in \mathcal{N}_{\lambda_n}^-$, we obtain

$$\begin{aligned} J_{\lambda_n}^- = \Phi_{\lambda_n}(w_{\lambda_n}) &= \left(\frac{1}{2} - \frac{1}{p+1}\right) ||w_{\lambda_n}||^2 + \lambda_n \left(\frac{1}{p+1} - \frac{1}{1-\gamma}\right) \int_{\mathbb{R}^N} a(x)|w_{\lambda_n}|^{1-\gamma} dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right) ||w_{\lambda_n}||^2 + C\left(\frac{1}{p+1} - \frac{1}{1-\gamma}\right) ||w_{\lambda_n}||^{1-\gamma}, \end{aligned}$$

which implies by Proposition 1.3.1 that $\infty > \lim \tilde{J}_{\lambda_n}^- \geq \infty$, which is a contradiction. Therefore $\{w_{\lambda_n}\}$ is bounded and we can assume that $w_{\lambda_n} \rightharpoonup w_{\lambda_*}$ in X,

$$\begin{aligned} &w_{\lambda_n} \to w_{\lambda_*} \text{ in } L^q(\mathbb{R}^N), \forall\, q \in [2, 2^*), \\ &w_{\lambda_n} \to w_{\lambda_*} \text{ a.e. } \mathbb{R}^N, \\ &\text{there exist } h_q \in L^q(\mathbb{R}^N) \text{ such that } |w_{\lambda_n}| \leqslant h_q \end{aligned}$$

with $w_{\lambda_*} \geqslant 0$.

Thus, once w_{λ_n} is a solution for Problem (P_{λ_n}) it follows that

$$(w_{\lambda_*}, \psi) - \int_{\mathbb{R}^N} b(x) w_{\lambda_*}^p \psi dx \geq \lambda_* \int_{\mathbb{R}^N} a(x) G(x) \psi dx \tag{1.20}$$

for all $\psi \in X_+$, where G is understood as $G(x) := w_{\lambda_*}^{-\gamma}(x)$ if $w_{\lambda_*}(x) \neq 0$ and $G(x) := \infty$ if $w_{\lambda_*}(x) = 0$. It follows that $0 \leq \int_{\mathbb{R}^N} a(x)G(x)\psi dx < \infty$, which implies $w_{\lambda_*}(x) > 0$ in $\mathbb{R}^N$ and

$$(w_{\lambda_*}, \psi) - \int_{\mathbb{R}^N} b(x) w_{\lambda_*}^p \psi dx \geq \lambda_* \int_{\mathbb{R}^N} a(x) w_{\lambda_*}^{-\gamma} \psi dx, \forall \psi \in X_+. \tag{1.21}$$

Moreover, it follows from Lemma 1.1.2 and Fatou's lemma that

$$\begin{aligned}\limsup_{n\to\infty}(w_{\lambda_n}, w_{\lambda_n} - w_{\lambda_*}) &\leq \limsup_{n\to\infty} \lambda_n \int_{\mathbb{R}^N} a(x) w_{\lambda_n}^{1-\gamma} dx + \limsup_{n\to\infty} -\lambda_n \int_{\mathbb{R}^N} a(x) w_{\lambda_n}^{-\gamma} w_{\lambda_*} dx \\ &= \lambda_* \int_{\mathbb{R}^N} a(x) w_{\lambda_*}^{1-\gamma} dx - \liminf \int_{\mathbb{R}^N} \lambda_n a(x) w_{\lambda_n}^{-\gamma} w_{\lambda_*} dx \\ &\leq \lambda_* \int_{\mathbb{R}^N} a(x) w_{\lambda_*}^{1-\gamma} dx - \int_{\mathbb{R}^N} \liminf \lambda_n a(x) w_{\lambda_n}^{-\gamma} w_{\lambda_*} dx \\ &= \lambda_* \int_{\mathbb{R}^N} a(x) w_{\lambda_*}^{1-\gamma} dx - \lambda_* \int_{\mathbb{R}^N} a(x) w_{\lambda_*}^{1-\gamma} dx = 0\end{aligned}$$

that is,

$$\limsup ||w_{\lambda_n} - w_{\lambda_*}||^2 \leq \limsup(w_{\lambda_n}, w_{\lambda_n} - w_{\lambda_*}) + \limsup -(w_{\lambda_*}, w_{\lambda_n} - w_{\lambda_*}) \leq 0,$$

which implies that $w_{\lambda_n} \to w_{\lambda_*}$ in X.

As a consequence of this, we have that

$$\phi'_{\lambda_*, w_{\lambda_*}}(1) = \lim \phi'_{\lambda_n, w_{\lambda_n}}(1) = 0 \text{ and } \phi''_{\lambda_*, w_{\lambda_*}}(1) = \lim \phi''_{\lambda_n, w_{\lambda_n}}(1) \leq 0$$

which implies, by the first equality, that $w_{\lambda_*} \in \mathcal{N}_{\lambda_*}$. We also have from Lemma 1.2.1 b), that

$$\begin{aligned}0 < (1+\gamma)||w_{\lambda_*}|| = (1+\gamma) \lim_{n\to\infty} ||w_{\lambda_n}|| &\leq (\gamma + p) \lim_{n\to\infty} \int_{\mathbb{R}^N} b(x) w_{\lambda_n}^{p+1} dx \\ &= (\gamma + p) \int_{\mathbb{R}^N} b(x) w_{\lambda_*}^{p+1} dx,\end{aligned}$$

that is, $\int_{\mathbb{R}^N} b(x) w_{\lambda_*}^{p+1} dx > 0$ and hence $w_{\lambda_*} \in \mathcal{N}_{\lambda_*}^- \cup \mathcal{N}_{\lambda_*}^0$.

By using that $w_{\lambda_*} \in \mathcal{N}_{\lambda_*}$, that is,

$$||w_{\lambda_*}||^2 - \lambda_* \int_{\mathbb{R}^N} a(x) |w_{\lambda_*}|^{1-\gamma} dx - \int_{\mathbb{R}^N} b(x) |w_{\lambda_*}|^{p+1} dx = 0$$

holds, taking $\Psi_\epsilon = (w_{\lambda_*} + \epsilon\psi)^+ \in X_+$, for $\psi \in X, \epsilon > 0$ given, as a test function in (1.21) and following similar arguments as done in the proof of the Proposition 1.2.1, we are able to conclude that w_{λ_*} is a solution of (P_{λ_*}). Moreover, $w_{\lambda_*} \in \mathcal{N}_{\lambda_*}^-$ due to Corollary 1.1.1. Finally, it follows from the strong convergence, Proposition 1.2.1, Proposition 1.3.1 and Proposition 1.1.3 $(iv), (v), (vi)$ that

$$\Phi_{\lambda_*}(w_{\lambda_*}) = \lim \Phi_{\lambda_n}(w_n) = \lim \tilde{J}_{\lambda_n}^- = \tilde{J}_{\lambda_*}^- = \inf\left\{\Phi_{\lambda_*}(t_{\lambda_*}(w)w) : w \in \mathcal{N}_{\lambda_*}^- \cup \mathcal{N}_{\lambda_*}^0\right\} \tag{1.22}$$

holds, that is, $w_{\lambda_*} \in \mathcal{N}_{\lambda_*}^-$ is a global minimum of Φ_{λ_*} constrained to $\mathcal{N}_{\lambda_*}^- \cup \mathcal{N}_{\lambda_*}^0$.

In order to show the existence of a second solution for Problem (P_{λ_*}), we proceed in a similar way, that is, pick a $\{\lambda_n\} \subset (0, \lambda_*)$ such that $\lambda_n \uparrow \lambda_*$ and $\{u_{\lambda_n}\} \subset \mathcal{N}^+_{\lambda_n}$ as given by Proposition 1.2.1. After some manipulations, we obtain that $u_{\lambda_n} \to u_{\lambda_*}$ in X for some $0 < u_{\lambda_*} \in \mathcal{N}^-_{\lambda_*} \cup \mathcal{N}^0_{\lambda_*}$, which is a solution for Problem (P_{λ_*}).

Besides this, if $\int_{\mathbb{R}^N} b(x)u^{p+1}_{\lambda_*}dx > 0$ and $\phi''_{\lambda_*,u_{\lambda_*}}(1) = 0$, then u_{λ_*} would be a solution for the problem (P_{λ_*}) in $\mathcal{N}^0_{\lambda_*}$, but this is impossible by Corollary 1.1.1. So we have $\phi''_{\lambda_*,u_{\lambda_*}}(1) > 0$ in this case. On the other side, if $\int_{\mathbb{R}^N} b(x)u^{p+1}_{\lambda_*}dx \leq 0$, then we have

$$\phi''_{\lambda_*,u_{\lambda_*}}(1) = ||u_{\lambda_*}||^2 + \gamma\lambda_* \int_{\mathbb{R}^N} a(x)u^{1-\gamma}_{\lambda_*}dx - p\int_{\mathbb{R}^N} b(x)u^{p+1}_{\lambda_*}dx > 0.$$

So, in both cases, we have $\phi''_{\lambda_*,u_{\lambda_*}}(1) > 0$ which implies that $u_{\lambda_*} \in \mathcal{N}^+_{\lambda_*}$. We also have that $u_{\lambda_*} \in \mathcal{N}^-_{\lambda_*}$ is a global minimum of Φ_{λ_*} constrained to $\mathcal{N}^+_{\lambda_*} \cup \mathcal{N}^0_{\lambda_*}$ as well. This ends the proof.

■

Before proving the multiplicity of solutions for Problem (P_λ) when $\lambda > \lambda_*$, let us gather further information on the sets

$$S^-_{\lambda_*} = \left\{w \in \mathcal{N}^-_{\lambda_*} : J^-_{\lambda_*}(w) = \tilde{J}^-_{\lambda_*}\right\} \text{ and } S^+_{\lambda_*} = \left\{u \in \mathcal{N}^+_{\lambda_*} : J^+_{\lambda_*}(u) = \tilde{J}^+_{\lambda_*}\right\}. \tag{1.23}$$

Corollary 1.3.1 *We have that:*

a) $S^-_{\lambda_*}$ *and* $S^+_{\lambda_*}$ *are non-empties,*

b) there exist $c_{\lambda_*}, C_{\lambda_*} > 0$ *such that* $c_{\lambda_*} \leq \|u\|, \|w\| \leq C_{\lambda_*}$ *for all* $u \in S^+_{\lambda_*}$ *and* $w \in S^-_{\lambda_*}$*,*

c) if $u \in S^-_{\lambda_*} \cup S^+_{\lambda_*}$*, then* u *is a solution for Problem* (P_{λ_*})*.*

Proof The item $a)$ follows immediately from (1.22), while $b)$ is a consequence of Lemma 1.2.1. Finally, the proof of the item $c)$ is similar to that of Proposition 1.3.2.

■

1.4 Multiplicity of solutions for $\lambda > \lambda_*$

In this section we show the existence of solutions for problem (P_λ) when λ is greater than λ_* but close to it. The idea is to minimize the energy functional Φ_λ over subsets of $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$, which are projections of subsets of $\mathcal{N}_{\lambda_*}^+$ and $\mathcal{N}_{\lambda_*}^-$ that have positive distances to $\mathcal{N}_{\lambda_*}^0$. To do this, we do a finer analysis on these sets and we obtain new estimates that are new even in the non-singular case as in [58].

Proposition 1.4.1 *Let $c < C$. Assume that $\lambda_n \downarrow \lambda_*$.*

a) suppose that $w_n \in \mathcal{N}_{\lambda_*}^-$ satisfies $c \leq ||w_n|| \leq C$. If $(t_{\lambda_n}^-(w_n))^2\phi''_{\lambda_n,w_n}(t_{\lambda_n}^-(w_n)) \to 0$, then $d(w_n, \mathcal{N}_{\lambda_*}^0) \to 0$ as $n \to \infty$,

b) suppose that $u_n \in \mathcal{N}_{\lambda_*}^+$ satisfies $c \leq ||u_n|| \leq C$. If $(t_{\lambda_n}^+(u_n))^2\phi''_{\lambda_n,u_n}(t_{\lambda_n}^+(u_n)) \to 0$, then $d(u_n, \mathcal{N}_{\lambda_*}^0) \to 0$ as $n \to \infty$.

Proof We prove only $a)$ since the proof of $b)$ follows the same strategy. It follows from Lemma 1.2.1 $b)$ that there exists a positive constant c such that $\int_{\mathbb{R}^N} b|w_n|^{p+1}dx \geq c$. We claim that the same holds for $\int_{\mathbb{R}^N} a|w_n|^{1-\gamma}dx$. To prove this, let us first prove that $t_{\lambda_n}^-(w_n) \to \theta \in (0,\infty)$.

Now, by applying Proposition 1.1.1, there exist $s_n := t_{\lambda_n}^+(w_n) < t_{\lambda_n}^-(w_n) := t_n$ such that

$$\begin{cases} t_n^2||w_n||^2 - t_n^{1-\gamma}\lambda_n \int_{\mathbb{R}^N} a|w_n|^{1-\gamma}dx - t_n^{p+1}\int_{\mathbb{R}^N} b|w_n|^{p+1}dx = 0, \\ t_n^2||w_n||^2 + t_n^{1-\gamma}\lambda_n\gamma \int_{\mathbb{R}^N} a|w_n|^{1-\gamma}dx - t_n^{p+1}p\int_{\mathbb{R}^N} b|w_n|^{p+1}dx = o(1), \\ s_n^2||w_n||^2 - s_n^{1-\gamma}\lambda_n \int_{\mathbb{R}^N} a|w_n|^{1-\gamma}dx - s_n^{p+1}\int_{\mathbb{R}^N} b|w_n|^{p+1}dx = 0, \end{cases}$$

where the second line is a consequence of the assumption $(t_{\lambda_n}^-(w_n))^2\phi''_{\lambda_n,w_n}(t_{\lambda_n}^-(w_n)) \to 0$.

So, by solving the system formed by the first and third equation of the above system treating the integrals as unknown, and substituting them into the second equation, we obtain

$$||w_n||^2 t_n^2 \left[\frac{(1+\gamma)\left(\frac{s_n}{t_n}\right)^{p+\gamma} + (p-1) - (\gamma+p)\left(\frac{s_n}{t_n}\right)^{1+\gamma}}{\left(\frac{s_n}{t_n}\right)^{p+\gamma} - 1}\right] = o(1), \; n \to \infty. \qquad (1.24)$$

Besides this, it follows from $C \geq ||w_n|| \geq c$, Lemma 1.2.1, the first and third equations of system above and $s_n < t_n$ that there exists positive constants $\tilde{c}, \tilde{C}, \theta, \alpha$ such that $t_n, s_n \in [\tilde{c}, \tilde{C}], \; t_n \to \theta, \; s_n \to \alpha$ and $||t_n w_n|| \geq \tilde{c}$. By using these informations and taking limit on (1.24), we conclude that $s_n/t_n \to 1$ and $\theta = \alpha$, because $t = 1$ is the only zero of the function

$$g(t) = (1+\gamma)t^{p+\gamma} + (p-1) - (\gamma+p)t^{1+\gamma}.$$

Once $s_n w_n \in \mathcal{N}^+_{\lambda_n}$, we obtain from Lemma 1.2.1 $a)$ that $\int a|w_n|^{1-\gamma}dx \geq c$. Follows that

$$\begin{cases} ||\theta w_n||^2 - \lambda_* \int_{\mathbb{R}^N} a|\theta w_n|^{1-\gamma}dx - \int_{\mathbb{R}^N} b|\theta w_n|^{p+1}dx = o(1), \\ ||\theta w_n||^2 + \gamma\lambda_* \int_{\mathbb{R}^N} a|\theta w_n|^{1-\gamma}dx - p\int_{\mathbb{R}^N} b|\theta w_n|^{p+1}dx = o(1) \end{cases}$$

and infer that

$$\frac{p-1}{\gamma+p}\frac{||\theta w_n||^2}{\int_{\mathbb{R}^N} a|\theta w_n|^{1-\gamma}dx} = \lambda_* + o(1),\ n \to \infty,$$

and

$$\frac{1+\gamma}{\gamma+p}\frac{||\theta w_n||^2}{\int_{\mathbb{R}^N} b|\theta w_n|^{p+1}dx} = 1 + o(1),\ n \to \infty.$$

Therefore, it follows from (1.2) and by 0-homogeneity that

$$\lambda(w_n) = \lambda(\theta w_n) = (1+o(1))^{\frac{1+\gamma}{p-1}}(\lambda_* + o(1)) \to \lambda_*,\ n \to \infty,$$

and w_n is a bounded minimizing sequence for λ_*. Moreover, by following similar arguments as done in the proof of Lemma 1.1.6, we obtain, up to a subsequence, that $w_n \to w \in \mathcal{N}^0_{\lambda_*}$ and consequently $d(w_n, \mathcal{N}^0_{\lambda_*}) \to 0$ as $n \to \infty$. This ends the proof.

■

Define

$$\mathcal{N}^-_{\lambda_*,d,C} = \left\{w \in \mathcal{N}^-_{\lambda_*} : d(w, \mathcal{N}^0_{\lambda_*}) > d, ||w|| \leq C\right\},$$

and

$$\mathcal{N}^+_{\lambda_*,d,c} = \left\{u \in \mathcal{N}^+_{\lambda_*} : d(u, \mathcal{N}^0_{\lambda_*}) > d, c \leq ||u||\right\},$$

for $c, C, d > 0$ given. As an immediately consequence of Proposition 1.4.1, we have.

Corollary 1.4.1 *Fix $c, C, d > 0$. Then there exist $\epsilon > 0$ satisfying:*

a) there exists $\delta < 0$ such that $(t^-_\lambda(w))^2\phi''_{\lambda,w}(t^-_\lambda(w)) < \delta$ for all $\lambda \in (\lambda_*, \lambda_* + \epsilon)$ and $w \in \mathcal{N}^-_{\lambda_*,d,C}$. In particular, we have that $t^-_\lambda(w)w \in \mathcal{N}^-_\lambda$ and $w \in \hat{\mathcal{N}}_\lambda$ for all $\lambda \in (\lambda_*, \lambda_* + \epsilon)$,

b) there exists $\delta > 0$ such that $(t^+_\lambda(u))^2\phi''_\lambda(t^+_\lambda(u)) > \delta$ for all $\lambda \in (\lambda_*, \lambda_* + \epsilon)$ and $u \in \mathcal{N}^+_{\lambda_*,d,c}$. In particular, we have that $t^+_\lambda(u)u \in \mathcal{N}^+_\lambda$ and $u \in \hat{\mathcal{N}}_\lambda \cup \hat{\mathcal{N}}^+_\lambda$ for all $\lambda \in (\lambda_*, \lambda_* + \epsilon)$.

To do a good choice of the parameter $d > 0$ in the last corollary, we prove the next result, where the sets $S^-_{\lambda_*}$ and $S^+_{\lambda_*}$ were defined at (1.23).

Proposition 1.4.2 *There holds:*

a) $d(S^-_{\lambda_*}, \mathcal{N}^0_{\lambda_*}) > 0$,

b) $d(S^+_{\lambda_*}, \mathcal{N}^0_{\lambda_*}) > 0$.

Proof We just prove *a*) because the proof of *b*) follows similar arguments. Assume by contradiction that $d(S^-_{\lambda_*}, \mathcal{N}^0_{\lambda_*}) = 0$. Then, there exist $w_n \in S^-_{\lambda_*}$ and $v_n \in \mathcal{N}^0_{\lambda_*}$ such that $\|w_n - v_n\| \to 0$ as $n \to \infty$ and

$$(w_n, \psi) = \lambda_* \int a w_n^{-\gamma} \psi dx + \int b w_n^p \psi dx, \ \forall \psi \in X, \ \forall n \in \mathbb{N}$$

holds, where this equality is a consequence of w_n being a solution for Problem (P_{λ_*}) as claimed in Corollary 1.3.1. Since $\mathcal{N}^0_{\lambda_*}$ is a compact set, see Lemma 1.1.6, we may assume that $v_n \to v \in \mathcal{N}^0_{\lambda_*}$ and hence $w_n \to v$ as well. From Fatou's Lemma we conclude that

$$(v, \psi) \geq \lambda_* \int a v^{-\gamma} \psi dx + \int b v^p \psi dx, \ \forall \psi \in X_+,$$

that is, we arrived in the same situation as in (1.20) with $v \in \mathcal{N}^0_{\lambda_*}$. So, by following the same arguments as done after (1.20), we are able to show that $v \in \mathcal{N}^0_{\lambda_*}$ is a solution for Problem (P_{λ_*}), but this is impossible by Corollary 1.1.1, which ends the proof. ■

After Corollaries 1.3.1, 1.4.1 and Proposition 1.4.2, we are in position to introduce

$$\tilde{J}^-_{\lambda,d^-,C} \equiv \inf \left\{ J^-_\lambda(w) : w \in \mathcal{N}^-_{\lambda_*,d^-,C} \right\} \quad \text{and} \quad \tilde{J}^+_{\lambda,d^+,c} \equiv \inf \left\{ J^+_\lambda(w) : w \in \mathcal{N}^+_{\lambda_*,d^+,c} \right\} \tag{1.25}$$

for each $0 < c < c_{\lambda_*}$, $C > C_{\lambda_*}$ (see Corollary 1.3.1 for both) $\lambda_* < \lambda < \lambda_* + \epsilon$ (see Corollary 1.4.1) and $0 < d^\pm < d(S^\pm_{\lambda_*}, \mathcal{N}^0_{\lambda_*})$ (see Proposition 1.4.2) which implies that $S^-_{\lambda_*} \subset \mathcal{N}^-_{\lambda_*,d^-,C}$ and $S^+_{\lambda_*} \subset \mathcal{N}^+_{\lambda_*,d^+,c}$. The proofs of the next propositions are similar to that of Propositions 4.5, 4.6, 4.7 in [58].

Proposition 1.4.3 *The λ-functions $\tilde{J}^-_{\lambda,d^-,C}$ and $\tilde{J}^+_{\lambda,d^+,C}$ are decreasing and there holds:*

a) $\lim_{\lambda \downarrow \lambda_*} \tilde{J}^-_{\lambda,d^-,C} = \tilde{J}^-_{\lambda_*}$,

b) $\lim_{\lambda \downarrow \lambda_*} \tilde{J}^+_{\lambda,d^+,c} = \tilde{J}^+_{\lambda_*}$.

Proposition 1.4.4 *There exists $\epsilon^- > 0$ such that J^-_λ constrained to $\mathcal{N}^-_{\lambda_*,d^-,C}$ has a minimizer $\tilde{w}_\lambda \in \mathcal{N}^-_{\lambda_*,d^-,C}$ for all $\lambda \in (\lambda_*, \lambda_* + \epsilon^-)$ given.*

Proposition 1.4.5 *There exists $\epsilon^+ > 0$ such that J_λ^+ constrained to $\mathcal{N}^+_{\lambda_*,d^+,c}$ has a minimizer $\tilde{u}_\lambda \in \mathcal{N}^+_{\lambda_*,d^+,c}$ for all $\lambda \in (\lambda_*, \lambda_* + \epsilon^+)$ given.*

Unlike the non-singular case, local or global minimizers for the energy functional constrained to Nehari sets, are not necessarily solutions for Problem (P_λ). In the next Proposition we will establish that this claim is true under our assumptions. The main point in order to prove that the minima found in Propositions 1.4.4, 1.4.5 are solutions of (P_λ) is to prove that $\tilde{w}_\lambda$ and $\tilde{u}_\lambda$ are interior points of $\mathcal{N}^-_{\lambda_*,d^-,C}$ and $\mathcal{N}^+_{\lambda_*,d^+,c}$ respectively.

Proposition 1.4.6 *There exists $\epsilon > 0$ such that the problem (P_λ) admits at least two solutions $w_\lambda \in \mathcal{N}^-_\lambda$ and $u_\lambda \in \mathcal{N}^+_\lambda$ for each $\lambda \in (\lambda_*, \lambda_* + \epsilon)$.*

Proof First, let us take advantage of the existence of the minimizer $\tilde{w}_\lambda \in \mathcal{N}^-_{\lambda_*,d^-,C}$ to build a solution for Problem (P_λ) in $\mathcal{N}^-_\lambda$. Let us do this by reminding that the definitions given at (1.25) and (1.13) implies that we can consider $w_\lambda := t^-_\lambda(\tilde{w}_\lambda)\tilde{w}_\lambda \in \mathcal{N}^-_\lambda$. Below, let us prove that w_λ is a solution for Problem (P_λ) if $\lambda > \lambda_*$ varies in an appropriate range. To this end, firstly we prove that $\tilde{w}_\lambda$ is a interior point of $\mathcal{N}^-_{\lambda_*,d^-,C}$ for λ close λ_*, which is equivalently to prove

Claim: there exists an $\epsilon_1 > 0$ such that

$$||\tilde{w}_\lambda|| < C, \ \forall \ \lambda \in (\lambda_*, \lambda_* + \epsilon_1), \tag{1.26}$$

where $C > C_{\lambda_*}$ and $C_{\lambda_*} > 0$ is given by Corollary 1.3.1.

Indeed, let $\lambda_n \downarrow \lambda_*$ and denote $\tilde{w}_{\lambda_n} = \tilde{w}_n$. Due to the boundedness of $\mathcal{N}^-_{\lambda_*,d^-,C}$, we may assume that $\tilde{w}_{\lambda_n} \rightharpoonup \tilde{w}$ in X. In fact, we have that $\tilde{w}_n \to \tilde{w}$ in X, otherwise we would have $||\tilde{w}|| < \liminf ||\tilde{w}_n||$ which implies

$$0 = \phi'_{\lambda_*,\tilde{w}}(t_{\lambda_*}(\tilde{w})) < \liminf \phi'_{\lambda_n,\tilde{w}_n}(t_{\lambda_*}(\tilde{w})),$$

where t_{λ_*} is given by Proposition 1.1.3 (iv). It follows that there exists k such that $\phi'_{\lambda_n,\tilde{w}_n}(t_{\lambda_*}(\tilde{w})) > 0$ for $n \geq k$, that is, $t^+_n(\tilde{w}_n) < t_{\lambda_*}(\tilde{w}) < t^-_n(\tilde{w}_n)$ by Proposition 1.1.1. Therefore

$$\|t_{\lambda_*}(\tilde{w})\tilde{w}\|^2 < \liminf_{n\to\infty} \|t_{\lambda_n}(\tilde{w}_n)\tilde{w}_n\|^2,$$

which lead us to

$$\Phi_{\lambda_*}(t_{\lambda_*}(\tilde{w})\tilde{w}) < \liminf_{\lambda_n \downarrow \lambda_*} \Phi_{\lambda_n}(t_{\lambda_*}(\tilde{w})\tilde{w}_n) \le \liminf_{\lambda_n \downarrow \lambda_*} \Phi_{\lambda_n}(t^-_{\lambda_n}(\tilde{w}_n)\tilde{w}_n) = \hat{J}^-_{\lambda_*}, \tag{1.27}$$

where the Proposition 1.4.3 *a*) was used to get the last equality. Moreover, it follows from Proposition 1.3.1 *b*), Proposition 1.2.1 and Corollary 1.1.2 that

$$\hat{J}^-_{\lambda_*} = \lim_{\lambda_n' \uparrow \lambda_*} \hat{J}^-_{\lambda_n'} \le \lim_{\lambda_n' \uparrow \lambda_*} \Phi_{\lambda_n'}(t^-_{\lambda_n'}(\tilde{w})\tilde{w}) = \Phi_{\lambda_*}(t_{\lambda_*}(\tilde{w})\tilde{w})$$

holds for any $\lambda_n' \uparrow \lambda_*$. By combining the last inequality with (1.27) we get a contradiction and hence $\tilde{w}_n \to \tilde{w}$ in X.

As a consequence of this strong convergence and Lemma 1.2.1 *b*), we obtain $\int b|\tilde{w}|^{p+1}dx > 0$ and $\phi'_{\lambda_*,\tilde{w}}(1) = 0$ and $\phi''_{\lambda_*,\tilde{w}}(1) \le 0$, which means by Proposition 1.1.1 that $\tilde{w} \in \mathcal{N}^-_{\lambda_*} \cup \mathcal{N}^0_{\lambda_*}$. Since

$$d(\tilde{w}, \mathcal{N}^0_{\lambda_*}) = \lim_{n\to\infty} d(\tilde{w}_n, \mathcal{N}^0_{\lambda_*}) \ge d^- > 0,$$

we have that $\tilde{w} \notin \mathcal{N}^0_{\lambda_*}$, that is, $\tilde{w} \in \mathcal{N}^-_{\lambda_*}$.

To conclude the proof of the claim, we just need to show that $\tilde{w} \in S^-_{\lambda_*}$. First note that similar arguments as done in the proof of Proposition 1.4.1-*a*) proves that $t^-_{\lambda_n}(\tilde{w}_n) \to t \in (0,\infty)$. From the strong convergence $\tilde{w}_n \to \tilde{w}$ in X, we get that $\phi'_{\lambda_*,\tilde{w}}(t) = 0$ and $\phi''_{\lambda_*,\tilde{w}}(t) \le 0$, which lead us to conclude that $t = 1$ since $\tilde{w} \in \mathcal{N}^-_{\lambda_*}$ and Proposition 1.1.1. From Proposition 1.4.3 and the strong convergence again, we obtain

$$\Phi_{\lambda_*}(\tilde{w}) = \lim_{\lambda_n \downarrow \lambda_*} \Phi_{\lambda_n}(t_{\lambda_n}(\tilde{w}_n)\tilde{w}_n) = \hat{J}^-_{\lambda_*},$$

which means that $\tilde{w} \in S^-_{\lambda_*}$. Therefore, from Corollary 1.3.1 we conclude that

$$\limsup_{\lambda \downarrow \lambda_*} ||\tilde{w}_\lambda|| \le ||\tilde{w}|| \le C_{\lambda_*}.$$

Since $C > C_{\lambda_*}$, the claim is true. This ends the proof of the claim.

To complete the proof that $w_\lambda := t^-_\lambda(\tilde{w}_\lambda)\tilde{w}_\lambda \in \mathcal{N}^-_\lambda$ is a solution to Problem (P_λ), let us perturb $\tilde{w}_\lambda$ by appropriate elements of X_+ and perform projections of it over $\mathcal{N}^-_{\lambda_*,d^-,C}$ and $\mathcal{N}^-_\lambda$. Let $\psi \in X_+$ and $\lambda \in (\lambda_*, \lambda_* + \epsilon_1)$. Since $\tilde{w}_\lambda \in \mathcal{N}^-_{\lambda_*}$, we are able to apply the implicit function Theorem, as done in Lemma 1.2.3 *b*), to prove that $t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi)$ (see Proposition 1.1.1) is well defined, is continuous for $\theta > 0$ small enough and $t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi) \longrightarrow 1$ as $\theta \longrightarrow 0$.

Thus, it follows from (1.26) and $d(\tilde{w}_\lambda, \mathcal{N}^0_{\lambda_*}) > d^-$ (see definition of $\mathcal{N}^-_{\lambda_*,d^-,C}$) that

$$||t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi)(\tilde{w}_\lambda + \theta\psi)|| < C \text{ and } d(t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi)(\tilde{w}_\lambda + \theta\psi), \mathcal{N}^0_{\lambda_*}) > d^-,$$

holds for $\theta > 0$ small enough, which implies

$$t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi)(\tilde{w}_\lambda + \theta\psi) \in \mathcal{N}^-_{\lambda_*,d^-,C}. \tag{1.28}$$

Therefore, by (1.28) and Corollary 1.4.1, we obtain

$$t_\lambda(\theta)t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi)(\tilde{w}_\lambda + \theta\psi) =: t^-_\lambda(t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi)(\tilde{w}_\lambda + \theta\psi))t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi)(\tilde{w}_\lambda + \theta\psi) \in \mathcal{N}^-_\lambda$$

By applying Proposition 1.4.4, we have

$$\Phi_\lambda(t_\lambda(\theta)t^-_{\lambda_*}(\tilde{w}_\lambda+\theta\psi)(\tilde{w}_\lambda+\theta\psi)) = J^-_\lambda(t^-_{\lambda_*}(\tilde{w}_\lambda+\theta\psi)(\tilde{w}_\lambda+\theta\psi)) \geq \tilde{J}^-_{\lambda,d^-,C} = \Phi_\lambda(t^-_\lambda(\tilde{w}_\lambda)\tilde{w}_\lambda),$$

which lead us to conclude that

$$\Phi_\lambda(t_\lambda(\theta)t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi)(\tilde{w}_\lambda + \theta\psi)) \geq \Phi_\lambda(t^-_\lambda(\tilde{w}_\lambda)t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi)\tilde{w}_\lambda), \tag{1.29}$$

holds for all $\theta > 0$ small enough, after using Proposition 1.1.1.

Again, due to the fact that $t^-_\lambda(\tilde{w}_\lambda)\tilde{w}_\lambda \in \mathcal{N}^-_\lambda$, we are able to apply the implicit function Theorem, as in Lemma 1.2.3 b) with the same function F at the point

$$\left(t^-_\lambda(\tilde{w}), ||\tilde{w}_\lambda||^2, \lambda\int_{\mathbb{R}^N} a|\tilde{w}_\lambda|^{1-\gamma}dx, \int_{\mathbb{R}^N} b|\tilde{w}_\lambda|^{p+1}dx\right)$$

to show that $t_\lambda(\theta) \to t^-_\lambda(\tilde{w})$ as $\theta \to 0$. Since (1.29) can be read as

$$\begin{aligned}
&(t_\lambda(\theta)t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi))^2\frac{[||\tilde{w}_\lambda + \theta\psi||^2 - ||\tilde{w}_\lambda||^2]}{\theta} \\
&-\frac{(t_\lambda(\theta)t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi))^{p+1}}{p+1}\int \frac{b(\tilde{w}_\lambda + \theta\psi)^{p+1} - b(\tilde{w}_\lambda)^{p+1}}{\theta}dx \\
\geq\ &\frac{(t_\lambda(\theta)t^-_{\lambda_*}(\tilde{w}_\lambda + \theta\psi))^{1-\gamma}}{1-\gamma}\lambda\int \frac{a(\tilde{w}_\lambda + \theta\psi)^{1-\gamma} - a(\tilde{w}_\lambda)^{1-\gamma}}{\theta}dx,
\end{aligned}$$

we can follow the arguments done in Lemma 1.2.4, Fatou's Lemma and $t_\lambda(\theta) \to t^-_\lambda(\tilde{w})$ as $\theta \to 0$, to infer that

$$(t^-_\lambda(\tilde{w}_\lambda))^2(\tilde{w}_\lambda, \psi) - (t^-_\lambda(\tilde{w}_\lambda))^{p+1}\int b\tilde{w}^p_\lambda\psi dx \geq (t^-_\lambda(\tilde{w}_\lambda))^{1-\gamma}\lambda\int a\tilde{w}^{-\gamma}_\lambda\psi dx,$$

that is,

$$(w_\lambda, \psi) - \int bw^p_\lambda\psi dx \geq \lambda\int aw^{-\gamma}_\lambda\psi dx.$$

To conclude that $w_\lambda \in \mathcal{N}_\lambda^-$ is a solution from (P_λ), we do as in Proposition 1.2.1. To complete the proof of Proposition 1.4.6, let us follow the arguments done just above with minors adjustments. First, by setting $u_\lambda = t_\lambda^+(\tilde{u}_\lambda)\tilde{u}_\lambda \in \mathcal{N}_\lambda^+$, with $\tilde{u}_\lambda \in \mathcal{N}_{\lambda_*,d^+,c}^-$ being the minimizer of J_λ^+ constrained to $\mathcal{N}_{\lambda_*,d^+,c}^+$ as given in Proposition 1.4.5, and adjusting the proof of the above claim, we also prove the below claim.

Claim: there exists an $\epsilon_2 > 0$ such that

$$||\tilde{u}_\lambda|| > c, \ \forall \ \lambda \in (\lambda_*, \lambda_* + \epsilon_2),$$

where $c < c_{\lambda_*}$ and $c_{\lambda_*} > 0$ is given by Corollary 1.3.1.

After this claim, by perturbing $\tilde{u}_\lambda$ by appropriate elements of X_+, performing projections of it over $\mathcal{N}_{\lambda_*,d^+,c}^+$ and $\mathcal{N}_\lambda^+$ and following the same strategy, we can prove that $u_\lambda \in \mathcal{N}_\lambda^+$ is a solution from (P_λ).

Finally, the proof of Proposition follows by taking $\epsilon = \min\{\epsilon_1, \epsilon_2\} > 0$, that is, for each $\lambda \in (\lambda_*, \lambda_* + \epsilon)$ the problem (P_λ) admits at least two solutions $u_\lambda \in \mathcal{N}_\lambda^+$ and $w_\lambda \in \mathcal{N}_\lambda^-$. This ends the proof. ∎

1.5 Proof of Theorems

In these section, we are going to prove the main Theorems of this Chapter.

Theorem 0.0.1 Suppose that $0 < \gamma < 1 < p < 2^* - 1; 0 < a \in L^{\frac{2}{1+\gamma}}(\mathbb{R}^N)$, $b \in L^\infty(\mathbb{R}^N)$, $b^+ \neq 0$, $(V)_0$ and $[a/b]^{\frac{1}{p+\gamma}} \notin X$ if $b > 0$ in $\mathbb{R}^N$ hold. Then there exists an $\epsilon > 0$ such that the problem (P_λ) has at least two positive solutions $w_\lambda, u_\lambda \in X$ for each $0 < \lambda < \lambda_* + \epsilon$ given. Besides this, we have:

a) $\frac{d^2\Phi_\lambda}{dt^2}(tu_\lambda)\big|_{t=1} > 0$ and $\frac{d^2\Phi_\lambda}{dt^2}(tw_\lambda)\big|_{t=1} < 0$ for all $0 < \lambda < \lambda_* + \epsilon$,

b) there exists a constant $c > 0$ such that $||w_\lambda|| \geq c$ for all $0 < \lambda < \lambda_* + \epsilon$,

c) u_λ is a ground state solution for all $0 < \lambda \leq \lambda_*$, $\Phi_\lambda(u_\lambda) < 0$ for all $0 < \lambda < \lambda_* + \epsilon$ and $\lim\limits_{\lambda \to 0} ||u_\lambda|| = 0$,

d) the applications $\lambda \longmapsto \Phi_\lambda(u_\lambda)$ and $\lambda \longmapsto \Phi_\lambda(w_\lambda)$ are decreasing for $0 < \lambda < \lambda_* + \epsilon$ and are left-continuous ones for $0 < \lambda < \lambda_*$, *e)* $\Phi_\lambda(w_\lambda) > 0$ for $0 < \lambda < \hat{\lambda}$, $\Phi_{\hat{\lambda}}(w_{\hat{\lambda}}) = 0$ and $\Phi_\lambda(w_\lambda) < 0$ for $\hat{\lambda} < \lambda < \lambda_* + \epsilon$ (see $\hat{\lambda}$ in (1)),

Proof First, we note that the multiplicity is given by Propositions 1.2.1, 1.3.2 and 1.4.6. About qualitative statements, we point out that *a)* is a consequence of Proposition 1.1.1. The statement *b)* follows from Lemma 1.2.1 and Sobolev embeddings. Let us prove *c)*. To prove that u_λ is a ground state solution for each $0 < \lambda \leq \lambda_*$, let us assume that w is another solution for Problem (P_λ). Then $w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ by either Lemma 1.1.4 *a)* or Corollary 1.1.1. If $w \in \mathcal{N}_\lambda^+$, then $\Phi_\lambda(u_\lambda) = \tilde{J}_\lambda^+ \leq \Phi_\lambda(w)$ by definition of $\tilde{J}_\lambda^+$. On the other hand, if $w \in \mathcal{N}_\lambda^-$, it follows from Proposition 1.1.1 and definition of $\tilde{J}_\lambda^+$ that $\Phi_\lambda(w) > \Phi_\lambda(t_\lambda^+ w) \geq \tilde{J}_\lambda^+ = \Phi_\lambda(u_\lambda)$ holds. So, combining both cases, we conclude that u_λ is a ground state solution for Problem (P_λ). Now, by (1.16) we have that $\Phi_\lambda(u_\lambda) < 0$ for all $0 < \lambda < \lambda_* + \epsilon$. From (1.14) and Sobolev embeddings, we have that $\lim_{\lambda \to 0} ||u_\lambda|| = 0$. The statement *d)* follow from Propositions 1.3.1, 1.4.3.

Finally, let us prove the item *e)*. First, we note that $\hat{\lambda}$ and λ_*, as defined at (1) and (2), respectively, are such that $\hat{\lambda} < \lambda_*$ and $\hat{\lambda} = \inf\{\hat{\lambda}(w) : w \in X_+ \text{ and } \int_{\mathbb{R}^N} b|w|^{p+1}dx > 0\}$, where $(\hat{\lambda}(w), \hat{t}(w))$ is the unique solution of the system $\phi_{\lambda,w}(t) = 0, \phi'_{\lambda,w}(t) = 0$. So, it follows from Proposition 1.2.1 that there exists a $w_{\hat{\lambda}} \in \mathcal{N}_{\hat{\lambda}}^-$ solution of Problem $(P_{\hat{\lambda}})$.

Now, by applying Proposition 1.1.1, we obtain that

$$\Phi_\lambda(w_\lambda) = \phi_{\lambda,w_\lambda}(1) \geq \phi_{\lambda,w_\lambda}(t(w_\lambda)) = \Phi_\lambda(t(w_\lambda)w_\lambda) > \Phi_{\hat{\lambda}(w_\lambda)}(t(w_\lambda)w_\lambda) = 0, \tag{1.30}$$

holds for each $0 < \lambda < \hat{\lambda}$ given, where $w_\lambda \in \mathcal{N}_\lambda^-$ is the solution of (P_λ) given by Proposition 1.2.1.

On the other hand, by proceeding as done in Lemma 1.1.3, we are able to prove that there exists a $w \in X_+$ such that $\hat{\lambda} = \hat{\lambda}(w)$ and $\Phi_{\hat{\lambda}}(w) = \phi_{\hat{\lambda},w}(1) = \phi'_{\hat{\lambda},w}(1) = 0$. Hence, the Proposition 1.1.1 imply that $t^-(w_{\hat{\lambda}}) = 1$, which lead us

$$0 = \Phi_{\hat{\lambda}}(w) \geq \Phi_{\hat{\lambda}}(w_{\hat{\lambda}}) = \tilde{J}_{\hat{\lambda}}^-. \tag{1.31}$$

As a consequence of (1.30) and of the fact that $\tilde{J}_\lambda^-$ is a decreasing and left-continuous function, we have that $\Phi_{\hat{\lambda}}(w_{\hat{\lambda}}) = \tilde{J}_{\hat{\lambda}}^- \geq 0$. So, this inequality together with (1.31) lead us to conclude that $\tilde{J}_{\hat{\lambda}}^- = \Phi_{\hat{\lambda}}(w_{\hat{\lambda}}) = 0$. The rest of the proof follows from the fact that the function $\tilde{J}_\lambda^- = \Phi_\lambda(w_\lambda)$ is decreasing for $0 < \lambda < \lambda_* + \epsilon$, as showed in Propositions 1.3.1, 1.4.3. ■

Below, we are going to prove Theorem 0.0.2.

Theorem 0.0.2 Suppose that the hypotheses of Theorem 0.0.1 hold. Moreover, assume that there exists a smooth bounded open set $\Omega \subset \mathbb{R}^N$ such that $b > 0$ in Ω and $a \in L^\infty(\Omega)$. Then there exists $\lambda^* > 0$ such that the problem (P_λ) has no solution at all for $\lambda > \lambda^*$. Moreover, we have the exact estimate

$$0 < \lambda_* < \lambda^* = \lambda_1^{\frac{p+\gamma}{p-1}} \left(\frac{\gamma+1}{p-1}\right)^{\frac{\gamma+1}{p-1}} \left(\frac{p-1}{p+\gamma}\right)^{\frac{p+\gamma}{p-1}},$$

where $\lambda_1 > 0$ is given in Lemma 1.5.1.

To prove the theorem we will need a preliminary lemma. Take a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and consider the eigenvalue problem

$$\begin{cases} -\Delta u + V(x)u = \lambda m(x)u \text{ in } \Omega \\ \quad u > 0 \text{ in } \Omega, \quad u \in H_0^1(\Omega), \end{cases} \tag{A_Ω}$$

where $m(x) = \min\{a(x), b(x)\}$. So, by a classical argument and Theorem 3 in Brezis-Nirenberg [12], we have.

Lemma 1.5.1 *The first eigenvalue λ_1 of the problem (A_Ω) is positive. Moreover, its associated eigenfunction e_1 is positive, $e_1 \in C^1(\overline{\Omega}) \cap H^2(\Omega)$ and $\partial e_1/\partial\nu \leqslant 0$ on $\partial\Omega$, where $\nu \in \mathbb{R}^N$ is the unit exterior normal to $\partial\Omega$.*

Proof [Proof of Theorem 0.0.2] Let us define $g : (0, \infty) \to \mathbb{R}$ by $g(t) = \lambda t^{-\gamma-1} + t^{p-1}$ and note that

$$t_\lambda = \left(\lambda\left(\frac{\gamma+1}{p-1}\right)\right)^{\frac{1}{p+\gamma}}, \ \lambda > 0,$$

is the its unique global minimum whose minimum value is given by

$$\tilde{g}(\lambda) := g(t_\lambda) = \lambda^{\frac{p-1}{p+\gamma}} \left(\frac{\gamma+1}{p-1}\right)^{\frac{-\gamma-1}{p+\gamma}} \left(\frac{p+\gamma}{p-1}\right),$$

which provides the existence of a $\lambda^* > 0$ such that $\tilde{g}(\lambda^*) = \lambda_1$, that is,

$$\lambda^* = \lambda_1^{\frac{p+\gamma}{p-1}} \left(\frac{\gamma+1}{p-1}\right)^{\frac{\gamma+1}{p-1}} \left(\frac{p-1}{p+\gamma}\right)^{\frac{p+\gamma}{p-1}}.$$

Assume that $u_\lambda \in X_+$ is a solution for Problem (P_λ). By Brezis-Nirenberg Theorem (see [12] Theorem 3 again), we have that $u_\lambda^{-\gamma} \in L^\infty(K)$ for every $K \subset\subset \Omega$ which implies by Theorem 12.2.2 (see J. Jost [43]) that $u_\lambda \in H^{2,\frac{2^*}{p}}(K)$ and

$$-\Delta u_\lambda = \lambda a(x) u_\lambda^{-\gamma} + b(x) u_\lambda^p - V(x) u_\lambda \text{ a. e. in } \Omega,$$

and after a classical bootstrap argument, we obtain that $u_\lambda \in H^2(\Omega) \cap C(\overline{\Omega})$. Now we apply Lemma 3.5 of Figueiredo-Gossez-Ubilla [23] to conclude that

$$\int_\Omega \nabla u_\lambda \nabla e_1 + V(x) e_1 u_\lambda dx \le \lambda_1 \int_\Omega m(x) e_1 u_\lambda dx. \tag{1.32}$$

So, it follows from the definition of λ^*, (1.32) and the fact that u_λ is a solution fo Problem (P_λ), that

$$\begin{aligned} \int_\Omega m(x)(\lambda^* u_\lambda^{-\gamma} + u_\lambda^p) e_1 dx \ge \lambda_1 \int_\Omega m(x) u_\lambda e_1 dx &\ge \int_\Omega \nabla e_1 \nabla u_\lambda + V(x) e_1 u_\lambda dx \\ &= \lambda \int_\Omega a(x) u_\lambda^{-\gamma} e_1 dx + \int_\Omega b(x) u_\lambda^p e_1 dx. \end{aligned}$$

Since $a(x), b(x) \ge m(x)$ in Ω, the last inequality lead us to

$$\begin{aligned} \lambda^* \int_\Omega a(x) u_\lambda^{-\gamma} e_1 dx + \int_\Omega b(x) u_\lambda^p e_1 dx &\ge \int_\Omega m(x)(\lambda^* u_\lambda^{-\gamma} + u_\lambda^p) e_1 dx \\ &\ge \lambda \int_\Omega a(x) u_\lambda^{-\gamma} e_1 dx + \int_\Omega b(x) u_\lambda^p e_1 dx, \end{aligned}$$

which implies that $\lambda^* \ge \lambda$. This ends the proof.

■

Chapter 2

Extremal regions and multiplicity of positive solutions for singular superlinear elliptic systems with indefinite-sign potential

In this chapter, we study the following elliptic system with singular nonlinearities

$$\begin{cases} -\Delta u + V(x)u = \lambda a(x)u^{-\gamma} + \dfrac{\alpha}{\alpha+\beta} b(x)u^{\alpha-1}v^{\beta} \ in\ \mathbb{R}^N, \\ -\Delta v + V(x)v = \mu c(x)v^{-\gamma} + \dfrac{\beta}{\alpha+\beta} b(x)u^{\alpha}v^{\beta-1} \ in\ \mathbb{R}^N, \\ u,v>0,\ \mathbb{R}^N,\ \displaystyle\int_{\mathbb{R}^N} Vu^2 + \int_{\mathbb{R}^N} Vv^2 < \infty,\ u,v \in H^1(\mathbb{R}^N), \end{cases} \qquad (\tilde{P}_{\lambda,\mu})$$

where $0 < a, c$ in $\mathbb{R}^N$, $b^+ \not\equiv 0$, $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous function; $0 < \gamma < 1 < \alpha, \beta$; $2 < \alpha + \beta < 2^*$; $N \geq 3$ and $\lambda, \mu \geq 0$ are real parameters.

To show the multiplicity of solutions for $(\tilde{P}_{\lambda,\mu})$, we will use the Nehari manifold method and the fibering method again. As in Chapter 1, the functional associated to the problem $(\tilde{P}_{\lambda,\mu})$ is not Gâteaux differentiable. As we already mentioned, by considering the problem with unrelated (λ, μ), as previous works have done, few information can be obtained about the set of parameters such that $(\tilde{P}_{\lambda,\mu})$ has solutions. Thus, the main idea to overcome this difficulty is to modify the problem $(\tilde{P}_{\lambda,\mu})$ to $(\tilde{P}_{\lambda,\theta\lambda})$ for each $\theta > 0$. With this modification, we are able to solve a similar system to that one considered in Chapter 1 (see (2.15)-(2.16)) and find $\lambda_*(\theta)$ as an extremal value in the sense of the applicability of Nehari method. By varying $\theta > 0$ we get a continuous curve $\tilde{\Gamma}(\theta) = (\lambda_*(\theta), \theta\lambda_*(\theta))$, which represents a part of the boundary of the set of the positive parameters (λ, μ) for which there is a solution for the system $(\tilde{P}_{\lambda,\mu})$, and this set is bigger than those considered by previous works. In addition, we obtain multiplicity of solution for parameters above $\tilde{\Gamma}(\theta)$, but close to it. These results generalize to the system $(\tilde{P}_{\lambda,\mu})$ the results obtained in the Chapter 1.

This chapter follows the following structure. In the first section, we present a new definition of critical points for non-differentiable functionals and prove a new abstract theorem for functionals of this type. We will also present some consequences of this abstract result and it will be applied in the next section. In Section 2.2, we use the abstract Theorem of Section 2.1 to show that some local minimizers over the Nehari manifold of functional associated with system $(\tilde{P}_{\lambda,\mu})$ are critical points in the sense of the abstract Theorem, and therefore, solutions of the system. After this, we study some topological structures associated to the energy functional associated with the modify problem $(\tilde{P}_{\lambda,\theta\lambda})$ for each $\theta > 0$. So, we introduce the Nehari manifold associated with the problem $(\tilde{P}_{\lambda,\theta\lambda})$ and study some of its properties as well, in a similar way to that done in Chapter 1. We also built the extremal curves claimed in the Theorem 0.0.3.

In Section 2.3, we show the multiplicity of solutions to the problem $(\tilde{P}_{\lambda,\theta\lambda})$ for $\lambda \in (0, \lambda_*(\theta))$, where $\theta > 0$ is fixed (see (2.20) for the definition of $\lambda_*(\theta)$). In Section 2.4, we show the multiplicity of solutions to $(\tilde{P}_{\lambda,\theta\lambda})$, when $\lambda = \lambda_*(\theta)$ and in Section 2.5, we show the multiplicity of solutions to $(\tilde{P}_{\lambda,\theta\lambda})$ when λ is greater than $\lambda_*(\theta)$, but close to it and at the end of this section we prove Theorem 0.0.3. Many results obtained in sections 2.2, 2.3, 2.4 and 2.5 are generalizations of those obtained in Chapter 1.

In the last section, we prove the supersolution Theorem 0.0.4 and the Theorem 0.0.5. To show the supersolution theorem we were inspired by Struwe [59], and combined a truncation argument with Perron's method to prove the existence of solution to the truncated problem. After this, through a fine analysis we obtain that the solutions of truncated problem converges to a solution of our problem. The next step is, through some preliminary lemmas, to show the existence of the function Γ^* stated in the Theorem 0.0.5 and finally proves the Theorem 0.0.5.

To state our main results, let us set

$$X = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 dx < \infty \right\}, \ E = X \times X,$$

and assume

$(V)_0$ $V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0$,

$(V)_1$ $1/V \in L^1(\mathbb{R}^N)$,

$(A1)$ $a, c \in L^\infty(\mathbb{R}^N) \cap L^{\frac{2}{1+\gamma}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$,

$(A2)$ $b^+ \not\equiv 0$ and $b \in L^\infty(\mathbb{R}^N) \cap L^{\frac{2^*}{2^*-\alpha-\beta}}(\mathbb{R}^N)$,

$(A3)$ $\left[\frac{a(x)}{b(x)}\right]^{\frac{1}{\alpha+\beta+\gamma-1}} \left[\frac{c(x)}{a(x)}\right]^{\frac{\beta}{(1-\gamma)(\alpha+\beta+\gamma-1)}} \notin X.$

As a consequence of these assumptions, we have well-defined the functionals

- $J(U) = \|U\|^2$,
- $K_{\lambda,\mu}(U) = \lambda \int_{\mathbb{R}^N} a(x)|u|^{1-\gamma} dx + \mu \int_{\mathbb{R}^N} c(x)|v|^{1-\gamma} dx$,
- $L(U) = \int_{\mathbb{R}^N} b(x)|u|^\alpha |v|^\beta dx$,
- $\langle J'(U), \Psi \rangle_E = \int_{\mathbb{R}^N} [\nabla u \nabla \varphi + V(x) u \varphi] dx + \int_{\mathbb{R}^N} [\nabla v \nabla \psi + V(x) v \psi] dx$,
- $\langle L'(U), \Psi \rangle_E = \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} b(x)|u|^{\alpha-2} u |v|^\beta \varphi dx + \dfrac{\beta}{\alpha+\beta} \int_{\mathbb{R}^N} b(x)|u|^\alpha |v|^{\beta-2} v \psi dx.$

and

- $\langle K'_{\lambda,\mu}(U), \Psi \rangle_E = \lambda \int_{\mathbb{R}^N} a(x)|u|^{-1-\gamma} u \varphi dx + \mu \int_{\mathbb{R}^N} c(x)|v|^{-1-\gamma} v \psi dx$,

if

$$\int_{\mathbb{R}^N} a(x)|u|^{-1-\gamma} u \varphi dx \in \mathbb{R} \text{ and } \int_{\mathbb{R}^N} c(x)|v|^{-1-\gamma} v \psi dx \in \mathbb{R}$$

hold, where $U = (u, v), \Psi = (\varphi, \psi) \in E$.

With these notations, a pair $U = (u, v) \in E$ is a solution of $(\tilde{P}_{\lambda,\mu})$ if

$$\langle J'(U), \Psi \rangle_E - \langle K'_{\lambda,\mu}(U), \Psi \rangle_E - \langle L'(U), \Psi \rangle_E = 0,$$

for all $\Psi \in E$.

2.1 An Abstract existence theorem for non-differentiable functionals on cones

In this section, we will give a new notion of critical points for non-differentiable functionals and prove a new abstract theorem. Throughout this section, we assume that F is a Banach space, $\mathcal{C} \subset F$ a cone with $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$ and $\leqq$ the partial order defined on $\mathcal{C}$. We also assume that $\mathcal{C}$ is reproducing, that is, $\mathcal{C} - \mathcal{C} = F$ (see Deimling [24] p. 219). So, for each $u \in F$ we have that $u = u^+ - u^-$, where $u^+, u^- \in \mathcal{C}$.

Let $I : F \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a functional such that $I = \Phi + \psi$, with $\Phi \in C^1(F, \mathbb{R})$ and $\psi : F \longrightarrow \mathbb{R} \cup \{+\infty\}$ be proper, that is, $D(\psi) = \{u \in F : \psi(u) < +\infty\} \neq \emptyset$. The set $D(\psi)$ is called the effective domain of ψ.

We state a new definition of critical point.

Definition 2.1.1 *A point $u \in D(\psi)$ is said to be a critical point of I if, for each $v \in F$, there exist an $\epsilon_0 > 0$ (which may depends on v) and a function $\xi : [0, \epsilon_0] \longrightarrow \mathbb{R}_0^+$ such that:*

i) $\xi(\epsilon) \to 1$ *as* $\epsilon \to 0$,

ii) $\xi(\epsilon) u \in D(\psi)$,

iii) the inequality

$$I(\xi(\epsilon)\,(u + \epsilon v)) - I(\xi(\epsilon) u) \geq 0, \tag{2.1}$$

holds for every $0 < \epsilon < \epsilon_0$.

We will now make some remarks about our definition of critical point.

Remark 2.1.1 *Some observations:*

a) if $u \in F$ is a local minimum point of I, then $I(u + \epsilon v) - I(u) \geq 0$ for all small $\epsilon > 0$, what is the definition (2.1) *by taking $\xi(\epsilon) \equiv 1$,*

b) assume that ψ is a convex and lower semicontinuous function and $u \in D(\psi)$ is a critical point in the sense of (2.1)*. Moreover, for each $v \in F$, assume that the function $\tilde{\psi} : [0, \epsilon_0] \mapsto \overline{\mathbb{R}}$ defined by $\tilde{\psi}(\epsilon) = \psi(\xi(\epsilon) v)$ is continuous. Then*

$$\langle \Phi'(u), v - u \rangle + \psi(v) - \psi(u) \geq 0,$$

that is, u is a critical point in Szulkin's sense (see [62]). In fact, it follows from (2.1)

$$\left[\frac{\Phi(\xi(\epsilon)(u + \epsilon(v - u))) - \Phi(\xi(\epsilon) u)}{\epsilon}\right] + \left[\frac{\psi(\xi(\epsilon)(u + \epsilon(v - u))) - \psi(\xi(\epsilon) u)}{\epsilon}\right] \geq 0,$$

which implies by the mean value theorem and convexity of ψ, that

$$\langle \Phi'(\kappa(\epsilon)), \xi(\epsilon)(v-u)\rangle + \psi(\xi(\epsilon)v) - \psi(\xi(\epsilon)u) \geq 0,$$

holds for $0 < \epsilon < \epsilon_0$, where $\kappa(\epsilon) \to u$ as $\epsilon \to 0$. Doing $\epsilon \to 0$ in the last inequality, using i), the continuity of $\psi(\xi(\epsilon)v)$, and lower semicontinuity of ψ, we obtain the claim,

c) *the above conclusion holds if we assume the function ψ be a homogeneous instead of $\tilde{\psi}$ being a continuous function,*

d) *a minimum point over the natural subsets of the Nehari manifold splitting are critical points in sense of Definition 2.1.1. We remember that this is true in spite of I being a non-differentiable functional.*

Keeping in mind the particular case of Remark 2.1.1 *b*), as pointed out in Moameni [49], it is well known that the solutions of (2.1) may not be solutions of equations of the type

$$\langle \Phi'(u), v\rangle + \langle \psi'(u), v\rangle = 0, \text{ for all } v \in F, \tag{2.2}$$

unless that $D(\psi) = F$. Therefore, in addition a point $u \in D(\psi)$ to be a critical point of I, additional hypotheses must be introduced for it be a solution of a equation of the type (2.2).

In this section, our aim is giving conditions for that a critical point in the sense of Definition 2.1.1 be a solution of equations of type

$$\langle \Phi'(u), v\rangle + L[u](v) = 0, \text{ for all } v \in F,$$

where $L[u] : F \to \mathbb{R}$ is a linear map.

Let us assume that $u \in F$ satisfies the condition:

$(L)_1$ there exists a linear map $L[u] : F \to \mathbb{R}$ such that

$$\limsup_{\epsilon \downarrow 0} \frac{\psi(\xi(\epsilon)(u+\epsilon v)) - \psi(\xi(\epsilon)u)}{\epsilon} \leq L[u]v,$$

for every $v \in \mathcal{C}$.

The main result of this section is the following theorem.

Theorem 2.1.1 *Let $u \in D(\psi) \cap \mathcal{C}$. Assume that for each $v \in \mathcal{C}$ there exist an $\epsilon_0 > 0$ and a function $\xi : [0, \epsilon_0] \longrightarrow \mathbb{R}_0^+$ satisfying the conditions i), ii) of Definition 2.1.1, and*

$$I(\xi(\epsilon)(u+\epsilon v)) - I(\xi(\epsilon)u) \geq 0,$$

for $0 < \epsilon < \epsilon_0$. Assume also $(L)_1$ and the conditions:

$(a)_1$ *the inequality* $L[u]v \leq 0$, *for every* $v \in \mathcal{C}$,

$(a)_2$ *the equality* $\langle \Phi'(u), u \rangle + L[u](u) = 0$,

$(a)_3$ *the inequality* $\limsup_{\epsilon \downarrow 0} \frac{1}{\epsilon} \langle \Phi'(u), (u + \epsilon v)^- \rangle \leq 0$

hold. Then, u is a solution of the equation

$$\langle \Phi'(u), w \rangle + L[u](w) = 0, \text{ for every } w \in F.$$

Proof Let $v \in \mathcal{C}$ and ϵ_0 and $\xi(\epsilon)$ as in the Definition 2.1.1. So,

$$\left[\frac{\Phi(\xi(\epsilon)(u + \epsilon v)) - \Phi(\xi(\epsilon)u)}{\epsilon}\right] + \left[\frac{\psi(\xi(\epsilon)(u + \epsilon v)) - \psi(\xi(\epsilon)u)}{\epsilon}\right] \geq 0$$

and using the mean value theorem in the functional Φ and $(L)_1$, and doing $\epsilon \downarrow 0$, we obtain that

$$\langle \Phi'(u), v \rangle + L[u](v) \geq 0, \tag{2.3}$$

for every $v \in \mathcal{C}$.

Let $w \in F$, $\epsilon > 0$ and remember that $u + \epsilon w = (u + \epsilon w)^+ - (u + \epsilon w)^-$. Since $(u + \epsilon w)^+ \in \mathcal{C}$, it follows from (2.3) that

$$\begin{aligned} 0 &\leq \langle \Phi'(u), (u + \epsilon w)^+ \rangle + L[u]((u + \epsilon w)^+) \\ &= \langle \Phi'(u), u + \epsilon w + (u + \epsilon w)^- \rangle + L[u](u + \epsilon w + (u + \epsilon w)^-) \\ &= \langle \Phi'(u), u \rangle + L[u](u) + \epsilon \left[\langle \Phi'(u), w \rangle + L[u](w) \right] \\ &+ \langle \Phi'(u), (u + \epsilon w)^- \rangle + L[u]((u + \epsilon w)^-). \end{aligned}$$

So, using this last inequality and $(a)_1 - (a)_2$, we have

$$0 \leq \epsilon \left[\langle \Phi'(u), w \rangle + L[u](w) \right] + \langle \Phi'(u), (u + \epsilon w)^- \rangle$$

and this implies by $(a)_3$ that

$$\begin{aligned} 0 \leq& \langle \Phi'(u), w \rangle + L[u](w) + \limsup_{\epsilon \downarrow 0} \frac{1}{\epsilon} \langle \Phi'(u), (u + \epsilon w)^- \rangle \\ \leq& \langle \Phi'(u), w \rangle + L[u](w), \end{aligned}$$

that is,

$$0 \leq \langle \Phi'(u), w \rangle + L[u](w),$$

for every $w \in F$. Replacing w by $-w$ in the last inequality above, we finally derive that

$$\langle \Phi'(u), w \rangle + L[u](w) = 0,$$

for every $w \in F$. ■

Looking at the proof of Theorem 2.1.1, it also proves the following Corollary.

Corollary 2.1.1 *Assume that u is a critical point of I and $(L)_1$ holds. Moreover, assume that the conditions:*

$(a)_1$ *the inequality $L[u]v \leq 0$, for every $v \in \mathcal{C}$,*

$(a)_2$ *the equation $\langle \Phi'(u), u \rangle + L[u](u) = 0$,*

$(a)_3$ *the inequality $\limsup_{\epsilon \downarrow 0} \frac{1}{\epsilon} \langle \Phi'(u), (u + \epsilon v)^- \rangle \leq 0$*

hold. Then, u is a solution of the equation

$$\langle \Phi'(u), w \rangle + L[u](w) = 0, \text{ for every } w \in F.$$

To state and prove the next corollary, let us remember the concepts of subdifferential and subgradient.

Let $\phi : F \to \overline{\mathbb{R}}$ be a proper functional and $u \in D(\phi)$. The subdifferential of ϕ at u is the subset $\partial\phi(u)$ of F^*, defined by

$$\partial\phi(u) = \{\eta \in F^* : \langle \eta, v - u \rangle \leq \phi(v) - \phi(u), \text{ for all } v \in F\}.$$

The elements of $\partial\phi(u)$ are called subgradients of ϕ at u.

Now we have the corollary.

Corollary 2.1.2 *Assume that ψ is a convex and lower semicontinuous functional. Moreover, assume $(L)_1$, the hypotheses $(a)_1, (a)_3$ of Theorem 2.1.1 and ψ satisfies*

$$\psi(tv) = t^\alpha \psi(v) \text{ and } \frac{1}{\alpha} L[u]u = \psi(u), \tag{2.4}$$

for every $t > 0$, $v \in E$ and some $\alpha \in \mathbb{R} \setminus \{0\}$. If $u \in \mathcal{C}$ is a critical point of I, then u is a solution of the equation

$$\langle \Phi'(u), w \rangle + L[u](w) = 0, \text{ for every } w \in F.$$

Proof It is sufficient to show that the condition $(a)_2$ of Theorem 2.1.1 is satisfied. We have from Remark 2.1.1 $b)$ that u is a critical point in Szulkin's sense, which implies together with Proposition 2.183 of Carl-Khoile-Montreanu [14] that $0 \in \{\Phi'(u)\} + \partial\psi(u)$, and therefore, there exists $\eta \in \partial\phi(u)$ such that

$$\langle \Phi'(u), w\rangle = -\langle \eta, w\rangle \tag{2.5}$$

for every $w \in F$.

Now, by definition of η and (2.4), for $0 < \epsilon < 1$, we have that

$$[(1-\epsilon)^\alpha - 1]\,\frac{1}{\alpha}L[u]u = [(1-\epsilon)^\alpha - 1]\,\psi(u) = \psi(u - \epsilon u) - \psi(u) \geq -\epsilon\langle \eta, u\rangle,$$

and dividing this last expression by ϵ, and doing $\epsilon \to 0$, we obtain

$$-L[u]u \geq -\langle \eta, u\rangle. \tag{2.6}$$

On the other hand, as u is critical point, by using the mean value theorem, we have that

$$\langle \Phi'(\kappa(\epsilon)), \epsilon v\rangle + \left[\frac{\psi(\xi(\epsilon)(u+\epsilon v)) - \psi(\xi(\epsilon)u)}{\epsilon}\right] \geq 0, \tag{2.7}$$

for every $v \in \mathcal{C}$, $0 < \epsilon < \epsilon_0$, where $\kappa(\epsilon) \to u$ as $\epsilon \to 0$, and ϵ_0 and $\xi(\epsilon)$ are as in the Definition 2.1.1. So, we may divide (2.7) by ϵ, do $\epsilon \to 0$ and use $(L)_1$ to obtain that

$$\langle \Phi'(u), v\rangle + L[u]v \geq 0,$$

for every $v \in \mathcal{C}$. In particular, taking $v = u$ in this last inequality and using (2.5), we conclude that

$$-\langle \eta, u\rangle \geq -L[u]u. \tag{2.8}$$

From (2.6) and (2.8) it follows that $-\langle \eta, u\rangle = -L[u]u$ and combining this with (2.5) we conclude that

$$\langle \Phi'(u), u\rangle + L[u]u = 0.$$

The proof is complete.

■

2.2 Reduction to one-parameter of $(\tilde{P}_{\lambda,\mu})$ and extremal region to applicability of Nehari's method

In this section, let us to prove some topological properties for the functional $\Phi_{\lambda,\mu}$ associated to the problem $(\tilde{P}_{\lambda,\mu})$. Let us denote by

$$X = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\}, \ E = X \times X$$

and endow X with the inner product

$$(u, w) = \int_{\mathbb{R}^N} \nabla u \nabla w + V(x) uw dx,$$

which turns X in a Hilbert space with induced norm given by $||u||^2 = (u, u)$. As a consequence, one deduces immediately from $(V)_0$ that X is embedded continuously into $H^1(\mathbb{R}^N)$. From these properties of X follow that E is a Hilbert space with the inner product $(U, W) = (u_1, w_1) + (u_2, w_2)$, where $U = (u_1, w_1), W = (u_2, w_2)$ and induced norm given by $||(u, w)||^2 = (u, u) + (w, w)$. The below Lemma was proved in [20].

Lemma 2.2.1 *Assume that $(V)_0 - (V)_1$ hold. The subspace E is continuously embedded into $L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$ for $q \in [1, 2^*]$ and compactly embedded for all $q \in [1, 2^*)$.*

After this Lemma, we have well-defined the energy functional $\Phi_{\lambda,\mu} : E \longrightarrow \mathbb{R}$ associated to the problem $(\tilde{P}_{\lambda,\mu})$ given by

$$\Phi_{\lambda,\mu}(U) = \frac{1}{2}J(U) - \frac{1}{1-\gamma}K_{\lambda,\mu}(U) - \frac{1}{\alpha+\beta}L(U).$$

We can rewrite it as

$$\Phi_{\lambda,\mu}(U) = \Phi(U) + \psi(U),$$

where $\Phi \in C^1(E, \mathbb{R})$ and ψ are defined by

$$\Phi(U) = \frac{1}{2}J(U) - \frac{1}{\alpha+\beta}L(U) \text{ and } \psi(U) = -\frac{1}{1-\gamma}K_{\lambda,\mu}(U).$$

Again, because of the singular terms $\Phi_{\lambda,\mu}$ is not Gâteaux differentiable. By using Lemma 2.2.1 the proof of the next lemma is very similar to the proof of lemma 1.1.2.

Lemma 2.2.2 *If $\lambda, \mu \geq 0$, then ψ is a continuous and weakly lower semicontinuous functional.*

After this result, we have that the functional $\Phi_{\lambda,\mu}$ has the same structure of functional I of Section 2.1. Our next goal will be to apply the Theorem 2.1.1.

2.2.1 An application of Theorem 2.1.1

To prove multiplicity of solutions for $(\tilde{P}_{\lambda,\mu})$, we will use a refinement of Nehari manifold and the fibering method again together with Theorem 2.1.1 to show that some local minimizers over the natural subsets of Nehari manifold are solutions of the system $(\tilde{P}_{\lambda,\mu})$. From now on, let us assume that $\lambda, \mu \geq 0$ with $\lambda + \mu > 0$.

Now, let us see that $E = X \times X$ has a cone reproducing. For each $u \in X$ we have that $u = u^+ - u^-$, where $u^+(x) = \max\{u(x), 0\} \geq 0$ and $u^-(x) = \max\{-u(x), 0\} \geq 0$, and hence, the cone $\mathcal{C} = X_+ = \left\{u \in X : u \geq 0 \text{ in } \mathbb{R}^N\right\}$ is such that $X = X_+ - X_+$, that is, X_+ is a cone reproducing in X. As a consequence of this, we have that the cone

$$E_+ = \{U \in E : U = (u, v) \geq (0, 0)\}$$

is a cone reproducing of E.

For each $U \in E_+$, we consider the fiber map $\phi_{U,(\lambda,\mu)} \in C^\infty((0,\infty), \mathbb{R})$, defined by $\phi_{U,(\lambda,\mu)}(t) = \Phi_{\lambda,\mu}(t)$, and the Nehari manifold associated to the problem $(\tilde{P}_{\lambda,\mu})$, defined by

$$\mathcal{N}_{\lambda,\mu} = \left\{U \in E_+ : ||U||^2 - K_{\lambda,\mu}(U) - L(U) = 0\right\} = \left\{U \in E_+ : \phi'_{U,(\lambda,\mu)}(1) = 0\right\}.$$

In order to find multiplicity of solutions for $(P)_{\lambda,\mu}$, as in Chapter 1, we have the following the decomposition: $\mathcal{N}_{\lambda,\mu} = \mathcal{N}^-_{\lambda,\mu} \cup \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$, where

$$\begin{aligned}\mathcal{N}^-_{\lambda,\mu} &= \left\{U \in \mathcal{N}_{\lambda,\mu} : ||U||^2 + \gamma K_{\lambda,\mu}(U) - (\alpha+\beta-1)L(U) < 0\right\} \\ &= \left\{U \in \mathcal{N}_{\lambda,\mu} : \phi''_{U,(\lambda,\mu)}(1) < 0\right\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}^+_{\lambda,\mu} &= \left\{U \in \mathcal{N}_{\lambda,\mu} : ||U||^2 + \gamma K_{\lambda,\mu}(U) - (\alpha+\beta-1)L(U) > 0\right\} \\ &= \left\{U \in \mathcal{N}_{\lambda,\mu} : \phi''_{U,(\lambda,\mu)}(1) > 0\right\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}^0_{\lambda,\mu} &= \left\{U \in \mathcal{N}_{\lambda,\mu} : ||U||^2 + \gamma K_{\lambda,\mu}(U) - (\alpha+\beta-1)L(U) = 0\right\} \\ &= \left\{U \in \mathcal{N}_{\lambda,\mu} : \phi''_{U,(\lambda,\mu)}(1) = 0\right\}.\end{aligned}$$

The next proposition is straightforward.

Proposition 2.2.1 *Let $U \in E_+ \backslash \{(0,0)\}$ and $\lambda + \mu > 0$. If $L(U) \leq 0$, then $\phi_{U,(\lambda,\mu)}$ has only one critical point at $t^+_{\lambda,\mu}(U) \in (0,\infty)$, which satisfies $\phi''_{U,(\lambda,\mu)}(t^+_{(\lambda,\mu)}(U)) > 0$. If $L(U) > 0$, then there are three possibilities:*

(I) *there are only two critical points for* $\phi_{U,(\lambda,\mu)}$. *The first of them is* $t^+_{(\lambda,\mu)}(U)$ *with* $\phi''_{U,(\lambda,\mu)}(t^+_{\lambda,\mu}(U)) > 0$ *and the second one is* $t^-_{U,(\lambda,\mu)}(U)$ *with* $\phi''_{U,(\lambda,\mu)}(t^-_{\lambda,\mu}(U)) < 0$. *Moreover,* $\phi_{U,(\lambda,\mu)}$ *is decreasing in the intervals* $[0, t^+_{(\lambda,\mu)}(U)], [t^-_{(\lambda,\mu)}(U), \infty)$ *and increasing in the interval* $[t^+_{(\lambda,\mu)}(U), t^-_{(\lambda,\mu)}(U)]$ *(evidently* $0 < t^+_{(\lambda,\mu)}(U) < t^-_{(\lambda,\mu)}(U)$*),*

(II) *there is only one critical point* $t^0_{U,(\lambda,\mu)}(U) > 0$ *for* $\phi_{U,(\lambda,\mu)}$*, which is an inflection point. Moreover,* $\phi_{U,(\lambda,\mu)}$ *is decreasing for* $t > 0$*,*

(III) *the function* $\phi_{U,(\lambda,\mu)}$ *is decreasing for* $t > 0$ *and hence there are no critical points.*

The fiber analysis of Proposition 2.2.1 and Theorem 2.1.1 allows us to prove the following proposition.

Proposition 2.2.2 *Suppose that* $U = (u, v) \in E_+ \setminus \{(0,0)\}$ *is a local minimizer for* $\Phi_{\lambda,\mu}$ *on* $\mathcal{N}_{\lambda,\mu}$ *such that* $U \notin \mathcal{N}^0_{\lambda,\mu}$ *and* $\lambda + \mu > 0$. *Then,* U *is a solution of* $(\tilde{P}_{\lambda,\mu})$.

Proof We just prove the case $\lambda, \mu > 0$, because the cases where, either $\lambda > 0$ and $\mu = 0$ or $\lambda = 0$ and $\mu > 0$ are very similar. Since $U \notin \mathcal{N}^0_{\lambda,\mu}$, then either $U \in \mathcal{N}^-_{\lambda,\mu}$ or $U \in \mathcal{N}^+_{\lambda,\mu}$. First, assume that $U \in \mathcal{N}^-_{\lambda,\mu}$. Then, there exists a $r > 0$ such that

$$\Phi_{\lambda,\mu}(U) \leq \Phi_{\lambda,\mu}(W), \ \forall \ W \in B_r(U) \cap \mathcal{N}^-_{\lambda,\mu}. \tag{2.9}$$

Let us show that the conditions of Theorem 2.1.1 are satisfied. First, let us consider the function $F \in C^\infty(\mathbb{R}^3 \times (0,\infty), \mathbb{R})$ defined by $F(e, f, g, t) = te - t^{-\gamma} f - t^{\alpha+\beta-1} g$. Since $U \in \mathcal{N}^-_{\lambda,\mu}$, we have

$$F(\|U\|^2, K_{\lambda,\mu}(U), L(U), 1) = 0 \text{ and } \frac{\partial F}{\partial t}\left(\|U\|^2, K_{\lambda,\mu}(U), L(U), 1\right) < 0, \tag{2.10}$$

which implies from the implicit function theorem that there exists an open set $\Omega \subset \mathbb{R}^3$ containing $(\|U\|^2,\ K_{\lambda,\mu}(U), L(U))$, an $\epsilon > 0$ and a function $t \in C^\infty(\Omega, (1-\epsilon, 1+\epsilon))$ such that $F((e, f, g), t(e, f, g)) = 0$ for $(e, f, g) \in \Omega$ and $((e, f, g), t(e, f, g))$ is the only solution to this equation in $\Omega \times (1-\epsilon, 1+\epsilon)$.

It follows from (2.10) and continuity of F that $\frac{\partial F}{\partial t}(e, f, g, t(e, f, g)) < 0$ holds for $(e, f, g) \in \Omega$. Besides this, we obtain from $\Psi = (\varphi, \psi) \in E_+$ that $(\|U + \epsilon\Psi\|^2, K_{\lambda,\mu}(U + \epsilon\Psi), L(U + \epsilon\Psi)) \in \Omega$ for $\epsilon > 0$ small enough. Hence, from Proposition 2.2.1 implies that

$$t(\|U + \epsilon\Psi\|^2, K_{\lambda,\mu}(U + \epsilon\Psi), L(U + \epsilon\Psi)) = t^-_{\lambda,\mu}(U + \epsilon\Psi) := \xi(\epsilon),$$

and since $\xi(\epsilon) = t^{-}_{\lambda,\mu}(U + \epsilon\Psi) \to 1$ as $\epsilon \to 0$, we have

$$\xi(\epsilon)(U + \epsilon\Psi) \in B_r(U) \cap \mathcal{N}^{-}_{\lambda,\mu},$$

for ϵ small enough, so that Proposition 2.2.1 and (2.9) lead us to

$$\Phi_{\lambda,\mu}(\xi(\epsilon)U) \le \Phi_{\lambda,\mu}(U) \le \Phi_{\lambda,\mu}(\xi(\epsilon)(U + \epsilon\Psi)),$$

that is, U is a critical point of $\Phi_{\lambda,\mu}$ in the sense of the Definition 2.1.1, and this implies that

$$\begin{aligned}(\xi(\epsilon))^{1-\gamma} \frac{[K_{\lambda,\mu}(U + \epsilon\Psi) - K_{\lambda,\mu}(U)]}{1-\gamma} \le (\xi(\epsilon))^2 \frac{[\|U + \epsilon\Psi\|^2 - \|U\|^2]}{2} \\ - (\xi(\epsilon))^{\alpha+\beta} \frac{[L(U + \epsilon\Psi) - L(U)]}{\alpha+\beta}.\end{aligned}$$

Dividing the last inequality above by $\epsilon > 0$, and doing $\epsilon \to 0$, we obtain that

$$\lambda \int a(x)G(x)\varphi(x)dx + \mu \int a(x)H(x)\psi(x)dx \le \langle \Phi'(U), \Psi\rangle, \tag{2.11}$$

where $\Phi' = J' - L'$,

$$G(x) = \begin{cases} u^{-\gamma}(x) & \text{if} \quad u(x) \ne 0 \\ \infty & \text{if} \quad u(x) = 0, \end{cases}$$

and

$$H(x) = \begin{cases} v^{-\gamma}(x), & \text{if} \quad v(x) \ne 0 \\ \infty & \text{if} \quad v(x) = 0. \end{cases}$$

So, by taking $\Psi = (\varphi, \psi) > (0, 0)$ in (2.11), we conclude that $U = (u, v) > 0$ a.e. in $\mathbb{R}^N$. Moreover, it follows from (2.11) and the continuity of Φ' that

$$\lambda \int a(x)u^{-\gamma}(x)\varphi(x)dx + \mu \int c(x)v^{-\gamma}(x)\psi(x)dx < \infty,$$

for each $(\varphi, \psi) \in E_+$. As a consequence of this, we may apply the Lebesgue's dominated convergence theorem and use the limit $\xi(\epsilon) \to 1$, as $\epsilon \to 0$, to conclude that

$$\begin{aligned} L[U](\Psi) :=& -\lambda \int a(x)u^{-\gamma}(x)\varphi(x)dx - \mu \int c(x)v^{-\gamma}(x)\psi(x)dx \\ =& -\lim_{\epsilon\downarrow 0} \xi(\epsilon)^{1-\gamma} \frac{[K_{\lambda,\mu}(U + \epsilon\Psi) - K_{\lambda,\mu}(U)]}{(1-\gamma)\epsilon} \\ =& \lim_{\epsilon\downarrow 0} \frac{-[K_{\lambda,\mu}(\xi(\epsilon)(U + \epsilon\Psi)) - K_{\lambda,\mu}(\xi(\epsilon)U)]}{(1-\gamma)\epsilon}, \end{aligned}$$

for every $\Psi = (\varphi, \psi) \in E_+$, whence we have well-defined the linear map $L[U] : E \to \mathbb{R}$ given by

$$L[U](\Psi) = -\lambda \int a(x)u^{-\gamma}(x)\varphi(x)dx - \mu \int c(x)v^{-\gamma}(x)\psi(x)dx, \tag{2.12}$$

where $\Psi = (\varphi, \psi)$, and so $L[U]\Psi \leq 0$ for every $\Psi \in E_+$. Therefore, the functional $\psi := \frac{-1}{1-\gamma}K_{\lambda,\mu}$ satisfies the conditions $(L)_1$ and $(a)_1$ of Theorem 2.1.1 with the linear map $L[U]$ defined in (2.12). Since $U \in \mathcal{N}_{\lambda,\mu}$, the condition $(a)_2$ is also satisfied. It only remains to show the condition $(a)_3$. Let us to do this. Let $\Psi = (\varphi, \psi) \in E$ and $\epsilon > 0$. So, by evaluating $\Phi'(U)$ at $\Psi^- = ((u + \epsilon\varphi)^-, (v + \epsilon\psi)^-) \in E_+$, we have

$$\begin{aligned}
\langle \Phi'(U), (U + \epsilon\Psi)^- \rangle &= \int \nabla u \nabla (u + \epsilon\varphi)^- + Vu(u + \epsilon\varphi)^- - \frac{\alpha}{\alpha+\beta} bu^{\alpha-1}v^{\beta}(u + \epsilon\varphi)^- dx \\
&+ \int \nabla v \nabla (v + \epsilon\psi)^- + Vv(v + \epsilon\psi)^- - \frac{\beta}{\alpha+\beta} bu^{\alpha}v^{\beta-1}(v + \epsilon\psi)^- dx \\
&= -\int_{\{u+\epsilon\varphi<0\}} \nabla u \nabla (u + \epsilon\varphi) + Vu(u + \epsilon\varphi) - \frac{\alpha}{\alpha+\beta} bu^{\alpha-1}v^{\beta}(u + \epsilon\varphi) dx \\
&- \int_{\{v+\epsilon\psi<0\}} \nabla v \nabla (v + \epsilon\psi) + Vv(v + \epsilon\psi) - \frac{\beta}{\alpha+\beta} bu^{\alpha}v^{\beta-1}(v + \epsilon\psi) dx \\
&\leq -\epsilon \int_{\{u+\epsilon\varphi<0\}} \nabla u \nabla \varphi + Vu\varphi - \frac{\alpha}{\alpha+\beta} bu^{\alpha-1}v^{\beta}(u + \epsilon\varphi) dx \\
&- \epsilon \int_{\{v+\epsilon\psi<0\}} \nabla v \nabla \psi + Vv\psi - \frac{\beta}{\alpha+\beta} bu^{\alpha}v^{\beta-1}(v + \epsilon\psi) dx.
\end{aligned}$$

Let

$$\Omega_1^{\epsilon} = \left\{ x \in \mathbb{R}^N : b(x) < 0, u(x) + \epsilon\varphi(x) < 0 \right\}$$

and

$$\Omega_2^{\epsilon} = \left\{ x \in \mathbb{R}^N : b(x) < 0, v(x) + \epsilon\psi(x) < 0 \right\}.$$

The last inequality above implies that

$$\begin{aligned}
\frac{1}{\epsilon}\left[\langle \Phi'(U), (U + \epsilon\Psi)^- \rangle \right] \leq &-\int_{\{u+\epsilon\varphi<0\}} \nabla u \nabla \varphi + Vu\varphi dx - \int_{\{v+\epsilon\psi<0\}} \nabla v \nabla \psi + Vv\psi dx \\
&+ \frac{\alpha}{\alpha+\beta} \int_{\Omega_1^{\epsilon}} bu^{\alpha-1}v^{\beta}\varphi dx + \frac{\beta}{\alpha+\beta} \int_{\Omega_2^{\epsilon}} bu^{\alpha}v^{\beta-1}\psi dx,
\end{aligned}$$

and since the measure of the domains of integration $\{u + \epsilon\varphi < 0\}$ and $\{v + \epsilon\psi < 0\}$ tends to zero as $\epsilon \to 0$, $\Omega_1^{\epsilon} \subset \{u + \epsilon\varphi < 0\}$ and $\Omega_2^{\epsilon} \subset \{v + \epsilon\psi < 0\}$, we have from the above inequality that

$$\limsup_{\epsilon \downarrow 0} \frac{1}{\epsilon}\left[\langle \Phi'(U), (U + \epsilon\Psi)^- \rangle \right] \leq 0,$$

holds. So, the condition $(a)_3$ is hold as well. Therefore, we may apply Theorem 2.1.1 to conclude that U is a solution of $(\tilde{P}_{\lambda,\mu})$.

When $U \in \mathcal{N}^+_{\lambda,\mu}$ the proof is very similar, so we omit it here. ■

Combining the proof of Theorem 2.1.1 and the end of the proof of Proposition 2.2.2 we can prove the following proposition.

Proposition 2.2.3 *Assume that $U = (u, v) \in E_+ \setminus \{(0, 0)\}$ satisfies the conditions:*

$$\langle J'(U), U\rangle_E - \langle K'_{\lambda,\mu}(U), U\rangle_E - \langle L'(U), U\rangle_E = \|U\|^2 - K_{\lambda,\mu}(U) - L(U) = 0, \quad (2.13)$$

$$0 \le \langle J'(U), \Psi\rangle_E - \langle K'_{\lambda,\mu}(U), \Psi\rangle_E - \langle L'(U), \Psi\rangle_E, \quad (2.14)$$

for every $\Psi = (\varphi, \psi) \in E_+$. Then, either $u > 0$ or $v > 0$ a.e. in $\mathbb{R}^N$, if either $\lambda > 0$ or $\mu > 0$, respectively, and U is a solution of $(\tilde{P}_{\lambda,\mu})$.

2.2.2 Reduction to one-parameter of the problem $(\tilde{P}_{\lambda,\mu})$

In this section, we are going to reduce the study of the problem $(\tilde{P}_{\lambda,\mu})$ to the problem $(\tilde{P}_{\lambda,\theta\lambda})$ for $\theta, \lambda > 0$, that is, we will consider the problem

$$\begin{cases} -\Delta u + V(x)u = \lambda a(x)u^{-\gamma} + \dfrac{\alpha}{\alpha+\beta} b(x)u^{\alpha-1}v^{\beta} \ in\ \mathbb{R}^N, \\ -\Delta v + V(x)v = \lambda\theta c(x)v^{-\gamma} + \dfrac{\beta}{\alpha+\beta} b(x)u^{\alpha}v^{\beta-1} \ in\ \mathbb{R}^N, \\ u, v > 0,\ \mathbb{R}^N, \displaystyle\int_{\mathbb{R}^N} Vu^2dx + \int_{\mathbb{R}^N} Vv^2dx < \infty,\ u, v \in H^1(\mathbb{R}^N), \end{cases} \quad (\tilde{P}_{\lambda,\theta\lambda})$$

denote by $\Phi_\lambda = \Phi_{\lambda,\lambda\theta}$ the functional associated with $(\tilde{P}_{\lambda,\lambda\theta})$, and by $\phi_{U,\lambda}(t) = \Phi_{\lambda,\lambda\theta}(tU)$, $t > 0$ its fiber map. For convenience of the notations, let us denote by

$t^+_{\lambda,\theta\lambda}(U) = t^+_\lambda(\theta, U)$, $\quad t^-_{\lambda,\theta\lambda}(U) = t^-_\lambda(\theta, U)$ and $t^0_{\lambda,\theta\lambda}(U) = t^0_\lambda(\theta, U)$ (see Proposition 2.2.1).

Let $P = \{U \in E_+ : L(U) > 0\}$. To find the pair $(\lambda, t^0_\lambda(\theta, U))$ satisfying the condition (II) of Proposition 2.2.1, for each $U \in P$, we have to solve the system $\phi'_{U,\lambda}(t) = \phi''_{U,\lambda}(t) = 0$, that is,

$$\begin{cases} t||U||^2 - t^{-\gamma}\lambda K_{1,\theta}(U) - t^{\alpha+\beta-1}L(U) = 0, \\ ||U||^2 + \gamma\lambda t^{-\gamma-1}K_{1,\theta}(U) - (\alpha+\beta-1)t^{\alpha+\beta-2}L(U) = 0. \end{cases} \quad (2.15)$$

This system has a unique solution, which is given by $(t(U), \lambda(\theta, U))$, where

$$\begin{cases} t(U) = \left(\dfrac{1+\gamma}{\alpha+\beta+\gamma-1}\right)^{\frac{1}{\alpha+\beta-2}} \left[\dfrac{||U||^2}{L(U)}\right]^{\frac{1}{\alpha+\beta-2}} \\ \lambda(\theta, U) = C(\gamma, \alpha, \beta)\dfrac{(||U||^2)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{[L(U)]^{\frac{1+\gamma}{\alpha+\beta-2}}\,[K_{1,\theta}(U)]}, \end{cases} \quad (2.16)$$

and

$$C(\gamma,\alpha,\beta) \equiv \left(\frac{1+\gamma}{\alpha+\beta+\gamma-1}\right)^{\frac{1+\gamma}{\alpha+\beta-2}} \left(\frac{\alpha+\beta-2}{\alpha+\beta+\gamma-1}\right). \tag{2.17}$$

Similarly, when either $\lambda \neq 0$ and $\mu = 0$ or $\lambda = 0$ and $\mu \neq 0$, for each $U \in P$, we may solve a similar system to the (2.16) to find

$$\lambda(U) = C(\gamma,\alpha,\beta)\frac{(||U||^2)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{[L(U)]^{\frac{1+\gamma}{\alpha+\beta-2}}\,[K_{1,0}(U)]} \tag{2.18}$$

and

$$\mu(U) = C(\gamma,\alpha,\beta)\frac{(||U||^2)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{[L(U)]^{\frac{1+\gamma}{\alpha+\beta-2}}\,[K_{0,1}(U)]}, \tag{2.19}$$

respectively.

From the definitions of $\lambda(\theta,U), \lambda(U)$ and $\mu(U)$, we conclude that the claim in below Proposition holds true.

Proposition 2.2.4 *Suppose that $U \in P$. If:*

(a) $\lambda \in (0, \lambda(\theta,U))$, *the fiber map $\phi_{U,\lambda}$ satisfies (I) of Proposition 2.2.1, while $\phi_{U,\lambda(\theta,U)}$ satisfies (II), and if $\lambda \in (\lambda(\theta,U),\infty)$ it must satisfies (III),*

(b) $\lambda \in (0, \lambda(U))$, *the fiber map $\phi_{U,(\lambda,0)}$ satisfies (I) of Proposition 2.2.1, while $\phi_{U,(\lambda(U),0)}$ satisfies (II), and if $\lambda \in (\lambda(U),\infty)$ it must satisfies (III),*

(c) $\mu \in (0, \mu(U))$, *the fiber map $\phi_{U,(0,\mu)}$ satisfies (I) of Proposition 2.2.1, while $\phi_{U,(0,\mu(U))}$ satisfies (II), and if $\lambda \in (\mu(U),\infty)$ it must satisfies (III).*

Now, define

$$\lambda_*(\theta) = \inf_{U\in P} \lambda(\theta,U),\ \theta > 0, \tag{2.20}$$

$$\lambda_* = \inf_{U\in P} \lambda(U), \tag{2.21}$$

$$\mu_* = \inf_{U\in P} \mu(U). \tag{2.22}$$

Lemma 2.2.3 *The function $\lambda(\theta,U)$ defined in (2.16) is continuous, 0-homogeneous and unbounded from above for each $\theta > 0$. Moreover, $\lambda_*(\theta) > 0$ and there exists $U \in P \cap \mathbb{S}$ such that $\lambda_*(\theta) = \lambda(\theta,U)$. The same statements are true for the functions $\lambda(U)$ and $\mu(U)$ defined in (2.18) and (2.19), respectively.*

Proof We just prove the properties of the function $\lambda(\theta, U)$, because the proof of the properties of functions $\lambda(U)$ and $\mu(U)$ are very similar. The first, second and third statements are proved as in Lemma 1.1.3 of Chapter 1. Let us prove the last statement. By the Young's inequality, $(A2)$ and Sobolev embedding, we have

$$\begin{aligned} 0 < L(U) &\leq \frac{\alpha}{\alpha+\beta}\int |b||u|^{\alpha+\beta}dx + \frac{\beta}{\alpha+\beta}\int |b||v|^{\alpha+\beta}dx \\ &\leq c_1||b||_{L^\infty(\mathbb{R}^N)}||U||^{\alpha+\beta} = c_1||b||_{L^\infty(\mathbb{R}^N)} = C_1, \end{aligned}$$

for all $U \in P \cap \mathbb{S}$ and for some constant $C_1 > 0$.

On the other hand, from $(A1)$, Hölder inequality and Sobolev embedding we obtain

$$\begin{aligned} K_{1,\theta}(U) \leq& c_1(||a||_{2/(1+\gamma)} + ||c||_{2/(1+\gamma)})||U||^{1-\gamma} \\ &= c_1(||a||_{2/(1+\gamma)} + ||c||_{2/(1+\gamma)}) = C_2, \end{aligned}$$

for all $U \in P \cap \mathbb{S}$ and some constant $C_2 > 0$.

As a consequence of these two last inequalities, using that $\lambda(\theta, U)$ is 0-homogeneous, we obtain

$$\lambda_*(\theta) = \inf_{U \in P \cap S} \lambda(\theta, U) \geq cC(\alpha, \beta, \gamma){C_1}^{-\frac{1+\gamma}{\alpha+\beta-2}}{C_2}^{-1} > 0.$$

To end the proof, take $\{U_n\} \subset P \cap \mathbb{S}$ such that $\lambda(\theta, U_n) \to \lambda_*(\theta)$. So, it follows from Lemma 2.2.1 that

$$U_n \rightharpoonup U = (u, v) \in E \text{ and } U_n(x) \to U(x) = (u(x), v(x)) \geq (0, 0) \text{ a.e. in } \mathbb{R}^N,$$

and by the Lemma 2.2.2 we have $K_{1,\theta}(U_n) \to K_{1,\theta}(U)$ and $L(U_n) \to L(U)$.

These convergences lead us to infer that $u \not\equiv 0$ and $v \not\equiv 0$. Otherwise, we would have $L(U) = 0$, and therefore,

$$\lambda_*(\theta) = \lim_{n\to\infty} \lambda(\theta, U_n) = \lim_{n\to\infty} \frac{C(\alpha, \beta, \gamma)}{[L(U_n)]^{\frac{1+\gamma}{\alpha+\beta-2}}[K_{1,\theta}(U_n)]} = \infty, \tag{2.23}$$

which is an absurd. Let $W = \frac{U}{\|U\|} \in P \cap \mathbb{S}$. If $U_n \not\to U$ in E, it would follow by the weak lower semi-continuity of the norm that

$$\lambda(\theta, W) = \lambda\left(\theta, \frac{U}{\|U\|}\right) = \lambda(\theta, U) < \liminf \lambda(\theta, U_n) = \lambda_*(\theta),$$

but this is impossible, that is, $U \in P \cap \mathbb{S}$ and $\lambda(\theta, U) = \lambda_*(\theta)$. This ends the proof. ■

As a consequence of Lemma 2.2.3, we have that the function $\tilde{\Gamma} : (0, \infty) \to \mathbb{R}^2$ defined by

$$\tilde{\Gamma}(\theta) = (\lambda_*(\theta), \mu_*(\theta)), \text{ where } \mu_*(\theta) = \theta\lambda_*(\theta),$$

is well defined. We will see in this chapter that this function plays a role similar to the extremal value introduced in the Chapter 1. Our next goal is to explore the properties of $\tilde{\Gamma}$. To do this, first note that from Lemma 2.2.3, for each $\theta > 0$, there exist $U_\theta \in P \cap \mathbb{S}$ and $t(\theta, U_\theta) > 0$ such that

$$\lambda(\theta, U_\theta) = \lambda_*(\theta) \text{ and } t(\theta, U_\theta)U_\theta \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)}. \tag{2.24}$$

The next lemma provides the main properties of the function $\tilde{\Gamma}$.

Lemma 2.2.4 *There holds:*

a) the function $\tilde{\Gamma}(\theta)$ is bounded,

b) the function $\lambda_*(\theta)$ is continuous, which implies that the function $\tilde{\Gamma}(\theta)$ is continuous as well. Moreover, $\tilde{\Gamma}(\theta)$ is injective,

c) $\lambda_*(\theta)$ is nonincreasing and $\mu_*(\theta)$ is nondecreasing,

d) $\lim_{\theta \to 0} \tilde{\Gamma}(\theta) = (\lambda_*, 0)$ and $\lim_{\theta \to \infty} \tilde{\Gamma}(\theta) = (0, \mu_*)$.

Proof To prove a), note that for each $\theta > 0$ the inequalities $K_{1,0}(U) < K_{1,\theta}(U)$ and $K_{0,1}(U) < K_{1/\theta,1}(U)$ are satisfied for every $U \in P \cap \mathbb{S}$, and by combining these inequalities with (2.16) and (2.18) we have

$$\lambda(\theta, U) \leq \lambda(U) \text{ and } \theta\lambda(\theta, U) \leq \mu(U),$$

and taking the infimum over $P \cap \mathbb{S}$, we obtain from Lemma 2.2.3 and (2.20)-(2.22) that $\lambda_*(\theta) \leq \lambda_*$ and $\theta\lambda_*(\theta) \leq \mu_*$. Hence, we have

$$|\tilde{\Gamma}(\theta)| \leq \sqrt{\lambda_*^2 + \mu_*^2},$$

for every $\theta > 0$, and therefore $\tilde{\Gamma}$ is bounded. The proof of $a)$ is complete.

Let us prove $b)$. First let $\tilde{\Gamma}(\theta_1) = \tilde{\Gamma}(\theta_2)$, then by the definition of $\tilde{\Gamma}$ we have $\lambda_*(\theta_1) = \lambda_*(\theta_2)$ and $\theta_1\lambda_*(\theta_1) = \theta_2\lambda_*(\theta_2)$, which implies that $\theta_1 = \theta_2$. Therefore, $\tilde{\Gamma}$ is injective.

Let us prove that $\tilde{\Gamma}$ is continuous. Consider $\theta_n \to \theta$ and U_{θ_n} as in (2.24). Since $U_{\theta_n} \in P \cap \mathbb{S}$, we have by Lemma 2.2.1 that there exists $U_\theta \in E$ such that $U_{\theta_n} \rightharpoonup U_\theta$ in E

and $U_{\theta_n} = (u_{\theta_n}, v_{\theta_n}) \longrightarrow U_\theta = (u_\theta, v_\theta)$ a.e. in $\mathbb{R}^N$, with $U_\theta = (u_\theta, v_\theta) \geq (0,0)$. Hence, by Lemma 2.2.2 we have $K_{1,\theta_n}(U_{\theta_n}) \longrightarrow K_{1,\theta}(U_\theta)$ and $L(U_{\theta_n}) \longrightarrow L(U_\theta)$, as $n \to \infty$. First, we claim that $u_\theta \not\equiv 0$ and $v_\theta \not\equiv 0$. Indeed, if not, we would have $L(U_\theta) = 0$ and using the item a) we obtain

$$\infty = \lim_{n\to\infty} C(\gamma, \alpha, \beta) \frac{1}{[L(U_{\theta_n})]^{\frac{1+\gamma}{\alpha+\beta-2}} [K_{1,\theta_n}(U_{\theta_n})]} = \lim_{n\to\infty} \lambda_*(\theta_n) \leq \sqrt{\lambda_*^2 + \mu_*^2},$$

which is an absurd. Hence, $u_\theta \not\equiv 0$ and $v_\theta \not\equiv 0$, and also $L(U_\theta) > 0$, that is, $U_\theta \in P$.

Now, we claim that $U_{\theta_n} \longrightarrow U_\theta$ in E. Indeed, if not, we would have that $\|U_\theta\| < \liminf_{n\to\infty} \|U_{\theta_n}\| = 1$. Remembering that for each $U \in P$,

$$\lambda_*(\theta_n) \leq \lambda(\theta_n, U), \tag{2.25}$$

$\|U_\theta\| < 1$, and $\|U_{\theta_n}\| = 1$, we get from (2.16), Lemma 2.2.2 and (2.25) that

$$\lambda(\theta, U_\theta) < \frac{C(\alpha, \beta, \gamma)}{[L(U_\theta)]^{\frac{1+\gamma}{\alpha+\beta-2}} [K_{1,\theta}(U_\theta)]} = \liminf_{n\to\infty} \lambda(\theta_n, U_{\theta_n}) = \liminf_{n\to\infty} \lambda_*(\theta_n) \leq \lambda(\theta, U), \tag{2.26}$$

for every $U \in P$. So, from (2.26) we obtain

$$\lambda_*(\theta) = \inf_{U\in P} \lambda(\theta, U) \leq \lambda(\theta, U_\theta) < \liminf_{n\to\infty} \lambda(\theta_n, U_{\theta_n}) \leq \inf_{U\in P} \lambda(\theta, U) = \lambda_*(\theta),$$

which is a contradiction. Therefore, $U_{\theta_n} \longrightarrow U_\theta$ and $\|U_\theta\| = 1$. As a consequence of this, (2.16) and (2.24), we have $\lambda_*(\theta_n) \longrightarrow \lambda(\theta, U_\theta)$ as $n \to \infty$, and taking the limit in (2.25) we conclude that

$$\lambda_*(\theta) \leq \lambda(\theta, U_\theta) \leq \lambda(\theta, U),$$

for every $U \in P$, which implies that

$$\lambda_*(\theta) \leq \lambda(\theta, U_\theta) \leq \inf_{U\in P} \lambda(\theta, U) = \lambda_*(\theta),$$

and therefore $\lambda_*(\theta) = \lambda(\theta, U_\theta)$. This equality together with the convergence $\lambda_*(\theta_n) \to \lambda(\theta, U_\theta)$ as $n \to \infty$, leads us to conclude that $\lambda_*(\theta_n) \longrightarrow \lambda_*(\theta)$, as $n \to \infty$. Therefore, the function $(0,\infty) \ni \theta \mapsto \lambda_*(\theta)$ is continuous. Since $\tilde{\Gamma}(\theta) = (\lambda_*(\theta), \theta\lambda_*(\theta))$, we conclude that $\tilde{\Gamma}$ is continuous. The proof of the item *b*) is complete.

Let us prove c). For each $\theta_1 < \theta_2$ and $U \in P$, we have from definition of $\lambda(\theta_1, U)$ and $\lambda(\theta_2, U)$ that

$$\lambda(\theta_2, U) < \lambda(\theta_1, U),$$

which implies $\lambda_*(\theta_2) \leq \lambda_*(\theta_1)$, by definition of $\lambda_*(\theta_2)$ and $\lambda_*(\theta_1)$. Hence, the function $\lambda_*(\theta)$ is monotone nonincreasing. Now, let us prove that the function $\mu_*(\theta)$ is monotone nondecreasing. Firstly, note that for every $\theta > 0$ and $U \in P$

$$\theta\lambda(\theta, U) = C(\gamma, \alpha, \beta)\frac{(||U||^2)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{[L(U)]^{\frac{1+\gamma}{\alpha+\beta-2}}\left[K_{1/\theta,1}(U)\right]}$$

holds, and this implies that

$$\theta_1\lambda(\theta_1, U) < \theta_2\lambda(\theta_2, U),$$

for $0 < \theta_1 < \theta_2$, which leads us to conclude that $\mu_*(\theta_1) = \theta_1\lambda_*(\theta_1) \leq \theta_2\lambda_*(\theta_2) = \mu_*(\theta_2)$, by definition of $\lambda_*(\theta_1)$ and $\lambda_*(\theta_2)$. This ends the proof of c).

Finally, let us prove d). The proof that $\lim\limits_{\theta\to 0}\lambda_*(\theta) = \lambda_*$ and $\lim\limits_{\theta\to\infty}\theta\lambda_*(\theta) = \mu_*$ is very similar to the proof of the item b), so we omit it here. By the item a) the function $\lambda_*(\theta)$ is bounded, therefore $\lim\limits_{\theta\to 0}\theta\lambda_*(\theta) = 0$, which implies that $\lim\limits_{\theta\to 0}\tilde{\Gamma}(\theta) = (\lambda_*, 0)$. To conclude the proof of item c), it is sufficient to prove that $\lim\limits_{\theta\to\infty}\lambda_*(\theta) = 0$. We have that $\lim\limits_{\theta\to\infty}\theta\lambda_*(\theta) = \mu_*$, and hence $\lim\limits_{\theta\to\infty}\lambda_*(\theta) = \lim\limits_{\theta\to\infty}\theta^{-1}(\theta\lambda_*(\theta)) = 0$, and this implies that $\lim\limits_{\theta\to\infty}\tilde{\Gamma}(\theta) = (0, \mu_*)$. The proof of lemma is complete. ■

Propositions 2.2.2, 2.2.4 and Lemma 2.2.3 leads us the following proposition.

Proposition 2.2.5 *For each $\theta > 0$, holds true:*

a) $\mathcal{N}^0_{\tilde{\Gamma}(\theta)} \neq \emptyset$ *and*

$$\mathcal{N}^0_{\tilde{\Gamma}(\theta)} = \left\{U \in \mathcal{N}_{\tilde{\Gamma}(\theta)} : U \in P, \lambda(\theta, U) = \lambda_*(\theta)\right\}.$$

Moreover, each $U \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ *satisfies*

$$2\langle J'(U), \Psi\rangle_E - \lambda_*(\theta)(1-\gamma)\langle K'_{1,\theta}(U), \Psi\rangle_E - (\alpha+\beta)\langle L'(U), \Psi\rangle_E = 0, \quad (2.27)$$

for all $\Psi \in E$.

b) $\mathcal{N}^0_{\lambda,\theta\lambda} = \emptyset$ *for each* $\lambda \in (0, \lambda_*(\theta))$ *and* $\mathcal{N}^0_{\lambda,\theta\lambda} \neq \emptyset$ *for each* $\lambda \in [\lambda_*(\theta), \infty)$.

Proof Let us prove a). From Lemma 2.2.3 there exists $U \in P \cap \mathbb{S}$ such that $\lambda(\theta, U) = \lambda_*(\theta)$, and hence $t^0_{\lambda_*(\theta)}(\theta, U)U \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$, which implies that $\mathcal{N}^0_{\tilde{\Gamma}(\theta)} \neq \emptyset$. This proves the first part of the statement of item a). Now, once that $\mathcal{N}^0_{\tilde{\Gamma}(\theta)} \neq \emptyset$, the equality $\mathcal{N}^0_{\tilde{\Gamma}(\theta)} = \left\{U \in \mathcal{N}_{\tilde{\Gamma}(\theta)} : U \in P, \lambda(\theta, U) = \lambda_*(\theta)\right\}$ is obvious.

The proof of (2.27) is similar to that done in Lemma 1.3, so let us summarize it here. Let $t > 0$ and $\Psi \in E_+$. Since U is the minimum point for $\lambda(\theta, U)$, we have that

$$\lambda(\theta, U + t\Psi) - \lambda(\theta, U) = \lambda(\theta, U + t\Psi) - \lambda_*(\theta) \geq 0,$$

for all $t \geq 0$ enough small, and applying the Mean Value Theorem, we have

$$\begin{aligned}
&\left[\frac{(\|U + t\Psi\|^2)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{(L(U + t\Psi))^{\frac{1+\gamma}{\alpha+\beta-2}}} - \frac{(\|U\|^2)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{(L(U))^{\frac{1+\gamma}{\alpha+\beta-2}}}\right] \frac{1}{K_{1,\theta}(U + t\Psi)} \\
&\geq -\frac{(\|U\|^2)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{(L(U))^{\frac{1+\gamma}{\alpha+\beta-2}}} \left[(K_{1,\theta}(U + t\Psi))^{-1} - (K_{1,\theta}(U))^{-1}\right] \\
&= \frac{(\|U\|^2)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{(L(U))^{\frac{1+\gamma}{\alpha+\beta-2}}} (v(t))^{-2} \left[K_{1,\theta}(U + t\Psi) - K_{1,\theta}(U)\right],
\end{aligned} \tag{2.28}$$

where the function $v(t) > 0$ satisfies $v(t) \longrightarrow K_{1,\theta}(U)$ as $t \to 0$. Now, we may use the Fatou's Lemma in (2.28) to conclude that

$$\left[\frac{2\left(\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}\right)\langle J'(U), \Psi\rangle_E - \left(\frac{(1+\gamma)(\alpha+\beta)}{\alpha+\beta-2}\right)(L(U))^{-1}\langle L'(U), \Psi\rangle_E \|U\|^2}{(L(U))^{\frac{1+\gamma}{\alpha+\beta-2}} (\|U\|^2)^{\frac{-1-\gamma}{\alpha+\beta-2}} K_{1,\theta}(U)}\right] \tag{2.29}$$

$$\geq \frac{(\|U\|^2)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{(L(U))^{\frac{1+\gamma}{\alpha+\beta-2}}} (K_{1,\theta}(U))^{-2} (1-\gamma) \left[\int a(x)G(x)\varphi(x)dx + \theta \int c(x)H(x)\psi(x)dx\right],$$

where

$$G(x) = \begin{cases} u^{-\gamma}(x), & \text{if } u(x) \neq 0 \\ \infty, & \text{if } u(x) = 0, \end{cases}$$

and

$$H(x) = \begin{cases} v^{-\gamma}(x), & \text{if } v(x) \neq 0 \\ \infty, & \text{if } v(x) = 0, \end{cases}$$

Taking $\Psi = (\varphi, \psi) > (0, 0)$ in (2.29) we conclude that $U = (u, v) > 0$ in $\mathbb{R}^N$ and $G(x) = u^{-\gamma}(x)$ and $H(x) = v^{-\gamma}(x)$ for all $x \in \mathbb{R}^N$. Hence, from (2.29) we have

$$\begin{aligned}
&\left[\frac{2\left(\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}\right)\langle J'(U), \Psi\rangle_E - \left(\frac{(1+\gamma)(\alpha+\beta)}{\alpha+\beta-2}\right)(L(U))^{-1}\langle L'(U), \Psi\rangle_E \|U\|^2}{(L(U))^{\frac{1+\gamma}{\alpha+\beta-2}} (\|U\|^2)^{\frac{-1-\gamma}{\alpha+\beta-2}} K_{1,\theta}(U)}\right] \\
&\geq \frac{(\|U\|^2)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{(L(U))^{\frac{1+\gamma}{\alpha+\beta-2}}} (K_{1,\theta}(U))^{-2} (1-\gamma) \langle K'_{1,\theta}(U), \Psi\rangle_E,
\end{aligned} \tag{2.30}$$

for every $\Psi \in E_+$.

Using homogeneity we may assume, without loss of generality, that $\|U\| = 1$ and from

$$1 - \lambda_*(\theta)K_{1,\theta}(U) - L(U) = 0 = 1 + \gamma\lambda_*(\theta)K_{1,\theta}(U) - (\alpha+\beta-1)L(U),$$

we produce the equalities

$$L(U) = \frac{1+\gamma}{\alpha+\beta+\gamma-1} \text{ and } K_{1,\theta}(U) = \frac{\alpha+\beta-2}{\lambda_*(\theta)(\alpha+\beta+\gamma-1)}$$

Now, replacing these equalities in (2.30), after some manipulations, we obtain

$$2\langle J'(U), \Psi\rangle_E - \lambda_*(\theta)\,(1-\gamma)\,\langle K'_{1,\theta}(U), \Psi\rangle_E - (\alpha+\beta)\,\langle L'(U), \Psi\rangle_E \geq 0, \tag{2.31}$$

for every $\Psi \in E_+$.

For $\Psi = (\varphi, \psi) \in E$ e $\epsilon > 0$ define $\Psi^+ = ((u+\epsilon\varphi)^+, (v+\epsilon\psi)^+) \in E_+$. Since $U \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ we have

$$2 - \lambda_*(\theta)(1-\gamma)K_{1,\theta}(U) - (\alpha+\beta)L(U) = 0,$$

which implies, by substituting Ψ^+ in (2.31), and following the same approach as done in the proof of Proposition 2.2.2 that

$$2\langle J'(U), \Psi\rangle_E - \lambda_*(\theta)\,(1-\gamma)\,\langle K'_{1,\theta}(U), \Psi\rangle_E - (\alpha+\beta)\,\langle L'(U), \Psi\rangle_E \geq 0$$

holds for every $\Psi \in E$. So, by changing Ψ by $-\Psi$ in the above inequality, we obtain

$$2\langle J'(U), \Psi\rangle_E - \lambda_*(\theta)(1-\gamma)\langle K'_{1,\theta}(U), \Psi\rangle_E - (\alpha+\beta)\langle L'(U), \Psi\rangle_E = 0,$$

for all $\Psi \in E$. That is, U satisfies (2.27). The proof of item a) is complete.

The item b) it is a consequence of the definition of $\lambda_*(\theta)$, Proposition 2.2.4 and Lemma 2.2.3. The proof is complete. ∎

Now we make the remark.

Remark 2.2.1 *The curve $\tilde{\Gamma}$ has the following property: if $(0,0) < (\lambda, \mu) < \tilde{\Gamma}(\theta)$, then $\mathcal{N}^0_{\lambda,\mu} = \emptyset$, while $(\lambda, \mu) \geq \tilde{\Gamma}(\theta)$ implies $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$. This is true because $(\lambda, \mu) > (0,0)$ can be rewritten as $(\lambda, \mu) = (\lambda, \theta\lambda)$, where $\theta = \mu/\lambda$, and $(0,0) < (\lambda, \mu) < \tilde{\Gamma}(\theta)$ is equivalent to claim that $\lambda < \lambda_*(\theta)$. So, Lemma 2.2.4 lead to the claimed.*

We are now in position to generalize Corollary 1.1.1 of Chapter 1.

Corollary 2.2.1 *Let $\theta > 0$. The problem $(\tilde{P}_{\tilde{\Gamma}(\theta)})$ has no solution $U \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$.*

Proof If there were a solution $U \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ for $(\tilde{P}_{\tilde{\Gamma}(\theta)})$, then it would follows from Proposition 2.2.5-(2.27) that

$$\int \left[-\lambda_*(\theta)(1+\gamma)au^{-\gamma} + \alpha \left(\frac{\alpha+\beta-2}{\alpha+\beta} \right) bu^{\alpha-1}v^{\beta} \right] \varphi = 0, \forall \varphi \in X$$

and

$$\int \left[-\theta\lambda_*(\theta)(1+\gamma)cv^{-\gamma} + \beta \left(\frac{\alpha+\beta-2}{\alpha+\beta} \right) bu^{\alpha}v^{\beta-1} \right] \psi = 0, \forall \psi \in X$$

hold, that is,

$$\lambda_*(\theta)(1+\gamma)au^{-\gamma} = \alpha \left(\frac{\alpha+\beta-2}{\alpha+\beta} \right) bu^{\alpha-1}v^{\beta} \text{ a.e. in } \mathbb{R}^N, \tag{2.32}$$

and

$$\theta\lambda_*(\theta)(1+\gamma)cv^{-\gamma} = \beta \left(\frac{\alpha+\beta-2}{\alpha+\beta} \right) bu^{\alpha}v^{\beta-1} \text{ a.e. in } \mathbb{R}^N. \tag{2.33}$$

Now, we consider two possible cases: If $b(x) \leq 0$ in Ω, for some $\Omega \subset \mathbb{R}^N$ with Lebesgue measure positive, then $(1+\gamma)a(x)u^{-\gamma} \leq 0$ a.e. in Ω, which is an absurd.

On the other side, assume that $b(x) > 0$ in $\mathbb{R}^N$. Then, multiplying by $u(x)$ and $v(x)$ in (2.32) and (2.33) respectively, we obtain

$$\beta\lambda_*(\theta)\left(1+\gamma\right)a(x)u^{1-\gamma}(x) = \alpha\theta\lambda_*(\theta)\left(1+\gamma\right)c(x)v^{1-\gamma}(x) \text{ a.e. in } \mathbb{R}^N,$$

that is,

$$v(x) = \left[\frac{\beta a(x)}{\theta\alpha c(x)} \right]^{\frac{1}{1-\gamma}} u(x) \text{ a.e. in } \mathbb{R}^N \tag{2.34}$$

and replacing (2.34) in (2.32) we have that

$$u(x) = \left[\frac{\lambda_*(\theta)(1+\gamma)(\alpha+\beta)}{\alpha(\alpha+\beta-2)} \right]^{\frac{1}{\alpha+\beta+\gamma-1}} \left[\frac{a(x)}{b(x)} \right]^{\frac{1}{\alpha+\beta+\gamma-1}} \left[\frac{\theta\alpha c(x)}{\beta a(x)} \right]^{\frac{\beta}{(1-\gamma)(\alpha+\beta+\gamma-1)}} \in X,$$

which is an absurd by $(A3)$. ■

The next lemma will be essential in order to prove the existence of multiple solutions for the system $(\tilde{P}_{\lambda,\mu})$ for $(\lambda,\mu) \geq \tilde{\Gamma}(\theta)$, with $\theta > 0$ fixed and (λ,μ) close to $\tilde{\Gamma}(\theta)$.

Lemma 2.2.5 *For each $\theta > 0$ the set $\mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ is compact.*

Proof First, observe that there exist positive constants c, C such that

$$c \leq \|U\| \leq C \text{ and } c \leq L(U). \tag{2.35}$$

for all $U \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$.

Let $\{U_n\} \subset \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$. From (2.35) we can assume that, up to a subsequence, $U_n \rightharpoonup U$ in E and $U_n \longrightarrow U = (u, v)$ a.e. in $\mathbb{R}^N$, with $u, v \geq 0$. From Lemma 2.2.2 and (2.35) we have $L(U) \geq c > 0$, which implies $U \in P$. Now, let us prove that $U_n \longrightarrow U$ in E. In fact, on the contrary, we would have that $\|U\| < \liminf_{n\to\infty} \|U_n\|$. Then from Proposition 2.2.5 a), Lemma 2.2.2 and defintion of $\lambda_*(\theta)$ we have

$$\lambda_*(\theta) \leq \lambda(\theta, U) < \liminf_{n\to\infty} \lambda(\theta, U_n) = \lambda_*(\theta),$$

which is an absurd. Therefore $U_n \longrightarrow U$ in E and consequently $\mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ is compact.

■

Now let us fix $\theta > 0$. Below, by taking advantage of Proposition 2.2.5 *b*), we define for each $\lambda > 0$ the non-empty set

$$\hat{\mathcal{N}}_{\lambda,\theta\lambda} = \left\{ U \in E_+ : L(U) = \int_{\mathbb{R}^N} b(x)|u|^\alpha |v|^\beta dx > 0, \phi_{U,\lambda} \text{ has two critical points} \right\},$$

and the set

$$\hat{\mathcal{N}}^+_{\lambda,\theta\lambda} = \left\{ U \in E_+ : L(U) = \int_{\mathbb{R}^N} b(x)|u|^\alpha |v|^\beta dx \leq 0 \right\}.$$

Let $\overline{\hat{\mathcal{N}}_{\lambda,\theta\lambda} \cup \hat{\mathcal{N}}^+_{\lambda,\theta\lambda}}$ be the closure of $\hat{\mathcal{N}}_{\lambda,\theta\lambda} \cup \hat{\mathcal{N}}^+_{\lambda,\theta\lambda}$ with respect to the topology norm.

As in Chapter 1, we have.

Proposition 2.2.6 *There holds:*

(*i*) *if* $\lambda_1, \lambda_2 \in (0, \lambda_*(\theta))$, *then* $\hat{\mathcal{N}}_{\lambda_1,\theta\lambda_1} = \hat{\mathcal{N}}_{\lambda_2,\theta\lambda_2}$,

(*ii*) *if* $U \in \hat{\mathcal{N}}_{\lambda,\theta\lambda} \cup \hat{\mathcal{N}}^+_{\lambda,\theta\lambda}$, *then* $tU \in \hat{\mathcal{N}}_{\lambda,\theta\lambda} \cup \hat{\mathcal{N}}^+_{\lambda,\theta\lambda}$ *for all* $t > 0$, *that is,* $\hat{\mathcal{N}}_{\lambda,\theta\lambda} \cup \hat{\mathcal{N}}^+_{\lambda,\theta\lambda}$ *is a positive cone generated by the set* $\mathcal{N}^+_{\lambda,\theta\lambda} \cup \mathcal{N}^-_{\lambda,\theta\lambda}$. *More specifically,*

$$\hat{\mathcal{N}}_{\lambda,\theta\lambda} \cup \hat{\mathcal{N}}^+_{\lambda,\theta\lambda} = \left\{ tU : t > 0,\ U \in \mathcal{N}^+_{\lambda,\theta\lambda} \cup \mathcal{N}^-_{\lambda,\theta\lambda} \right\},$$

(*iii*) *there holds*

$$\overline{\hat{\mathcal{N}}_{\tilde{\Gamma}(\theta)} \cup \hat{\mathcal{N}}^+_{\tilde{\Gamma}(\theta)}} = \hat{\mathcal{N}}_{\tilde{\Gamma}(\theta)} \cup \hat{\mathcal{N}}^+_{\tilde{\Gamma}(\theta)} \cup \left\{ tU : t > 0,\ U \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)} \right\} \cup \{0\},$$

(*iv*) *the function* $t_{\lambda_*(\theta)}$ *is continuous and* $P^- : \mathbb{S} \cap \overline{\hat{\mathcal{N}}_{\tilde{\Gamma}(\theta)}} \to \mathcal{N}^-_{\tilde{\Gamma}(\theta)} \cup \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ *defined by* $P^-(W) = t_{\lambda_*(\theta)}(W)W$ *is a homeomorphism, where*

$$t_{\lambda_*(\theta)}(W) = \begin{cases} t^-_{\lambda_*(\theta)}(\theta, W) & \text{if } W \in \hat{\mathcal{N}}_{\tilde{\Gamma}(\theta)}, \\ t^0_{\lambda_*(\theta)}(\theta, W) & \text{otherwise,} \end{cases} \tag{2.36}$$

(*v*) *the function* $s_{\lambda_*(\theta)}$ *is continuous and* $P^+ : \mathbb{S} \to \mathcal{N}^+_{\tilde{\Gamma}(\theta)} \cup \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ *defined by* $P^+(u) = s_{\lambda_*(\theta)}(U)U$ *is a homeomorphism, where*

$$s_{\lambda_*(\theta)}(U) = \begin{cases} t^+_{\lambda_*(\theta)}(\theta, U) & \text{if } U \in \hat{\mathcal{N}}_{\tilde{\Gamma}(\theta)} \cup \hat{\mathcal{N}}^+_{\tilde{\Gamma}(\theta)} \\ t^0_{\lambda_*(\theta)}(\theta, U) & \text{otherwise,} \end{cases} \tag{2.37}$$

(*vi*) *the set* $\mathcal{N}^0_{\tilde{\Gamma}(\theta)} \subset \mathcal{N}_{\tilde{\Gamma}(\theta)}$ *has empty interior, where* $\mathcal{N}_{\tilde{\Gamma}(\theta)}$ *is endowed with the induced topology of the norm on* E.

As a fundamental ingredient to show multiplicity of solutions for Problem $(\tilde{P}_{\lambda,\theta\lambda})$ beyond the extremal curve $\tilde{\Gamma}(\theta)$, we have to prove the continuity and monotonicity of the energy functional constrained on $\mathcal{N}^+_{\lambda,\theta\lambda}$ and $\mathcal{N}^-_{\lambda,\theta\lambda}$. To do these, let us define $J^+_\lambda : \hat{\mathcal{N}}_{\lambda,\theta\lambda} \cup \hat{\mathcal{N}}^+_{\lambda,\theta\lambda} \to \mathbb{R}$ and $J^-_\lambda : \hat{\mathcal{N}}_{\lambda,\theta\lambda} \to \mathbb{R}$ by

$$J^+_\lambda(U) = \Phi_\lambda(t^+_\lambda(\theta, U)U) \text{ and } J^-_\lambda(U) = \Phi_\lambda(t^-_\lambda(\theta, U)U) \tag{2.38}$$

and denote their infimum by

$$\tilde{J}^+_\lambda = \inf\left\{J^+_\lambda(U) : U \in \mathcal{N}^+_{\lambda,\theta\lambda}\right\} \text{ and } \tilde{J}^-_\lambda = \inf\left\{J^-_\lambda(U) : U \in \mathcal{N}^-_{\lambda,\theta\lambda}\right\},$$

respectively.

The same proof of Lemma 1.1.7 of Chapter 1 also shows the following lemma.

Lemma 2.2.6 *Let* $U \in E_+$ *and* $I \subset \mathbb{R}$ *be an open interval such that* $t^\pm_\lambda(\theta, U)$ *are well defined for all* $\lambda \in I$. *Then:*

a) *the functions* $I \ni \lambda \to t^\pm_\lambda(\theta, U)$ *are* $C^\infty(I)$. *Moreover,* $I \ni \lambda \to t^-_\lambda(\theta, U)$ *is decreasing while* $I \ni \lambda \to t^+_\lambda(\theta, U)$ *is increasing.*

b) *the functions* $I \ni \lambda \to J^\pm_\lambda(U)$ *are* $C^\infty(I)$ *and decreasing.*

In particular, both claims hold true for $I = (0, \lambda_*(\theta))$ *and all* $U \in E_+$ *given.*

As a consequence of the monotonicity above, we have.

Corollary 2.2.2 *Suppose that* $U \notin \hat{\mathcal{N}}^{+}_{\tilde{\Gamma}(\theta)}$. *Then*

$$\lim_{\lambda \uparrow \lambda_*(\theta)} t_\lambda^-(\theta, U) = t_{\lambda_*(\theta)}(U), \quad \lim_{\lambda \uparrow \lambda_*(\theta)} t_\lambda^+(\theta, U) = s_{\lambda_*(\theta)}(U)$$

$$\lim_{\lambda \uparrow \lambda_*(\theta)} J_\lambda^-(U) = \Phi_{\lambda_*(\theta)}(t_{\lambda_*(\theta)}(U)U), \quad \lim_{\lambda \uparrow \lambda_*} J_\lambda^+(U) = \Phi_{\lambda_*(\theta)}(s_{\lambda_*(\theta)}(U)U),$$

where $t_{\lambda_*(\theta)}(U)$ *and* $s_{\lambda_*(\theta)}(U)$ *are defined at* (2.36) *and* (2.37), *respectively.*

Let us finish this section by introducing a curve that will play a role similar to parameter defined at (1) of Chapter 1. To find the region where the system has solution with its energy being positive we will consider the system $\phi_{U,\lambda}(t) = \phi'_{U,\lambda}(t) = 0$, that is,

$$\begin{cases} \dfrac{t^2}{2}\|U\|^2 - \dfrac{t^{1-\gamma}}{1-\gamma}\lambda K_{1,\theta}(U) - \dfrac{t^{\alpha+\beta}}{\alpha+\beta}L(U) = 0 \\ \quad t\|U\|^2 - \lambda t^{-\gamma}K_{1,\theta}(U) - t^{\alpha+\beta-1}L(U) = 0. \end{cases} \tag{2.39}$$

for each $U \in P$.

This last system has a unique solution which is given by $(\hat{t}(U), \hat{\lambda}(\theta, U))$, where

$$\begin{cases} \hat{t}(\theta, U) = \left[\dfrac{(1+\gamma)(\alpha+\beta)}{2(\alpha+\beta+\gamma-1)}\right]^{\frac{1}{\alpha+\beta-2}} \left[\dfrac{||U||^2}{L(U)}\right]^{\frac{1}{\alpha+\beta-2}} \\ \hat{\lambda}(\theta, U) = \hat{C}(\gamma, \alpha, \beta)\dfrac{\left(||U||^2\right)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{[L(U)]^{\frac{1+\gamma}{\alpha+\beta-2}}\,[K_{1,\theta}(U)]}, \end{cases}$$

$$\hat{C}(\gamma, \alpha, \beta) \equiv \frac{(1-\gamma)}{2}\left[\frac{\alpha+\beta}{2}\right]^{\frac{1+\gamma}{\alpha+\beta-2}} C(\gamma, \alpha, \beta)$$

and $C(\gamma, \alpha, \beta)$ is defined in (2.17).

So, similarly to (2.18) and (2.19), when either $\lambda \neq 0$ and $\mu = 0$ or $\lambda = 0$ and $\mu \neq 0$, for each $U \in P$, we may solve a system similar to (2.39) to find

$$\hat{\lambda}(U) = \hat{C}(\gamma, \alpha, \beta)\frac{\left(||U||^2\right)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{[L(U)]^{\frac{1+\gamma}{\alpha+\beta-2}}\,[K_{1,0}(U)]}$$

and

$$\hat{\mu}(U) = \hat{C}(\gamma, \alpha, \beta)\frac{\left(||U||^2\right)^{\frac{\alpha+\beta+\gamma-1}{\alpha+\beta-2}}}{[L(U)]^{\frac{1+\gamma}{\alpha+\beta-2}}\,[K_{0,1}(U)]},$$

respectively. So, we can define

$$\hat{\lambda}_*(\theta) = \inf_{U \in P}\hat{\lambda}(\theta, U) = \frac{(1-\gamma)}{2}\left[\frac{\alpha+\beta}{2}\right]^{\frac{1+\gamma}{\alpha+\beta-2}} \lambda_*(\theta),$$

$$\hat{\lambda}_* = \inf_{U \in P}\hat{\lambda}(U) = \frac{(1-\gamma)}{2}\left[\frac{\alpha+\beta}{2}\right]^{\frac{1+\gamma}{\alpha+\beta-2}} \lambda_*,$$

$$\hat{\mu}_* = \inf_{U \in P} \hat{\mu}(U) = \frac{(1-\gamma)}{2} \left[\frac{\alpha+\beta}{2}\right]^{\frac{1+\gamma}{\alpha+\beta-2}} \mu_*.$$

and the function $\Gamma_0 : (0,\infty) \to \mathbb{R}^2$ defined by

$$\Gamma_0(\theta) = (\hat{\lambda}_*(\theta), \hat{\mu}_*(\theta)), \text{ where } \hat{\mu}_*(\theta) = \theta\hat{\lambda}_*(\theta).$$

We have that the inequality $\hat{\lambda}_*(\theta) < \lambda_*(\theta)$ holds, which implies that $\Gamma_0(\theta) < \Gamma(\theta)$ for every $\theta > 0$.

The following lemma is a consequence of the Lemmas 2.2.3 and 2.2.4.

Lemma 2.2.7 *There holds:*

a) there exists $U \in P \cap \mathbb{S}$ such that $\hat{\lambda}_*(\theta) = \hat{\lambda}(\theta, U)$. The same statements are true for the functions $\hat{\lambda}(U)$ and $\hat{\mu}(U)$,

b) the function $\Gamma_0(\theta)$ is bounded,

c) the function $\hat{\lambda}_*(\theta)$ is continuous, which implies that the function $\Gamma_0(\theta)$ is continuous. Moreover $\Gamma_0(\theta)$ is injective,

d) $\hat{\lambda}_*(\theta)$ is monotone nondecreasing and $\hat{\mu}_*(\theta)$ is monotone nonincreasing

e) $\lim_{\theta\to 0} \Gamma_0(\theta) = (\hat{\lambda}_*, 0)$ and $\lim_{\theta\to\infty} \Gamma(\theta) = (0, \hat{\mu}_*)$.

2.3 Multiplicity of solutions in the extremal region to the applicability of the Nehari method

In this section we show the existence of two solutions for problem $(\tilde{P}_{\lambda,\theta\lambda})$ when $\lambda \in (0, \lambda_*(\theta))$, for each $\theta > 0$ fixed. To achieve this we will need some preliminary results. After introducing the modified problem, we will use in this section the approach of Chapter 1.

Let us continue to use the notation $\Phi_\lambda = \Phi_{\lambda,\theta\lambda}$. We are going beginning to prove the next Lemma.

Lemma 2.3.1 *Let $\lambda > 0$. Then:*

a) for all $U \in \mathcal{N}^+_{\lambda,\theta\lambda}$, we have that

$$\|U\|^2 < \frac{\lambda(\alpha+\beta+\gamma-1)}{\alpha+\beta-2} \left[\int_{\mathbb{R}^N} a(x)|u|^{1-\gamma}dx + \theta \int_{\mathbb{R}^N} c(x)|v|^{1-\gamma}dx\right] \qquad (2.40)$$

holds. In particular $\sup\left\{\|U\| : U \in \mathcal{N}^+_{\lambda,\theta\lambda}\right\} < \infty$.

b) for all $W \in \mathcal{N}^-_{\lambda,\theta\lambda}$*, we have that*

$$\|W\|^2 < \frac{(\alpha+\beta+\gamma-1)}{(1+\gamma)} \int_{\mathbb{R}^N} b|u|^\alpha |v|^\beta dx = \frac{(\alpha+\beta+\gamma-1)}{(1+\gamma)} L(W) \qquad (2.41)$$

holds and $\sup\left\{\|W\| : W \in \mathcal{N}^-_{\lambda,\theta\lambda}, \Phi_\lambda(W) \leq M\right\} < \infty$ *for each* $M > 0$ *given. Moreover*

$$\inf\left\{\|W\| : W \in \mathcal{N}^-_{\lambda,\theta\lambda}\right\} > 0.$$

Furthermore,

$$0 > \tilde{J}^+_\lambda := \inf_{U \in \mathcal{N}^+_{\lambda,\theta\lambda}} \Phi_\lambda(U) > -\infty \ \ \text{and} \ \ \tilde{J}^-_\lambda := \inf_{W \in \mathcal{N}^-_{\lambda,\theta\lambda}} \Phi_\lambda(W) > -\infty. \qquad (2.42)$$

Proof The item $a)$ is a consequence of $\phi''_{U,\lambda}(1) > 0$, Hölder inequality and Sobolev embedding. The inequalities (2.41) of $b)$ and $\inf\left\{||W|| : W \in \mathcal{N}^-_\lambda\right\} > 0$ are direct consequences of $\phi''_{W,\lambda}(1) < 0$, Hölder and Young inequalities and Sobolev embedding. Now fix $M > 0$ and $W \in \mathcal{N}^-_{\lambda,\theta\lambda}$ such that $\Phi_\lambda(W) \leq M$. By using Hölder and Sobolev embeddings, we obtain

$$\left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\|W\|^2 + \lambda\left(\frac{1}{\alpha+\beta} - \frac{1}{1-\gamma}\right) C\|W\|^{1-\gamma} \leq \Phi_\lambda(W) \leq M,$$

where C is a positive constant. Since $0 < 1-\gamma < 2$, we have

$$\sup\left\{\|W\| : W \in \mathcal{N}^-_{\lambda,\theta\lambda}, \Phi_\lambda(W) \leq M\right\} < \infty.$$

Now, let us prove the two first inequalities in (2.42). First, let $U_n \subset \mathcal{N}^+_{\lambda,\theta\lambda}$ such that $\Phi_\lambda(U_n) \to \tilde{J}^+_\lambda$. Thus, it follows from the boundedness of $\mathcal{N}^+_{\lambda,\theta\lambda}$, proved in $a)$, that, up to a subsequence, $U_n \rightharpoonup U$ in E and hence $-\infty < \Phi_\lambda(U) \leq \liminf \Phi_\lambda(U_n) = \tilde{J}^+_\lambda$. To show the first inequality, we use (2.40) in the expression of $\Phi_\lambda(U)$ to infer that

$$\begin{aligned}
\Phi_\lambda(U) &= \left(\frac{\alpha+\beta-2}{2(\alpha+\beta)}\right)\|U\|^2 - \lambda\left(\frac{\alpha+\beta+\gamma-1}{(\alpha+\beta)(1-\gamma)}\right) K_{1,\theta}(U) \\
&< \left(\frac{\alpha+\beta-2}{2(\alpha+\beta)}\right)\|U\|^2 - \left(\frac{(\alpha+\beta-2)}{(\alpha+\beta)(1-\gamma)}\right)\|U\|^2 \\
&= -\left(\frac{(1+\gamma)(\alpha+\beta-2)}{2(1-\gamma)(\alpha+\beta)}\right)\|U\|^2 < 0
\end{aligned}$$

holds, that is, $\tilde{J}^+_\lambda < 0$.

In a similar way, we can prove that $-\infty < \Phi_\lambda(W) \leq \liminf \Phi_\lambda(W_n) = \tilde{J}^-_\lambda$. This ends the proof. ■

Now we show that the infimum value is achieved in both Nehari manifolds $\mathcal{N}^+_{\lambda,\theta\lambda}$ and $\mathcal{N}^-_{\lambda,\theta\lambda}$.

Lemma 2.3.2 *Let* $0 < \lambda < \lambda_*(\theta)$. *Then there exist* $U_\lambda \in \mathcal{N}^+_{\lambda,\theta\lambda}$ *and* $W_\lambda \in \mathcal{N}^-_{\lambda,\theta\lambda}$ *such that* $\Phi_\lambda(U_\lambda) = \tilde{J}^+_\lambda$ *and* $\Phi_\lambda(W_\lambda) = \tilde{J}^-_\lambda$.

Proof First, we will show that there exists $U_\lambda \in \mathcal{N}^+_{\lambda,\theta\lambda}$ such that $\Phi_\lambda(U_\lambda) = \tilde{J}^+_\lambda$. Let $\{U_n\} \subset \mathcal{N}^+_{\lambda,\theta\lambda}$ such that $\Phi_\lambda(U_n) \to \tilde{J}^+_\lambda$. So, it follows from Lemma 2.3.1 a) that, up to a subsequence, $U_n \rightharpoonup U_\lambda$ in E and $U_\lambda \geq 0$. Suppose, on the contrary, that $U_\lambda = 0$. Then $0 = \Phi_\lambda(U_\lambda) \leq \liminf \Phi_\lambda(U_n) = \tilde{J}^+_\lambda < 0$, which is impossible, that is, $U_\lambda \neq 0$ and so $U_\lambda \in E_+$.

Let us prove that $U_\lambda \in \mathcal{N}^+_{\lambda,\theta\lambda}$. First, we claim that $\{U_n\}$ converges strongly to U_λ in E. On the contrary, we would have that $\|U_\lambda\| < \liminf \|U_n\|$ and thus

$$\liminf_{n\to\infty} \phi'_{U_n,\lambda}(t^+_\lambda(\theta, U_\lambda)U_n) > \phi'_{U_\lambda,\lambda}(t^+_\lambda(\theta, U_\lambda)U_\lambda) = 0,$$

which implies that $\phi'_{U_n,\lambda}(t^+_\lambda(\theta, U_\lambda)U_n) > 0$ for sufficiently large n. It follows from Proposition 2.2.1 and Proposition 2.2.5 *b*) applied to the fiber map $\phi_{U_n,\lambda}$ that $1 = t^+_\lambda(\theta, U_n) < t^+_\lambda(\theta, U_\lambda)$ holds for large n. Therefore, by coming back to the fiber map $\phi_{U_n,\lambda}$, we obtain from Proposition 2.2.1 again that $\Phi_\lambda(t^+_\lambda(\theta, U_\lambda)U_\lambda) < \Phi_\lambda(U_\lambda)$ and consequently

$$\tilde{J}_\lambda \leq J^+_\lambda(U_\lambda) = \Phi_\lambda(t^+_\lambda(U_\lambda)U_\lambda) < \liminf \Phi_\lambda(U_n) = \tilde{J}^+_\lambda,$$

which is an absurd, that is, $U_n \to U_\lambda$ in E and hence

$$\phi'_{U_\lambda,\lambda}(1) = \lim_{n\to\infty} \phi'_{U_n,\lambda}(1) = 0 \quad \text{and} \quad \phi''_{U_\lambda,\lambda}(1) = \lim_{n\to\infty} \phi''_{U_n,\lambda}(1) \geq 0. \tag{2.43}$$

Since from Lemma 2.2.5 *b*) holds, we have that $\mathcal{N}^0_{\lambda,\theta\lambda} = \emptyset$ for $0 < \lambda < \lambda_*(\theta)$, which oblige us to conclude that $U_\lambda \in \mathcal{N}^+_{\lambda,\theta\lambda}$ and $\Phi_\lambda(U_\lambda) = \tilde{J}^+_\lambda$.

Next, let us prove that there exists $W_\lambda \in \mathcal{N}^-_{\lambda,\theta\lambda}$ that satisfies $\Phi_\lambda(W_\lambda) = \tilde{J}^-_\lambda$. Let $\{W_n\} \subset \mathcal{N}^-_{\lambda,\theta\lambda}$ be such that $\Phi_\lambda(W_n) \to \tilde{J}^-_\lambda$. As above, we have that $W_n \rightharpoonup W_\lambda$ in E and $W_\lambda \geq 0$. Assume on the contrary that $W_\lambda = 0$. Then, from Lemma 2.3.1 *b*), we obtain the absurd

$$0 < \inf\left\{\|W\|^2 : W \in \mathcal{N}^-_{\lambda,\theta\lambda}\right\} \leq \liminf_{n\to\infty} \|W_n\|^2 \leq \liminf_{n\to\infty} \frac{(\alpha+\beta+\gamma-1)}{(1+\gamma)} L(W_n) = 0,$$

where the last equality follows from the compact embedding E into $L^{\alpha+\beta}(\mathbb{R}^N)$ so that $W_\lambda \neq 0$ and thus $W_\lambda \in E_+$. By repeating the above arguments, we have $L(W_\lambda) > 0$.

We claim that $\{W_n\}$ converges strongly to W_λ in E. Suppose not. Then we may assume that $||W_n - W_\lambda|| \to \kappa > 0$. So, by Brezis-Lieb Lemma, we infer that

$$\tilde{J}_\lambda^- = \Phi_\lambda(W_\lambda) + \frac{\kappa^2}{2},\ \phi'_{W_\lambda,\lambda}(1) + \kappa^2 = 0,\ \text{and } \phi''_{W_\lambda\lambda} + \kappa^2 \le 0.$$

So, we would have $\phi'_{W_\lambda,\lambda}(1) < 0$ and $\phi''_{W_\lambda,\lambda}(1) < 0$. As a consequence of Proposition 2.2.1 and Lemma 2.3.1 b), there exists a $t_\lambda^- \in (0,1)$ such that $\phi'_{W_\lambda\lambda}(t_\lambda^-) = 0$, $\phi''_{W_\lambda,\lambda}(t_\lambda^-) < 0$ and $t_\lambda^- W_\lambda \in \mathcal{N}^-_{\lambda,\theta\lambda}$.

By setting $g(t) = \phi_{W_\lambda,\lambda}(t) + \frac{\kappa^2 t^2}{2}$ for $t > 0$, we conclude that $0 < t_\lambda^- < 1$, $g'(1) = 0$ and $g'(t_\lambda^-) = \kappa^2 t_\lambda^- > 0$, which together with Proposition 2.2.1 lead us to conclude that g is increasing on $[t_\lambda^-, 1]$. Thus, we have

$$\tilde{J}_\lambda^- = \lim \Phi_\lambda(W_n) = g(1) > g(t_\lambda^-) > \phi_{W_\lambda,\lambda}(t_\lambda^-) = \Phi_\lambda(t_\lambda^- W_\lambda) \ge \tilde{J}_\lambda^-,$$

which is a contradiction, that is $\kappa = 0$ and $\{W_n\}$ converges strongly to W_λ in E. After this, we obtain that $W_\lambda \in \mathcal{N}^-_{\lambda,\theta\lambda}$ and $\Phi_\lambda(W_\lambda) = \tilde{J}_\lambda^-$, as done at (2.43). This ends the proof. ■

As a consequence of Proposition 2.2.2 we have the following proposition.

Proposition 2.3.1 *Let $0 < \lambda < \lambda_*(\theta)$. Then $U_\lambda \in \mathcal{N}^+_{\lambda,\theta\lambda}$ and $W_\lambda \in \mathcal{N}^-_{\lambda,\theta\lambda}$ are solutions of Problem $(\tilde{P}_{\lambda,\theta\lambda})$.*

2.4 Multiplicity of solutions on boundary of the extremal region to applicability of Nehari method

In this section we prove the existence of at least two solutions for Problem $(\tilde{P}_{\lambda,\mu})$ on the curve $\tilde{\Gamma}$. To do this, it suffices to show that the problem $(\tilde{P}_{\tilde{\Gamma}(\theta)})$ has at least two solutions for each $\theta > 0$ fixed. We will take advantage of the multiplicity result given in Proposition 2.3.1 for $0 < \lambda < \lambda_*(\theta)$ and perform a limit process. The next proposition is a consequence of monotonicities and regularities of the functions $t_\lambda^+(\theta, U), t_\lambda^-(\theta, U)$, J_λ^+ and J_λ^- given by Lemma 2.2.6.

Proposition 2.4.1 *There holds:*

a) the functions $(0, \lambda_*(\theta)] \ni \lambda \to \tilde{J}_\lambda^\pm$ are decreasing and left-continuous for $\lambda \in (0, \lambda_*(\theta))$, *b)* $\lim\limits_{\lambda \uparrow \lambda_*(\theta)} \tilde{J}_\lambda^\pm = \tilde{J}^\pm_{\lambda_*(\theta)}$.

Proposition 2.4.2 *The problem $(P_{\tilde{\Gamma}(\theta)})$ admits at least two solutions $W_{\lambda_*(\theta)} \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)}$ and $U_{\lambda_*(\theta)} \in \mathcal{N}^+_{\tilde{\Gamma}(\theta)}$ for each $\theta > 0$.*

Proof First, let us show that there exists a solution $W_{\lambda_*(\theta)} \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)}$ for $(P_{\tilde{\Gamma}(\theta)})$. Let $\{\lambda_n\} \subset (0, \lambda_*(\theta))$ be such that $\lambda_n \uparrow \lambda_*(\theta)$ and $\{W_{\lambda_n}\} = \{(u_n, v_n)\} \subset \mathcal{N}^-_{\lambda_n,\theta\lambda_n}$ as in Proposition 2.3.1. Suppose on the contrary that $||W_{\lambda_n}|| \to \infty$. Hence after applying the Hölder inequality, Sobolev embedding and the fact that $W_{\lambda_n} \in \mathcal{N}^-_{\lambda_n,\theta\lambda_n}$, we obtain

$$
\begin{aligned}
J^-_{\lambda_n} = \Phi_{\lambda_n}(W_{\lambda_n}) &= \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right) ||W_{\lambda_n}||^2 + \lambda_n \left(\frac{1}{\alpha+\beta} - \frac{1}{1-\gamma}\right) K_{1,\theta}(W_n) \\
&\geq \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right) ||W_{\lambda_n}||^2 + C\lambda_n \left(\frac{1}{\alpha+\beta} - \frac{1}{1-\gamma}\right) ||W_{\lambda_n}||^{1-\gamma},
\end{aligned}
$$

which implies by Proposition 2.4.1 that $\infty > \lim \tilde{J}^-_{\lambda_n} \geq \infty$, which is a contradiction. Therefore $\{W_{\lambda_n}\}$ is bounded and we can assume that $W_{\lambda_n} \rightharpoonup W_{\lambda_*}$ in E,

$$
\begin{aligned}
&W_{\lambda_n} \to W_{\lambda_*(\theta)} = (u_{\lambda_*(\theta)}, v_{\lambda_*(\theta)}) \text{ in } L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \forall\ q \in [2, 2^*), \\
&W_{\lambda_n} \to W_{\lambda_*} \text{ a.e. in } \mathbb{R}^N, \\
&\text{there exist } h_q \in L^q(\mathbb{R}^N) \text{ such that } |W_{\lambda_n}| \leqslant h_q
\end{aligned}
$$

with $W_{\lambda_*(\theta)} = (u_{\lambda_*(\theta)}, v_{\lambda_*(\theta)}) \geqslant 0$.

Thus, once W_{λ_n} is a solution for Problem $(P_{\lambda_n,\theta\lambda_n})$, we may use Fatou's Lemma to show that

$$
0 \leq \langle J'(W_{\lambda_*(\theta)}), \Psi\rangle_E - \lambda_*(\theta)\langle K'_{1,\theta}(W_{\lambda_*(\theta)}), \Psi\rangle_E - \langle L'(W_{\lambda_*(\theta)}), \Psi\rangle_E, \tag{2.44}
$$

for all $\Psi \in E_+$, that is, the condition (2.14) of Proposition 2.2.3 is satisfied.

Moreover, from Lebesgue's dominated convergence Theorem and Fatou's Lemma we have

$$
\begin{aligned}
&\limsup \int_{\mathbb{R}^N} [\nabla u_n \nabla(u_n - u_{\lambda_*(\theta)}) + V(x)u_n(u_n - u_{\lambda_*(\theta)})dx \\
&= \limsup \left[\lambda_n \int_{\mathbb{R}^N} a(x)u_n(x)^{-\gamma}(u_n - u_{\lambda_*(\theta)})dx + \int_{\mathbb{R}^N} b(x)u_n^{\alpha-1}v_n^\beta(u_n - u_{\lambda_*(\theta)})dx\right] \\
&= \limsup \left[\lambda_n \int_{\mathbb{R}^N} a(x)u_n(x)^{-\gamma}(u_n - u_{\lambda_*(\theta)})dx\right] \\
&= \limsup \left[\lambda_n \int_{\mathbb{R}^N} a(x)u_n^{1-\gamma}(x)dx - \lambda_n \int_{\mathbb{R}^N} a(x)u_n(x)^{-\gamma}u_{\lambda_*(\theta)}dx\right]
\end{aligned}
$$

$$
\begin{aligned}
\leq & \lambda_*(\theta)\int_{\mathbb{R}^N} a(x)u^{1-\gamma}_{\lambda_*(\theta)}(x)dx - \liminf \lambda_n \int_{\mathbb{R}^N} a(x)u_n(x)^{-\gamma}u_{\lambda_*(\theta)}dx \\
\leq & \lambda_*(\theta)\int_{\mathbb{R}^N} a(x)u^{1-\gamma}_{\lambda_*(\theta)}(x)dx - \lambda_*(\theta)\int_{\mathbb{R}^N} a(x)u^{1-\gamma}_{\lambda_*(\theta)}(x)dx = 0,
\end{aligned}
$$

which implies that $u_n \to u_{\lambda_*(\theta)}$ in X. A similar argument show that $v_n \to v_{\theta\lambda^*(\theta)}$ in X. So, it follows $W_n \to W_{\lambda_*(\theta)}$ in E and this yields

$$
\|W_{\lambda_*(\theta)}\|^2 - \lambda_*(\theta)K_{1,\theta}(W_{\lambda_*(\theta)}) - L(W_{\lambda_*(\theta)}) = 0,
$$

that is, the condition (2.13) of Proposition 2.2.3 is satisfied. Therefore, by Proposition 2.2.3, we obtain $W_{\lambda_*(\theta)}$ is a solution of problem $(\tilde{P}_{\tilde{\Gamma}(\theta)})$.

Moreover, we have that

$$
\phi'_{W_{\lambda_*(\theta),\lambda_*(\theta)}}(1) = \lim \phi'_{\lambda_n,w_{\lambda_n}}(1) = 0 \text{ and } \phi''_{W_{\lambda_*(\theta)},\lambda_*(\theta)}(1) = \lim \phi''_{W_{\lambda_n},\lambda_n}(1) \leq 0,
$$

which implies, by the first equality, that $W_{\lambda_*(\theta)} \in \mathcal{N}_{\tilde{\Gamma}(\theta)}$. We also have, from Lemma 2.3.1 $b)$, that

$$
\begin{aligned}
0 < (1+\gamma)||W_{\lambda_*(\theta)}|| &\leq \lim_{n\to\infty} \frac{(\alpha+\beta+\gamma-1)}{(1+\gamma)}\int_{\mathbb{R}^N} b|u_n|^{\alpha}|v_n|^{\beta}dx \\
&= \frac{(\alpha+\beta+\gamma-1)}{(1+\gamma)}L(W_n) = \frac{(\alpha+\beta+\gamma-1)}{(1+\gamma)}L(W_{\lambda_*(\theta)}),
\end{aligned}
$$

that is, $L(W_{\lambda_*(\theta)}) > 0$ and hence $W_{\lambda_*(\theta)} \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)} \cup \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$. We point out that $W_{\lambda_*(\theta)} \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)}$ due to Lemma 2.2.5.

Finally, it follows from the strong convergence, Proposition 2.3.1, Proposition 2.4.1 and Proposition 2.2.6 $(iv), (v), (vi)$ that

$$
\begin{aligned}
\Phi_{\lambda_*(\theta)}(W_{\lambda_*(\theta)}) &= \lim \Phi_{\lambda_n}(W_{\lambda_n}) = \lim \tilde{J}^-_{\lambda_n} = \tilde{J}^-_{\lambda_*(\theta)} \\
&= \inf\left\{\Phi_{\lambda_*(\theta)}(t_{\lambda_*(\theta)}(\theta,w)w) : w \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)} \cup \mathcal{N}^0_{\tilde{\Gamma}(\theta)}\right\}
\end{aligned} \tag{2.45}
$$

holds, that is, $W_{\lambda_*(\theta)} \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)}$ is a global minimum of $\Phi_{\lambda_*(\theta)}$ constrained to $\mathcal{N}^-_{\tilde{\Gamma}(\theta)} \cup \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$.

In order to show the existence of a second solution for Problem $(\tilde{P}_{\tilde{\Gamma}(\theta)})$, we proceed in a similar way, that is, pick a $\{\lambda_n\} \subset (0, \lambda_*(\theta))$ such that $\lambda_n \uparrow \lambda_*(\theta)$ and $\{U_{\lambda_n}\} \subset \mathcal{N}^+_{\lambda_n,\theta\lambda_n}$ as given by Proposition 2.3.1. After some manipulations, we obtain that $U_{\lambda_n} \to U_{\lambda_*(\theta)}$ in E for some $0 < U_{\lambda_*} \in \mathcal{N}^+_{\tilde{\Gamma}(\theta)} \cup \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$, which is a solution for Problem $(\tilde{P}_{\tilde{\Gamma}(\theta)})$.

Besides this, if $L(U_{\lambda_*(\theta)}) > 0$ and $\phi''_{U_{\lambda_*(\theta)},\lambda_*(\theta)}(1) = 0$, then $U_{\lambda_*(\theta)}$ would be a solution for the problem $(\tilde{P}_{\tilde{\Gamma}(\theta)})$ in $\mathcal{N}^0_{\tilde{\Gamma}(\theta)}$, but this is impossible by Proposition 2.2.5.

So we have $\phi''_{U_{\lambda_*(\theta)},\lambda_*(\theta)}(1) > 0$ in this case. On the other side, if $L(U_{\lambda_*(\theta)}) \leq 0$, then we have

$$\phi''_{U_{\lambda_*(\theta)},\lambda_*(\theta)}(1) = ||U_{\lambda_*(\theta)}||^2 + \gamma\lambda_*(\theta)K_{1,\lambda_*(\theta)}(U_{\lambda_*(\theta)}) - (\alpha+\beta-2)L(U_{\lambda_*(\theta)}) > 0.$$

So, in both cases, we have $\phi''_{U_{\lambda_*(\theta)},\lambda_*(\theta)}(1) > 0$, which implies that $U_{\lambda_*(\theta)} \in \mathcal{N}^+_{\tilde{\Gamma}(\theta)}$. We also have that $U_{\lambda_*(\theta)} \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)}$ is a global minimum of $\Phi_{\lambda_*(\theta)}$ constrained to $\mathcal{N}^+_{\tilde{\Gamma}(\theta)} \cup \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ as well. This ends the proof. ■

Before proving the multiplicity of solutions for Problem $(\tilde{P}_{\lambda,\theta\lambda})$ when $\lambda > \lambda_*(\theta)$, let us gather further information on the sets

$$S^-_{\lambda_*(\theta)} = \left\{ W \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)} : J^-_{\lambda_*(\theta)}(W) = \tilde{J}^-_{\lambda_*(\theta)} \right\} \tag{2.46}$$

and

$$S^+_{\lambda_*(\theta)} = \left\{ U \in \mathcal{N}^+_{\tilde{\Gamma}(\theta)} : J^+_{\lambda_*(\theta)}(U) = \tilde{J}^+_{\lambda_*(\theta)} \right\}. \tag{2.47}$$

Corollary 2.4.1 *We have that:*

a) $S^-_{\lambda_*(\theta)}$ *and* $S^+_{\lambda_*(\theta)}$ *are non-empties,*

b) there exist $c_{\lambda_*(\theta)}, C_{\lambda_*(\theta)} > 0$ *such that* $c_{\lambda_*(\theta)} \leq \|U\|, \|W\| \leq C_{\lambda_*(\theta)}$ *for all* $U \in S^+_{\lambda_*(\theta)}$ *and* $W \in S^-_{\lambda_*(\theta)}$*,*

c) if $U \in S^-_{\lambda_*(\theta)} \cup S^+_{\lambda_*(\theta)}$*, then* U *is a solution for Problem* $(\tilde{P}_{\tilde{\Gamma}(\theta)})$*.*

Proof The item $a)$ follows immediately from (2.45), while $b)$ is a consequence of Lemma 2.3.1. Finally, the proof of the item $c)$ follows of Proposition 2.2.2. ■

2.5 Multiplicity of solutions beyond the extremal region to the applicability of the Nehari method

In this section we show the existence of solutions for problem $(\tilde{P}_{\lambda,\theta\lambda})$ when λ is greater than $\lambda_*(\theta)$ but close to it. The idea is to minimize the energy functional Φ_λ over subsets of $\mathcal{N}^+_{\lambda,\theta\lambda}$ and $\mathcal{N}^-_{\lambda,\theta\lambda}$, which are projections of subsets of $\mathcal{N}^+_{\tilde{\Gamma}(\theta)}$ and $\mathcal{N}^-_{\tilde{\Gamma}(\theta)}$ that have positive distances to $\mathcal{N}^0_{\lambda_*(\theta),\theta\lambda_*(\theta)}$.

Proposition 2.5.1 *Let* $c < C$*. Assume that* $\lambda_n \downarrow \lambda_*(\theta)$*.*

a) suppose that $W_n \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)}$ *satisfies* $c \leq ||W_n|| \leq C$. *If*

$$(t^-_{\lambda_n}(\theta, W_n))^2 \phi''_{W_n,\lambda_n}(t^-_{\lambda_n}(\theta, W_n)) \to 0,$$

then $d(W_n, \mathcal{N}^0_{\tilde{\Gamma}(\theta)}) \to 0$ *as* $n \to \infty$,

b) suppose that $U_n \in \mathcal{N}^+_{\tilde{\Gamma}(\theta)}$ *satisfies* $c \leq ||U_n|| \leq C$. *If*

$$(t^+_{\lambda_n}(\theta, U_n))^2 \phi''_{U_n,\lambda_n}(t^+_{\lambda_n}(\theta, U_n)) \to 0,$$

then $d(U_n, \mathcal{N}^0_{\tilde{\Gamma}(\theta)}) \to 0$ *as* $n \to \infty$.

Proof We prove only $a)$ since the proof of $b)$ follows the same strategy. It follows from Lemma 2.3.1 $b)$ that there exists a positive constant c such that $L(W_n) \geq c$. We claim that the same holds for $K_{1,\theta}(W_n)$. To prove this, let us first prove that $t^-_{\lambda_n}(\theta, W_n) \to \rho \in (0, \infty)$.

Now, by applying Proposition 2.2.1, there exist $s_n := t^+_{\lambda_n}(\theta, W_n) < t^-_{\lambda_n}(\theta, W_n) := t_n$ such that

$$\begin{cases} t_n^2||W_n||^2 - t_n^{1-\gamma}\lambda_n K_{1,\theta}(W_n) - t_n^{\alpha+\beta}L(W_n) = 0, \\ t_n^2||W_n||^2 + t_n^{1-\gamma}\lambda_n \gamma K_{1,\theta}(W_n) - t_n^{\alpha+\beta}(\alpha+\beta-1)L(W_n) = o(1), \\ s_n^2||W_n||^2 - s_n^{1-\gamma}\lambda_n K_{1,\theta}(W_n) - s_n^{\alpha+\beta}L(W_n) = 0, \end{cases} \tag{2.48}$$

where the second line is a consequence of the assumption

$$(t^-_{\lambda_n}(\theta, W_n))^2 \phi''_{W_n,\lambda_n}(t^-_{\lambda_n}(\theta, W_n)) \to 0.$$

So, by solving the system formed by the first and third equation of the above system, considering the integrals as unknown, and substituting them into the second equation, we obtain

$$||W_n||^2 t_n^2 \left[\frac{(1+\gamma)\left(\frac{s_n}{t_n}\right)^{\alpha+\beta+\gamma-1} + (\alpha+\beta-2) - (\alpha+\beta+\gamma-1)\left(\frac{s_n}{t_n}\right)^{1+\gamma}}{\left(\frac{s_n}{t_n}\right)^{p+\gamma} - 1} \right] = o(1), \tag{2.49}$$

as $n \to \infty$.

Besides this, it follows from $C \geq ||W_n|| \geq c$, Lemma 2.3.1, the first and third equations of system above and $s_n < t_n$ that there exist positive constants $\tilde{c}, \tilde{C}, \rho, \alpha$ such that $t_n, s_n \in [\tilde{c}, \tilde{C}]$, $t_n \to \theta$, $s_n \to \alpha$ and $||t_n W_n|| \geq \tilde{c}$. By using these information and taking limit on (2.49), we conclude that $s_n/t_n \to 1$ and $\rho = \alpha$, because $t = 1$ is the only zero of the function

$$g(t) = (1+\gamma)t^{p+\gamma} + (\alpha+\beta-2) - (\alpha+\beta+\gamma-1)t^{1+\gamma}.$$

Using the above information and manipulating in the first and second equations of (2.48), we obtain

$$\begin{cases} ||\rho W_n||^2 - \lambda_*(\theta)K_{1,\theta}(\rho W_n) - L(\rho W_n) = o(1), \\ ||\rho W_n||^2 + \gamma\lambda_*(\theta)K_{1,\theta}(\rho W_n) - (\alpha+\beta-1)L(\rho W_n) = o(1). \end{cases}$$

Since $s_nW_n \in \mathcal{N}^+_{\lambda_n,\theta\lambda_n}$, we obtain from Lemma 2.3.1 *a)* that $K_{1,\theta}(W_n) \geq c$. So coming back in the above system and using this positive boundedness from below, we have

$$\frac{\alpha+\beta-2}{\alpha+\beta+\gamma-1}\frac{||\rho W_n||^2}{K_{1,\theta}(\rho W_n)} = \lambda_*(\theta) + o(1),\ n \to \infty,$$

and

$$\frac{1+\gamma}{\alpha+\beta+\gamma-1}\frac{||\rho W_n||^2}{L(\rho W_n)} = 1 + o(1),\ n \to \infty.$$

Therefore, it follows from (2.16) and 0-homogeneity of $\lambda(\theta,\cdot)$ that

$$\lambda(\theta, W_n) = \lambda(\theta, \rho W_n) = (1+o(1))^{\frac{1+\gamma}{\alpha+\beta-2}}(\lambda_*(\theta) + o(1)) \to \lambda_*(\theta),\ n \to \infty,$$

and W_n is a bounded minimizing sequence for $\lambda_*(\theta)$. Moreover, by following similar arguments as done in the proof of Lemma 2.2.5, we obtain, up to a subsequence, that $W_n \to W \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ and consequently $d(W_n, \mathcal{N}^0_{\tilde{\Gamma}(\theta)}) \to 0$ as $n \to \infty$. This ends the proof.

■

Define

$$\mathcal{N}^-_{\lambda_*(\theta),d,C} = \left\{W \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)} : d(W, \mathcal{N}^0_{\tilde{\Gamma}(\theta)}) > d, ||W|| \leq C\right\},$$

and

$$\mathcal{N}^+_{\lambda_*(\theta),d,c} = \left\{U \in \mathcal{N}^+_{\tilde{\Gamma}(\theta)} : d(U, \mathcal{N}^0_{\tilde{\Gamma}(\theta)}) > d, c \leq ||U||\right\}$$

for $c, C, d > 0$ given. As an immediately consequence of Proposition 2.5.1, we have.

Corollary 2.5.1 *Fix $c, C, d > 0$. Then there exist $\epsilon > 0$ satisfying:*

a) there exists $\delta < 0$ such that $(t^-_\lambda(\theta,W))^2\phi''_{W,\lambda}(t^-_\lambda(\theta,W)) < \delta$ for all $\lambda \in (\lambda_*(\theta), \lambda_*(\theta)+\epsilon)$ and $W \in \mathcal{N}^-_{\lambda_*(\theta),d,C}$. In particular, we have that $t^-_\lambda(\theta,W)W \in \mathcal{N}^-_{\lambda,\theta\lambda}$ and $W \in \hat{\mathcal{N}}_{\lambda,\theta\lambda}$ for all $\lambda \in (\lambda_*(\theta), \lambda_*(\theta)+\epsilon)$, *b) there exists* $\delta > 0$ *such that* $(t_\lambda^+(\theta,U))^2\phi_\lambda''(t_\lambda^+(\theta,U)) > \delta$ *for all* $\lambda \in (\lambda_*(\theta), \lambda_*(\theta)+\epsilon)$ *and* $U \in \mathcal{N}^+_{\lambda_*(\theta),d,c}$*. In particular, we have that* $t_\lambda^+(\theta,U)U \in \mathcal{N}^+_{\lambda,\theta\lambda}$ *and* $U \in \hat{\mathcal{N}}_{\lambda,\theta\lambda} \cup \hat{\mathcal{N}}^+_{\lambda,\theta\lambda}$ *for all* $\lambda \in (\lambda_*(\theta), \lambda_*(\theta)+\epsilon)$*.*

To do a good choice of the parameter $d > 0$ in the last corollary, we prove the next result, where the sets $S^-_{\lambda_*(\theta)}$ and $S^+_{\lambda_*(\theta)}$ were defined at (2.46) and (2.47).

Proposition 2.5.2 *There holds:*

a) $d(S^-_{\lambda_*(\theta)}, \mathcal{N}^0_{\tilde{\Gamma}(\theta)}) > 0,$

b) $d(S^+_{\lambda_*(\theta)}, \mathcal{N}^0_{\tilde{\Gamma}(\theta)}) > 0.$

Proof We just prove $a)$ because the proof of $b)$ follows similar arguments. Assume by contradiction that $d(S^-_{\lambda_*(\theta)}, \mathcal{N}^0_{\tilde{\Gamma}(\theta)}) = 0$. Then, there exist $W_n \in S^-_{\lambda_*(\theta)}$ and $V_n \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ such that $\|W_n - V_n\| \to 0$ as $n \to \infty$ and

$$(W_n, \Psi) = \lambda_*(\theta)\langle dK_{1,\theta}(W_n), \Psi\rangle_E + \langle dL(W_n), \Psi\rangle_E, \ \forall \psi \in E, \ \forall n \in \mathbb{N}$$

holds, where this equality is a consequence of W_n be a solution for Problem $(P_{\tilde{\Gamma}(\theta)})$ as claimed in Corollary 2.4.1. Since $\mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ is a compact set, see Lemma 2.2.5, we may assume that $V_n \to V \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ and hence $W_n \to V$ as well. From Fatou's Lemma we conclude that

$$(V, \Psi) \geq \lambda_*(\theta)\langle dK_{1,\theta}(V), \Psi\rangle_E + \langle dL(V), \Psi\rangle_E, \ \forall \Psi \in E_+,$$

that is, we arrived in the same situation as in Proposition 2.2.3 with $V \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$. So, by Proposition 2.2.3 follow that $V \in \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$ is a solution for Problem $(P_{\tilde{\Gamma}(\theta)})$, but this is impossible by Corollary 2.2.1, which ends the proof. ∎

After Corollaries 2.4.1, 2.5.1 and Proposition 2.5.2, we are in position to introduce

$$\tilde{J}^-_{\lambda,d^-,C} \equiv \inf\left\{J^-_\lambda(W) : W \in \mathcal{N}^-_{\lambda_*(\theta),d^-,C}\right\} \text{ and } \tilde{J}^+_{\lambda,d^+,c} \equiv \inf\left\{J^+_\lambda(W) : W \in \mathcal{N}^+_{\lambda_*(\theta),d^+,c}\right\} \tag{2.50}$$

for each $0 < c < c_{\lambda_*}$, $C > C_{\lambda_*}$ (see Corollary 2.4.1 for both) $\lambda_*(\theta) < \lambda < \lambda_*(\theta)+\epsilon$ (see Corollary 2.5.1) and $0 < d^\pm < d(S^\pm_{\lambda_*(\theta)}, \mathcal{N}^0_{\tilde{\Gamma}(\theta)})$ (see Proposition 2.5.2) which implies that $S^-_{\lambda_*(\theta)} \subset \mathcal{N}^-_{\lambda_*(\theta),d^-,C}$ and $S^+_{\lambda_*(\theta)} \subset \mathcal{N}^+_{\lambda_*(\theta),d^+,c}$. The proofs of the next propositions are similar to those of Propositions 1.4.3, 1.4.4, 1.4.5 of Chapter 1.

Proposition 2.5.3 *The λ-functions $\tilde{J}^-_{\lambda,d^-,C}$ and $\tilde{J}^+_{\lambda,d^+,C}$ are decreasing and there holds:*

a) $\lim_{\lambda\downarrow\lambda_*} \tilde{J}^-_{\lambda,d^-,C} = \tilde{J}^-_{\lambda_*(\theta)}$,

b) $\lim_{\lambda\downarrow\lambda_*} \tilde{J}^+_{\lambda,d^+,c} = \tilde{J}^+_{\lambda_*(\theta)}$.

Proposition 2.5.4 *There exists $\epsilon^- > 0$ such that J^-_λ constrained to $\mathcal{N}^-_{\lambda_*(\theta),d^-,C}$ has a minimizer $\tilde{W}_\lambda \in \mathcal{N}^-_{\lambda_*(\theta),d^-,C}$ for all $\lambda \in (\lambda_*(\theta), \lambda_*(\theta)+\epsilon^-)$ given.*

Proposition 2.5.5 *There exists $\epsilon^+ > 0$ such that J^+_λ constrained to $\mathcal{N}^+_{\lambda_*(\theta),d^+,c}$ has a minimizer $\tilde{U}_\lambda \in \mathcal{N}^+_{\lambda_*(\theta),d^+,c}$ for all $\lambda \in (\lambda_*(\theta), \lambda_*(\theta)+\epsilon^+)$ given.*

The main point in order to prove that the minima found in Propositions 2.5.4, 2.5.5 are solutions of $(\tilde{P}_{\lambda,\theta\lambda})$ is to prove that $\tilde{W}_\lambda$ and $\tilde{U}_\lambda$ are interior points of $\mathcal{N}^-_{\lambda_*(\theta),d^-,C}$ and $\mathcal{N}^+_{\lambda_*(\theta),d^+,c}$ respectively.

Proposition 2.5.6 *There exists $\epsilon > 0$ such that the problem $(\tilde{P}_{\lambda,\theta\lambda})$ admits at least two solutions $W_\lambda \in \mathcal{N}^-_{\lambda,\theta\lambda}$ and $U_\lambda \in \mathcal{N}^+_{\lambda,\theta\lambda}$ for each $\lambda \in (\lambda_*(\theta), \lambda_*(\theta)+\epsilon)$.*

Proof First, let us take advantage of the existence of the minimizer $\tilde{W}_\lambda \in \mathcal{N}^-_{\lambda_*(\theta),d^-,C}$ to build a solution for Problem $(\tilde{P}_{\lambda,\theta\lambda})$ in $\mathcal{N}^-_{\lambda,\theta\lambda}$. Let us do this by reminding that the definitions given at (2.50) and (2.38) implies that we can consider $W_\lambda := t^-_\lambda(\theta,\tilde{W}_\lambda)\tilde{W}_\lambda \in \mathcal{N}^-_{\lambda,\theta\lambda}$. Below, let us prove that W_λ is a solution for Problem $(\tilde{P}_{\lambda,\theta\lambda})$ if $\lambda > \lambda_*(\theta)$ varies in an appropriate range. To this end, firstly we prove that $\tilde{W}_\lambda$ is a interior point of $\mathcal{N}^-_{\lambda_*(\theta),d^-,C}$ for λ close $\lambda_*(\theta)$, which is equivalently to prove

Claim: there exists an $\epsilon_1 > 0$ such that

$$||\tilde{W}_\lambda|| < C, \ \forall \ \lambda \in (\lambda_*(\theta), \lambda_*(\theta)+\epsilon_1), \tag{2.51}$$

where $C > C_{\lambda_*(\theta)}$ and $C_{\lambda_*(\theta)} > 0$ is given by Corollary 2.4.1.

Indeed, let $\lambda_n \downarrow \lambda_*(\theta)$ and denote $\tilde{W}_{\lambda_n} = \tilde{W}_n$. Due to the boundedness of $\mathcal{N}^-_{\lambda_*(\theta),d^-,C}$, we may assume that $\tilde{W}_{\lambda_n} \rightharpoonup \tilde{W}$ in E. In fact, we have that $\tilde{W}_n \to \tilde{W}$ in E, otherwise we would have $||\tilde{W}|| < \liminf ||\tilde{W}_n||$ which implies

$$0 = \phi'_{\tilde{W},\lambda_*(\theta)}(t_{\lambda_*(\theta)}(\theta,\tilde{W})) < \liminf \phi'_{\tilde{W}_n,\lambda_n}(t_{\lambda_*(\theta)}(\theta,\tilde{W})),$$

where $t_{\lambda_*(\theta)}$ is given by Proposition 2.2.6 (iv). It follows that there exists k such that $\phi'_{\tilde{W}_n,\lambda_n}(t_{\lambda_*(\theta)}(\theta,\tilde{W})) > 0$ for $n \geq k$, that is, $t^+_{\lambda_n}(\theta,\tilde{W}_n) < t_{\lambda_*(\theta)}(\tilde{W}) < t^-_{\lambda_n}(\theta,\tilde{W}_n)$ by

Proposition 2.2.1. For convenience we will denote by

$$t^+_{\lambda_n}(\tilde{W}_n) = t^+_{\lambda_n}(\theta, \tilde{W}_n) \text{ and } t^-_{\lambda_n}(\tilde{W}_n) = t^-_{\lambda_n}(\theta, \tilde{W}_n).$$

Therefore

$$\|t_{\lambda_*(\theta)}(\tilde{W})\tilde{W}\|^2 < \liminf_{n\to\infty} \|t_{\lambda_n}(\tilde{W}_n)\tilde{W}_n\|^2,$$

which lead us to

$$\Phi_{\lambda_*(\theta)}(t_{\lambda_*(\theta)}(\tilde{W})\tilde{W}) < \liminf_{\lambda_n \downarrow \lambda_*(\theta)} \Phi_{\lambda_n}(t_{\lambda_*(\theta)}(\tilde{W})\tilde{W}_n) \leq \liminf_{\lambda_n \downarrow \lambda_*(\theta)} \Phi_{\lambda_n}(t^-_{\lambda_n}(\tilde{W}_n)\tilde{W}_n) = \hat{J}^-_{\lambda_*(\theta)}, \tag{2.52}$$

where Proposition 2.5.3 a) was used to get the last equality. Moreover, it follows from Proposition 2.4.1 b), Proposition 2.3.1 and Corollary 2.2.2 that

$$\hat{J}^-_{\lambda_*(\theta)} = \lim_{\lambda'_n \uparrow \lambda_*(\theta)} \hat{J}^-_{\lambda'_n} \leq \lim_{\lambda'_n \uparrow \lambda_*(\theta)} \Phi_{\lambda'_n}(t^-_{\lambda'_n}(\tilde{W})\tilde{W}) = \Phi_{\lambda_*(\theta)}(t_{\lambda_*(\theta)}(\tilde{W})\tilde{W})$$

holds for any $\lambda'_n \uparrow \lambda_*(\theta)$. By combining the last inequality with (2.52), we get a contradiction and hence $\tilde{W}_n \to \tilde{W}$ in E.

As a consequence of this strong convergence and Lemma 2.3.1 b), we obtain $L(\tilde{W}) > 0$ and $\phi'_{\tilde{W},\lambda_*(\theta)}(1) = 0$ and $\phi''_{\tilde{W},\lambda_*(\theta)}(1) \leq 0$, which means by Proposition 2.2.1 that $\tilde{W} \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)} \cup \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$. Since

$$d(\tilde{W}, \mathcal{N}^0_{\tilde{\Gamma}(\theta)}) = \lim_{n\to\infty} d(\tilde{W}_n, \mathcal{N}^0_{\tilde{\Gamma}(\theta)}) \geq d^- > 0,$$

we have that $\tilde{W} \notin \mathcal{N}^0_{\tilde{\Gamma}(\theta)}$, that is, $\tilde{W} \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)}$.

To conclude the proof of the claim, we just need to show that $\tilde{W} \in S^-_{\lambda_*(\theta)}$. First note that similar arguments as done in the proof of Proposition 2.5.1-a) prove that $t^-_{\lambda_n}(\tilde{W}_n) \to t \in (0, \infty)$. From the strong convergence $\tilde{W}_n \to \tilde{W}$ in E, we get that $\phi'_{\tilde{W},\lambda_*(\theta)}(t) = 0$ and $\phi''_{\tilde{W},\lambda_*(\theta)}(t) \leq 0$, which lead us to conclude that $t = 1$ since $\tilde{W} \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)}$ and Proposition 2.2.1. From Proposition 2.5.3 and the strong convergence again, we obtain

$$\Phi_{\lambda_*(\theta)}(\tilde{W}) = \lim_{\lambda_n \downarrow \lambda_*(\theta)} \Phi_{\lambda_n}(t^-_{\lambda_n}(\tilde{W}_n)\tilde{W}_n) = \hat{J}^-_{\lambda_*(\theta)},$$

which means that $\tilde{W} \in S^-_{\lambda_*(\theta)}$. Therefore, from Corollary 2.4.1 we conclude that $\limsup_{\lambda \downarrow \lambda_*(\theta)} ||\tilde{W}_\lambda|| \leq ||\tilde{W}|| \leq C_{\lambda_*(\theta)}$. Since $C > C_{\lambda_*(\theta)}$, the claim is true. This ends the proof of the claim.

To complete the proof that $W_\lambda := t^-_\lambda(\tilde{W}_\lambda)\tilde{W}_\lambda \in \mathcal{N}^-_{\lambda,\theta\lambda}$ is a solution to Problem $(\tilde{P}_{\lambda,\theta\lambda})$, let us perturb $\tilde{W}_\lambda$ by appropriate elements of E_+ and perform projections of it over $\mathcal{N}^-_{\lambda_*(\theta),d^-,C}$ and $\mathcal{N}^-_{\lambda,\theta\lambda}$. Let $\Psi \in E_+$ and $\lambda \in (\lambda_*(\theta), \lambda_*(\theta)+\epsilon_1)$. Since $\tilde{W}_\lambda \in \mathcal{N}^-_{\tilde{\Gamma}(\theta)}$, we are able to apply the implicit function Theorem, as done in the Proposition 2.2.2, to prove that $t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi)$ (see Proposition 2.2.1) is well defined, is continuous for $\rho > 0$ small enough and $t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi) \longrightarrow 1$ as $\rho \longrightarrow 0$.

Thus, it follows from (2.51) and $d(\tilde{W}_\lambda, \mathcal{N}^0_{\tilde{\Gamma}(\theta)}) > d^-$ (see definition of $\mathcal{N}^-_{\lambda_*(\theta),d^-,C}$) that

$$||t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi)(\tilde{W}_\lambda + \rho\Psi)|| < C \text{ and } d(t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi)(\tilde{W}_\lambda + \rho\Psi), \mathcal{N}^0_{\tilde{\Gamma}(\theta)}) > d^-$$

holds for $\rho > 0$ small enough, which implies

$$t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi)(\tilde{W}_\lambda + \rho\Psi) \in \mathcal{N}^-_{\lambda_*(\theta),d^-,C}. \tag{2.53}$$

Therefore, by (2.53) and Corollary 2.5.1, we obtain

$$t_\lambda(\rho)t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi)(\tilde{W}_\lambda + \rho\Psi) \in \mathcal{N}^-_{\lambda,\theta\lambda},$$

where

$$t_\lambda(\rho) =: t^-_\lambda(t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi)(\tilde{W}_\lambda + \rho\Psi)).$$

By applying Proposition 2.5.4, we have

$$\begin{aligned}\Phi_\lambda(t_\lambda(\rho)t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi)(\tilde{W}_\lambda + \rho\Psi)) &= J^-_\lambda(t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi)(\tilde{W}_\lambda + \rho\Psi)) \\ &\geq \tilde{J}^-_{\lambda,d^-,C} = \Phi_\lambda(t^-_\lambda(\tilde{W}_\lambda)\tilde{W}_\lambda),\end{aligned}$$

which lead us to conclude that

$$\Phi_\lambda(t_\lambda(\rho)t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi)(\tilde{W}_\lambda + \rho\Psi)) \geq \Phi_\lambda(t^-_\lambda(\tilde{W}_\lambda)t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi)\tilde{W}_\lambda), \tag{2.54}$$

holds for all $\rho > 0$ small enough, after using Proposition 2.2.1.

Again, due to the fact that $t^-_\lambda(\tilde{W}_\lambda)\tilde{W}_\lambda \in \mathcal{N}^-_{\lambda,\theta\lambda}$, we are able to apply the implicit function Theorem, as in the Proposition 2.2.2 with the same function F at the point

$$\left(t^-_\lambda(\tilde{W}), ||\tilde{W}_\lambda||^2, \lambda K_{1,\theta}(\tilde{W}_\lambda), L(\tilde{W}_\lambda)\right)$$

to show that $t_\lambda(\rho) \to t_\lambda^-(\tilde{W})$ as $\rho \to 0$. Since (2.54) can be read as

$$
\begin{aligned}
&(t_\lambda(\rho) t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi))^2 \frac{\left[||\tilde{W}_\lambda + \rho\Psi||^2 - ||\tilde{W}_\lambda||^2\right]}{\rho} \\
&- \frac{(t_\lambda(\rho) t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi))^{\alpha+\beta}}{\alpha+\beta} \frac{L(\tilde{W}_\lambda + \rho\Psi) - L(\tilde{W}_\lambda)}{\rho} \\
&\geq \frac{(t_\lambda(\rho) t^-_{\lambda_*(\theta)}(\tilde{W}_\lambda + \rho\Psi))^{1-\gamma}}{1-\gamma} \frac{K_{1,\theta}(\tilde{W}_\lambda + \rho\Psi) - K_{1,\theta}(\tilde{W}_\lambda)}{\rho},
\end{aligned}
$$

we can follow the arguments done in Lemma 2.2.2, Fatou's Lemma and $t_\lambda(\rho) \to t_\lambda^-(\tilde{W})$ as $\rho \to 0$, to infer that

$$0 \leq (t_\lambda^-(\tilde{W}_\lambda))^2 \langle J'(\tilde{W}_\lambda), \Psi\rangle_E - (t_\lambda^-(\tilde{W}_\lambda))^{1-\gamma} \lambda \langle K'_{1,\theta}(\tilde{W}_\lambda), \Psi\rangle_E - (t_\lambda^-(\tilde{W}_\lambda))^{\alpha+\beta} \langle L'(\tilde{W}_\lambda), \Psi\rangle_E,$$

for every $\Psi = (\varphi, \psi) \in E_+$, that is,

$$0 \leq \langle J'(W_\lambda), \Psi\rangle_E - \lambda \langle K'_{1,\theta}(W_\lambda), \Psi\rangle_E - \langle L'(W_\lambda), \Psi\rangle_E.$$

To conclude that $W_\lambda \in \mathcal{N}^-_{\lambda,\theta\lambda}$ is a solution from $(\tilde{P}_{\lambda,\theta\lambda})$, we applied the Proposition 2.2.3.

To complete the proof of Proposition 2.5.6, let us follow the arguments done just above with minors adjustments. First, by setting $U_\lambda = t_\lambda^+(\theta, \tilde{U}_\lambda)\tilde{U}_\lambda \in \mathcal{N}^+_{\lambda,\theta\lambda}$, with $\tilde{U}_\lambda \in \mathcal{N}^-_{\lambda_*(\theta),d^+,c}$ being the minimizer of J_λ^+ constrained to $\mathcal{N}^+_{\lambda_*(\theta),d^+,c}$ as given in Proposition 2.5.5, and adjusting the proof of the above claim, we also prove the below claim.

Claim: there exists an $\epsilon_2 > 0$ such that

$$||\tilde{U}_\lambda|| > c, \ \forall \ \lambda \in (\lambda_*(\theta), \lambda_*(\theta) + \epsilon_2),$$

where $c < c_{\lambda_*(\theta)}$ and $c_{\lambda_*(\theta)} > 0$ is given by Corollary 2.4.1.

After this claim, by perturbing $\tilde{U}_\lambda$ by appropriate elements of E_+, performing projections of it over $\mathcal{N}^+_{\lambda_*(\theta),d^+,c}$ and $\mathcal{N}^+_{\lambda,\theta\lambda}$ and following the same strategy, we can prove that $U_\lambda \in \mathcal{N}^+_{\lambda,\theta\lambda}$ is a solution from $(\tilde{P}_{\lambda,\theta\lambda})$.

Finally, the proof of Proposition follows by taking $\epsilon = \min\{\epsilon_1, \epsilon_2\} > 0$, that is, for each $\lambda \in (\lambda_*(\theta), \lambda_*(\theta) + \epsilon)$ the problem $(\tilde{P}_{\lambda,\theta\lambda})$ admits at least two solutions $U_\lambda \in \mathcal{N}^+_{\lambda,\theta\lambda}$ and $W_\lambda \in \mathcal{N}^-_{\lambda,\theta\lambda}$. This ends the proof. ■

Now let us prove the Theorem 0.0.3.

Theorem 0.0.3 Suppose that $0 < \gamma < 1 < \alpha, \beta$; $2 < \alpha + \beta < 2^*$; $0 < a, c$ in $\mathbb{R}^N$, $(A1)-(A2)$, $(V)_0-(V)_1$ and $(A3)$ if $b > 0$ in $\mathbb{R}^N$ hold. Then there exist two continuous simple arc $\Gamma_0 = \{(\hat{\lambda}(\theta), \hat{\mu}(\theta)) : \theta > 0\}, \tilde{\Gamma} = \{(\lambda_*(\theta), \mu_*(\theta)) : \theta > 0\} \subset \mathbb{R}_0^+ \times \mathbb{R}_0^+$, with $\Gamma_0(\theta) < \tilde{\Gamma}(\theta)$ for all $\theta > 0$; $\hat{\lambda}(\theta), \lambda_*(\theta)$ non-increasing; $\hat{\mu}(\theta), \mu_*(\theta)$ non-decreasing and $\hat{\mu}(\theta) = \theta\hat{\lambda}(\theta), \mu_*(\theta) = \theta\lambda_*(\theta)$ satisfying the property: for each $\theta > 0$ there exists an $\epsilon = \epsilon(\theta) > 0$ such that the problem $(\tilde{P}_{\lambda,\mu})$ has at least two positive solutions $W_\lambda, U_\lambda \in E$ for each $(\lambda, \mu) \in](0,0), \tilde{\Gamma}(\theta) + (\epsilon, \theta\epsilon)[$ given. Besides this, writing $(\lambda, \mu) = (\lambda, \theta\lambda)$ we have:

a) $\frac{d^2\Phi_{\lambda,\theta\lambda}}{dt^2}(tU_\lambda)\Big|_{t=1} > 0$ and $\frac{d^2\Phi_{\lambda,\theta\lambda}}{dt^2}(tW_\lambda)\Big|_{t=1} < 0$ for all $(\lambda, \mu) \in](0,0), \tilde{\Gamma}(\theta) + (\epsilon, \theta\epsilon)[$,

b) there exists a constant $c > 0$ such that $||W_\lambda|| \geq c$ for all $(\lambda, \mu) \in](0,0), \tilde{\Gamma}(\theta) + (\epsilon, \theta\epsilon)[$,

c) U_λ is a ground state solution for all $(\lambda, \mu) \in](0,0), \tilde{\Gamma}(\theta)]$, $\Phi_{\lambda,\theta\lambda}(U_\lambda) < 0$ for all $(\lambda, \mu) \in](0,0), \tilde{\Gamma}(\theta) + (\epsilon, \theta\epsilon)[$ and $\lim_{\lambda\to 0} ||U_\lambda|| = 0$,

d) the applications $\lambda \longmapsto \Phi_{\lambda,\theta\lambda}(U_\lambda)$ and $\lambda \longmapsto \Phi_{\lambda,\theta\lambda}(W_\lambda)$ are decreasing for $0 < \lambda < \lambda_*(\theta) + \epsilon$ and are left-continuous ones for $0 < \lambda < \lambda_*(\theta)$,

e) $\Phi_{\lambda,\theta\lambda}(W_\lambda) > 0$ for $(\lambda, \mu) \in](0,0), \Gamma_0(\theta)[$, $\Phi_{\Gamma_0(\theta)}(W_{\hat{\lambda}(\theta)}) = 0$ and $\Phi_{\lambda,\theta\lambda}(W_\lambda) < 0$ for $(\lambda, \mu) \in]\Gamma_0(\theta), \tilde{\Gamma}(\theta) + (\epsilon, \theta\epsilon)[$.

Proof For each $(\lambda, \mu) > (0,0)$ we can write $(\lambda, \mu) = (\lambda, \theta\lambda)$, where $\theta = \frac{\mu}{\lambda}$. Now, after introducing the family of modify problems $(\tilde{P}_{\lambda,\theta\lambda})$, with $\lambda > 0$, and considering the $\epsilon = \epsilon(\theta) > 0$ given in Proposition 2.5.6, the curves $\Gamma(\theta), \Gamma_0(\theta)$ given in Lemmas 2.2.4,2.2.7, the results obtained in the Sections 3.2, 3.3, 3.4 and in this current section, the proof of Theorem follows in a similar way as done in the proof of Theorem 0.0.1 of Chapter 1. ■

2.6 The extremal region for the existence of positive solutions

In this section, we will prove the supersolution Theorem 0.0.4 and Theorem 0.0.5. Let us start remembering the definition of supersolution.

Definition 0.0.1 Let $(\lambda,\mu) > (0,0)$. A function $\overline{U} = (\overline{u},\overline{v}) \in E$ is said to be a supersolution of $(\tilde{P}_{\lambda,\mu})$ if $\overline{u},\overline{v} > 0$ a.e. in $\mathbb{R}^N$ and

$$\begin{aligned}
&\int_{\mathbb{R}^N}[\nabla\overline{u}\nabla\varphi + V(x)\overline{u}\varphi]dx + \int_{\mathbb{R}^N}[\nabla\overline{v}\nabla\psi + V(x)\overline{v}\psi]dx\\
&\geq \lambda\int_{\mathbb{R}^N} a(x)\overline{u}^{-\gamma}\varphi dx + \mu\int_{\mathbb{R}^N} c(x)\overline{v}^{-\gamma}\psi dx\\
&+\frac{\alpha}{\alpha+\beta}\int_{\mathbb{R}^N} b(x)\overline{u}^{\alpha-1}\overline{v}^{\beta}\varphi dx + \frac{\beta}{\alpha+\beta}\int_{\mathbb{R}^N} b(x)\overline{u}^{\alpha}\overline{v}^{\beta-1}\psi dx
\end{aligned}$$

holds for all $\Psi = (\varphi,\psi) \in E_+$.

Now, for each $n \in \mathbb{N}$ let us consider the truncated problem

$$\begin{cases} -\Delta u + V(x)u = \lambda a(x)g_n(u) + \frac{\alpha}{\alpha+\beta}b(x)u^{\alpha-1}v^{\beta} \; in \; \mathbb{R}^N,\\ -\Delta v + V(x)v = \mu c(x)g_n(v) + \frac{\beta}{\alpha+\beta}b(x)u^{\alpha}v^{\beta-1} \; in \; \mathbb{R}^N,\\ (u,v) \in E, \end{cases} \tag{$\tilde{P}^n_{\lambda,\mu}$}$$

where

$$g_n(t) = \begin{cases} (t+\frac{1}{n})^{-\gamma} & \text{if} \quad t \geq 0\\ n^{\gamma} & \text{if} \quad t < 0, \end{cases}$$

is a continuous function. The energy functional associated to the problem $(\tilde{P}^n_{\lambda,\mu})$ is the functional $\Phi_{\lambda,\mu,n} \in C^1(E,\mathbb{R})$ defined by

$$\Phi_{\lambda,\mu,n}(U) = \frac{J(U)}{2} - K_{\lambda,\mu,n}(U) - \frac{L(U)}{\alpha+\beta},$$

where

$$K_{\lambda,\mu,n}(U) = \lambda\int_{\mathbb{R}^N} a(x)G_n(u)dx + \mu\int_{\mathbb{R}^N} c(x)G_n(v)dx.$$

and

$$G_n(t) = \begin{cases} \frac{1}{1-\gamma}(t+\frac{1}{n})^{1-\gamma} - \frac{1}{1-\gamma}(\frac{1}{n})^{1-\gamma} & \text{if} \quad t \geq 0\\ n^{\gamma}t & \text{if} \quad t < 0, \end{cases}$$

Note that if $U = (u,v) \gneqq (0,0)$ is a solution of $(\tilde{P}^n_{\lambda,\mu})$, then it satisfies,

$$\begin{cases} -\Delta u + V(x)u = \lambda a(x)(u+\frac{1}{n})^{-\gamma} + \frac{\alpha}{\alpha+\beta}b(x)u^{\alpha-1}v^{\beta} \; in \; \mathbb{R}^N,\\ -\Delta v + V(x)v = \mu c(x)(v+\frac{1}{n})^{-\gamma} + \frac{\beta}{\alpha+\beta}b(x)u^{\alpha}v^{\beta-1} \; in \; \mathbb{R}^N,\\ u,v \in X. \end{cases}$$

With these considerations we are already in position to prove our supersolution theorem.

Theorem 0.0.4 Suppose that $0 < \gamma < 1 < \alpha,\beta$; $2 < \alpha+\beta < 2^*$; $0 < a,c$ in

$\mathbb{R}^N$, $(A1)-(A2)$ and $(V)_0-(V)_1$ hold. Assume that the problem $(\tilde{P}_{\overline{\lambda},\overline{\mu}})$ admits a supersolution for some $(\overline{\lambda},\overline{\mu}) > (0,0)$. Then the problem $(\tilde{P}_{\overline{\lambda},\overline{\mu}})$ has at least one solution $U_{\overline{\lambda},\overline{\mu}} = (u_{\overline{\lambda}}, v_{\overline{\mu}})$ with $\Phi_{\overline{\lambda},\overline{\mu}}(U_{\overline{\lambda},\overline{\mu}}) < 0$. In particular, we have that the problem $(\tilde{P}_{\lambda,\mu})$ has at least one solution $U_{\lambda,\mu}$ satisfying $\Phi_{\lambda,\mu}(U_{\lambda,\mu}) < 0$ for all $(0,0) \lneqq (\lambda,\mu) \leq (\overline{\lambda},\overline{\mu})$.

Proof Let show that the problem $(\tilde{P}_{\lambda,\mu})$ has at least one solution $U_{\lambda,\mu}$ for all $(0,0) \lneqq (\lambda,\mu) \leq (\overline{\lambda},\overline{\mu})$ with $\Phi_{\lambda,\mu}(U_{\lambda,\mu}) < 0$ and thus, by taking $(\lambda,\mu) = (\overline{\lambda},\overline{\mu})$ in the first statement of the theorem, we have the claimed.

By assumption there is a supersolution $\overline{U} = (\overline{u},\overline{v})$ of the problem $(\tilde{P}_{\overline{\lambda},\overline{\mu}})$ and it satisfies

$$\begin{aligned} &\int_{\mathbb{R}^N} [\nabla\overline{u}\nabla\varphi + V(x)\overline{u}\varphi]dx + \int_{\mathbb{R}^N} [\nabla\overline{v}\nabla\psi + V(x)\overline{v}\psi]dx \\ &\geq \lambda \int_{\mathbb{R}^N} a(x)(\overline{u}+\frac{1}{n})^{-\gamma}\varphi dx + \mu \int_{\mathbb{R}^N} c(x)(\overline{v}+\frac{1}{n})^{-\gamma}\psi dx \\ &+\frac{\alpha}{\alpha+\beta}\int_{\mathbb{R}^N} b(x)\overline{u}^{\alpha-1}\overline{v}^{\beta}\varphi dx + \frac{\beta}{\alpha+\beta}\int_{\mathbb{R}^N} b(x)\overline{u}^{\alpha}\overline{v}^{\beta-1}\psi dx, \end{aligned}$$

for every $(\varphi,\psi) = \Psi \in E_+$, that is,

$$\langle \Phi'_{\lambda,\mu,n}(\overline{U}), \Psi\rangle \geq 0, \tag{2.55}$$

for every $(\varphi,\psi) = \Psi \in E_+$. For simplicity let us denote by $\Phi_{\lambda,\mu,n} = \Phi_n$. Now note that

$$\langle \Phi'_n((0,0)), \Psi\rangle \leq 0, \tag{2.56}$$

for every $(\varphi,\psi) = \Psi \in E_+$.

Now the proof will be done in some steps. The first one is as follows.

Step 1. The problem $(\tilde{P}^n_{\lambda,\mu})$ has a solution U_n satisfying $(0,0) \lneqq U_n = (u_n,v_n) \leq \overline{U} = (\overline{u},\overline{v})$ a.e. $\mathbb{R}^N$.

The solution will be obtained by minimizing the functional Φ_n over the set

$$M := \left\{U \in E : (0,0) \leq U = (u,v) \leq \overline{U} = (\overline{u},\overline{v})\right\}.$$

We first observe that M is convex and closed with respect to the E-topology. Furthermore, using the inequaility $G_n(t) \leq \frac{t^{1-\gamma}}{1-\gamma}$ if $t \geq 0$, for all $U \in M$, we have

$$\Phi_n(U) \geq \frac{J(U)}{2} - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N} a(x)|\overline{u}|^{1-\gamma}dx - \frac{\mu}{1-\gamma}\int_{\mathbb{R}^N} c(x)|\overline{v}|^{1-\gamma}dx - \int_{\mathbb{R}^N} |b(x)||\overline{u}|^{\alpha}|\overline{v}|^{\beta}dx,$$

which implies that Φ_n is coercive on M. To apply Theorem 1.2 of Struwe [59] we need to show that Φ_n is weakly lower semicontinuous on M. To this aim, let $\{U_k\} \subset M$ be an arbitary sequence that converges weakly to U in M. Then, $U_k \to U$ almost everywhere in $\mathbb{R}^N$ as $k \to \infty$. Due $(0,0) \leq U_k = (u_k, v_k) \leq \overline{U} = (\overline{u}, \overline{v})$ in $\mathbb{R}^N$ and Lebesgue's dominated convergence Theorem, we have

$$\Phi_n(U) \leq \liminf_{k\to\infty} \Phi_n(U_k),$$

which implies that Φ_n is weakly lower semicontinuous on M. So, by Theorem 1.2 of [59], there exists $U_n = (u_n, v_n) \in M$ such that

$$\Phi_n(U_n) = \inf_{U \in M} \Phi_n(U).$$

Now let us prove that U_n is a solution of $(\tilde{P}^n_{\lambda,\mu})$. Let $(\varphi, \psi) = \Psi \in E$, $\epsilon > 0$, and consider

$$w^\epsilon := (u_n + \epsilon\varphi - \overline{u})^+ \,, \; w_\epsilon := (u_n + \epsilon\varphi)^- \,,$$

$$z^\epsilon := (v_n + \epsilon\psi - \overline{v})^+ \,, \; z_\epsilon := (v_n + \epsilon\psi)^- \,.$$

Set

$$\eta_\epsilon := u_n + \epsilon\varphi - w^\epsilon + w_\epsilon \text{ and } \nu_\epsilon := v_n + \epsilon\psi - z^\epsilon + z_\epsilon.$$

Then

$$U_\epsilon := (\eta_\epsilon, \nu_\epsilon) = U_n + \epsilon\Psi - (w^\epsilon, z^\epsilon) + (w_\epsilon, z_\epsilon) \in M,$$

which implies that $U_n + t\,(U_\epsilon - U_n) \in M$, for all $0 < t < 1$. Since U_n minimizes Φ_n in M, this yields

$$0 \leq \langle \Phi_n'(U_n, (U_\epsilon - U_n)\rangle = \epsilon\langle \Phi_n'(U_n), (\varphi,\psi)\rangle - \langle \Phi_n'(U_n), (w^\epsilon, z^\epsilon)\rangle + \langle \Phi_n'(U_n), (w_\epsilon, z_\epsilon)\rangle,$$

so that

$$\langle \Phi_n'(U_n), (\varphi,\psi)\rangle \geq \frac{1}{\epsilon}\left[\langle \Phi_n'(U_n), (w^\epsilon, z^\epsilon)\rangle - \langle \Phi_n'(U_n), (w_\epsilon, z_\epsilon)\rangle\right]. \tag{2.57}$$

Now, for convenience of the notations, set

$$H_1(x,s,t) = \lambda a(x) g_n(s) + \frac{\alpha}{\alpha+\beta} b(x) s^{\alpha-1} t^\beta$$

and

$$H_2(x,s,t) = \mu c(x) g_n(t) + \frac{\beta}{\alpha+\beta} b(x) s^{\alpha} t^{\beta-1},$$

that is, by using (2.55), we have

$$
\begin{aligned}
\langle \Phi_n'(U_n), (w^\epsilon, z^\epsilon)\rangle =& \langle \Phi_n'(\overline{U}), (w^\epsilon, z^\epsilon)\rangle + \langle \Phi_n'(U_n) - \Phi_n'(\overline{U}), (w^\epsilon, z^\epsilon)\rangle \\
\geq& \langle \Phi_n'(U_n) - \Phi_n'(\overline{U}), (w^\epsilon, z^\epsilon)\rangle \\
=& \int_{\Omega_\epsilon} \nabla(u_n - \overline{u})\nabla(u_n + \epsilon\varphi - \overline{u}) + V(u_n - \overline{u})(u_n + \epsilon\varphi - \overline{u})dx \\
-& \int_{\Omega_\epsilon} [H_1(x, u_n, v_n) - H_1(x, \overline{u}, \overline{v})]\,\varphi dx \\
+& \int_{\Omega^\epsilon} \nabla(v_n - \overline{v})\nabla(v_n + \epsilon\psi - \overline{v}) + V(v_n - \overline{v})(v_n + \epsilon\psi - \overline{v})dx \\
-& \int_{\Omega^\epsilon} [H_2(x, u_n, v_n) - H_2(x, \overline{u}, \overline{v})]\,\psi dx \\
\geq& \epsilon \int_{\Omega_\epsilon} \nabla(u_n - \overline{u})\nabla\varphi + V(u_n - \overline{u})\varphi dx \\
-& \epsilon \int_{\Omega_\epsilon} |H_1(x, u_n, v_n) - H_1(x, \overline{u}, \overline{v})||\varphi| dx \\
& \epsilon \int_{\Omega^\epsilon} \nabla(v_n - \overline{v})\nabla\psi + V(v_n - \overline{v})\psi dx \\
-& \epsilon \int_{\Omega^\epsilon} |H_2(x, u_n, v_n) - H_2(x, \overline{u}, \overline{v})||\psi| dx,
\end{aligned}
$$

where

$$
\Omega_\epsilon = \left\{x \in \mathbb{R}^N : u_n + \epsilon\varphi \geq \overline{u} > u_n\right\} \text{ and } \Omega^\epsilon = \left\{x \in \mathbb{R}^N : v_n + \epsilon\psi \geq \overline{v} > v_n\right\}.
$$

Note that $\mathcal{L}(\Omega_\epsilon) \to 0$ and $\mathcal{L}(\Omega^\epsilon) \to 0$ as $\epsilon \to 0$. Hence by absolute continuity of the Lebesgue integral, we obtain that

$$
\frac{\langle \Phi_n'(U_n), (w^\epsilon, z^\epsilon)\rangle}{\epsilon} \geq o(\epsilon) \text{ where } o(\epsilon) \to 0, \text{ as } \epsilon \to \infty. \tag{2.58}
$$

Now, using that $(0, 0)$ satisfies (2.56) and following similar arguments as done in the proof of (2.58), we obtain

$$
\frac{\langle \Phi_n'(U_n), (w_\epsilon, z_\epsilon)\rangle}{\epsilon} \leq o(\epsilon) \text{ where } o(\epsilon) \to 0, \text{ as } \epsilon \to \infty, \tag{2.59}
$$

which implies, together with (2.57), (2.58) and (2.59), that

$$
\langle \Phi_n'(U_n), \Psi\rangle \geq 0
$$

for all $\Psi = (\varphi, \psi) \in E$. Reversing the sign of Ψ we see that $\langle \Phi_n'(U_n), \Psi\rangle = 0$ for all $\Psi \in E$, that is, U_n is a solution of $(\tilde{P}^n_{\lambda,\mu})$.

Now let us go to the second step.

Step 2. The sequence $\{U_n\} = \{(u_n, v_n)\} \subset E$ is bounded in E.

Let us prove that the sequence $\{u_n\}$ is bounded in X. To prove the boundedness of $\{u_n\}$, note that $0 \leq (u_n, v_n) \leq (\overline{u}, \overline{v})$ for every $n \in \mathbb{N}$. Hence,

$$\begin{aligned}
\|u_n\|^2 =&\lambda \int_{\mathbb{R}^N} a(x)(u_n + \frac{1}{n})^{-\gamma} u_n dx + \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} b(x) u_n^{\alpha} v_n^{\beta} dx \\
\leq&\lambda \int_{\{x\in\mathbb{R}^N : u_n(x)>0\}} a(x) u_n^{1-\gamma} dx + \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} |b(x)| u_n^{\alpha} v_n^{\beta} dx \\
\leq&\lambda \int_{\{x\in\mathbb{R}^N : u_n(x)>0\}} a(x) \overline{u}^{1-\gamma} dx + \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} |b(x)| \overline{u}^{\alpha} \overline{v}^{\beta} dx \\
\leq&\lambda \int_{\mathbb{R}^N} a(x) \overline{u}^{1-\gamma} dx + \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} |b(x)| \overline{u}^{\alpha} \overline{v}^{\beta} dx,
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in X. The proof of boundedness of $\{v_n\}$ follows similarly. Therefore $\{U_n\} = \{(u_n, v_n)\} \subset E$ is bounded in E.

Using the step 2 let us go to the last one.

Step 3. Existence of solution for $(\tilde{P}_{\lambda,\mu})$.

By step 2 the sequence $\{U_n\} = \{(u_n, v_n)\} \subset E$ is bounded in E and therefore there is a function $U_{\lambda,\mu} = (u_\lambda, v_\mu) \geq (0,0)$ a.e. $\mathbb{R}^N$ such that

$$U_n \rightharpoonup U_{\lambda,\mu} \text{ in } E,\ U_n \to L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N), s \in [1, 2^*),\ U_n \to U_{\lambda,\mu} \text{ a.e. } \mathbb{R}^N.$$

Let us show that $U_{\lambda,\mu} = (u_\lambda, v_\mu) > (0,0)$ a.e. $\mathbb{R}^N$. Using that U_n is solution of $(\tilde{P}^n_{\lambda,\mu})$ and Fatou's Lemma, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^N} [\nabla u_\lambda \nabla \varphi + V(x) u_\lambda \varphi] dx - \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} b(x) u_\lambda^{\alpha-1} v_\mu^{\beta} \varphi dx \\
&\geq \lambda \liminf \int_{\mathbb{R}^N} a(x)(u_n + \frac{1}{n})^{-\gamma} \varphi dx \geq \lambda \int_{\mathbb{R}^N} a(x) H(x) \varphi dx
\end{aligned}$$

for all $\varphi \geq 0$, where

$$H(x) = \begin{cases} u_\lambda^{-\gamma}(x), & \text{if } u_\lambda(x) \neq 0 \\ \infty, & \text{if } u_\lambda(x) = 0, \end{cases}$$

So, by taking $\varphi > 0$, $\varphi \in X_+$ above, we obtain that $H(x) = u_\lambda^{-\gamma}(x)$ for all $x \in \mathbb{R}^N$, that is, $u_\lambda > 0$ in $\mathbb{R}^N$. This implies that $0 < \int_{\mathbb{R}^N} a u_\lambda^{-\gamma} \varphi dx < \infty$ for all $\varphi \in X_+$. As a consequence, we have

$$\int_{\mathbb{R}^N} [\nabla u_\lambda \nabla \varphi + V(x) u_\lambda \varphi] dx - \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} b(x) u_\lambda^{\alpha-1} v_\mu^{\beta} \varphi dx \geq \lambda \int_{\mathbb{R}^N} a(x) u_\lambda^{-\gamma}(x) \varphi dx \quad (2.60)$$

for all $\varphi \in X_+$. In a similar way we can prove that $v_\mu > 0$ in $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} [\nabla v_\mu \nabla \psi + V(x) v_\mu \psi] dx - \frac{\beta}{\alpha+\beta} \int_{\mathbb{R}^N} b(x) u_\lambda^\alpha v_\mu^{\beta-1} \psi dx \geq \mu \int_{\mathbb{R}^N} c(x) v_\mu^{-\gamma}(x) \psi dx \quad (2.61)$$

for all $\psi \in X_+$.

Now, our next goal is to prove that the sequence $\{U_n\} = \{(u_n, v_n)\} \subset E$ converges strongly to $U_{\lambda,\mu}$ in E and that $U_{\lambda,\mu}$ is a solution of $(\tilde{P}_{\lambda,\mu})$. To this aim, note that $(0,0) \leq (u_n, v_n) \leq (\overline{u}, \overline{v})$, and

$$a(x) u_n^{1-\gamma} \leq a(x) \overline{u}^{1-\gamma} \text{ a.e. } \mathbb{R}^N \text{ and } |b(x)||u_n|^\alpha |v_n|^\beta \leq |b(x)||\overline{u}|^\alpha |\overline{v}|^\beta \text{ a.e. } \mathbb{R}^N,$$

where $a(x)\overline{u}^{1-\gamma} \in L^1(\mathbb{R}^n)$ and $|b(x)||\overline{u}|^\alpha |\overline{v}|^\beta \in L^1(\mathbb{R}^n)$. Therefore by Lebesgue's dominated convergence Theorem and Fatou's Lemma, we have

$$\begin{aligned}
&\limsup \int_{\mathbb{R}^N} [\nabla u_n \nabla (u_n - u_\lambda) + V(x) u_n (u_n - u_\lambda)] dx \\
=& \limsup \left[\lambda \int_{\mathbb{R}^N} a(x) \left(u_n(x) + \frac{1}{n} \right)^{-\gamma} (u_n - u_\lambda) dx + \int_{\mathbb{R}^N} b(x) u_n^{\alpha-1} v_n^\beta (u_n - u_\lambda) dx \right] \\
=& \limsup \left[\lambda \int_{\mathbb{R}^N} a(x) \left(u_n(x) + \frac{1}{n} \right)^{-\gamma} (u_n - u_\lambda) dx \right] \\
=& \limsup \left[\lambda \int_{\mathbb{R}^N} a(x) \left(u_n(x) + \frac{1}{n} \right)^{-\gamma} u_n dx - \lambda \int_{\mathbb{R}^N} a(x) \left(u_n(x) + \frac{1}{n} \right)^{-\gamma} u_\lambda dx \right] \\
=& \limsup \left[\lambda \int_{\{x \in \mathbb{R}^N : u_n(x) > 0\}} a(x) \left(u_n(x) + \frac{1}{n} \right)^{-\gamma} u_n dx - \lambda \int_{\mathbb{R}^N} a(x) \left(u_n(x) + \frac{1}{n} \right)^{-\gamma} u_\lambda dx \right] \\
\leq& \limsup \left[\lambda \int_{\{x \in \mathbb{R}^N : u_n(x) > 0\}} a(x) u_n^{1-\gamma}(x) dx - \lambda \int_{\mathbb{R}^N} a(x) \left(u_n(x) + \frac{1}{n} \right)^{-\gamma} u_\lambda dx \right] \\
\leq& \limsup \left[\lambda \int_{\mathbb{R}^N} a(x) u_n^{1-\gamma}(x) dx - \lambda \int_{\mathbb{R}^N} a(x) \left(u_n(x) + \frac{1}{n} \right)^{-\gamma} u_\lambda dx \right] \\
=& \lambda \int_{\mathbb{R}^N} a(x) u_\lambda^{1-\gamma}(x) dx - \liminf \lambda \int_{\mathbb{R}^N} a(x) \left(u_n(x) + \frac{1}{n} \right)^{-\gamma} u_\lambda dx \\
\leq& \lambda \int_{\mathbb{R}^N} a(x) u_\lambda^{1-\gamma}(x) dx - \lambda \int_{\mathbb{R}^N} a(x) u_\lambda^{1-\gamma}(x) dx = 0.
\end{aligned}$$

This information together with $u_n \rightharpoonup u_\lambda$ in X imply that

$$\|u_n - u_\lambda\|^2 = (u_n, u_n - u_\lambda) - (u_\lambda, u_n - u_\lambda) \to 0$$

as $n \to \infty$, that is $u_n \to u_\lambda$ in X. In a similar way we can prove that $v_n \to v_\mu$ in X. Since $(0,0) \leq (u_n, v_n) \leq (\overline{u}, \overline{v})$, we have that

$$a(x)(u_n + \frac{1}{n})^{-\gamma} u_n \leq a(x) u_n^{1-\gamma} \leq a(x) \overline{u}^{1-\gamma} \text{ a.e. } \mathbb{R}^N,$$

and

$$|b(x)||u_n|^{\alpha}|v_n|^{\beta} \leq |b(x)||\overline{u}|^{\alpha}|\overline{v}|^{\beta} \text{ a.e. } \mathbb{R}^n$$

hold. So, the convergence $u_n \to u_\lambda$ in X and Lebesgue's dominated convergence Theorem applied to

$$\|u_n\|^2 = \lambda \int_{\mathbb{R}^N} a(x)(u_n + \frac{1}{n})^{-\gamma} u_n dx + \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} b(x) u_n^{\alpha} v_n^{\beta} dx,$$

lead us to obtain

$$\|u_\lambda\|^2 = \lambda \int_{\mathbb{R}^N} a(x) u_\lambda^{1-\gamma} dx + \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} b(x) u_\lambda^{\alpha} v_\mu^{\beta} dx.$$

Similarly we may show that

$$\|v_\mu\|^2 = \mu \int_{\mathbb{R}^N} c(x) v_\mu^{1-\gamma} dx + \frac{\beta}{\alpha+\beta} \int_{\mathbb{R}^N} b(x) u_\lambda^{\alpha} v_\mu^{\beta} dx$$

and therefore

$$\|U_{\lambda,\mu}\|^2 - K_{\lambda,\mu}(U_{\lambda,\mu}) - L(U_{\lambda,\mu}) = 0 \tag{2.62}$$

As a consequence of (2.60), (2.61) and (2.62), we conclude that function $U_{\lambda,\mu}$ satisfies the conditions (2.13)-(2.14) of Proposition 2.2.3 and so Proposition 2.2.3 implies that $U_{\lambda,\mu}$ is a solution of $(\tilde{P}_{\lambda,\mu})$.

To prove that $\Phi_{\lambda,\mu}(U_{\lambda,\mu}) < 0$, notice that $\Phi_n(U_n) \leq \Phi_n(U)$ for every $n \in \mathbb{N}$ and $U \in M$. Thus, this inequality, the convergence $U_n \to U_{\lambda,\mu}$ in E, and Lebesgue's dominated convergence, lead to $\Phi_{\lambda,\mu}(U_{\lambda,\mu}) \leq \Phi_{\lambda,\mu}(U)$ for all $U \in M$, that is,

$$\Phi_{\lambda,\mu}(U_{\lambda,\mu}) = \inf_{U \in M} \Phi_{\lambda,\mu}(U). \tag{2.63}$$

Since $t\overline{U} \in M$ and $\Phi_{\lambda,\mu}(t\overline{U}) < 0$ for $0 < t$ small enough, we have from (2.63) that $\Phi_{\lambda,\mu}(U_{\lambda,\mu}) \leq \Phi_{\lambda,\mu}(t\overline{U}) < 0$ for $0 < t$ small enough. The proof of this Theorem is complete. ■

After proving Theorem 0.0.4, we are going to study the structure of the set

$$\Upsilon = \left\{ (\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+ : (\tilde{P}_{\lambda,\mu}) \textit{ admits solution} \right\}.$$

First, using the classic Nehari manifold method for functional of class C^1, it is well known that problem $(\tilde{P}_{0,0})$ has a positive solution, therefore $(0,0) \in \Upsilon$. Also, it is well known that for every $\lambda > 0$ and $\mu > 0$ the purely singular problems $(\tilde{P}_{\lambda,0})$ and $(\tilde{P}_{0,\mu})$ have a positive solution, and thus $(\lambda, 0), (0, \mu) \in \Upsilon$ for every $\lambda > 0$ and $\mu > 0$.

Again, for each $\theta > 0$ let us consider the system $(\tilde{P}_{\lambda,\theta\lambda})$ and define the sets

$$\Upsilon_\theta = \Big\{\lambda > 0 : (\tilde{P}_{\lambda,\theta\lambda}) \text{ admits solution}\Big\} \subset \Upsilon,$$
$$\Upsilon_0 = \Big\{\lambda > 0 : (\tilde{P}_{\lambda,0}) \text{ admits solution}\Big\} \subset \Upsilon,$$
$$\Upsilon_\infty = \Big\{\mu > 0 : (\tilde{P}_{0,\mu}) \text{ admits solution}\Big\} \subset \Upsilon$$

and the extended function

$$\Gamma^*(\theta) = (\lambda^*(\theta), \mu^*(\theta)), \text{ where } \mu^*(\theta) = \theta\lambda^*(\theta) \text{ and } \lambda^*(\theta) = \sup(\Upsilon_\theta) \leq \infty. \tag{2.64}$$

Since, we already know from Proposition 2.3.1 and Proposition 2.5.6 that $0 < \lambda_*(\theta) < \lambda^*(\theta) \leq \infty$, we have $\tilde{\Gamma}(\theta) < \Gamma^*(\theta)$ for every $\theta > 0$. Moreover, for each $0 < \lambda < \lambda^*(\theta)$ it follows from the definition of $\lambda^*(\theta)$ and Theorem 0.0.4 that problem $(\tilde{P}_{\lambda,\theta\lambda})$ has a solution, that is, $(0, \lambda^*(\theta)) \subset \Upsilon_\theta$. Notice that $\Gamma^*(\theta) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ when $\lambda^*(\theta) < \infty$.

Moreover, we have the following lemma.

Lemma 2.6.1 *Assume $\lambda^*(\theta) < \infty$. Then the problem $(\tilde{P}_{\Gamma^*(\theta)})$ has at least one solution $U_{\Gamma^*(\theta)}$ satisfying $\Phi_{\Gamma^*(\theta)}(U_{\Gamma^*(\theta)}) \leq 0$.*

Proof Let $\lambda_n \in \Upsilon_\theta \subset (0, \lambda^*(\theta)]$ be an increasing sequence such that $\lambda_n \to \lambda^*(\theta)$, and $U_n = (u_n, v_n) := U_{\lambda_n,\theta\lambda_n}$ be the solution of $(\tilde{P}_{\lambda_n,\theta\lambda_n})$ obtained in the Theorem 0.0.4. Then,

$$\Phi_{\lambda_n,\theta\lambda_n}(U_n) = \frac{J(U_n)}{2} - \frac{\lambda_n K_{1,\theta}(U_n)}{1-\gamma} - \frac{L(U_n)}{\alpha+\beta} < 0$$

and

$$J(U_n) - \lambda_n K_{1,\theta}(U_n) - L(U_n) = 0,$$

which implies together with Hölder inequality and Sobolev embedding that there exists a constant $c > 0$ such that

$$\|U_n\| = J(U_n) \leq C,$$

that is, the sequence $\{U_n\}$ is bounded in E.

Thus, we can assume that there is a subsequence, still denoted by $\{U_n\}$, and a function $U_{\Gamma^*(\theta)} = (u_{\lambda^*(\theta)}, v_{\theta\lambda^*(\theta)}) \geq (0,0)$ a.e. $\mathbb{R}^N$ such that $U_n \rightharpoonup U_{\Gamma^*(\theta)}$ in E, $U_n \to U_{\Gamma^*(\theta)}$ in $L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N)$, $s \in [0, 2^*)$ and pointwise a.e. in $\mathbb{R}^N$. By letting $n \to \infty$ in the equality

$$\langle J'(U_n), \Psi\rangle - \langle L'(U_n), \Psi\rangle = \lambda_n \langle K'_{1,\theta}(U_n), \Psi\rangle,$$

for each $\Psi = (\varphi, \psi) \in E_+$, and following as in the proof of Theorem 0.0.4, we obtain that $U_{\Gamma^*(\theta)} = (u_{\lambda^*(\theta)}, v_{\theta\lambda^*(\theta)}) > (0,0)$ a.e. in $\mathbb{R}^N$ and

$$\langle J'(U_{\Gamma^*(\theta)}), \Psi\rangle - \lambda^*(\theta)\langle K'_{1,\theta}(U_{\Gamma^*(\theta)})\rangle - \langle L'(U_{\Gamma^*(\theta)}), \Psi\rangle \geq 0 \tag{2.65}$$

hold for every $\Psi = (\varphi, \psi) \in E_+$. Moreover, from Lebesgue's dominated convergence Theorem and Fatou's Lemma, we have

$$\begin{aligned}
&\limsup \int_{\mathbb{R}^N} [\nabla u_n \nabla(u_n - u_{\lambda^*(\theta)}) + V(x)u_n(u_n - u_{\lambda^*(\theta)})]dx \\
=&\limsup \left[\lambda_n \int_{\mathbb{R}^N} a(x)u_n(x)^{-\gamma}(u_n - u_{\lambda^*(\theta)})dx + \frac{\alpha}{\alpha+\beta}\int_{\mathbb{R}^N} b(x)u_n^{\alpha-1}v_n^{\beta}(u_n - u_{\lambda^*(\theta)})dx\right] \\
=&\limsup \left[\lambda_n \int_{\mathbb{R}^N} a(x)u_n(x)^{-\gamma}(u_n - u_{\lambda^*(\theta)})dx\right] \\
=&\limsup \left[\lambda_n \int_{\mathbb{R}^N} a(x)u_n^{1-\gamma}(x)dx - \lambda_n \int_{\mathbb{R}^N} a(x)u_n(x)^{-\gamma}u_{\lambda^*(\theta)}dx\right] \\
\leq&\lambda^*(\theta)\int_{\mathbb{R}^N} a(x)u_{\lambda^*(\theta)}^{1-\gamma}(x)dx - \liminf \lambda_n \int_{\mathbb{R}^N} a(x)u_n(x)^{-\gamma}u_{\lambda^*(\theta)}dx \\
\leq&\lambda^*(\theta)\int_{\mathbb{R}^N} a(x)u_{\lambda^*(\theta)}^{1-\gamma}(x)dx - \lambda^*(\theta)\int_{\mathbb{R}^N} a(x)u_{\lambda^*(\theta)}^{1-\gamma}(x)dx = 0,
\end{aligned}$$

which implies that $u_n \to u_{\lambda^*(\theta)}$ in X. A similar argument show that $v_n \to v_{\theta\lambda^*(\theta)}$ in X as well. So, we have $U_n \to U_{\Gamma^*(\theta)}$ in E and this yields

$$\|U_{\Gamma^*(\theta)}\|^2 - \lambda^*(\theta)K_{1,\theta}(U_{\Gamma^*(\theta)}) - L(U_{\Gamma^*(\theta)}) = 0. \tag{2.66}$$

Hence, we obtain from (2.65) and (2.66) that $U_{\Gamma^*(\theta)}$ satisfies the conditions (2.13)-(2.14) of Proposition 2.2.3 and therefore $U_{\Gamma^*(\theta)}$ is a solution of $(\tilde{P}_{\Gamma^*(\theta)})$. The proof of this lemma is completed. ■

As a consequence of Lemma 2.6.1 we have.

Corollary 2.6.1 *Assume that $\lambda^*(\theta) < \infty$ for all $\theta > 0$. Then the set Υ is closed.*

Proof Let $\{(\lambda_n, \mu_n)\} \subset \Upsilon$ be a sequence such that $(\lambda_n, \mu_n) \to (\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+$. We have three possibilities to consider now. If $(\lambda, \mu) = \Gamma^*(\theta)$, for some $\theta > 0$, we have by Lemma 2.6.1 that $(\lambda, \mu) = \Gamma^*(\theta) \in \Upsilon$. Now, if $\lambda > 0$ and $\mu > 0$, we can assume that $\lambda_n > 0$ and $\mu_n > 0$ for all n and rewrite $(\lambda_n, \mu_n) = (\lambda_n, \theta_n\lambda_n)$ for $\theta_n = \mu_n/\lambda_n$. So, by definition of $\lambda^*(\theta_n)$ and $(\lambda_n, \mu_n) \in \Upsilon_{\theta_n}$ we have $\lambda_n \leq \lambda^*(\theta_n)$ which implies by Theorem 0.0.4 and Lemma 2.6.1 that there exists a solution U_n of $(\tilde{P}_{\lambda_n,\mu_n})$ satisfying $\Phi_{\lambda_n,\mu_n}(U_n) < 0$. Thus, as in the Lemma 2.6.1, we can show that there exists a solution

$U_{\lambda,\mu}$ of problem $(\tilde{P}_{\lambda,\mu})$, that is, $(\lambda,\mu) \in \Upsilon$. Finally, when $\lambda = 0$ or $\mu = 0$, we have from $\Upsilon_0 = (0,\infty)$ and $\Upsilon_\infty = (0,\infty)$ that $(\lambda,\mu) \in \Upsilon$. This ends the proof. ■

Our goal now is providing conditions for $\lambda^*(\theta)$ be finite for all $\theta > 0$.

Assume that $0 < m \in L^\infty(\Omega)$ and consider the eigenvalue problem

$$\begin{cases} -\Delta u + V(x)u = \lambda m(x)u \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u \in H_0^1(\Omega), \end{cases} \tag{A_Ω}$$

So, by a classical argument and Theorem 3 in Brezis-Nirenberg [12], we have.

Lemma 2.6.2 *The first eigenvalue λ_1 of the problem (A_Ω) is positive. Moreover, its associated eigenfunction e_1 is positive, $e_1 \in C^1(\overline{\Omega}) \cap H^2(\Omega)$ and $\partial e_1/\partial\nu \leqslant 0$ on $\partial\Omega$, where $\nu \in \mathbb{R}^N$ is the unit exterior normal to $\partial\Omega$.*

We have the following lemma.

Lemma 2.6.3 *Assume that there exists a smooth bounded open set $\Omega \subset \mathbb{R}^N$ such that $b > 0$ in Ω. Then there exists $\eta^* = \eta^*(\theta) > 0$ such that the problem $(\tilde{P}_{\lambda,\theta\lambda})$ has no solution for all $\lambda > \eta^*$.*

Proof First we intend to regularize the solutions of the problem $(\tilde{P}_{\lambda,\theta\lambda})$ in Ω using interior regularity. Assume that $U_\lambda = (u_\lambda, v_\lambda) \in E_+$ is a solution for Problem $(\tilde{P}_{\lambda,\theta\lambda})$. By Brezis-Nirenberg Theorem (see [12] Theorem 3 again), we have that there exists a constant c such that $u_\lambda, v_\lambda \geq cd(x) = cd(x,\partial\Omega)$ in Ω and therefore $u_\lambda^{-\gamma}, v_\lambda^{-\gamma} \in L^\infty(K)$ for every $K \subset\subset \Omega$. Using this information and Young's inequality we have that

$$\lambda a(x)u_\lambda^{-\gamma} + \frac{\alpha}{\alpha+\beta}b(x)u_\lambda^{\alpha-1}v_\lambda^\beta - V(x)u_\lambda \in L^{\frac{2^*}{\alpha+\beta}}(K),$$

and

$$\lambda\theta c(x)v_\lambda^{-\gamma} + \frac{\alpha}{\alpha+\beta}b(x)u_\lambda^{\alpha}v_\lambda^{\beta-1} - V(x)v_\lambda \in L^{\frac{2^*}{\alpha+\beta}}(K),$$

which implies by Theorem 12.2.2 of J. Jost [43] that $u_\lambda, v_\lambda \in H^{2,\frac{2^*}{\alpha+\beta}}(K)$ and

$$-\Delta u_\lambda = \lambda a(x)u_\lambda^{-\gamma} + \frac{\alpha}{\alpha+\beta}b(x)u_\lambda^{\alpha-1}v_\lambda^\beta - V(x)u_\lambda \text{ a. e. in } \Omega,$$

$$-\Delta v_\lambda = \lambda\theta c(x)v_\lambda^{-\gamma} + \frac{\alpha}{\alpha+\beta}b(x)u_\lambda^{\alpha}v_\lambda^{\beta-1} - V(x)v_\lambda \text{ a. e. in } \Omega.$$

After a classical bootstrap argument, we obtain that $u_\lambda, v_\lambda \in H^2(K) \cap C(\overline{K})$ for every $K \subset\subset \Omega$. Without loss of generality we may assume that $u_\lambda, v_\lambda \in H^2(\Omega) \cap C(\overline{\Omega})$.

For $\lambda > 1$ by the comparison principle of Gonçalves-Carvalho-Santos (see Theorem 1.2 of [36]) we have that $u_\lambda \geq u$ and $v_\lambda \geq v$ in Ω, where $u, v \in C(\overline{\Omega}) \cap H_0^1(\Omega)$ are the solutions of

$$\begin{cases} -\Delta u + V(x)u = a(x)u^{-\gamma} \text{ in } \Omega, \\ \qquad\qquad\qquad\quad u > 0 \text{ in } \Omega, \end{cases}$$

and

$$\begin{cases} -\Delta v + V(x)v = \theta c(x)v^{-\gamma} \text{ in } \Omega, \\ \qquad\qquad\qquad\quad v > 0 \text{ in } \Omega, \end{cases}$$

respectively, and therefore, by Brezis-Nirenberg Theorem there exists a constant $c > 0$ independent of λ such that $u_\lambda, v_\lambda \geq \min\{u(x), v(x)\} \geq cd(x) = cd(x, \partial\Omega)$ in Ω for every $\lambda > 1$.

From now on let us assume that $\lambda > 1$. After regularizing the solutions, we may apply Lemma 3.5 of Figueiredo-Gossez-Ubilla [23] to conclude that

$$\int_\Omega \nabla u_\lambda \nabla e_1 + V(x)e_1 u_\lambda dx \leq \lambda_1 \int_\Omega \tilde{m}(x) e_1 u_\lambda dx, \tag{2.67}$$

and

$$\int_\Omega \nabla v_\lambda \nabla e_1 + V(x)e_1 v_\lambda dx \leq \lambda_1 \int_\Omega \tilde{m}(x) e_1 v_\lambda dx, \tag{2.68}$$

where

$$\tilde{m}(x) = \min\left\{a(x), \frac{\beta c^{\beta-1}}{\alpha+\beta} b(x) d^{\beta-1}(x), \theta c(x), \frac{\alpha c^{\alpha-1}}{\alpha+\beta} b(x) d^{\alpha-1}(x)\right\},$$

c is a constant independent of λ and λ_1 is the first eigenvalue of (A_Ω) with the weight function $\tilde{m}$ given above.

Now, let us define $g_\alpha, g_\beta : (0, \infty) \to \mathbb{R}$ by $g_\alpha(t) = \lambda t^{-\gamma-1} + t^{\alpha-1}$, $g_\beta(t) = \lambda t^{-\gamma-1} + t^{\beta-1}$ and note that

$$t_\lambda = \left(\lambda\left(\frac{\gamma+1}{\alpha-1}\right)\right)^{\frac{1}{\alpha+\gamma}} \text{ and } s_\lambda = \left(\lambda\left(\frac{\gamma+1}{\beta-1}\right)\right)^{\frac{1}{\beta+\gamma}}, \ \lambda > 0,$$

are the uniques global minimum of g_α and g_β respectively, whose minimum value is given by

$$\tilde{g}_\alpha(\lambda) := g_\alpha(t_\lambda) = \lambda^{\frac{\alpha-1}{\alpha+\gamma}} \left(\frac{\gamma+1}{\alpha-1}\right)^{\frac{-\gamma-1}{\alpha+\gamma}} \left(\frac{\alpha+\gamma}{\alpha-1}\right),$$

and

$$\tilde{g}_\beta(\lambda) := g_\beta(s_\lambda) = \lambda^{\frac{\beta-1}{\alpha+\gamma}} \left(\frac{\gamma+1}{\beta-1}\right)^{\frac{-\gamma-1}{\beta+\gamma}} \left(\frac{\beta+\gamma}{\beta-1}\right),$$

which provides the existence of a $\eta^* = \eta^*(\theta) > 0$ such that

$$\tilde{g}_\alpha(\eta^*), \tilde{g}_\beta(\eta^*) \geq \lambda_1 = \lambda_1(\theta).$$

So, it follows from the definition of η^*, (2.67), (2.68) and the fact that (u_λ, v_λ) is a solution for Problem $(\tilde{P}_{\lambda,\theta\lambda})$, that

$$\begin{aligned}\int_\Omega m(x)(\eta^* u_\lambda^{-\gamma} + u_\lambda^\alpha + \eta^* v_\lambda^{-\gamma} + v_\lambda^\beta)e_1 dx &\geq \lambda_1 \int_\Omega m(x)(u_\lambda + v_\lambda)e_1 dx \\ &\geq \int_\Omega \nabla e_1 \nabla u_\lambda + V(x)e_1 u_\lambda dx + \int_\Omega \nabla e_1 \nabla v_\lambda + V(x) e_1 v_\lambda dx \\ = \int_\Omega (\lambda a(x) u_\lambda^{-\gamma} + \frac{\alpha}{\alpha+\beta} b(x) u_\lambda^{\alpha-1} v_\lambda^\beta &+ \lambda\theta c(x) v_\lambda^{-\gamma} + \frac{\beta}{\alpha+\beta} b(x) u_\lambda^\alpha v_\lambda^{\beta-1}) e_1 dx \end{aligned} \tag{2.69}$$

holds. Since

$$a(x), b(x), \frac{\alpha}{\alpha+\beta} b(x) u_\lambda^{\alpha-1}, \theta c(x), \frac{\beta}{\alpha+\beta} b(x) v_\lambda^{\beta-1} \geq \tilde{m}(x) \text{ in } \Omega,$$

we have from (2.69)

$$\begin{aligned}\int_\Omega (\eta^* a(x) u_\lambda^{-\gamma} + \frac{\alpha}{\alpha+\beta} b(x) u_\lambda^{\alpha-1} v_\lambda^\beta + \eta^*\theta c(x) v_\lambda^{-\gamma} + \frac{\beta}{\alpha+\beta} b(x) u_\lambda^\alpha v_\lambda^{\beta-1}) e_1 dx \\ \geq \int_\Omega \tilde{m}(x)(\eta^* u_\lambda^{-\gamma} + u_\lambda^\alpha + \eta^* v_\lambda^{-\gamma} + v_\lambda^\beta) e_1 dx \\ \geq \int_\Omega (\lambda a(x) u_\lambda^{-\gamma} + \frac{\alpha}{\alpha+\beta} b(x) u_\lambda^{\alpha-1} v_\lambda^\beta + \lambda\theta c(x) v_\lambda^{-\gamma} + \frac{\beta}{\alpha+\beta} b(x) u_\lambda^\alpha v_\lambda^{\beta-1}) e_1 dx,\end{aligned}$$

which implies that $\eta^* \geq \lambda$. This ends the proof. ■

As a consequence of the Lemma 2.6.3 we have the following Corollary.

Corollary 2.6.2 *Assume that there exists a smooth bounded open set $\Omega \subset \mathbb{R}^N$ such that $b > 0$ in Ω. Then $\lambda^*(\theta) < \infty$ for all $\theta > 0$.*

Let us prove some properties of the function Γ^* in the next lemma.

Lemma 2.6.4 *Assume that there exists a smooth bounded open set $\Omega \subset \mathbb{R}^N$ such that $b > 0$ in Ω. Then,*

a) $\Gamma^* : (0, \infty) \to \mathbb{R}^2$ is a continuous function and injective,

b) $\lambda^*(\theta)$ is nonincreasing and $\mu^*(\theta)$ is nondecreasing,

c) the $\partial\Upsilon \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+) = \Gamma^* = \{\Gamma^*(\theta) = (\lambda^*(\theta), \mu^*(\theta)) : \theta > 0\}$.

Proof Firstly let us prove a). It is sufficient to prove that $\lambda^*(\theta)$ is a continuous function. If $\lambda^*(\theta)$ were discontinuous, at say a point θ, then there would exist an $\epsilon > 0$ and a sequence $\theta_n \longrightarrow \theta$ such that $|\lambda^*(\theta_n) - \lambda^*(\theta)| \geq \epsilon$. So, up to a subsequence, there would have two possibilities:

$$\lambda^*(\theta_n) < \lambda^*(\theta) \text{ or } \lambda^*(\theta_n) > \lambda^*(\theta),$$

for n sufficiently large. Assume that the first one holds. Let $\lambda_1 < \lambda_2$ such that $\lambda^*(\theta_n) < \lambda_1 < \lambda_2 < \lambda^*(\theta)$. Since $\theta\lambda_1 < \theta\lambda_2$, then

$$\theta_n\lambda^*(\theta_n) < \theta_n\lambda_1 < \theta\lambda_2 < \theta\lambda^*(\theta),$$

for n large enough. Thus, by the definition of $\Gamma^*(\theta)$ and Theorem 0.0.4 the system $(\tilde{P}_{\lambda_2,\theta\lambda_2})$ has a solution (u, v), which is a supersolution of $(\tilde{P}_{\lambda_1,\theta_n\lambda_1})$. So, Theorem 0.0.4 implies that the system $(\tilde{P}_{\lambda_1,\theta_n\lambda_1})$ admits a solution $(\tilde{u}, \tilde{v})$, which lead us to conclude that $\lambda_1 \leqslant \lambda^*(\theta_n)$, but this is a contradiction. The second case runs in a similar manner.

Let us show that Γ^* is injective. If $\Gamma^*(\theta) = \Gamma(\rho)$, then $\lambda^*(\theta) = \lambda^*(\rho)$ and $\theta\lambda^*(\theta) = \rho\lambda^*(\rho)$ that implies $\theta = \rho$. Therefore, Γ^* is injective and this completes the proof of a).

Now, let us to prove b). Suppose by contradiction that there exists $\theta_1, \theta_2 \in (0, \infty)$ with $\theta_1 < \theta_2$ and $\lambda^*(\theta_1) < \lambda^*(\theta_2)$. Then, we would have $\mu^*(\theta_1) = \theta_1\lambda^*(\theta_1) < \theta_2\lambda^*(\theta_2) = \mu^*(\theta_2)$ and from $\Gamma^*(\theta_2) \in \Upsilon$ (see Lemma 2.6.1) and Theorem 0.0.4 there would exist $(\lambda, \theta_1\lambda) \in \Upsilon_{\theta_1}$ such that $\Gamma(\theta_1) < (\lambda, \theta_1\lambda) < \Gamma(\theta_2)$. By the definition of $\lambda^*(\theta_1)$ it follows that $\lambda \leq \lambda^*(\theta_1)$ which is a contradiction, because $\lambda^*(\theta_1) < \lambda$. The proof that $\mu^*(\theta)$ is nondecreasing runs in a similar manner.

Proof of c). We first prove that $\Gamma^* \subset \partial\Upsilon \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+)$. To this aim, note that from Lemma 2.6.1 and item a) we have that $\Gamma^*(\theta) \in (\Upsilon \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+))$ for every $\theta > 0$, which implies $B_\epsilon(\Gamma^*(\theta)) \cap (\Upsilon \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+)) \neq \emptyset$ for all $\epsilon > 0$. Now, fix $\theta > 0$ and let $\epsilon > 0$ be arbitrary. Take a sequence $\{\lambda_k\}$ such that $\lambda_k \to \lambda^*(\theta)$, with $\lambda_k > \lambda^*(\theta)$ for every k. Since the problem $(\tilde{P}_{\lambda_k,\theta\lambda_k})$ has no solution (by definition of $\lambda^*(\theta)$), and $(\lambda_k, \theta\lambda_k) \in B_\epsilon(\Gamma^*(\theta))$ for sufficiently large k, we can conclude that $B_\epsilon(\Gamma^*(\theta)) \cap ((\mathbb{R}_0^+ \times \mathbb{R}_0^+)\backslash\Upsilon) \neq \emptyset$. These arguments prove that $\Gamma^* \subset \partial\Upsilon \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+)$.

To complete the proof of c), it suffices to show $(\partial\Upsilon \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+))\backslash \subset \Gamma^*$. To do this, we take $(\lambda, \mu) \in (\partial\Upsilon \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+))$ and apply Corollary 2.6.1 to obtain that $(\lambda, \mu) \in \Upsilon_\theta \subset \Upsilon$ for $\theta = \mu/\lambda$. This implies, by the definition of $\lambda^*(\theta)$, that $\lambda \leq \lambda^*(\theta)$, which leads us to infer that $(\lambda, \mu) \leq \Gamma^*(\theta)$. Now, we claim that $(\lambda, \mu) = \Gamma^*(\theta)$. Indeed, if were $(\lambda,\mu) \neq \Gamma^*(\theta) = (\lambda^*(\theta), \theta\lambda^*(\theta))$, then $(\lambda,\mu) < \Gamma^*(\theta)$, which would imply, by the Lemma 2.6.1 and Theorem 0.0.4, that the problem $(\tilde{P}_{a,b})$ admits a solution for every $(0,0) \leq (a,b) \leq \Gamma^*(\theta)$, and this imply that $(\lambda,\mu) \in int(\Upsilon \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+))$, but this is a contradiction. Therefore, $\partial\Upsilon \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+) = \Gamma^*$. This concludes the proof of Lemma. ■

Finally, let us prove the Theorem 0.0.5.

Theorem 0.0.5 Suppose that $0 < \gamma < 1 < \alpha, \beta$; $2 < \alpha + \beta < 2^*$; $0 < a, c$ in $\mathbb{R}^N$, $(A1) - (A2)$, $(V)_0 - (V)_1$ and $(A3)$ if $b > 0$ in $\mathbb{R}^N$ hold. Then:

a) there exists an extended function $\Gamma^* : (0,\infty) \to \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ ($\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$), with $\Gamma^*(\theta) = (\lambda^*(\theta), \mu^*(\theta))$ and $\mu^*(\theta) = \theta\lambda^*(\theta)$ such that system $(\tilde{P}_{\lambda,\mu})$ has at least one solution $U_{\lambda,\mu}$ for $(\lambda,\mu) \in \Theta$ and no solution for $(\lambda,\mu) \notin \Theta$, where

$$\Theta = \{(\lambda,\mu) : (0,0) < (\lambda,\mu) \leq \Gamma^*(\theta),\ \theta > 0\} \cup \{(\lambda,0) : \lambda \in [0,\infty)\} \cup \{(0,\mu) : \mu \in [0,\infty)\}.$$

Moreover, we have $\Phi_{\lambda,\mu}(U_{\lambda,\mu}) < 0$ if $(\lambda,\mu) \in \Theta \setminus \{\Gamma^*(\theta) : \theta > 0\}$ and $\Phi_{\lambda,\mu}(U_{\lambda,\mu}) \leq 0$ if $(\lambda,\mu) \in \Gamma^*(\theta)$ for $\theta > 0$ if $\Gamma^*(\theta) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$,

b) if in addition there exists a smooth bounded open set $\Omega \subset \mathbb{R}^N$ such that $b > 0$ in Ω, then $\Gamma^* \subset \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and $\Gamma^* : (0,\infty) \to \mathbb{R}_0^+ \times \mathbb{R}_0^+$ is a continuous curve, with $0 < \lambda^*(\theta)$ non-increasing and $0 < \mu^*(\theta)$ non-decreasing. In particular, $\{\Gamma^*(\theta) = (\lambda^*(\theta), \mu^*(\theta)) : \theta > 0\} = \partial\Theta \cap (\mathbb{R}_0^+ \times \mathbb{R}_0^+)$ and $(\tilde{P}_{\Gamma^*(\theta)})$ has at least one solution for all $\theta > 0$.

Proof Let Γ^* defined by (2.64) and consider

$$\Theta = \{(\lambda,\mu) : (0,0) < (\lambda,\mu) \leq \Gamma^*(\theta),\ \theta > 0\} \cup \{(\lambda,0) : \lambda \in [0,\infty)\} \cup \{(0,\mu) : \mu \in [0,\infty)\}.$$

Let us prove that $\Upsilon = \Theta$ what implies that the problem $(\tilde{P}_{\lambda,\mu})$ has at least one solution for $(\lambda,\mu) \in \Theta$ and it has no solution for $(\lambda,\mu) \notin \Theta$, by definition of $\Upsilon = \Theta$. First, consider $(\lambda,\mu) \in \Upsilon$. We have three cases: $(\lambda,\mu) > (0,0)$, $(\lambda,0)$ for $\lambda > 0$, and $(0,\mu)$ for $\mu > 0$. Assume that $(\lambda,\mu) > (0,0)$ and set $\theta = \frac{\mu}{\lambda}$. So, we have that $(\lambda,\mu) = (\lambda,\theta\lambda)$ and the problem $(\tilde{P}_{\lambda,\theta\lambda})$ admits solution, by definition of Υ, that implies by the definition of $\lambda^*(\theta)$ that $\lambda \leq \lambda^*(\theta)$. As a consequence of this inequality, we obtain $(\lambda, \mu) = (\lambda, \theta\lambda) \leq \Gamma^*(\theta)$, so that $(\lambda, \mu) \in \Theta$. The other cases follow from definition of Θ. That is, $\Upsilon \subset \Theta$.

To show that $\Theta \subset \Upsilon$, let $(\lambda, \mu) \in \Theta$. If either $\lambda = 0$ or $\mu = 0$, then $(\lambda, \mu) \in \Upsilon$, because the problems $(\tilde{P}_{0,\mu})$ and $(\tilde{P}_{\lambda,0})$ have a solutions. Assume that $(\lambda, \mu) > (0, 0)$. Since $(\lambda, \mu) \in \Theta$, we have $(\lambda, \mu) \leq \Gamma^*(\theta) = (\lambda^*(\theta), \theta\lambda^*(\theta))$ for some $\theta > 0$. If $\lambda^*(\theta) < \infty$, follows of Theorem 0.0.4 and Lemma 2.6.1 that problem $(\tilde{P}_{\lambda,\mu})$ has a solution, that is, $(\lambda, \mu) \in \Upsilon$. If $\lambda^*(\theta) = \infty$, there exists a $\xi > 0$ such that $(\lambda, \mu) < (\xi, \theta\xi)$ and follows from Theorem 0.0.4 that problem $(\tilde{P}_{\lambda,\mu})$ has a solution, that is, $(\lambda, \mu) \in \Upsilon$ that implies $\Theta \subset \Upsilon$ and $\Upsilon = \Theta$. The property of the solution stated in the item $a)$ follows of Theorem 0.0.4 and Lemma 2.6.1. This ends the proof of $a)$.

The item $b)$ is a consequence of Lemma 2.6.4 and item $a)$. The proof of theorem is now complete. ■

Chapter 3

Extremal curves for existence of positive solutions for multi-parameter elliptic systems in $\mathbb{R}^N$

In this chapter, we are going to study the elliptic system

$$\begin{cases} -\Delta u = \lambda w(x) f_1(u) g_1(v) \ in \ \mathbb{R}^N, \\ -\Delta v = \mu w(x) f_2(v) g_2(u) \ in \ \mathbb{R}^N, \\ u, v > 0 \ in \ \mathbb{R}^N \text{ and } u(x), v(x) \overset{|x|\to\infty}{\longrightarrow} 0. \end{cases} \tag{$P_{\lambda,\mu}$}$$

with respect to the parameters $\lambda, \mu \in \mathbb{R}^+$, where $N \geq 3$ and $\mathbb{R}^+ = [0, \infty)$.

System $(P_{\lambda,\mu})$ has no variational structure, so variational techniques do not apply here. The techniques used to prove the main results of existence of solutions of this chapter are the Leray-Schauder Degree and the sub-supersolution methods. We emphasize here that the representation of Riesz given in (3.1) played a fundamental role in proving some results. In fact, it allows us to obtain some estimates that replace the famous Agmon-Douglis-Nirenberg Theorem and the Schauder estimates, both for bounded domains.

Besides this, we will show how changing the hypotheses on nonlinearities impact the shape of regions of existence and non-existence of solution.

This chapter has the following structure. In the first section, we will introduce the spaces where we will work and prove the sub-supersolution theorem which will be our main tool to show the existence of solutions. This Theorem extends to the whole space the Theorem 1.2 of Cheng-Zhang [17]. In Section 3.2, we will prove some preliminary lemmas and build the extremal curves as claimed in the main Theorems. In the last section we prove our main Theorems.

3.1 Sub-Supersolution Theorem

In this section we will give some definitions and prove a sub-supersolution theorem that will be essential to prove the multiplicity of positive solutions to our problem. Since we are working in the whole space, one of the main difficulties to prove it is to find a suitable open set in which the degree of Leray-Schauder of solution operator associated to the problem be equal to 1.

Throughout this section, we will assume $(W)_1 - (W)_4$:

$(W)_1$: $w \in C^{\alpha}_{loc}(\mathbb{R}^N, \mathbb{R}^+_0)$ for some $\alpha \in (0,1)$ and there exists $W \in C(\mathbb{R}^+_0, \mathbb{R}^+_0)$ such that $0 < w(x) \leq W(|x|)$ for all $x \in \mathbb{R}^N \setminus \{0\}$,

$(W)_2$: $\displaystyle\int_{\mathbb{R}^N} |x|^{2-N} W(|x|) dx < \infty$,

$(W)_3$: $W \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,

$(W)_4$: $\displaystyle\int_{\mathbb{R}^N} \frac{W(|y|)}{|x-y|^{N-2}} dy \leq \frac{C}{|x|^{N-2}}$ for all $x \in \mathbb{R}^N \setminus \{0\}$ and for some constant $C > 0$.

In particular, they permit us to find solutions vanishing at infinity with a velocity of order least $|x|^{2-N}$ and gradient of the solution in $L^2(\mathbb{R}^N)$. To do this, let us set our settings to work. We begin by remembering that

$$D^{1,2}(\mathbb{R}^N) = \left\{u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\right\}$$

is a Hilbert space endowed with the inner product

$$(u, v) = \int_{\mathbb{R}^N} \nabla u \nabla v dx,$$

where $2^* = 2N/(N-2)$ is the critical Sobolev exponent. Hereafter, $\|.\|_2$ designates the norm associated with the inner product $(,)$.

We know that $D^{1,2}(\mathbb{R}^N)$ is not compactly embedded into any Lebesgue space, which prevent us to have spectral theory on these spaces. However, under our hypotheses, we have well-defined the weighted Lebesgue space

$$L^2_w(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \to \mathbb{R} \ : \ u \text{ is Lebesgue measurable and } \int_{\mathbb{R}^N} w(x)|u(x)|^2 dx < \infty\right\}$$

that yields the embedding of $D^{1,2}(\mathbb{R}^N)$ into it be compact (see [5]). Besides this, $L^2_w(\mathbb{R}^N)$ is a Hilbert space endowed with the inner product

$$(u,v)_{2,w} = \int_{\mathbb{R}^N} w(x)u(x)v(x)dx,\ \forall u,v \in L^2_w(\mathbb{R}^N).$$

Aiming to find solutions that are continuous too, we introduce the Banach spaces

$$E = \left\{ u \in C(\mathbb{R}^N,\mathbb{R}) : \sup_{x\in\mathbb{R}^N} |u(x)| < \infty \right\} \text{ and } E_r = \left\{ u \in E : u(x) = u(|x|),\ \forall x \in \mathbb{R}^N \right\}$$

endowed with the norm $\|u\| = \sup_{x\in\mathbb{R}^N} |u(x)|$ for $u \in E$. As proved in [5], we know that the embeddings $E, E_r \hookrightarrow L^2_w(\mathbb{R}^N)$ are continuous as well. Besides this, we know from [5] or [55] that there exists a unique weak solution $u := S(v) \in D^{1,2}(\mathbb{R}^N)$ of the problem

$$\begin{cases} -\Delta u = w(x)v \ in\ \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N) \end{cases} \tag{L}$$

that satisfies $\lim_{|x|\to\infty} u(x) = 0$ for each $v \in E \subset L^2_w(\mathbb{R}^N)$. More specifically, besides this vanishing property, the function $u := S(v)$ satisfies

$$\int_{\mathbb{R}^N} \nabla u \nabla \phi dx = \int_{\mathbb{R}^N} w(x)v\phi dx \ \forall\ \phi \in D^{1,2}(\mathbb{R}^N).$$

They also proved that $S : E \longrightarrow E_1 \subset E$ and that Riesz representation of u is given by

$$u(x) = S(v)(x) = C_N \int_{\mathbb{R}^N} \frac{w(y)}{|y-x|^{N-2}} v(y)dy, \tag{3.1}$$

where $C_N = (N(N-2)|B_1(0)|)^{-1}$ and

$$E_1 = \left\{ u \in E : \sup_{x\in\mathbb{R}^N} |x|^{N-2}|u(x)| < \infty \right\}.$$

In addition, we have:

- S is a compact linear operator in E (by $(W)_1 - (W)_4$),
- $S(C^{\alpha}_{loc}(\mathbb{R}^N,\mathbb{R}) \cap E) \subset C^{2,\alpha}_{loc}(\mathbb{R}^N,\mathbb{R})$, for some $\alpha \in (0,1)$ (by $(W)_1$).

As a consequence of the above information, we have the following lemma.

Lemma 3.1.1 *The function $\Psi : \mathbb{R}^N \longrightarrow \mathbb{R}$, defined by*

$$\Psi(x) = C_N \int_{\mathbb{R}^N} \frac{w(y)}{|y-x|^{N-2}} dy, \tag{3.2}$$

belongs to $C^2(\mathbb{R}^N,\mathbb{R}) \cap E_1$. In particular, $\Psi \in L^\infty(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} \Psi(x) = 0$.

Below, with the help of the compact embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^2_w(\mathbb{R}^N)$, the definition of the operator S and its properties, let us build a solution operator associated to the problem

$$\begin{cases} -\Delta u = w(x)F(u,v) \text{ in } \mathbb{R}^N, \\ -\Delta v = w(x)G(u,v) \text{ in } \mathbb{R}^N, \\ u(x), v(x) \overset{|x|\to\infty}{\longrightarrow} 0, \end{cases} \tag{R}$$

where $F, G : \mathbb{R}^2 \to \mathbb{R}$ are such that $F, G \in C^{\alpha(r)}((-r,r)\times(-r,r),\mathbb{R})$ for each $r > 0$ and some $\alpha(r) \in (0,1)$.

To do this, first define $\tilde{F}, \tilde{G} : E \times E \longrightarrow E$ by $\tilde{F}(u,v) = F \circ (u,v)$ and $\tilde{G}(u,v) = G \circ (u,v)$. So, we obtain from the locally Hölder continuity assumption on F e G that $\tilde{F}, \tilde{G}$ are continuous and $\tilde{F}(A), \tilde{G}(A) \subset E$ are bounded sets for any bounded set $A \subset E \times E$ given, that is, the operator $\tilde{S} : E \times E \longrightarrow E \times E$, given by

$$\tilde{S}(u,v) = (S\tilde{F}(u,v), S\tilde{G}(u,v)),$$

is compact, due to the compactness of S.

When we constrain to w radially symmetric, the above conclusions are still true.

Lemma 3.1.2 *If w is a radially symmetric function, then $S(E_r) \subset E_r$. Therefore, $\tilde{S}(E_r \times E_r) \subset E_r \times E_r$ and $\Psi \in E_r$, where Ψ is defined at* (3.2).

Proof To prove that $S(E_r) \subset E_r$, let $v \in E_r$ and $O : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be an orthogonal linear operator. Then by the Riesz representation (3.1), we have

$$\begin{aligned} S(v)(x) &= \int_{\mathbb{R}^N} \frac{C_N w(y)v(y)}{|y-x|^{N-2}}dy = \int_{\mathbb{R}^N} \frac{C_N w(O^{-1}(z))v(O^{-1}(z))}{|O^{-1}(z)-x|^{N-2}}dz \\ &= \int_{\mathbb{R}^N} \frac{C_N w(z)v(z)}{|z-O(x)|^{N-2}}dz = S(v)(O(x)), \end{aligned}$$

after proceeding to the change of variable $y = O^{-1}(z)$. Therefore, $S(v)(x) = S(v)(O(x))$ for all $x \in \mathbb{R}^N$ and $O : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ orthogonal linear operator, which implies that $S(v)$ is a radially symmetric function. In particular, since $\Psi = S(1)$ and $1 \in E_r$, we have $\Psi \in E_r$. ∎

Now, we are in position to state and prove the sub-supersolution Theorem. Before these, let us do two definitions.

Definition 3.1.1 *A pair $(\underline{u}, \underline{v}) \in (C^2(\mathbb{R}^N,\mathbb{R}) \cap E)^2$ is said to be a subsolution (strict subsolution) of* (R) *if*

$$\begin{cases} -\Delta u \le (<) w(x)F(u,v) \text{ in } \mathbb{R}^N, \\ -\Delta v \le (<) w(x)G(u,v) \text{ in } \mathbb{R}^N, \end{cases}$$

while $(\overline{u}, \overline{v}) \in (C^2(\mathbb{R}^N, \mathbb{R}) \cap E)^2$ *is said to be a supersolution (strict supersolution) if the both inequalities above are reversed.*

and

Definition 3.1.2 *A function* $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is said to be quasi-monotone non-decreasing with respect to* t *(or* s*) if,*

$$H(s, t_1) \leq H(s, t_2) \text{ as } t_1 \leq t_2 \text{ (or } H(s_1, t) \leq H(s_2, t) \text{ as } s_1 \leq s_2).$$

In the proof of the theorem below, a key point is to have an open set spanned by the sub and supersolution. Unlike to the case in what Ω is a bounded domain, the set

$$\langle \underline{u}, \overline{u} \rangle = \left\{ u \in C(\overline{\Omega}, \mathbb{R}) : \underline{u} < u < \overline{u} \text{ in } \overline{\Omega} \right\}$$

is not open anymore, when Ω is unbounded. In order to apply the degree theory, the set $\langle \underline{u}, \overline{u} \rangle$ has to be modified.

The main result of this section is the next one.

Theorem 3.1.1 *Assume that* $F, G \in C^{\alpha(r)}((-r, r) \times (-r, r), \mathbb{R})$ *for every* $r > 0$ *and some* $\alpha(r) \in (0, 1)$ *and* $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$ *be a subsolution and a supersolution of (R), respectively, such that:*

(i) $(\underline{u}(x), \underline{v}(x)) \leq (\overline{u}(x), \overline{v}(x))$ *for every* $x \in \mathbb{R}^N$,

$$\lim_{|x| \to \infty} (\underline{u}(x), \underline{v}(x)) = (a_1, a_2) \leq (0, 0) \text{ and } \lim_{|x| \to \infty} (\overline{u}(x), \overline{v}(x)) = (b_1, b_2) \geq (0, 0)$$

for some $a_i, b_i \in \mathbb{R}$ *with* $i = 1, 2$,

(ii) $F(s, t)$ *is quasi-monotone non-decreasing with respect to* t *and* $G(s, t)$ *is quasi-monotone non-decreasing with respect to* s.

Then:

a) the degree

$$deg(I - \tilde{S}, \mathcal{W}, 0) = 1 \tag{3.3}$$

if additionally $(\underline{u}, \underline{v})$ *and* $(\overline{u}, \overline{v})$ *are strict subsolution and supersolution of (R), respectively, and all inequalities in (i) are strict, where*

$$\mathcal{W} := \left\{ (u, v) \in E \times E : (\underline{u}, \underline{v}) < (u, v) < (\overline{u}, \overline{v}) \text{ in } \mathbb{R}^N, M(u, v) > 0 \right\} \tag{3.4}$$

and

$$M(u, v) = \min \left\{ dist(u, \underline{u}), dist(u, \overline{u}), dist(v, \underline{v}), dist(v, \overline{v}) \right\}.$$

In particular, the system (R) has at least one solution (u, v) *in* $\mathcal{W}$, *b) the system* (R) *has at least one solution* $(u,v) \in \overline{\mathcal{W}}$*, where*

$$\overline{\mathcal{W}} := [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}] = \left\{(u,v) \in E \times E : (\underline{u}, \underline{v}) \leq (u,v) \leq (\overline{u}, \overline{v}) \text{ in } \mathbb{R}^N\right\},$$

that is, $\overline{\mathcal{W}}$ *is the closure of* $\mathcal{W}$ *in the* $\| \cdot \|$.

Proof We begin proving the item a), that is, (3.3). To do this, first we are going to prove that $\mathcal{W} \subset E \times E$ is an open set in $E \times E$. We note that it suffices to prove

$$\langle \underline{u}, \overline{u} \rangle = \left\{u \in E : \underline{u} < u < \overline{u} \text{ in } \mathbb{R}^N \text{ and } \min\{dist(u, \underline{u}), dist(u, \overline{u})\} > 0\right\} \subset E,$$

and

$$\langle \underline{v}, \overline{v} \rangle = \left\{v \in E : \underline{v} < v < \overline{v} \text{ in } \mathbb{R}^N \text{ and } \min\{dist(v, \underline{v}), dist(v, \overline{v})\} > 0\right\} \subset E,$$

are open sets in E. We will just prove that $\langle \underline{u}, \overline{u} \rangle$ is an open set, because of the proof of $\langle \underline{v}, \overline{v} \rangle$ be an open set is similar. Let $u \in \langle \underline{u}, \overline{u} \rangle$ and denote by

$$\theta := \min\{dist(u, \underline{u}), dist(u, \overline{u})\} > 0 \text{ and } r = \frac{\theta}{2}.$$

So, by considering $\psi \in B(u,r)$, we have

$$|\psi(x) - u(x)| \leq ||\psi - u|| < r, \forall x \in \mathbb{R}^N, \tag{3.5}$$

which implies that

$$|\underline{u}(x) - \psi(x)| \geq |\underline{u}(x) - u(x)| - |u(x) - \psi(x)| > dist(\underline{u}, u) - \frac{\theta}{2} \geq \frac{\theta}{2} > 0,$$

and therefore $dist(\underline{u}, \psi) = \inf\limits_{x \in \mathbb{R}^N} |\underline{u}(x) - \psi(x)| > 0$. Similarly we have $dist(\overline{u}, \psi) = \inf\limits_{x \in \mathbb{R}^N} |\overline{u}(x) - \psi(x)| > 0$. Besides this, after some manipulations, definition of r and (3.5), we have $\underline{u}(x) < \psi(x) < \overline{u}(x)$ for $x \in \mathbb{R}^N$. These show that $B(u,r) \subset \langle \underline{u}, \overline{u} \rangle$ and, in particular, $\langle \underline{u}, \overline{u} \rangle$ is an open set as claimed.

Now, we define the modified functions $F^*, G^* : \mathbb{R}^N \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$F^*(x,y,z) = F(p_1(x,y,z), P_1(x,y,z)) \text{ and } G^*(x,y,z) = G(p_1(x,y,z), P_1(x,y,z)),$$

where p_1, P_1 are given by

$$p_1(x,y,z) = \max\{\underline{u}(x), \min\{y, \overline{u}(x)\}\} \text{ and } P_1(x,y,z) = \max\{\underline{v}(x), \min\{z, \overline{v}(x)\}\}.$$

So, we have:

- $\underline{u}(x) \leq p_1(x,y,z) \leq \overline{u}(x)$, $\underline{v}(x) \leq P_1(x,y,z) \leq \overline{v}(x)$ and therefore F^*, G^* are continuous and bounded due to the assumptions on F, G,

- $|p_1(x,y_1,z_1) - p_1(x,y_2,z_2)| \leq |y_1 - y_2|$ and

$$|P_1(x,y_1,z_1) - P_1(x,y_2,z_2)| \leq |z_1 - z_2|, \forall (x,y_1,z_1),(x,y_2,z_2) \in \mathbb{R}^N \times \mathbb{R}^2.$$

These imply that the operators $F^*, G^* : E \times E \longrightarrow E$, defined by $F^*(u,v) = F^* \circ (x,u,v)$ and $G^*(u,v) = G^* \circ (x,u,v)$, are continuous and bounded, that is, $\tilde{T} : E \times E \longrightarrow E \times E$, defined by

$$\tilde{T}(u,v) = (SF^*(u,v), SG^*(u,v)),$$

is a compact operator. Moreover, $\tilde{T}$ is the solution operator of the problem

$$\begin{cases} -\Delta u = w(x) F^*(\phi,\psi) \ in \ \mathbb{R}^N, \\ -\Delta v = w(x) G^*(\phi,\psi) \ in \ \mathbb{R}^N, \\ u(x), v(x) \overset{|x|\to\infty}{\longrightarrow} 0, \end{cases} \qquad (M)$$

which means that (u,v) is a solution of the problem (M) in $C^2(\mathbb{R}^N,\mathbb{R})^2$ whenever $\tilde{T}(u,v) = (u,v)$. The $C^2(\mathbb{R}^N,\mathbb{R})^2$-regularity is a consequence of the standard elliptic regularity theory.

To end the proof of the theorem we will prove four claims. The first one is:

Claim 1. $\tilde{T} = \tilde{S}$ in $\mathcal{W}$.

Indeed, if $(\phi,\psi) \in \mathcal{W}$, then

$$p_1(x,\phi(x),\psi(x)) = \max\{\underline{u}(x), \phi(x)\} = \phi(x)$$

and

$$P_1(x,\phi(x),\psi(x)) = \max\{\underline{v}(x), \psi(x)\} = \psi(x).$$

Therefore, $F^*(\phi,\psi) = F(\phi,\psi)$, $G^*(\phi,\psi) = G(\phi,\psi)$ and so $\tilde{T}(\phi,\psi)$ is a solution of (R) as well, that is, $\tilde{T} = \tilde{S}$ in $\mathcal{W}$.

Let us do the second claim.

Claim 2. If (u,v) is a fixed point of $\tilde{T}$, then $(u,v) \in \mathcal{W}$.

Let us just prove that $\underline{u} < u$ in $\mathbb{R}^N$, because of the proof to the other three cases are similar. First, we show that $\underline{u} \leq u$ in $\mathbb{R}^N$ by assuming that there were $x_0 \in \mathbb{R}^N$

such that $u(x_0) < \underline{u}(x_0)$. Since $\lim_{|x|\to\infty}(\underline{u}(x) - u(x)) = a_1 < 0$ by (i), it would follows from our contradiction assumption and continuity of $\underline{u} - u$ that there exists an open and bounded set $\Omega_0 \subset \mathbb{R}^N$ such that

$$u < \underline{u} \text{ in } \Omega_0 \text{ and } u = \underline{u} \text{ on } \partial\Omega_0, \tag{3.6}$$

which implies that $\underline{u} - u$ has a positive maximum on Ω_0.

On the other hand, we obtain from the property of P_1 and assumption (ii), that

$$\begin{aligned} \Delta(\underline{u}(x) - u(x)) > & - w(x)F(\underline{u}(x), \underline{v}(x)) + w(x)F^*(u(x), v(x)) \\ = & w(x)F(\underline{u}(x), P_1(x, u(x), v(x)) - w(x)F(\underline{u}(x), \underline{v}(x)) \\ \geq & w(x)F(\underline{u}(x), \underline{v}(x)) - w(x)F(\underline{u}(x), \underline{v}(x)) = 0, \end{aligned} \tag{3.7}$$

for all $x \in \Omega_0$. By the maximum principle (see Gilbarg-Trudinger [32], Theorem 2.3), we have that $\sup_{\Omega_0}(\underline{u}-u) = \sup_{\partial\Omega_0}(\underline{u}-u) = 0$, which leads us to a contradiction with (3.6). Therefore $\underline{u} \leq u$ in $\mathbb{R}^N$.

Next, we prove that $\underline{u} < u$ in $\mathbb{R}^N$. Again, by contradiction, assume that there were a $x^* \in \mathbb{R}^N$ such that $\underline{u}(x^*) = u(x^*)$. Then, we would have $\Delta(\underline{u} - u)(x^*) \leq 0$, which implies by (3.7) that $0 \geq \Delta(\underline{u} - u)(x^*) > 0$. Therefore $\underline{u} < u$.

To end the proof of the claim, it is sufficient to prove that

$$\min\{dist(u, \underline{u}), dist(u, \overline{u})\} > 0.$$

Since

$$\lim_{|x|\to\infty} |u(x) - \underline{u}(x)| = |a_1| > 0 \text{ and } \lim_{|x|\to\infty} |u(x) - \overline{u}(x)| = |a_2| > 0,$$

there exists $R > 0$ such that

$$|u(x) - \underline{u}(x)| > \frac{|a_1|}{2} \text{ and } |u(x) - \overline{u}(x)| > \frac{|a_2|}{2},$$

for $|x| > R$. These inequalities and the fact that $\underline{u} < u < \overline{u}$ imply, after some manipulations, that $\min\{dist(u, \underline{u}), dist(u, \overline{u})\} > 0$. Hence by definition we have that $u \in \langle \underline{u}, \overline{u} \rangle$.

Claim 3. There exists an open ball $B(0, r)$ such that $\tilde{T}(E \times E) \subset B(0, r)$ and $\mathcal{W} \subset B(0, r)$. Since F^* and G^* are bounded, there exists a constant $c_1 > 0$ such that

$$\|(F^*(\phi, \psi), G^*(\phi, \psi))\| \leq c_1$$

for each $(\phi, \psi) \in E \times E$, which implies by Riesz representation (3.1), that

$$\|\tilde{T}(\phi, \psi)\| \le c_1 \|\Psi\|,$$

for each $(\phi, \psi) \in E \times E$, where Ψ is defined at (3.2).

Besides this, since $\mathcal{W}$ is a bounded set, we are able to take a $r > c_1\|\Psi\|$ such that $\tilde{T}(E \times E) \subset B(0, r)$ and $\mathcal{W} \subset B(0, r)$. This ends the proof of the claim.

After these claims, we are in position to prove (3.3). To do this, first we note that by the claim 1, we have that $\tilde{T} = \tilde{S}$ in $\mathcal{W}$, which leads us to

$$deg(I - \tilde{S}, \mathcal{W}, 0) = deg(I - \tilde{T}, \mathcal{W}, 0). \tag{3.8}$$

If there were $(u, v) \in B(0, r)\backslash\mathcal{W}$ such that $\tilde{T}(u, v) = (u, v)$, then (u, v) would be a solution of (M) such that $(u, v) \notin \mathcal{W}$, but this is a contradiction with the claim 2. Hence,

$$0 \notin (I - \tilde{T})\big(B(0, r)\backslash\mathcal{W}\big)$$

and

$$deg(I - \tilde{T}, \mathcal{W}, 0) = deg(I - \tilde{T}, B(0, r), 0), \tag{3.9}$$

by the excision property of the Leray-Schauder degree.

Now, define the homotopy

$$J(t, (u, v)) = I(u, v) - t\tilde{T}(u, v),\ (t, (u, v)) \in [0, 1] \times \overline{B(0, r)}.$$

Suppose that there were a $(t, (u, v)) \in [0, 1] \times \partial B(0, r)$ such that $t\tilde{T}(u, v) = (u, v)$. If $t = 1$, then $\tilde{T}(u, v) = (u, v)$ and (u, v) would be a solution of (M) such that $(u, v) \notin \mathcal{W}$, which is a contradiction with the claim 2. If $0 \le t < 1$, then, by the claim 3, we would have

$$r = \|(u, v)\| = t\|\tilde{T}(u, v)\| \le tr < r,$$

which is a contradiction again. That is, $0 \notin J([0, 1] \times \partial B(0, r))$.

Hence, by the invariance of the homotopy of the Leray-Schauder degree, we have

$$deg(I - \tilde{T}, B(0, r), 0) = deg(I, B(0, r), 0) = 1, \tag{3.10}$$

whence, combined with (3.8), (3.9), (3.10), we obtain

$$deg(I - \tilde{S}, \mathcal{W}, 0) = deg(I - \tilde{T}, \mathcal{W}, 0) = deg(I - \tilde{T}, B(0, r), 0) = 1.$$

So, by the property of solution of Leray-Schauder degree, the problem (R) admits a solution $(u, v) \in \mathcal{W}$, that completes the proof of item a).

To finish the proof of the theorem, we just point out that just minors adjustments are necessary in the approach of the proof of item a) to prove the item b), more specifically, we have just to adjust the proof of above Claims to $\overline{\mathcal{W}}$ and apply to Leray-Schauder degree to the ball $B(0, r)$ as given in Claim 3. These end the proof of the theorem 3.1.1. ■

As a consequence of the Lemma 3.1.2 we have the following corollary.

Corollary 3.1.1 *If w is a radially symmetric function, then all the conclusions of Theorem 3.1.1 are still true if we change E by E_r.*

3.2 An extremal curve on the parameters for existence of one solution for the problem $(P_{\lambda,\mu})$

In this section, we will build the extremal curves $\tilde{\Gamma}$ and Γ claimed in the main Theorems and study their structures. One of the key points to prove the results of this section is the choice of the appropriated spaces that permit us to apply some ideas found in [17] to whole space. To ease our statement, let us assume that $w(x)$ satisfies $(W)_1 - (W)_4$ throughout this section. Also, for completeness, below we recall once again all the assumptions required in the nonlinearities throughout this chapter for $i \in \{1, 2\}$:

$(H)_1$: $f_i, g_i \in C^{\alpha(r)}((-r, r), \mathbb{R}_0^+)$, for each $r > 0$ and some $\alpha(r) \in (0, 1)$,

$(H)_2$: $0 < \inf_{s \in \mathbb{R}} g_i(s) \leq \sup_{s \in \mathbb{R}} g_i(s) < \infty$,

$(H)_3$: $g_i(s_1) \leq g_i(s_2)$ for $s_1 \leq s_2$,

$(H)_4$: $\dfrac{\delta_1}{g_i(0)} < \liminf_{s \to \infty} \dfrac{f_i(s)}{s} \leq \infty$, where $\delta_1 > 0$ is the first eigenvalue of (A),

$(H)_5$: there exist $p_1, p_2 > 0$ and $q_1, q_2 \in (1, \frac{N}{N-2})$ such that

$$\lim_{s \to \infty} \frac{f_i(s)}{s^{q_i}} = p_i.$$

$(H)_6$: $f_i(s_1) \leq f_i(s_2)$ for $s_1 \leq s_2$,

$(H)_7$: $0 < \lim_{t\to\infty} \dfrac{g_i(t)}{t} \leq \infty$,

$(H)_8$: $\lim_{t\to\infty} \dfrac{f_i(t)}{t} = 0$.

We begin by denoting the set

$$P = \left\{ u \in E : u(x) \geq 0, \forall x \in \mathbb{R}^N \right\}$$

and reminding that $S : E \longrightarrow E$ is the solution operator of the linear problem (L).

After this, by denoting $h_1(u, v) = f_1(u)g_1(v)$, $h_2(u, v) = f_2(v)g_2(u)$ and defining

$$\begin{cases} A^\tau_\lambda(u, v) = \lambda S[\tau h_1(u, v) + (1-\tau)h_1(u, 0)], \\ B^\tau_\mu(u, v) = \mu S[\tau h_2(u, v) + (1-\tau)h_2(u, 0)], \end{cases}$$

for each $\tau \in [0, 1]$ and $u, v \in E$, we obtain from Riesz representation (3.1) and $(H)_1$, that

$$A^\tau_\lambda(u, v)(x) = C_N \int_{\mathbb{R}^N} \frac{w(y)}{|y-x|^{N-2}} \lambda[\tau h_1(u(y), v(y)) + (1-\tau)h_1(u(y), 0)]dy > 0,$$

for $(u, v) \in E \times E$, which implies that $A^\tau_\lambda(u, v) \in P$. Similarly, we have $B^\tau_\mu(u, v) \in P$. Hence, it follows from these information that $T^\tau_{\lambda,\mu} : E \times E \longrightarrow E \times E$, defined by

$$T^\tau_{\lambda,\mu}(u, v) = (A^\tau_\lambda(u, v), B^\tau_\lambda(u, v)), \tag{3.11}$$

is well defined. Besides this, by using the assumption $(H)_1$, we have that $T^\tau_{\lambda,\mu}$ is a compact operator for each $\tau \in [0, 1]$, which implies that $T_{\lambda,\mu}(u, v, \tau) := T^\tau_{\lambda,\mu}(u, v)$ is a compact operator as well.

With these, let us denote by

$$\Upsilon := \left\{ (\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+ : T^1_{\lambda,\mu} \text{ has a fixed point in } E \times E \right\},$$

$$int(\Upsilon) = \text{the interior of } \Upsilon,$$

$$\Upsilon_{rad} := \left\{ (\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+ : T^1_{\lambda,\mu} \text{ has a fixed point in } E_r \times E_r \right\}$$

and

$$int(\Upsilon_{rad}) = \text{the interior of } \Upsilon_{rad}.$$

The next Lemma shows in particular that $\Upsilon \neq \emptyset$.

Lemma 3.2.1 *Assume that* $(H)_1$ *holds for* $i = 1, 2$. *For any* $r > 0$ *there exists a* $(\lambda_r, \mu_r) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ *such that:*

(*i*) $[0, \lambda_r] \times [0, \mu_r] \backslash \{(0,0)\} \subset \Upsilon$,

(*ii*) *for each* $(\lambda, \mu) \in [0, \lambda_r] \times [0, \mu_r] \backslash \{(0,0)\}$, $T^1_{\lambda,\mu}$ *has a nonzero fixed point in* $B(0, r) \subset E \times E$.

Proof Firstly, let us define the functions

$$\tilde{h}_i(s,t) = \begin{cases} h_i(s,t) & \text{if} \quad (s,t) \in [-r,r] \times [-r,r], \\ h_i(r,r) & \text{if} \quad (s,t) \notin [-r,r] \times [-r,r], \end{cases}$$

for any $r > 0$ and $i = 1, 2$, and set the positive numbers

$$\gamma = \sup_{(s,t)\in\mathbb{R}\times\mathbb{R}} |\tilde{h}_1(s,t)| > 0 \text{ and } \eta = \sup_{(s,t)\in\mathbb{R}\times\mathbb{R}} |\tilde{h}_2(s,t)| > 0.$$

In the sequel, let us build (λ_r, μ_r) depending on r, γ and η. To do this, consider the problem

$$\begin{cases} -\Delta u = \lambda w \tilde{h}_1(u,v), \\ -\Delta v = \mu w \tilde{h}_2(u,v), \\ u(x), v(x) \stackrel{|x|\to\infty}{\longrightarrow} 0 \end{cases} \qquad (\tilde{Q}_{\lambda,\mu})$$

and denote by $\tilde{S}_{\lambda,\mu}$ the solution operator associated to $(\tilde{Q})_{\lambda,\mu}$. So, we know from Riesz representation (3.1) and definition of γ that

$$0 \leq u(x) = \lambda C_N \int_{\mathbb{R}^N} \frac{w(y)\tilde{h}_1(\phi(y), \psi(y))}{|y-x|^{N-2}} dy \leq \lambda\gamma\|\Psi\|,$$

for all $(\phi, \psi) \in E \times E$ such that $(u, v) = \tilde{S}_{\lambda,\mu}(\phi, \psi)$, where Ψ is defined at (3.2). This implies that $0 \leqslant u(x) \leqslant r$, $x \in \mathbb{R}^N$ and $\lambda \geq 0$ such that $\lambda\gamma\|\Psi\| \leqslant r$. In similar way, we have $0 \leqslant v(x) \leqslant r$, $x \in \mathbb{R}^N$ and $\mu \geq 0$ such that $\mu\eta\|\Psi\| \leqslant r$. So, for such $\lambda, \mu \geq 0$ with $\lambda + \mu > 0$, we have $\tilde{S}_{\lambda,\mu}(B(0,r)) \subset B(0,r)$.

Besides this, it follows from definition of $\tilde{h}_i$ in $[-r,r] \times [-r,r]$ and Riesz representation (3.1), that

$$\tilde{S}_{\lambda,\mu}(u,v) = T^1_{\lambda,\mu}(u,v), \ \forall (u,v) \in B(0,r), \qquad (3.12)$$

whence, together with $\tilde{S}_{\lambda,\mu}(B(0,r)) \subset B(0,r)$, become well defined the Homotopy

$$J(t,(u,v)) := I(u,v) - t\tilde{S}_{\lambda,\mu}(u,v) \text{ for } (t,(u,v)) \in [0,1] \times \overline{B(0,r)}$$

and so by Homotopy invariance of Leray-Schauder degree, we obtain

$$deg(I - \tilde{S}_{\lambda,\mu}, B(0,r), 0) = deg(I, B(0,r), 0) = 1,$$

for each $(\lambda,\mu) \leqslant (\lambda_r,\mu_r)$, where $(\lambda_r,\mu_r) := (r/2\gamma\|\Psi\|, r/2\eta\|\Psi\|)$.

Hence, by the solution property of the Leray-Schauder degree, there exists a $(u,v) \in B(0,r)$ such that $(u,v) = \tilde{S}_{\lambda,\mu}(u,v)$ whence implies by (3.12) that $(u,v) = T^1_{\lambda,\mu}(u,v)$. Therefore $T^1_{\lambda,\mu}$ has a nonzero fixed point in $B(0,r)$ for all $(\lambda,\mu) \in [0,\lambda_r] \times [0,\mu_r] \setminus \{(0,0)\}$. This completes the proof of Lemma. ■

We note that in the proof of the next lemma it is very important that the solutions of the system $(P_{\lambda,\mu})$ satisfies the conditions $u(0) = \max_{x\in\mathbb{R}^N} u(x)$ and $v(0) = \max_{x\in\mathbb{R}^N} v(x)$ to allow us to apply the blow up method. Since $\liminf_{|x|\to\infty} w(x) = 0$, we are not able to use such method in general.

Lemma 3.2.2 *Assume that $(H)_1 - (H)_3, (H)_5$ hold for $i = 1,2$ and w is radially symmetric. One has*

$$S_{u,v} = \left\{(u,v) : T^\tau_{\lambda,\mu}(u,v) = (u,v), (\lambda,\mu) \in I_1 \times I_2, \tau \in [0,1] \text{ and } (u,v) \in E_r \times E_r\right\},$$

is a bounded set, where $I_i = [a_i,b_i]$ for some constants $b_i > a_i > 0$ and $i = 1,2$.

Proof Assume by the contradiction that there were sequences $\{(\lambda_k,\tau_k)\} \subset I_1 \times [0,1]$ and $\{(u_k,v_k)\} \subset E_r \times E_r$ such that $T^{\tau_k}_{\lambda_k,\mu_k}(u_k,v_k) = (u_k,v_k)$ and $\lim_{k\to\infty} \|(u_k,v_k)\| = \infty$. So, by using that u_k, v_k are positive, continuous and decreasing, we have

$$M_k := \sup_{x\in\mathbb{R}^N} u_k(x) = u_k(0), \qquad N_k := \sup_{x\in\mathbb{R}^N} v_k(x) = v_k(0)$$

and $M_k + N_k \to \infty$ as $k \to \infty$.

Without loss of generality, we may suppose that $M_k \geq N_k$. In this case, there would be two sequences of numbers $\{\lambda_k\} \subset I_1, \{\tau_k\} \subset [0,1]$ and a sequence of positive solutions $\{(u_k,v_k)\}$ of the family of equations

$$\begin{cases} -\Delta u_k = \lambda_k w(x)[\tau_k h_1(u_k,v_k) + (1-\tau_k)h_1(u_k,0)], \\ \lim_{|x|\to\infty} u_k(x) = 0 \end{cases}$$

satisfying $\lim_{k\to\infty} \|u_k\| = \infty$.

By rescaling the functions u_k and v_k by

$$\overline{u}_k(y) = \sigma_k^{\frac{2}{q_1-1}} u_k(\sigma_k y) \text{ and } \overline{v}_k(y) = \sigma_k^{\frac{2}{q_1-1}} v_k(\sigma_k y) \text{ for } y \in \mathbb{R}^N,$$

where

$$\sigma_k^{\frac{2}{q_1-1}} M_k = 1, \tag{3.13}$$

we obtain from $M_k \geq N_k$ and (3.13) that $0 < \overline{v}_k(y) \leq 1$ for every $y \in \mathbb{R}^N$, $\sigma_k \to 0$ as $k \to \infty$ and

$$\sup_{y \in \mathbb{R}^N} \overline{u}_k(y) = \sup_{y \in B_R(0)} \overline{u}_k(y) = \overline{u}_k(0) = 1 \tag{3.14}$$

for all $R > 0$. Moreover, $\overline{u}_k$ satisfies

$$-\Delta \overline{u}_k(y) = F_k(y) \text{ in } B_R(0), \tag{3.15}$$

for all $R > 0$, where

$$F_k(y) := \lambda_k w(\sigma_k y) \sigma_k^{\frac{2q_1}{q_1-1}} [\tau_k h_1(\sigma_k^{\frac{-2}{q_1-1}} \overline{u}_k(y), \sigma_k^{\frac{-2}{q_1-1}} \overline{v}_k(y)) + (1-\tau_k) h_1(\sigma_k^{\frac{-2}{q_1-1}} \overline{u}_k(\sigma_k y), 0)].$$

So, by taking $R > 0$ and $R' > R$, we obtain from $(H)_5$, (3.14) and some manipulations, that

$$|\sigma_k^{\frac{2q_1}{q_1-1}} f_1(\sigma_k^{\frac{-2}{q_1-1}} \overline{u}_k(y))| \leq c(\overline{u}_k(y))^{q_1} + \sigma_k^{\frac{2q_1}{q_1-1}} C \leq C$$

for some constant C independently of $k \in \mathbb{N}$ and $y \in \mathbb{R}^N$. This inequality, together with $(H)_2$ and $(W)_3$, imply

$$|F_k(y)| < c, \ y \in B_{R'}(0) \text{ and } k \in \mathbb{N} \tag{3.16}$$

for some constant $c > 0$.

Since $\overline{u}_k \in C^2(\mathbb{R}^N, \mathbb{R})$ and $F_k \in L^\infty(B_{R'}(0))$, we have that $\overline{u}_k \in W^{1,1}(B_{R'}(0))$ and $F_k \in L^m(B_{R'}(0))$ for all $m \in (1, \infty)$. Then by Theorem 10.2.2 in [43], we have $\overline{u}_k \in W^{2,m}(B_R(0))$ and

$$||\overline{u}_k||_{W^{2,m}(B_R(0))} \leq C[||\overline{u}_k||_{L^m(B_{R'}(0))} + ||F_k||_{L^m(B_{R'}(0))}],$$

where $C = C(m, N, B_{R'}(0), B_R(0))$. Hence, by combining (3.14), (3.16) and the last inequality, we get

$$||\overline{u}_k||_{W^{2,m}(B_R(0))} \leq C[|B_{R'}(0)|^{\frac{1}{m}} + c] := C,$$

where $C = C(m, N, B_{R'}(0), B_R(0))$ again.

Now, choose $m > N$ large. By Sobolev compact embedding theorem, we obtain that $\{\overline{u}_k\}$ is precompact in $C^{1,\alpha}(B_R(0), \mathbb{R})(0 < \alpha < 1)$, which implies that there exists a subsequence $\overline{u}_{k_j}$ converging to $\overline{u}_R$ in $W^{2,m}(B_R(0)) \cap C^{1,\alpha}(B_R(0), \mathbb{R})$ satisfying

$\overline{u}_R(0) = 1$. Besides this, by using that $u_R \neq 0$, the maximum principle and $\sigma_k \to 0$ we obtain

$$\sigma_{k_j}^{\frac{-2}{q_1-1}} \overline{u}_{k_j}(y) \to \infty \text{ with } k_j \to \infty \text{ in } B_R(0). \tag{3.17}$$

Therefore, it follows from (3.17) and $(H)_5$ that

$$\lim_{k_j\to\infty} |\sigma_{k_j}^{\frac{2q_1}{q_1-1}} f_1(\sigma_{k_j}^{\frac{-2}{q_1-1}} \overline{u}_{k_j}(y)) - (\overline{u}_{k_j}(y))^{q_1} p_1| = 0$$

uniformly in $B_R(0)$, which leads us to conclude that

$$\lim_{k_j\to\infty} \sigma_{k_j}^{\frac{2q_1}{q_1-1}} f_1(\sigma_{k_j}^{\frac{-2}{q_1-1}} \overline{u}_{k_j}(y)) = (\overline{u}_R(y))^{q_1} p_1 \text{ for all } y \in B_R(0). \tag{3.18}$$

After this, we are almost in position to pass to the limit in (3.15). By using $(H)_3$, we have that

$$g_1(\sigma_{k_j}^{\frac{-2}{q_1-1}} \overline{v}_{k_j}(y)) \geq g_1(0) > 0 \text{ for all } y \in B_R(0) \tag{3.19}$$

and for every $k_j \in \mathbb{N}$.

Finally, we may assume that $\lambda_k \to \lambda \in I_1$ and $\tau_k \to \tau \in [0,1]$ as $k \to \infty$ and infer by (3.15), (3.18) and (3.19) that $\overline{u}_R$ satisfies

$$-\Delta \overline{u}_R \geq a \overline{u}_R^{q_1} \text{ in } B_R \text{ and } \overline{u}_R(0) = 1,$$

where $a = \lambda g_1(0) w(0) p_1 > 0$.

Hence, the above argument together with a classical diagonal principle approach lead us to obtain a $0 < \overline{u} \in C^1(\mathbb{R}^N, \mathbb{R}) \cap W^{2,m}_{loc}(\mathbb{R}^N)$ that satisfies

$$-\Delta \overline{u} \geq a \overline{u}^{q_1} \text{ in } \mathbb{R}^N \text{ and } \overline{u}(0) = 1, \tag{3.20}$$

and so by setting $z(x) = \overline{u}(x/\sqrt{a})$ for $x \in \mathbb{R}^N$, we obtain from (3.20) that z satisfies

$$-\Delta z \geq z^{q_1} \text{ in } \mathbb{R}^N \text{ and } z(0) = 1,$$

which is impossible by Corollary *II* of Serrin-Zhou [56]. The proof is complete. ■

In the next lemma we will prove that Υ is a connected set.

Lemma 3.2.3 *Assume that $(H)_1$ and $(H)_3$ hold for $i = 1, 2$. Suppose that $T^1_{\overline{\lambda},\overline{\mu}}$ has a non null fixed point $(\overline{u}, \overline{v}) \in E \times E$ for some $(\overline{\lambda}, \overline{\mu}) \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus \{(0, 0)\}$. Then $T^1_{\lambda,\mu}$ has a non null fixed point in $E \times E$ for any $(\lambda, \mu) \in [0, \overline{\lambda}] \times [0, \overline{\mu}] \setminus \{(0, 0)\}$. In particular, Υ is a connected set. The same statements are true if $(H)_1, (H)_3$ and $(H)_6$ hold for $i = 1, 2$.*

Proof For any $(\lambda, \mu) \in [0, \overline{\lambda}] \times [0, \overline{\mu}] \setminus \{(0,0)\}$, it is easy to verify that $(\underline{u}, \underline{v}) = (0,0)$ and $(\overline{u}, \overline{v})$ are a pair of subsolutions and supersolutions to the system $(P_{\lambda,\mu})$. So, Theorem 3.1.1 implies that the system $(P_{\lambda,\mu})$ has at least one solution for $(\lambda, \mu) \in [0, \overline{\lambda}] \times [0, \overline{\mu}] \setminus \{(0,0)\}$, that is, $T^1_{\lambda,\mu}$ has a non null fixed point in $E \times E$ for such (λ, μ). To prove that Υ is a connected set, let us take $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \Upsilon$. Without loss of generality, we may assume that $(\lambda_1, \mu_1) \neq (0,0)$ and $(\lambda_2, \mu_2) \neq (0,0)$.

As we just proved, we have $[0, \lambda_1] \times [0, \mu_1], [0, \lambda_2] \times [0, \mu_2] \subset \Upsilon$ and so there exists a $(\lambda, \mu) \in ([0, \lambda_1] \times [0, \mu_1]) \cap ([0, \lambda_2] \times [0, \mu_2])$. As a consequence of this, we are able to connect $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \Upsilon$ by a polygonal path in Υ, which shows that Υ is a connected set. The proof is complete. ■

In the next lemma, let us prove the main topological properties of set Υ.

Lemma 3.2.4 *Assume that $(H)_1$ holds for $i = 1, 2$. The following conclusions are valid:*

a) $\{(0,0)\} \subsetneq \Upsilon$ *and* $int(\Upsilon)$ *is nonempty,*

b) Υ *is bounded if we assume* $(H)_2 - (H)_4$ *for* $i = 1, 2$. *If in addition we assume* $(H)_5$ *and* w *radially symmetric the set* Υ_{rad} *is closed,*

c) the set Υ *is unbounded in both directions if we assume* $(H)_3, (H)_6, (H)_8$ *for* $i = 1, 2$. *Moreover,* $int(\Upsilon)$ *is an unbounded set in both directions under the same assumptions,*

d) $int(\Upsilon)$ *is a bounded set in the direction* λ *and an unbounded one in the direction* μ *if we assume* $(H)_3, (H)_6$ *for* $i = 1, 2$, $(H)_2, (H)_4$ *for* $i = 1$ *and* $(H)_8$ *for* $i = 2$,

e) $int(\Upsilon)$ *is an unbounded set in the direction* λ *and a bounded one in the direction* μ *if we assume* $(H)_3, (H)_6$ *for* $i = 1, 2$, $(H)_8$ *for* $i = 1$ *and* $(H)_2, (H)_4$ *for* $i = 2$.

Proof First we notice that $(0,0) \in \Upsilon$, because of $T^1_{0,0}(0,0) = (0,0)$. Let us prove the item $a)$. By the Lemma 3.2.1, given $r > 0$ there exists $(\lambda_r, \mu_r) > (0,0)$ such that the operator $T^1_{\lambda,\mu}$ has a fixed point in $E \times E$ for all $(\lambda, \mu) \in (0, \lambda_r) \times (0, \mu_r)$. Therefore $(0, \lambda_r) \times (0, \mu_r) \subset \Upsilon$ and $\Upsilon \setminus \{(0,0)\}$ is nonempty. After these information and Lemma 3.2.1, we have $int(\Upsilon) \neq \emptyset$. The prove of item $a)$ is complete.

Now let us prove the item $b)$. First we will to prove that Υ_{rad} is a closed set. To do this let $\{(\lambda_n, \mu_n)\} \subset \Upsilon_{rad}$ such that $(\lambda_n, \mu_n) \to (\lambda, \mu) \geq (0,0)$ as $n \to \infty$. If

$(\lambda, \mu) = (0, 0)$, we know from a similar statement of item $a)$ that $(\lambda, \mu) \in \Upsilon_{rad}$. Then, we have two cases to consider, namely: $\lambda\mu > 0$ or $\lambda\mu = 0$. Let $\{(u_n, v_n)\} \subset E_r \times E_r$ such that $T^1_{\lambda_n,\mu_n}(u_n, v_n) = (u_n, v_n)$. Consider first that $\lambda\mu > 0$. In this case we may use Lemma 3.2.2 and the compactness of $T^1_{\lambda_n,\mu_n}$ to prove that up to subsequences $(u_n, v_n) \to (u, v)$ in $E_r \times E_r$ as $n \to \infty$ and $(u, v) = T^1_{\lambda,\mu}(u, v)$, that is, $(\lambda, \mu) \in \Upsilon_{rad}$.

In the other case, that is, $\lambda\mu = 0$, we may assume without loss of generality that $\lambda \neq 0$ and $\mu = 0$. By Lemma 3.2.3, for each $n \in \mathbb{N}$, there exits $u_n \in E_r$ such that $T^1_{\lambda_n,0}(u_n, 0) = (u_n, 0)$. Since $\lambda > 0$ it is easy to see that the proof of Lemma 3.2.2 may be applied to prove that $\{u_n\}$ is a bounded sequence, which implies by the compactness of $T^1_{\lambda,\mu}$ that $u_n \to u$ in E_r as $n \to \infty$, up to subsequences, and $(u, 0) = T^1_{\lambda,0}(u, 0)$, that is, $(\lambda, 0) \in \Upsilon_{rad}$. Therefore, Υ_{rad} is a closed set.

Now, we show that Υ is bounded. If Υ were unbounded, then there would be sequences $\{(u_n, v_n)\} \subset E \times E$ and $\{(\lambda_n, \mu_n)\} \subset \mathbb{R}^+ \times \mathbb{R}^+$ such that $T^1_{\lambda_n,\mu_n}(u_n, v_n) = (u_n, v_n)$ and either $\lim \lambda_n = \infty$ or $\lim \mu_n = \infty$.

Without loss of generality, suppose that $\lim \lambda_n = \infty$. By combining this assumption with $(H)_1 - (H)_4$, there exists an $\epsilon > 0$ and a sufficiently large $k \in \mathbb{N}$ such that

$$\lambda_k f_1(r) g_1(s) \geq \lambda_k f_1(r) g_1(0) > r(\delta_1 + \epsilon),$$

for all $s, r \in \mathbb{R}^+$, which implies that

$$-\Delta u_k = \lambda_k w(x) f_1(u_k) g_1(v_k) > (\delta_1 + \epsilon) w(x) u_k \tag{3.21}$$

due to the fact that $T^1_{\lambda_k,\mu_k}(u_k, v_k) = (u_k, v_k)$.

Finally we can multiply (3.21) by the eigenfunction $\phi_1 > 0$, corresponding to δ_1, to get

$$\delta_1 \int w(x) u_k \phi_1 dx = \int \nabla u_k \nabla \phi_1 dx > (\delta_1 + \epsilon) \int w(x) u_k \phi_1 dx$$

that leads to $\delta_1 > \delta_1 + \epsilon$, which is impossible. Thus Υ is a bounded set.

To prove $c)$ is suffices to show that for each $\lambda > 0$ there exists a $\mu > 0$ such that $T^1_{\lambda,\mu}$ has a nonzero fixed point and vice versa. To do this, let us fix $\lambda > 0$. First let us find an appropriated μ and build a supersolution for $(P_{\lambda,\mu})$. To do this, let $t_0 > 0$ be fixed. So, by defining the continuous function $h : (0, \infty) \to \mathbb{R}$ by

$$h(s) = \frac{f_1(s)}{s} - \frac{1}{\lambda \|\Psi\| g_1(t_0)}$$

and applying $(H)_1$ and $(H)_8$, we obtain $\lim_{s\to 0^+} h(s) = \infty$ and $\lim_{s\to\infty} h(s) = -1/\lambda\|\Psi\|g_1(t_0) < 0$, which implies that there exists $s_0 = s_0(\lambda, t_0) > 0$ such that

$$\lambda\|\Psi\|f_1(s_0)g_1(t_0) = s_0. \tag{3.22}$$

Let $\phi \in E$ be the solution of

$$\begin{cases} -\Delta\phi = w(x)f_2(t_0)g_2(s_0) \text{ in } \mathbb{R}^N, \\ \phi > 0 \text{ in } \mathbb{R}^N \text{ and } \phi(x) \overset{|x|\to\infty}{\longrightarrow} 0, \end{cases}$$

the parameter $\mu > 0$ such that $v := \mu\phi \le t_0$ in $\mathbb{R}^N$ and $u \in E$ be the solution of

$$\begin{cases} -\Delta u = \lambda w(x)f_1(s_0)g_1(v) \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N \text{ and } u(x) \overset{|x|\to\infty}{\longrightarrow} 0. \end{cases}$$

So, by the Riesz representation (3.2), $v \le t_0$, $(H)_3$ and (3.22), we obtain

$$u(x) = \lambda C_N \int_{\mathbb{R}^N} \frac{w(y)f_1(s_0)g_1(v)}{|y-x|^{N-2}}dy \le \lambda\|\Psi\|f_1(s_0)g_1(t_0) \le s_0. \tag{3.23}$$

Finally, it follows from $v \le t_0$ again, (3.23), $(H)_3$ and $(H)_6$, that

$$\begin{cases} -\Delta u = \lambda w(x)f_1(s_0)g_1(v) \ge \lambda w(x)f_1(u)g_1(v) \text{ in } \mathbb{R}^N, \\ -\Delta v = \mu w(x)f_2(t_0)g_1(s_0) \ge \mu w(x)f_2(v)g_2(u) \text{ in } \mathbb{R}^N \end{cases}$$

holds, which implies that (u, v) is a supersolution of $(P_{\lambda,\mu})$. Since $(0, 0)$ is a subsolution of $(P_{\lambda,\mu})$ and $(0, 0) < (u, v)$, we obtain from Theorem 3.1.1 that $(P_{\lambda,\mu})$ admits a solution. An analogous statement to μ is justified in a similar way. This proves the item $c)$.

The proofs of the items $d)$ and $e)$ follow from arguments done to prove the items $b)$ and $c)$. The proof of lemma is complete. ■

Before stating the next lemma, we need to set the notations:

$$\begin{aligned} \partial(int(\Upsilon)) &:= \text{ the boundary of } int(\Upsilon), \\ d(int(\Upsilon)) &:= \text{ the derived set of } int(\Upsilon), \\ \overline{int(\Upsilon)} &:= \text{ the closure of } int(\Upsilon) \end{aligned}$$

and apply the assumptions $(H)_1, (H)_3$ and $(H)_6 - (H)_7$ to obtain that

$$f_1(s)g_1(t) > \rho_1 t \text{ and } f_2(t)g_2(s) > \rho_2 s, \ \forall s, t \in \mathbb{R}_0^+ \tag{3.24}$$

hold, for some constants $\rho_1, \rho_2 > 0$. Besides these, let us denote by

$$\rho = \delta_1^2/\rho_1\rho_2. \tag{3.25}$$

After these, we have.

Lemma 3.2.5 *Assume* $(H)_1, (H)_3$ *holds for* $i = 1, 2$. *If in addition:*

a) the assumptions $(H)_2$ *and* $(H)_4$ *hold for* $i = 1, 2$, *then there exists a* $(\lambda_*, \mu_*) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ *such that*

$$\{(\lambda, 0) : \lambda \in [0, \lambda_*]\} \cup \{(0, \mu) : \mu \in [0, \mu_*]\} \subset \partial(int(\Upsilon)) \text{ and } \overline{int(\Upsilon)} \subset [0, \lambda_*] \times [0, \mu_*], \tag{3.26}$$

b) the assumption $(H)_6 - (H)_7$ *hold for* $i = 1, 2$, *then*

$$\Upsilon \subset \left\{(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+ : \lambda\mu < \rho\right\}, \tag{3.27}$$

where ρ *is defined at* (3.25),

c) the hypothesis $(H)_6$ *hold for* $i = 1, 2$; $(H)_2, (H)_4$ *hold for* $i = 1$ *and* $(H)_7$ *hold for* $i = 2$, *then there exists* $\lambda_* > 0$ *such that*

$$\Upsilon \subset [0, \lambda_*] \times [0, \infty),$$

d) the assumption $(H)_6$ *holds for* $i = 1, 2$; $(H)_2, (H)_4$ *hold for* $i = 2$ *and* $(H)_7$ *hold for* $i = 1$, *then there exists* $\mu_* > 0$ *such that*

$$\Upsilon \subset [0, \infty) \times [0, \mu_*].$$

Proof Let us prove a). First we will prove that

$$\left\{\lambda \in \mathbb{R}_0^+ : (\lambda, 0) \in \partial(int(\Upsilon))\right\} \text{ and } \left\{\mu \in \mathbb{R}_0^+ : (0, \mu) \in \partial(int(\Upsilon))\right\} \tag{3.28}$$

are nonempty sets. By Lemma 3.2.4 c), there exists a $(\lambda_0, \mu_0) \in int(\Upsilon) \subset \mathbb{R}_0^+ \times \mathbb{R}_0^+$. So, by combining this information with Lemma 3.2.3, we have that $(0, \lambda_0) \times (0, \mu_0) \subset int(\Upsilon)$, which implies that $(\lambda_0, 0) \in \partial(int(\Upsilon))$ and $(0, \mu_0) \in \partial(int(\Upsilon))$. Therefore, the sets defined in (3.28) are nonempty and bounded by Lemma 3.2.4 b). In particular, we have that the numbers

$$\lambda_* = \sup\left\{\lambda \in \mathbb{R}_0^+ : (\lambda, 0) \in \partial(int(\Upsilon))\right\} \text{ and } \mu_* = \sup\left\{\mu \in \mathbb{R}_0^+ : (0, \mu) \in \partial(int(\Upsilon))\right\} \tag{3.29}$$

are finite, which helps us to show that $\{(\lambda_*, 0), (0, \mu_*)\} \subset \partial(int(\Upsilon)) \cap d(int(\Upsilon))$. Indeed, by definition of λ_* there exists $\{(\lambda_k, 0)\} \subset \partial(int(\Upsilon))$ such that $(\lambda_k, 0) \to (\lambda_*, 0)$. Since $\partial(int(\Upsilon))$ is a closed set, we obtain $(\lambda_*, 0) \in \partial(int(\Upsilon))$. Now, if $(\lambda, 0) \in \partial(int(\Upsilon))$, then $(\lambda, 0) \in d(int(\Upsilon))$, because clearly $(\lambda, 0) \notin int(\Upsilon)$. Hence, $(\lambda_*, 0) \in d(int(\Upsilon))$. Similarly we have $(0, \mu_*) \in \partial(int(\Upsilon)) \cap d(int(\Upsilon))$.

As $\{(\lambda_*, 0), (0, \mu_*)\} \subset \partial(int(\Upsilon))$, to end the proof it is enough to prove that

$$\{(\lambda, 0) : \lambda \in [0, \lambda_*)\} \cup \{(0, \mu) : \mu \in [0, \mu_*)\} \subset \partial(int(\Upsilon)).$$

In fact the above inclusion holds true. For any $\lambda_0 \in [0, \lambda_*)$, we obtain from $(\lambda_*, 0) \in d(int(\Upsilon))$ that there exists $\tilde{\lambda}, \tilde{\mu}$ such that

$$(\tilde{\lambda}, \tilde{\mu}) \in (int(\Upsilon)) \cap B_{\lambda_* - \lambda_0}((\lambda_*, 0)),$$

which implies by Lemma 3.2.3 that $(0, \tilde{\lambda}) \times (0, \tilde{\mu}) \subset (int(\Upsilon))$. Since

$$\lambda_* - \tilde{\lambda} \leq |\tilde{\lambda} - \lambda_*| < \lambda_* - \lambda_0,$$

we have $\lambda_0 < \tilde{\lambda}$. Again by Lemma 3.2.3, we have $(\lambda_0, 0) \in \partial(int(\Upsilon))$. Therefore, $\{(\lambda, 0) : \lambda \in [0, \lambda_*)\} \subset \partial(int(\Upsilon))$. Similarly, one can show that $\{(0, \mu) : \mu \in [0, \mu_*)\} \subset \partial(int(\Upsilon))$ holds as well.

Next, we prove that $\overline{int(\Upsilon)} \subset [0, \lambda_*] \times [0, \mu_*]$. In fact, if

$$(\overline{\lambda}, \overline{\mu}) \in \overline{int(\Upsilon)} \text{ and } (\overline{\lambda}, \overline{\mu}) \notin [0, \lambda_*] \times [0, \mu_*],$$

we have either $\overline{\lambda} > \lambda_*$ or $\overline{\mu} > \mu_*$. Without loss of generality, by supposing that $\overline{\lambda} > \lambda_*$, we obtain that there exists a sequence $\{(\lambda_k, \mu_k)\} \subset int(\Upsilon)$ such that $(\lambda_k, \mu_k) \to (\overline{\lambda}, \overline{\mu})$ and $\overline{\lambda} \geq \lambda_k > \lambda_*$ for every $k > k_0$ and some $k_0 > 0$. By Lemma 3.2.3 we have $(0, \lambda_k) \times (0, \mu_k) \subset int(\Upsilon)$. Thus $\{(\lambda_k, 0)\} \subset \partial(int(\Upsilon))$. Since $\lambda_k \to \overline{\lambda}$ and $\partial(int(\Upsilon))$ is a closed set, we obtain that $(\overline{\lambda}, 0) \in \partial(int(\Upsilon))$ and so by combining this information with (3.29) we conclude that $\overline{\lambda} \leq \lambda_*$, but this is a contradiction.

Let us prove of b). Let $(\lambda, \mu) \in \Upsilon$. Then $(u, v) = T^1_{\lambda,\mu}(u, v)$ for some $u, v \in P$. Since ϕ_1 is the first eigenfunction of (A), we obtain from (3.24) that

$$\delta_1 \int w(x) u \phi_1 dx = \int \nabla u \nabla \phi_1 dx > \rho_1 \lambda \int w(x) v \phi_1 dx$$

and

$$\delta_1 \int w(x) v \phi_1 dx = \int \nabla v \nabla \phi_1 dx > \rho_2 \mu \int w(x) u \phi_1 dx,$$

which implies that $\delta_1^2 > \rho_1 \rho_2 \lambda \mu$. Hence $\Upsilon \subset \{(\lambda, \mu)\mathbb{R}^+ \times \mathbb{R}^+ : \lambda\mu < \rho\}$, where ρ is defined in (3.25).

Proof of c). As in the item a), we can prove that

$$\lambda_* = \sup\left\{\lambda \in \mathbb{R}_0^+ : (\lambda, 0) \in \partial(int(\Upsilon))\right\} < \infty$$

and $\Upsilon \subset [0, \lambda_*] \times [0, \infty)$.

Proof of d). As in the item a), we can prove that

$$\mu_* = \sup \left\{ \mu \in \mathbb{R}_0^+ : (0, \mu) \in \partial(int(\Upsilon)) \right\} < \infty$$

and $\Upsilon \subset [0, \infty) \times [0, \mu_*]$. This ends the proof of Lemma. ■

Our next goal is to make a detailed study of the boundary $\partial(int(\Upsilon))$ of the set $int(\Upsilon)$. To do this, first we define a family of straight lines

$$L(t) = \{(\lambda, t\lambda) : \lambda \in (0, \infty)\}, \ t \in (0, \infty)$$

and

$$\lambda(t) = \sup \left\{ \lambda : (\lambda, t\lambda) \in \overline{int(\Upsilon)} \right\}, \ \mu(t) = t\lambda(t) \text{ and } \Gamma(t) = (\lambda(t), \mu(t)).$$

The next lemma ensures that $\Gamma(t)$ is well defined for every $t > 0$.

Lemma 3.2.6 *Assume $(H)_1$ and $(H)_3$ hold for $i = 1, 2$. Then:*

a) *$\lambda(t) \leq \lambda_*$ for every $t > 0$ if we assume in addition that $(H)_2$ and $(H)_4$ hold for $i = 1, 2$,*

b) *the estimate*
$$\Gamma(t) \leq \mathcal{H}(\sqrt{\rho/t}) \tag{3.30}$$
holds if we also assume $(H)_6 - (H)_8$ for $i = 1, 2$, where $\mathcal{H} : (0, \infty) \to \mathbb{R}$ is defined by $\mathcal{H}(t) = (t, \rho/t)$,

c) *$\lambda(t) \leq \lambda_*$ for every $t > 0$ if in addition we assume $(H)_6$ hold for $i = 1, 2$; $(H)_2, (H)_4$ hold for $i = 1$ and $(H)_7 - (H)_8$ hold for $i = 2$,*

d) *$\lambda(t) \leq \mu_*/t$ for every $t > 0$ if also we assume $(H)_6$ hold for $i = 1, 2$; $(H)_2, (H)_4$ hold for $i = 2$ and $(H)_7 - (H)_8$ hold for $i = 1$.*

Proof The statement of a) is a consequence of Lemmas 3.2.3, 3.2.4 a) and 3.2.5 a). Let us prove the item b). First we note that $\mathcal{H}(\sqrt{\rho/t}) \in L(t)$ for every $t > 0$ that implies together with Lemmas 3.2.3, 3.2.5 b) and definition of $\lambda(t)$, that $\Gamma(t) \leq \mathcal{H}(\sqrt{\rho/t})$ for all $t > 0$. Now the items c) and d) are consequence of Lemmas 3.2.3, 3.2.4 a), 3.2.5 c), d), the definitions of $\lambda(t)$ and $\Gamma(t)$. ■

Now, we have the following.

Lemma 3.2.7 *Assume that* $(H)_1, (H)_3$ *hold for* $i = 1, 2$. *Then,* $\Gamma(t) \in \partial(int(\Upsilon))$ *for every* $t > 0$ *if:*

a) $(H)_2$ *and* $(H)_4$ *are also satisfied for* $i = 1, 2$,

b) *when* $(H)_6 - (H)_8$ *are also satisfied for* $i = 1, 2$,

c) $(H)_6$ *is satisfied for* $i = 1, 2$, $(H)_2, (H)_4$ *are satisfied for* $i = 1$ *and* $(H)_7 - (H)_8$ *are satisfied for* $i = 2$,

d) $(H)_6$ *is satisfied for* $i = 1, 2$, $(H)_7 - (H)_8$ *hold for* $i = 1$ *and* $(H)_2, (H)_4$ *hold true for* $i = 2$.

Proof Let us prove just the item $a)$, because of the proofs of the other items are very similar. For any $t > 0$ given, by the definition of $\lambda(t)$ there exists a sequence $\{(\lambda_k, \mu_k)\} \subset L(t) \cap \overline{int(\Upsilon)}$ that converge to $(\lambda(t), \mu)$, for some $\mu \in \mathbb{R}^+$. Now, by the definition of $L(t)$ and this convergence, we have $\mu_k = t\lambda_k$ and $\mu = \lim_{k\to\infty} \mu_k = t\lambda(t) = \mu(t)$. Hence, $(\lambda(t), \mu(t)) = (\lambda(t), \mu) \in \overline{int(\Upsilon)}$, that is, $\Gamma(t) \in \overline{int(\Upsilon)}$. We claim that $\Gamma(t) \notin int(\Upsilon)$. Indeed, if $\Gamma(t) \in int(\Upsilon)$, then there would be a $r > 0$ such that $B_r(\Gamma(t)) \subset int(\Upsilon)$. Since $B_r(\Gamma(t)) \cap L(t) \neq \emptyset$ and $f(\lambda) = t\lambda$ is an increasing function, there exists $\lambda > \lambda(t)$ such that $(\lambda, t\lambda) \in B_r(\Gamma(t))$ and so $(\lambda, t\lambda) \in \overline{int(\Upsilon)}$, which is a contradiction with the definition of $\lambda(t)$. Therefore $\Gamma(t) \in \partial(int(\Upsilon))$. ■

The next lemma give us a full description of the boundary of $int(\Upsilon)$. Furthermore, it establishes the region of existence and nonexistence of positive solution for the problem $(P_{\lambda,\mu})$.

Lemma 3.2.8 *Assume that* $(H)_1$ *and* $(H)_3$ *hold for* $i = 1, 2$. *Then the following conclusions hold true:*

a) $\Gamma : (0, \infty) \longrightarrow \mathbb{R}^2$ *is a continuous function if we also assume either* $(H)_2, (H)_4$ *or* $(H)_6 - (H)_8$,

b) $\lambda(t)$ *is nonincreasing and* $\mu(t)$ *is nondecreasing if we assume either* $(H)_2, (H)_4$ *or* $(H)_6 - (H)_8$ *as well,*

c) $\lim_{t\to 0} \Gamma(t) = (\lambda_*, 0)$ *and* $\lim_{t\to\infty} \Gamma(t) = (0, \mu_*)$ *if we assume* $(H)_2$ *and* $(H)_4$, *too,*

d) $\Gamma(t)$ *is injective if in addition we assume either* $(H)_2, (H)_4$ *or* $(H)_6 - (H)_8$, *e) the* $\partial(int(\Upsilon))$ *is a simple closed curve and*

$$\partial(int(\Upsilon)) = \{\Gamma(t) : t \in (0,\infty)\} \cup \{(\lambda, 0) : \lambda \in [0, \lambda_*]\} \cup \{(0, \mu) : \mu \in [0, \mu_*]\} \tag{3.31}$$

if $(H)_2$ *and* $(H)_4$ *are satisfied as well,*

f) the

$$\overline{int(\Upsilon)} = \bigcup_{t\in(0,\infty)} \{(\lambda, \mu) \in L(t) : (0, 0) \leq (\lambda, \mu) \leq \Gamma(t)\} \tag{3.32}$$
$$\cup \{(\lambda, 0) : \lambda \in [0, \lambda_*]\} \cup \{(0, \mu) : \mu \in [0, \mu_*]\}$$

if $(H)_2$ *and* $(H)_4$ *are also satisfied,*

g) $\lim_{t\to 0} \Gamma(t) = (\infty, 0)$ *and* $\lim_{t\to\infty} \Gamma(t) = (0, \infty)$ *if in addition we assume* $(H)_6 - (H)_8$*,*

h) the statements (3.31) *and* (3.32) *hold with the bounded intervals changed by unbounded ones of the form* $[0,\infty)$ *if we also assume that* $(H)_6 - (H)_8$ *hold true.*

All the above additional assumptions are made for $i = 1, 2$*.*

Proof Firstly let us prove a). It is sufficient to prove that $\lambda(t)$ is a continuous function. If $\lambda(t)$ were discontinuous at, say, a point t, then there would exist an $\epsilon > 0$ and a sequence $t_n \longrightarrow t$ such that $|\lambda(t_n) - \lambda(t)| \geq \epsilon$. So, up to a subsequence, there would have two possibilities:

$$\lambda(t_n) < \lambda(t) \text{ or } \lambda(t_n) > \lambda(t),$$

for n sufficiently large. Assume that the first one holds. Let $\lambda_1 < \lambda_2$ such that $\lambda(t_n) < \lambda_1 < \lambda_2 < \lambda(t)$. Since $t\lambda_1 < t\lambda_2$, then

$$t_n\lambda(t_n) < t_n\lambda_1 < t\lambda_2 < t\lambda(t),$$

for n large enough. Thus, by the definition of $\Gamma(t)$ the system $(P_{\lambda_2,t\lambda_2})$ has a solution (u, v), which is a supersolution of $(P_{\lambda_1,t_n\lambda_1})$. So, Theorem 3.1.1 implies that the system $(P_{\lambda_1,t_n\lambda_1})$ admits a solution $(\tilde{u}, \tilde{v})$, which lead us to conclude that $\lambda_1 \leqslant \lambda(t_n)$, but this is a contradiction. The second case runs in a similar manner.

Now, let us to prove b). Suppose by contradiction that there exists $t_1, t_2 \in (0, \infty)$ with $t_1 < t_2$ and $\lambda(t_1) < \lambda(t_2)$. Then, we would have $\mu(t_1) = t_1\lambda(t_1) < t_2\lambda(t_2) = \mu(t_2)$ and from $\Gamma(t_2) \in \partial(int(\Upsilon))$ there would exist $(\lambda, \mu) \in int(\Upsilon)$ such that $\Gamma(t_1) < (\lambda, \mu) < \Gamma(t_2)$. By Lemma 3.2.3 we have $\Gamma(t_1) \in (0, \lambda) \times (0, \mu) \subset int(\Upsilon)$ which is a contradiction, because of Lemma 3.2.7 implies that $\Gamma(t_1) \in \partial(int(\Upsilon))$. Similarly, if there were $t_1, t_2 \in (0,\infty)$ with $t_1 < t_2$ and $\mu(t_1) > \mu(t_2)$, then the definition of $\mu(t)$ would lead us to infer that $\lambda(t_1) > \lambda(t_2)$ and this implies that $\Gamma(t_2) \in int(\Upsilon)$, which is a contradiction again.

Let us prove the first statement of item c). To do this, first we note that the item b) and Lemma 3.2.5 imply that

$$\lambda_* \geq \lim_{t\to 0} \lambda(t) = \sup_{t>0} \lambda(t) := \tilde{\lambda} > 0$$

holds. We claim that $\lambda_* = \tilde{\lambda}$. If were $\lambda_* > \tilde{\lambda}$, there would exist a $(\lambda, \mu) \in int(\Upsilon)$ such that $(\lambda,\mu) \in B_{\lambda_* - \tilde{\lambda}}((\lambda_*, 0))$ and $(0,\lambda) \times (0,\mu) \subset int(\Upsilon)$ due to the definition of λ_*. Therefore, these information together with the definition of $\lambda(t)$ imply that

$$\tilde{\lambda} < \lambda \leqslant \lambda(t)$$

holds for all $t > 0$ small enough due to the fact that $L(t) \cap \{(\lambda,\theta) : 0 < \theta < \mu\} \neq \emptyset$ for all t small enough. So, we obtain that $\tilde{\lambda} = \lim\limits_{t\to 0} \lambda(t) \geq \lambda > \tilde{\lambda}$, which is impossible. Hence, $\lim\limits_{t\to 0} \lambda(t) = \tilde{\lambda} = \lambda_*$ and $\lim\limits_{t\to 0} \mu(t) = \lim\limits_{t\to 0} t\lambda(t) = 0$, that is, $\lim\limits_{t\to 0} \Gamma(t) = (\lambda_*, 0)$. This proves the first statement of the item c).

To prove the second statement, first we note that the proof of $\lim\limits_{t\to\infty} \mu(t) = \mu_*$ is similar to the proof of $\lim\limits_{t\to 0} \lambda(t) = \lambda_*$. Now, let us prove that $\lim\limits_{t\to\infty} \lambda(t) = 0$. Indeed, it follows from Lemma 3.2.5 a) and the definition of the norm $|\Gamma(t)|$ that

$$0 < \lambda(t) \leqslant \frac{\sqrt{\lambda_*^2 + \mu_*^2}}{\sqrt{1+t^2}}$$

that lead to $\lim\limits_{t\to\infty} \lambda(t) = 0$ and therefore $\lim\limits_{t\to\infty} \Gamma(t) = (0, \mu_*)$. This completes the proof of the item c).

Now, let us prove d). If $\Gamma(t) = \Gamma(s)$, then $\lambda(t) = \lambda(s)$ and $t\lambda(t) = s\lambda(s)$ that implies $t = s$. Therefore, Γ is injective and this completes the proof of d).

Proof of e). Firstly, we will prove (3.31). It follows from Lemmas 3.2.5 a) and 3.2.7 a) that

$$\{\Gamma(t) : t \in (0,\infty)\} \cup \{(\lambda, 0) : \lambda \in [0, \lambda_*]\} \cup \{(0,\mu) : \mu \in [0,\mu_*]\} \subset \partial(int(\Upsilon))$$

and so, to complete the proof, it suffices to show

$$\partial(int(\Upsilon)) \subset \{\Gamma(t) : t \in (0,\infty)\} \cup \{(\lambda, 0) : \lambda \in [0, \lambda_*]\} \cup \{(0,\mu) : \mu \in [0,\mu_*]\}.$$

To do this, by letting

$$(a,b) \in \partial(int(\Upsilon)) \setminus \{(\lambda,0) : \lambda \in [0,\lambda_*]\} \cup \{(0,\mu) : \mu \in [0,\mu_*]\}, \tag{3.33}$$

we have that

$$(a,b) \in L(t_0)$$

for $t_0 = b/a$, whence together with (3.33), we obtain $(a,b) \in L(t_0) \cap \overline{int(\Upsilon)}$. Besides this, just by definition of $\lambda(t_0)$, we have that $a \leq \lambda(t_0)$. Therefore,

$$\{(a,b),(\lambda(t_0),\mu(t_0))\} \subset L(t_0) \text{ and } a \leq \lambda(t_0).$$

We are going to proof that $a = \lambda(t_0)$. If $a < \lambda(t_0)$, then $b < \mu(t_0)$. By definition of $\Gamma(t_0)$, there exists $\{(\lambda_k,\mu_k)\} \subset int(\Upsilon)$ such that $\lambda_k \to \lambda(t_0)$ and $\mu_k \to \mu(t_0)$ with $k \to +\infty$. Hence, there exists $k_0 \in \mathbb{N}$ such that

$$a < \lambda_{k_0} < \lambda(t_0) \text{ and } b < \mu_{k_0} < \mu(t_0),$$

which implies, together with the Lemma 3.2.3, that

$$(a,b) \in (0,\lambda_{k_0}) \times (0,\mu_{k_0}) \subset int(\Upsilon),$$

that is, $(a,b) \in int(\Upsilon)$, but this is a contradiction with (3.33). So

$$(a,b) = (\lambda(t_0),\mu(t_0)) \in \{\Gamma(t) : t \in (0,\infty)\}$$

that shows (3.31).

Finally, we show that $\partial(int(\Upsilon))$ is a simple closed curve. It is clear that

$$\{(\lambda,0) : \lambda \in [0,\lambda_*]\} \cup \{(0,\mu) : \mu \in [0,\mu_*]\}$$

is a continuous simple arc. In addition, by items $c)$ and $d)$ of Lemma 3.2.8, we have that

$$\{\Gamma(t) : t \in (0,\infty)\} \text{ and } \{(\lambda,0) : \lambda \in [0,\lambda_*]\} \cup \{(0,\mu) : \mu \in [0,\mu_*]\}$$

has just their end points $\{(\lambda_*,0),(0,\mu_*)\}$ in common. So, this information, together with the fact that $\Gamma(t)$ is a simple arc, imply by (3.31) that $\partial(int(\Upsilon))$ is a simple closed curve.

Proof of f). By definition, for any

$$(a,b) \in \bigcup_{t\in(0,\infty)} \{(\lambda,\mu) \in L(t) : 0 \le (\lambda,\mu) \le \Gamma(t)\}$$

given, there exists a $t \in (0,\infty)$ such that

$$(a,b) \in L(t),\ 0 \le a \le \lambda(t) \text{ and } 0 \le b \le \mu(t). \tag{3.34}$$

In view of Lemma 3.2.7 a), $(\lambda(t),\mu(t)) \in L(t) \cap \partial(int(\Upsilon))$. Let $(0,0) < (\lambda,\mu) < (\lambda(t),\mu(t))$. So, there exists $(\kappa,\xi) \in int(\Upsilon)$ such that $(\lambda,\mu) < (\kappa,\xi)$, which implies by Lemma 3.2.3 that $(\lambda,\mu) \in int(\Upsilon) \subset \overline{int(\Upsilon)}$. Therefore $[0,\lambda(t)] \times [0,\mu(t)] \subset \overline{int(\Upsilon)}$ and by (3.34) we have $(a,b) \in \overline{int(\Upsilon)}$. This means that

$$\bigcup_{t\in(0,\infty)} \{(\lambda,\mu) \in L(t) : 0 \le (\lambda,\mu) \le \Gamma(t)\} \subset \overline{int(\Upsilon)}. \tag{3.35}$$

Besides this, we have from Lemma 3.2.5 a) that

$$\{(\lambda,0) : \lambda \in [0,\lambda_*]\} \cup \{(0,\mu) : \mu \in [0,\mu_*]\} \subset \overline{int(\Upsilon)} \tag{3.36}$$

holds. Hence, it follows from (3.35) and (3.36) that

$$\bigcup_{t\in(0,\infty)} \{(\lambda,\mu) \in L(t) : 0 \le (\lambda,\mu) \le \Gamma(t)\} \cup \{(\lambda,0) : \lambda \in [0,\lambda_*]\} \cup \{(0,\mu) : \mu \in [0,\mu_*]\} \subset \overline{int(\Upsilon)}.$$

To end the proof, we claim that

$$\overline{int(\Upsilon)} \subset \bigcup_{t\in(0,\infty)} \{(\lambda,\mu) \in L(t) : (0,0) \le (\lambda,\mu) \le \Gamma(t)\} \cup \{(\lambda,0) : \lambda \in [0,\lambda_*]\} \cup \{(0,\mu) : \mu \in [0,\mu_*]\}. \tag{3.37}$$

Indeed, for any $(a,b) \in \overline{int(\Upsilon)}$, we obtain from Lemma 3.2.5 a) that $(a,b) \in [0,\lambda_*] \times [0,\mu_*]$. If $a = 0$ or $b = 0$, we have

$$(a,b) \in \{(\lambda,0) : \lambda \in [0,\lambda_*]\} \cup \{(0,\mu) : \mu \in [0,\mu_*]\}. \tag{3.38}$$

Assume that $a, b > 0$. Let $t = b/a$. Then $(a,b) = (a,ta) \in L(t)$ so $(a,b) \in L(t) \cap \overline{int(\Upsilon)}$. By the definitions of $\lambda(t)$ and $\mu(t)$, we have $a \le \lambda(t)$ and $b \le \mu(t)$. Hence, $(a,b) \in L(t), 0 < a \le \lambda(t)$ and $0 < b \le \mu(t)$, that is,

$$(a,b) \in \bigcup_{t\in(0,\infty)} \{(\lambda,\mu) \in L(t) : (0,0) < (\lambda,\mu) \le \Gamma(t)\}. \tag{3.39}$$

Thus, the claim (3.37) is a consequence of (3.38) and (3.39).

Let us prove the item g). First, we are going to prove $\lim_{t\to 0}\lambda(t) = \infty$. To do this, fix a $\lambda > 0$. By Lemma 3.2.4 c), there exists a $\mu > 0$ such that $(\lambda,\mu) \in \overline{int(\Upsilon)}$. Since $(\lambda,\mu) \in L(t_0)$, where $t_0 = \mu/\lambda$, we obtain from the properties of $\lambda(t)$ that $\lambda(t) \geq \lambda(t_0) \geq \lambda$ for all $t \in (0,t_0)$, that is, $\lim_{t\to 0}\lambda(t) = \infty$. Now, it follows from (3.30) that $\lim_{t\to 0}\mu(t) = 0$. Hence, $\lim_{t\to 0}\Gamma(t) = (\infty, 0)$. The proof of $\lim_{t\to\infty}\Gamma(t) = (0,\infty)$ follows in a similar way.

The proof of item h) is very similar to the proof of items e) and f) and we omit it here. The proof of lemma is now complete. ■

Corollary 3.2.1 (of the demonstration) *Assume* $(H)_1, (H)_3$ *and* $(H)_6$ *hold for* $i = 1, 2$.

i) *if* $(H)_2$ *and* $(H)_4$ *hold for* $i = 1$ *and* $(H)_7 - (H)_8$ *hold for* $i = 2$, *then the conclusions of items the* $a), b), d)$ *of Lemma* 3.2.8 *are still valid. Moreover, we have* $\lim_{t\to 0}\Gamma(t) = (\lambda_*, 0)$, $\lim_{t\to\infty}\Gamma(t) = (0,\infty)$,

$$\partial(int(\Upsilon)) = \{\Gamma(t) : t \in (0,\infty)\} \cup \{(\lambda, 0) : \lambda \in [0,\lambda_*]\} \cup \{(0,\mu) : \mu \in [0,\infty)\}$$

and

$$\overline{int(\Upsilon)} = \bigcup_{t\in(0,\infty)} \{(\lambda,\mu) \in L(t) : (0,0) \leq (\lambda,\mu) \leq \Gamma(t)\} \cup \{(\lambda,0) : \lambda \in [0,\lambda_*]\}$$
$$\cup \{(0,\mu) : \mu \in [0,\infty)\},$$

where λ_* *is given in Lemma* 3.2.5 c).

ii) *if* $(H)_2$ *and* $(H)_4$ *hold for* $i = 2$ *and* $(H)_7 - (H)_8$ *hold for* $i = 1$, *then the conclusions of items* $a), b), d)$ *of Lemma* 3.2.8 *hold true. Besides these, we have* $\lim_{t\to 0}\Gamma(t) = (\infty, 0)$, $\lim_{t\to\infty}\Gamma(t) = (0,\mu_*)$,

$$\partial(int(\Upsilon)) = \{\Gamma(t) : t \in (0,\infty)\} \cup \{(\lambda, 0) : \lambda \in [0,\infty)\} \cup \{(0,\mu) : \mu \in [0,\mu_*]\}$$

and

$$\overline{int(\Upsilon)} = \bigcup_{t\in(0,\infty)} \{(\lambda,\mu) \in L(t) : (0,0) \leq (\lambda,\mu) \leq \Gamma(t)\} \cup \{(\lambda,0) : \lambda \in [0,\infty)\}$$
$$\cup \{(0,\mu) : \mu \in [0,\mu_*]\},$$

where μ_* *is given in Lemma* 3.2.5 d).

The next lemma give us a full picture of the boundary of Υ.

Lemma 3.2.9 *Assume that* $(H)_1$ *and* $(H)_3$ *hold for* $i = 1, 2$. *Then:*

a) *there exist* $\lambda^* \geq \lambda_*$ *and* $\mu^* \geq \mu_*$ *such that*

$$\partial(\Upsilon) = \{\Gamma(t) : t \in (0, \infty)\} \cup \{(\lambda, 0) : \lambda \in [0, \lambda^*]\} \cup \{(0, \mu) : \mu \in [0, \mu^*]\}$$

if we also assume $(H)_2$ *and* $(H)_4$ *for* $i = 1, 2$,

b) $\partial(\Upsilon) = \partial(int(\Upsilon))$ *if we assume* $(H)_6 - (H)_8$, *for* $i = 1, 2$, *as well,*

c) *there exists* $\lambda^* \geq \lambda_*$ *such that*

$$\partial(\Upsilon) = \{\Gamma(t) : t \in (0, \infty)\} \cup \{(\lambda, 0) : \lambda \in [0, \lambda^*]\} \cup \{(0, \mu) : \mu \in [0, \infty)\}$$

if we assume in addition $(H)_6$ *for* $i = 1, 2$; $(H)_2, (H)_4$ *for* $i = 1$ *and* $(H)_7 - (H)_8$ *for* $i = 2$,

d) *there exists* $\mu^* \geq \mu_*$ *such that*

$$\partial(\Upsilon) = \{\Gamma(t) : t \in (0, \infty)\} \cup \{(\lambda, 0) : \lambda \in [0, \infty)\} \cup \{(0, \mu) : \mu \in [0, \mu^*]\}$$

if we also assume $(H)_6$ *for* $i = 1, 2$; $(H)_7 - (H)_8$ *for* $i = 1$ *and* $(H)_2, (H)_4$ *for* $i = 2$.

Proof Let us prove *a*). If $\overline{int(\Upsilon)} = \Upsilon$, then $\partial(int(\Upsilon)) = \partial(\overline{int(\Upsilon)})$ due to the fact that $int(\Upsilon)$ be an open set. So, by (3.31), the lemma follows. If $\overline{int(\Upsilon)} \subsetneq \Upsilon$, then we claim that

$$\emptyset \neq \Upsilon \backslash \overline{int(\Upsilon)} \subset \left\{(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+ : \lambda\mu = 0\right\}. \tag{3.40}$$

In fact, if $(\lambda, \mu) \in \Upsilon \backslash \overline{int(\Upsilon)}$ and $(\lambda, \mu) > (0, 0)$, then by Lemma 3.2.3 we would have $[0, \lambda] \times [0, \mu] \subset \overline{int(\Upsilon)}$, which implies that $(\lambda, \mu) \in \overline{int(\Upsilon)}$, but this is a contradiction. Thus (3.40) is satisfied.

So, by denoting

$$\lambda^* = \sup\{(\lambda, 0) : (\lambda, 0) \in \partial(\Upsilon)\} \text{ and } \mu^* = \sup\{(0, \mu) : (0, \mu) \in \partial(\Upsilon)\},$$

we get from (3.40) and (3.29) that $\lambda^* \geq \lambda_*$ and $\mu^* \geq \mu_*$. Hence, this information combined with Lemmas 3.2.4 *b*) and 3.2.3, lead us to

$$\{(\lambda, 0) : \lambda \in [0, \lambda^*]\} \cup \{(0, \mu) : \mu \in [0, \mu^*]\} \subset \partial(\Upsilon) \tag{3.41}$$

and

$$[\{(\lambda, 0) : \lambda > \lambda^*\} \cup \{(0, \mu) : \mu > \mu^*\}] \cap \partial(\Upsilon) = \emptyset. \tag{3.42}$$

Thus, it follows from Lemmas 3.2.7 a), 3.2.8 e) and (3.41) that

$$\{\Gamma(t) : t \in (0,\infty)\} \cup \{(\lambda,0) : \lambda \in [0,\lambda^*]\} \cup \{(0,\mu) : \lambda \in [0,\mu^*]\} \subset \partial(\Upsilon). \qquad (3.43)$$

On the other hand, we obtain from Lemma 3.2.7 a) and (3.42) that

$$\partial(\Upsilon\backslash(\overline{int(\Upsilon)})) \subset \{(\lambda,0) : \lambda \in (\lambda_*,\lambda^*]\} \cup \{(0,\mu) : \lambda \in (\mu_*,\mu^*]\}$$

and $\partial(\Upsilon) = \partial(int(\Upsilon)) \cup \partial(\Upsilon\backslash(\overline{int(\Upsilon)}))$. This equality together with the Lemma 3.2.8 e) imply that

$$\partial(\Upsilon) \subset \{\Gamma(t) : t \in (0,\infty)\} \cup \{(\lambda,0) : \lambda \in [0,\lambda^*]\} \cup \{(0,\mu) : \lambda \in [0,\mu^*]\} \qquad (3.44)$$

and so the item a) follows from (3.43) and (3.44).

Now, let us prove b). By using Lemmas 3.2.8 h) and 3.2.3, we have that $\partial(int(\Upsilon)) \subset \partial(\Upsilon)$. On the other hand, if $(\lambda,\mu) \in \partial(\Upsilon)$, we may take a sequence in Υ converging to (λ,μ). So, by using Lemma 3.2.3 and the fact that $\partial(int(\Upsilon))$ is a closed set, we obtain that $\partial(\Upsilon) \subset \partial(int(\Upsilon))$.

The proof of the items c) and d) follow from similar arguments as those done to prove the previous items a) and b). The proof of the lemma is now complete. ■

Let us end this section by doing the following observation:

Remark 3.2.1 *We note that when w is radially symmetric the properties of the sets Υ and $int(\Upsilon)$, proved in the previous Lemmas, remain valid for the sets Υ_{rad} and $int(\Upsilon_{rad})$ just redoing the equivalents proofs with the operator $T^1_{\lambda,\mu}\mid_{E_r\times E_r}$, using Lemma* 3.1.2 *and Corollary* 3.1.1. *However, the extremal curves and parameters may be different from the non-radial case. In this case we will denote the extremal curves by $\tilde{\Gamma}$ and the parameters by $\tilde{\lambda}_*, \tilde{\lambda}^*, \tilde{\mu}_*$ and $\tilde{\mu}^*$.*

3.3 Proof of the main results

In this section we are going to prove our main results. First let us prove Theorem 0.0.6 and use the notation set in Remark 3.2.1.

Theorem 0.0.6 Assume $(W)_1-(W)_4, (H)_1-(H)_5$ for $i = 1,2$ and that w is radially symmetric. Then:

a) there exists a continuous simple arc $\tilde{\Gamma} = \{(\lambda(t),\mu(t)) : \ t > 0\}$, with $0 < \lambda(t)$ non-increasing, $0 < \mu(t)$ non-decreasing and $\mu(t) = t\lambda(t)$, connecting $(\tilde{\lambda}_*, 0)$ and

$(0, \tilde{\mu}_*)$, for some $\tilde{\lambda}_*, \tilde{\mu}_* > 0$, that separates $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ into two disjoint open subsets $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ such that system $(P_{\lambda,\mu})$ has no radially symmetric positive solutions, at least one or at least two radially symmetric positive solutions according to (λ, μ) belongs to $\tilde{\Theta}_2, \tilde{\Gamma}$ or $\tilde{\Theta}_1$, respectively. Moreover, $\tilde{\Gamma} \cup [0, \tilde{\lambda}_*] \cup [0, \tilde{\mu}_*] = \partial\tilde{\Theta}_1$,

b) there exists $\tilde{\lambda}^* \geq \tilde{\lambda}_*$ and $\tilde{\mu}^* \geq \tilde{\mu}_*$ such that the system $(P_{\lambda,\mu})$ has no radially symmetric positive solution for $(\lambda, \mu) \in \{(\lambda, 0) : \lambda > \tilde{\lambda}^*\} \cup \{(0, \mu) : \mu > \tilde{\mu}^*\}$, at least one semi-trivial radially symmetric positive solution for $(\lambda, \mu) \in \{(\tilde{\lambda}^*, 0), (0, \tilde{\mu}^*)\}$ or at least two semi-trivial radially symmetric positive solutions for $(\lambda, \mu) \in \{(\lambda, 0) : \lambda < \tilde{\lambda}^*\} \cup \{(0, \mu) : \mu < \tilde{\mu}^*\}$.

Proof We just prove the item *a*), because of the proof of *b*) is very similar. We know from Lemma 3.2.8 *a*), *c*), *d*) and *e*) that $\{\tilde{\Gamma}(t) : t \in (0, \infty)\}$ is a continuous simple arc that separates $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ into two disjoint open subsets $int(\Upsilon_{rad})$ and $\mathbb{R}_0^+ \times \mathbb{R}_0^+ \setminus \overline{int(\Upsilon_{rad})}$. Let us denote by

$$\tilde{\Theta}_1 = int(\Upsilon_{rad}) \text{ and } \tilde{\Theta}_2 = \mathbb{R}_0^+ \times \mathbb{R}_0^+ \setminus \overline{int(\Upsilon_{rad})}.$$

After these, it is direct application of Lemmas 3.2.4 *b*), 3.2.8 *e*), *f*) and 3.2.9 *a*) that $\tilde{\Gamma} \cup \tilde{\Theta}_1 \subset \overline{int(\Upsilon_{rad})} \subset \Upsilon_{rad}$. Besides this, we claim that $\tilde{\Theta}_2 \cap \Upsilon_{rad} = \emptyset$. In fact, if there were $(\lambda, \mu) \in \tilde{\Theta}_2 \cap \Upsilon_{rad}$, then we would obtain from Lemma 3.2.3 that $(\lambda, \mu) \in [0, \lambda] \times [0, \mu] \subset \overline{int(\Upsilon_{rad})}$, which is a contradiction. So $\tilde{\Theta}_2 \cap \Upsilon_{rad} = \emptyset$. Since $\tilde{\Theta}_1 \cup \tilde{\Gamma} \subset \Upsilon_{rad}$, it follows from definition of Υ_{rad} that the system $(P_{\lambda,\mu})$ admits at least one nontrivial positive solution for $(\lambda, \mu) \in \tilde{\Theta}_1 \cup \tilde{\Gamma}$.

We will prove the existence of the second solution of the system $(P_{\lambda,\mu})$ for $(\lambda, \mu) \in \tilde{\Theta}_1$. To do this, by fixing a $(\lambda, \mu) \in \tilde{\Theta}_1$, we obtain from the fact that $\tilde{\Theta}_1$ is an open set that there exist a $(\overline{\lambda}, \overline{\mu}) \in \tilde{\Theta}_1$ such that $\lambda < \overline{\lambda}$ and $\mu < \overline{\mu}$. So, by definition of Υ_{rad}, there exist $(\tilde{u}, \tilde{v}) \in E_r \times E_r$ such that $T^1_{\overline{\lambda},\overline{\mu}}(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{v}) > (0, 0)$, that is, $(\tilde{u}, \tilde{v})$ is a nontrivial positive solution to system $(P_{\overline{\lambda},\overline{\mu}})$.

After this, let us build a supersolution to the problem $(P_{\lambda,\mu})$. To do this, we first claim that there exists an $\varepsilon \in (0, 1)$ such that

$$\begin{cases} \lambda[h_1(\tilde{u}(x) + \epsilon, \tilde{v}(x) + \epsilon) - h_1(\tilde{u}(x), \tilde{v}(x))] < (\overline{\lambda} - \lambda)\eta_1, x \in \mathbb{R}^N, \\ \mu[h_2(\tilde{u}(x) + \epsilon, \tilde{v}(x) + \epsilon) - h_2(\tilde{u}(x), \tilde{v}(x))] < (\overline{\mu} - \mu)\eta_2, x \in \mathbb{R}^N \end{cases} \tag{3.45}$$

hold for all $\epsilon \in (0, \varepsilon)$, where

$$\eta_1 = \min\{h_1(s,0) : s \in [0, ||\tilde{u}||]\} \text{ and } \eta_2 = \min\{h_2(0,t) : t \in [0, ||\tilde{v}||]\} \tag{3.46}$$

are positives due to the assumption $(H)_1$.

If the claim were not true, then there would exist sequences $\{\varepsilon_n\} \subset (0,1)$ and $\{x_n\} \subset \mathbb{R}^N$ satisfying $\varepsilon_n \to 0$ and

$$\lambda[h_1(\tilde{u}(x_n) + \epsilon_n, \tilde{v}(x_n) + \epsilon_n) - h_1(\tilde{u}(x_n), \tilde{v}(x_n))] \geq (\overline{\lambda} - \lambda)\eta_1 > 0. \tag{3.47}$$

Since $h_1 \in C^{\alpha(r)}((-r,r) \times (-r,r), \mathbb{R}_0^+)$ for some $\alpha(r) \in (0,1)$, where

$$r = \max\{\|\tilde{u}\| + 1, \|\tilde{v}\| + 1\},$$

we obtain from (3.47) that there exists a constant $\kappa = \kappa(r) > 0$ such that

$$\lambda\kappa 2\varepsilon_n^{\alpha(r)} \geq (\overline{\lambda} - \lambda)\eta_1 > 0$$

and this implies that $0 = \lim_{n\to\infty} \lambda\kappa 2\varepsilon_n^{\alpha(r)} \geq (\overline{\lambda} - \lambda)\eta_1 > 0$, which is impossible. Thus there exist $\varepsilon_1 > 0$ such that the first inequality in (3.45) is satisfied for $\epsilon \in (0, \varepsilon_1)$.

Similarly, we are able to find an $\varepsilon_2 > 0$ such that the second inequality of (3.45) is satisfied for any $\epsilon \in (0, \varepsilon_2)$. To finish the proof of the claim it is enough to take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

So, it follows from (3.45), (3.46) and $(H)_3$ that

$$\begin{cases} \lambda h_1(\tilde{u}+\epsilon, \tilde{v}+\epsilon) - \overline{\lambda} h_1(\tilde{u}, \tilde{v}) & < (\overline{\lambda} - \lambda)\eta_1 - (\overline{\lambda} - \lambda)h_1(\tilde{u}, \tilde{v}) \\ & \leq (\overline{\lambda} - \lambda)[\eta_1 - h_1(\tilde{u}, 0)] \leq 0, x \in \mathbb{R}^N, \\ \mu h_2(\tilde{u}+\epsilon, \tilde{v}+\epsilon) - \overline{\mu} h_2(x, \tilde{u}, \tilde{v}) & < (\overline{\mu} - \mu)[\eta_2 - h_2(0, \tilde{v})] \leq 0, x \in \mathbb{R}^N, \end{cases}$$

for any $\epsilon \in (0, \varepsilon)$, whence we conclude that $(\overline{u}, \overline{v}) = (\tilde{u}+\epsilon, \tilde{v}+\epsilon)$ is a supersolution for $(P_{\lambda,\mu})$ for any $\epsilon \in (0, \varepsilon)$ given.

On the other hand, the pair $(\underline{u}, \underline{v}) = (-\epsilon, -\epsilon)$ is a subsolution of the system $(P_{\lambda,\mu})$. Moreover, it is clear that $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ satisfy the condition (i) of the Corollary 3.1.1, which implies that

$$deg(I - T^1_{\lambda,\mu}, \mathcal{W}, 0) = 1, \tag{3.48}$$

where $\mathcal{W} \subset \langle \underline{u}, \overline{u} \rangle \times \langle \underline{v}, \overline{v} \rangle$ is defined at (3.4).

Let us do a new claim. There exists a $R > 0$ large enough such that

$$\begin{cases} \overline{\mathcal{W}} \subsetneq B(0,R), \\ deg(I - T^1_{\lambda,\mu}, B(0,R), 0) = 0. \end{cases}$$

In fact, let $(\tilde{\lambda}, \tilde{\mu}) \in \tilde{\Theta}_2$ with $\tilde{\lambda} > \lambda$ and $\tilde{\mu} > \mu$. Consider $R > 0$ large enough such that $R > C_{I_1}, C_{I_2}$, where C_{I_1} and C_{I_2} are the constants given in Lemma 3.2.2 with $I_1 = [\lambda, \tilde{\lambda}]$ and $I_2 = [\mu, \tilde{\mu}]$. In addition, due to the boundedness of $\mathcal{W}$, we may assume that $\overline{\mathcal{W}} \subset B(0, R/2) \times B(0, R/2)$. Then, by combining Lemma 3.2.2 with Homotopy invariance, we have that

$$deg(I - T^1_{\lambda,\mu}, B(0,R), 0) = deg(I - T^1_{\lambda,\tilde{\mu}}, B(0,R), 0) = 0, \tag{3.49}$$

which implies by the additivity of Leray-Schauder degree, (3.48) and (3.49) that

$$deg(I - T^1_{\lambda,\mu}, B(0,R) \setminus \overline{\mathcal{W}}, 0) = deg(I - T^1_{\lambda,\mu}, B(0,R), 0) - deg(I - T^1_{\lambda,\mu}, \mathcal{W}, 0) = -1. \tag{3.50}$$

Therefore, by (3.48) and (3.50) the operator $T^1_{\lambda,\mu}$ has at least two nontrivial fixed points in $E_r \times E_r$, that is, the system $(P_{\lambda,\mu})$ admits at least two positive solutions for $(\lambda, \mu) \in \tilde{\Theta}_1$. The proof is now complete. ■

Now let us prove Theorem 0.0.7.

Theorem 0.0.7 Assume that $(H)_1 - (H)_4$ for $i = 1, 2$ and $(W)_1 - (W)_4$ hold. Then:

a) there exists a continuous simple arc Γ, with the same properties as those one in Theorem 0.0.6, which separates $\mathbb{R}^+_0 \times \mathbb{R}^+_0$ into two disjoint open subsets Θ_1 and Θ_2 such that system $(P_{\lambda,\mu})$ has no positive solution and has at least one according to (λ, μ) belongs to Θ_2 and Θ_1, respectively. Moreover, $\Gamma \cup [0, \lambda_*] \cup [0, \mu_*] = \partial\Theta_1$ for some $\lambda_*, \mu_* > 0$,

b) there exists $\lambda^* \geq \lambda_*$ and $\mu^* \geq \mu_*$ such that the system $(P_{\lambda,\mu})$ has no positive solutions for $(\lambda, \mu) \in \{(\lambda, 0) : \lambda > \lambda^*\} \cup \{(0, \mu) : \mu > \mu^*\}$ and at least one for $(\lambda, \mu) \in \{(\lambda, 0) : \lambda < \lambda^*\} \cup \{(0, \mu) : \mu < \mu^*\}$.

Proof We know from Lemma 3.2.8 $a), c), d)$ and $e)$ that $\{\Gamma(t) : t \in (0, \infty)\}$ is a continuous simple arc that separates $\mathbb{R}^+_0 \times \mathbb{R}^+_0$ into two disjoint open subsets $int(\Upsilon)$ and $\mathbb{R}^+_0 \times \mathbb{R}^+_0 \setminus \overline{int(\Upsilon)}$. By denoting by

$$\Gamma = \{\Gamma(t) : t \in (0, \infty)\}, \ \Theta_1 = int(\Upsilon) \text{ and } \Theta_2 = \mathbb{R}^+_0 \times \mathbb{R}^+_0 \setminus \overline{int(\Upsilon)},$$

we obtain that $\Theta_1 \subset \overline{int(\Upsilon)}$. Besides this, we have $\Theta_2 \cap \Upsilon = \emptyset$. In fact, if there were $(\lambda, \mu) \in \Theta_2 \cap \Upsilon$, then we would have by Lemma 3.2.3 that $(\lambda, \mu) \in [0, \lambda] \times [0, \mu] \subset \overline{int(\Upsilon)}$, which is a contradiction. So $\Theta_2 \cap \Upsilon = \emptyset$, that is, the system $(P_{\lambda,\mu})$ has no solution for $(\lambda, \mu) \in \Theta_2$. Since $\Theta_1 \subset \Upsilon$, we obtain by definition of Υ that the system $(P_{\lambda,\mu})$ admits at least one nontrivial positive solution for $(\lambda, \mu) \in \Theta_1$. This ends the proof of Theorem. ■

Let us prove the Corollary 0.0.1.

Corollary 0.0.1 Assume that $(W)_1 - (W)_4$, $(H)_1 - (H)_5$ for $i = 1, 2$ hold and w is radially symmetric. Let $\tilde{\Theta}_1, \tilde{\Gamma}, \Theta_1$ and Θ_2 as in Theorems 0.0.6 and 0.0.7. If $\Theta_1 \backslash \overline{\tilde{\Theta}}_1 \neq \emptyset$, then the system $(P_{\lambda,\mu})$ has no positive solution, at least one and at least two ones according to (λ, μ) in $\Theta_2, \tilde{\Gamma}$ or $\Theta_1 \setminus \tilde{\Gamma}$, respectively.

Proof Firstly we note by Theorem 0.0.7 that the system $(P_{\lambda,\mu})$ has no positive solution for $(\lambda, \mu) \in \Theta_2$ and by Theorem 0.0.6 the system $(P_{\lambda,\mu})$ admits at least one positive solution for $(\lambda, \mu) \in \tilde{\Gamma}$. To prove the multiplicity of solutions as statement in Corollary, let us write $(\Theta_1 \setminus \tilde{\Gamma}) = \tilde{\Theta}_1 \cup (\Theta_1 \setminus \overline{\tilde{\Theta}}_1)$. If $(\lambda, \mu) \in \tilde{\Theta}_1$, then the statement follows from Theorem 0.0.6. Otherwise, if $(\lambda, \mu) \in (\Theta_1 \setminus \overline{\tilde{\Theta}}_1)$, we obtain from definition of Υ_{rad} that $(P_{\lambda,\mu})$ has no radially symmetric positive solution, which implies that the solution (u, v) obtained in Theorem 0.0.7 satisfies $(u, v) \notin E_r \times E_r$, that is, either u is not radially symmetric or v is not radially symmetric as well. Assume that $u \notin E_r$. So, we are able to build a second solution. In fact, since $u \notin E_r$, there exist an orthogonal map O and $x_0 \in \mathbb{R}^N$ such that

$$u(O(x_0)) \neq u(x_0). \tag{3.51}$$

Now, by defining $j, k : \mathbb{R}^N \longrightarrow \mathbb{R}$ by $j(x) = u(O(x))$ and $k(x) = v(O(x))$, we obtain from (3.51) that $(j, k) \neq (u, v)$. Besides this, by using Riesz representation (3.1), combining with the change of variables $z = O^{-1}(y)$ and $w(O(z)) = w(z)$ for each

$z \in \mathbb{R}^N$, we obtain

$$\begin{aligned} j(x) = u(O(x)) &= C_N\lambda \int_{\mathbb{R}^N} \frac{w(y)f_1(u(y))g_1(v(y))}{|y-O(x)|^{N-2}}dy \\ &= C_N\lambda \int_{\mathbb{R}^N} \frac{w(O(z))f_1(u(O(z)))g_1(v(O(z)))}{|z-x|^{N-2}}dz \\ &= C_N\lambda \int_{\mathbb{R}^N} \frac{w(z)f_1(u(O(z)))g_1(v(O(z)))}{|z-x|^{N-2}}dz \\ &= C_N\lambda \int_{\mathbb{R}^N} \frac{w(z)f_1(j(z))g_1(k(z))}{|z-x|^{N-2}}dz = A^1_\lambda(j,k)(x) \end{aligned}$$

holds for each $x \in \mathbb{R}^N$. Similarly, we have $k(x) = B^1_\mu(j,k)(x)$ for each $x \in \mathbb{R}^N$. That is, due to (3.11), we have $T^1_{\lambda,\mu}(j,k) = (A^1_\lambda(j,k), B^1_\mu(j,k)) = (j,k)$, which proves that (j,k) is a positive solution of the problem $(P_{\lambda,\mu})$ as well. This completes the proof of Corollary. ■

Now let us prove the Theorem 0.0.8.

Theorem 0.0.8 Assume that $(W)_1-(W)_4$, $(H)_1,(H)_3$ and $(H)_6-(H)_8$ for $i=1,2$ hold. Then there exists a continuous simple arc $\Gamma = \{(\lambda(t),\mu(t)) : \ t>0\}$, with $0<\lambda(t)$ non-increasing, $0<\mu(t)$ non-decreasing and $\mu(t)=t\lambda(t)$, satisfying $\lim_{t\to 0}\Gamma(t) = (\infty,0)$ and $\lim_{t\to\infty}\Gamma(t) = (0,\infty)$ that separates $\mathbb{R}^+_0 \times \mathbb{R}^+_0$ into two disjoint open subsets Θ_1 and Θ_2 such that the system $(P_{\lambda,\mu})$ has no positive solution and has at least one according to (λ,μ) belongs to Θ_2 and Θ_1, respectively.

Proof We know from Lemma 3.2.8 $a),d),g),h)$ that $\{\Gamma(t) : t \in (0,\infty)\}$ is a continuous simple arc that separates $\mathbb{R}^+_0 \times \mathbb{R}^+_0$ into two disjoint unbounded open subsets $int(\Upsilon)$ and $\mathbb{R}^+_0 \times \mathbb{R}^+_0 \setminus \overline{int(\Upsilon)}$ satisfying $\lim_{t\to 0}\Gamma(t) = (\infty,0)$ and $\lim_{t\to\infty}\Gamma(t) = (0,\infty)$. So, by denoting

$$\Theta_1 = int(\Upsilon) \text{ and } \Theta_2 = \mathbb{R}^+_0 \times \mathbb{R}^+_0 \setminus \overline{int(\Upsilon)},$$

we obtain from Lemma 3.2.8 $h)$ that $\Gamma \cup \Theta_1 \subset \overline{int(\Upsilon)}$ holds. We claim that $\Theta_2 \cap \Upsilon = \emptyset$. In fact, if there were $(\lambda,\mu) \in \Theta_2 \cap \Upsilon$, then we would have by Lemma 3.2.3 that $(\lambda,\mu) \in [0,\lambda]\times[0,\mu] \subset \overline{int(\Upsilon)}$, which is a contradiction. So $\Theta_2 \cap \Upsilon = \emptyset$, that is, the system $(P_{\lambda,\mu})$ has no solution for $(\lambda,\mu) \in \Theta_2$. Since $\Theta_1 \subset \Upsilon$, we obtain by definition of Υ that the system $(P_{\lambda,\mu})$ admits at least one nontrivial positive solution for $(\lambda,\mu) \in \Theta_1$. This ends the proof of Theorem. ■

Finally, we just note that the proofs of Theorems 0.0.9 and 0.0.10 are very similar to the proof of previous Theorems.

Conclusion

In this work we constructed a region that produces a result of global existence of positive solutions to problem $(\tilde{P}_{\lambda,\mu})$. From our point of view this result is interesting due to the loss of comparison principle and improves the results already existing in the literature. However, we were unable to obtain the behavior of curve Γ^* to $\theta \to 0$ and $\theta \to \infty$, which would lead us to boundedness or not of the extremal region Θ. With respect to Chapter 1, the approach of Section 2.6 of Chapter 2 can be applied to obtain the existence of a parameter $\Lambda_* > 0$ such that problem (P_λ) has at least one positive solution u_λ with $\Phi_\lambda(u_\lambda) \leq 0$ for $0 < \lambda \leq \Lambda_*$, and problem (P_λ) has no solution for $\lambda > \Lambda_*$.

Now let us point out some open questions. It is an open question when the system $(\tilde{P}_{\lambda,\mu})$ (and equation (P_λ)) admits multiplicity of solutions, even on the positive semi-axes. Other open questions are about the boundedness or not of extremal curve Γ^* and how smooth it is. Is it C^1 or C^2?

Related to Chapter 3, we constructed multiple extremal curves that produce different regions of existence and non-existence of positive solutions. Under the assumptions of radiality of the potential w and $(H)_5$, we proved global multiplicity results in Theorem 0.0.6 and Corollary 0.0.1. We also concluded that different combinations of the hypotheses $(H)_4$ and $(H)_8$ lead to different shapes of the extremal curve $\Gamma(t)$. Besides these, it is not usual to use topological arguments to prove directly sub-supersolution theorems in the whole space without approaching the problem by auxiliary problems in bounded domains. We were able to do this and obtain information about the Leray-Schauder degree of the compact operator associated with the problem.

Now let us make some comments and point out some open questions. It is an open question when the system $(P_{\lambda,\mu})$ admits multiplicity of solutions under the hypotheses $(H)_1 - (H)_5$ and w being not necessarily a radially symmetric potential. Other open questions are to find appropriated assumptions to obtain global multiplicity results in Theorems 0.0.8, 0.0.9 and 0.0.10. Our results answer partially the these questions by establishing extremal curves and a complete study of the properties of the regions of existence and nonexistence of positive solutions for elliptic systems in $\mathbb{R}^N$.

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