Intransitive state-closed groups

Alex Carrazedo Dantas

Universidade de Brasília, Departamento de Matemática

Joint work with Tulio Marcio Gentil dos Santos and Said Najati Sidki

Algebra seminar

September 2, 2020

1. Automorphism group $\mathcal{A}_m$

The automorphism group $\mathcal{A}_m$ of the 1-rooted m-regular tree $\mathcal{T}_m$ is isomorphic to the restricted wreath product recursively defined as

$$\mathcal{A}_m = \mathcal{A}_m \wr S_m = \left(\prod_{i=0}^{m-1} \mathcal{A}_m\right) \rtimes S_m,$$

where S_m is the symmetric group of degree m. Then each $\alpha \in \mathcal{A}_m$ has the form

$$\alpha = (\alpha_0, \alpha_1, ..., \alpha_{m-1})\sigma(\alpha)$$

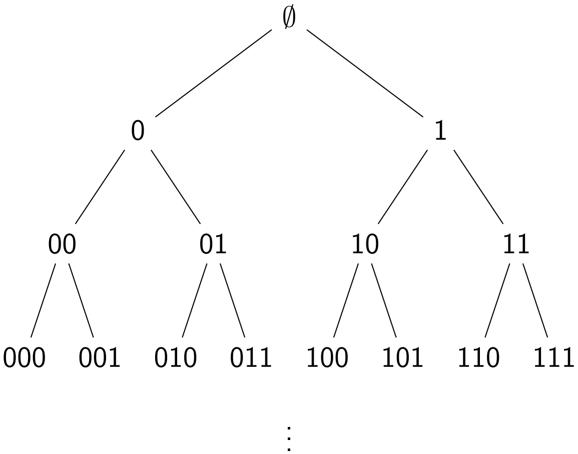
where $\alpha_0, \alpha_1, ..., \alpha_{m-1} \in \mathcal{A}_m$ and $\sigma(\alpha) \in S_m$.
For $\alpha \in \mathcal{A}_m$, the set of automorphisms

$$Q(\alpha) = \{\alpha, \alpha_0, ..., \alpha_{m-1}\} \cup_{i=1}^{m-1} Q(\alpha_i)$$

is called the set of states of α and this automorphism is said to be finite-state provided $Q(\alpha)$ is finite.

1. Automorphism group $\mathcal{A}_m$

The binary regular tree $\mathcal{T}_2$:



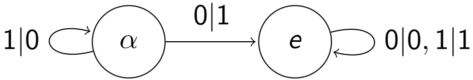
If $\alpha \in \mathcal{A}_2 = \mathcal{A}_2 \wr S_2 = (\mathcal{A}_2 \times \mathcal{A}_2) \rtimes S_2$ then $\alpha = (\alpha_0, \alpha_1)\sigma(\alpha)$, where $\alpha_0, \alpha_1 \in \mathcal{A}_2$ and $\sigma(\alpha) \in S_2$.

2. State-closed groups

A subgroup G of $\mathcal{A}_m$ is state-closed (or, self-similar) if $Q(\alpha)$ is a subset of G for all α in G.

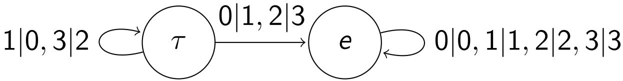
Example

a) (The binary adding machine) The group $G_1 = \langle \alpha = (e, \alpha)(0\,1) \rangle$ is a state-closed subgroup of $\mathcal{A}_2$.



;

b) The group $G_2 = \langle \tau = (e, \tau, e, \tau)(0\,1)(2\,3) \rangle$ is a state-closed subgroup of $\mathcal{A}_4$.



;

2. State-closed groups

c) (Grigorchuck, 1980) The finitely generated 2-group

$$\langle \sigma = (0\,1), u = (e, v), v = (\sigma, w), w = (\sigma, u) \rangle$$

is a state-closed subgroup of $\mathcal{A}_2$ (Grigorchuck group);

d) (Gupta, Sidki, 1983) The finitely generated p-group

$$\langle \sigma = (0\,1\,...\,p-1), \gamma = (\gamma, e, ..., e, \sigma, \sigma^{p-1}) \rangle$$

is a state-closed subgroup of $\mathcal{A}_p$, p odd prime (Gupta-Sidki group);

2. State-closed groups

e) (Muntyan, Savchuk) The free product $C_2 * C_2 * C_2$ and the free group $\mathbb{F}_2$ of rank 2 are isomorphics to the state-closed groups

$$C_2 * C_2 * C_2 \simeq \langle a = (b, b)(0\,1), b = (a, c), c = (c, a)\rangle$$

and

$$\mathbb{F}_2 = \langle ab = (bc, ba)(0\,1), bc = (ac, ca)\rangle.$$

d) (Grigorchuck, Zuk, 2000) The Lamplighter group $C_2 \wr \mathbb{Z} \simeq (\oplus_{\mathbb{Z}} C_2) \rtimes \mathbb{Z}$ is isomorphic to the state-closed group

$$\langle \alpha = (\alpha, \beta), \beta = (\alpha, \beta)\sigma\rangle.$$

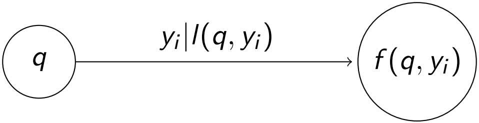
3. Finite Mealy automata

A finite Mealy automata is a Turing machine defined by a quadruple $A = (Q, Y, f, l)$ where:

- Q is a finite set of states;
- Y is a finite set of letters (alphabet);
- $f : Q \times Y \to Q$ is the state transition function;
- $l : Q \times Y \to Y$ is the output function.

If $l(q, *)$ is invertible for any $q \in Q$ we said that the Mealy automata A is invertible and each state $q \in Q$ can be interpreted as an automorfism of an 1-rooted m-regular tree $\mathcal{T}_{|Y|}$:

$$q = (f(q, y_1), ..., f(q, y_{|Y|}))l(q, *)$$



4. Transitive state-closed groups

A group which is finitely generated, state-closed and finite-state in $\mathcal{A}_m$ is called an automata group of degree m.

A subgroup G of $\mathcal{A}_m$ is called transitive if its is transitive on the tree's first level.

A virtual endomorphism is a homomorphism $f : H \to G$ where H is a subgroup with finite index in G

(Nekrashevych and Sidki, 2004) A subgroup G of $\mathcal{A}_m$ is transitive state-closed if, and only if, there exist a subgroup H with index m in G and a virtual endomorphism $f : H \to G$ such that

$$\langle K \leq H \mid K \lhd G, K^f \leq K \rangle = 1.$$

4. Transitive state-closed groups

(Silva, Steinberg - 2005, IJAC) The Lamplighter group $C_n \wr \mathbb{Z}$ is a transitive automata group of degree n.

(Sidki, - 2017, JA) The Lamplighter group $C_p \wr \mathbb{Z}^d$ ($d \geq 2$ and p prime) is a transitive automata group of degree p^2. More that, $C_p \wr \mathbb{Z}^d$ is not transitive state-closed of a prime degree q.

(Sidki, - 2018, GGD) Let $G = B \wr X$ be a transitive state-closed wretah product of abelian groups. If X is torsion free then B is a torsion group of finite exponent. In particular, $\mathbb{Z} \wr \mathbb{Z}$ cannot a transitive state-closed group.

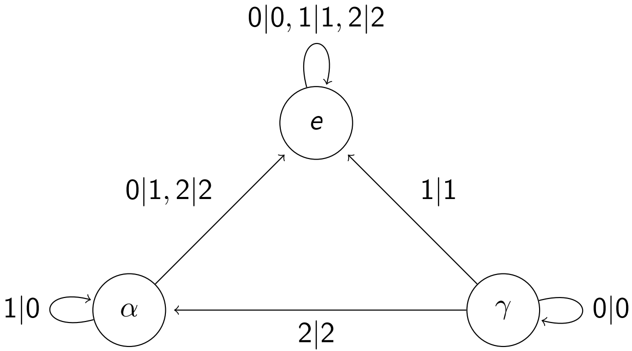
(Bartholdi, Sidki - 2020, GGD) Let B be a finite abelian group. Then $B \wr \mathbb{Z}^d$ is a transitive automata group of degree $2|B|$.

5. Intransitive state-closed groups

Example

(dos Santos, Sidki, - 2020, submitted) The wreath product $\mathbb{Z} \wr \mathbb{Z}$ is isomorphic to the automata group of degree 3

$$\langle \gamma = (\gamma, e, \alpha), \alpha = (e, \alpha, e)(0\,1) \rangle \simeq \langle \gamma \rangle \wr \langle \alpha \rangle$$



5. Intransitive state-closed groups

A subgroup G of $\mathcal{A}_m$ is state-closed if, and only if, there exist subgroups $H_1, ..., H_s$ of finite index in G and virtual endomorphisms $f_1 : H_1 \to G$, ..., $f_s : H_s \to G$ such that $[G : H_1] + ... + [G : H_s] = m$ and

$$\langle K \leq \cap_{i=1}^{s} H_i \mid K \lhd G, K^{f_i} \leq K, \forall i = 1, .., s\rangle = 1.$$

Theorem

Let A be a finitely generated abelian group and $B = Tor(A)$. Then $G = A \wr \mathbb{Z}^d$ is an automata group of degree $2|B| + 4$. In particular, for $A = \mathbb{Z}^l$ the degree can be reduced to 4.

5. Intransitive state-closed groups

Theorem

Let p a prime number then $C_p \wr \mathbb{Z}^2$ is a state-closed group of degree $p+1$. Indeed, $C_p \wr \mathbb{Z}^2$ is generated by

$$\alpha = (\alpha, \alpha\sigma, ..., \alpha\sigma^{p-1}, \alpha\beta),$$

$$\sigma = (e, ..., e, \sigma)(0\,1\,...\,p-1)$$

and

$$\beta = (e, ..., e, \alpha).$$

In particular, the group $C_2 \wr \mathbb{Z}^2$ is state-closed of degree 3. Note that this representation is not finite-state.

5. Intransitive state-closed groups

Theorem

Let G be a state-closed group of degree m. Then the following hold.

1) $G^{(\omega)}$ *is a state-closed group of degree* $m+1$*; in particular, for* $G=\mathbb{Z}$*, the representation of the group* $\mathbb{Z}^{(\omega)}$ *is of degree* 3 *and is in addition finite-state.*

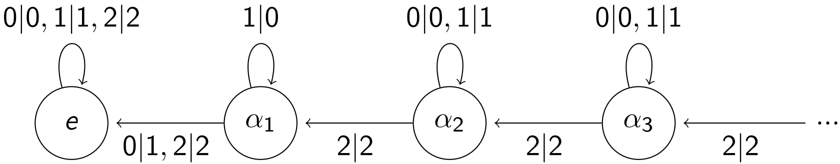
2) *Let* K *be a regular subgroup of* $Sym(\{1,...,s\})$*. Then the restricted wreath product* $G \wr K$ *is a transitive state-closed group of degree* $n.s$*, for some integer* n*; in particular, the group* $(\mathbb{Z} \wr \mathbb{Z}) \wr C_2$ *is a transitive automata group of degree* 4*.*

5. Intransitive state-closed groups

1) The group $\mathbb{Z}^{(\omega)}$ is isomorphic to

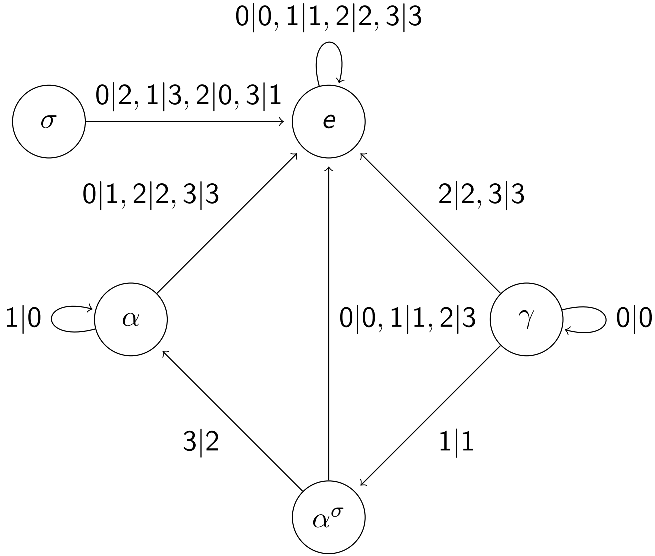
$$\langle \alpha_1 = (e, \alpha_1, e)(0\,1), \alpha_i = (\alpha_i, \alpha_i, \alpha_{i-1}) \mid i = 2, 3, 4, \ldots \rangle$$

which is faithful and finite-state.



5. Intransitive state-closed groups

2) The group $(\mathbb{Z} \wr \mathbb{Z}) \wr C_2$ is generated by the following automaton:



5. Intransitive state-closed groups

Questions

1. Is the group $C_2 \wr (\mathbb{Z} \wr \mathbb{Z})$ transitive state-closed?

2. Is there a faithful state-closed representation of degree 3 *for the group* $\mathbb{Z}^l \wr \mathbb{Z}^d$ *when* $l, d \geq 2$*?*

3. Is there a faithful finite-state and state-closed representation for the group $C_2 \wr \mathbb{Z}^2$ *of degree* 3*?*

6. Main References

- L. Bartholdi and S. N. Sidki, *Self-similar products of groups.* Groups, Geometry and Dynamics **14(1)**, 107-115, 2020.
- A. M. Brunner and S. N. Sidki, *Wreath operations in the group of automorphisms of the binary tree.* Journal of Algebra, **257**, 51 - 64, 2002.
- A. C. Dantas and S. N. Sidki, *On self-similarity of wreath products of abelian groups.* Groups, Geometry, and Dynamics, **12**, 1061 - 1068, 2018.
- E. I. Khukhro and V. D. Mazurov, *THE KOUROVKA NOTEBOOK - Unsolved problems in group theory.* arXiv:1401.0300v14, **19**, 2018.
- A. Woryna, *The concept of self-similar automata over a changing alphabet and lamplighter groups generated by such automata.* Theoretical Computer Science, **482**, 96 - 110, 2013.

P. Silva, B. Steinberg, *On a class of automata groups generalizing Lamplighter groups.* Int. J. Algebra Comut. **15** (2005), 1213 - 1234.

V. Nekrashevych, *Self-similar groups.* Math. Surveys and Monographs, 117, American Mathematical Society, Providence, RI, (2005).

V. Nekrachevych and S. N. Sidki, *Automorphisms of the binary tree: state-closed subgroups and dynamics of 1/2 endomorphisms.* In: Groups: London Mathematical Lecture Notes Series, Topological, Combinatorial and Arithmetic Aspects, Muller, T. W., (Ed.) **311** (2004), 375 - 404.

A. C. Dantas and S. N. Sidki, *On state-closed representations of restricted wreath product of groups of type $G_{p,d} = C_p wr C^d$.* Journal of Algebra, **500**, 335 - 361, 2018.

D. J. S. Robinson, *A course in the theory of groups.* Springer-Verlag, New York, (2000).

S. N. Sidki, *Regular trees and their automorphisms.* Monográfias de Matemática, vol 56, Instituto de Matemática Pura e Aplicada, **15** (1998).

A. C. Dantas, T. M. G. dos Santos, and S. N. Sidki, *Intransitive state-closed groups.* Submitted in Journal of Algebra (2020).

Obrigado!