Closure Properties of the non-abelian Tensor Product and its Applications

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August 21, 2020

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Some generalisations of Schur's and Baer's theorem and their connection with homological algebra
To appear in Mathematische Nachrichten

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Definition

Let G and H be groups acting on each other. The mutual actions are said to be *compatible* if

$$ {}^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h')) \text{ and } {}^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')), $$

for each $g, g' \in G$ and $h, h' \in H$.

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Definition

(Brown-Loday) Let G and H be two groups that act compatibly on each other. Then the *non-abelian tensor product* $G \otimes H$ is the group generated by the symbols $g \otimes h$ for $g \in G$ and $h \in H$ with relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h),$$
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There is well-defined homomorphism $\lambda_2^G : G \otimes G \to [G, G]$ given by

$$g \otimes g' \mapsto [g, g'].$$

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Theorem

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A class of groups $\mathcal{X}$ is said to be Schure class, if for any central extension $1 \to N \to H \to G \to 1$ of $G \in \mathcal{X}$, $[H,H] \in \mathcal{X}$.

Schure Theorem

Lemma

Let $1 \to N \to H \to G \to 1$ *be a central extension of groups. Then, there exists a surjective homomorphism* $G \otimes G \to [H,H]$ *making the following diagram commutative:*

$$\begin{CD} G \otimes G @>{\mathrm{id}}>> G \otimes G \\ @VVV @VV{\lambda_2^G}V \\ [H,H] @>>> G \end{CD}$$

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$$^{(\cdots((g_1 \otimes g_2) \otimes g_3) \otimes \cdots \otimes g_{n-1})}g_n = {}^{[\cdots[[g_1, g_2], g_3], \cdots, g_{n-1}]}g_n.$$

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Lemma

Let $1 \to N \to H \to G \to 1$ be an extension of groups such that $N \leq Z_n(H)$ for a fixed positive integer n. Then, there exists a surjective homomorphism $G^{\otimes n+1} \to \gamma_{n+1}(H)$ making the following diagram commutative:

$$\begin{array}{ccc} G^{\otimes n+1} & \xrightarrow{\mathrm{id}} & G^{\otimes n+1} \\ \downarrow & & \downarrow \lambda_{n+1}^G \\ \gamma_{n+}(H) & \longrightarrow & G \end{array}$$

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Definition

Let G and H be groups with H acting of G. The *derivative* of G by H is the subgroup of G defined by

$$D_H(G) = \langle g\, {}^{h}g^{-1} \mid g \in G, h \in H \rangle.$$

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G. Donadze, M. Ladra, V. Z. Thomas, On some closure properties of the non-abelian tensor product, J. Algebra **472**, 399-413 (2017)

Baer Theorem for finitely generated groups

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Let G be a finitely generated group. Then the following are equivalent:

(i) $\gamma_{n+1}(G)$ *is a finitely generated group;*

(ii) $\gamma_{n+1}(H)$ *is a finitely generated group for any extension of groups* $1 \to N \to H \to G \to 1$ *with* $N \leq Z_n(H)$.

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If G is a superperfect group, then $H_2^{T_n}(G)$ is trivial.