Seminário de Álgebra

The Alternating Group as product of four conjugacy classes.

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Abstract.

In 1992, László Babai and Ákos Seress conjectured that if G is a non-abelian simple group then the diameter of any Cayley graph of G is bounded above by $(\log |G|)^c$ for some absolute constant c [1, Conjecture 1.7]. Babai's conjecture can be reformulated by saying that $(\log |G|)^c$ is an upper bound for the minimal positive integer m such that $(S \cup S^{-1})^m = G$ for every generating set S of G. For example, solving the Rubik's cube (cubo mágico) in the smallest number of moves is equivalent to finding the diameter of some Cayley graph of some group.

A normal subset of a group G is a subset S such that $gsg^{-1} \in S$ whenever $s \in S$, $g \in G$. Clearly, a subset is normal if and only if it is a union of conjugacy classes. Note that any normal subset generates a normal subgroup, therefore, in a non-abelian simple group, any normal subset distinct from $\{1\}$ is a generating set and it makes sense to consider Babai's conjecture in this particular situation. In 2001, Liebeck and Shalev famously proved that, if S is a nontrivial normal subset of G, then an even better bound holds, namely $c \cdot \log |G| / \log |S|$ [4, Theorem 1.1].

In other words, if $|S|^k \geq |G|^c$ then $S^k = G$.

It is natural to ask whether the same thing holds when instead of k copies of S we have distinct normal sets. Gill, Pyber and Szabó proposed a conjecture [3, Conjecture 2] later proved by Maróti and Pyber [5, Theorem 1.2], the following:

There exists a constant c such that, if $S_1, \ldots, S_k$ are normal subsets of a non-abelian finite simple group G satisfying $\prod_{i=1}^k |S_i| \geq |G|^c$, then $S_1 \cdots S_k = G$.

In simple (but less formal) words, this says that if we have "large" normal subsets of a non-abelian simple group G, then their product equals G. In this spirit, we proved the following result [2, Theorem 1.1].

Theorem 1 (G., Maróti) *Let* $\mathrm{Alt}(n)$ *be the alternating group on* n *letters. For any* $\varepsilon > 0$ *there exists* $N = N(\varepsilon) \in \mathbb{N}$ *such that whenever* $n \geq N$ *and* A, B, C, D *are normal subsets of* $\mathrm{Alt}(n)$ *of size at least* $|\mathrm{Alt}(n)|^{1/2+\varepsilon}$, *then* $ABCD = \mathrm{Alt}(n)$. *Moreover, the constant* $1/2$ *is best possible.*

To prove this, one quickly reduces to assuming that A, B, C, D are conjugacy classes and then uses character theory. I will introduce the basic concepts that allow to prove this.

This was the content of a paper which was submitted in May 2020 and it was later accepted. If there is time, I will talk a little bit about the project I'm currently involved in, which concerns Babai's conjecture for classical simple groups.

Joint work with A. Maróti

References

[1] L. Babai, Á. Seress, On the diameter of permutation groups. European J. Combin. 13 (1992) 231–243.

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[5] A. Maróti, L. Pyber, A generalization of the diameter bound of Liebeck and Shalev for finite simple groups. arXiv: 2003.14270v2.