Universidade de Brasília
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On the Maximal Eigenspace of the Ruelle Operator

por

Leonardo Cavalcanti de Melo

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por

Leonardo Cavalcanti de Melo †

Tese de Doutorado apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática da Universidade de Brasília, PPGMat–UnB, como parte dos requisitos necessários para obtenção do título de Doutor em Matemática.

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Comissão Examinadora:

Leandro Martins Cioletti
____

Prof. Dr. Leandro Martins Cioletti - MAT/UnB (Orientador)

Eduardo Antonio da Silva
____

Prof. Dr. Eduardo Antonio da Silva – MAT/UnB (Membro)

____

Prof. Dr. Artur Oscar Lopes – UFRGS (Membro)

Paulo Varandas
____

Prof. Dr. Paulo César Rodrigues Pinto Varandas – UFBA (Membro)

A Amyrthes Fernandes de Moraes Rego, meu modelo de cientista.

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Abstract

In this PhD dissertation we analyze the spectral data of the Ruelle operator $\mathscr{L}$ and its extension to $L^1(\nu)$, here denoted by $\mathbb{L}$. To do so, we use as the main route the L^1 theory of Markov processes introduced by Eberhard Hopf. This technique provides us with sufficient tools to extract information on the eigenspace of the operator, even for low-regularity potentials presenting phase transition. The theory we develop here comprises compact metric alphabets. In this setting and for an arbitrary continuous potential, we show that the eigenspace of $\mathscr{L}$ associated with its spectral radius is at most one-dimensional and, when there is a continuous maximal eigenfunction, it must have a definite sign. These properties are known to hold for finite alphabets and some classes of regular potentials, such as those fulfilling the hypothesis of the Ruelle-Perron-Frobenius Theorem. We demonstrate that those properties are only related to the positivity of the operator and full support of the eigenmeasures and do not depend on the finite alphabet or on the regularity of the potential. As for the extension $\mathbb{L}$ to $L^1(\nu)$, we give conditions on ν to have a well-posed extension. When the chosen ν is a conformal measure (maximal eigenmeasure), we prove that ν is fully supported if and only if the *a priori* measure p is fully supported. In this case, we demonstrate that the dimension of the maximal eigenspace of the extended operator is upper bounded by the number of extreme measures whose convex combination yields ν. This gives us a new criterion for phase transition, since a multidimensional maximal eigenspace can only emerge in the case of multiple extreme conformal measures. We also construct an example inspired on the Currie-Weiss model that exhibits phase transition with a bi-dimensional maximal eigenspace.

Keywords: Ruelle operator, transfer operator, Ruelle-Perron-Frobenius Theorem, spectral analysis, harmonic functions, invariant functions, conformal measures, ergodic theory, Markov processes, Currie-Weiss model, mean-field model.

Resumo

Nesta tese de doutorado analisamos os dados espectrais do operador de Ruelle $\mathscr{L}$ e de sua extensão ao espaço $L^1(\nu)$, aqui denotada por $\mathbb{L}$. Para tal, nossa abordagem principal é a teoria L^1 de processos de Markov introduzida por Eberhard Hopf. Essa técnica nos provê de ferramentas suficientes para extrair informações sobre o autoespaço maximal do operador, mesmo para potenciais com baixa regularidade, ainda que apresentem transição de fase. A teoria desenvolvida aqui abrange alfabetos métricos compactos. Nesse contexto, e para um potencial contínuo arbitrário, mostramos que o autoespaço associado ao raio espectral é no máximo unidimensional e, quando há uma autofunção maximal contínua, ela tem sinal definido. A validade dessas propriedades era sabida para alfabetos finitos ou potenciais em algumas classes de regularidade, como aqueles satisfazendo as hipóteses do Teorema de Ruelle-Perron-Frobenius. Mostramos que essas propriedades são consequência apenas da positividade do operador e do suporte total das automedidas, não dependendo da finitude do alfabeto ou da regularidade do potencial. Sobre a extensão $\mathbb{L}$ do operador de Ruelle $\mathscr{L}$ ao espaço $L^1(\nu)$, especificamos quais são as condições necessárias para se ter um operador em $L^1(\nu)$ bem definido. Quando a medida ν é uma medida conforme (automedida maximal), provamos que ν é totalmente suportada se e somente se a medida *a priori* é totalmente suportada. Nesse caso, demonstramos que o autoespaço maximal do operador extendido tem dimensão limitada superiormente pelo número de medidas extremais cuja combinação convexa gera ν. Isso nos dá um novo critério para detectar transição de fase, já que um autoespaço maximal multidimensional só pode aparecer se existirem múltiplas medidas extremais. Também construímos um exemplo, inspirado no modelo de Currie-Weiss, que apresenta transição de fase e cujo autoespaço maximal é bidimensional.

Palavras-chave: Operador de Ruelle, operador de transferência, teorema de Ruelle-Perron-Frobenius, análise espectral, funções harmônicas, funções invariantes, medidas conformes, teoria ergódica, processos de Markov, modelo de Currie-Weiss, modelo de campo médio.

CONTENTS

CHAPTER 1

INTRODUCTION

This dissertation presents the L^1 approach to Markov processes, introduced by Eberhard Hopf in 1954 in the article [30], as a tool to study the Ruelle operator. In that approach, a Markov process is seen as a continuous linear operator $T : L^1(\mu) \to L^1(\mu)$, which satisfies two properties, namely it is a positive operator and a weak contraction.

The Ruelle transfer operator $\mathscr{L}_f$ is defined as an operator acting on the continuous functions on a metric space X, to be described in Section 1.2. In order to use Hopf's formalism to study the operator, it must be extended to the space $L_1(\nu)$, of the almost everywhere equivalence classes of integrable functions with respect to a measure ν. Such a measure will be chosen as a maximal eigenmeasure of the transposed operator $\mathscr{L}_f^*$, here called a *conformal measure.*

With the extension to $L^1(\mu)$ of the Ruelle operator, here denoted by $\mathbb{L}_f$, we will show that, when divided by its spectral radius, the operator can be seen as a Markov process in the L^1 sense.

We organize our findings in three classes:

- those concerning the proper existence and good definition of the extension,
- those related to its spectral data, and
- an example of a discontinuous potential that gives rise to a Ruelle-like operator with multidimensional eigenspace.

Regarding the existence of the extended operator, we show that, when ν is a conformal measure, the only necessary condition to have a well-defined extension of the continuous Ruelle operator to $L^1(X, \mathscr{B}(X), \nu)$ is a fully supported *a priori* measure. The proof holds when $X = E^{\mathbb{N}}$, for E a compact metric space. On the path to the construction of the extended operator, we show that a conformal measure is fully supported if and only if the a priori measure implicit on the operator is fully supported. We describe how to explicitly represent this operator when it acts on the integrable functions and show that, even without a fully supported *a priori* measure, this representation is well defined and preserves $L^1(\nu)$ classes.

With the extension in hand, we use the L^1 theory of Markov process to analyze its properties. We believe that this procedure produces the most relevant outcome in this dissertation. We show that the positivity of the maximal eigenfunction and one-dimensionality of the eigenspace of the Ruelle operator on the continuous functions, are related only to the positivity of the operator. Indeed, for any continuous potential, the eigenspace is at most one-dimensional and, when there is an eigenfunction, it is positive on a full ν-measure set for any conformal measure ν, even for infinite compact alphabets.

When the analyzed operator is the extended operator to $L^1(\nu)$, if ν is an extreme conformal measure, the eigenspace of the operator also has its dimension limited to one and, if there is a $L^1(\nu)$ eigenfunction, it has ν-a.e. defined sign, which is analogous to the continuous case. However, circumstances change when ν is a non-extreme conformal measure, as it is possible for continuous potentials exhibiting phase transition. In this case, and when the chosen conformal measure is a nontrivial convex combination of N extreme ones, we prove that the dimension of the eigenspace of the L^1 operator will be limited by N, the number of extreme measures in the decomposition.

To illustrate the generality of Hopf's approach and yield some intuition on the theory to be developed ahead, following this introductory chapter, we present an example of a potential with some nontrivial properties. Specifically, it exhibits phase transition, with two extreme conformal measures, and the dimension of the eigenspace of the L^1 operator is also equal to two. This example is inspired by the Currie-Weiss or mean-field model of Statistical Mechanics. This model will be described as a transfer operator on the L^1 that does not come from a Ruelle operator associated with a continuous potential. To do so, we must develop a definition of conformal measure compatible with a discontinuous potential, since it is the case in our example. This also motivates the extension of some subsequent results to the class of bounded potentials, when possible. Due to the simplicity of calculating the spectral data associated with the operator representing the Currie-Weiss model, such a model can be useful as a *"toy model"* representative of other statistical mechanics systems with ferromagnetic interactions. The analogy includes the ferromagnetic models exhibiting phase transition since this phenomenon is also present in the Currie-Weiss model.

1.1 Overview

In Chapter 1 we introduce the theoretical framework to be used throughout the dissertation, as the definition of the Ruelle operator and some of its properties. We also show a general picture of related results on the literature and how they relate to our constructions.

In Chapter 2 we sketch an example of a Ruelle-like operator with a discontinuous potential inspired by the Currie-Weiss model. We show that, by properly choosing a parameter, it exhibits phase transition and the eigenspace of the operator is also multidimensional. The example is constructed to give some intuition on the general theory developed in the chapters that follow, since most of the results are replicated for continuous potentials on the subsequent examination.

In Chapter 3 we investigate the necessary conditions to extend a Ruelle operator acting on the space of the continuous functions to $L^1(\mu)$, for an arbitrary probability measure μ. We also show how this extension can be represented as an operator on the integrable functions and show the relation between a fully supported conformal measure and a fully supported *a priori* measure.

In Chapter 4 we present the general theory of L^1 Markov processes first developed by E. Hopf. The chapter is focused on the duality between the analytic (L^1) point of view and the probabilistic point of view on Markov processes.

In Chapter 5 we continue to present the general Hopf's theory, now focused on the dissipative-conservative decomposition of Markov processes.

In Chapter 6 we show that the L^1 extension of the Ruelle operator can be seen as a Markov process in Hopf's sense and apply the theory developed on the previous chapters to understand some properties of its eigendata.

In Chapter 7 we conclude our work revisiting the main results presented here and pose some unanswered questions that can be investigated on subsequent works.

Throughout the text, to make explicit in the notation in which class we are taking our functions, we use the Greek letters φ, ψ for continuous functions, u, v, w for L^1 functions (classes of equivalence), f, g for bounded or L^∞ functions. For measures we use μ, ν and m.

1.2 The Ruelle Operator

One-dimensional dynamical systems described by Gibbs measures occupy a very unique position in the broad spectrum of ergodic theory and statistical mechanics models. They include the class of Markov chains and the whole family of finite memory discrete-time stochastic processes, but they also encompass a broader class of processes presenting long-range dependency.

In a certain way, they are in the frontier between the too complex to deal with and the too simple to yield interesting phenomena. On the one hand, their relative simplicity allows the application of a wide range of techniques to infer properties, sometimes in an explicit fashion. On the other hand, they are complex enough to exhibit some characteristics of higher dimensional systems.

The Ruelle operator, or transfer operator, was created to analyze these one-dimensional systems. It arises as an extension of the transfer-matrix method. The latter can only deal with finite interaction systems and the operator can be seen as an "infinite transfer-matrix".

To define the operator, we will need to specify the space on which it acts. Take E a metric space and $X = E^{\mathbb{N}}$ the right-infinite product space endowed with any metric that induces the product topology $\mathscr{B}(X)$. E plays the role of the alphabet and X is the space of right-infinite words. The operator will act on some spaces of functions (or their equivalence classes) on X. Classically, it acts on $C(X, \mathbb{R})$, the space of real-valued functions on X continuous on the product topology and endowed with the supremum norm.

The original version of the Ruelle operator was created to deal with the Ising model so, in this case, we have $X = \{-1,+1\}^{\mathbb{N}}$. But a slightly more general definition of the operator would comprise a finite alphabet $E = \{a_1, ...a_n\}$, $n \in \mathbb{N}$. In this setting, the Ruelle operator is defined as shown below:

$$\mathscr{L}_\psi u := \sum_{a \in E} \exp(\psi(ax))\varphi(ax), \quad \forall \varphi \in C(X,\mathbb{R}),$$

The continuous function ψ is called the *potential* defining the operator $\mathscr{L}_\psi$.

Going further in generality, one can begin with a compact metric alphabet E. Tychonoff's Theorem implies that X is compact, since E is already compact. This means that $C(X,\mathbb{R})$ coincides with the family of continuous bounded functions. See [52] for examples where this setting is chosen. In this case,

$$\mathscr{L}_\psi u := \int_E \exp(\psi(ax))\varphi(ax)dp(a), \quad \forall \varphi \in C(X,\mathbb{R}). \tag{1.1}$$

Once more, the potential $\psi \in C(X,\mathbb{R})$. The set function p is a probability measure on the alphabet E denoted as an *a priori* measure and is replaced by the counting measure on the previous definition.

More general settings are under development, and one can choose a non-compact alphabet E, as in [15], or a dynamic not restricted to the left shift, as in [28], but our focus in this dissertation is on the generality implicit on the operator given by 1.1.

Since $\mathscr{L}_\psi$ is a continuous operator acting on the space of continuous bounded functions, its dual, $\mathscr{L}_\psi^*$, acts on the topological dual of $C(X,\mathbb{R})$, which is the space of signed measures on the Borel-measurable space $(X, \mathscr{B}(X))$, here denoted by $\mathscr{M}_s(X)$. Note that $\mathscr{L}_\psi$ is a positive operator, i.e., taken a non-negative function φ, its image $\mathscr{L}_\psi(\varphi)$ is also non-negative. This implies that its dual $\mathscr{L}_\psi^*$ is also positive. An argument based on the compactness of the probability measures space $\mathscr{M}_1(X)$, positivity of $\mathscr{L}_\psi^*$, and the Schauder-Tychonoff fixed point theorem shows that at least one probability measure satisfies $\mathscr{L}_\psi^* \nu = \rho(\mathscr{L}_\psi)\nu$. See [10], section 2. Following the nomenclature established by Denker and Urbański in [20], we call these maximal probability eigenmeasures conformal measures, as on the definition bellow.

Definition 1.2.1 (Conformal measure)**.** *Let $\mathscr{L}_\psi$ be the Ruelle operator defined in 1.1, we say that a probability measure ν is a conformal measure associated with the potential ψ if it is a maximal eigenmeasure of $\mathscr{L}_\psi^*$, in this case, $\mathscr{L}_\psi^* \nu = \rho(\mathscr{L}_\psi)\nu$.*

We denote the set of conformal measures associated with ψ by $\mathscr{G}^*(\psi)$, or $\mathscr{G}$, when there is no risk of ambiguity. When $\mathscr{G}^*(\psi)$ is not a singleton, we say that the potential, or the system defined by the potential, exhibits *phase transition*.

1.3 Main Results and Related Literature

The Ruelle operator was first defined on Ruelle's seminal work [46]. In that work, he proved there is no phase transition for a class of potentials on the space $X = \{0,1\}^{\mathbb{N}}$ describing the lattice gas. In the nomenclature used in this dissertation, he showed there is a unique conformal measures for those systems.

This was the starting point of a theory which is nowadays called Thermodynamic Formalism. Actually with the advent of Markov partitions many asymptotic properties of smooth dynamics and limit theorems such as the Central Limit Theorem where established using the technology of transfer operators. The literature is vast and we refer the interested reader to [2, 5, 6, 17, 20, 28, 32, 39, 40, 44, 46, 47, 48, 49, 51, 52, 53, 54] and references therein.

The classical approach to prove existence and uniqueness of the conformal measure is to require some regularity on the potential. For Hölder-continuous potentials, Ruelle has shown that the spectral gap phenomenon is present, in the sense that there is a maximal isolated real eigenvalue $\rho(\mathscr{L}_\psi)$ and all of the remaining spectrum is contained in a disk with radius strictly smaller than $\rho(\mathscr{L}_\psi)$. Other properties of these systems defined by regular potentials are the nonexistence of phase transition, as well as the uniqueness, positivity, and Hölder continuity of the eigenfunction.

These facts are an extension of the Perron-Frobenius Theorem, which asserts that every square matrix with only positive entries has a maximal positive real eigenvalue and the corresponding eigenspace is one-dimensional with an eigenvector that can be chosen with only positive entries. This fact motivates the nomenclature often used to address the Ruelle operator by the Ruelle-Perron-Frobenius operator. Because of this close relation, Ruelle's theorem is also called Ruelle-Perron-Frobenius Theorem.

Some of these results were also developed for classes of less regular potentials, as those satisfying Walters' and Bowen's conditions. Peter Walters has shown in [52] that, for a class now known as satisfying Walters' regularity condition, there is no phase transition and there is a unique positive eigenfunction that is also Walters' regular, so it is continuous. He also asked in [53] if this would be the case for a wider class of potentials satisfying Bowen's condition. It was already known (see [6]) that there is no phase transition in this case. In [53] Walters showed there is a bounded "eigenfunction" for Bowen's potentials. However, the question about the existence of a continuous eigenfunction for potentials in this regularity class is still open.

We have listed some conditions that lead to a single conformal measure, but there are potentials that give rise to a multiplicity of them. This phenomenon is known as *phase transition* and the classical example are the Dyson potentials of Statistical Mechanics. See [21] for the original work, or [41] for a master's thesis in Portuguese on the theme. Other examples of potentials giving rise to phase transition can be seen in [4, 7, 11, 23, 29, 31]. See Remark 6.1.11 for an application of them.

All of the works cited in this section thus far deal with potentials defined on $X = E^{\mathbb{N}}$ and E a finite alphabet. On the matter of infinite alphabets in [3], Baraviera, Cioletti, Lopes, Mohr, and Souza investigate the XY model where the alphabet is $E = \mathbb{S}^1$, the unit circle. This paper introduced the idea of a priori measure on Thermodynamic Formalism. Afterward, in [37], Lopes, Mengue, Mohr, and Souza generalized the ideas of [3] to the case where E is a general compact metric space. This paper also introduced a suitable notion of entropy which takes the a priori measure into account and in turn opened up the possibility to talk about equilibrium states in such a general setting.

In [14], Cioletti and Silva established the Ruelle-Perron-Frobenius theorem for potentials in the Walters' class on compact spaces. They also showed that for potentials in this class that the spectral gap property is in general absent. Another important generalization of these results, in the setting of compact alphabets, were obtained recently by Cioletti, Lopes and Stadlbauer in [10]. Among other things they proved a version of the Ruelle-Perron-Frobenius theorem for potentials in the Bowen's class, which is defined by a property also known as bounded distortion. This paper also discuss the relations between the Gibbs measures in Statistical Mechanics and Thermodynamic Formalism. A very general theory was developed recently by Kloeckner in [35], where a new and interesting Ruelle operator is introduced. To construct the Ruelle operator, instead of considering an a priori measure on the fibers the auhtor consider a Markov chain on the phase space, which could be a full shift on a compact alphabet. Among other things a

Ruelle-Perron-Frobenius Theorem is obtained, in this new setting, for generalized Hölders potentials. The author also establishes sharp upper bounds for the decay of correlations.

The Thermodynamic Formalism for full shifts on non-compact alphabets endowed with a priori measure is also studied in some recent works. For example, in [15] the authors considered a very general setting where the alphabet can be any standard Borel spaces, thus generalizing a lot of previously cited works. In particular, in the work [15], Cioletti, Silva, and Stadlbauer showed that, for Hölder potentials and the Ruelle operator defined as in 3.1, a Ruelle-Perron-Frobenius Theorem also holds. Recently, Lopes and Vargas [38] studied similar issues on $\mathbb{R}^{\mathbb{N}}$ using a one-point compactification technique.

On the extension of the operator to $L^1(\nu)$, where ν is a conformal measure, Cioletti, Lopes, and Stadlbauer have shown in [10] that, given the hypothesis that ν is fully supported, the operator on the continuous functions $\mathscr{L}_\psi$ can be extended to $L^1(\nu)$. In Proposition 3.1.8 we show that every conformal measure of a continuous potential is fully supported, so the hypothesis in [10] is always fulfilled.

In [16], Cioletti, van Enter, and Ruviaro use the cited extended operator, here denoted by $\mathbb{L}_\psi$, to demonstrate that, under certain abstract conditions, it has an $L^1(\nu)$ eigenfunction. They also build a counterexample, based on a Manneville-Pomeau map, of a continuous potential with no L^1 eigenfunction. On Chapter 6 we revisit the issue of the conditions for the existence of L^1 eigenfunctions.

The results in this dissertation are a continuation of these works, as we also study the maximal eigenspace of the operator. Our approach to investigate it is to extend the operator to a space bigger then $C(X, \mathbb{R})$, describe the maximal eigenspace of the extended operator and then check which of these characteristics can be translated back to the operator on the continuous functions.

Following [9] the natural candidate spaces to search for eigenfunctions are the Lebesgue spaces $L^p(\nu) \equiv L^p(X, \mathscr{B}(X), \nu)$, where $\nu \in \mathscr{G}^*(\psi)$ and $1 \leqslant p \leqslant \infty$. But, in the generality considered here, it is not always true that $L^1(\nu)$ is larger than $C(X)$. More precisely, depending on the support of ν, it might happen that there is no linear embedding of $C(X)$ in $L^1(\nu)$ having a trivial kernel. For instance, if E is not a countable set and the support of the *a priori* measure is a finite set, such embedding does not exist. Therefore, our first goal is to establish sufficient conditions for the existence of such embedding. The following theorem proved in Section 3.1 poses the conditions to have a proper embedding,

Proposition 3.1.8. *Let $X = E^{\mathbb{N}}$ endowed with the usual metric, with E a compact metric space, and $\psi \in C(X, \mathbb{R})$. Let ν be a conformal measure associated with ψ and a priori measure p fully supported on E. Then ν is also fully supported on X.*

A fully supported *a priori* measure will actually be the only necessary condition to have a well defined extended operator $\mathbb{L}_\psi$, as we detail in Chapter 3.

An additional reason to chose $L^1(\nu)$, for ν a conformal measure, is that, for continuous potentials, when there is an eigenfunction h such that $\mathscr{L}_\psi h = \rho(\mathscr{L}_\psi)h$, it is know that h is a Radon-Nikodym derivative of a equilibrium measure with respect to the conformal measure ν, i. e., $d\mu = hd\nu$, and the measure μ is a equilibrium measure with respect to the potential ψ. See [44] for a better explanation on this relation.

Once we have constructed $\mathbb{L}_\psi$, the extension of the operator to $L^1(\nu)$, dividing it by its operator norm, we end up with a Markov process in Hopf's sense. Then, in Chapter 6, we apply the theory we present in Chapters 4 and 5 (based on [8], [24] and [43]) to look to $\mathbb{L}_\psi$ as a Markov process and extract information on its maximal eigenspace. The main results we obtain arise from this approach. The following theorem associates the dimension of the maximal eigenspace of $\mathbb{L}_\psi$ with the structure of the conformal measures space.

Theorem 6.1.10. *Let f be a bounded potential and $m \in \mathscr{G}^*$ a generalized conformal measure (not necessarily extreme). Then the eigenspace of $\mathbb{L}_f$ (acting on $L^1(m)$) associated to its operator norm has dimension not bigger than the cardinality of the set of extreme points in $\mathscr{G}^*$.*

In particular, this theorem holds for continuous potentials f, as they are bounded, and conformal measures m, as we prove they are generalized conformal measures in Chapter 3. It allows the existence of multidimensional maximal eigenspaces for a extended Ruelle operator $\mathbb{L}_f$ when there is phase transition. Indeed, in Chapter 2 we give an example of a discontinuous potentials exhibiting phase transition and with a multidimensional maximal eigenspace.

Theorem 6.1.10 has two interesting consequences. The first one is an application on the study of phase transition in Equilibrium Statistical Mechanics. In [10], Ciolleti, Lopes and Stadlbauer have shown that, for continuous potentials, the set $\mathscr{G}^*$ coincides with a set of measures that takes place in Equilibrium Statistical Mechanics, called DLR

Gibbs measures. Then, if we have a multidimensional eigenspace for an operator $\mathbb{L}_\psi$, Theorem 6.1.10 implies the existence of multiple extreme elements in $\mathscr{G}^*(\psi)$. Thus, by the results in [10], there is phase transition in the DLR sense for a potential suitably translated to their setting. This gives us a criterion to identify phase transition in the DLR sense only by looking to the eigenspace of $\mathbb{L}_\psi$. For more details, see [10, 27, 50].

The second application of Theorem 6.1.10 concerns its consequences to the continuous eigenfunctions. Since, for a continuous potential ψ, $\mathbb{L}_\psi$ is a extension of $\mathscr{L}_\psi$, a continuous eigenfunction of $\mathscr{L}_\psi$ (or its equivalence class) is also an eigenfunction of $\mathbb{L}_\psi$. This observation and the theory developed for the maximal eigenspace of $\mathbb{L}_\psi$ will imply the following corollary.

Corollary 6.1.6. *Let ψ be any continuous potential and $\mathscr{L}_\psi : C(X) \to C(X)$ be a transfer operator constructed from this potential and a fully supported a priori probability measure p on E. Then the eigenspace of $\mathscr{L}_\psi$ associated to its spectral radius has either dimension zero or one. If that eigenspace is one-dimensional, any eigenfunction in it has definite sign and it can vanish at most in a ν-null set, for ν any conformal measure with respect to ψ.*

These results were proved by Parry and Pollicott in [44] for a finite alphabet E, but their methods are not suitable to deal with infinite ones. Corollary 6.1.6 extends these to an arbitrary compact metric space.

Maximal eigenspaces similar to ours appear in other settings. In the theory of Markov processes, as the maximal eigenvalue is the unity, an eigenfunction associated to it is called an *invariant function*. Neveu in [43] developed a theory giving necessary and sufficient conditions for the existence of such invariant functions for general Markov processes. On Section 6.2 we apply his findings to the specific case of the extended Ruelle operator $\mathbb{L}_f$ associated to a bounded potential f.

In an abstract setting of functional equations, Conze and Raugi in [18] investigate the maximal eigenspace of some transfer operators. They denote the functions in that space by *harmonic functions* with respect to the transfer operator. In their setting, the action of the operator on a continuous test function $\varphi : [0,1] \to \mathbb{R}$ is defined by

$$\mathscr{T}_\phi \varphi(x) = \phi\left(\frac{x}{2}\right)\varphi\left(\frac{x}{2}\right) + \phi\left(\frac{x}{2}+\frac{1}{2}\right)\varphi\left(\frac{x}{2}+\frac{1}{2}\right),$$

where $\phi : [0,1] \to [0,1]$ is a fixed continuous function satisfying $\phi(x/2) + \phi(x/2 + 1/2) = 1$.

Translating to a setting closer to ours, their setting is analogous to a transfer operator given, for every continuous function φ, by

$$\mathscr{L}_\phi \varphi = \sum_{a \in E} \phi(ax)\varphi(ax), \quad \forall x \in X = E^{\mathbb{N}}.$$

Where $E = \{0,1\}$ is a finite alphabet and $\phi \geq 0$ is the continuous transfer function. Note that their transfer function ϕ is allowed to vanish. This provides the emergence of multiple closed subsets A_i in the space X. Each orbit of the underlying dynamic is restrained to just one of the sets A_i and can not "jump" to another A_j because the transfer function that connects any pair of points in different A_i vanish.

In their original setting, of functions on $[0,1]$, they prove that the dimension of the harmonic eigenspace of $\mathscr{T}_\phi$ is bounded by the number of disjoint closed subsets $A_i \subseteq X$ where transitive subsystems can be defined. Thus, we can see a similar behaviour of the operator $\mathscr{T}_\phi$, on the continuous functions, and our $\mathbb{L}_\psi$, on $L^1(\nu)$. Both of them have the dimension of the harmonic eigenspace bounded by the number of disjoint sets that are invariant by the dynamic in some sense.

But the mechanism behind the multiplicity of the space of harmonic functions here is also different from the one in [18]. In [18] the multidimentionality of the maximal eigenspace arises solely because the transfer function, that is allowed to vanish, forbidding the transitivity. In our case, as our transfer function $\phi = e^f > 0$ is always positive, the system is always transitive. The only closed set with positive measure invariant with respect to the dynamic is the whole space X. This implies that there is at most a one-dimensional **continuous** harmonic eigenspace of the operator $\mathscr{L}_\psi$, as stated in Corollary 6.1.6.

On the other hand, when we search for the possibly discontinuous functions that make up the maximal eigenspace of $\mathbb{L}_\psi$, relevant sets B_i appear once more (see the demonstration of Theorem 6.1.10). And, as in [18], they are invariant with respect to the underlying dynamic (the left shift $\sigma(ax) = x$). But they can not be closed in the case of multiple extreme conformal measures, as each one of the B_i's must be dense in X. This separates in one side the operator on the continuous functions $\mathscr{L}_\psi$, that have at most a one-dimensional maximal eigenspace, from $\mathbb{L}_\psi$ and $\mathscr{T}_u$ in the other, as they allow multidimensional ones. This distinction is important as it makes the space of harmonic functions of $\mathscr{L}_\psi$ insensitive to phase transition, in opposition to $\mathbb{L}_\psi$ or $\mathscr{T}_u$.

CHAPTER 2

THE CURRIE-WEISS MODEL

As mentioned in the introduction, there is a vast literature on the subject of the spectral data of transfer operators when the potential under analysis presents some degree of regularity. On the other hand, very little is known on the behaviour of the subspace of eigenfunctions when the potential is wildly irregular, such as the ones allowing phase transition. In this case, is the dimension of the eigenfunction space again the same as the one of the eigenmeasures? How these spaces change as one change a multiplicative parameter β (the temperature inverse) of the potential?

In this chapter we try to shed some light on those questions by bringing an example of an irregular potential that allows explicit calculations of the eigendata. This example is inspired on the Currie-Weiss model (or mean field model). To do so, we introduce a Ruelle operator formalism to deal with discontinuous potentials. This structure will naturally give rise to the eingenmeasures that are classically known as related to the Currie-Weiss model. See Ivan Velenik's book [25] for a general introduction to statistical mechanics and a classical description of the Currie-Weiss model, or [36] for a similar treatment, including some related systems, as the generalized Currie-Weiss models.

Despite it may look artificial to choose a discontinuous potential to analyse, we believe that this model, having strong similarities with other ferromagnetic continuous systems, can give interesting clues about how the eigendata behaves in a general ferromagnetic system presenting phase transition.

We think that the application of the theory to a potential describing a model classically known by the statistical mechanics community, like the Currie-Weiss, also strengthens the idea that dealing with bounded potentials is not just an unnecessary continuation of the theory.

Some definitions given on this chapter are limited to the minimum necessary to define the operator describing the Currie-Weiss model. Part of the presentation is also rather imprecise. The intention with the example on this chapter is to provide some intuition on the structures that will appear on a more general and formal setting ahead.

2.1 Constructing a Ruelle Operator for a Bounded Potential

The natural choice of a measurable space to define the transfer operator associated to the mean-field model is $(X, \mathscr{B}(X))$ with $X = \{-1, +1\}^{\mathbb{N}}$ and $\mathscr{B}(X)$ the Borelian sets generated by the product topology.

The term mean-field naming the model implies that each "spin" interacts only with the average value of the others. A Ruelle operator should describe the interaction of the spin in the first coordinate, x_0, with each one of the following coordinates. Since we have infinite spins and the first one should interact the same way with each of the following, the only continuous solution is to have zero interaction between each pair. This is a zero potential, which obviously does not describe the mean-field model. To choose a potential capable of describing it, we propose the discontinuous family of functions $\beta f : X \to \mathbb{R}$ given by

$$\beta f(x) := \beta x_1 \limsup_{N\to\infty} \left[\frac{1}{N} \sum_{k=2}^{N+1} x_k \right], \tag{2.1}$$

where β is a positive multiplicative parameter which also indexes the elements in the family. We will omit the parameter β when possible. f has the desired property, since the function is a multiplication of x_0 by the average o the subsequent x_k.

Note that f indeed defines a discontinuous function. Suppose, on the contrary, that f is continuous. Then, for each $\epsilon > 0$, there is a δ sufficiently small such that $d(x,y) < \delta \implies |f(x) - f(y)| < \epsilon$. Rephrasing it, there is an $N \in \mathbb{N}$ sufficiently large such that $y \in [x_1 ... x_N] \implies |f(x) - f(y)| < \epsilon$. On the other hand, we can take two points $x, y \in X$ arbitrarily close, i. e., $x = (x_1, ..., x_N, -1, -1, ...)$ and $y = (x_1, ..., x_N, +1, +1, ...)$ with

$d(x,y) = 2^{-N-1}$, but with different, and independent of N, values of f, $|f(x) - f(y)| = 2\beta$. From this we can conclude that f is not a continuous function with respect to this metric.

If one defines $\mathscr{L}$ in the usual way for the potential βf, as a map with domain $C(X,\mathbb{R})$, then $\mathscr{L}(C(X,\mathbb{R})) \notin C(X,\mathbb{R})$. To see this, define $\alpha : X \to \mathbb{R}$ as

$$\alpha(x) = \limsup_N \sum_{k=1}^{N} x_k/N. \tag{2.2}$$

Then, f can be written as $f(x) = x_1\alpha(x)$. As an example, take $\varphi = \mathbb{1}$ to calculate

$$(\mathscr{L}\mathbb{1})(x) = \beta \sum_{x_1\in\{-1,+1\}} \exp(x_1\alpha(x)) = 2\beta\cosh(\alpha(x)).$$

Note that α is a tail function, that is, it is independent of any finite set of coordinates. Since it is also non-constant, it is discontinuous. For the same reason, $\mathscr{L}\mathbb{1} = 2\cosh\alpha$ is discontinuous. Thus, $\mathscr{L}$ does not define an operator on $C(X,\mathbb{R})$.

With this in mind, we will search for a larger space where an extension of $\mathscr{L}$ will act as an operator. The chosen space will be $L^1(\nu) := L^1(X,\mathscr{B}(X),\nu)$, the space of ν-equivalence classes of ν integrable functions, for a measure ν analogous to a maximal eigenmeasure of $\mathscr{L}^*$. We use the notation $M(\mathscr{B}(X))$ for the space of $\mathscr{B}(X)$-measurable real functions on X and $\mathcal{L}^1(X,\mathscr{B}(X),\mu)$ the space of $M(\mathscr{B}(X))$ functions which are also μ integrable.

First, lets check for which measures the map $\mathbb{L}_{\beta f} : \mathcal{L}^1(X,\mathscr{B}(X),\mu) \to M(\mathscr{B}(X))$ given by

$$(\mathbb{L}_{\beta f}u)(x) = \sum_{a\in\{-1,+1\}} \exp[\beta f(ax)]u(ax), \tag{2.3}$$

for all $x \in X$ and $u \in \mathcal{L}^1(X,\mathscr{B}(X),\mu)$, is in fact an operator in $\mathcal{L}^1(X,\mathscr{B}(X),\mu)$ and preserves μ-equivalence classes. When this is the case, we can say, incurring in a small abuse of notation, that $\mathbb{L}_{\beta f}$ defines an operator in $L^1(X,\mathscr{B}(X),\mu)$.

So, back to the characteristics of f, despite discontinuous, it is a measurable function. To show this, note that functions depending on a finite set of coordinates are continuous, therefore measurable. Thus, for every natural number N, defining $f_N(x) = x_1\sum_{k=2}^{N}[x_k/N]$, f_N is continuous (so measurable in $\mathscr{B}(X)$). It is easy to see that $f = \limsup_N f_N$. Since f is a limit superior of a sequence of measurable functions, it is also measurable. So is $\mathbb{L}_{\beta f}u$ given by 2.3.

Now, choosing a probability measure μ on $(X, \mathscr{B}(X))$, we suppose there is a constant $K > 0$ such that, for every bounded function g, the following equation holds

$$\int_{x \in X} \sum_{a \in \{-1,+1\}} g(ax) d\mu(x) \leqslant K \int_{x \in X} g(x) d\mu(x).$$

If this is the case for our measure μ, we have

$$\|\mathbb{L}_{\beta f} g\|_{L^1(\mu)} = \int_{x \in X} \sum_{a \in \{-1,+1\}} \exp[\beta f(ax)] g(ax) d\mu(x) \leqslant N \exp(\beta \|f\|_\infty) \|g\|_{L^1(\mu)}.$$

Which means that $\mathbb{L}_{\beta f}$ is a bounded linear operator on the space of the bounded functions endowed with the $L^1(\mu)$ norm. Since the bounded functions are dense in the $L^1(\mu)$ space (with the $L^1(\mu)$ norm), $\mathbb{L}_{\beta f}$ can be seen as an also linear bounded operator acting on $L^1(\mu)$, which we denote by the same symbol. Note that the inequality above also shows that $\mathbb{L}_{\beta f}$ preserves μ-equivalence classes as, taking two functions in the same class, g_1 and g_2, and substituting $g = g_1 - g_2$ on that inequality, one have $\|\mathbb{L}_{\beta f} g_1 - \mathbb{L}_{\beta f} g_2\|_{L^1(\mu)} = 0$.

So, the operator $\mathbb{L}_{\beta f}$ defined as above is a well posed bounded linear operator acting on $L^1(\mu)$ for every fixed probability measure μ on $(X, \mathscr{B}(X))$. But which measures μ would have interesting properties?

2.2 Currie-Weiss Eigenmeasures

A theoretical barrier arising from trying to choose a suitable "eigenmeasure" for a discontinuous potential is that $\mathscr{L}^*$ does not define an operator on the measures space. Indeed, if $\mathscr{L} := \mathbb{L}_{\beta f}|_{C(X,\mathbb{R})}$ was a well defined operator on $C(X, \mathbb{R})$, it would be easy to choose an eigenvector of $\mathscr{L}^*$, but it is not the case.

To construct an alternative definition of eigenmeasure, observe that, if $\psi \in C(X, \mathbb{R})$ with $\rho(\mathscr{L}_\psi) = 1$, it is always possible to find at least one probability measure for which $\mathscr{L}_\psi^* \nu = \nu$. We can find the demonstration of that fact in [16], for example. Then, writing $\langle \cdot, \cdot \rangle$ for the duality between $C(X, \mathbb{R})$ (right side) and its dual, the Radon Measures space (left side), we have for the case of a continuous potential,

$$\langle \nu, \mathscr{L}_\psi \varphi \rangle = \langle \mathscr{L}_\psi^* \nu, \varphi \rangle = \langle \nu, \varphi \rangle, \quad \forall \varphi \in C(X, \mathbb{R}).$$

If there is a well defined extension of $\mathscr{L}_\psi$ to $L^1(\nu)$, that we will denote by $\mathbb{L}_\psi$, then the expression above can be rewritten using the notation $\langle\cdot,\cdot\rangle_\nu$ for the duality action of $L^\infty(\nu)$ (left) on $L^1(\nu)$ (right) and remembering that $\langle\nu,\varphi\rangle = \langle\mathbb{1},\varphi\rangle_\nu$, we have

$$\begin{aligned}\langle\mathbb{L}_\psi^*\mathbb{1},\varphi\rangle_\nu =&\langle\mathbb{1},\mathbb{L}_\psi\varphi\rangle_\nu = \langle\mathbb{1},\mathscr{L}_\psi\varphi\rangle_\nu\\ =&\langle\nu,\mathscr{L}_\psi\varphi\rangle = \langle\nu,\varphi\rangle = \langle\mathbb{1},\varphi\rangle_\nu, \quad \forall\varphi\in C(X,\mathbb{R}).\end{aligned}$$

And supposing once more that the continuous functions $C(X,\mathbb{R})$ are dense in $L^1(\nu)$, then we have $\mathbb{L}^*{}_\psi\mathbb{1} = \mathbb{1}$ ν-a.e. . The conditions for the existence of the extension $\mathbb{L}_\psi$ and density of the continuous functions in $L^1(\nu)$ will be detailed on a much general setting on the following chapter. For now, suppose they hold.

Thus, the reciprocal statement is also true. Take a measure ν for which

$$\mathbb{L}_\psi^*\mathbb{1} = \mathbb{1} \quad \mu-\text{a.e.} \tag{2.4}$$

In this case, rewriting the duality in the same way,

$$\begin{aligned}\langle\mu,\varphi\rangle =&\langle\mathbb{1},\varphi\rangle_\mu = \langle\mathbb{L}_\psi^*\mathbb{1},\varphi\rangle_\mu\\ =&\langle\mathbb{1},\mathbb{L}_\psi\varphi\rangle_\mu = \langle\mathbb{1},\mathscr{L}_\psi\varphi\rangle_\mu = \langle\mu,\mathscr{L}_\psi\varphi\rangle \quad \forall\varphi\in C(X,\mathbb{R})\end{aligned}$$

also holds.

So, for continuous potentials,

$$\mathscr{L}_\psi^*\nu = \nu \Leftrightarrow \mathbb{L}_\psi^*\mathbb{1} = \mathbb{1} \quad \nu - a.e.$$

This equivalence motivates the following definition.

Definition 2.2.1 (Generalized Eigenmeasure - for bounded potentials)**.** *Let $X = E^{\mathbb{N}}$ with $E = \{a_1,...,a_N\}$ a finite set and $g: X\to\mathbb{R}$ a bounded $\mathscr{B}(X)$-measurable function, we say that μ is a generalized eigenmeasure associated to the potential g if the operator $\mathbb{L}_g: L^1(\mu)\to L^1(\mu)$ which acts on the integrable functions as L_g below,*

$$(L_g\bar{u})(x) = \sum_{a\in E}\exp[f(ax)]\bar{u}(ax) \quad \forall x\in X, \bar{u}\in\mathcal{L}^1(\mu),$$

satisfies $\mathbb{L}_g^*\mathbb{1} = \|\mathbb{L}_g\|_{op}\mathbb{1}$ μ*-a.e.* .

From the previous definition, supposing that μ is a generalized eigenmeasure associated with g, as $\mathbb{L}_g$ is a positive operator, we have

$$\begin{aligned}\|\mathbb{L}_g\|_{op} &:= \sup_{\substack{u\in X\\ \|u\|_1=1}} \|\mathbb{L}_g u\|_1 = \sup_{\substack{u\in X\\ \|u\|_1=1}} \langle \mathbb{1}, |\mathbb{L}_g u|\rangle_\mu = \sup_{\substack{u\in X\\ \|u\|_1=1}} \langle \mathbb{1}, \mathbb{L}_g u\rangle_\mu \\ &= \sup_{\substack{u\in X\\ \|u\|_1=1}} \langle \mathbb{L}_g^*\mathbb{1}, u\rangle_\mu = \sup_{\substack{u\in X\\ \|u\|_1=1}} \langle \|\mathbb{L}_g\|_{op}\mathbb{1}, u\rangle_\mu = \|\mathbb{L}_g\|_{op} \sup_{\substack{u\in X\\ \|u\|_1=1}} \|u\|_1 = \|\mathbb{L}_g\|_{op}.\end{aligned}$$

The third equality above follows from the positivity of $\mathbb{L}_g$. So, there is no danger of inconsistency by defining $\mathbb{1}$ as the maximal eigenvector of $\mathbb{L}_g^*$ associated with the eigenvalue $\|\mathbb{L}_g\|_{op}$.

With Definition 2.2.1, we can check if the measures that are classically known to be associated to the Currie-Weiss model are also generalized eigenmeasures. The traditional way to get to these measures is to take a weak limit of a sequence of measures. A very good statistical mechanical introduction to the Currie-Weiss model where that weak limit is taken can be found in Chapter 2 of Ivan Velenik's book [25]. Another presentation which includes a wide variety of related models is found in [36]. But, in our case, we are interested only on checking that these measures are compatible with the formalism developed here.

The measures that arise from a statistical mechanical approach to the Currie-Weiss model are product measures where each coordinate is described by a Bernoulli distribution independent of each other and with same parameter γ. Thus, for each coordinate x_k, we have

$$\mu_\gamma(x_k = +1) = p; \quad \mu_\gamma(x_k = -1) = 1 - p; \quad \gamma = 2p - 1.$$

Which means that γ is taken in a way that $\mathbb{E}_{\mu_\gamma}[x_k] = \gamma$, for every natural number k. More generally, μ_γ can be described by the probability of the cylinders, as below

$$\mu_\gamma([x_0...x_N]) = p^{\#\{k:x_k=+1\}}(1-p)^{\#\{k:x_k=-1\}}.$$

Since we defined $\alpha(x) := \limsup_N[(\sum_{k=0}^N x_k)/N] = \alpha(ax)$, by the Law of Large Numbers for independent and identically distributed variables, $\alpha = \gamma$, μ_γ-a.e.

The aim of the rest of this section is to show that, for every fixed β, we can properly choose γ in a way that μ_γ is an eigenmeasure of $\mathbb{L}_{\beta f}$.

Next, we compute the operator norm of $L_{\beta f}$. By definition

$$\|\mathbb{L}_{\beta f}\| = \int_X \sum_{a\in\{-1,1\}} \exp(\beta f(ax))\, d\mu_\gamma(x) = \int_X \sum_{a\in\{-1,1\}} \exp(\beta\, a\, \alpha(ax))\, d\mu_\gamma(x)$$

And we remember that, for any $a \in \{-1,1\}$ we have $\alpha(ax) = \alpha(x) = \gamma$, $\mu - \gamma$-a.e, therefore

$$\|\mathbb{L}_{\beta f}\| = \int_X e^{-\beta\gamma} + e^{\beta\gamma}\, d\mu_\gamma(x) = 2\cosh(\beta\gamma).$$

Thus far the parameter γ is free, but in order to μ_γ to be a generalized conformal measure for βf the equality $\langle \mathbb{1}, \mathbb{L}_{\beta f} u\rangle_{\mu_\gamma} = 2\cosh(\beta\gamma)\langle \mathbb{1}, u\rangle_{\mu_\gamma}$ must hold for any $u \in L^1(\mu_\gamma)$. For this, it is enough that, $\langle \mathbb{1}, \mathbb{L}_{\beta f}\mathbb{1}_B\rangle_{\mu_\gamma} = \cosh(\beta\gamma)\langle \mathbb{1}, \mathbb{1}_B\rangle_{\mu_\gamma}$ for any indicator function $\mathbb{1}_B$, where $B \in \mathscr{B}(X)$. By computing the left hand side of the last equation we get that

$$\begin{aligned}
\int_X \mathbb{L}_{\beta f}\mathbb{1}_B\, d\mu_\gamma &= \int_X \sum_{a\in\{-1,+1\}} \exp(a\beta m(ax))\mathbb{1}_B(ax)\, d\mu_\gamma(x) \\
&= \int_X \exp(-\beta\gamma)\mathbb{1}_B(-1x) + \exp(\beta\gamma)\mathbb{1}_B(+1x)\, d\mu_\gamma(x) \\
&= \frac{e^{-\beta\gamma}\mu_\gamma(B\cap[-1])}{\mu_\gamma([-1])} + \frac{e^{\beta\gamma}\mu_\gamma(B\cap[+1])}{\mu_\gamma([+1])}.
\end{aligned}$$

By taking $B = [+1]$ and next $B = [-1]$, in the previous identity, and using $\langle \mathbb{1}, \mathbb{L}_{\beta f}\mathbb{1}_B\rangle_{\mu_\gamma} = \cosh(\beta\gamma)\langle \mathbb{1}, \mathbb{1}_B\rangle_{\mu_\gamma}$, we see that the following relations must be satisfied

$$e^{\pm\beta\gamma} = \int_X \mathbb{L}_{\beta f}\mathbb{1}_{[\pm 1]} d\mu_\gamma = 2\cosh(\beta\gamma)\int_X \mathbb{1}_{[\pm1]} d\mu_\gamma = 2\cosh(\beta\gamma)\mu_\gamma([\pm 1]).$$

Since $p = \mu_\beta([+1]) = e^{\beta\gamma}/2\cosh(\beta\gamma)$ and $1 - p = \mu_\beta([-1]) = e^{-\beta\gamma}/2\cosh(\beta\gamma)$ we finally get that γ has to be a solution to the following equation

$$\gamma = 2p - 1 = \frac{e^{\beta\gamma} - e^{-\beta\gamma}}{2\cosh(\beta\gamma)} = \tanh(\beta\gamma).$$

The equation $\gamma = \tanh(\beta\gamma)$ has either one or three solutions, depending on the value of β. If $0 < \beta \leqslant 1$ then $\gamma = 0$ is the unique solution. Otherwise, if $\beta > 1$ then there is some $\gamma(\beta) \in (0,1)$ such that $-\gamma(\beta), 0$ and $\gamma(\beta)$ are all the solutions to the equation.

Fixed a γ that satisfies $\gamma = \tanh(\beta\gamma)$, we can go back to the calculations of $\langle \mathbb{1}, \mathbb{L}_{\beta\gamma}\mathbb{1}_B\rangle_{\mu_\gamma}$ and replace the computed values of $\mu([-1])$ and $\mu([+1])$ with the following result

$$\begin{aligned}\int_X \mathbb{L}_{\beta f}\mathbb{1}_B d\mu_\gamma &= \frac{e^{-\beta\gamma}\mu_\gamma(B\cap[-1])}{\mu_\gamma([-1])} + \frac{e^{\beta\gamma}\mu_\gamma(B\cap[+1])}{\mu_\gamma([+1])}\\ &= \cosh(\beta\gamma)[\mu_\gamma(B\cap[-1]) + \mu_\gamma(B\cap[+1])] = \cosh(\beta\gamma)\mu_\gamma(B)\\ &= \cosh(\beta\gamma)\int_X \mathbb{1}_B d\mu_\gamma.\end{aligned}$$

I.e., $\langle \mathbb{1}, \mathbb{L}_{\beta f}\mathbb{1}_B\rangle_{\mu_\gamma} = \langle \mathbb{1}, \mathbb{1}_B\rangle_{\mu_\gamma}$ and μ_γ is indeed a generalized conformal measure to the potential βf, result which we consolidate on the following proposition.

Proposition 2.2.2 (Generalized Eigenmeasures – Curie-Weiss Model)**.** *Let βf be the potential defined by* (2.1) *and for each $\gamma \in (-1,1)$ let μ_γ be a Bernoulli measure as defined above. Then*

- *μ_γ is a generalized conformal measure, if and only if, γ is a solution of the equation $\gamma = \tanh(\beta\gamma)$;*
- *for any solution γ of the above equation, $2\cosh(\beta\gamma)$ is an eigenvalue of $\mathbb{L}^*_{\beta f}$.*

By Proposition 2.2.2 if $0 < \beta < 1$, then μ_0 the symmetric Bernoulli measure, with parameter $p = 1/2$ is a generalized conformal measure associated to the eigenvalue 2. But on the other hand, if $\beta > 1$, this measure still is an eigenmeasure associated to the eigenvalue 2, but now there are two other Bernoulli measures $\mu_{\pm\gamma(\beta)}$ associate to a strictly bigger eigenvalue $2\cosh(\beta\gamma(\beta))$.

Note that, even though μ_0 still is an eigenmeasure, it is associated to a smaller eigenvalue. So, it is not a maximal eigenmeasure for $\beta > 1$.

2.3 Multidimensional Eigenspace

Now let us move the discussion to the eigenfunctions of $\mathbb{L}_{\beta f}$. We first observe that, for any fixed $\beta > 0$, the operator $\mathbb{L}_{\beta f} : L^1(\mu_0) \to L^1(\mu_0)$, has the constant (modulo μ_0) function as eigenfunction associated to the eigenvalue $\lambda = 2$, that is, $\mathbb{L}_{\beta f}\mathbb{1} = 2\mathbb{1}$.

But for $\beta > 1$, which is above the critical point of the original Curie-Weiss model, we can see more interesting phenomena, as for example, multidimensional eigenspaces. Since β is fixed, in what follows we will write $\mu_\pm \equiv \mu_{\pm\gamma(\beta)}$ to ligthen the notation. Now we consider the operator $\mathbb{L}_{\beta f} : L^1(\nu(t)) \to L^1(\nu(t))$, where $\nu(t) \equiv t\mu_+ + (t-1)\mu_-$ is a nontrivial convex combination of $\mu_{\pm}$. The measurable sets $B_{+} = \{x \in X : \alpha(x) = +\gamma(\beta)\}$ and $B_{-} = \{x \in X : \alpha(x) = -\gamma(\beta)\}$ are chosen in such way they form a measurable partition of the space $X = B_{+} \cup B_{-} \cup N$ up to a $\nu(t)$-null set N. Furthermore, they are disjoint and $\mu_{+}(B_{+}) = 1$ and $\mu_{-}(B_{-}) = 1$. This means that in a certain way these sets separate μ_{+} and μ_{-}. Note that they are also invariant by the addition of a symbol since $\alpha(ax) = \alpha(x)$, for every $a \in E, x \in X$.

Regarding the eigenspace of $\mathbb{L}_{\beta f}$, it turns out that the characteristic functions $\mathbb{1}_{B_{\pm}}$ are also eigenfunctions, more precisely, $\mathbb{L}_{\beta f}\mathbb{1}_{B_{+}} = 2\cosh(\beta\gamma(\beta))\mathbb{1}_{B_{+}}$ and $\mathbb{L}_{\beta f}\mathbb{1}_{B_{-}} = 2\cosh(\beta\gamma(\beta))\mathbb{1}_{B_{-}}$. To see this, remember that in every point $x \in B_{+}$, $\alpha(x) = \gamma(\beta)$, and so

$$L\mathbb{1}_{B_{+}}(x) = \sum_{a \in \{+1,-1\}} \mathbb{1}_{B_{+}}(x)\exp(\beta a\alpha(ax)) = (2\cosh\beta\gamma)\mathbb{1}_{B_{+}}(x).$$

The same is true for B_{-}, since $\alpha(B_{-}) = -\gamma(\beta)$.

We summarize this discussion with the following theorem.

Theorem 2.3.1. *Let $\beta > 0$, f be the potential given by (2.1), μ_{γ} the Bernoulli measure given by (2.2) and $\gamma(\beta)$ the positive solution of the equation $\gamma = \tanh(\beta\gamma)$, then the following hold:*

1. *the operator $\mathbb{L}_{\beta f} : L^1(\mu_0) \to L^1(\mu_0)$ has norm $\|\mathbb{L}_{\beta f}\| = 2$ and the symmetric Bernoulli measure μ_0 is a generalized eigenmeasure, associated to βf, in the sense of Definition 2.2.1.*

2. *for $0 < \beta \leqslant 1$:*

   (a) *the eigenspace of $\mathbb{L}_{\beta f}$, associated to the eigenvalue 2, is one-dimensional and is spanned by $\mathbb{1}$;*

3. *for $\beta > 1$:*

   (a) *the operator $\mathbb{L}_{\beta f} : L^1(\nu) \to L^1(\nu)$, where $\nu = t\mu_{\gamma(\beta)} + (t-1)\mu_{-\gamma(\beta)}$ and $t \in (0,1)$, has operator norm $\|\mathbb{L}_{\beta f}\| = 2\cosh(\beta\gamma(\beta)) > 2$ and ν is a generalized eigenmeasure associated to βf, in the sense of Definition 2.2.1.*

   (b) *for any non-trivial convex combination $\nu = t\mu_{\gamma(\beta)} + (t-1)\mu_{-\gamma(\beta)}$, the eigenspace of $\mathbb{L}_{\beta f}$ is two-dimensional and is spanned by $\{\mathbb{1}_{B^{+}}, \mathbb{1}_{B^{-}}\}$, where $B_{\pm} = \{x \in X : \limsup[(\sum_{k=1}^{N} x_k)/N] = \pm\gamma(\beta)\}$.*

This give us an example of a (discontinuous) potential for which the eigenspace associated to its Ruelle operator on $L^1(\nu)$ has dimension bigger then one. We will see on Chapter 6 this is characteristic of systems exhibiting phase transition and that, despite they are not always equal, the dimension of the maximal eigenmesures limit the dimension of the proper eigenspace. This means that, potentials with similar behavior could happen only if we have phase transition, in the sense of multiple extreme points in $\mathscr{G}^*$. This is the case in our example for $\beta > 1$ if we interpret μ_+ and μ_- as our extreme points.

Another relevant structure that appears here and will play an important role on the general case developed in Chapter 6 are the sets B_+ and B_-. They have μ_+ and μ_- full measure, respectively and, at the same time, they are the support of the eigenfunctions $\mathbb{1}_{B_+}$ and $\mathbb{1}_{B_-}$. We will see ahead that this is not specific to this system, but a general property of the L^1 extension of Ruelle operator.

CHAPTER 3

THE RUELLE OPERATOR ON L^1

As already mentioned, the classical Ruelle operator acts on the continuous functions defined on a metric space X. The expression for the operator is

$$\mathscr{L}_\varphi u := \int_E \exp(\varphi(ax))u(ax)\,dp(a), \quad \forall u \in \mathcal{C}, \tag{3.1}$$

with $\mathcal{C} = C(X,\mathbb{R})$. Keeping fixed expression 3.1, one can ask when this formula gives a well defined operator on $L^1(\mu)$, for μ an arbitrary measure on $(X,\mathscr{B}(X))$. This chapter deals with this question. Restating it, under which conditions on φ and μ expression 3.1 with $\mathcal{C} = L^1(\mu)$ gives an operator on $L^1(\mu)$?

Theorem 3.1.2 provides conditions on μ and p, the a priori measure implicit in 3.1, under which it is a well defined operator. For a matter of clarity, in this dissertation the symbol $\mathscr{L}_\varphi$ will be used for the Ruelle operator defined on $C(X,\mathbb{R})$ and $\mathbb{L}_\varphi$ for the analogous operator defined on $L^1(X,\mathscr{B}(X),\mu)$. In Theorem 3.1.2, the potential f can be seen as a bounded function on $(X,\mathscr{B}(X))$, opposed to the classical operator on $C(X,\mathbb{R})$, where φ has to be also continuous. The possibility to choose a discontinuous potential will be useful to formalize the rather incomplete description we gave to the Currie-Weiss model in the last chapter.

The following question to be dealt with refers to when μ is chosen to be an eigenmeasure of the adjoint of the Ruelle operator associated to its maximal eigenvalue, i. e., take a continuous potential φ and $\mu = \nu_\varphi$, such that $\mathscr{L}_\varphi^* \nu_\varphi = \rho(\mathscr{L}_\varphi)\nu_\varphi$, where $\rho(\mathscr{L}_\varphi)$ is the spectral radius of $\mathscr{L}_\varphi^*$. Theorem 3.1.5 shows that, in this case, $\mathbb{L}_\varphi$ always is a well defined operator on $L^1(\nu_\varphi)$.

The last question addressed on Section 3.1 is about the relation between $\mathscr{L}_\varphi$ and $\mathbb{L}_\varphi : L^1(\nu_\varphi) \to L^1(\nu_\varphi)$. In particular, when $\mathbb{L}_\varphi$ is the extension of $\mathscr{L}_\varphi$ to $L^1(\nu_\varphi)$? Since Theorem 3.1.5 gives the good definition of $\mathbb{L}_\varphi$, it only lacks to show that two different continuous functions are represented by different elements of $L^1(\nu_\varphi)$. On the contrary, it would not be possible for $\mathbb{L}_\varphi$ to be the $\mathscr{L}_\varphi$ extension. A sufficient and necessary condition for every two different continuous functions be represented by different equivalence classes in $L^1(\nu_\varphi)$ is ν_φ having the entire space X as its support (ν_φ is fully supported). Theorem 3.1.8 attests that it is sufficient to have an a priori measure p fully supported on E to end with an eigenmeasure ν_φ fully supported on X.

It is possible to conclude that, if p has full support on E, fixed an eigenmeasure ν_φ of $\mathscr{L}_\varphi^*$, $\mathbb{L}_\varphi$ is the extension to $L^1(\nu_\varphi)$ of $\mathscr{L}_\varphi$. That result is stated in Theorem 3.1.9.

This chapter resulted from an attempt to prove Theorem 3.1.9. A version of this theorem was already known to hold by the scientific community. Indeed, in [12] Cioletti and Lopes show that such an operator exists if it is given that the conformal measure is fully supported. In [16] the authors study the double adjoint of the operator and take conclusions related to its spectrum. The novelties in our approach, expressed in this chapter, is to show that an a fully supported a priori measure implies full support on the conformal measure; to give an explicit formula for the Ruelle operator in $\mathcal{L}^1(X, \mathscr{B}(X), \nu_\varphi)$, the space of ν_φ-integrable functions; and to show that that expression preserves classes in $L^1(\nu)$. Alternatively stated, what is shown is that, under the condition of a fully supported a priori measure, the operator proved to exist in [12] is compatible with the dynamic implicit in the classical formula 3.1.

In Section 3.2 we study how to define a Ruelle-like operator associated to a bounded potential f. We define the generalized conformal measures, that will play the role of the conformal measures for bounded potentials. We also show that these measures are fully supported in the space.

In this chapter E denotes a compact metric space, $X = E^{\mathbb{N}}$ will be equipped with the usual metric, and $f \in B(X)$ is a bounded function. When φ is continuous, μ_φ is again an eigenmeasure of the dual of the Ruelle operator $\mathscr{L}_\varphi$ associated to the potential φ and a priori measure p.

3.1 Extension of Ruelle operator to L^1

To define an operator in $L^1(\nu)$ we will need the following compatibility hypothesis.

Hypothesis H1. Let $\mu \in \mathscr{M}_1(X)$ be an arbitrary Borel probability measure on X. By using the product structure of X we can also consider the product measure $p \times \mu$ as an element of $\mathscr{M}_1(X)$, which is defined on the cylinder sets in a natural way. We will say that a pair (p, μ), where $\mu \in \mathscr{M}_1(X)$ and $p \in \mathscr{M}_1(E)$, satisfies the hypothesis (H1) if

$$\text{(H1)} \qquad \exists K > 0 \text{ such that } \quad (p \times \mu)(B) \leq K\mu(B), \ \forall B \in \mathscr{B}(X).$$

And when this hypothesis holds we can define a mapping on a dense subset of the μ-integrable functions with the same formula as the Ruelle operator.

Proposition 3.1.1. *Let (p, μ) be a pair satisfying hypothesis* (H1) *and f a bounded potential on X, then there is a positive linear transformation $L_f : \mathrm{dom}(L_f) \subset \mathcal{L}^1(\mu) \to \mathcal{L}^1(\mu)$ given by*

$$L_f\bar{u}(x) := \int_E \exp(f(ax))\bar{u}(ax)\, dp(a), \quad \forall x \in X, \bar{u} \in \mathrm{dom}(L_f) \tag{3.2}$$

and moreover

$$\int_X |L_f\bar{u}|\, d\nu \leq Ke^{\|f\|_\infty} \int_X |\bar{u}|\, d\nu, \quad \forall \bar{u} \in \mathrm{dom}(L_f). \tag{3.3}$$

Proof. To lighten the notation take $\mathscr{D} := \mathrm{dom}(L_f)$. We begin by proving that $L_f(\mathscr{D}) \subseteq \mathcal{L}^1(\mu)$. Note that that integral $L_f\bar{u}$ exists for every bounded function $\bar{u}$ and, in particular, for every continuous function. So we have that $\mathscr{D}$ is a dense subset of $\mathcal{L}^1(\mu)$ in the $L^1(\mu)$ norm. Note also, that for every element $\bar{u} \in \mathscr{D}$, $L_f\bar{u}$ is a $\mathscr{B}(X)$-measurable function, by definition of the Lebesgue integral.

The work here is to show that, taken an arbitrary function $\bar{u} \in \mathscr{D}$, its image $L_f\bar{u}$ is also μ-integrable and that L_f preserves μ-a.e. classes of equivalence.

Let $(p \times \mu) : \mathscr{B}(X) \to [0, 1]$ be the product measure (where p is computed on the first coordinate and μ on the following vector). By hypothesis H1, there is $K > 0$ such that $(p \times \mu)(B) \leqslant K\mu(B)$ holds for every $B \in \mathscr{B}(X)$. Then, taken a simple function $S = \sum_{n=1}^{N} a_n \mathbb{1}_{B_n}$,

$$
\begin{aligned}
\int_X |L_f 1_B| d\mu =& \int_X \int_E \exp(f(ax)) 1_B(ax) dp(a) d\mu(x) \\
\leqslant& e^{\|f\|_\infty} \int_X \int_E 1_B(ax) dp(a) d\mu(x) \\
=& e^{\|f\|_\infty} (p \times \mu)(B) \\
\leqslant& K e^{\|f\|_\infty} \mu(B) = K e^{\|f\|_\infty} \int_X |1_B| d\mu.
\end{aligned}
$$

The above inequality shows that, for any simple function S, $L_f S$ is a μ-integrable function. Since it holds with the same constant $K e^{\|f\|_\infty}$, by density of simple functions on the space of μ-integrable functions, it follows, that for every $\bar{u} \in \mathscr{D}$,

$$
\int_X |L_f \bar{u}| d\mu \leqslant K e^{\|f\|_\infty} \int_X |\bar{u}| d\mu
$$

holds. □

Theorem 3.1.2 (Transfer operator on $L^1(\mu)$)**.** *Let (p, μ) be a pair satisfying hypothesis* (H1)*, f a bounded potential on X and L_f the map given by Proposition 3.1.1, then*

$$
\mathbb{L}_f[\bar{u}]_\mu := [L_f \bar{u}]_\mu, \quad \forall \bar{u} \in \mathscr{D} = dom(L_f)
$$

defines by density a continuous linear operator $\mathbb{L}_f : L^1(\mu) \to L^1(\mu)$.

Proof. Using the inequality 3.3 we see that, for every $\bar{u} \in \mathscr{D}$, $L_f \bar{u} \in \mathcal{L}^1(\mu)$. Then $\mathbb{L}_f[\bar{u}]_\mu := [L_f \bar{u}]_\mu \in L^1(\mu)$. Again by 3.3, L_f preserves equivalence classes in $L^1(\mu)$ because, taking two μ-integrable functions $\bar{u}$ and $\hat{u}$ in $\mathscr{D}$ with $\|\bar{u} - \hat{u}\|_{L^1(\mu)} = 0$,

$$
\int_X |L_f \bar{u} - L_f \hat{u}| d\mu = \int_X |L_f(\bar{u} - \hat{u})| d\mu \leqslant K e^{\|f\|_\infty} \int_X |\bar{u} - \hat{u}| d\mu = 0.
$$

So $\mathbb{L}_f$ is well-defined.

Once more by inequality 3.3, $\|\mathbb{L}_f\|_{op} \leqslant K e^{\|f\|_\infty}$.

□

Remark 3.1.3 (Tranfer operator on L^q)**.** *An analogous operator $\mathbb{L}$ can be defined on L^q. To see this, take an arbitrary element $u \in L^\infty(\mu)$ and $\bar{u} \in \mathscr{D}$ a bounded element in the*

μ-equivalence class u.

$$\|\mathbb{L}_\varphi u\|_\infty = \left\| \int_E \exp(\varphi(ax))\bar{u}(ax)dp(a) \right\|_\infty \leqslant e^{\|f\|_\infty} \|u\|_\infty.$$

So $\mathbb{L}_\varphi$ preserves $L^\infty(\mu)$. We can now apply the Riesz-Thorin Theorem to conclude that $\mathbb{L}_\varphi$ preserves L^q for $1 \leqslant q \leqslant \infty$.

Remark 3.1.4 (The Currie-Weiss eigenmeasures satisfy H1)**.** *Note that in the example given in Chapter 2, every eigenmeasure satisfies H1, then the operator on $L^1(\mu_\gamma)$ is well defined for every γ.*

An important case to be analyzed is when $f = \varphi \in C(X, \mathbb{R})$ is a continuous function. Since X is compact, φ is bounded. To use Theorem 3.1.2 to show that $\mathbb{L}_\varphi$ is a well defined operator on $L^q(\nu)$ it is sufficient to prove that ν satisfies H1. This can be stated as below.

Proposition 3.1.5. *Let $\mu = \nu \in \mathscr{G}^*$ be a conformal measure and p the a priori measure used to define $\mathscr{L}_\varphi$. Then the pair (p, ν) satisfies hypothesis* (H1).

Proof. The goal is to prove the inequality in (H1) for every Borel set $B \in \mathscr{B}(X)$. We first show its validity for a family of rectangles

$$\mathscr{R} = \{U \times V : U \subseteq E \text{ and } V \subseteq X \text{ are open sets}\}.$$

Let $B \in \mathscr{R}$ of the form $B = U \times V$ (a rectangle with open sides). Since U is open in E, there is an increasing sequence of continuous functions $\psi_n : E \to [0, 1]$ such that, for every $n \in \mathbb{N}$, $\psi_n \uparrow \mathbb{1}_U$ pointwisely and, therefore, in $L^1(p)$. Similarly, there is an increasing sequence of continuous functions $\phi_n : X \to [0, 1]$ (Urysohn functions) such that $\phi_n \uparrow \mathbb{1}_V$ again pointwisely and in $L^1(\nu)$. Therefore for any $x \in X$ we have that $\Psi_n(x) := \psi_n(x_1)\phi_n(\sigma(x)) \uparrow \mathbb{1}_B(x)$. Clearly $\Psi_n \in C(X)$ and we have

$$\begin{aligned}
\nu(B) &= \int_X 1_B d\nu \geqslant \int_X \Psi_n d\nu \\
&= \frac{1}{\rho(\mathscr{L}_\varphi)} \int_X \Psi_n \, d[\mathscr{L}_\varphi^* \nu] = \frac{1}{\rho(\mathscr{L}_\varphi)} \int_X (\mathscr{L}_\varphi \Psi_n) \, d\nu \\
&= \frac{1}{\rho(\mathscr{L}_\varphi)} \int_X \left[\int_E \exp(\varphi(ay)) \, \Psi_n(ax) \, dp(a) \right] d\nu(x) \\
&= \frac{1}{\rho(\mathscr{L}_\varphi)} \int_X \int_E \exp(\varphi(ax)) \, \psi_n(a)\phi_n(\sigma(ax)) \, dp(a) d\nu(x)
\end{aligned}$$

$$= \frac{e^{-\|f\|_\infty}}{\rho(\mathscr{L}_\varphi)} \int_E \psi_n(a)\, dp(a) \int_X \phi_n(x)\, d\nu(x).$$

Thus, taking the limit when $n \to \infty$, one can conclude that

$$\nu(B) \geqslant \frac{e^{-\|f\|_\infty}}{\rho(\mathscr{L}_\varphi)} p(U)\nu(V) = \frac{e^{-\|f\|_\infty}}{\rho(\mathscr{L}_\varphi)} (p \times \mu)(B).$$

That is, inequality in (H1) holds for any open rectangle and $K = \rho(\mathscr{L}_\varphi)e^{\|f\|_\infty}$.

Since the inequality in (H1) holds for any element of $\mathscr{R}$ (which generates the Borel sigma-algebra $\mathscr{B}(X)$), it would be natural to expect that the same should be true for every Borel set of X. This is actually true, but a careful argument is required to give a rigorous proof of this fact. In the sequel we show why this is not a completely trivial statement at least in the generality considered in this paper. □

Now we want to discuss the validity of the inequality $(p \times \nu)(B) \leqslant K\nu(B)$, for every Borel set, using its validity for the family of open rectangles, which is a subbase for the product topology.

As we will see the validity of this inequality on the open rectangles is enough, but the reason is not because this family generate the product topology and consequently $\mathscr{B}(X)$. In fact, the family of all open balls also form a subbase for the product topology on X (if E is a finite alphabet, the open balls and the cylinders sets coincides) but in a famous paper [19] Davies constructed two distinct Borel probability measures μ_1 and μ_2 on a compact metric space Y that coincides in every open ball. That is, $\mu_1(B(y,r)) = \mu_2(B(y,r))$, for all $y \in Y$ and $r > 0$, but $\mu_1 \neq \mu_2$. This result provides a direct counterexample for the following statement: if an inequality between two Borel measures holds on a subbase of the topology, then it also holds for every Borel set.

On the other hand, it is obviously true that, if the inequality holds on two disjoint sets, it also holds on their union. The issue here is, then, to cover a Borel set B with a **disjoint collection** of open rectangles which approximates the measure of this Borelian arbitrarily well. This is the spirit of the Vitali Covering Theorem, which roughly speaking states that, under suitable hypothesis, it is possible to approximate the measure of a Borelian (up to an ϵ) by the measure of a finite disjoint union of sets taken from a Vitali Covering.

Unfortunately, the Vitali Covering Theorem does not hold for every compact metric space. Indeed, again in [19], Davies constructs a compact metric space and a Borel probability measure where any disjoint collection of open balls have measure at most 1/2. This is a clear example where the Vitali Covering Theorem does not hold when the covering is taken as the family of disjoint open balls, see also [34].

The task to be undertaken on the next proposition is to show that, when the chosen underlying space is a product of two compact metric spaces, the family of rectangles with open sides can act as a Vitali Covering and, given a measurable set $B \in \mathscr{B}(X)$, its measure can be approximated arbitrarily well by the measure of a finite disjoint union in this family of rectangles. This will be enough to show that the inequality in (H1), valid initially only for the rectangles (Proposition 3.1.5), can be generalized to any Borel set.

To complete the proof we show first that the rectangles of open sides approximate the rectangles of measurable sides. Next we use that the family of finite disjoint unions of rectangles with measurable sides form an algebra and conclude by applying Carathéodory's Extension Theorem.

The following proposition summarizes what was discussed on the last paragraphs. This should be a very-well known result. We prove it here because we did not found a precise reference for this inequality.

Proposition 3.1.6. *Let E and F be two compact metric spaces and μ, ν two Borel measures on the product space $(E \times F, \mathscr{B}(E \times F))$. If $\nu(U \times V) \leqslant \mu(U \times V)$ for every open rectangle $U \times V$, then $\nu(B) \leqslant \mu(B)$ for every $B \in \mathscr{B}(E \times F)$.*

Proof. The first step towards this generalization is to approximate an arbitrary rectangle $R = C \times D$ with measurable sides, i. e., $R \in \mathscr{B}(E \times F)$ by a sequence of open rectangles.

Since every Borel measure on a metric space is regular [45], the set function $\nu(C \times \cdot)$ defines a regular measure on $(F, \mathscr{B}(F))$. Hence, for every $\epsilon = 1/n$, it is possible to find an open set $V_n \supseteq D$ such that $\nu(C \times V_n) \leqslant \nu(C \times D) + 1/n$. Again, by regularity of $\nu(\cdot \times V_n)$, it is possible to find U_n open such that $\nu(U_n \times V_n) \leqslant \nu(C \times V_n) + 1/n$. Piecing together the last two inequalities, we get that $\nu(U_n \times V_n) \leqslant \nu(C \times D) + 2/n$. This construction gives a sequence of open rectangles $(U_n \times V_n)$ which approximates $(C \times D)$ from above and it is such that $\nu(C \times D) = \lim \nu(U_n \times V_n)$.

Using the above result for open rectangles we get that

$$\nu(R) = \inf_{\substack{U \times V \subseteq X \text{ open} \\ R \subseteq U \times V}} \nu(U \times V)$$

$$\geqslant \inf_{\substack{U\times V \subseteq X \text{ open} \\ R\subseteq U\times V}} \mu(U\times V)$$

$$\geqslant \inf_{\substack{W \subseteq X \text{ open} \\ R\subseteq W}} \mu(W)$$

$$= \mu(R).$$

This means that the desired inequality holds for every mensurable rectangle $R = C \times B$.

It is clear that, if the above inequality holds separately for two disjoint measurable rectangles R_1 and R_2, it also holds for their union $R_1 \cup R_2$ and more generally for any finite pairwise disjoint union of rectangles. Recall that the family $\mathscr{C}$ of unions of pairwise disjoint mensurable rectangles forms an algebra of sets.

From the last paragraph we conclude that $\mu|_{\mathscr{C}} \leq \nu|_{\mathscr{C}}$. Therefore the outer-measures associated to them will satisfy $(\mu|_{\mathscr{C}})^* \leq (\nu|_{\mathscr{C}})^*$. Since $\mu|_{\mathscr{C}}$ and $\nu|_{\mathscr{C}}$ are countable-additive pre-measures it follows from Carathéodory's Extension Theorem that $\mu = (\mu|_{\mathscr{C}})^* \leq (\nu|_{\mathscr{C}})^* = \nu$ on $\sigma(\mathscr{C}) = \mathscr{B}(E \times F)$. □

The following lemma is an exercise in set theory showing that the family of pairwise disjoint unions of rectangles is actually an algebra, as used on the previous proof.

Lemma 3.1.7 (The family of pairwise disjoint unions of rectangles is an algebra)**.** *Let* $(E, \mathcal{F}), (F, \mathcal{G})$ *be two measurable spaces, the family* $\mathcal{C}$ *of finite unions of pairwise disjoint rectangles of measurable sides,* $R = C \times D$ *with* $C \in \mathscr{B}(E)$ $D \in \mathscr{B}(F)$*, is an algebra of sets.*

Proof. Since $\varnothing \in \mathcal{F}$, it is clear that $\varnothing = (\varnothing \times F) \in \mathcal{C}$.

Now take two generic elements $G, H \in \mathcal{C}$ given by $G = \dot\bigcup_{j\in J} R_j$, $H = \dot\bigcup_{k\in K} T_k$ with $\{R_j\}_{j\in J}$ a finite family of disjoint rectangles, and the same for $\{T_k\}_{k\in K}$. Their intersection can be written as

$$G \bigcap H = \left[\dot\bigcup_{j\in J} R_j\right] \bigcap \left[\dot\bigcup_{k\in K} T_k\right] = \dot\bigcup_{\substack{j\in J\\k\in K}} [R_j \bigcap T_k]$$

and it is easy to see that the intersection of two rectangles is also a rectangle. Than, $\mathcal{C}$ is closed under intersections.

For the matter of closure under complementation, going back to the generic $G \in \mathcal{C}$,

$$G^c = (\dot\bigcup_{j\in J} R_j)^c = \bigcap_{j\in J} R_j^c.$$

Since it was already proved that $\mathcal{C}$ is closed under intersection, it only remains to show that $R^c \in \mathcal{C}$ for an arbitrary rectangle $R = C \times D$. That complement can be expressed as

$$R^c = (C \times D)^c = (C^c \times F) \bigcup\!\!\!\cdot\, (C \times D^c).$$

Which again is a disjoint union of two rectangles, then $\mathcal{C}$ is also closed under complementation.

Those three results, $\varnothing \in \mathcal{C}$, and closure under intersections and complementation, are enough to show that $\mathcal{C}$ is an algebra of sets. □

To say something about an extension of the operator $\mathscr{L}_\varphi$ it is necessary to have a "copy" of the space $C(X,\mathbb{R})$ in $L^1(\mu)$. This is true if the measure μ is fully supported. On the contrary, two different continuous functions would be represented in $L^1(\mu)$ by the same equivalence class and there is nothing to say about a possible extension. The following proposition elucidates when this is the case for a conformal measure with respect to a continuous potential. Therefore, it will make sense under the hypothesis of the proposition to look for an extension of $\mathscr{L}_\varphi$ to $L^1(\mu_\varphi)$.

Proposition 3.1.8 (Conformal measures are fully supported)**.** *Let $X = E^{\mathbb{N}}$ endowed with the usual metric, with E a compact metric space, and $\varphi \in C(X,\mathbb{R})$. Let ν be a conformal measure with respect to φ and a priori measure p fully supported on E. Then ν is also fully supported on X.*

Proof. Take $x \in X$ and $r > 0$ to calculate $\mu(B(x,r))$. Due to the chosen metric, there are $n(r) \in \mathbb{N}$ and $R(r) \in \mathbb{R}$ such that $B_X(x,r) \supseteq B_E(x_1,R) \times \ldots \times B_E(x_{n(r)},R) \times E^{\mathbb{N}} =: B$. Ahead it will be useful to define, for $a \in E$, the continuous function $\psi_a : E \to [0,1]$ as $\psi_a(x) = \max\{1 - \frac{2}{R} d[a, B_E(a,\frac{R}{2})], 0\}$. Realize that, in addition to be continuous, ψ_a is in between the indicators of two balls centered on a, i.e., $1_{B_E(a,\frac{R}{2})} \leqslant \psi_a \leqslant 1_{B_E(a,R)}$. Another useful function in the following discussion will be $\Psi_x \in C(X,\mathbb{R})$ given by $\Psi_x(y) = \prod_{k=1}^{n(r)} \psi_{x_k}(y_k)$. Calculating the measure of the set B, one have,

$$\begin{aligned}
\nu(B) &= \int_X 1_B d\nu(y) \geqslant \int_X \Psi_x d\nu \\
&= \int_X \Psi_x d(\mathscr{L}_\varphi^*)^n \nu(y) = \int_X \left(\mathscr{L}_\varphi^n \Psi_x\right)(y) d\nu(y) \\
&= \int_X \mathscr{L}_\varphi^{n-1} \left[\int_E \exp\left(\varphi(a_1 \cdot)\right) \Psi(a_1 \cdot) dp(a_1)\right](y) d\nu(y) \\
&= \int_X \int_{E^n} \exp\left[\sum_{k=1}^n \varphi(a_k \ldots a_n y)\right] \prod_{k=1}^n \psi_k(a_k) dp^n(a) d\nu(y) \\
&= \left(\min_{x \in X}\{e^{f(x)}\}\right)^n \cdot \prod_{k=1}^n \left[\int_{E^n} \psi(a_k) dp(a_k)\right] \\
&\geqslant \prod_{k=1}^n p(B_E(x_k, R/2)) \cdot \left(\min_{x \in X}\{e^{\varphi(x)}\}\right)^n > 0.
\end{aligned}$$

The existence of a positive minimum follows from the compactness of X and the continuity of φ. $p(B_E(x_k, R/2)) > 0$ because p is fully supported on the space E. □

The following theorem summarizes all of the results in this chapter.

Theorem 3.1.9 ($\mathbb{L}_\varphi$ extends $\mathscr{L}_\varphi$)**.** *Let φ be a continuous potential defined on $X = E^{\mathbb{N}}$, where E is a compact metric space. Let $\mathscr{L}_\varphi$ be the Ruelle operator with a priori measure p. If p is fully supported on E, then, for any fixed conformal ν with respect to $\mathscr{L}_\varphi^*$:*

1. *ν is fully supported on X; and*
2. *$\mathbb{L}_\varphi : L^1(\nu) \to L^1(\nu)$ given by Theorem 3.1.2 is the extension of $\mathscr{L}_\varphi$ to $L^1(\nu)$.*

3.2 Bounded Potentials

Now we extend the idea of conformal measures in a way that it can also make sense for discontinuous bounded potentials f. In this way we formalize most of the theory implicit in Chapter 2 in a wider setting of discontinuous potentials and compact metric alphabets E. This discussion would be unnecessary for finite alphabets, since in this case the calculations would follow easily.

So, let $f : X \to [-\|f\|_\infty, +\|f\|_\infty] \subset \mathbb{R}$ be a bounded function and p an a priori measure defined on the measurable space $(E, \mathscr{B}(X))$, with E a compact metric space, as we have done on most of this text. Then, for every measure $\nu \in \mathcal{M}_1(X)$ such that $H1$ holds, we can define a transfer operator $\mathbb{L}_f$ on $L^1(\nu)$. Its representation on the finite

ν-integrable functions is given by Equation (3.2). This gives us the necessary background to set down the following definition.

Definition 3.2.1 (Generalized Conformal Measure)**.** *Let f be a bounded potential and $\mathscr{H}_1$ the family of probability measures for which H1 holds. We say that $\nu \in \mathscr{H}_1$ is a* ***generalized conformal measure*** *with respect to the potential f if $\mathbb{L}_f^* \mathbb{1} = \|\mathbb{L}_f\|_{op} \mathbb{1}$, ν a. e. .*

On the definition above, $\|\cdot\|_{op}$ is the operator norm i.e., $\|\mathbb{L}_f\|_{op} = \sup_{\substack{u \in L^1(\nu) \\ \|u\|_1 = 1}} \|\mathbb{L}_f u\|_{L^1(\nu)}$. Note that if $f = \varphi$ is continuous, by positivity of the extended operator $\mathbb{L}_\varphi$, $\rho(\mathscr{L}_\varphi) = \|\mathbb{L}_\varphi\|_{op}$ because

$$\|\mathbb{L}_\varphi u\|_{L^1(\nu)} \leqslant \|\mathbb{L}_\varphi \mathbb{1}\|_{L^1(\nu)} = \int_X \mathscr{L}_\varphi 1 d\nu = \rho(\mathscr{L}_f).$$

When ν is a conformal measure to φ, it is clear that

$$\int_X \mathscr{L} \varphi \, d\nu = \rho(\mathscr{L}_\varphi) \int_X \varphi \, d\nu \quad \forall \varphi \in C(X, \mathbb{R}).$$

As $\mathbb{L}_\varphi$ extends $\mathscr{L}_\varphi$, this duality relation can be rewritten as $\langle \mathbb{1}, \mathbb{L}\varphi \rangle_\nu = \rho(\mathscr{L}_\varphi) \langle \mathbb{1}, \varphi \rangle_\nu$ for every $\varphi \in C(X, \mathbb{R})$, where $\langle \cdot, \cdot \rangle_\nu$ is the usual duality conjugation of $L^\infty(\nu)$ (left entry) and $L^1(\nu)$ (right entry), and $\mathbb{1} \in L^\infty(\nu)$ is the ν-equivalence class to which the constant function 1 belongs. As $\rho(\mathscr{L}_\varphi) = \|\mathbb{L}_\varphi\|_{op}$, we also have $\langle \mathbb{1}, \mathbb{L}\varphi \rangle_\nu = \|\mathbb{L}_\varphi\|_{op} \langle \mathbb{1}, \varphi \rangle_\nu$

From the continuity of $\mathbb{L}_\varphi$ and density of $C(X)$ on $L^1(\nu)$, we get that

$$\langle \mathbb{L}_\varphi^* \mathbb{1}, u \rangle_\nu = \|\mathbb{L}_\varphi\|_{op} \langle \mathbb{1}, \mathbb{L}_\varphi u \rangle_\nu = \|\mathbb{L}_\varphi\|_{op} \langle \mathbb{1}, u \rangle_\nu \quad \forall u \in L^1(\nu).$$

Which means that $\mathbb{L}_\varphi^* \mathbb{1} = \mathbb{1}$, that is, the way $\mathbb{L}_\varphi$ was constructed $\mathbb{1}$ is always a maximal eigenfunction of its dual. This is equivalent to say that for continuous potentials every conformal measure is a generalized conformal measure. The opposite inclusion is evident, then conformal measures and generalized conformal measures are synonyms if we have a continuous potential.

Back to a discontinuous potential f, we can substitute the duality relation $\mathscr{L}_\varphi \nu = \rho(\mathscr{L}_\varphi)\nu$ by the analogous relation $\mathbb{L}_f^* \mathbb{1} = \|\mathbb{L}_f\|_{op} \mathbb{1}$ to prove the following theorem.

Proposition 3.2.2 (Generalized conformal measures are fully supported)**.** *Let $X = E^{\mathbb{N}}$ endowed with the usual metric, with E a compact metric space, and f a bounded potential.*

Let ν a generalized conformal measure with respect to f and a priori measure p fully supported on E. Then ν is also fully supported on X.

Proof. The proof is totally analogous to the proof of Proposition 3.1.8. With the same notation as there, but with ν a generalized conformal measure with respect to the bounded potential f, we can calculate $\nu(B)$ as bellow

$$\begin{aligned}
\nu(B) &= \int_X \mathbb{1}_B d\nu \geqslant \int_X \Psi_x d\nu = \langle \mathbb{1}, \Psi_x \rangle_\nu \\
&= \langle (\mathbb{L}_f^*)^n \mathbb{1}, \Psi_x \rangle_\nu = \langle \mathbb{1}, \mathbb{L}_f^n \Psi_x \rangle_\nu \\
&= \int_X L_f^{n-1} \left[\int_E \exp\left(f(a_1 \cdot)\right) \Psi(a_1 \cdot) dp(a_1) \right] (y) d\nu(y) \\
&= \int_X \int_{E^n} \exp\left[\sum_{k=1}^n f(a_k \ldots a_n y) \right] \prod_{k=1}^n \psi_k(a_k) dp^n(a) d\nu(y) \\
&= \left(\min_{x \in X} \{ e^{f(x)} \} \right)^n \cdot \prod_{k=1}^n \left[\int_{E^n} \psi(a_k) dp(a_k) \right] \\
&\geqslant \prod_{k=1}^n p(B_E(x_k, R/2)) \cdot \left(\min_{x \in X} \{ e^{f(x)} \} \right)^n > 0.
\end{aligned}$$

The last inequality is a consequence of f being bounded. □

Remark 3.2.3. *The theorem above also sheds light on the support of conformal measures in some examples of systems with compact alphabets investigated in other works. In [37], Lopes, Mengue, Mohr and Souza study the XY model, where the alphabet is the compact $\mathbb{S}^1$. Our theorem complement their investigation in some sense.*

Remark 3.2.4. *Note that the full support appears in every generalized conformal measure for the Currie-Wiess potential in Chapter 2.*

CHAPTER 4

MARKOV PROCESSES

In this chapter and the next one we follow closely some unpublished lecture notes on the Ergodic Theory of Markov Processes by L. Cioletti, [8] which, by its turn, is based on the classical references [24, 43].

The main theme of this chapter is the dual nature of the Markov processes first presented in the seminal paper [30] by Eberhard Hopf. In a classical probabilistic description, a Markov process is defined as a measure kernel on a sigma-finite measure space with some properties to be specified ahead. Hopf linked this description to an equivalent one where the main role is played by a positive contraction on the L^1 space. In this text this is the analytic description of a Markov process. We begin the next section presenting this view.

In Section 4.2 we present the probabilistic description of Markov processes and follow it by a prove of the equivalence of these descriptions in Section 4.3. Then, in Section 4.4 we show that a Markov process can be extended to spaces beyond L^1. We finish the chapter in Section 4.5 bringing a variety of examples of processes that will be referenced on the analysis to be made on the following chapters.

As the theory of Markov processes holds for a an arbitrary σ-algebra, in this chapter and the next one, instead of working on a metric space and the Borelian σ-albegra, we work on a general measurable space $(X, \mathscr{F})$.

4.1 Analytic Description of Markov Processes

Definition 4.1.1 (Markov Process)**.** *A Markov process is defined as an ordered quadruple* $(X, \mathscr{F}, \mu, T)$*, where the triple* $(X, \mathscr{F}, \mu)$ *is a sigma-finite measure space with a positive measure* μ *and* T *is a bounded linear operator acting on* $L^1(X, \mathscr{F}, \mu)$ *satisfying:*

i) T *is a contraction:* $\sup\{\|Tu\|_1 : \|u\|_1 \le 1\} \equiv \|T\| \le 1$*;*

ii) T *is a positive operator, that is, if* $u \ge 0$*, then* $Tu \ge 0$*.*

As usual functions which are equal almost everywhere will be identified. Thus all inequalities are to hold almost everywhere. The Banach spaces $L^1(X, \mathscr{F}, \mu)$ and $L^\infty(X, \mathscr{F}, \mu)$ form a natural dual pair. We emphasize this duality writing

$$\langle f, u \rangle \equiv \int_X f u \, d\mu,$$

where $f \in L^\infty(X, \mathscr{F}, \mu)$ and $u \in L^1(X, \mathscr{F}, \mu)$

The operator adjoint to T is denoted by T^* and will be considered as an operator acting on $L^\infty(X, \mathscr{F}, \mu)$. To be more precise, the operator T^* is the unique bounded linear operator, satisfying $\langle T^* f, u \rangle = \langle f, Tu \rangle$ for all $f \in L^\infty(X, \mathscr{F}, \mu)$ and $u \in L^1(X, \mathscr{F}, \mu)$.

Recall that an element in $L^\infty(X, \mathscr{F}, \mu)$ is an equivalence class. Two real-valued $\mathscr{F}$-measurable functions $\bar{f}, \hat{f} \in [f] \in L^\infty(X, \mathscr{F}, \mu)$ if $\mu(\{x \in X : \bar{f}(x) \neq \hat{f}(x)\}) = 0$. So strictly speaking, an operator $T^* : L^\infty(X, \mathscr{F}, \mu) \to L^\infty(X, \mathscr{F}, \mu)$ is a map taking equivalence classes into equivalence classes. Sometimes we will abuse notation and write $T^* f$ instead of $T^*[f]$. And this notation should not trick us into believing that $T^* f$ defines a function. As emphasized above, $T^* f$ is simply a notation for an equivalence class of functions.

For any $A \in \mathscr{F}$ we have that its indicator function $1_A \in L^\infty(X, \mathscr{F}, \mu)$. Take any function in $[T^* 1_A]$, namely $T^* 1_A$ and consider the function $P : X \times \mathscr{F} \to \mathbb{R}$ defined as follows

$$P(x, A) = (T^* 1_A)(x). \tag{4.1}$$

The main properties of this function are summarized in the proposition bellow.

Proposition 4.1.2. *The function* P *defined by* (4.1) *has the following properties:*

i) for μ-almost all $x \in X$ and $A \in \mathscr{F}$ we have $0 \leq P(x, A) \leq 1$;

ii) for any fixed $A \in \mathscr{F}$, the mapping $x \longmapsto P(x, A)$ is a $\mathscr{F}$-measurable function;

iii) for μ-almost all $x \in X$, the set-function $A \longmapsto P(x, A)$ is a non-negative measure on $(X, \mathscr{F})$;

iv) if $\mu(A) = 0$, then $P(x, A) = 0$, for μ-almost all $x \in X$.

Proof. i). We first prove the lower bound. Let $A \in \mathscr{F}$ and $u \in L^1(X, \mathscr{F}, \mu)$ be a positive function. Then from the definition of adjoint and positivity of T we have that

$$0 \leq \langle 1_A, Tu \rangle = \langle T^* 1_A, u \rangle = \int_X (T^* 1_A) u \, d\mu.$$

Consider the set $N \equiv \{x \in X : (T^* 1_A)(x) < 0\}$. Note that to prove $0 \leq P(x, A)$ it is enough to prove that $\mu(N) = 0$. Since $(X, \mathscr{F}, \mu)$ is a sigma-finite measure space, there is a non-decreasing sequence $(N_k)_{k \in \mathbb{N}}$ in $\mathscr{F}$ such that $N_k \uparrow N$, when $k \to \infty$, and $\mu(N_k) < +\infty$, for all $k \in \mathbb{N}$. Therefore for all $k \in \mathbb{N}$, we have that $1_{N_k} \in L^1(X, \mathscr{F}, \mu)$. Note that the above inequality implies $\mu(N_k) = 0$. Indeed, if $\mu(N_k) > 0$, then by taking $u = 1_{N_k}$, we get this contradiction

$$0 \leq \int_X (T^* 1_A) 1_{N_k} \, d\mu < 0.$$

The continuity of μ implies that $0 = \lim_{k \to \infty} \mu(N_k) = \mu(N)$.

A similar idea works to get the upper bound. But now, instead of only using the positivity of T, we will need its contraction property. The proof is as follows. We first observe that the contraction property of T and the Hölder inequality imply

$$\langle (T^* 1_A), u \rangle = \langle 1_A, Tu \rangle = \int_X 1_A Tu \, d\mu \leq \|Tu\|_1 \leq \|u\|_1.$$

Let $U \equiv \{x \in X : (T^* 1_A)(x) > 1\}$. To obtain the inequality $P(x, A) \leq 1$ it is enough to prove that $\mu(U) = 0$. Suppose, by contradiction, that $\mu(U) > 0$. Since $(X, \mathscr{F}, \mu)$ is a sigma-finite measure space, there is a sequence $(U_k)_{k \in \mathbb{N}}$ of μ-finite measure sets such that $U_k \uparrow U$, when $k \to \infty$. The continuity of μ implies that, for some $k \in \mathbb{N}$, we have $0 < \mu(U_k) < +\infty$. Therefore $1_{U_k} / \mu(U_k)$ is a norm one element in $L^1(X, \mathscr{F}, \mu)$. By using the definition of U and the above inequality with $u = 1_{U_k}/\mu(U_k)$ we get

$$\begin{aligned}
1 &< \frac{1}{\mu(U_k)} \int_X (T^* 1_A)\, 1_{U_k}\, d\mu \\
&= \frac{1}{\mu(U_k)} \langle (T^* 1_A), 1_{U_k} \rangle \\
&= \langle (T^* 1_A), 1_{U_k}/\mu(U_k) \rangle \\
&\leq 1
\end{aligned}$$

which is an absurd. So $\mu(U) = 0$ and the proof of item $i)$ is complete.

$ii)$. For any $A \in \mathscr{F}$ we have that $1_A \in L^\infty(X, \mathscr{F}, \mu)$. Since T^* sends $L^\infty(X, \mathscr{F}, \mu)$ to itself it, $x \longmapsto (T^* 1_A)(x) = P(x, A)$ is a $\mathscr{F}$-measurable function.

$iii)$. Let $(A_k)_{k\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathscr{F}$, and $u \in L^1(X, \mathscr{F}, \mu)$ a fixed non-negative function. Then

$$\begin{aligned}
\langle T^* 1_{\cup_{k=1}^\infty A_k}, u \rangle &= \langle 1_{\cup_{k=1}^\infty A_k}, Tu \rangle \\
&= \int_X 1_{\cup_{k=1}^\infty A_k}\, Tu\, d\mu = \int_X \sum_{k=1}^\infty 1_{A_k}\, Tu\, d\mu = \sum_{k=1}^\infty \int_X 1_{A_k}\, Tu\, d\mu \\
&= \sum_{k=1}^\infty \langle 1_{A_k}, Tu \rangle = \sum_{k=1}^\infty \langle T^* 1_{A_k}, u \rangle = \sum_{k=1}^\infty \int_X T^* 1_{A_k}\, u\, d\mu \\
&= \int_X \sum_{k=1}^\infty T^* 1_{A_k}\, u\, d\mu \\
&= \langle \sum_{k=1}^\infty T^* 1_{A_k}, u \rangle,
\end{aligned}$$

where in the fourth equality we have used the positivity of T to ensure that $Tu \geq 0$ to apply the Lebesgue Dominated Convergence Theorem. In the eighth equality we used item $i)$ to ensure the non-negativity of $T^* 1_{A_k}(x) = P(x, A_k)$ and then we applied the Fatou Lemma. To prove the above identity for a general u in $L^1(X, \mathscr{F}, \mu)$, just decompose it on its positive and negative parts, use the previous identity for each of them and conclude that

$$\langle T^* 1_{\cup_{k=1}^\infty A_k}, u \rangle = \langle \sum_{k=1}^\infty T^* 1_{A_k}, u \rangle, \qquad \forall u \in L^1(X, \mathscr{F}, \mu).$$

This proves

$$P(x, \cup_{k=1}^\infty A_k) \equiv T^* 1_{\cup_{k=1}^\infty A_k}(x) = \sum_{k=1}^\infty T^* 1_{A_k}(x) = \sum_{k=1}^\infty P(x, A_k),$$

for μ-almost all $x \in X$.

iv). Suppose that $\mu(A) = 0$ and let $u \in L^1(X, \mathscr{F}, \mu)$ be an arbitrary integrable function. Then we have $\langle T^* 1_A, u \rangle = \langle 1_A, Tu \rangle = 0$. Therefore, $T^* 1_A$ is the null-vector in $L^\infty(X, \mathscr{F}, \mu)$, and so $P(x, A) \equiv (T^* 1_A)(x) = 0$ for μ-almost all $x \in X$. □

4.2 Probabilistic Description of Markov Processes

In probability theory it is not common to introduce a Markov process as above. Normally, one begins with a sigma-finite measure space $(X, \mathscr{F}, \mu)$ and a measure kernel $P : X \times \mathscr{F} \to [0, \infty]$ satisfying the following four properties:

i) for μ-almost all $x \in X$ and $A \in \mathscr{F}$, we have $0 \le P(x, A) \le 1$;

ii) for any fixed $A \in \mathscr{F}$, the mapping $x \longmapsto P(x, A)$ is a $\mathscr{F}$-measurable function;

iii) for μ-almost all $x \in X$, the set-function $A \longmapsto P(x, A)$ is a non-negative measure on $(X, \mathscr{F})$;

iv) if $\mu(A) = 0$, then $P(x, A) = 0$, for μ-almost all $x \in X$.

Next, we prove that the measure Kernel P induces a positive linear operator $L : L^\infty(X, \mathscr{F}, \mu) \to L^\infty(X, \mathscr{F}, \mu)$ defined in this way

$$Lf(x) = \int_X f(y)\, dP(x, y) \equiv \int_X f(y)\, P(x, dy).$$

To prove that L is well-defined it is enough to check that

1. if $f \in L^\infty(X, \mathscr{F}, \mu)$ then the mapping $x \longmapsto \int_X f(y)\, P(x, dy)$ is a $\mathscr{F}$-measurable function;

2. for any pair $f, g \in L^\infty(X, \mathscr{F}, \mu)$ satisfying $\mu(\{x \in X : f(x) \ne g(x)\}) = 0$ we have $Lf(x) = Lg(x)$ for μ-almost all $x \in X$;

3. $|L(f)(x)| \le \|f\|_\infty$, μ-almost all $x \in X$.

We begin by proving (1). Let us first assume that f is a real-valued simple function. In this case, (1) is a straightforward consequence of ii).

If f is an arbitrary positive element of $L^\infty(X, \mathscr{F}, \mu)$, then we can decompose it as $f = f1_Z + f1_{X\setminus Z}$, where Z is a $\mathscr{F}$-measurable subset of X such that $\mu(Z) = 0$ and $|f(x)| \leq \|f\|_\infty$ for all $x \in X \setminus Z$. From iv) we have that $P(x, Z) = 0$, for μ-almost all $x \in X$. Therefore

$$x \longmapsto \int_X f(y)\, P(x, dy) = \int_X 1_Z(y) f(y) + 1_{X\setminus Z}(y) f(y)\, P(x, dy)$$
$$= \int_X 1_{X\setminus Z}(y) f(y)\, P(x, dy).$$

Since $1_{X\setminus Z} f$ is non-negative and everywhere bounded $\mathscr{F}$-measurable function, we known that there is a non-decreasing sequence of simple functions $(\phi_n)_{n\in\mathbb{N}}$ converging uniformly to $1_{X\setminus Z} f$. From the Monotone Convergence Theorem

$$\int_X 1_{X\setminus Z}(y) f(y)\, P(x, dy) = \lim_{n\to\infty} \int_X \phi_n(y)\, P(x, dy).$$

Recalling that pointwise limit of $\mathscr{F}$-measurable functions is also a $\mathscr{F}$-measurable function then (1) is proved when $f \geq 0$. If f is a general element in $L^\infty(X, \mathscr{F}, \mu)$ and $f = f^+ - f^-$ is its decomposition on positive and negative parts, we have that

$$\int_X f^\pm(y)\, P(x, dy) < +\infty.$$

This implies

$$x \longmapsto \int_X f(y)\, P(x, dy) = \int_X f^+(y)\, P(x, dy) - \int_X f^-(y)\, P(x, dy)$$

is a difference of two $\mathscr{F}$-measurable maps, thus completing the proof of (1).

Now we prove (2). Suppose that $f, g \in L^\infty(X, \mathscr{F}, \mu)$ and they are equal μ-almost everywhere. Item iv) implies that $P(x, \{y \in X : f(y) \neq g(y)\}) = 0$, for μ-almost all $x \in X$. Therefore for μ-almost all $x \in X$ we have

$$L(f)(x) - L(g)(x) = \int_X f(y) - g(y)\, P(x, dy) = 0,$$

which proves (2).

Finally, we prove (3). Similarly to the proof of item (1), we can decompose $|f| = 1_Z|f| + 1_{X\setminus Z}|f|$, where Z is a $\mathscr{F}$-measurable subset of X such that $\mu(Z) = 0$ and $|f(x)| \le \|f\|_\infty$ for all $x \in X \setminus Z$. By using iv) again we have that $P(x, Z) = 0$ for μ-almost all $x \in X$. Therefore for μ-almost all $x \in X$, we have

$$|Lf(x)| \le \int_X |f(y)|\, P(x, dy) = \int_Z |f(y)|\, P(x, dy) \le \|f\|_\infty,$$

where in the last inequality we used, for the first time in this proof, the condition i), that is, $0 \le P(x, A) \le 1$.

4.3 Equivalence of the Analytic and Probabilistic Descriptions

In this section we show that it is possible to use the measure Kernel P to construct a bounded linear operator $T : L^1(X, \mathscr{F}, \mu) \to L^1(X, \mathscr{F}, \mu)$ so that $T^* = L$ and the measure Kernel induced by T^* is exactly P. In other words, we want to show that L has a pre-transpose operator, see the diagram bellow

$$\begin{array}{ccc} L^\infty(X, \mathscr{F}, \mu) & \xleftarrow{\quad L \quad} & L^\infty(X, \mathscr{F}, \mu) \\ \text{duality}\uparrow & & \uparrow\text{duality} \\ L^1(X, \mathscr{F}, \mu) & \xrightarrow{\quad T \quad} & L^1(X, \mathscr{F}, \mu). \end{array}$$

Moreover, we will show that T has the properties required in Definition 4.1.1 and the Kernel it induces is exactly P.

To construct T from P we employ a useful identification. By the Radon-Nikodym theorem the space of all signed measures absolutely continuous with respect to μ, with the induced total variation norm, is isometrically isomorphic as a Banach space and as a Banach lattice to $L^1(X, \mathscr{F}, \mu)$:

$$\nu \text{ signed measure } |\nu| \ll \mu, \qquad \nu \longmapsto \frac{d\nu}{d\mu} \in L^1(X, \mathscr{F}, \mu)$$

$$u \in L^1(X, \mathscr{F}, \mu), \qquad u \longmapsto u^+ d\mu - u^- d\mu \equiv \nu, \quad |\nu| \ll \mu.$$

The first step in the construction of the operator T from the measure kernel P consists in introducing a new linear operator $\mathcal{T} : M(\mu) \to M(\mu)$, where

$$M(\mu) \equiv \{\nu \in \mathscr{M}_s(X, \mathscr{F}) : \nu \ll \mu \text{ and } |\nu|(X) < +\infty\}$$

and then use the above isomorphism.

For each $\nu \in M(\mu)$ consider the set-function $\mathscr{F} \ni A \mapsto \mathcal{T}\nu(A)$ given by

$$\mathcal{T}\nu(A) \equiv \int_X P(x, A)\, d\nu(x).$$

We claim that $\mathcal{T}(M(\mu)) \subset M(\mu)$. Let $\nu \in M(\mu)$ such that $\nu(A) \geq 0$ for all $A \in \mathscr{F}$. From property i) of P,

$$0 \leq \mathcal{T}\nu(A) = \int_X P(x, A)\, d\nu(x) \leq \nu(X) < +\infty. \tag{4.2}$$

Next, we prove that $\mathscr{F} \ni A \mapsto \mathcal{T}\nu(A)$ defines a countably additive set-function. Indeed, if $(A_k)_{k\in\mathbb{N}}$ is a pairwise disjoint sequence and $A = \cup_{k\in\mathbb{N}} A_k$, then

$$\begin{aligned}\mathcal{T}\nu(A) &= \int_X P(x, \cup_{k\in\mathbb{N}} A_k)\, d\nu(x) = \int_X \sum_{k\in\mathbb{N}} P(x, A_k)\, d\nu(x) = \sum_{k\in\mathbb{N}} \int_X P(x, A_k)\, d\nu(x) \\ &= \sum_{k\in\mathbb{N}} \mathcal{T}\nu(A_k),\end{aligned}$$

where in the second equality we used property iii) of P and to obtain the third equality we applied Fatou's Lemma. These observations prove that $\mathcal{T}\nu$ is actually a non-negative finite measure on X.

From property iv) of P, we have that if $\mu(A) = 0$, then $P(A, x) = 0$ for μ-almost all $x \in X$. Since $\nu \in M(\mu)$, and so $\nu \ll \mu$, we have $P(A, x) = 0$ for ν-almost all $x \in X$. Therefore from definition of $\mathcal{T}$ we have $\mathcal{T}\nu(A) = 0$, thus showing that $\mathcal{T}\nu \ll \mu$.

Piecing together the information on the last two paragraphs we conclude that $\mathcal{T}(\nu) \in M(\mu)$ for any positive measure ν in $M(\mu)$. Let ν be a generic signed measure in $M(\mu)$. The Jordan Decomposition Theorem ensures the existence of a pair (ν^+, ν^-) of positive measures such that $\nu = \nu^+ - \nu^-$. Since ν has finite total variation, $\nu^\pm$ are finite positive measures. Since (ν^+, ν^-) is a Jordan decomposition of ν we know that its variation is given by $|\nu| = \nu^+ + \nu^-$. From definition of $M(\mu)$ we have that $|\nu| \ll \mu$ and so

$\nu^{\pm} \ll \mu$. Applying once more property i) of P, that is, $0 \le P(x,A) \le 1$, we get

$$\begin{aligned}\mathcal{T}\nu(A) = \int_X P(x,A)\,d\nu(x) &= \int_X P(x,A)\,d\nu^+(x) - \int_X P(x,A)\,d\nu^-(x)\\ &= \mathcal{T}\nu^+(A) - \mathcal{T}\nu^-(A),\end{aligned}$$

for all $A \in \mathscr{F}$. This implies $\mathcal{T}\nu = \mathcal{T}\nu^+ - \mathcal{T}\nu^-$. From the above discussions, we have concluded that $\mathcal{T}\nu^{\pm} \in M(\mu)$. As $M(\mu)$ is a vector space, $\mathcal{T}\nu \in M(\mu)$, thus proving the claim.

Now we prove that $\mathcal{T} : M(\mu) \to M(\mu)$ is a contraction. Since $(\mathcal{T}\nu^+, \mathcal{T}\nu^-)$ is a pair of finite positive measures such that $\mathcal{T}\nu = \mathcal{T}\nu^+ - \mathcal{T}\nu^-$ and from the *minimality property* of the Jordan decomposition,

$$(\mathcal{T}\nu)^+ \le \mathcal{T}\nu^+ \qquad \text{and} \qquad (\mathcal{T}\nu)^- \le \mathcal{T}\nu^-$$

These inequalities together with (4.2) imply

$$\begin{aligned}\|\mathcal{T}\| &\equiv \sup\{|\mathcal{T}\nu|(X) : |\nu|(X) \le 1\}\\ &= \sup\{(\mathcal{T}\nu)^+(X) + (\mathcal{T}\nu)^-(X) : |\nu|(X) \le 1\}\\ &\le \sup\{(\mathcal{T}\nu^+)(X) + (\mathcal{T}\nu^-)(X) : |\nu|(X) \le 1\}\\ &\le \sup\{\nu^+(X) + \nu^-(X) : |\nu|(X) \le 1\}\\ &= 1,\end{aligned}$$

which proves that $\mathcal{T}$ is a linear contraction.

Now, we finally define the operator T. Let $\Psi : L^1(X, \mathscr{F}, \mu) \to M(\mu)$ denote the isometric lattice isomorphism mentioned above. For each $u \in L^1(X, \mathscr{F}, \mu)$ we define

$$Tu \equiv \frac{d\nu_u}{d\mu}, \quad \text{where } \nu_u = \mathcal{T} \circ \Psi(u).$$

Clearly, the map $u \longmapsto Tu$ defines a linear operator on $L^1(X, \mathscr{F}, \mu)$. Note that if $u \ge 0$ then $\Psi(u) \ge 0$. Positivity of $\mathcal{T}$ implies that ν_u is a positive measure in $M(\mu)$. Therefore $d\nu_u/d\mu \ge 0$, which implies T is a positive operator.

In order to get the contraction property, it is enough to observe that

$$\begin{aligned}
\|T\| &= \sup_{\|u\|\le 1} \|Tu\|_{L^1} = \sup_{\|u\|\le 1} \|\frac{d\nu_u}{d\mu}\|_{L^1} \\
&= \sup_{\|u\|\le 1} \int_X \left|\frac{d\nu_u}{d\mu}\right| d\mu = \sup_{\|u\|\le 1} \int_X \frac{d\nu_u^+}{d\mu} + \frac{d\nu_u^-}{d\mu}\, d\mu \\
&= \sup_{\|u\|\le 1} \nu_u^+(X) + \nu_u^-(X) \\
&= \sup_{\|u\|\le 1} (\mathcal{T}\circ\Psi(u))^+(X) + (\mathcal{T}\circ\Psi(u))^-(X) \\
&= \sup_{\substack{|\nu|(X)\le 1\\ \nu\in M(\mu)}} (\mathcal{T}\nu)^+(X) + (\mathcal{T}\nu)^-(X) \\
&\le 1,
\end{aligned}$$

where the last inequality is a consequence of the contraction property of $\mathcal{T}$.

Next, we prove that for all $A\in\mathscr{F}$ we have $(T^*1_A)(x) = P(x,A)$, for μ-almost all $x\in X$. Consider $u\in L^1(X,\mathscr{F},\mu)$ and $A\in\mathscr{F}$. Then

$$\langle T^*1_A, u\rangle = \langle 1_A, Tu\rangle = \int_X 1_A Tu\, d\mu = \int_X 1_A \frac{d\nu_u}{d\mu}\, d\mu = \nu_u(A) = (\mathcal{T}\circ\Psi(u))(A).$$

Recall that the measure $\Psi(u) = u^+d\mu - u^-d\mu$. By using this observation and the linearity of $\mathcal{T}$ we get

$$\begin{aligned}
\langle T^*1_A, u\rangle = (\mathcal{T}\circ\Psi(u))(A) &= \mathcal{T}(u^+d\mu - u^-d\mu)(A) \\
&= \mathcal{T}(u^+d\mu)(A) - \mathcal{T}(u^-d)\mu(A) \\
&= \int_X P(x,A)\, d[u^+\mu](x) - \int_X P(x,A)\, d[u^-\mu](x) \\
&= \int_X P(x,A)u^+(x)\, d\mu(x) - \int_X P(x,A)u^-(x)\, d\mu(x) \\
&= \int_X P(x,A)u(x)\, d\mu(x).
\end{aligned}$$

The last equality is simply

$$\int_X (T^*1_A)(x)u(x)\, d\mu(x) = \int_X P(x,A)u(x)\, d\mu(x), \qquad \forall u\in L^1(X,\mathscr{F},\mu).$$

Therefore, $(T^*1_A)(x) = P(x,A)$ for μ-almost all $x\in X$. Since this equality holds for any $A\in\mathscr{F}$ it follows from the continuity of T^* and a similar reasoning as before that for each

$f \in L^\infty(X, \mathscr{F}, \mu)$ we have $\langle T^* f, u\rangle = \langle Lf, u\rangle$ for all $u \in L^1(X, \mathscr{F}, \mu)$. And so $T^* = L$.

4.4 Extension of a Markov Process

Now we discuss how to extend both a Markov process T on $(X, \mathscr{F}, \mu)$ as its adjoint beyond the Lebesgue spaces $L^1(X, \mathscr{F}, \mu)$ and $L^\infty(X, \mathscr{F}, \mu)$, respectively.

Recall that if $u : X \to [0, +\infty]$ and $f : X \to [0, +\infty]$ are $\mathscr{F}$-measurable functions then there are two monotone non-decreasing sequences of non-negative μ-integrable simple functions $(u_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ on X such that for μ-almost all $x \in X$, we have $u_k(x) \uparrow u(x)$ and $f_k(x) \uparrow f(x)$. Since T and T^* are both positive operators the following limits are well defined for μ-almost all $x \in X$

$$\overline{T}u(x) = \lim_{k\to\infty} Tu_k(x) \quad \text{and} \quad \overline{T^*}f(x) = \lim_{k\to\infty} T^* f_k(x).$$

Note that these limits may be infinity at some $x \in X$. To prove that $\overline{T}$ and $\overline{T^*}$ are well-defined we have to show that their defining limits are independent of the choice of the approximating sequences $(u_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$, respectively. Suppose that $(v_k)_{k\in\mathbb{N}}$ is a monotone non-decreasing sequence of non-negative $\mathscr{F}$-measurable simple functions on X such that $v_k(x) \uparrow u(x)$ for μ-almost all $x \in X$. For each pair $k, n \in \mathbb{N}$, the function

$$w_{k,n}(x) \equiv \min(v_k(x), u_n(x)),$$

is well-defined μ-almost everywhere. If we fix $k \in \mathbb{N}$ then we have $w_{k,n}(x) \uparrow v_k(x)$ for μ-almost all $x \in X$, when $n \to \infty$. Applying Fatou's Lemma we get

$$\int_X v_k \, d\mu = \int_X \liminf_{n\to\infty} w_{k,n} \, d\mu \le \liminf_{n\to\infty} \int_X w_{k,n} \, d\mu = \lim_{n\to\infty} \int_X w_{k,n} \, d\mu.$$

Since $w_{k,n} \le v_k$ μ-almost everywhere we conclude that

$$\lim_{n\to\infty} \int_X w_{k,n} \, d\mu = \int_X v_k \, d\mu.$$

Moreover

$$\lim_{n\to\infty} \int_X |v_k - w_{k,n}| \, d\mu = \lim_{n\to\infty} \int_X v_k - w_{k,n} \, d\mu = 0.$$

Thus proving that $w_{k,n} \to v_k$, when $n \to \infty$, in the $L^1(X, \mathscr{F}, \mu)$-norm. Since T is a positive and bounded operator we get that $Tw_{k,n} \le Tv_k$ and $Tw_{k,n} \to Tv_k$, when $n \to \infty$, in the $L^1(X, \mathscr{F}, \mu)$-norm. Therefore a classical result of measure theory implies the existence of a subsequence $(Tw_{k,n_m})_{m\in\mathbb{N}}$ such that $Tw_{k,n_m}(x) \to Tv_k(x)$ for μ-almost all $x \in X$, when $m \to \infty$. From the monotonicity it follows that $Tw_{k,n}(x) \to Tv_k(x)$ for μ-almost all $x \in X$, when $n \to \infty$. A similar reasoning shows that $Tw_{k,n}(x) \to Tu_n(x)$ for μ-almost all $x \in X$, when $k \to \infty$. Recalling that μ-almost everywhere we have $Tw_{k,n} \le Tv_k$, then

$$Tu_n(x) = \lim_{k\to\infty} Tw_{k,n}(x) \le \lim_{k\to\infty} Tv_k(x) \quad \Longrightarrow \quad \lim_{n\to\infty} Tu_n(x) \le \lim_{k\to\infty} Tv_k(x).$$

Analogously, we prove the reverse inequality and so we have that $\overline{T}$ is well-defined.

Now we prove that $\overline{T^*}f$ is well-defined. Let $(g_k)_{k\in\mathbb{N}}$ be a sequence of positive functions in $L^\infty(X, \mathscr{F}, \mu)$ such that $g_k(x) \uparrow f(x)$ for μ-almost all $x \in X$. Define μ-almost everywhere the function $h_{k,n}(x) = \min(g_k(x), f_n(x))$. Note that for any fixed $k \in \mathbb{N}$, we have $h_{k,n}(x) \uparrow g_k(x)$ for μ-almost all $x \in X$, when $n \to \infty$ and another application of Fatou's Lemma shows that, for any positive function $u \in L^1(X, \mathscr{F}, \mu)$ we have

$$\begin{aligned} 0 \le \lim_{n\to\infty} \langle u, T^*(g_k - h_{k,n})\rangle &= \liminf_{n\to\infty} \langle u, T^*(g_k - h_{k,n})\rangle \\ &= \liminf_{n\to\infty} \langle Tu, (g_k - h_{k,n})\rangle \\ &\le \langle Tu, \liminf_{n\to\infty}(g_k - h_{k,n})\rangle = 0. \end{aligned}$$

Applying the above reasoning to the positive and negative parts of an arbitrary $u \in L^1(X, \mathscr{F}, \mu)$ we get

$$\lim_{n\to\infty} \langle u, T^*(g_k - h_{k,n})\rangle = 0.$$

Next we prove that the above equality implies, for any fixed $k \in \mathbb{N}$, that $T^*h_{k,n}(x) \to T^*g_k(x)$, when $n \to \infty$, for μ-almost all $x \in X$. Positivity of T^* implies that $0 \le T^*g_k(x) - T^*h_{k,n}(x)$ and the sequence $T^*g_k(x) - T^*h_{k,n}(x)$ is non-increasing in n almost everywhere. For each fixed $k \in \mathbb{N}$ consider the set

$$Z_k \equiv \{x \in X : \liminf_{n\to\infty} \Big(T^*g_k(x) - T^*h_{k,n}(x)\Big) > 0\}.$$

We claim that $\mu(Z_k) = 0$. This will be proved by contradiction. Suppose that $\mu(Z_k) > 0$

and consider the following decomposition of the set

$$Z_k = \bigcup_{p\in\mathbb{N}} \left\{ x \in X : \liminf_{n\to\infty} \Big(T^* g_k(x) - T^* h_{k,n}(x) \Big) > \frac{1}{p} \right\} \equiv \bigcup_{p\in\mathbb{N}} Z_k(p).$$

Since $0 < \mu(Z_k) \le \sum_{p\in\mathbb{N}} \mu(Z_k(p))$, there is some $p_0 \in \mathbb{N}$ so that $\mu(Z_k(p_0)) > 0$. Taking $u = 1_{Z_k(p_0)}$ on the equation above, we get the following contradiction

$$\begin{aligned} \frac{1}{p_0}\mu(Z_k(p_0)) &< \int_X 1_{Z_k(p_0)} \liminf_{n\to\infty} T^*(g_k - h_{k,n})\, d\mu \\ &= \langle 1_{Z_k(p_0)}, \liminf_{n\to\infty} T^*(g_k - h_{k,n}) \rangle \\ &\le \liminf_{n\to\infty} \langle 1_{Z_k(p_0)}, T^*(g_k - h_{k,n}) \rangle \\ &\le \lim_{n\to\infty} \langle 1_{Z_k(p_0)}, T^*(g_k - h_{k,n}) \rangle \\ &= 0, \end{aligned}$$

where in the second inequality we used Fatou's lemma.

Since $\mu(Z_k) = 0$, we have that, for μ-almost all $x \in X$,

$$\lim_{n\to\infty} T^* h_{k,n}(x) = T^* g_k(x).$$

From positivity we also have

$$T^* g_k(x) = \lim_{n\to\infty} T^* h_{k,n}(x) \le \lim_{n\to\infty} T^* f_n(x)$$

Using again the monotonicity and taking the limit when $k \to \infty$ we get

$$\lim_{k\to\infty} T^* g_k(x) \le \lim_{n\to\infty} T^* f_n(x) \equiv \overline{T^*} f(x)$$

By using similar reasoning, we get the reverse inequality thus proving that $\overline{T^*}$ is well-defined.

The way we extended T^* to $\overline{T^*}$ give us a result that will be very useful ahead. It can be stated as follows. If $(f_n)_{n\in\mathbb{N}}$ is sequence of non-negative functions in $L^\infty(X, \mathscr{F}, \mu)$, then

$$\overline{T^*}\Big(\sum_{n=1}^{\infty} f_n\Big) = \sum_{n=1}^{\infty} \overline{T^*} f_n,$$

even if $\sum_{n=1}^{\infty} f_n$ is not in $L^\infty(X, \mathscr{F}, \mu)$.

4.5 Examples of Markov Processes

Example 4.5.1. *Let X be the closed interval $[0,1] \subset \mathbb{R}$ endowed with the usual metric, m the Lebesgue measure, and T the Markov process defined on $(X, \mathscr{B}(X), m)$ which acts on an integrable function $\bar{u}$ as bellow*

$$T\bar{u}(x) = \begin{cases} 2\bar{u}(2x), & \text{if } x \leqslant 1/2; or \\ 0, & \text{otherwise.} \end{cases}$$

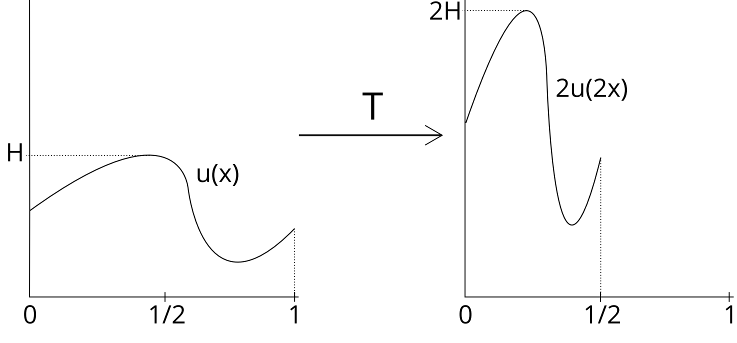
Then $(X, \mathscr{B}(X), m, T)$ is a Markov process. The positivity of T is clear. To see that T is a contraction note that a simple change of variables $y = 2x$ gives us

$$\int_X (Tu)(x)dm = 2\int_X (u)(2x)dm(x) = \int_X u(y)dm(y).$$

This shows that T "preserves area", in the sense that $\|Tu\|_{L^1} = \|u\|_{L^1}$. So,

$$\|T\| = \sup_{\|u\|_{L^1}=1} \|Tu\|_{L^1} = \sup_{\|u\|_{L^1}} \|u\|_{L^1} = 1.$$

Figure 4.1: Example of how T transforms a continous function. Note that the area under the graph is preserved



Processes that preserve area in this manner will play a major role on the following chapters. The equation $\|Tu\|_{L^1} = \|u\|_{L^1}$ can be rewritten as $\langle \mathbb{1}, Tu\rangle = \langle T^*\mathbb{1}, u\rangle = \langle \mathbb{1}, u\rangle$, for *all* $u \in L^1(X, \mathscr{F}, \mu)$*. So those processes are characterized by the property* $T^*\mathbb{1} = \mathbb{1}$*.*

Example 4.5.2. *If* T_1, T_2 *are Markov processes on* $(X, \mathscr{F}, \mu)$*, then the composition* $T_1 \circ T_2$ *is also a Markov process. It is clear that* $T_1 \circ T_2$ *is a positive operator and also a contraction since for all for all* $u \in L^1(X, \mathscr{F}, \mu)$ *such that* $\|u\|_1 \leq 1$ *we have* $\|T_1 \circ T_2 u\|_1 \leq \|T_2 u\|_1 \leq 1$*. Let* P_1, P_2 *be the measure kernels induced by* T_1 *and* T_2*, respectively. Recall that the each measure Kernel* $P_i, i = 1, 2$ *induces a positive linear operator* $L_i : L^\infty(X, \mathscr{F}, \mu) \to L^\infty(X, \mathscr{F}, \mu)$ *defined as follows*

$$L_i f(x) = \int_X f(y)\, dP_i(x, y) \equiv \int_X f(y)\, P_i(x, dy).$$

We also proved that $L_i = T_i^*$*. Let* P *be the measure kernel induced by* $T_1 \circ T_2$ *and* $L \equiv (T_1 \circ T_2)^*$*. Then for any* $f \in L^\infty(X, \mathscr{F}, \mu)$ *we have*

$$\begin{aligned} Lf(x) = (T_2^* \circ T_1^*)f(x) &= T_2^*(T_1^* f)(x) \\ &= \int_X T_1^* f(y)\ P_2(x, dy) \\ &= \int_X \int_X f(z)\, P_1(y, dz)\, P_2(x, dy). \end{aligned}$$

If we take $f = 1_A$ *then the measure kernel* P *is given by*

$$P(x, A) = \int_X P_1(y, A)\, P_2(x, dy).$$

Because of this identity, we sometimes use the suggestive notation $P = P_1 P_2$*. In particular, from the above observations,* T^n *is a Markov process for every positive integer* n*, and*

$$P^{n+m}(x, A) = \int_X P^n(y, A)\, P^m(x, dy).$$

Example 4.5.3. *Suppose that* T *is a Markov Process on* $(X, \mathscr{F}, \mu)$ *and* P *the associated measure kernel. Then for any* $A \in \mathscr{F}$ *and* $x \in X$ *we have*

$$\begin{aligned} P^4(x, A) = P^3 P(x, A) &= \int_X P^3(y_1, A)\, P(x, dy_1) \\ &= \int_X \int_X P^2(y_2, A)\, P(y_1, dy_2)\, P(x, dy_1) \\ &= \int_X \int_X \int_X P(y_3, A)\, P(y_2, dy_3)\, P(y_1, dy_2)\, P(x, dy_1). \end{aligned}$$

Example 4.5.4. *In this example the measurable space* $(X,\mathscr{F}) = (\mathbb{Z},\mathscr{P}(\mathbb{Z}))$*, where* $\mathscr{P}(\mathbb{Z})$ *is the set of parts of* $\mathbb{Z}$*. The measure* μ *is the counting measure, that is* $\mu(\{x\}) = 1$*, for all* $x \in \mathbb{Z}$*. The operator defining the Markov Process will be the linear operator* $T : L^1(\mathbb{Z},\mathscr{P}(\mathbb{Z}),\mu) \to L^1(\mathbb{Z},\mathscr{P}(\mathbb{Z}),\mu)$ *sending* $u \longmapsto Tu$ *given by*

$$Tu(y) = \int_{\mathbb{Z}} P_{x,y}\ u(x)\ d\mu(x),$$

where $(P_{x,y})_{x,y\in\mathbb{Z}}$ *is a family of real numbers satisfying* $P_{x,y} \geq 0$*, for all* $x, y \in \mathbb{Z}$*, and for any fixed* $x \in \mathbb{Z}$ *we have* $\sum_{y\in\mathbb{Z}} P_{x,y} \leq 1$*.*

Let us prove that T *is well-defined, that is,* $Tu \in L^1(\mathbb{Z},\mathscr{P}(\mathbb{Z}),\mu)$ *for all* $u \in L^1(X,\mathscr{F},\mu)$*. Indeed, it follows from basic properties of the Lebesgue integral and Tonelli's theorem that*

$$\begin{aligned}
\int_{\mathbb{Z}} |Tu(y)|\, d\mu(y) &= \int_{\mathbb{Z}} \Big| \int_{\mathbb{Z}} P_{x,y}\ u(x)\ d\mu(x) \Big|\, d\mu(y) \\
&\leq \int_{\mathbb{Z}} \int_{\mathbb{Z}} P_{x,y}\ |u(x)|\ d\mu(x)\, d\mu(y) \\
&= \int_{\mathbb{Z}} \int_{\mathbb{Z}} P_{x,y}\ |u(x)|\ d\mu(y)\, d\mu(x) \\
&= \int_{\mathbb{Z}} |u(x)| \Big[\int_{\mathbb{Z}} P_{x,y}\ d\mu(y) \Big]\ d\mu(x) \\
&\leq \int_{\mathbb{Z}} |u(x)| \Big[\sum_{y\in\mathbb{Z}} P_{x,y} \Big]\ d\mu(x) \\
&\leq \int_{\mathbb{Z}} |u(x)|\, d\mu(x).
\end{aligned}$$

From the last inequality it should be clear that T *is well-defined operator in* $L^1(X,\mathscr{F},\mu)$ *and its operator norm is less or equal than one. Of course,* T *is a positive operator and therefore* T *defines a Markov process on* $(\mathbb{Z},\mathscr{P}(\mathbb{Z}),\mu)$*.*

Next we compute the adjoint of T*. Let* $u \in L^1(X,\mathscr{F},\mu)$ *and* $f \in L^\infty(X,\mathscr{F},\mu)$*. Then*

$$\begin{aligned}
\langle f, Tu\rangle = \int_{\mathbb{Z}} f(y)\, Tu(y)\, d\mu(y) &= \int_{\mathbb{Z}} f(y) \Big[\int_{\mathbb{Z}} P_{x,y} u(x)\, d\mu(x) \Big]\, d\mu(y) \\
&= \int_{\mathbb{Z}} \int_{\mathbb{Z}} P_{x,y} u(x) f(y)\ d\mu(x)\, d\mu(y) \\
&= \int_{\mathbb{Z}} \int_{\mathbb{Z}} P_{x,y} u(x) f(y)\ d\mu(y)\, d\mu(x) \\
&= \int_{\mathbb{Z}} \Big[\int_{\mathbb{Z}} P_{x,y} f(y)\, d\mu(y) \Big] u(x)\, d\mu(x)
\end{aligned}$$

$$= \langle T^* f, u \rangle,$$

where $T^* f(x) \equiv \int_{\mathbb{Z}} P_{x,y} f(y)\, d\mu(y)$. *To confirm that this is actually the adjoint of* T, *it is enough to show that* T^* *is a bounded linear operator on* $L^\infty(X, \mathscr{F}, \mu)$. *Indeed, for any* $f \in L^\infty(X, \mathscr{F}, \mu)$ *we have*

$$\sup_{x \in \mathbb{Z}} |T^* f(x)| \le \sup_{x \in \mathbb{Z}} \int_{\mathbb{Z}} P_{x,y} |f(y)|\, d\mu(y) \le \|f\|_\infty \sup_{x \in \mathbb{Z}} \int_{\mathbb{Z}} P_{x,y}\, d\mu(y) \le \|f\|_\infty.$$

The linearity is obvious.

Now we can give an explicit expression for the kernel $P : \mathbb{Z} \times \mathscr{P}(\mathbb{Z}) \to \mathbb{R}$. *For any subset* $A \subset \mathbb{Z}$ *and* $x \in \mathbb{Z}$ *the measure kernel* P *is given by*

$$P(x, A) \equiv T^* 1_A(x) = \int_{\mathbb{Z}} P_{x,y} 1_A(y)\, d\mu(y) = \sum_{y \in A} P_{x,y}.$$

This discrete Markov process is called a Markov Chain. We can think of the family $(P_{x,y})_{x,y \in \mathbb{Z}}$ *as an infinite matrix. In this case the matrix is called Markov matrix.*

Now let us compute the "fourth power" of P

$$\begin{aligned}
P^4(x, A) &= \int_{\mathbb{Z}} \int_{\mathbb{Z}} \int_{\mathbb{Z}} P(y_3, A)\, P(y_2, dy_3)\, P(y_1, dy_2)\, P(x, dy_1) \\
&= \int_{\mathbb{Z}} \int_{\mathbb{Z}} \int_{\mathbb{Z}} \left[\sum_{a \in A} P_{y_3,a} \right] P(y_2, dy_3)\, P(y_1, dy_2)\, P(x, dy_1) \\
&= \int_{\mathbb{Z}} \int_{\mathbb{Z}} \left[\sum_{y_3 \in \mathbb{Z}} \sum_{y \in A} P_{y_3,a}\, P_{y_2,y_3} \right] P(y_1, dy_2)\, P(x, dy_1) \\
&= \sum_{y_1 \in \mathbb{Z}} \sum_{y_2 \in \mathbb{Z}} \sum_{y_3 \in \mathbb{Z}} \sum_{a \in A} P_{y_3,a} P_{y_2,y_3} P_{y_1,y_2} P_{x,y_1} \\
&= \sum_{a \in A} \sum_{y_3 \in \mathbb{Z}} \sum_{y_2 \in \mathbb{Z}} \sum_{y_1 \in \mathbb{Z}} P_{x,y_1} P_{y_1,y_2} P_{y_2,y_3} P_{y_3,a}.
\end{aligned}$$

Example 4.5.5. *This example is simply a generalization of the previous example. We will omit some details since the computations are completely similar to the previous example.*

Now $(X, \mathscr{F}, \mu)$ *is a general* σ*-finite measure space. Let* $K : X \times X \to [0, +\infty]$ *a measurable function with respect to the product sigma algebra. Suppose that for any fixed* $x \in X$ *this function satisfies*

$$\int_X K(x, y)\, d\mu(y) \le 1.$$

Consider the linear operator $T : L^1(X, \mathscr{F}, \mu) \to L^1(X, \mathscr{F}, \mu)$ *taking* $u \longmapsto Tu$ *given by*

$$Tu(y) = \int_X u(x)\, K(x,y)\, d\mu(x).$$

To prove that T *is well-defined as an operator on* $L^1(X, \mathscr{F}, \mu)$ *it is enough to proceed as in the previous example by using elementary results of measure theory. As before, the contraction property is a consequence of the inequalities bellow.*

$$\begin{aligned}\int_X |Tu(y)|\, d\mu(y) &\le \int_X \int_X |u(x)| K(x,y)\, d\mu(x)\, d\mu(y) \\ &\le \int_X |u(x)| \int_X K(x,y)\, d\mu(y)\, d\mu(x) \\ &\le \int_X |u(x)|\, d\mu(x).\end{aligned}$$

To compute the adjoint of T *one can proceed as follows. Take an arbitrary* $f \in L^\infty(X, \mathscr{F}, \mu)$ *and* $u \in L^1(X, \mathscr{F}, \mu)$*. Then*

$$\begin{aligned}\langle f, Tu\rangle = \int_X f(y)\, Tu(y)\, d\mu(y) &= \int_X f(y) \left[\int_X u(x) K(x,y)\, d\mu(x)\right] d\mu(y) \\ &= \int_X \int_X u(x) K(x,y) f(y)\ d\mu(y)\, d\mu(x) \\ &= \int_X \left[\int_X K(x,y) f(y)\, d\mu(y)\right] u(x)\, d\mu(x) \\ &= \langle T^* f, u\rangle,\end{aligned}$$

where we used that the expression inside the brackets defines a bounded linear operator on $L^\infty(X, \mathscr{F}, \mu)$*. Indeed, for any* $f \in L^\infty(X, \mathscr{F}, \mu)$ *we have*

$$\int_X K(x,y) |f(y)|\, d\mu(y) \le \|f\|_\infty \int_X K(x,y)\, d\mu(y) \le \|f\|_\infty.$$

Therefore the measure kernel defining the Markov process on $(X, \mathscr{F}, \mu)$ *is given by*

$$P(x, A) \equiv T^* 1_A(x) = \int_X K(x,y) 1_A(y)\, d\mu(y) = \int_A K(x,y)\, d\mu(y).$$

Example 4.5.6. *Now we present an example of a different nature which appears in the context of Ergodic Theory.*

Let $(X, \mathscr{F}, \mu)$ *be a* σ*-finite measure space, and* $S : X \to X$ *a measurable transfor-*

mation of X *or a measurable dynamical system. Suppose that* μ *and* S *are chosen so that if* $\mu(A) = 0$*, then* $\mu(S^{-1}(A)) = 0$.

In this example the definition of the operator $T : L^1(X, \mathscr{F}, \mu) \to L^1(X, \mathscr{F}, \mu)$ *is somewhat involved. For each* $u \in L^1(X, \mathscr{F}, \mu)$ *its image by* T *is the element of* $L^1(X, \mathscr{F}, \mu)$ *given by*

$$Tu \equiv \frac{d\nu_u}{d\mu},$$

the Radon-Nikodym derivative of ν_u *with respect to* μ*, where the signed measure* ν_u *is given by the expression*

$$\nu_u(A) = \int_X 1_{S^{-1}(A)}(x) u(x)\ d\mu(x).$$

It is not trivial to verify that T *is well-defined. First step is to prove that* $\nu_u \ll \mu$*. Suppose that* A *is a measurable set and* $\mu(A) = 0$*. Then*

$$\nu_u(A) = \int_X 1_{S^{-1}(A)}(x) u(x)\ d\mu(x) = \int_{S^{-1}(A)} u(x)\ d\mu(x) = 0,$$

where the last equality follows from the hypothesis we are considering on S *and* μ*. The uniqueness of the Radon-Nikodym derivative ensures that if* $u_1 = u_2$ μ*-almost everywhere then* $d\nu_{u_1}/d\mu = d\nu_{u_2}/d\mu$ *and so the mapping* $u \longmapsto d\nu_u/d\mu$ *defines a function on* $L^1(X, \mathscr{F}, \mu)$*. It remains to prove that the image of this functions is also an element of* $L^1(X, \mathscr{F}, \mu)$*. Indeed, for any* $u \in L^1(X, \mathscr{F}, \mu)$ *we have* $|d\nu_u/d\mu| = d\nu_{u^+}/d\mu + d\nu_{u^-}/d\mu$ *and so*

$$\int_X \left|\frac{d\nu_u}{d\mu}\right| d\mu = \int_X \frac{d\nu_{u^+}}{d\mu}\, d\mu + \int_X \frac{d\nu_{u^-}}{d\mu}\, d\mu = \nu_{u^+}(X) + \nu_{u^-}(X) = \int_X |u|\, d\mu.$$

Of course, T *is linear and positive (which are simple consequences of the the Radon-Nikodym Theorem). From the above identity,* T *is contraction and therefore* T *defines a Markov process on* $(X, \mathscr{F}, \mu)$.

For this example, the computation of T^* *is more delicate. In general, it does not make sense to compose an element of* $L^\infty(X, \mathscr{F}, \mu)$ *(which is a set of equivalence classes) with a measurable function* $S : X \to X$*. The problem is the following. Suppose we want to make sense of* $[f] \circ S$*, where* $[f] \in L^\infty(X, \mathscr{F}, \mu)$*. Since* $[f]$ *is an equivalence class a natural meaning for* $[f] \circ S$ *would be the equivalence class* $[f \circ S]$*. This can be done if for any pair* $f_1, f_2 \in [f]$ *we have* $f_1 \circ S$ *and* $f_2 \circ S$ *in same equivalence class. In general, to* *know that* $\mu(\{x \in X : f_1(x) \neq f_2(x)\}) = 0$ *do not give us any information on the* μ *measure of the set* $\{x \in X : f_1 \circ S(x) \neq f_2 \circ S(x)\}$. *The hypothesis we are considering on* μ *and* S *in this example put us on a better shape. Since we are assuming that* $\mu(A) = 0$ *implies* $\mu(S^{-1}(A)) = 0$ *for any measurable set* A, *we can guarantee that* $\mu(\{x \in X : f_1 \circ S(x) \neq f_2 \circ S(x)\}) = 0$. *Indeed, let* $g \equiv f_1 - f_2$. *The set* $\{x \in X : f_1(x) \neq f_2(x)\} = g^{-1}(\mathbb{R} \setminus \{0\})$ *and the set* $\{x \in X : f_1 \circ S(x) \neq f_2 \circ S(x)\} = S^{-1} \circ g^{-1}(\mathbb{R} \setminus \{0\})$. *Therefore our hypothesis implies that whenever* $\mu(g^{-1}(\mathbb{R} \setminus \{0\})) = 0$ *we have* $\mu(S^{-1} \circ g^{-1}(\mathbb{R} \setminus \{0\})) = 0$. *This argument shows that in our case we can define* $[f] \circ S \equiv [f \circ S]$. *Moreover (omitting brackets from notation) the mapping taking* $f \mapsto f \circ S$ *defines a bounded linear operator from* $L^\infty(X, \mathscr{F}, \mu)$ *to itself. We will use this operator to compute* T^* *but before we need to present a preliminary computation.*

Suppose that $f \in L^\infty(X, \mathscr{F}, \mu)$ *is a simple function written in its standard form as* $f = \alpha_1 1_{A_1} + \ldots \alpha_n 1_{A_n}$. *Then for any* $u \in L^1(X, \mathscr{F}, \mu)$ *we have*

$$\int_X f \, d\nu_u = \sum_{j=1}^{n} \alpha_j \int_X 1_{S^{-1}(A_j)} u \, d\mu = \sum_{j=1}^{n} \alpha_j \int_X (1_{A_j} \circ S) \, u \, d\mu = \int_X (f \circ S) \, u \, d\mu.$$

From the above discussion we have concluded that the hypothesis on μ *and* S *implies that the linear operator on* $L^\infty(X, \mathscr{F}, \mu)$ *taking* $f \longmapsto f \circ S$ *is a continuous map. Let* $f \in L^\infty(X, \mathscr{F}, \mu)$ *be an arbitrary element. Without loss of generality, we can assume that* $|f(x)| \leq \|f\|_\infty$ *for all* $x \in X$. *In this case there is a sequence* $(f_n)_{n \in \mathbb{N}}$ *of simple functions in* $L^\infty(X, \mathscr{F}, \mu)$ *such that* $\|f_n - f\|_\infty \to 0$, *when* $n \to \infty$. *From Hölder inequality we get*

$$\int_X |(f_n \circ S) - (f \circ S)||u| \, d\mu \leq \|f_n \circ S - f \circ S\|_\infty \|u\|_1$$

which implies

$$\int_X (f \circ S) u \, d\mu = \lim_{n \to \infty} \int_X (f_n \circ S) u \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\nu_u = \int_X f \, d\nu_u,$$

where in the last equality we used that ν_u *is a finite measure and the Lebesgue Dominated Convergence Theorem.*

We claim that $T^* f = f \circ S$ *for all* $f \in L^\infty(X, \mathscr{F}, \mu)$. *Indeed, for any* $f \in L^\infty(X, \mathscr{F}, \mu)$

and $u \in L^1(X, \mathscr{F}, \mu)$ *we have*

$$\langle f, Tu \rangle = \int_X f\, Tu\, d\mu = \int_X f \frac{d\nu_u}{d\mu}\, d\mu = \int_X f\, d\nu_u = \int_X (f \circ S) u\, d\mu,$$

thus showing that $T^* f = f \circ S$.

The measure kernel associated to the Markov process T *is given by the expression*

$$P(x, A) = T^* 1_A(x) = 1_A \circ S(x) = 1_{S^{-1}(A)}(x).$$

Before computing, as in the previous examples, the fourth power of P, *we make a remark which is an application of the results obtained in Section 4.3, about the equivalence between the descriptions of Markov processes,*

$$\int_X f(y) P(x, dy) = Lf(x) = T^* f(x) = (f \circ S)(x).$$

Therefore we have the expression

$$\begin{aligned} P^4(x, A) &= \int_X \int_X \int_X P(y_3, A)\, P(y_2, dy_3)\, P(y_1, dy_2)\, P(x, dy_1) \\ &= \int_X \int_X \int_X [(1_A \circ S)(y_3)]\, P(y_2, dy_3)\, P(y_1, dy_2)\, P(x, dy_1) \\ &= \int_X \int_X (1_A \circ S^2)(y_2) P(y_1, dy_2)\, P(x, dy_1) \\ &= (1_A \circ S^4)(x). \end{aligned}$$

Example 4.5.7. *Let* $(X, \mathscr{F}, \mu)$ *a* σ*-finite measure space. Fix a measurable set* B *such that* $\mu(B) > 0$. *Consider the operator* $T : L^1(X, \mathscr{F}, \mu) \to L^1(X, \mathscr{F}, \mu)$ *given by* $Tu(x) = 1_B(x) u(x)$. *It is clear that* T *is a well-defined linear operator acting on* $L^1(X, \mathscr{F}, \mu)$, *contractive and positive.*

In this case is simple to compute T^*. *In fact, for any* $f \in L^\infty(X, \mathscr{F}, \mu)$ *and* $u \in L^1(X, \mathscr{F}, \mu)$ *we have*

$$\langle f, Tu \rangle = \int_X f \cdot 1_B \cdot u\, d\mu = \int_X T^* f \cdot u\, d\mu = \langle T^* f, u \rangle,$$

where $T^* f(x) = 1_B(x) f(x)$.

In this case the measure kernel P is given by

$$P(x, A) = T^*1_A(x) = 1_B(x)1_A(x) = 1_{A\cap B}(x).$$

Following the lines of the previous example we remark that

$$\int_X f(y)P(x, dy) = Lf(x) = T^*f(x) = 1_B(x)f(x).$$

Now it is simple to compute the powers of P. As before let us compute fourth power

$$\begin{aligned}
P^4(x, A) &= \int_X \int_X \int_X P(y_3, A)\, P(y_2, dy_3)\, P(y_1, dy_2)\, P(x, dy_1) \\
&= \int_X \int_X \int_X 1_{A\cap B}(y_3)\ P(y_2, dy_3)\, P(y_1, dy_2)\, P(x, dy_1) \\
&= \int_X \int_X 1_{A\cap B}(y_2)\ P(y_1, dy_2)\, P(x, dy_1) \\
&= 1_{A\cap B}(x).
\end{aligned}$$

Example 4.5.8. *Let $(X, \mathscr{F}, \mu)$ be a probability space, and let $\mathscr{B}$ a sub-sigma-algebra of $\mathscr{F}$. Define $T : L^1(X, \mathscr{F}, \mu) \to L^1(X, \mathscr{F}, \mu)$ as bellow*

$$Tu(x) = \mathbb{E}_\mu[u|\mathscr{A}](x),$$

where $\mathbb{E}_\mu[u|\mathscr{A}](x)$ is the conditional expectation of u with respect to the sub-sigma-algebra $\mathscr{A}$ on the probability space $(X, \mathscr{F}, \mu)$. From the elementary properties of the conditional expectation it is simple to see that T is a linear contractive positive operator on $L^1(X, \mathscr{F}, \mu)$.

In order to compute T^* we use the tower property of conditional expectation. For any $f \in L^\infty(X, \mathscr{F}, \mu)$ and $u \in L^1(X, \mathscr{F}, \mu)$ we have

$$\begin{aligned}
\langle f, Tu\rangle = \int_X f \cdot \mathbb{E}_\mu[u|\mathscr{A}]\, d\mu &= \mathbb{E}_\mu\big[f \cdot \mathbb{E}_\mu[u|\mathscr{A}]\big] \\
&= \mathbb{E}_\mu\big[f \cdot \mathbb{E}_\mu[u|\mathscr{A}]\big] = \mathbb{E}_\mu\Big[\mathbb{E}_\mu\big[f \cdot \mathbb{E}_\mu[u|\mathscr{A}]\big|\mathscr{A}\big]\Big] \\
&= \mathbb{E}_\mu\big[\mathbb{E}_\mu[f|\mathscr{A}] \cdot \mathbb{E}_\mu[u|\mathscr{A}]\big] = \mathbb{E}_\mu\big[\mathbb{E}_\mu[u \cdot \mathbb{E}_\mu[f|\mathscr{A}]|\mathscr{A}]\big] \\
&= \mathbb{E}_\mu\big[u \cdot \mathbb{E}_\mu[f|\mathscr{A}]\big] \\
&= \int_X u \cdot \mathbb{E}_\mu[f|\mathscr{A}]\, d\mu
\end{aligned}$$

$$= \langle \mathbb{E}_\mu[f|\mathscr{A}], u\rangle = \langle T^* f, u\rangle,$$

where $T^* f = \mathbb{E}_\mu[f|\mathscr{A}]$. *To guarantee that this expression is indeed the formula for* T^* *one have to check that* $f \longmapsto \mathbb{E}_\mu[f|\mathscr{A}]$ *defines a positive bounded linear operator on* $L^\infty(X, \mathscr{F}, \mu)$. *But this is an immediate consequence of the elementary properties of the conditional expectation.*

The expression for the measure kernel is the following:

$$P(x, A) = T^* 1_A(x) = \mathbb{E}_\mu[1_A|\mathscr{A}](x)$$

and therefore

$$\int_X f(y) P(x, dy) = Lf(x) = T^* f(x) = \mathbb{E}_\mu[f|\mathscr{A}](x).$$

which implies

$$\begin{aligned}
P^4(x, A) &= \int_X \int_X \int_X P(y_3, A)\, P(y_2, dy_3)\, P(y_1, dy_2)\, P(x, dy_1) \\
&= \int_X \int_X \int_X \mathbb{E}_\mu[1_A|\mathscr{A}](y_3)\ P(y_2, dy_3)\, P(y_1, dy_2)\, P(x, dy_1) \\
&= \int_X \int_X \mathbb{E}_\mu\big[\mathbb{E}_\mu[1_A|\mathscr{A}]\big|\mathscr{A}\big](y_2)\ P(y_1, dy_2)\, P(x, dy_1) \\
&= \mathbb{E}_\mu[f|\mathscr{A}](x).
\end{aligned}$$

Example 4.5.9 (Restrictions of a Process)**.** *In this example* $(X, \mathscr{F}, \mu, T)$ *will be a Markov process, and we assume that there is some* $Y \subset X$ *with* $Y \in \mathscr{F}$ *such that* $\mu(Y) > 0$, *and* $T^* 1_{Y^c}$ *is supported on* Y^c, *that is,* $1_Y \cdot T^* 1_{Y^c} = 0$ *in* $L^\infty(X, \mathscr{F}, \mu)$.

We say that $u \in L^1(X, \mathscr{F}, \mu)$ *is supported in* Y, *if* $1_{Y^c} \cdot u = 0$ *in* $L^1(X, \mathscr{F}, \mu)$. *Note that* u *is supported in* Y *if and only if* $|u|$ *is supported in* Y. *Indeed, if* $|u|$ *is supported in* Y, *then* $1_{Y^c}|u| = 1_{Y^c}(u^+ + u^-) = 0$. *Therefore* $0 \leq 1_{Y^c} u^\pm \leq 1_{Y^c}|u| = 0$ *and so* $1_{Y^c} \cdot u = 0$. *Conversely, if* $0 = 1_{Y^c} \cdot u$, *then* $0 = |1_{Y^c} \cdot u| = 1_{Y^c}|u|$.

We claim that if u *is supported in* Y, *then* Tu *is also supported in* Y. *Indeed, if* u *is supported in* Y, *then* $|u|$ *is also supported in* Y. *And so we have* $\langle 1_{Y^c}, T|u|\rangle = \langle T^* 1_{Y^c}, |u|\rangle = \langle (1_Y + 1_{Y^c}) \cdot T^* 1_{Y^c}, |u|\rangle = \langle 1_{Y^c} \cdot T^* 1_{Y^c}, |u|\rangle = \langle T^* 1_{Y^c}, 1_{Y^c} \cdot |u|\rangle = 0$, *where in the last equality we have used that* $|u|$ *is supported in* Y. *Therefore we have* $\langle 1_{Y^c}, T|u|\rangle = 0$ *which implies that* $1_{Y^c} T|u| = 0$ *in* $L^1(X, \mathscr{F}, \mu)$. *To prove the claim we will show that both* $1_{Y^c}(Tu)^\pm = 0$

in $L^1(X,\mathscr{F},\mu)$. Since T is a positive operator we have $0 \le 1_{Y^c}(Tu)^{\pm} = 1_{Y^c}\max\{\pm Tu, 0\} \le 1_{Y^c}\max\{T|u|, 0\} = 1_{Y^c}T|u| = 0$. Thus showing that $1_{Y^c}|Tu| = 0$ is in $L^1(X,\mathscr{F},\mu)$, which is equivalent to say that $1_{Y^c}Tu = 0$ is in $L^1(X,\mathscr{F},\mu)$ finishing the proof of the claim.

The aim of this example is to show that if Y is a measurable subset of X such that $\mu(Y) > 0$ and $T^*1_{Y^c}$ is supported on Y^c, then $(X,\mathscr{F},\mu,T)$ induces a Markov process $(Y,\mathscr{F}_Y,\mu_Y,T_Y)$, where $\mathscr{F}_Y = \{Y \cap F :$ where $F \in \mathscr{F}\}$, $\mu_Y \equiv \mu|_{\mathscr{F}_Y}$ and $T_Y : L^1(Y,\mathscr{F}_Y,\mu_Y) \to L^1(Y,\mathscr{F}_Y,\mu_Y)$ is a Markov process conjugated to $T|_{V_Y}$, where V_Y is the proper T-invariant subspace of $L^1(X,\mathscr{F},\mu)$ given by $V_Y \equiv \{u \in L^1(X,\mathscr{F},\mu) : u$ is supported on $Y\}$.

From the previous claim, we have $TV_Y \subset V_Y$. Note that the mapping $V_Y \ni u \longmapsto u|_Y \in L^1(Y,\mathscr{F}_Y,\mu_Y)$ is well-defined, since equivalence classes of functions modulo-μ are mapped to equivalence class of functions modulo-μ_Y. It is clearly an one-to-one mapping and onto. The surjectivity is proved by extending the elements of $L^1(Y,\mathscr{F}_Y,\mu_Y)$ as zero outside Y, that is, if $v : Y \to \mathbb{R}$ is an μ_Y-integrable function then

$$u(x) \equiv \begin{cases} v(x) & \text{if } x \in Y; \\ 0 & \text{otherwise}, \end{cases}$$

defines a $\mathscr{F}$-measurable function. Moreover, if $u_1 = u_2$ μ_Y-almost everywhere, then their extensions belong to same equivalence class in V_Y, thus showing that there is a linear isomorphism $i : L^1(Y,\mathscr{F}_Y,\mu_Y) \to V_Y$. In fact, this isomorphism is a linear isometry as shown below

$$\int_X |u|\, d\mu = \int_X (1_{Y^c} + 1_Y)|u|\, d\mu = \int_X 1_Y|u|\, d\mu = \int_Y |\, u|_Y\,|\, d\mu_Y.$$

Note that the following diagram induces a map T_Y from $L^1(Y,\mathscr{F}_Y,\mu_Y)$ to itself, which is a Markov process

$$\begin{array}{ccc} V_Y & \xrightarrow{\quad T \quad} & V_Y \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle i^{-1}} \\ L^1(Y,\mathscr{F}_Y,\mu_Y) & \xrightarrow[\quad T_Y \quad]{} & L^1(Y,\mathscr{F}_Y,\mu_Y). \end{array}$$

In fact, the mapping $T_Y = i^{-1} \circ T \circ i$ is positive one, since it is a composition of positive maps. Contractiveness is a consequence of the contractive property of both maps i, T and

i^{-1}.

Before finish this example we provide a necessary and sufficient condition to ensure that $T^*1_{Y^c}$ is supported on Y^c.

Let T be a Markov operator on $L^1(X, \mathscr{F}, \mu)$ and $Y \in \mathscr{F}$. Then $T^*1_{Y^c}$ is supported on Y^c if and only if $T^*1_{Y^c} \leq 1_{Y^c}$. Indeed, let us first assume that $T^*1_{Y^c}$ is supported on Y^c, that is, $1_Y \cdot T^*1_{Y^c} = 0$. So, $T^*1_{Y^c} = (1_Y + 1_{Y^c})T^*1_{Y^c} = 1_{Y^c}T^*1_{Y^c} \leq 1_{Y^c}$. Conversely, if $T^*1_{Y^c} \leq 1_{Y^c}$, then we have $0 \leq 1_Y T^*1_{Y^c} \leq 1_Y 1_{Y^c} = 0$, meaning that $T^*1_{Y^c}$ is supported on Y^c.

A last remark. The inequality $T^*1_{Y^c} \leq 1_{Y^c}$ is a consequence of $1_Y \leq T^*1_Y$. Indeed, the contraction property of T^* implies that $T^*1_Y + T^*1_{Y^c} \leq 1 = 1_Y + 1_{Y^c}$ and therefore $0 \leq T^*1_Y - 1_Y \leq 1_{Y^c} - T^*1_{Y^c}$ thus showing that $T^*1_{Y^c} \leq 1_{Y^c}$.

Example 4.5.10. *If T_1 and T_2 are Markov process on $(X, \mathscr{F}, \mu)$, then for all $0 \leq \alpha, \beta \leq 1$ such that $\alpha + \beta \leq 1$ we have that $\alpha T_1 + \beta T_2$ is a Markov process on $(X, \mathscr{F}, \mu)$.*

Example 4.5.11. *Let $(X, \mathscr{F}, \mu, T)$ be an arbitrary Markov process and consider the collection $\mathscr{F}_d \equiv \{A \in \mathscr{F} : (T^*)^n 1_A = 1_{B_n}, \text{where } B_n \in \mathscr{F}, \ \forall n \in \mathbb{N}\}$. Then*

i) $\mathscr{F}_d$ is closed for countable unions and differences;

ii) if $A \in \mathscr{F}_d$ then $T^*1_A = 1_B$ for some $B \in \mathscr{F}_d$.

Let us first prove item i). For each $k \in \mathbb{N}$ consider the collection of measurable sets $\mathscr{F}^k \equiv \{A \in \mathscr{F} : (T^*)^k 1_A = 1_B, \text{ for some } B \in \mathscr{F}\}$. Of course, $\mathscr{F}_d = \cap_{k \in \mathbb{N}} \mathscr{F}^k$. Therefore it is enough to prove that $\mathscr{F}^k$, for all $k \in \mathbb{N}$ is closed for countable unions and differences. We begin by proving, for any fixed $k \in \mathbb{N}$, that $\mathscr{F}^k$ is closed by finite unions. So we fix A_1 and A_2 in $\mathscr{F}^k$ and let $B_1, B_2 \in \mathscr{F}$ be so that $(T^*)^k 1_{A_i} = 1_{B_i}$. Then from definitions and positivity of T^* we get these inequalities:

$$\begin{aligned} 1_{B_1 \cup B_2} &= \max\{1_{B_1}, 1_{B_2}\} = \max\{(T^*)^k 1_{A_1}, (T^*)^k 1_{A_2}\} \\ &\leq (T^*)^k 1_{A_1 \cup A_2} \leq (T^*)^k 1_{A_1} + (T^*)^k 1_{A_2} = 1_{B_1} + 1_{B_2}. \end{aligned} \tag{4.3}$$

From (4.3), we see that $1_{B_1 \cup B_2} \leq (T^*)^k 1_{A_1 \cup A_2} \leq 1$. And so for μ almost all $x \in B_1 \cup B_2$, we have $((T^*)^k 1_{A_1 \cup A_2})(x) = 1$. From the inequalities in (4.3) we also have $0 \leq (T^*)^k 1_{A_1 \cup A_2} \leq$

$1_{B_1} + 1_{B_2}$ and therefore if $x \notin B_1 \cup B_2$ then $((T^*)^k 1_{A_1 \cup A_2})(x) = 0$, thus proving that $(T^*)^k 1_{A_1 \cup A_2} = 1_{B_1 \cup B_2}$, and consequently that $\mathscr{F}^k$ is closed for finite unions.

Next, we prove that $\mathscr{F}^k$ is closed by differences. Since $A_1 - A_2 = (A_1 \cup A_2) - A_2$ there is no loss of generality in assuming that $A_2 \subset A_1$. Under this assumption, we have $1_{B_2} = (T^*)^k 1_{A_2} \leq (T^*)^k 1_{A_1} = 1_{B_1}$, which implies $B_2 \subset B_1$. From the identity $1_{A_1 - A_2} = 1_{A_1} - 1_{A_2}$ and last observation, $(T^*)^k 1_{A_1 - A_2} = 1_{B_1} - 1_{B_2} = 1_{B_1 - B_2}$.

Now in order to prove that $\mathscr{F}^k$ is closed by countable unions it is enough to prove that if $(A_n)_{n \in \mathbb{N}}$ is an increasing sequence of sets in $\mathscr{F}^k$ such that $A_n \uparrow A$, then $A \in \mathscr{F}^k$. Indeed, if $A_n \uparrow A$ then the sequence $(B_n)_{n \in \mathbb{N}}$ given by $1_{B_n} = (T^*)^k 1_{A_n}$ is an increasing sequence (same argument as in previous paragraph) and converges to $B \equiv \cup_{n \in \mathbb{N}} B_n$. From positivity, we can immediately check that up to μ-zero sets, we have pointwise

$$(T^*)^k 1_A = \lim_{n \to \infty} (T^*)^k 1_{A_n} = \lim_{n \to \infty} 1_{B_n} = 1_B.$$

So, $\mathscr{F}^k$ is closed for countable unions and differences, which in turn implies, as mentioned before, that $\mathscr{F}_d$ has these two properties.

Note that the collection $\mathscr{F}_d$ needs not to be a sigma-algebra. Because, in general, there is no guarantee that X belongs to $\mathscr{F}_d$.

In the notation $\mathscr{F}_d$, the letter d stands for deterministic. A Markov process $(X, \mathscr{F}, \mu, T)$ is called deterministic if $\mathscr{F}_d = \mathscr{F}$. The Markov process defined in Example 4.5.6 is a deterministic Markov process, in this sense.

Example 4.5.12. *Let $(X, \mathscr{F}, \mu, T)$ be an arbitrary Markov process and consider the collection $\mathscr{F}_i \equiv \{A \in \mathscr{F} : T^* 1_A = 1_A\}$. These are called invariant sets. The same reasoning as above shows that $\mathscr{F}_i$ is closed for countable unions and differences.*

CHAPTER 5

CONSERVATIVE/DISSIPATIVE DECOMPOSITION

In this chapter we present the conservative/dissipative decomposition of a Markov Processes and its consequence on the conditions for existence of invariant functions. The main tool to prove those facts is the Hopf Maximal Ergodic Lemma, presented in Section 5.1 and first proved by Hopf in [30]. The proof presented here is an elegant one due to A. Garsia in [26].

As consequence of this lemma, we will show how to obtain the so-called convervative-dissipative decomposition of a Markov process. The approach to this decomposition is the same adopted by Jacques Neveu in [42]. We will see that a invariant function, or maximal eigenfunction, must be supported on the conservative part of the process.

The last section about subinvariant functions (and measures) is based on the work of Jacob Feldman [22]. We will see that a subinvariant function supported on the conservative part is actually an invariant function.

5.1 Hopf Maximal Ergodic Lemma

Throughout this chapter $(X, \mathscr{F}, \mu, T)$ is a fixed Markov process.

Lemma 5.1.1 (Hopf Maximal Ergodic Lemma)**.** *Let* $(X, \mathscr{F}, \mu, T)$ *be a fixed Markov* *process, and* $u \in L^1(X, \mathscr{F}, \mu)$*. Consider the measurable set*

$$E \equiv \left\{ x \in X : \sup_{n \in \mathbb{N}} \big(u(x) + Tu(x) + \ldots + T^{n-1}u(x) \big) > 0 \right\}.$$

Then $\int_E u \, d\mu \geq 0$.

Proof. The delicate issue here is that u may take negative values. For each non-negative integer k, we define a bounded linear operator $S_k : L^1(X, \mathscr{F}, \mu) \to L^1(X, \mathscr{F}, \mu)$ as follows. Set $S_0 u = 0$, for all $u \in L^1(X, \mathscr{F}, \mu)$, that is, the null operator, and for $k \geq 1$

$$S_k u \equiv u + Tu + \ldots + T^{k-1}u.$$

For $u \in L^1(X, \mathscr{F}, \mu)$, we define

$$M_n u \equiv \max_{0 \leq k \leq n} S_k u.$$

Since $S_0 u = 0$ for any $u \in L^1(X, \mathscr{F}, \mu)$, we have $M_n u \geq 0$. We remark that $M_n u \in L^1(X, \mathscr{F}, \mu)$, for all $u \in L^1(X, \mathscr{F}, \mu)$. For each fixed $u \in L^1(X, \mathscr{F}, \mu)$ consider the set

$$E_n \equiv \{ x \in X : M_n u(x) > 0 \}.$$

Note that E_n actually depends on u, but we will omit this dependence because there is no danger of confusion.

The definition of M_n implies that $M_n u \leq M_{n+1} u$ and therefore if $x \in E_n$ we have that $x \in E_{n+1}$, from which $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of subsets of X. Moreover, the set E defined on the statement of the lemma is the limit of this sequence, that is, $E = \lim_{n \to \infty} E_n$. From Lebesgue Dominated Convergence Theorem,

$$\int_E u \, d\mu = \int_X 1_E u \, d\mu = \int_X \lim_{n \to \infty} 1_{E_n} u \, d\mu = \lim_{n \to \infty} \int_X 1_{E_n} u \, d\mu = \lim_{n \to \infty} \int_{E_n} u \, d\mu$$

and therefore it suffices to show that $\int_{E_n} u \, d\mu \geq 0$ for all $n \in \mathbb{N}$.

From the definition, for any $0 \leq k \leq n$, we have $S_k u \leq M_n u$. Since T is a positive operator, $T S_k u \leq T M_n u$. By summing u to both sides of this inequality we get $u + T S_k u \leq u + T M_n u$. From this last inequality and the definition of the sequence $(S_n)_{n \in \mathbb{N}}$ it follows that

$$S_{k+1} u \leq u + T M_n u, \qquad \text{for } 0 \leq k \leq n. \tag{5.1}$$

If $x \in E_n$, then $M_n u(x) > 0$ and so

$$M_n u(x) = \max_{1 \le k \le n} S_k u(x)$$

since $S_0 u(x) = 0$. By taking $x \in E_n$ and applying the inequality (5.1) for $1 \le k+1 \le n$ we get

$$M_n(u)(x) = \max_{0 \le k \le n} S_k u(x) \le u(x) + T M_n u(x)$$

which implies

$$M_n(u)(x) - T M_n u(x) \le u(x).$$

Integrating by sides on E_n we get

$$\int_{E_n} (M_n(u) - T M_n u)\, d\mu \le \int_{E_n} u \, d\mu.$$

By definition, if $x \notin E_n$, then $M_n u(x) = 0$. From the positivity of T and the definition of M_n, for any $x \in X$, we have $T M_n u(x) \ge 0$. These two observations can be used to get a lower bound for the left hand side above which is

$$\int_X (M_n(u) - T M_n u)\, d\mu \le \int_{E_n} (M_n(u) - T M_n u)\, d\mu \le \int_{E_n} u \, d\mu. \tag{5.2}$$

From the non-negativity of $M_n u$, and positivity and the contraction property of T, we have

$$\int_X T M_n u \, d\mu = \|T M_n u\|_1 \le \|M_n u\|_1 = \int_X M_n u \, d\mu.$$

This implies that the left hand side of (5.2) is non-negative, which finishes the proof. □

The first important consequence of the Maximal Ergodic Lemma concerns the convergence properties of the sums of the iterates of the process. To be more precise, fix a pair $u, v \in L^1(X, \mathscr{F}, \mu)$ with $u, v \ge 0$ and consider the following set

$$A_{u,v} \equiv \left\{ x \in X : \quad \sum_{n=0}^{\infty} T^n u(x) = +\infty, \quad \sum_{n=0}^{\infty} T^n v(x) < +\infty \right\}.$$

Therefore, for any $\alpha > 0$ and $x \in A_{u,v}$, we have

$$\sum_{n=0}^{\infty} T^n (u - \alpha u)(x) = +\infty.$$

In particular, for any $\alpha > 0$, we have

$$A_{u,v} \subset \left\{ x \in X : \sup_{n \in \mathbb{N}} \sum_{j=0}^{n} T^j (u - \alpha v)(x) > 0 \right\} \equiv B_\alpha(u,v).$$

By the Maximal Ergodic Lemma, positivity of u, v, and the above continence we have, for any $\alpha > 0$, the following inequality

$$0 \leq \int_{B_\alpha(u,v)} (u - \alpha v)\, d\mu \leq \int_X u\, d\mu - \alpha \int_{A_{u,v}} v\, d\mu.$$

Since this inequality holds as $\alpha \to +\infty$, the second integral on the right hand side above has to be zero. By using a similar argument we get, for any $k \in \mathbb{N}$, the inequality below

$$0 \leq \int_{B_\alpha(u,T^k v)} (u - \alpha T^k v)\, d\mu \leq \int_X u\, d\mu - \alpha \int_{A_{u,T^k v}} T^k v\, d\mu.$$

By taking the limit when $\alpha \to +\infty$ we have that the last integral above is zero. Note that for any $k \in \mathbb{N}$ we have the identity $A_{u,v} = A_{u,T^k v}$. Therefore

$$\int_{A_{u,v}} T^k v\, d\mu = 0, \qquad \forall k \in \mathbb{N}.$$

By the Monotone Convergence Theorem,

$$\int_{A_{u,v}} \sum_{n=0}^{\infty} T^n v\ d\mu = 0.$$

And we can conclude that

$$\sum_{n=0}^{\infty} T^n v(x) = 0, \qquad \forall x \in A_{u,v}. \tag{5.3}$$

As a consequence of (5.3) we have the following proposition, which will be used to construct the dissipative-conservative decomposition of X.

Proposition 5.1.2. *Let $u, v \in L^1(X, \mathscr{F}, \mu)$ with $u, v \geq 0$. For any $x \in X$ such that $\sum_{n=0}^{\infty} T^n u(x) = +\infty$, we have either*

$$\sum_{n=0}^{\infty} T^n v(x) = +\infty \qquad \textit{or} \qquad \sum_{n=0}^{\infty} T^n v(x) = 0. \tag{5.4}$$

Proof. Let $x \in X$ be such that $\sum_{n=0}^{\infty} T^n u(x) = +\infty$. If $x \in A_{u,v}$, then, from (5.3), $\sum_{n=0}^{\infty} T^n v(x) = 0$. On the other hand, if $x \notin A_{u,v}$, from the definition of $A_{u,v}$, we have $\sum_{n=0}^{\infty} T^n v(x) = +\infty$. □

Theorem 5.1.3. *Let $(X, \mathscr{F}, \mu, T)$ be a Markov Process and $u, v \in L^1(X, \mathscr{F}, \mu)$ an arbitrary pair of strictly positive functions. Then*

$$\left\{x \in X : \sum_{n=0}^{\infty} T^n u(x) = +\infty\right\} \equiv \mathscr{C}(u) = \mathscr{C}(v) \equiv \left\{x \in X : \sum_{n=0}^{\infty} T^n v(x) = +\infty\right\}.$$

Proof. We begin by proving that in any sigma-finite measure space $(X, \mathscr{F}, \mu)$ there are an infinite number of strictly positive functions in $L^1(X, \mathscr{F}, \mu)$. Indeed, if $\mu(X) < +\infty$, then any positive constant function is in $L^1(X, \mathscr{F}, \mu)$. If $\mu(X) = +\infty$, let $(B_n)_{n \in \mathbb{N}}$ be a sequence of increasing sets such that $B_n \uparrow X$, $\mu(B_n) < +\infty$ and $\mu(B_{n+1} \setminus B_n) > 0$. Then the function $u : X \to \mathbb{R}$ given by

$$u(x) \equiv \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\mu(B_{n+1} \setminus B_n)} 1_{B_{n+1} \setminus B_n}(x)$$

is strictly positive and in $L^1(X, \mathscr{F}, \mu)$. By considering the family $\{\alpha u\}_{\alpha > 0}$ the claim follows.

Now let u, v a pair of strictly positive functions in $L^1(X, \mathscr{F}, \mu)$. Consider the set $\mathscr{C}(u) \equiv \{x \in X : \sum_{n=0}^{\infty} T^n u(x) = +\infty\}$. If $x \in \mathscr{C}(u)$ then we have from Proposition 5.1.2 that either $\sum_{n=0}^{\infty} T^n v(x) = +\infty$ or $\sum_{n=0}^{\infty} T^n v(x) = 0$. Since $v > 0$ we must have $\sum_{n=0}^{\infty} T^n v(x) = +\infty$, which in turn implies $x \in \mathscr{C}(v)$ and therefore $\mathscr{C}(u) \subset \mathscr{C}(v)$. By switching u and v in the above argument we get $\mathscr{C}(u) = \mathscr{C}(v)$. □

Definition 5.1.4. *Let $(X, \mathscr{F}, \mu, T)$ be a Markov process and $u \in L^1(X, \mathscr{F}, \mu)$ be an arbitrary strictly positive function. We call $\mathscr{C}(u)$ and $\mathscr{D}(u) \equiv X \setminus \mathscr{C}(u)$ the conservative and dissipative parts of the process, respectively. Theorem 5.1.3 implies that the there is a set $\mathscr{C} = \mathscr{C}(u)$ which is independent of the choice of u. And so we have a well-defined decomposition $X = \mathscr{C} \cup \mathscr{D}$ called conversative-dissipative decomposition of the Markov process T.*

Proposition 5.1.5. *Let $(X, \mathscr{F}, \mu, T)$ be a Markov process, $u \in L^1(X, \mathscr{F}, \mu)$ be an arbitrary non-negative function, and $X = \mathscr{C} \cup \mathscr{D}$ the conservative-dissipative decomposition of X with respect to T. Then*

1. *for all* $x \in \mathscr{D}$ *we have* $\sum_{n=0}^{\infty} T^n u(x) < +\infty$*;*

2. *for all* $x \in \mathscr{C}$*, we have* $\sum_{n=0}^{\infty} T^n u(x) = 0$ *or* $\sum_{n=0}^{\infty} T^n u(x) = +\infty$*;*

Proof. Item *(1)*. Let $0 < v \in L^1(X, \mathscr{F}, \mu)$ and $x \in \mathscr{D}$. Proposition 5.1.2 implies that, if $\sum_{n=0}^{\infty} T^n u(x) = +\infty$, then $\sum_{n=0}^{\infty} T^n v(x) = +\infty$. This means that $x \in \mathscr{C}$, which is a contradiction since $\mathscr{C} \cap \mathscr{D} = \varnothing$. Therefore, if $x \in \mathscr{D}$, we must have $\sum_{n=0}^{\infty} T^n u(x) < +\infty$.

Item *(2)*. If $x \in \mathscr{C}$ and $0 < v \in L^1(X, \mathscr{F}, \mu)$, then $\sum_{n=0}^{\infty} T^n v(x) = +\infty$. From Proposition 5.1.2 we get that $\sum_{n=0}^{\infty} T^n u(x) = 0$ or $\sum_{n=0}^{\infty} T^n u(x) < +\infty$. □

We can also characterize the conservative-dissipative decomposition in terms of T^* acting on $L^\infty(X, \mathscr{F}, \mu)$. For example, condition *(2)* of Proposition 5.1.5 has a "dual condition" which is stated as follows.

Proposition 5.1.6. *Let* $(X, \mathscr{F}, \mu, T)$ *be a Markov process and* $f \in L^\infty(X, \mathscr{F}, \mu)$ *be an arbitrary non-negative function. Then, for all* $x \in \mathscr{C}$ *we have either*

$$\sum_{n=0}^{\infty} (T^*)^n f(x) = 0 \quad \text{or} \quad \sum_{n=0}^{\infty} (T^*)^n f(x) = +\infty.$$

Proof. We fix $0 \le f \in L^\infty(X, \mathscr{F}, \mu)$ and assume that there is a subset $A \subset \mathscr{C}$ with $\mu(A) > 0$ on which

$$\sum_{n=0}^{\infty} (T^*)^n f(x) \le M < \infty, \qquad \forall x \in A;$$

for some positive constant M. Take a strictly positive function $u \in L^1(X, \mathscr{F}, \mu)$ and consider the function $v \equiv 1_A u$. Then we have, for any $k \in \mathbb{N}$, the following uniform estimate

$$\left\langle (T^*)^k f,\ 1_A \sum_{n=0}^{\infty} T^n v \right\rangle \le \left\langle (T^*)^k f,\ \sum_{n=0}^{\infty} T^n v \right\rangle = \left\langle \sum_{n=k}^{\infty} (T^*)^n f, v \right\rangle \le M \|v\|_1 < +\infty$$

Since $A \subset \mathscr{C}$, it follows that $1_A \sum_{n=0}^{\infty} T^n v$ is a constant function equal to either zero or plus infinity. As $v > 0$ on A, this series is divergent. And so the above upper bound forces $(T^*)^k f = 0$ on A, for all $k \in \mathbb{N}$, and consequently

$$\sum_{n=0}^{\infty} (T^*)^n f(x) = 0, \qquad \forall x \in A. \tag{5.5}$$

Let us assume that $\mu(\mathscr{C}) > 0$, otherwise there is nothing to prove. For each $n \in \mathbb{N}$ define the set

$$A_n \equiv \left\{ x \in \mathscr{C} : \sum_{k=0}^{\infty} (T^*)^k f(x) \leq n \right\} \tag{5.6}$$

and put

$$A_\infty \equiv \left\{ x \in \mathscr{C} : \sum_{k=0}^{\infty} (T^*)^k f(x) = +\infty \right\}. \tag{5.7}$$

Note that $A_\infty \cup (\cup_{n\in\mathbb{N}} A_n) = \mathscr{C}$. Clearly, $(A_n)_{n\in\mathbb{N}}$ defines a sequence of increasing sets. And so if $\mu(A_n) = 0$ for all $n \in \mathbb{N}$ then $A_\infty = \mathscr{C}$ modulo a set of μ-measure zero. Otherwise, there is some $n_0 \in \mathbb{N}$ such that $\mu(A_{n_0}) > 0$. In this case, of course, $\mu(A_n) > 0$ for all $n \geq n_0$. Therefore, for any $x \in \mathscr{C} \setminus A_\infty$, there is $n \geq n_0$ such that $x \in A_n$ and $\mu(A_n) > 0$. So we can apply (5.5) to guarantee that

$$\sum_{n=0}^{\infty} (T^*)^n f(x) = 0, \qquad \forall x \in \mathscr{C} \setminus A_\infty.$$

Thus proving the proposition. □

The condition in item *(1)* of Proposition 5.1.5 has also a "dual condition" but the corresponding statement is more subtle. In general, it is false that if $0 \leq f \in L^\infty(X, \mathscr{F}, \mu)$ and $x \in \mathscr{D}$ then $\sum_{n=0}^{\infty} (T^*)^n f(x) < +\infty$. In fact, we can construct a Markov process T for which $X = \mathscr{D}$ and $T^* 1 = 1$. Example 4.5.1 gives a process with those characteristics.

Proposition 5.1.7. *Let $(X, \mathscr{F}, \mu, T)$ be a Markov process. Assume that the dissipative set satisfies $\mu(\mathscr{D}) > 0$. Then there is an increasing sequence of measurable sets $(D_n)_{n\in\mathbb{N}}$ such that $D_n \uparrow \mathscr{D}$ and for any fixed $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{\infty} (T^*)^k 1_{D_n}(x) \equiv \sum_{k=0}^{\infty} P^k(x, D_n) < +\infty, \qquad \forall x \in X.$$

Proof. Take $0 < u \in L^1(X, \mathscr{F}, \mu)$. From the definition of the dissipative set we have $\sum_{n=0}^{\infty} T^n u(x) < +\infty$ for all $x \in \mathscr{D}$. Let $f \in L^\infty(X, \mathscr{F}, \mu)$ be chosen so that $f|_{\mathscr{C}} \equiv 0$, $f(x) > 0$ for all $x \in \mathscr{D}$ and

$$\left\langle f, \sum_{n=0}^{\infty} T^n u \right\rangle < +\infty. \tag{5.8}$$

An explicit example of such a function can be constructed as follows. Suppose that $\mu(X) = +\infty$ and let $(B_n)_{n\in\mathbb{N}}$ be an increasing sequence of measurable sets so that $B_n \uparrow X$,

$\mu(B_n) < +\infty$, and $\mu(B_{n+1} \setminus B_n) > 0$, for all $n \in \mathbb{N}$. Define $f : X \to \mathbb{R}$ by the expression

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{2^n} \min\left(1, \ \frac{1}{\mu(B_{n+1} \setminus B_n)} \frac{1}{\sum_{n=0}^{\infty} T^n u(x)}\right) 1_{B_{n+1} \setminus B_n}(x).$$

Since we are assuming $u > 0$, from the basic properties of the Lebesgue integral and the integrability condition (5.8), we have $\sum_{k=0}^{\infty}(T^*)^k f(x) < +\infty, \forall x \in X$. Now, for each $n \in \mathbb{N}$, consider the set $D_n := \{x \in X : f(x) \geq 1/n\}$. Clearly, $(1/n)1_{D_n} \leq f$ and $D_n \uparrow \mathscr{D}$. Hence given $n \in \mathbb{N}$ we have, for all $x \in X$, the inequality below

$$\frac{1}{n} \sum_{k=0}^{\infty} (T^*)^k 1_{D_n}(x) \leq \sum_{k=0}^{\infty} (T^*)^k f(x) < +\infty.$$

Thus there is an increasing sequence of measurable sets $D_n \uparrow \mathscr{D}$ so that for each fixed $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{\infty} P^k(x, D_n) < +\infty, \qquad \forall x \in X.$$ □

As an example of the preceding discussion, we consider a Markov process T on $(X, \mathscr{F}, \mu)$ defined as in Example 4.5.6 where $T^* f(x) = f(S(x))$, and the transformation $S : X \to X$ is not singular with respect to μ, meaning that $\mu(A) = 0$ implies $\mu(S^{-1}(A)) = 0$. Let us assume that $\mu(\mathscr{D}) > 0$. From the previous proposition we know that there is some measurable subset $D \subset \mathscr{D}$ such that $\mu(D) > 0$ and for all $x \in D$ we have

$$\sum_{k=0}^{\infty} 1_{S^{-k}(D)}(x) = \sum_{k=0}^{\infty} P^k(x, D) < +\infty.$$

Since each term in the series on the left hand side above is either zero or one, we have, for a fixed $x \in \mathscr{D}$, that the set $\{k \in \mathbb{N} : x \in S^{-k}(D)\}$ has finite cardinality. Therefore

$$\bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} S^{-k}(D) = \varnothing.$$

This in turn implies that the set

$$E \equiv \bigcup_{n=0}^{\infty} S^{-k}(D)$$

is such that $D \subset E$, $S^{-1}(E) \subset E$, and $S^{-n}(E) \downarrow \varnothing$. In particular, $E \neq X$ since $S^{-n}(E) \downarrow \varnothing$. We also remark that $\mu(E \setminus S^{-1}(E)) > 0$. Indeed, since S is not singular with respect to μ, if $\mu(E \setminus S^{-1}(E)) = 0$ then $\mu(E) = \mu(S^{-1}(E))$, and $0 = \mu(S^{-1}(E \setminus S^{-1}(E))) = \mu(S^{-1}(E) \setminus S^{-2}(E)))$, which implies $\mu(E) = \mu(S^{-1}(E)) = \mu(S^{-2}(E))$. Proceeding by induction we get that $\mu(E) = \mu(S^{-n}(E)) \to 0$, when $n \to \infty$. But $D \subset E$ and $\mu(D) > 0$ and so we have a contradiction.

To summarize, we have proved that if $T^* f = f(S(x))$ and $\mu(\mathscr{D}) > 0$, then there exists a measurable set E such that $S^{-1}(E) \subset E$ and $\mu(E \setminus S^{-1}(E)) > 0$. In the next section we will show that if $X = \mathscr{C}$ then such a set E does not exist.

5.2 Subinvariant Measures and Functions

This section also deals with consequences of the Maximal Ergodic Lemma. We now explore this lemma to obtain the behavior of subinvariant functions and measures which are defined in the sequel.

Let $(X, \mathscr{F}, \mu, T)$ be a Markov process. A function $f \in L^\infty(X, \mathscr{F}, \mu)$ will be called subinvariant if $T^* f \leq f$. We are mostly interested in conditions under which subinvariance implies the strong statement $T^* f = f$.

Assume that all the hypothesis of Proposition 5.1.7 holds. Then, from the positivity of T^*, for each fixed $n \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{k=0}^{\infty} (T^*)^k (T^* 1_{D_n})(x) &= \sum_{k=0}^{\infty} (T^*)^{k+1} 1_{D_n}(x) = \sum_{k=1}^{\infty} (T^*)^k 1_{D_n}(x) \\ &\leq \sum_{k=0}^{\infty} (T^*)^k 1_{D_n}(x) = \sum_{k=0}^{\infty} P^k(x, D_n) < +\infty,\end{aligned}$$

for all $x \in X$. So, from Proposition 5.1.6, for any $x \in \mathscr{C}$, the series in left hand side above is zero. In particular, its first term $T^* 1_{D_n}(x) = 0$ for all $x \in \mathscr{C}$. Since $D_n \uparrow \mathscr{D}$ and from the final result in Section 4.4, $T^* 1_{\mathscr{D}}(x) = 0$ for all $x \in \mathscr{C}$. Therefore

$$T^* 1_{\mathscr{D}} \leq 1_{\mathscr{D}} \iff 1_{\mathscr{C}} T^* 1_{\mathscr{D}} = 0.$$

This is precisely the condition used in Example 4.5.9 to define a restriction of a Markov process. The restricted process $(\mathscr{C}, \mathscr{F}_{\mathscr{C}}, \mu_{\mathscr{C}}, T_{\mathscr{C}})$ has no dissipative part, that is, if $0 < u \in$

$L^1(\mathscr{C}, \mathscr{F}_{\mathscr{C}}, \mu_{\mathscr{C}})$ then

$$\sum_{k=0}^{\infty}(T_{\mathscr{C}})^k u(x) = +\infty, \qquad \text{for} \mu\text{-almost every } x \in \mathscr{C}.$$

For the next result, instead of working with the Markov process T on $L^1(X, \mathscr{F}, \mu)$ or its Banach transpose T^* on $L^\infty(X, \mathscr{F}, \mu)$, we will consider their extensions $\overline{T}$ and $\overline{T^*}$, respectively, as defined in Section 4.4. But to lighten the notation we will denote theses extensions simply by T and T^*. We are making this comment here to emphasize that we will not require that the function f in the next theorem is in $L^\infty(X, \mathscr{F}, \mu)$, it is only a non-negative and almost surely finite function.

Theorem 5.2.1. *Let $(X, \mathscr{F}, \mu, T)$ be a Markov process, and assume that the conservative-dissipative decomposition $X = \mathscr{C} \cup \mathscr{D}$ is so that $\mu(\mathscr{D}) = 0$. Suppose that there is some $\mathscr{F}$-measurable function $f : X \to \mathbb{R}$ such that $0 \le f < +\infty$ and $T^* f \le f$. Then $T^* f = f$.*

Proof. Let $f : X \to \mathbb{R}$ be $\mathscr{F}$-measurable function such that $0 \le f < +\infty$ and take $0 < u \in L^1(X, \mathscr{F}, \mu)$ satisfying

$$\int_X f u \, d\mu < +\infty.$$

Although f is not necessarily an element in $L^\infty(X, \mathscr{F}, \mu)$, we still denote the above integral by $\langle f, u\rangle$. By using a telescopic argument we get, uniformly in n, the following estimate

$$\left\langle f - T^* f, \sum_{k=0}^{n} T^k u \right\rangle = \langle f, u\rangle - \left\langle (T^*)^{n+1} f, u \right\rangle < \langle f, u\rangle < +\infty.$$

Since we are assuming $T^* f \le f$ we can apply the Fatou Lemma on the left hand side above obtaining

$$\left\langle f - T^* f, \sum_{k=0}^{\infty} T^k u \right\rangle \le \langle f, u\rangle < +\infty.$$

From the hypothesis $\mu(\mathscr{D}) = 0$, we know that $\sum_{k=0}^{\infty} T^k u(x) = +\infty$ for μ-almost all $x \in X$. This observation together with the above inequality imply $f - T^* f = 0$ μ-almost surely, which completes the proof. □

Corollary 5.2.2. *Let $(X, \mathscr{F}, \mu, T)$ be a Markov process, and $X = \mathscr{C} \cup \mathscr{D}$ its conservative-dissipative decomposition. Suppose that there is some $\mathscr{F}$-measurable function $f : X \to \mathbb{R}$ such that $0 \le f < +\infty$ and $T^* f \le f$ on $\mathscr{C}$. Then $T^* f = f$ on $\mathscr{C}$.*

Proof. If $\mu(\mathscr{D}) = 0$ then we have all the hypothesis of Theorem 5.2.1 and there is nothing to do. To deal with the case $\mu(\mathscr{D}) > 0$ the idea is to consider the restriction of the Markov process to the conservative part. To construct this restricted process, we need the inequality $T^* 1_{\mathscr{D}} \leq 1_{\mathscr{D}}$. In case $\mu(\mathscr{D}) = 0$ this is obvious and if $\mu(\mathscr{D}) > 0$ this is a consequence of Proposition 5.1.6. We already explained this in the beginning of this section. As shown in Example 4.5.9, the restricted Markov process $(\mathscr{C}, \mathscr{F}_{\mathscr{C}}, \mu_{\mathscr{C}}, T_{\mathscr{C}})$ has no dissipative part. Of course, the condition $T^* f \leq f$ on $\mathscr{C}$ is equivalent to $1_{\mathscr{C}} T^* f \leq 1_{\mathscr{C}} f$. And we get, from the positivity of T^*, that $1_{\mathscr{C}} T^*(1_{\mathscr{C}} f) \leq 1_{\mathscr{C}} T^*(1_{\mathscr{C}} f + 1_{\mathscr{D}} f) = 1_{\mathscr{C}} T^* f \leq 1_{\mathscr{C}} f$. From this inequality it follows that $T^*_{\mathscr{C}}(1_{\mathscr{C}} f)|_{\mathscr{C}} \leq (1_{\mathscr{C}} f)|_{\mathscr{C}}$. Now, we can apply Theorem 5.2.1 to conclude that $T^*_{\mathscr{C}}(1_{\mathscr{C}} f)|_{\mathscr{C}} = (1_{\mathscr{C}} f)|_{\mathscr{C}}$. This last equality implies $T^*(1_{\mathscr{C}} f) = 1_{\mathscr{C}} f$ and so $1_{\mathscr{C}} T^*(1_{\mathscr{C}} f) = 1_{\mathscr{C}} f$. Since $T^* 1_{\mathscr{D}} \leq 1_{\mathscr{D}}$ then $1_{\mathscr{C}} T^*(1_{\mathscr{D}} f) = 0$. Indeed, take a sequence $(f_n)_{n \in \mathbb{N}}$ of non-negative functions in $L^\infty(X, \mathscr{F}, \mu)$ such that $f_n \uparrow f$. For each $n \in \mathbb{N}$ we have $0 \leq 1_{\mathscr{C}} T^*(1_{\mathscr{D}} f_n) \leq 1_{\mathscr{C}} T^*(1_{\mathscr{D}} \| f_n \|_\infty) \leq 1_{\mathscr{C}} 1_{\mathscr{D}} \| f_n \|_\infty = 0$, and we obtain the desired identity by taking the limit, when $n \to \infty$. By summing the two established above identities, $1_{\mathscr{C}} T^*(1_{\mathscr{C}} f) = 1_{\mathscr{C}} f$ and $1_{\mathscr{C}} T^*(1_{\mathscr{D}} f) = 0$, we get $1_{\mathscr{C}} T^*(f) = 1_{\mathscr{C}} f$ and therefore $T^* f = f$ on $\mathscr{C}$. □

Corollary 5.2.3. *Let $(X, \mathscr{F}, \mu, T)$ be a Markov process, and $X = \mathscr{C} \cup \mathscr{D}$ its conservative-dissipative decomposition. Then $1_{\mathscr{C}}(x) \leq T^* 1_{\mathscr{C}}(x)$ (equivalently $T^* 1_{\mathscr{C}} = 1_{\mathscr{C}}$ on $\mathscr{C}$), for all $x \in X$. In particular, if $T^* 1(x) < 1$, then $x \in \mathscr{D}$. If, in addition, we assume $\mathscr{C} = X$, then $T^* 1 = 1$.*

Proof. The contractive property of T^* implies that $T^* 1_{\mathscr{C}} \leq 1$. So $1_{\mathscr{C}} T^* 1_{\mathscr{C}} \leq 1_{\mathscr{C}}$ which means that $T^* 1_{\mathscr{C}} \leq 1_{\mathscr{C}}$ on $\mathscr{C}$. Hence we apply Corollary 5.2.2 and conclude that $T^* 1_{\mathscr{C}} = 1_{\mathscr{C}}$ on $\mathscr{C}$ which is equivalent to $1_{\mathscr{C}}(x) \leq T^* 1_{\mathscr{C}}(x)$, for all $x \in X$.

Let us prove now that if $T^* 1(x) < 1$, then $x \in \mathscr{D}$. Indeed, from the previous result and positivity of T^*, we have $1_{\mathscr{C}}(x) \leq T^* 1_{\mathscr{C}}(x) \leq T^*(1_{\mathscr{C}} + 1_{\mathscr{D}})(x) = T^* 1(x) < 1$ and so $1_{\mathscr{C}}(x) < 1$, which implies $x \in \mathscr{D}$.

When $\mathscr{C} = X$, the inequality $1_{\mathscr{C}}(x) \leq T^* 1_{\mathscr{C}}(x)$, proved above, and the contraction property implies immediately that $1 \leq T^* 1 \leq 1$, or simply $T^* 1 = 1$. □

Next we show a result which is complementary to Theorem 5.2.1 for superinvariant functions in $L^\infty(X, \mathscr{F}, \mu)$. Before giving its statement and proof we want to emphasize that the most important difference between the next result and Corollary 5.2.2 is that in

Corollary 5.2.2 the subinvariant function is only required to be non-negative and μ-almost surely pointwise finite, while in the next result we work with superinvariant functions and they are required to be in $L^\infty(X, \mathscr{F}, \mu)$.

Corollary 5.2.4. *Let $(X, \mathscr{F}, \mu, T)$ be a Markov process, $X = \mathscr{C} \cup \mathscr{D}$ its conservative-dissipative decomposition, and $f \in L^\infty(X, \mathscr{F}, \mu)$ a non-negative function. Suppose that $f \le T^* f$ on $\mathscr{C}$. Then $T^* f = f$ on $\mathscr{C}$.*

Proof. Let $0 \le f \in L^\infty(X, \mathscr{F}, \mu)$ with $f \le T^* f$ on $\mathscr{C}$, that is, $1_{\mathscr{C}} f \le 1_{\mathscr{C}} T^* f$. Note that this inequality together with the contraction property give $1_{\mathscr{C}} T^*(\|f\|_\infty - f) = 1_{\mathscr{C}} \cdot \|f\|_\infty \cdot T^*(1) - 1_{\mathscr{C}} \cdot T^* f \le 1_{\mathscr{C}}(\|f\|_\infty - f)$, which means that $\|f\|_\infty - f$ is a subinvariant function on $\mathscr{C}$. Now we can apply Corollary 5.2.2 to conclude $T^*(\|f\|_\infty - f) = \|f\|_\infty - f$ on $\mathscr{C}$. Note that $1_{\mathscr{C}} T^*(\|f\|_\infty) = 1_{\mathscr{C}} T^*\big((1_{\mathscr{D}} + 1_{\mathscr{C}})\|f\|_\infty\big) = 1_{\mathscr{C}} T^*\big(1_{\mathscr{D}}\|f\|_\infty\big) + 1_{\mathscr{C}} T^*\big(1_{\mathscr{C}}\big)\|f\|_\infty\big) = 1_{\mathscr{C}}\|f\|_\infty$, where in the last equality we used that $1_{\mathscr{C}} T^* 1_{\mathscr{D}} = 0$ and $1_{\mathscr{C}} T^* 1_{\mathscr{C}} = 1_{\mathscr{C}}$ (Corollary 5.2.3). Therefore the identity $T^*(\|f\|_\infty - f) = \|f\|_\infty - f$ on $\mathscr{C}$ together with this last observation actually imply $T^* f = f$ on $\mathscr{C}$. □

Corollary 5.2.2 has a "dual relation", which is particularly simple when $T^* 1 = 1$. In this case, if $0 \le u \in L^1(X, \mathscr{F}, \mu)$ and $Tu \le u$, then $Tu = u$. Indeed, the proof is immediate $\langle 1, u\rangle = \langle T^* 1, u\rangle = \langle 1, Tu\rangle$ and so $\langle 1, u - Tu\rangle = 0$. Since $0 \le u - Tu$ it follows that $u - Tu = 0$.

In general, a similar conclusion holds only on the conservative set and we remark that it is not necessary to require u to be in $L^1(X, \mathscr{F}, \mu)$. This is stated in the following proposition.

Proposition 5.2.5. *Let $(X, \mathscr{F}, \mu, T)$ be a Markov process, $X = \mathscr{C} \cup \mathscr{D}$ its conservative-dissipative decomposition, and $f : X \to \mathbb{R}$ a $\mathscr{F}$-measurable function such that $0 \le u < +\infty$ μ-almost surely. If $Tu \le u$ on $\mathscr{C}$, then $Tu = u$ on $\mathscr{C}$.*

The proof can be found in

Proof. Let $0 \le u < +\infty$ satisfying $Tu \le u$ on $\mathscr{C}$. Take $f \in L^\infty(X, \mathscr{F}, \mu)$ such f is supported $(1_{\mathscr{D}} f = 0)$ and strictly positive $(1_{\mathscr{C}} f > 0)$ on $\mathscr{C}$, and satisfying $\langle f, u\rangle < +\infty$. Then,

$$\langle \sum_{k=0}^{n} (T^*)^k f, u - Tu\rangle = \langle f, u\rangle - \langle (T^*)^{n+1} f, u\rangle \leqslant \langle f, u\rangle.$$

Now take the limit $n \to \infty$ to have

$$\langle \sum_{k=0}^{\infty} T^k f, u - Tu \rangle \leqslant \langle f, u \rangle < \infty.$$

And remember that $\sum_{k=0}^{\infty} T^k f$ diverge on $\mathscr{C}$, which forces $u - Tu = 0$ on $\mathscr{C}$, otherwise the left hand side of the above inequality would diverge. □

The last theorem in this chapter, also related to functions invariant with respect to T^*, actually gives their explicit form in the sense that they have to be measurable with respect to the sigma-algebra $\mathscr{F}_i$ of the sets $A \in \mathscr{F}$ for which $T^* 1_A = 1_A$ holds. See Example 4.5.12 for a better description of $\mathscr{F}_i$.

Theorem 5.2.6 (Maximal eigenfunctions of T^* on $\mathscr{C}$)**.** *Let $X = \mathscr{C}$ and $f \in L^\infty(\mu)$. Then $T^* f = f$ if and only if f is $\mathscr{F}_i$-measurable.*

Proof. Theorem A, pg. 21 in [24]. □

Remark 5.2.7. *It is clear that, a process with a positive invariant function $Tu = u$ is purely conservative ($X = \mathscr{C}$). Corollary 5.2.3, by its turn, implies that a purely conservative process satisfies $T^* 1 = 1$. We summarize those implications on the expression below.*

$$Tu = u \text{ with } u > 0 \implies X = \mathscr{C} \implies T^* 1 = 1.$$

We have already noted that Example 4.5.1 proves that the reciprocal of the second implication is false, since in that case $T^* 1 = 1$ but $X = \mathscr{D}$. Fogel in [24], pg 95, cites a counterexample showing that the reciprocal of the first implication is also false, as there is a conservative process with no invariant function.

CHAPTER 6

THE EIGENSPACE OF $\mathbb{L}_f$

In this chapter we go back to $\mathbb{L}_f$, the Ruelle operator on $L^1(\nu)$ as defined in Chapter 3. When divided by its operator norm, this operator is actually a Markov process. So we apply the theory presented on Chapters 4 and 5 to find relevant information concerning the eigendata of $\mathbb{L}_f$. If f is a continuous potential and the a priori measure p is fully supported, then $\mathbb{L}_f$ is the extension to $L^1(\nu)$ of the continuous operator $\mathscr{L}$, for ν a conformal measure with respect to f. But most of the theory developed in this chapter can be applied to the case where f is a bounded operator and ν is a generalized conformal measure, as described in Section 3.2. The most important results at the end of the chapter are restricted to a continuous potential ψ. as we resort to properties of the conformal measures developed in other works.

We believe that Section 6.1 is the most important in this dissertation, since it synthesizes all the theory developed up to here. In that section we apply Hopf's theory to get some properties of the eigendata of $\mathbb{L}_\psi$ and, by consequence, also of $\mathscr{L}_\psi$. Most of the demonstrations are self contained and we could be more succinct by referencing to results in Chapters 4 and 5 more often. We preferred to repeat some passages to improve readability.

6.1 Markov theory of the Ruelle operator

In this chapter f is a bounded potential and $\mathbb{L}_f$ is the Ruelle operator as defined in chapter $L^1(\nu)$ for ν a generalized conformal with respect to f. To avoid intricacies, the reader may suppose that the a priori measure is fully supported, and then consider f a continuous potential and ν a conformal measure for f.

In this setting, $\mathbb{L}_f$ divided by its operator norm $\|\mathbb{L}_f\|_{op}$ actually is a Markov process in $(X, \mathscr{F}, \nu)$, for $\mathscr{F} = \mathscr{B}(X)$. We drop the potential on the notation for the operator and write $\mathbb{L}$ instead of $\mathbb{L}_f$. The positivity property of $\mathbb{L}$ comes from the positivity of $\mathscr{L}$ and dividing it by its operator norm, we have that the second condition for a Markov Processes in the sense of Hopf is satisfied. Thus, without loss of generality, in this section we will assume that $\|\mathbb{L}\|_{\text{op}} = 1$. By analogy with the continuous potential case, we define $\mathscr{G}^*$ as the set of generalized conformal (probability) measures.

Since we have chosen to work with a generalized conformal measure for f, it is clear that $\mathbb{L}^*\mathbb{1} = \mathbb{1}$. We remember that this is equivalent to

$$\int_X \mathscr{L}\varphi \, d\nu = \int_X \varphi \, d\nu \quad \forall \varphi \in C(X, \mathbb{R}),$$

for f continuous potential and p a fully supported a priori measure. The reader can refer to Section 3.2 for more details on this equivalence.

We already have a maximal eigenfunction for $\mathbb{L}^*$, which is the constant function $\mathbb{1}$. On the other hand, the existence of a positive or nonnegative eigenfunction for $\mathbb{L}$ itself is a much more delicate issue. In the remaining of this section we discuss this problem and provide some sufficient conditions for its existence and uniqueness.

Proposition 6.1.1. *If $\mathbb{L}^*\mathbb{1}_B = \mathbb{1}_B$ for some $B \in \mathscr{B}(X)$ with $\nu(B) \neq 0$, then the Borel measure given by $A \longmapsto \nu(A \cap B)/\nu(B) \equiv \nu(A|B)$ is an element of $\mathscr{G}^*$.*

Proof. The proof is immediate, as, for every $u \in L^1(\nu)$,

$$\langle \mathbb{1}, \mathbb{L}u \rangle_{\nu(\cdot|B)} = \langle \mathbb{1}_B, \mathbb{L}u \rangle_\nu = \langle \mathbb{L}^*\mathbb{1}_B, u \rangle_\nu = \langle \mathbb{1}_B, u \rangle_\nu = \langle \mathbb{1}, u \rangle_{\nu(\cdot|B)}.$$

Meaning that $\mathbb{L}^*\mathbb{1} = \mathbb{1}$ if the underlying measure is $\nu(\cdot|B)$. This is exactly the condition for $\nu(\cdot|B)$ be in $\mathscr{G}^*$. □

Lemma 6.1.2. *If* $\nu \in \mathscr{G}^*$ *is an extreme element, then there is no* $B \in \mathscr{B}(X)$ *such that* $0 < \nu(B) < 1$ *and* $\mathbb{L}^* \mathbb{1}_B = \mathbb{1}_B$.

Proof. Suppose, by contradiction, that such a Borel set U does exist. From Proposition 6.1.1 we know that $\nu(\cdot|B)$ is a generalized conformal measure. Since $\mathbb{L}^*\mathbb{1} = \mathbb{1}$, linearity of $\mathbb{L}$ implies that $\mathbb{L}^*\mathbb{1}_{B^c} = \mathbb{1}_{B^c}$. By applying again Proposition 6.1.1 we get that $\nu(\cdot|B^c)$ is again conformal. Clearly $\nu(\cdot|B) \neq \nu(\cdot|B^c)$. But $\nu = \nu(B)\nu(\cdot|B) + \nu(B^c)\nu(\cdot|B^c)$ which contradicts the assumption that ν is extreme. □

The next result show that any eigenfunction of $\mathbb{L}$, associated to its operator norm, satisfies a kind of identity principle. More precisely, it says that if a non-negative eigenfunction vanishes on a set of positive ν-measure ($\nu \in \mathrm{ex}(\mathscr{G}^*)$), then it should vanish ν-almost everywhere.

Theorem 6.1.3. *Let* ν *be an extreme point in* $\mathscr{G}^*$ *and* $u \geqslant 0$ *a maximal eigenfunction (therefore not identically zero) of* $\mathbb{L} : L^1(\nu) \to L^1(\nu)$*, associated to its operator norm. Then* $u > 0$ ν*-a.e.*

Proof. Suppose, by contradiction, there is a set $B \in \mathscr{B}(X)$ for which $0 < \nu(B) < 1$, $u|_B = 0$ and $u_{B^c} > 0$. Since $\mathbb{L}u = u$, we get that

$$\langle \mathbb{L}^*\mathbb{1}_{B^c}, u\rangle_\nu = \langle \mathbb{1}_{B^c}, \mathbb{L}u\rangle_\nu = \langle \mathbb{1}_{B^c}, u\rangle_\nu. \tag{6.1}$$

Note that $\mathbb{L}^*\mathbb{1}_{B^c} \leqslant \mathbb{1}$, because the adjoint of a positive contraction is also a positive contraction, so $\mathbb{L}^*$ is a contraction in $L^\infty(\nu)$. Since u is non-negative and supported in B^c, and $0 \leqslant \mathbb{L}^*\mathbb{1}_{B^c} \leqslant 1$, it follows from 6.1 that $\mathbb{1}_{B^c}\mathbb{L}^*\mathbb{1}_{B^c} = \mathbb{1}_{B^c}$. From these observations, we get that $\mathbb{L}^*\mathbb{1}_{B^c} \geqslant \mathbb{1}_{B^c}$, since $\mathbb{L}^*$ is positive. Therefore

$$\|\mathbb{L}^*\mathbb{1}_{B^c}\|_1 \geqslant \|\mathbb{1}_{B^c}\|_1.$$

As we already mentioned $\mathbb{L}^*$ is a contraction with respect to the $L^\infty(\nu)$-norm. Moreover, the operator $\mathbb{L}^*$ acts as a contraction, with respect to the $L^1(\nu)$-norm, on the linear manifold spanned by the characteristic functions . Indeed,

$$\|\mathbb{L}^*\mathbb{1}_{B^c}\|_1 = \langle \mathbb{L}^*\mathbb{1}_{B^c}, \mathbb{1}\rangle_\nu = \langle \mathbb{1}_{B^c}, \mathbb{L}\mathbb{1}\rangle_\nu \leqslant \langle \mathbb{1}_{B^c}, \mathbb{1}\rangle_\nu = \|\mathbb{1}_{B^c}\|_1.$$

Since $0 \leqslant \mathbb{1}_{B^c} \leqslant \mathbb{L}^*\mathbb{1}_{B^c}$ and $\|\mathbb{L}^*\mathbb{1}_{B^c}\|_1 = \|\mathbb{1}_{B^c}\|_1$, follows that $\mathbb{L}^*\mathbb{1}_{B^c} = \mathbb{1}_{B^c}$. On the other hand, since $0 < \nu(B^c) < 1$ and ν is an extreme point in $\mathscr{G}^*$ Proposition 6.1.1 applies and we get a contradiction. □

In what follows we show that these set of ideas can also be used to handle the simplicity of the eigenspace associated to the eigenvalue one, when the conformal measure is an extreme point of $\mathscr{G}^*$.

Lemma 6.1.4. *If ν is an extreme point in $\mathscr{G}^*$ and $u \in L^1(\nu)$ is eigenfunction of $\mathbb{L}$, associated to one, then u has a definite sign ν-almost everywhere.*

Proof. The proof is by contradiction. Assume that u has non-trivial decomposition on its positive and negative parts, that is, $0 < \nu(\{x \in X : u(x) > 0\}) < 1$ and $0 < \nu(\{x \in X : u(x) \leqslant 0\}) < 1$. We call $B = \{x \in X : u(x) > 0\}$. By using the linearity of $\mathbb{L}$ and that $\mathbb{L}u = u$, we get

$$
\begin{aligned}
\mathbb{L}(u^+) &= \mathbb{L}(u + u^-) = u + \mathbb{L}(u^-) \\
&= u^+ + (\mathbb{L}u^- - u^-) \\
&= (u^+ + (\mathbb{L}u^- - u^-))\mathbb{1}_B + (u^+ + (\mathbb{L}u^- - u^-))\mathbb{1}_{B^c} \\
&= (u^+ + \mathbb{L}u^-)\mathbb{1}_B + (\mathbb{L}u^- - u^-))\mathbb{1}_{B^c}
\end{aligned}
\tag{6.2}
$$

By multiplying both sides of (6.2) by $\mathbb{1}_{B^c}$, we get from the positivity of $\mathbb{L}$ that $0 \leq \mathbb{1}_{B^c}\mathbb{L}(u^+) = (\mathbb{L}u^- - u^-))\mathbb{1}_{B^c}$. Therefore

$$
\begin{aligned}
0 \leq \int_X (\mathbb{L}u^- - u^-))\mathbb{1}_{B^c}\, d\nu = \int_X \mathbb{1}_{B^c}\mathbb{L}u^- - u^-\, d\nu \\
\leq \int_X \mathbb{L}u^-\, d\nu - \int_X u^-\, d\nu \\
= \|\mathbb{L}u^-\|_1 - \|u^-\|_1 \leq 0,
\end{aligned}
$$

where in the last inequality we used the contraction property of $\mathbb{L}$. This shows that $(\mathbb{L}u^- - u^-))\mathbb{1}_{B^c} = 0$ ν-a.e.. Replacing this in (6.2) we get the following equality $\mathbb{L}u^+ = (u^+ + \mathbb{L}u^-)\mathbb{1}_B$. Now, we integrate both sides of this equality obtaining $\|\mathbb{L}u^+\|_1 = \|u^+\|_1 + \|\mathbb{L}u^-\mathbb{1}_B\|_1$. Applying one more time the contraction property we have that $\|\mathbb{L}u^-\mathbb{1}_B\|_1 = 0$. This implies $\mathbb{L}u^-\mathbb{1}_B = 0$ ν-a.e.. Finally, from the identity $\mathbb{L}u^+ = (u^+ + \mathbb{L}u^-)\mathbb{1}_B$, it follows that $\mathbb{L}u^+ = \mathbb{1}_B u^+ = u^+$.

Since we are assuming that ν is an extreme point in $\mathscr{G}^*$ and we have shown that u^+ is a not identically zero non-negative maximal eigenfunction of $\mathbb{L}$, we can apply Theorem 6.1.3 to get that $u^+ > 0$ ν-a.e. which implies that $u = u^+$ contradicting that assumption that u has non-trivial positive and negative parts. □

Theorem 6.1.5. *If ν is an extreme point in $\mathscr{G}^*$ then the dimension of the eigenspace associated to the operator norm of $\mathbb{L} : L^1(\nu) \to L^1(\nu)$ is at most one.*

Proof. The proof is by contradiction. Suppose that there are two linearly independent maximal eigenfunctions u and v for $\mathbb{L}$. By Lemma 6.1.4, we can assume that they are both positive almost everywhere. Let $\int_X u d\nu \equiv \nu(u)$ be the mean value of u, with respect to ν. Consider the following linear combination $w \equiv u - (\nu(u)/\nu(v))v$. Since w is a linear combination of two maximal eigenfunctions, it is also a maximal eigenfunction. Clearly, $\nu(w) = 0$, and so it is either identically zero or it has non-trivial positive and a negative parts. Note that it can not be identically zero, since w is a linear combination of two linear independent functions. Nor w can have a positive and a negative part, since it would contradict Lemma 6.1.4 and therefore we have a contradiction. □

Now we have the tools to prove the following Corollary that show the consequences for the operator $\mathscr{L}$ of the theory developed for the operator on $L^1(\nu)$ when the potential is continuous.

Corollary 6.1.6. *Let ψ be any continuous potential and $\mathscr{L} : C(X) \to C(X)$ be a transfer operator constructed from this potential and a fully supported a priori probability measure p on E. Then the eigenspace of $\mathscr{L}$ associated to its spectral radius has either dimension zero or one. If that eigenspace is one-dimensional, any eigenfunction in it has definite sign and it can vanish at most in a ν-null set, for ν any conformal measure with respect to f.*

Proof. Up to addition of a constant to f, we can assume $\rho(\mathscr{L}) = 1$. Since X is a compact space we have that $\mathscr{G}^*$ is convex and compact. By Krein-Milman Theorem the set of extreme points of $\mathscr{G}^*$ is necessarily not empty. Take any $\nu \in \text{ex}(\mathscr{G}^*)$. Suppose that φ and ψ are two linearly independent continuous eigenfunctions of $\mathscr{L}$, associated to the eigenvalue one. Note that we are in conditions to apply Theorem 3.1.8 and therefore we have that the probability measure ν is fully supported. From this fact, $[\varphi]_\nu \neq [\psi]_\nu$. Actually, they are linearly independent in $L^1(\nu)$.

Now, we consider the extension $\mathbb{L} : L^1(\nu) \to L^1(\nu)$ of the transfer operator $\mathscr{L}$ provided by Theorem 3.1.9. Of course, $[\varphi]_\nu$ and $[\psi]_\nu$ are eigenfunctions of $\mathbb{L}$, associated to the eigenvalue one. Then Lemma 6.1.4 ensure that they have definite signs, which can be assumed to be positive. But Theorem 6.1.5 implies $[\varphi]_\nu = \lambda[\psi]_\nu$, for some real number λ, which is a contradiction. □

When we deal with a extreme conformal measure ν, the operators $\mathbb{L}$ defined on $L^1(\nu)$ is also simple in terms of the conservative-dissipative decomposition, as we read below.

Theorem 6.1.7. *If ν is an extreme point in $\mathscr{G}^*$ then the conservative-dissipative decomposition of $\mathbb{L} : L^1(\nu) \to L^1(\nu)$ is trivial.*

Proof. Suppose, by contradiction, that the decomposition is not trivial. Then we have two sets with positive ν-measure $\mathscr{C}$ and $\mathscr{D}$. By Corollary 5.2.3, $\mathbb{L}^*\mathbb{1}_{\mathscr{C}} = \mathbb{1}_{\mathscr{C}}$, but it contradicts Lemma 6.1.2, because ν is extreme. So, the decomposition must be trivial. □

We end this section presenting a result on the problem of the dimension of the maximal eigenspace of $\mathbb{L} : L^1(m) \to L^1(m)$, when m is as before a generalized conformal measure, but not necessarily an extreme point in $\mathscr{G}^*$. But first we investigate the question of how we can represent extreme conformal measures on the L^∞ space of a non-extreme one. To do so, we resort to some results from the statistical mechanics and restrict ourselves to continuous potentials, as the works we cite are developed in this setting.

Lemma 6.1.8. *Let ψ be a continuous potential such that there are $\nu, \mu \in \mathrm{ex}(\mathscr{G}^*)$, distinct conformal extreme measures associated with ψ. Let also m be a non-trivial convex combination of them, $m = t\nu + (1-t)\mu$. Then there is a $\mathscr{B}(X)$-measurable set, B, for which $\nu(B) = 1$, $\mu(B^c) = 1$ and the following equations hold*

$$\mathbb{L}^*\mathbb{1}_B = \mathbb{1}_B \quad \textit{and} \quad \mathbb{L}^*\mathbb{1}_{B^c} = \mathbb{1}_{B^c}, \tag{6.3}$$

where $\mathbb{L}$ is the extension of $\mathscr{L}$ to $L^1(m)$. Moreover if $D \in \mathscr{B}(X)$ is another set satisfying $\mathbb{L}^*\mathbb{1}_D = \mathbb{1}_D$ and $0 < m(D) < 1$, then either $m(D \triangle B) = 0$ or $m(D \triangle B^c) = 0$, where $D \triangle B$ denotes the symmetric difference between D and B.

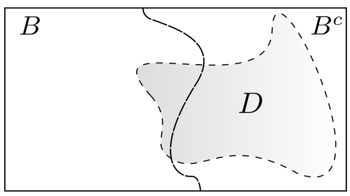
Proof. In the appendix of reference [10] the Theorem 7.7 item (c) of [27] is generalized for the Thermodynamic Formalism setting on general compact metric alphabets. This result says that any extreme point in $\mathscr{G}^*$ is uniquely determined by the values it takes on the elements of the tail sigma-algebra $\mathscr{T}$. Since ν and μ are distinct and determined by the values taken on $\mathscr{T}$, there is at least one element $B \in \mathscr{T}$ such that $\mu(B) \neq \nu(B)$. But from Corollary 10.5 in [10] we also know that any extreme point in $\mathscr{G}^*$ is trivial on $\mathscr{T}$, meaning that, for every $B \in \mathscr{T}$, $\nu(B) = 0$ or $\nu(B) = 1$. Then, supposing that $\mu(B) = 0$, we have that $\nu(B) = 1$. We have now two disjoint sets, B and B^c with $\nu(B) = 1$ and $\mu(B^c) = 1$.

Following the computations of Proposition 6.1.1, we will show that $\mathbb{L}^*\mathbb{1}_B = \mathbb{1}_B$. Actually, it is enough to prove that $\langle \mathbb{L}^*\mathbb{1}_B, \varphi\rangle_m = \langle \mathbb{1}_B, \varphi\rangle_m$ for every $\varphi \in C(X,\mathbb{R})$. Indeed, for an arbitrary continuous function φ we have

$$\begin{aligned}\langle \mathbb{L}^*\mathbb{1}_B, \varphi\rangle_m &= \langle \mathbb{1}_B, \mathbb{L}\varphi\rangle_m = \langle \mathbb{1}_B, \mathscr{L}\varphi\rangle_m = \int_X \mathbb{1}_B \mathscr{L}\varphi \, dm \\ &= \int_X \mathscr{L}\varphi \, dm(\cdot \cap B) = \int_X \mathscr{L}\varphi \, d(t\nu) = \int_X \varphi \, d(t\mathscr{L}^*\nu) \\ &= \int_X \varphi \, d(t\nu) = \int_X \mathbb{1}_B \varphi \, dm = \langle \mathbb{1}_B, \varphi\rangle_m\end{aligned}$$

The third equality above holds because $m(\cdot \cap B) = t\nu$. Again by using the density of $C(X)$ in $L^1(m)$, we conclude that $\mathbb{L}^*\mathbb{1}_B = \mathbb{1}_B$. Recalling that $\mathbb{L}^*\mathbb{1} = \mathbb{1}$ we get from previous identity that $\mathbb{L}^*\mathbb{1}_{B^c} = \mathbb{1}_{B^c}$.

It remains to prove the m-almost everywhere uniqueness of B. More precisely, suppose that there exist $D \in \mathscr{B}(X)$ with $0 < m(D) < 1$ such that $\mathbb{L}^*\mathbb{1}_D = \mathbb{1}_D$ and $m(D \triangle B) > 0$. Then we have to show that $m(D \triangle B^c) = 0$.



It follows from our assumption that $m(D \cap B^c) > 0$ or $m(D^c \cap B) > 0$. The analysis of both cases are similar and so we can assume that $m(D \cap B^c) > 0$.

We claim that $\mathbb{L}^*\mathbb{1}_{D\cap B^c} = \mathbb{1}_{D\cap B^c}$. Indeed, $\mathbb{1}_D = \mathbb{L}^*(\mathbb{1}_D\mathbb{1}_B + \mathbb{1}_D\mathbb{1}_{B^c})$. By multiplying both sides by $\mathbb{1}_{B^c}$ we get $\mathbb{1}_D\mathbb{1}_{B^c} = \mathbb{1}_{B^c}\mathbb{L}^*(\mathbb{1}_D\mathbb{1}_B) + \mathbb{1}_{B^c}\mathbb{L}^*(\mathbb{1}_D\mathbb{1}_{B^c})$. To prove the claim it is enough to show that $\mathbb{1}_{B^c}\mathbb{L}^*(\mathbb{1}_D\mathbb{1}_B) = 0$ and $\mathbb{1}_{B^c}\mathbb{L}^*(\mathbb{1}_D\mathbb{1}_{B^c}) = \mathbb{L}^*(\mathbb{1}_D\mathbb{1}_{B^c})$. These two statements follow immediately from the positiveness of $\mathbb{L}^*$. In fact, $0 \leqslant \mathbb{1}_{B^c}\mathbb{L}^*(\mathbb{1}_D\mathbb{1}_B) \leqslant \mathbb{1}_{B^c}\mathbb{L}^*(\mathbb{1}_B) = \mathbb{1}_{B^c}\mathbb{1}_B = 0$.

Since $m(D \cap B^c) > 0$ it follows from definitions of m, B and $t - 1 > 0$ that $0 <$

$\mu(D \cap B^c) \leqslant 1$. We claim that $\mathbb{L}_\mu^* \mathbb{1}_{D\cap B^c} = \mathbb{1}_{D\cap B^c}$ and $\mathbb{L}_\nu^* \mathbb{1}_{D\cap B} = \mathbb{1}_{D\cap B}$. Once the claim is established, we can use Lemma 6.1.2 to ensure that $\mu(D \cap B^c) = 1$, because the measure of this set has to be positive. The equality $\nu(D \cap B) = 0$ follows from the fact that it could be zero or one. If it were one, we would have immediately $m(D) = 1$, which is a contradiction. These information, together with the definition of B and elementary properties of probability measures, implies what we wanted to show $m(D \triangle B^c) = t\nu(D^c \cap B^c) + (1-t)\mu(D^c \cap B^c) + t\nu(D \cap B) + (1-t)\mu(D \cap B) = 0$.

Now, we prove the last claim. Since both equalities have a similar proof, it is enough to prove the first one, that is, $\mathbb{L}_\mu^* \mathbb{1}_{D\cap B^c} = \mathbb{1}_{D\cap B^c}$. Indeed, we already know that $\mathbb{L}^* \mathbb{1}_{D\cap B^c} = \mathbb{1}_{D\cap B^c}$. From the results in the proof of Theorem 3.1.2, we have that $\mathbb{L}^* \mathbb{1}_{D\cap B^c}(x) = \mathbb{1}_{D\cap B^c}(x)$ for almost all $x \in X$, with respect to m. Since $\mu \ll m$ we get the same conclusion, of the last equation, but for almost all x, with respect to μ. And by taking the μ-a.e. equivalence classes, again as in Theorem 3.1.2, we get that $\mathbb{L}_\mu^* \mathbb{1}_{D\cap B^c} = \mathbb{1}_{D\cap B^c}$, thus finally completing the proof. □

Note that, when $m = \sum_{i=1}^2 t_i \nu_i$, each measure ν_i is absolutely continuous with respect to m and we can represent them as elements of $L^1(m)$. In this case, $dm = \sum_{i=1}^2 h_i dm$ and $h_i dm = t_i d\nu_i$, for $h_i \in L^1(m)$. But, since each h_i is a non-negative function bounded by $\mathbb{1}$, all of them also belong to $L^\infty(m)$. And as ν_i are conformal measures, this will imply that $\mathbb{L}^* h_i = h_i$ for each h_i. The last Lemma above shows that these h_i must be very simple. Indeed, $h_i = \mathbb{1}_{B_i}$ and the sets B_i make partition of the whole X, modulo null-m-measure sets.

Remark 6.1.9. *Consider a phase transition case, where $m = \sum_{i=1}^n t_i \nu_i$, for each $i \in 1, \ldots, n$, $t_i \in (0,1)$, $\sum_{i=1}^n t_i = 1$ and $\nu_i \in \mathrm{ex}(\mathscr{G}^*)$ are distinct conformal measures. By applying Lemma 6.1.8 we can find a Borel set B_{1i} such that $\nu_1(B_{1i}) = 1$ and $\nu_i(B_{1i}^c) = 1$. Set $B_1 = \cap_{i=2}^n B_{1i}$, it is clear that $\nu_1(B_1) = 1$. On the other hand, since $B_1^c = \cup_{i=2}^n B_{1i}^c \supseteq B_{1i}^c$ and $\nu_i(B_{1i}^c) = 1$, we have $\nu_i(B_1^c) = 1$, for all $i \in 2, \ldots, n$. By considering the set B_1 and repeating the arguments in the proof of Lemma 6.1.8 we get that $\mathbb{L}^* \mathbb{1}_{B_1} = \mathbb{1}_{B_1}$ and $\mathbb{L}^* \mathbb{1}_{B_1^c} = \mathbb{1}_{B_1^c}$. Its almost surely uniqueness is obtained similarly.*

We use the lemma above to prove the following theorem. In doing so, we restrict ourselves to continuous potentials.

Theorem 6.1.10. *Let* ψ *be an arbitrary continuous potential and* $p : \mathscr{B}(E) \to [0,1]$ *a fully supported probability measure, and* $m \in \mathscr{G}^*$ *be an arbitrary conformal measure. Then the eigenspace of* $\mathbb{L}_m$ *associated to its spectral radius has dimension not bigger than the cardinality of the set of extreme points in* $\mathscr{G}^*$*.*

Proof. The arguments in this proof involves, simultaneously, different extensions of transfer operator $\mathscr{L} : C(X) \to C(X)$. To avoid confusions these extensions will be indexed by the conformal measure as in notation $\mathbb{L}_\nu$. It has the advantage of let clear on which Lebesgue space the extension acts.

As before, there is no loss of generality in assuming that $\rho(\mathscr{L}) = 1$. Therefore for each conformal measure $m, \nu, \mu \in \mathscr{G}^*$ we have that the extensions $\mathbb{L}_m$, $\mathbb{L}_\nu$ and $\mathbb{L}_\mu$, provided by Theorem 3.1.9, define themselves Markov processes.

Of course, there is nothing to prove if $\#\mathrm{ex}(\mathscr{G}^*) = +\infty$, thus in what follows we assume that the cardinality of the set of extreme points of $\mathscr{G}^*$ is finite.

In case $\mathrm{ex}(\mathscr{G}^*)$ is a singleton the conclusion follows immediately from Theorem 6.1.5. In the sequel we will assume that $\#\mathrm{ex}(\mathscr{G}^*) = 2$. The generalization of the following argument to the case of a convex combination of a finite number of extreme measures is straightforward and involves the application of Remark 6.1.9. It is omitted to avoid an unnecessary cumbersome notation.

We denote by $\mathbb{L}_m$ be the extension of $\mathscr{L} : C(X) \to C(X)$, provided by Theorem 3.1.9, to $L^1(m)$, corresponding to the measure $m = t\nu + (1-t)\mu$. Lemma 6.1.8 implies that there is a unique (modulo-m) set $B \in \mathscr{B}(X)$ such that $\nu(B) = 1$, $\mu(B^c) = 1$, $\mathbb{L}_m^* \mathbb{1}_B = \mathbb{1}_B$ and $\mathbb{L}_m^* \mathbb{1}_{B^c} = \mathbb{1}_{B^c}$.

Note that one of the following three possibilities occurs:

i) the eigenvalue problem $\mathbb{L}_m[u]_m = [u]_m$ has only the trivial solution, i.e., $[u]_m = 0$;

ii) any maximal eigenfunction $[u]_m$ for $\mathbb{L}_m$ is such that $[\mathbb{1}_B u]_m \neq 0$, but $[\mathbb{1}_{B^c} u]_m = 0$ and vice-versa;

iii) there is a maximal eigenfunction $[u]_m$ such that both $[\mathbb{1}_B u]_m \neq 0$, and $[\mathbb{1}_{B^c} u]_m \neq 0$.

Of course, in the first case the dimension of the maximal eigenspace is zero and the theorem is proved. We will show next that in the second case, the maximal eigenspace is one-dimensional. In this case we will say that the eigenfunctions are supported on either B or B^c, depending on where u does not vanish. Finally, in the third case we will show that the maximal eigenspace is spanned by two linearly independent functions $\{[\mathbb{1}_B u]_m, [\mathbb{1}_{B^c} u]_m\}$, and therefore will be a two-dimensional space subspace of $L^1(m)$.

Let us assume that *iii*) holds. We are choosing to handle this case firstly because the arguments involved in it works similarly in case *ii*).

We are going to show that if $[v]_m$ is any other maximal eigenfunction then $[v]_m = \alpha[\mathbb{1}_B u]_m + \beta[\mathbb{1}_{B^c} u]_m$, for some $\alpha, \beta \in \mathbb{R}$.

Firstly, we will show that both $[\mathbb{1}_B u]_m$ and $[\mathbb{1}_{B^c} u]_m$ are two linearly independent maximal eigenfunctions of $\mathbb{L}_m$. The linear independence of these two functions is obvious. Lets us show that $[\mathbb{1}_B u]_m$ is a maximal eigenfunction of $\mathbb{L}_m$. Note that

$$\begin{aligned} \mathbb{L}_m[\mathbb{1}_B u]_m =& \mathbb{L}_m[u]_m - \mathbb{L}_m[\mathbb{1}_{B^c} u]_m = [u]_m - \mathbb{L}_m[\mathbb{1}_{B^c} u]_m \\ =& [\mathbb{1}_B u]_m + [\mathbb{1}_{B^c} u]_m - \mathbb{L}_m[\mathbb{1}_{B^c} u]_m \end{aligned} \tag{6.4}$$

Recalling that $\mathbb{L}_m^* \mathbb{1}_B = \mathbb{1}_B$ and using the above equality, we obtain

$$\begin{aligned} \|\mathbb{1}_B u\|_{L^1(m)} &= \langle \mathbb{1}_B, [\mathbb{1}_B u] \rangle_m = \langle \mathbb{1}_B, \mathbb{L}_m[\mathbb{1}_B u] \rangle_m \\ &= \langle \mathbb{1}_B, [\mathbb{1}_B u] + [\mathbb{1}_{B^c} u] - \mathbb{L}_m[\mathbb{1}_{B^c} u] \rangle_m \\ &= \|u \mathbb{1}_B\|_{L^1(m)} - \langle \mathbb{1}_B, \mathbb{L}_m[\mathbb{1}_{B^c} u] \rangle_m, \end{aligned}$$

which implies $\mathbb{L}_m[\mathbb{1}_{B^c} u]_m = 0$ in B. Similarly, we get $\mathbb{L}_m[\mathbb{1}_B u]_m = 0$ in B^c. By plugging this back in 6.4 we get that $\mathbb{L}_m[\mathbb{1}_B u]_m = [\mathbb{1}_B u]_m$ and consequently $\mathbb{L}_m[\mathbb{1}_{B^c} u]_m = [\mathbb{1}_{B^c} u]_m$.

From definition of m, $\mu(B) = 0$, and $\mathbb{L}_m[\mathbb{1}_B u]_m = [\mathbb{1}_B u]_m$ it follows that $\mathbb{L}_\nu[\mathbb{1}_B u]_\nu = [\mathbb{1}_B u]_\nu$. The conformal measure $\nu \ll m$ and therefore we get from item *iii*) that $[\mathbb{1}_B u]_\nu \neq 0$. Since $\nu \in \mathrm{ex}(\mathscr{G}^*)$ we can apply Theorem 6.1.3 to ensure that $[\mathbb{1}_B u]_\nu$ is positive ν-almost everywhere.

Now, let $[v]_m$ be an arbitrary maximal eigenfunction of $\mathbb{L}_m$. By repeating the above steps we conclude that $[\mathbb{1}_B v]_\nu$ is also a ν-almost everywhere positive eigenfunction of $\mathbb{L}_\nu$. But Theorem 6.1.5 states that there is some $\alpha \in \mathbb{R}$ such that $[\mathbb{1}_B v]_\nu = \alpha[\mathbb{1}_B u]_\nu$. From the definition of B and m we conclude that the last equality actually implies $[\mathbb{1}_B v]_m = \alpha[\mathbb{1}_B u]_m$. By repeating this argument for $[\mathbb{1}_{B^c} v]_\nu$ we get that $[v]_m = \alpha[\mathbb{1}_B u]_m + \beta[\mathbb{1}_{B^c} u]_m$, which finishes the proof of the theorem. □

Remark 6.1.11 (Continuous potentials with multidimensional maximal eigenspace)**.** *In* *chapter 2 we presented an example of a potential exhibiting phase transition. It has the advantage to allow explicit calculations, which can yield some intuition on the phenomenon of phase transition and its impact on the maximal eigenspace of the operator $\mathbb{L}$. On the other hand, one could argue that this relation is artificially generated by some peculiarity intrinsic of the discontinuous nature of the potential. We show that this is not the case by resorting to a class of continuous potentials exhibiting phase transition.*

The g-functions form a class of continuous and strictly positive functions. They were first defined by M. Keane in the article [33]. Its logarithm defines a normalized potential, i. e., $\mathscr{L}1 = 1$. It means that all of the Ruelle operators associated to them have spectral radius equal to one and the constant functions generates the maximal eigenspace of the operator $\mathscr{L}$. Since $\mathbb{L}_m$ is the extension of $\mathscr{L}$ to $L^1(m)$, for any conformal measure m, the m-equivalence class which the constant function 1 belongs to is also a maximal eigenfunction of $\mathbb{L}_m$.

There are some classical cases of g-measures exhibiting phase transition. They include the Hofbauer-type potentials; the Bramson-Kalikow and Berger-Hoffman-Sidoravicius examples; and the Fisher-Felderhof renewal-type examples. See [4, 7, 11, 23, 29, 31].

Take one of these examples and m a non-extreme conformal measures composed by the convex combination of n (finite) extreme ones. By Lemma 6.1.8, there are n disjoint measurable sets B_k, $1 \leqslant k \leqslant n$, each of them with total measure with respect to a extreme conformal ν_k. By the structure arising on the proof of the above theorem, we know that the restriction of an eigenfunction to one of these B_k is also an eigenfunction. This means that the eigenspace of $\mathbb{L}_m$, the extension to $L^1(m)$ of the continuous operator $\mathscr{L}$, is generated by the restrictions of the continuous maximal eigenfunction 1 to these sets B_k. We can write it in an equation as

$$\mathbb{L}_m u = u, \quad \forall u \in \operatorname{span}\{\mathbb{1}_{B_1}, \ldots, \mathbb{1}_{B_n}\}.$$

This is a class of examples of continuous potentials where the operator $\mathbb{L}_m$ has a multidimensional maximal eigenspace.

6.2 Conditions for the existence of an invariant function

As mentioned in Remark 5.2.7, the following chain of implications is valid for an arbitrary Markov process $(X, \mathscr{F}, \mu, T)$.

$$Tu = u \text{ with } u > 0 \implies X = \mathscr{C} \implies T^*\mathbb{1} = \mathbb{1}.$$

We also have seen that, for the particular case of the class of operators $\mathbb{L}$ as defined in Chapter 3, the condition $\mathbb{L}^*\mathbb{1} = \mathbb{1}$ is precisely the definition of a generalized conformal measure (we suppose $\|\mathbb{L}\|_{op} = 1$). But we also cited on 5.2.7, that each step in the chain of implications above is strict. Then, there is no reason to believe that every operator $\mathbb{L}_\psi$ that is an extension of a $\mathscr{L}_\psi$ defined by a continuous potential ψ has a $L^1(\nu)$ eigenfuntion. On the opposite, a classical example of Ruelle operator based on the Maneville-Pomeau map and given by a continuous potential is known to have no $L^1(\nu)$ eigenfunction. See [16] for details.

The question to be investigated in this section is this: if $\mathscr{L}_\psi$ is a a Ruelle operator given by a continuous potential ψ and $\mathbb{L}$ its extension to $L^1(\nu)$, for ν a conformal measure, are there necessary and sufficient conditions for the existence of a invariant function to $\mathbb{L}$? The answer is positive and it relies on an important theorem that is valid for a general Markov process. The main reference on this matter is "Existence of Bounded Invariant Measures", by J. Neveu, [43]. We one more time follow the lines of Foguel in Theorem E of [24, p.45] and adapt his conclusions in his Chapter IV to our setting.

So, we apply Neveu's theory on invariant functions to the operator $\mathbb{L}$, the extension of the classical transfer operator $\mathscr{L}_\psi$, defined by a fully supported a priori measure p on a compact metric alphabet E and continuous potential ψ. As multiplication by a constant does not change the eigenfunctions of an operator, we choose $\|\mathbb{L}\|_{\text{op}} = 1$. From the discussion of the previous section it should be clear that the problem of the existence of a maximal eigenfunctions can analyzed on each of the supporting sets B_k with $\nu_k(B_k) = 1$ and $B_j \cap B_k = \varnothing$. Therefore as in the proof of Theorem 6.1.10, we will restrict ourselves, without loss of generality, to extensions associated to extreme elements of $\mathscr{G}^*$.

This is an outstanding result in the theory of Markov processes and roughly speaking it says that there exists a measurable partition of the space $X = A_0 \cup A_1$, called Hopf decomposition, for which there is at least one maximal eigenfunction which is positive on A_1, but there is no maximal eigenfunction having a positive part in any subset of A_0 (ν-a.e).

Recall that, whenever $\nu \in \mathrm{ex}(\mathscr{G}^*)$, Lemma 6.1.4 guarantees that any non-negative maximal eigenfunction is actually positive ν-a.e. and Theorem 6.1.5 says us that if there is an eigenfunction, it is unique up to multiplication by a constant. In terms of the theory of Markov processes this is the same as saying that the $A_0 - A_1$ decomposition is trivial in the sense that $X = A_1$. On the other hand, when no maximal eigenfunction exists, $X = A_0$, and so the decomposition is always trivial.

Note that these are exactly the hypothesis of Corollaries 1 and 2 in [24, p.45–46], which, translated back to our notation gives us the following conclusion.

Lemma 6.2.1. *Let* ψ *be a continuous potential and* $\nu \in \mathscr{G}^*$ *an extreme conformal measure. Then the following are equivalent:*

1. *there exists a unique, up to multiplication by a constant,* $0 \not\equiv u \in L^1(\nu)$ *such that* $\mathbb{L}u = u$;

2. *if* $0 \leqslant v \in L^\infty(\nu)$, $v \not\equiv 0$, *then* $\liminf_n \langle v, \mathbb{L}^n 1\rangle_\nu > 0$;

3. *if* $0 \leqslant v \in L^\infty(\nu)$, $v \not\equiv 0$, *then*

$$\liminf_N \frac{1}{N}\sum_{n=0}^{N-1} \langle v, \mathbb{L}^n 1\rangle_\nu > 0;$$

4. *There is no set* A *of positive* ν*-measure for which*

$$\lim_N \frac{1}{N}\sum_{n=0}^{N-1} (\mathbb{L}^*)^n \mathbb{1}_A = 0 \quad (\nu - \text{a.e.});$$

5. *There is no set* A *of positive measure for which*

$$\lim_N \frac{1}{N}\sum_{n=0}^{N-1} (\mathbb{L}^*)^n \mathbb{1}_A = 0 \text{ uniformly } (\nu\text{-a.e.}).$$

CHAPTER 7

CONCLUSION

We studied the spectral data of the Ruelle operator $\mathscr{L}$ and its extension $\mathbb{L}$ by means of the theory of Markov processes. In the last chapter, we have shown in Corollary 6.1.6 that if the Ruelle operator $\mathscr{L}$ has an eigenfunction (given a fully supported a priori measure), it must be unique up to multiplication by a constant and it must be positive, touching zero at most in a ν-null set.

We also discussed the eigenspace of the extension $\mathbb{L}$ to $L^1(\nu)$ and showed that there is a strong relation between the extreme conformal measures composing ν and the dimension of the maximal eigenspace of $\mathbb{L}$. In Chapter 2 we present an example where this relation is highlighted.

Many questions related to our work are still unanswered and can give way to future investigations. The conservative-dissipative classification of Markov processes $\mathbb{L}$, defined as extensions of Ruelle operators acting on continuous functions, for example, is unknown to us. For many regularity classes, it is known that there is a continuous eigenfuntion. All of these case are conservative when seen as Markov processes. On the other hand, we have the example based on the Maneville-Pomeau map of a continuous potential with no L^1 eigenfunction. The relation between a process being conservative and this same process having a eigenfunction in the case of the operators $\mathbb{L}$ is not completely understood to us and some effort to elucidate this question may be fruitful. In particular, we do not know if there is a continuous potential that gives rise to a dissipative process $(X, \mathscr{B}(X), \nu, \mathbb{L})$.

We know that the conditions given in Section 6.2 are optimal for a general Markov process T. But we are not certain if they are also optimal for the specific case of the Ruelle-like $\mathbb{L}$, or they can be simplified due to some characteristics of $\mathbb{L}$ not shared by general Markov processes. During the presentation of this work to the dissertation committee, Professor Artur Lopes suggested that an involution kernel technique, as in [13], can be used to construct the L^1 eigenfunctions in association with our methods. He also suggested that the techniques developed in [1] can be useful to establish sharp lower and upper bounds on the spectral radius. We think that future investigations on this theme can generate interesting results.

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