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Conformal solitons to the mean curvature flow: minimal submanifolds and stability

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por

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Abstract

Estudamos o artigo [2] por C. Arezzo e J. Sun. Apresentamos uma correspondência entre sólitons conformes para o fluxo da curvatura média em uma variedade Riemanniana ambiente N e subvariedades mínimas em um produto warped $N \times \mathbb{R}$. A demonstração dessa correspondência nos fornece uma função potencial para o campo vetorial conforme do sóliton conforme, que nos possibilita apresentar uma correspondência entre estabilidade de subvariedades mínimas associadas aos sólitons conformes em um produto warped e estabilidade de subvariedades mínimas como pontos críticos para um funcional volume com peso, em que o peso é dado em termos da função pontencial. Na sequência, apresentamos uma demonstração que os self-shrinkers compactos em $\mathbb{R}^{n+1}$ não são estáveis e, seguindo C. Arezzo e J. Sun, apresentamos uma demonstração de que o cilindro grim reaper é um sóliton de translação estável em $\mathbb{R}^{n+1}$. Finalmente, apresentamos uma correspondência entre sólitons conformes em $\mathbb{R}^{n+p}$ e subvariedades totalmente geodésicas em $\mathbb{R}^{n+p+1}$ por C. Arezzo e J. Sun.

Palavras-chave: estabilidade, subvariedades mínimas, produto warped, fluxo da curvatura média, sólitons conformes.

Abstract

We study the paper [2] by C. Arezzo and J. Sun. We present a correspondence between conformal solitons to the mean curvature flow in a Riemannian ambient manifold N and minimal submanifolds in a warped product $N \times \mathbb{R}$. The proof of this correspondence provide us a potential function for the conformal vector field of the conformal soliton, which enable us to present a proof of a correspondence between stable minimal submanifolds associated to the conformal solitons in a warped product and the stability of minimal submanifolds as critical points to a weighted volume functional where the weight depends on the potential function. In the sequence, we give a proof that compact self-shrinkers in $\mathbb{R}^{n+1}$ are unstable and, following C. Arezzo and J. Sun, we present a proof that the cylinder grim reaper is a stable translating soliton in $\mathbb{R}^{n+1}$. Finally, we present a correspondence between conformal solitons in $\mathbb{R}^{n+p}$ and totally geodesic submanifolds in $\mathbb{R}^{n+p+1}$ by C. Arezzo and J. Sun.

Keywords: stability, minimal submanifolds, warped products, mean curvature flow, conformal solitons.

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Introduction

One of the most amazing themes to work in geometry is the study of geometric flows. These flows have shown themselves a powerful feature to answer big questions in topology, e.g., the Ricci Flow was used by Hamilton and Perelman to prove the Poincaré's conjecture, one of the Millenium Prize Problems and one of the most difficult problems in topology which remained open for almost one century.

The mean curvature flow stands out among the geometric flows for its various topological consequences (see [6], [17], [18] and [22] for some examples) and its applications in other areas outside of mathematics (see [5], [9], [13] and [24] for some examples). The first theoretical mathematical approach to the mean curvature flow was developed by Brakke in a geometric measure point of view in [4], but only with the advent of the Ricci Flow the mean curvature flow gained more prominence when Huisken adapted the techniques of the Ricci Flow to the mean curvature flow which culminated in his famous work [20]. Formally, the mean curvature flow is a family of smooth immersions such that the initial Riemannian hypersurface M evolves by its mean curvature over the time in the Riemannian ambient manifold N, i.e.,

$$F : M \times [0,T) \longrightarrow N$$

such that

$$\begin{cases} \dfrac{\partial F}{\partial t}(p,t) & = H(p,t)\nu(p,t) \\ F(M,0) & = M, \end{cases}$$

where $\nu(\cdot,t)$ is the unit normal to $F(\cdot,t)$ pointing inward and $H(\cdot,t)$ its mean curvature.

Intuitively, the mean curvature flow is a way to deform into its normal direction and with velocity given by its mean curvature. For example, round spheres are deformed into "points" and cylinders are deformed into "straight lines", both preserving their shape along with the flow. On the other hand, planes are preserved by the mean curvature flow (see Fig. 1).

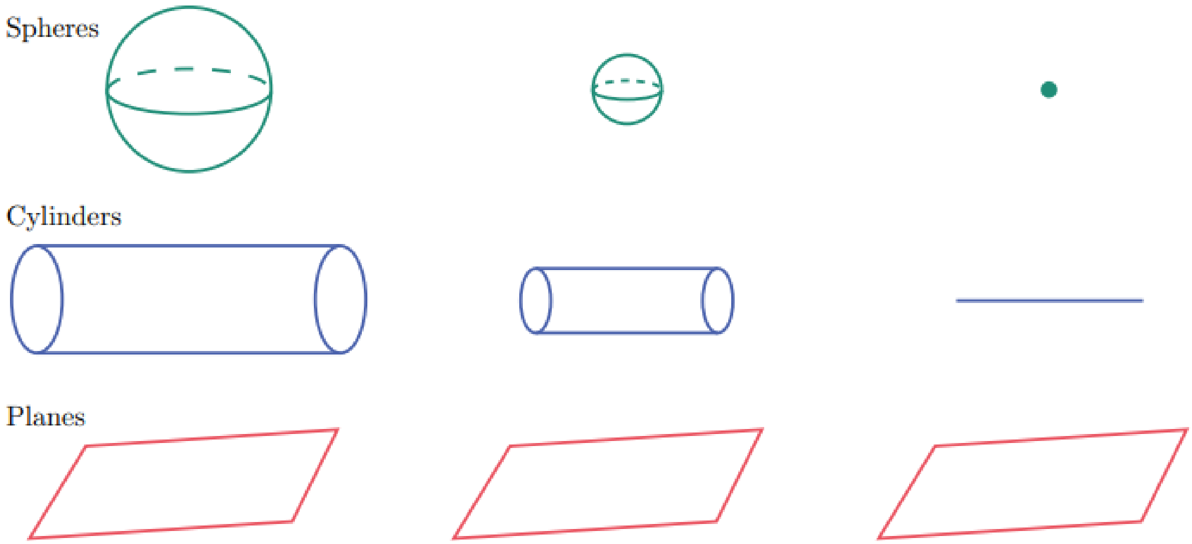

Fig. 1 Figure extracted from [10].

There are substantial works on the mean curvature flow when $N = \mathbb{R}^{n+1}$. Among these works, special solutions received a lot of attention recently. One of these special solutions are the solitons solutions, associated to a given vector field X. Such solutions provides interesting geometric structures on the initial data given by the hypersurface M. Following definition 1.1 in [1], a hypersurface $f : M \longrightarrow N$ is a soliton of the mean curvature flow with respect to a vector field $\boldsymbol{X}$ on N if $c\boldsymbol{X}^{\perp} = \boldsymbol{H}$ for some constant c and where $\boldsymbol{H}$ is the mean curvature vector field. When N is the Euclidean space, we have interesting structures by taking particular cases of c and [23]: when $\boldsymbol{X}$ is a constant vector field the soliton is called translating soliton. When [23] is the position vector field, we have the self-shrinkers if $c < 0$ and the self-expanders if $c > 0$. In a general setup, we say that the soliton is a conformal soliton if X is a conformal vector field. Besides of being related with mean curvature flow, translating solitons, self-shrinkers and self-expanders can also be view as weighted minimal surfaces or the so called f-minimal surfaces, when we consider a conformal metric $e^f g$ in $\mathbb{R}^n$, where g is the Euclidean metric. For this reason, such structures have also their own interest and have been intensively studied in the last years (see [3], [7], [8], [23] and [26] for some examples).

Another motivation for the study of solitons is the close relation with singularities. If a hypersurface develops a singularity under the evolution by the mean curvature flow such that the norm of the second fundamental form has the growth rate

$$\max_{p\in M}|A(p,t)|^2 \leq \frac{C_0}{2(T-t)},\ \forall t\in[0,T),\ C_0>0,$$

the singularity is called Type I singularity. Otherwise, the singularity is called Type II singularity. Type I singularities are close related to self-shirinkers, as we can see in Huisken [19], whereas a relation between translating solitons and Type II singularity was obtained by Huisken and Sinestrari [21] (see also Corollary 9.4 in [28]).

These solutions motivate the study of self-similar solutions evolving by mean curvature in a setting more abstract, by considering conformal solitons for the mean curvature flow in arbitrary ambient spaces, as we can see in Smoczyk [30] and in Lira [1].

The present thesis is based in [2] by Arezzo and Sun, where the ideas of Smoczyk in [30] are extended for the case of submanifolds and a study of stable self-similar solutions in the sense of stability for conformal solitons is presented. As some proofs are extensions of Smoczyk's ideas for higher codimension, many of the proofs in [2] are omitted or not given in details. We present full proofs of the results, including omitted proofs, e.g., the proof that grim reaper cylinder in $\mathbb{R}^{n+1}$ is stable. One of the main contributions of this dissertation was to fix a mistaken computation of the curvature tensor component (2.8) in [2] and, consequently, the computations of the components of the Ricci tensor were fixed. The results of the section 3 in [2] holds with some corrections in the sign of the conformal factor (see definition 1.3) and in the sign of the last term in the right side of the equality in the lemma 2.4 as well as some corrections in the tensor curvature in the definitions 2.1 and 3.1. Other important correction is in the sign of the second derivative in the lemma 3.2 in [30] which is the lemma 2.2 in this dissertation. These corrections have some interesting consequences listed below:

1. there is only one stability operator and the notions of stability for submanifolds obtained in the section 3 in [2] and by variational principle for the weighted volume are equivalent;

2. following Colding and Minicozzi [11], it is given a proof that that every compact self-shrinker in $\mathbb{R}^{n+1}$ is unstable, when considered as critical points for the weighted volume. This result seems to contrast with Theorem 5.2 in [2], which states that a self-shrinker is stable if and only if is the sphere $S^n(\sqrt{2n})$. However, it is important to note that such Theorem is stated following the notion of stability with the wrong sign mentioned above. Moreover, as observed in the beginning of the section 4 in [12], every critical point of the F functional is unstable if you fix x_0 and t_0 and vary Σ alone, but the F functional with $x_0 = 0$ and $t_0 = 1$ fixed is exactly the weighted functional. Therefore, we believe that these observations help to clarify the stability of self-shrinkers considered in [2] and [30].

This master thesis is organized as follows. The chapter 1 contains some preliminary results. It is also given a geometric interpretation of the stability of minimal hypersurfaces with respect to the functional volume to familiarize the reader with the notion of stability and the variational principle that will studied in the chapter 3.

The first two sections of the chapter 2 is based on the ideas of Smoczyk in [30] and extended for higher codimension as done in [2]. Some geometric quantities are computed for a warped product metric that are useful to establish a correspondence between conformal solitons of the mean curvature flow and minimal submanifolds in a warped Riemannian manifold. Also, they are useful to establish a correspondence between minimal hypersurfaces in a warped Riemannian manifold and an inequality of stability. Arezzo and Sun extend these results to submanifolds and characterize conformal solitons in $\mathbb{R}^{n+p}$ endowed with the Euclidean metric through tottaly geodesic submanifolds in $\mathbb{R}^{n+p+1}$ endowed with a warped product metric in the last section.

The first section of the chapter 3 provides the computations of the First and Second Variations of a weighted functional volume following [2] and [11]. This motivates the definition of a stability operator for conformal solitons and the local minimum of this functional provides a different proof of the stability inequality for conformal solitons obtained in the chapter 3. It is presented a proof that the conformal solitons are the only one critical points of this functional and some examples of stable conformal solitons are given. Finally, we present a proof that the grim reaper in $\mathbb{R}^2$ and the grim reaper cylinders in $\mathbb{R}^{n+1}$ are stable translating solitons given by Arezzo and Sun.

Chapter 1

Preliminary results

In this chapter, we will introduce basic notions and results of smooth manifolds and Riemannian geometry. We will also present a brief introduction to the variational approach for minimal hypersurfaces, in order to familiarize the reader with the notion of stability. We will end the chapter with some basic concepts on the mean curvature flow that will be useful throughout this dissertation.

1.1 Tensors and Lie derivative

We give a short introduction to tensors following section 5 of the chapter 4 of [14] and we introduce the Lie derivative as a consequence of the Corollary 12.33 in [25]. This is not a geometric introduction to these concepts, but we do in this way to give an easy and quick introduction to the structures to prove the proposition 1.1. The curious reader can read a geometric introduction in the chapter 12 of [25].

Definition 1.1. A tensor T of order r on a Riemannian manifold M is a multilinear mapping

$$T : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r \text{ times}} \longrightarrow \mathcal{C}^{\infty}(M)$$

This means that given $Y_1, \cdots, Y_r \in \mathfrak{X}(M)$, $T(Y_1, \cdots, Y_r)$ is a differentiable function on M, and that T is linear in each argument, that is,

$$T(Y_1,\cdots,fX+gY,\cdots,Y_r)=fT(Y_1,\cdots,X,\cdots,Y_r)+gT(Y_1,\cdots,Y,\cdots,Y_r)$$

for all $X,Y\in\mathfrak{X}(M), f,g\in\mathcal{C}^\infty(M)$.

A tensor T is a pointwise object in a sense that we now explain. Fix a point $p\in M$ and let U be a neighborhood of $p\in M$ on which it is possible to define vector fields $E_1,\cdots,E_n\in\mathfrak{X}(M)$, i.e., the vectors $\{E_i(q)\}_{i=1}^n$ form a basis of T_qM at each $q\in U$; we say, in this case, that $\{E_i\}_{i=1}^n$ is a moving frame on U. Let

$$Y_j=\sum_{i_j=1}^n y_{i_j}E_{i_j},\ j=1,\cdots,r.$$

be the restrictions to U of the vector fields $Y_1,\cdots,Y_r$ expressed in the moving frame $\{E_i\}$. By linearity,

$$T(Y_1,\cdots,Y_r)=\sum_{i_1,\cdots,i_r=1}^n y_{i_1}\cdots y_{i_r}T(E_{i_1},\cdots,E_{i_r}).$$

The functions $T(E_{i_1},\cdots,E_{i_r})=T_{i_1\cdots i_r}$ on U are called the components of T in the frame $\{E_i\}$.

Example 1.1. The Riemannian metric is a tensor of order 2.

Definition 1.2. Let T be a tensor of order r. The Lie derivative $\mathscr{L}$ of T in the direction of a vector field $Z\in\mathfrak{X}(M)$ is a tensor of order $(r+1)$ given by

$$(\mathscr{L}_ZT)(Y_1,\cdots,Y_r):=Z(T(Y_1,\cdots,Y_r))-T([Z,Y_1],\cdots,Y_r)-\cdots-T(Y_1,\cdots,Y_{r-1},[Z,Y_r]).$$

1.2 Basic results on Riemannian manifolds

We introduce the Einstein's sum convention that will used throughout this work without do mentions posteriori. The Einstein's sum convention consists to omit the sum always that appears upper and lower indexes repeated. See an example below.

Example 1.2. Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface. The mean curvature of M is written as

$$H = \sum_{i,j=1}^{n} g^{ij} h_{ij}.$$

The mean curvature of M is written in the Einstein's sum convention as

$$H = g^{ij} h_{ij}.$$

Proposition 1.1 (The Lie derivative in terms of the connection)**.** If (M,g) is a Riemannian manifold, then

$$(\mathscr{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i,$$

where ∇ denotes the Levi-Civita connection of the metric g and X is any vector field defined on M.

Proof. We follow the proof presented on page 14 of [29]. Let ω be an 1-form dual to the vector field X, i.e., ω is an 1-form which satisfies $\omega_p(Y) = g_p(X,Y)$ for each $p \in M$. Omitting $p \in M$ for simplicity, using the compatibility of the metric and the symmetry of the connection,

$$\begin{aligned}
(\mathscr{L}_X g)(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g([X,Y],Z) - g(Y,[X,Z]) \\
&= g(\nabla_X Y - [X,Y], Z) + g(Y, \nabla_X Z - [X,Z]) \\
&= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \\
&= Y(g(X,Z)) - g(X, \nabla_Y Z) + Z(g(Y,X)) - g(\nabla_Z Y, X) \\
&= Y(\omega(Z)) - \omega(\nabla_Y Z) + Z(\omega(Y)) - \omega(\nabla_Z Y) \\
&= (\nabla_Y \omega)(Z) + (\nabla_Z \omega)(Y),
\end{aligned}$$

which is free-coordinate, expressing the identity desired.

□

Definition 1.3. A smooth vector field $\mathbf{X}$ on a Riemannian manifold (M,g) is said to be a conformal vector field if there exists a smooth function λ on M, which is called the conformal factor of the conformal vector field $\mathbf{X}$, that satisfies $\mathscr{L}_{\mathbf{X}}g = 2\lambda g$, where $\mathscr{L}_{\mathbf{X}}g$ is the Lie derivative of g with respect $\mathbf{X}$.

Remark 1.1. Although conformal vector fields are most commonly defined as above, proposition 1.1 give us an equivalent definition that can be stated as follows:

A smooth vector field $\mathbf{X}$ on a Riemannian manifold (M,g) is said to be a conformal vector field if there exists a smooth function λ on M, which is called the conformal factor of the conformal vector field $\mathbf{X}$, that satisfies $\nabla_i X_j + \nabla_j X_i = 2\lambda g$.

The advantage of see conformal vector fields in such way will be clear in the proof of the theorem 2.1 and the lemma 2.3.

Definition 1.4 (Hodge star operation)**.** Given a k-form ω in a smooth n-dimensional manifold M, define an $(n-k)$-form $*\omega$ by setting

$$*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) := (-1)^{\sigma}(dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}})$$

and extending it linearly, where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k}$, $(i_1 \cdots i_k\ j_1 \cdots j_{n-k})$ is a permutation of $(1\ 2, \cdots n)$ and σ is 0 or 1 according to the permutation is even or odd, respectively.

Example 1.3. Let $M = \mathbb{R}^3$.

a) If $\omega = dx_1$, then the permutation $(1\ 2\ 3)$ is even and $*\omega = dx_2 \wedge dx_3$;

b) If $\omega = dx_2 \wedge dx_3$, then the permutation $(2\ 3\ 1)$ is even and $*\omega = dx_1$;

c) If $\omega = dx_2$, then the permutation $(2\ 1\ 3)$ is odd and $*\omega = -(dx_1 \wedge dx_3)$.

Definition 1.5. Let (M,g) be a n-dimensional Riemannian manifold, let ∇ be its connection and let $\mathfrak{X}(M)$ be the set of smooth vector fields defined on M. The Riemannian curvature tensor is

$$\begin{aligned} R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (X,Y,Z) &\longmapsto R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \end{aligned}$$

Let us define the tensor

$$\overline{R}(X,Y,Z,W) := g\left(R(X,Y)Z,W\right)$$

and the Ricci tensor Ric as the trace of the map $X \mapsto R(X,Y)Z$. If $\{e_i\}_{i=1}^n$ is an orthonormal basis, then

$$\mathrm{Ric}\,(Y,Z) = \sum_{i=1}^{n} g\left(R(e_i,Y)Z,e_i\right) = \sum_{i=1}^{n} \overline{R}(e_i,Y,Z,e_i).$$

Proposition 1.2 (Local expressions of the Riemannian connection and the Lie Bracket)**.** Let (M,g) be a n-dimensional Riemannian manifold and let $\mathfrak{X}(M)$ be the space of smooth vector fields defined on M, then

$$\begin{cases} \nabla_X Y = \displaystyle\sum_{k=1}^{n} \left(X(Y^k) + \sum_{i,j=1}^{n} X^i Y^j \Gamma_{ij}^k \right) e_k \\ [X,Y] = \displaystyle\sum_{i,j=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x_i} \right) e_j, \end{cases}$$

where $\{e_i\}_{i=1}^n$ is a local coordinate basis.

Definition 1.6. Let (M,g) be a Riemannian submanifold immersed on a Riemannian manifold $(N,\overline{g})$. Let $f : M \longrightarrow N$ be such immersion, let ∇ and $\overline{\nabla}$ be the connections of M and N respectively. We denote by $\mathfrak{X}(N)$ be the space of smooth vector fields defined on N and and by $\mathfrak{X}(N)^\perp$ be the space of smooth vector fields defined on N orthogonal to $f(M)$. The second fundamental form of M in N is the bilinear map

$$\begin{aligned} B : \mathfrak{X}(N) \times \mathfrak{X}(N) &\longrightarrow \mathfrak{X}(N)^\perp \\ (X,Y) &\longmapsto B(X,Y) := \overline{\nabla}_X Y - \nabla_X Y. \end{aligned}$$

Definition 1.7. A Riemannian submanifold M of N is totally geodesic provided its the second fundamental form vanishes.

Proposition 1.3. For $M \subset \overline{M}$ the following are equivalent.

1. M is totally geodesic in $\overline{M}$.

2. Every geodesic of M is also a geodesic of $\overline{M}$.

3. If $v \in T_p\overline{M}$ is tangent to M, then the $\overline{M}$ geodesic γ_v lies initially in M.

4. If α is a curve in M and $v \in T_{\alpha(0)}M$, then parallel translation of v along α is the same for M and $\overline{M}$.

Proof. See proposition 13 on page 105 of [27]. $\square$

Lemma 1.1. Let M and N be complete, connected, totally geodesic Riemannian submanifolds of $\overline{M}$. If there is a point $p \in M \cap N$ at which $T_pM = T_pN$, then $M = N$.

Proof. See lemma 14 on page 105 of [27]. $\square$

Remark 1.2. Although the last two results are proved for semi-Riemannnian manifolds, they hold for Riemannian manifolds.

Theorem 1.1. The only one complete, connected and totally geodesic Riemannian submanifolds of $\mathbb{R}^n$ are the linear subspaces of $\mathbb{R}^n$ and their translations.

Proof. From the first and second items of the previous proposition, linear subspaces of $\mathbb{R}^n$ are totally geodesic and, consequently, their translations are also totally geodesic. Let M be a complete, connected and totally geodesic Riemannian k-submanifold of $\mathbb{R}^n$ and N be a k-linear subspace of $\mathbb{R}^n$. Recalling that the spaces T_pM, $(\mathbb{R}^k)'$ and $\mathbb{R}^k$ are isomorphic for every Riemannian k-submanifold M of $\mathbb{R}^n$, we can suppose, applying rigid motions if necessary, that the hypothesis of the previous lemma are fulfilled less isomorphism, then $M = N$, i.e., M is a linear subspace of $\mathbb{R}^n$. $\square$

Remark 1.3. The hypothesis of completeness and connectedness in the theorem can be removed if we add in the theorem the possibility that the totally geodesic Riemannian submanifolds M of $\mathbb{R}^n$ can be also linear subspaces of $\mathbb{R}^n$ or its translations in each connected component of M.

Theorem 1.2 (Inverse Function Theorem)**.** Suppose M and N are smooth manifolds, and $F : M \longrightarrow N$ is a smooth map. If $p \in M$ is a point such that dF_p is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \longrightarrow V_0$ is a diffeomorphism.

Proof. See theorem 4.5 on page 79 of [25].

$\square$

Theorem 1.3 (Stokes's Theorem)**.** Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$-form on M. Then

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. See theorem 16.11 on page 411 of [25].

□

Definition 1.8. Let $\{E_i\}_{i=1}^n$ an orthonormal frame at a point $p \in M$ of a n-dimensional Riemannian manifold (M,g). The divergence of a smooth vector field V on M is

$$\text{div } V = \sum_{i=1}^{n} \langle \nabla_{E_i} V, E_i \rangle.$$

Theorem 1.4 (Divergence Theorem)**.** Let (M,g) be an oriented Riemannian manifold with boundary. For any compactly supported smooth vector field X on M,

$$\int_M (\text{div } X) dV_g = \int_{\partial M} \langle X, N \rangle_g dV_{\tilde{g}},$$

where N is the outward pointing unit normal vector field along ∂M and $\tilde{g}$ is the induced Riemannian metric on ∂M.

Proof. See theorem 16.32 on page 424 of [25].

□

For our purposes in this work, we refere to the following corollary as the Divergence Theorem.

Corollary 1.1. Let (M,g) be an oriented closed Riemannian manifold without boundary. For any smooth vector field X on M,

$$\int_M (\text{div } X) dV_g = 0.$$

Definition 1.9. Let M be a smooth manifold and ω a n-form defined on M.

a) ω is a closed form if $d\omega = 0$;

b) ω is an exact form if there exists a smooth function $f : M \longrightarrow \mathbb{R}$ such that $\omega = df$.

Theorem 1.5. On a simply connected smooth manifold, every closed 1-form is exact.

Proof. See Corollary 16.27 on page 421 of [25].

□

The next result will be necessary to prove theorems 3.4 and 3.5.

Proposition 1.4 (Wirtinger's inequality)**.** If $f : [a,b] \longrightarrow \mathbb{R}$ is differentiable on (a,b) with $f(a) = f(b) = 0$, then

$$\int_a^b (f(x))^2 dx \leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b (f'(x))^2 dx;$$

the constant $\left(\dfrac{b-a}{\pi}\right)^2$ is optimal.

Proof. See page 47 of [15]. □

1.3 Geometric interpretation of the stability of minimal hypersurfaces

We follow [16] in this section. Let (N,g) be a Riemannian manifold and M a Riemannian submanifold of dimension n, with boundary and with the metric induced by N. Denote by ∇ and $\overline{\nabla}$ the connections of M and N, respectively. Consider a variation of M on N with fixed boundary:

$$f : M \times I \longrightarrow N, f_0 = id_M, f|_{\partial M \times \{t\}} = id_{\partial M}, \forall t \in I.$$

Assume that $f_t : M \longrightarrow N$ is an embedding for each $t \in I$ and let $M_t := f_t(M)$, ω_t the element of volume induced on M_t and $\omega_0 = dx_1 \wedge \cdots \wedge dx_n$ the element of volume induced on M_0, then

$$\mathrm{vol}(M_t) = \int_M f_t^* \omega_t, \ \mathrm{vol}(M) = \int_M \omega_0.$$

The variational vector field associated is $V = \dfrac{\partial f}{\partial t}$. Observe that $V|_{\partial M} \equiv 0$ since the boundary of M is fixed throught the variation.

1.3.1 The First Variational Formula for the functional area.

Lemma 1.2. Let E be a vector space of dimension n equipped with an inner product, oriented and with a positive basis $\{v_1, \cdots, v_n\}$, then

$$\det(v_1, \cdots, v_n) = \sqrt{\det(\langle v_i, v_j \rangle)}.$$

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal positive basis of E. Observe that the matrix in the left of the equality of the lemma has entries $a_{ij} = \langle v_j, e_i \rangle$, while the matrix P with the entries $p_{ij} = \langle v_i, v_j \rangle$ is $P = A^T A$ as verified below:

$$p_{ij} = \langle v_i, v_j \rangle = \sum_{k=1}^{n} \langle v_i, e_k \rangle \langle e_k, v_j \rangle = \sum_{k=1}^{n} a_{ki} a_{kj}.$$

Thus,

$$\det P = \det(A^T A) = (\det A^T)(\det A) = (\det A)^2,$$

i.e.,

$$\det(v_1, \cdots, v_n) = \sqrt{\det(\langle v_i, v_j \rangle)}.$$

□

Lemma 1.3. If $A(t)$ is a family of linear and invertible functions defined on a vector space E of dimension n so that $A(0) = Id$, then

$$\frac{d}{dt}(\det A(t))\bigg|_{t=0} = \operatorname{tr} A'(0).$$

Proof. Writting as a n-form applied in vectors,

$$\det(A(t)) = \omega(A(t)e_1, \cdots, A(t)e_n) \text{ with } A'(0)e_i = \sum_{j=1}^{n} a_i^j e_j.$$

Thus,

$$\begin{aligned}
\frac{d}{dt}(\det A(t))\Big|_{t=0} &= \sum_{i=1}^{n} \omega(e_1,\cdots,A'(0)e_i,\cdots,e_n) = \sum_{i=1}^{n} \omega\left(e_1,\cdots,\sum_{j=1}^{n} a_i^j e_j,\cdots,e_n\right) \\
&= \sum_{i=1}^{n} \omega(e_1,\cdots,a_i^i e_i,\cdots,e_n) = \sum_{i=1}^{n} a_i^i \omega(e_1,\cdots,e_i,\cdots,e_n) \\
&= \sum_{i=1}^{n} a_i^i \det(A(0)) = \sum_{i=1}^{n} a_i^i = \operatorname{tr} A'(0).
\end{aligned}$$

□

Lemma 1.4. If $\alpha = *(V^*)$ is the $(n-1)$-form defined on T_pM where V^* is the 1-form dual to the field V defined in the beginning of this section, then $d\alpha = \sum_{i=1}^{n} e_i(\langle V,e_i\rangle)\omega_0$.

Proof.

$$\begin{aligned}
\alpha = *(V^*) &= *\left(\left(\sum_{k=1}^{n}\langle V,e_k\rangle e_k\right)^*\right) = *\left(\sum_{k=1}^{n}\langle V,e_k\rangle e_k^*\right) \\
&= *\left(\sum_{k=1}^{n}\langle V,e_k\rangle dx^k\right) = \sum_{k=1}^{n}\langle V,e_k\rangle * dx^k \\
&= \sum_{k=1}^{n}\langle V,e_k\rangle(-1)^{k-1} dx^1\wedge\cdots\wedge\widehat{dx^k}\wedge\cdots\wedge dx^n,
\end{aligned}$$

which implies

$$\begin{aligned}
d\alpha &= d\left(\sum_{k=1}^{n}\langle V,e_k\rangle(-1)^{k-1} dx^1\wedge\cdots\wedge\widehat{dx^k}\wedge\cdots\wedge dx^n\right) \\
&= \sum_{k=1}^{n}(-1)^{k-1}\left(\sum_{i=1}^{n} e_i(\langle V,e_k\rangle)dx^i\wedge dx^1\wedge\cdots\wedge\widehat{dx^k}\wedge\cdots\wedge dx^n\right) \\
&= \sum_{k=1}^{n}(-1)^{k-1} e_k(\langle V,e_k\rangle)dx^k\wedge dx^1\wedge\cdots\wedge\widehat{dx^k}\wedge\cdots\wedge dx^n \\
&= \sum_{k=1}^{n} e_k(\langle V,e_k\rangle)dx^1\wedge\cdots\wedge dx^k\wedge\cdots\wedge dx^n = \sum_{k=1}^{n} e_k(\langle V,e_k\rangle)\omega_0.
\end{aligned}$$

□

Theorem 1.6 (First Variational Formula for the functional area.)**.** Considering the hypothesis made in the beginning of this section, then

$$\frac{d}{dt}\mathrm{vol}(M_t)\Big|_{t=0} = -\int_M \langle V, \overrightarrow{H}\rangle \omega_0.$$

Proof. Fix $p \in M$ and a system of coordinates on a neighborhood of $(p,t) \in M \times I$ with coordinate vector fields ∂_t and $\partial_i|_p = e_i|_p$ for each $i = 1, \cdots, n$ where e_i are orthonormal vector fields and tangent to M in a normal neighborhood of $p \in N$ and geodesic on p and we extend they to be tangent to M_t, which imply $\nabla_{e_i} e_j(p) = 0$.

Observe that $[V, df_{(p,t)}(e_i)] = [df_{(p,t)}(\partial_t), df_{(p,t)}(e_i)] = df_{(p,t)}([\partial_t, e_i]) = df_{(p,t)}(0) = 0$. This imply, in p, that

$$\begin{aligned}
\frac{1}{2}\frac{\partial g_{ii}}{\partial t}(p,t)\Big|_{t=0} &= \langle \overline{\nabla}_V df_{(p,t)}(e_i), df_{(p,t)}(e_i)\rangle\Big|_{t=0} = \langle \overline{\nabla}_{df_{(p,t)}(e_i)} V, df_{(p,t)}(e_i)\rangle\Big|_{t=0} \\
&= e_i(\langle V, e_i\rangle) - \langle V, \overline{\nabla}_{df_{(p,t)}(e_i)} df_{(p,t)}(e_i)\rangle\Big|_{t=0} \\
&= e_i(\langle V, e_i\rangle) - \langle V, B(e_i, e_i)\rangle - \langle V, \nabla_{e_i} e_i\rangle \\
&= e_i(\langle V, e_i\rangle) - \langle V, B(e_i, e_i)\rangle.
\end{aligned}$$

From Lemma 1.3,

$$\frac{1}{2}\frac{dg}{dt}(0) = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial g_{ii}}{\partial t}(p,t)\Big|_{t=0} = -\langle V, \overrightarrow{H}\rangle + \sum_{i=1}^{n} e_i(\langle V, e_i\rangle). \tag{1.1}$$

Observing that

$$g_{ij}(t) = \langle df_t(e_i), df_t(e_j)\rangle,\ g(t) = \det(g_{ij}(t)),\ df_t = \begin{pmatrix} df_t(e_1) & \cdots & df_t(e_n)\end{pmatrix}$$

and using Lemma 1.2, we find $\det(df_t) = \sqrt{g(t)}$. From this and from (1.1),

$$\frac{d}{dt}\mathrm{vol}(M_t)\Big|_{t=0} = \int_M \frac{d}{dt}\det(df_t)\Big|_{t=0}\omega_0 = \int_M \frac{d}{dt}\sqrt{g(t)}\Big|_{t=0}\omega_0 = \int_M \frac{1}{2\sqrt{g(0)}}\frac{dg}{dt}(0)\omega_0$$
$$= \int_M \frac{1}{2}\frac{dg}{dt}(0)\omega_0 = \int_M \left(-\langle V, \overrightarrow{H}\rangle + \sum_{i=1}^n e_i(\langle V, e_i\rangle)\right)\omega_0.$$

Then, from Lemma 1.4 and the Stokes' theorem we get

$$\frac{d}{dt}\mathrm{vol}(M_t)\Big|_{t=0} = \int_M \left(-\langle V, \overrightarrow{H}\rangle + \sum_{i=1}^n e_i(\langle V, e_i\rangle)\right)\omega_0 = \int_M -\langle V, \overrightarrow{H}\rangle\omega_0 + d\alpha$$
$$= -\int_M \langle V, \overrightarrow{H}\rangle\omega_0 + \int_{\partial M}\alpha = -\int_M \langle V, \overrightarrow{H}\rangle\omega_0,$$

where the last equality follows from the fact that $\alpha = *(V^*)$ and $V|_{\partial M} \equiv 0$. □

1.3.2 The Second Variational Formula for the functional area.

Theorem 1.7. Suppose $H \equiv 0$ on M and $V = \partial_t f = u\nu$ where u is a function with compact support on M, then

$$\frac{d^2}{dt^2}\mathrm{vol}(M_t)\Big|_{t=0} = -\int_M (\Delta_M u + |A|^2 u + \mathrm{Ric}_N u)u\omega_0,$$

where Δ_M is the Beltrami-Laplace operator and Ric_N is the Ricci curvature of N in the direction ν.

Proof. Let $p \in M$ and consider $\{e_i\}_{i=1}^n$ a local frame on a neighborhood of p, orthonormal on M and geodesic in p. Assume that $\{e_i\}_{i=1}^n$ is transportated by df_t and define $g_{ij} := \langle e_i, e_j\rangle$, in particular, $[V, e_i] = 0$.

From the First Variational Formula for the functional area,

$$
\begin{aligned}
\frac{d^2}{dt^2}\mathrm{vol}(M_t)\Big|_{t=0} &= -\left(\int_M \frac{d}{dt}\langle V,\overrightarrow{H}\rangle\Big|_{t=0}\omega_0 + \langle V,\overrightarrow{H}\rangle\frac{d\omega_t}{dt}\Big|_{t=0}\right)\\
&= -\int_M \frac{d}{dt}\langle V,\overrightarrow{H}\rangle\Big|_{t=0}\omega_0\\
&= -\int_M \frac{d}{dt}\langle V, g^{ij}\overline{\nabla}_{e_i}e_j\rangle\Big|_{t=0}\omega_0\\
&= -\left(\int_M \left\langle \frac{d}{dt}\Big|_{t=0} V, g^{ij}\overline{\nabla}_{e_i}e_j\right\rangle\omega_0 + \left\langle V, \frac{d}{dt}(g^{ij}\overline{\nabla}_{e_i}e_j)\Big|_{t=0}\right\rangle\omega_0\right)\\
&= -\left(\int_M \frac{dg^{ij}}{dt}\Big|_{t=0}\left\langle V,\overline{\nabla}_{e_i}e_j\right\rangle\omega_0 + g^{ij}\left\langle V,\frac{d}{dt}\Big|_{t=0}\overline{\nabla}_{e_i}e_j\right\rangle\omega_0\right)\\
&= -\left(\int_M \frac{dg^{ij}}{dt}\Big|_{t=0}\left\langle V,\overline{\nabla}_{e_i}e_j\right\rangle\omega_0 + g^{ij}\left\langle V,\overline{\nabla}_V\overline{\nabla}_{e_i}e_j\right\rangle\omega_0\right)
\end{aligned}
$$

Observing that $g_{ij}=\delta_{ij}$, we have, in $t=0$ and in p

$$
\begin{aligned}
\frac{dg^{ij}}{dt} &= -\frac{dg_{ij}}{dt} = -V(\langle e_i,e_j\rangle)\\
&= -\langle\overline{\nabla}_V e_i,e_j\rangle - \langle e_i,\overline{\nabla}_V e_j\rangle = -\langle\overline{\nabla}_{e_i}V,e_j\rangle - \langle e_i,\overline{\nabla}_{e_j}V\rangle\\
&= -\Big(e_i(\langle V,e_j\rangle) - \langle V,\overline{\nabla}_{e_i}e_j\rangle\Big) - \Big(e_j(\langle e_i,V\rangle) - \langle\overline{\nabla}_{e_j}e_i,V\rangle\Big)\\
&= -\Big(e_i(\langle u\nu,e_j\rangle) - \langle V,\overline{\nabla}_{e_i}e_j\rangle\Big) - \Big(e_j(\langle e_i,u\nu\rangle) - \langle\overline{\nabla}_{e_j}e_i,V\rangle\Big)\\
&= \langle V,\overline{\nabla}_{e_i}e_j\rangle + \langle\overline{\nabla}_{e_j}e_i,V\rangle\\
&= \langle V,B(e_i,e_j)\rangle + \langle B(e_j,e_i),V\rangle = 2\langle B(e_i,e_j),V\rangle,
\end{aligned}
$$

where the penultimate equality is true because $\{e_i\}_{i=1}^n$ is geodesic in p. Thus,

$$
\frac{dg^{ij}}{dt}\Big|_{t=0}\left\langle V,\overline{\nabla}_{e_i}e_j\right\rangle\omega_0 = 2\sum_{i,j=1}^n\langle B(e_i,e_j),V\rangle^2\omega_0 = 2u^2|A|^2\omega_0 \tag{1.2}
$$

for normal variations of hypersurfaces.

Recalling that $[V, e_i] = 0$ and considering,

$$\begin{aligned}\left\langle V, \overline{\nabla}_V \overline{\nabla}_{e_i} e_i \right\rangle &= \left\langle V, \overline{\nabla}_{e_i} \overline{\nabla}_V e_i \right\rangle - \overline{R}_N(e_i, V, e_i, V) \\ &= e_i(\langle V, \overline{\nabla}_V e_i \rangle) - \langle \overline{\nabla}_{e_i} V, \overline{\nabla}_V e_i \rangle + \overline{R}_N(e_i, V, V, e_i) \\ &= e_i(\langle V, \overline{\nabla}_{e_i} V \rangle) - |\overline{\nabla}_{e_i} V|^2 + \overline{R}_N(e_i, V, V, e_i)\end{aligned}$$

In the case that codimension is 1 with $V = u\nu$,

$$|\overline{\nabla}_{e_i} V|^2 = |e_i(u)\nu + u\overline{\nabla}_{e_i}\nu|^2 = e_i(u)^2 + u^2|\overline{\nabla}_{e_i}\nu|^2.$$

Taking the sum on i,

$$\sum_{i=1}^{n} |\overline{\nabla}_{e_i} V|^2 = |\nabla u|^2 + u^2 \sum_{i,j=1}^{n} \langle \overline{\nabla}_{e_i}\nu, e_j \rangle = |\nabla u|^2 + u^2 \sum_{i,j=1}^{n} B(e_j, e_i)^2 = |\nabla u|^2 + u^2|A|^2,$$

while

$$e_i(\langle V, \overline{\nabla}_{e_i} V \rangle) = e_i(\langle u\nu, e_i(u)\nu + u\overline{\nabla}_{e_i}\nu \rangle) = e_i(ue_i(u)) = e_i(u)^2 + ue_i(e_i(u)),$$

then

$$\sum_{i=1}^{n} e_i(\langle V, \overline{\nabla}_{e_i} V \rangle) = \sum_{i=1}^{n} ue_i(e_i(u)) + e_i(u)^2 = \sum_{i=1}^{n} u(e_i(e_i(u)) - (\overline{\nabla}_{e_i} e_i)u) + e_i(u)^2 = u\Delta_M u + |\nabla u|^2.$$

Combining the sums above, we get

$$\begin{aligned}\sum_{i=1}^{n} \left\langle V, \overline{\nabla}_V \overline{\nabla}_{e_i} e_i \right\rangle &= \sum_{i=1}^{n} e_i(\langle V, \overline{\nabla}_{e_i} V \rangle) - |\overline{\nabla}_{e_i} V|^2 + \overline{R}_N(e_i, V, V, e_i) \\ &= (u\Delta_M u + |\nabla u|^2) - (|\nabla u|^2 + u^2|A|^2) + \sum_{i=1}^{n} \overline{R}_N(e_i, V, V, e_i) \\ &= u\Delta_M u + \sum_{i=1}^{n} \overline{R}_N(e_i, u\nu, u\nu, e_i) - u^2|A|^2 \\ &= u\Delta_M u + u^2 \mathrm{Ric}_N - u^2|A|^2,\end{aligned}$$

i.e.,

$$\sum_{i=1}^{n}\left\langle V,\overline{\nabla}_V\overline{\nabla}_{e_i}e_i\right\rangle = u\Delta_M u + u^2\mathrm{Ric}_N - u^2|A|^2. \tag{1.3}$$

Substituting (1.2) and (1.3) in the second derivative of $\mathrm{vol}(M_t)$ in $t=0$,

$$\frac{d^2}{dt^2}\mathrm{vol}(M_t)\bigg|_{t=0} = -\int_M(\Delta_M u + |A|^2u + u\mathrm{Ric}_N)u\omega_0.$$

□

Definition 1.10. A hypersurface is stable if it is a local minimum of $\mathrm{vol}(M_t)$, i.e., $\frac{d^2}{dt^2}\mathrm{vol}(M_t)\Big|_{t=0} \geq 0$.

Corollary 1.2. Let M be a hypersuperface without boundary. M is stable if and only if

$$\int_M(|A|^2+\mathrm{Ric}_N)u^2\omega_0 \leq \int_M|\nabla u|^2\omega_0.$$

Proof. From the previous theorem, the definition of stability and the Green's identities over Riemannian manifolds,

$$\begin{aligned}
\frac{d^2}{dt^2}\mathrm{vol}(M_t)\bigg|_{t=0} \geq 0 &\iff -\int_M(\Delta_M u + |A|^2u + u\mathrm{Ric}_N)u\omega_0 \geq 0\\
&\iff \int_M(\Delta_M u + |A|^2u + u\mathrm{Ric}_N)u\omega_0 \leq 0\\
&\iff \int_M(|A|^2u + u\mathrm{Ric}_N)u\omega_0 \leq \int_M -u\Delta_M u\omega_0\\
&\iff \int_M(|A|^2+\mathrm{Ric}_N)u^2\omega_0 \leq \int_M|\nabla u|^2\omega_0.
\end{aligned}$$

□

1.4 Mean Curvature Flow

Definition 1.11. Let M be a smooth Riemannian hypersurface without boundary of $\mathbb{R}^{n+1}$ endowed with a Riemannian metric. The mean curvature flow of M in $\mathbb{R}^{n+1}$ is a family of immersions $F : M \times [0,T) \longrightarrow \mathbb{R}^{n+1}$ which satisfies

$$\begin{cases} \dfrac{\partial F}{\partial t}(p,t) & = \mathbf{H}(p,t) \\ F(M,0) & = M, \end{cases}$$

where $\mathbf{H}(\cdot,t) = H(\cdot,t)\nu(\cdot,t)$ denotes the mean curvature vector field of $F(\cdot,t) \subset \mathbb{R}^{n+1}$ for each $t \in [0,T)$ and $\nu(\cdot,t)$ is the unit normal to $F(\cdot,t)$ pointing inward.

Results regarding the existence of solutions of equation $\dfrac{\partial F}{\partial t}(p,t) = \mathbf{H}(p,t)$ for a short time are well known (see for example Theorem 3.1 in [20]). It may ocurrs that mean curvature flow becomes singular at some time T. The study of singularities constitutes an important research branch in mean curvature flow. When the flow develops a singularity in a time T such that the norm of the second fundamental form has the growth rate

$$\max_{p\in M} |A(p,t)|^2 \leq \frac{C_0}{2(T-t)},\ \forall t \in [0,T),\ C_0 > 0$$

we say that the singularity is a Type I singularity. Otherwise, the singularity is called Type II singularity. Both Type I and Type II singularities are related to solutions of the mean curvature flow called, self-similar solutions. Among such solutions we have the self-shrinkers, self-expanders and the translating solitons. Type I singularities are close related to self-shirinkers, as we can see in Huisken [19], whereas a relation between translating solitons and Type II singularity was obtained by Huisken and Sinestrari [21] (see also Corollary 9.4 in [28]). Roughly speaking, a self-similar solution of the mean curvature flow is a solutions that preserve its shape along the flow, as we can see below in the description of self-shrinkers, self-expanders and the translating solitons.

Definition 1.12. Let $M_{-1} \subset \mathbb{R}^{n+1}$ be a Riemannian hypersurface evolving by mean curvature such that the evolved hypersurfaces are

$$M_t := \sqrt{-t}M_{-1},\ t \in [-1,0).$$

M_{-1} is said a self-shrinker.

Proposition 1.5. A self-shrinker M_{-1} satisfies $H_{M_{-1}}(x) = -\frac{1}{2}\langle x, \nu\rangle$, where $H_{M_{-1}}$ is the mean curvature, $x \in M_{-1}$ and ν is the unit normal.

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal frame at $x \in M_{-1}$. If $F(x,t) = \sqrt{-t}x$ flow by mean curvature, then

$$g_{ij}(x,t) = \langle \sqrt{-t}e_i, \sqrt{-t}e_j\rangle = -t\delta_{ij} \Longrightarrow g^{ij}(x,t) = -\frac{\delta_{ij}}{t},$$

$$\begin{aligned}
H_{M_t}(x,t) &= \sum_{i,j=1}^n g^{ij}(x,t)\left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}(x,t), \nu(x,t)\right\rangle \\
&= \sum_{i,j=1}^n -\frac{\delta_{ij}}{t}\left\langle \sqrt{-t}\frac{\partial^2 F}{\partial x_i \partial x_j}(x,-1), \nu(x,-1)\right\rangle \\
&= \frac{1}{\sqrt{-t}} H_{M_{-1}}(x), \\
\nu(x,t) &= \nu(x,-1)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\sqrt{-t}} H_{M_{-1}}(x) &= H_{M_t}(x,t) \\
&= \left\langle \frac{\partial F}{\partial t}(x,t), \nu(x,t)\right\rangle \\
&= \left\langle \frac{\partial}{\partial t}\left(\sqrt{-t}x\right), \nu(x,-1)\right\rangle \\
&= -\frac{1}{2\sqrt{-t}}\langle x, \nu\rangle.
\end{aligned}$$

Thus,

$$H_{M_{-1}}(x) = -\frac{1}{2}\langle x, \nu\rangle.$$

□

Definition 1.13. Let $M_1 \subset \mathbb{R}^{n+1}$ be a Riemannian hypersurface evolving by mean curvature such that the evolved hypersurfaces are

$$M_t := \sqrt{t}M_1,\ t \in [1, +\infty).$$

M_1 is said a self-expander.

Proposition 1.6. A self-expander M_1 satisfies $H_{M_1}(x) = \frac{1}{2}\langle x, \nu \rangle$, where H_{M_1} is the mean curvature, $x \in M_1$ and ν is the unit normal.

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal frame at $x \in M_1$. If $F(x,t) = \sqrt{t}x$ flow by mean curvature, then

$$g_{ij}(x,t) = \langle \sqrt{t}e_i, \sqrt{t}e_j \rangle = t\delta_{ij} \Longrightarrow g^{ij}(x,t) = \frac{\delta_{ij}}{t},$$

$$\begin{aligned}
H_{M_t}(x,t) &= \sum_{i,j=1}^n g^{ij}(x,t) \left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}(x,t), \nu(x,t) \right\rangle \\
&= \sum_{i,j=1}^n \frac{\delta_{ij}}{t} \left\langle \sqrt{t} \frac{\partial^2 F}{\partial x_i \partial x_j}(x,1), \nu(x,1) \right\rangle \\
&= \frac{1}{\sqrt{t}} H_{M_1}(x), \\
\nu(x,t) &= \nu(x,1)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\sqrt{t}} H_{M_1}(x) &= H_{M_t}(x,t) \\
&= \left\langle \frac{\partial F}{\partial t}(x,t), \nu(x,t) \right\rangle \\
&= \left\langle \frac{\partial}{\partial t}\left(\sqrt{t}x\right), \nu(x,1) \right\rangle \\
&= \frac{1}{2\sqrt{t}} \langle x, \nu \rangle.
\end{aligned}$$

Thus,

$$H_{M_1}(x) = \frac{1}{2}\langle x, \nu \rangle.$$

□

Definition 1.14. Let $M_0 \subset \mathbb{R}^{n+1}$ be a Riemannian hypersurface evolving by mean curvature such that the evolved hypersurfaces are

$$M_t := M_0 + tT,\ t \in [0, +\infty)$$

for some $T \in \mathbb{R}^{n+1}$. M_0 is said a translating soliton.

Proposition 1.7. A translating soliton M_0 satisfies $H = \langle T, \nu \rangle$, where H is the mean curvature, $x \in M_0$ and ν is the unit normal.

Proof. If $F(x,t) = F(x,0) + tT$ flow by mean curvature, then

$$\begin{aligned} H_{M_t}(x,t) &= \sum_{i=1}^{n} \left\langle \frac{\partial^2 F}{\partial x_i^2}(x,t), \nu(x,t) \right\rangle \\ &= \sum_{i=1}^{n} \left\langle \frac{\partial^2 F}{\partial x_i^2}(x,0), \nu(x,0) \right\rangle \\ &= H_{M_0}(x,0). \end{aligned}$$

and

$$\begin{aligned} H_{M_0}(x,0) &= H_{M_t}(x,t) \\ &= \left\langle \frac{\partial F}{\partial t}(x,t), \nu(x,t) \right\rangle \\ &= \left\langle \frac{\partial}{\partial t}(x + tT), \nu(x,t) \right\rangle \\ &= \langle T, \nu \rangle. \end{aligned}$$

□

Chapter 2

Conformal solitons and submanifolds

Let M be a smooth n-dimensional manifold without boundary and (N,g) a smooth $n+p$-dimensional Riemannian manifold. We are interesting in understand special solutions for the mean curvature flow, i.e., a family of immersions $F : M \times [0,T) \longrightarrow N$ satisfying

$$\begin{cases} \dfrac{\partial F}{\partial t}(p,t) & = \mathbf{H}(p,t) \\ F(M,0) & = M, \end{cases}$$

where $\mathbf{H}(\cdot,t)$ denotes the mean curvature vector field of $F(\cdot,t)$ for each $t \in [0,T)$.

This class of solutions was studied previously by Smoczyk in [30] and can be defined as follows: given a conformal vector field $\mathbf{X}$, M is a conformal soliton to the mean curvature flow if M satisfies the following equation

$$\mathbf{H} = \mathbf{X}^{\perp}, \tag{2.1}$$

where $\mathbf{H}$ denotes the mean curvature vector of M and $\perp$ denotes the projection on the normal bundle. Recall that a vector field $\mathbf{X}$ is conformal if in local coordinate, $\overline{\nabla}_j X_i + \overline{\nabla}_i X_j = 2\lambda g_{ij}$ for some smooth function λ. Following the paper by Smoczyk, we consider the special conformal vector fields $\mathbf{X}$ which satisfy

$$\overline{\nabla}_j X_i = \overline{\nabla}_i X_j$$

or, in other words,

$$\overline{\nabla}_j X_i = \lambda g_{ij} \tag{2.2}$$

for some smooth function λ and where $\overline{\nabla}$ denotes the Levi-Civita connection on N.

As we will see in the next chapter (Proposition 3.2), this particular class of conformal solitons generalizes a class of solutions of the mean curvature flow, namely, the self-similar solutions. This fact is one of the motivations for studying the conformal solitons to the mean curvature flow. This chapter is dedicated to obtain some correspondences for conformal solitons and minimal submanifolds in a warped product Riemannian manifold as well as a notion of stability for conformal solitons in higher codimensions.

2.1 Warped product metric.

Let (N,g) be a Riemannian manifold and $f : N \longrightarrow \mathbb{R}$ a smooth function. The warped product metric $\tilde{g}$ on $\tilde{N} = \mathbb{R} \times N$ is defined by

$$\tilde{g}(s,x) := e^{2f(x)} ds^2 + g(x), \tag{2.3}$$

where $x \in N$ and ds^2 is the standard metric on $\mathbb{R}$. The projection

$$\begin{aligned} \pi : \tilde{N} &\longrightarrow N \\ (s,x) &\mapsto \pi(s,x) := x \end{aligned}$$

is a Riemannian submersion $\pi : (\tilde{N}, \tilde{g}) \longrightarrow (N,g)$ with fibers $\pi^{-1}(x) =: [x] = \mathbb{R} \times \{x\}$.

Consider now $M^n \subset N^{n+p}$ a Riemannian submanifold. Denote by $\tilde{M}$ a submanifold on $\tilde{N}$ given by $\tilde{M} = \mathbb{R} \times M$ the submanifold associated to M. Let $\{e_1, \cdots, e_n, \nu_{n+1}, \cdots, \nu_{n+p}\}$ be such that $\{e_1, \cdots, e_n\}$ are in the tangent bundle of M and $\{\nu_{n+1}, \cdots, \nu_{n+p}\}$ are in the normal bundle of M. Throughout this dissertation, we are going to assume the following convention:

- $1 \le i, j, \cdots \le n,\ n+1 \le \alpha, \beta, \cdots \le n+p,\ 0 \le a, b, \cdots \le n;$

- $1 \leq A, B, \cdots \leq n+p$, $0 \leq \tilde{A}, \tilde{B}, \cdots \leq n+p$.

Defining

$$\tilde{e}_0 := \frac{\partial}{\partial s} = (1,0),\ \tilde{e}_i := (0, e_i),\ \tilde{\nu}_\alpha := (0, \nu_\alpha), \tag{2.4}$$

we see that $\{\tilde{e}_0, \tilde{e}_1, \cdots, \tilde{e}_n\}$ spans the tangent space of $\tilde{M}$ and $\{\nu_{n+1}, \cdots, \nu_{n+p}\}$ spans the normal space of $\tilde{M}$ at every point of $\tilde{M}$. By the choice of the warped product metric (2.3), $\tilde{g}(\tilde{e}_0, \tilde{e}_0) = e^{2f(x)}$. Thus, $\{e^{-f(x)}\tilde{e}_0, \tilde{e}_1, \cdots, \tilde{e}_n, \tilde{\nu}_{n+1}, \cdots, \tilde{\nu}_{n+p}\}$ is an orthonormal frame of $\tilde{N}$.

From now on, $\overline{\Gamma}$, $\tilde{\overline{\Gamma}}$, Γ and $\tilde{\Gamma}$ are going to denote the Christoffel symbols on N, $\tilde{N}$, M and $\tilde{M}$, respectively. The Levi-Civita connection and the Laplacian operator on N will be denoted by $\overline{\nabla}$ and $\overline{\Delta}$.

Proposition 2.1. The mean curvature vector $\mathbf{H}_{[x]}$ of $[x]$ in $\tilde{N}$ at (s,x) is given by $(0, -\overline{\nabla} f(x))$. Furthermore,

$$(\tilde{g}_{\tilde{A}\tilde{B}}) = \begin{pmatrix} e^{2f} & 0 \\ 0 & g_{AB} \end{pmatrix}, \quad (\tilde{g}^{\tilde{A}\tilde{B}}) = \begin{pmatrix} e^{-2f} & 0 \\ 0 & g^{AB} \end{pmatrix}, \tag{2.5}$$

$$\tilde{\overline{\Gamma}}^A_{BC} = \overline{\Gamma}^A_{BC}, \quad \tilde{\overline{\Gamma}}^0_{BC} = \tilde{\overline{\Gamma}}^A_{B0} = \tilde{\overline{\Gamma}}^0_{00} = 0, \quad \tilde{\overline{\Gamma}}^0_{B0} = \frac{\partial f}{\partial x^B}, \quad \tilde{\overline{\Gamma}}^A_{00} = -e^{2f} g^{AD} \frac{\partial f}{\partial x^D}. \tag{2.6}$$

$$\tilde{R}_{ABCD} = R_{ABCD}, \tag{2.7}$$

$$\tilde{R}_{0ABC} = 0, \tag{2.8}$$

$$\tilde{R}_{0A0B} = e^{2f}\left(\overline{\nabla}_A \overline{\nabla}_B f + \overline{\nabla}_A f \overline{\nabla}_B f\right). \tag{2.9}$$

The Ricci curvature tensor is given by

$$(\tilde{R}_{\tilde{A}\tilde{B}}) = \begin{pmatrix} -e^{2f}\left(\overline{\Delta} f + |\overline{\nabla} f|^2\right) & 0 \\ 0 & R_{AB} - \overline{\nabla}_A \overline{\nabla}_B f - \overline{\nabla}_A f \overline{\nabla}_B f \end{pmatrix} \tag{2.10}$$

Proof. By (2.3), we compute the entries of $(\tilde{g}_{\tilde{A}\tilde{B}})$.

$$\tilde{g}_{\tilde{A}\tilde{B}} := \tilde{g}(\tilde{e}_{\tilde{A}}, \tilde{e}_{\tilde{B}}) = \begin{cases} e^{2f}, \text{ if } \tilde{A} = \tilde{B} = 0 \\ 0, \text{ if } \tilde{A} = 0 \text{ and } \tilde{B} \neq 0 \text{ or } \tilde{A} \neq 0 \text{ and } \tilde{B} = 0 \\ g(e_A, e_B), \text{ if } \tilde{A} \neq 0 \text{ and } \tilde{B} \neq 0 \end{cases},$$

We also compute the entries of $(\tilde{g}^{\tilde{A}\tilde{B}})$.

a) $1 = \sum_{\tilde{A}=0}^{n+p} \tilde{g}_{0\tilde{A}} \tilde{g}^{\tilde{A}0} = \tilde{g}_{00}\tilde{g}^{00} = e^{2f}\tilde{g}^{00}$, which implies $\tilde{g}^{00} = e^{-2f}$.

b) $0 = \sum_{B=0}^{n+p} \tilde{g}_{0\tilde{B}} \tilde{g}^{\tilde{B}A} = \tilde{g}_{00}\tilde{g}^{0A} = e^{2f}\tilde{g}^{0A}$ for $A \neq 0$, which implies $\tilde{g}^{0A} = 0$. Thus, $\tilde{g}^{A0} = 0$ by the symmetry of the metric $\tilde{g}$.

c) $\delta_A^B = \sum_{\tilde{C}=0}^{n+p} \tilde{g}_{A\tilde{C}} \tilde{g}^{\tilde{C}B} = \sum_{C=1}^{n+p} g_{AC}\tilde{g}^{CB}$ for $A \neq 0$ and $B \neq 0$, which implies $\tilde{g}^{CB} = g^{CB}$.

We obtain the relations for the Christoffel symbols in $(\tilde{N}, \tilde{g})$ analyzing each case.

1. Case $1 \leq \tilde{A}, \tilde{B}, \tilde{C} \leq n+p$:

$$\begin{aligned} \tilde{\overline{\Gamma}}^{\tilde{A}}_{\tilde{B}\tilde{C}} &= \frac{1}{2}\sum_{\tilde{D}=0}^{n+p} \left(\frac{\partial \tilde{g}_{\tilde{D}\tilde{C}}}{\partial x_{\tilde{B}}} + \frac{\partial \tilde{g}_{\tilde{D}\tilde{B}}}{\partial x_{\tilde{C}}} - \frac{\partial \tilde{g}_{\tilde{B}\tilde{C}}}{\partial x_{\tilde{D}}} \right) \tilde{g}^{\tilde{D}\tilde{A}} \\ &= \frac{1}{2}\sum_{D=1}^{n+p} \left(\frac{\partial g_{DC}}{\partial x_B} + \frac{\partial g_{DB}}{\partial x_C} - \frac{\partial g_{BC}}{\partial x_D} \right) g^{DA} = \overline{\Gamma}^{A}_{BC}, \end{aligned}$$

where the second equality was obtained by the computations of the entries of the matrix $(\tilde{g}_{\tilde{A}\tilde{B}})$ did previously.

2. Case $\tilde{A} = 0$, $\tilde{B} \neq 0$, $\tilde{C} \neq 0$ or $\tilde{A} \neq 0$, $\tilde{B} \neq 0$, $\tilde{C} = 0$ or $\tilde{A} = \tilde{B} = \tilde{C} = 0$:

$$\tilde{\overline{\Gamma}}^{\tilde{A}}_{\tilde{B}\tilde{C}} = \frac{1}{2}\sum_{\tilde{D}=0}^{n+p} \left(\frac{\partial \tilde{g}_{\tilde{D}\tilde{C}}}{\partial x_{\tilde{B}}} + \frac{\partial \tilde{g}_{\tilde{D}\tilde{B}}}{\partial x_{\tilde{C}}} - \frac{\partial \tilde{g}_{\tilde{B}\tilde{C}}}{\partial x_{\tilde{D}}} \right) \tilde{g}^{\tilde{D}\tilde{A}}.$$

   (a) Subcase $\tilde{A} = 0$, $\tilde{B} \neq 0$, $\tilde{C} \neq 0$:

$$\tilde{g}^{\tilde{D}\tilde{A}} = 0$$

for each $\tilde{D} \in \{1, \cdots, n+p\}$ and

$$\frac{\partial \tilde{g}_{\tilde{D}\tilde{C}}}{\partial x_{\tilde{B}}} + \frac{\partial \tilde{g}_{\tilde{D}\tilde{B}}}{\partial x_{\tilde{C}}} - \frac{\partial \tilde{g}_{\tilde{B}\tilde{C}}}{\partial x_{\tilde{D}}} = 0$$

for $\tilde{D} = 0$ by (2.5).

(b) Subcase $\tilde{A} = \tilde{B} = \tilde{C} = 0$:

$$\tilde{g}^{\tilde{D}\tilde{A}} = \tilde{g}_{\tilde{D}\tilde{C}} = \tilde{g}_{\tilde{D}\tilde{B}} = \tilde{g}_{\tilde{B}\tilde{C}} = 0$$

for each $\tilde{D} \in \{1, \cdots, n+p\}$ by (2.5) and

$$\frac{\partial \tilde{g}_{\tilde{D}\tilde{C}}}{\partial x_{\tilde{B}}} + \frac{\partial \tilde{g}_{\tilde{D}\tilde{B}}}{\partial x_{\tilde{C}}} - \frac{\partial \tilde{g}_{\tilde{B}\tilde{C}}}{\partial x_{\tilde{D}}} = \frac{\partial(e^{2f})}{\partial s} + \frac{\partial(e^{2f})}{\partial s} - \frac{\partial(e^{2f})}{\partial s} = 0$$

for $\tilde{D} = 0$ by (2.5) and from the fact that f is not defined on $\tilde{N}$, but it is defined on N.

In any of these subcases, $\tilde{\tilde{\Gamma}}^{\tilde{A}}_{\tilde{B}\tilde{C}} = 0$.

(c) Subcase $\tilde{A} \neq 0$, $\tilde{B} \neq 0$, $\tilde{C} = 0$:

$$\tilde{\tilde{\Gamma}}^{\tilde{A}}_{\tilde{B}\tilde{C}} = \frac{1}{2} \sum_{\tilde{D}=0}^{n+p} \left(\frac{\partial \tilde{g}_{\tilde{D}\tilde{C}}}{\partial x_{\tilde{B}}} + \frac{\partial \tilde{g}_{\tilde{D}\tilde{B}}}{\partial x_{\tilde{C}}} \right) \tilde{g}^{\tilde{D}\tilde{A}},$$

but

$$\tilde{g}^{\tilde{D}\tilde{A}} = \tilde{g}_{\tilde{B}\tilde{C}} = 0$$

for each $\tilde{D} \in \{1, \cdots, n+p\}$ by (2.5) and

$$\frac{\partial \tilde{g}_{\tilde{D}\tilde{C}}}{\partial x_{\tilde{B}}} + \frac{\partial \tilde{g}_{\tilde{D}\tilde{B}}}{\partial x_{\tilde{C}}} = \frac{\partial(e^{2f})}{\partial s} + \frac{\partial(e^{2f})}{\partial s} = 0.$$

for $\tilde{D} = 0$ by (2.5) and from the fact that f is not defined on $\tilde{N}$, but it is defined on N. Thus, $\tilde{\tilde{\Gamma}}^{\tilde{A}}_{\tilde{B}\tilde{C}} = 0$ also in this subcase.

3. $\tilde{A} = \tilde{C} = 0$, $\tilde{B} \neq 0$:

By (2.5), $\tilde{g}^{\tilde{D}\tilde{A}} = 0$ for each $\tilde{D} \in \{1, \cdots, n+p\}$ and $\tilde{g}_{\tilde{D}\tilde{B}} = \tilde{g}_{\tilde{B}\tilde{C}} = 0$ for $\tilde{D} = 0$. Thus,

$$\tilde{\overline{\Gamma}}^{\tilde{A}}_{\tilde{B}\tilde{C}} = \frac{1}{2}\sum_{\tilde{D}=0}^{n+p}\left(\frac{\partial \tilde{g}_{\tilde{D}\tilde{C}}}{\partial x_{\tilde{B}}} + \frac{\partial \tilde{g}_{\tilde{D}\tilde{B}}}{\partial x_{\tilde{C}}} - \frac{\partial \tilde{g}_{\tilde{B}\tilde{C}}}{\partial x_{\tilde{D}}}\right)\tilde{g}^{\tilde{D}\tilde{A}} = \frac{1}{2}\frac{\partial \tilde{g}_{00}}{\partial x_{\tilde{B}}}\tilde{g}^{00} = \frac{1}{2}\left(2\frac{\partial f}{\partial x_{\tilde{B}}}e^{2f}\right)e^{-2f} = \frac{\partial f}{\partial x_{\tilde{B}}}.$$

4. $\tilde{A} \neq 0, \tilde{B} = \tilde{C} = 0$:

   By (2.5), $\tilde{g}^{\tilde{D}\tilde{A}} = 0$ for $\tilde{D} = 0$ and $\tilde{g}_{\tilde{D}\tilde{B}} = \tilde{g}_{\tilde{D}\tilde{C}} = 0$ for each $\tilde{D} \in \{1, \cdots, n+p\}$. Thus,

$$\begin{aligned}
\tilde{\overline{\Gamma}}^{\tilde{A}}_{\tilde{B}\tilde{C}} &= \frac{1}{2}\sum_{\tilde{D}=0}^{n+p}\left(\frac{\partial \tilde{g}_{\tilde{D}\tilde{C}}}{\partial x_{\tilde{B}}} + \frac{\partial \tilde{g}_{\tilde{D}\tilde{B}}}{\partial x_{\tilde{C}}} - \frac{\partial \tilde{g}_{\tilde{B}\tilde{C}}}{\partial x_{\tilde{D}}}\right)\tilde{g}^{\tilde{D}\tilde{A}} \\
&= -\frac{1}{2}\sum_{\tilde{D}=1}^{n+p}\frac{\partial \tilde{g}_{\tilde{B}\tilde{C}}}{\partial x_{\tilde{D}}}\tilde{g}^{\tilde{D}\tilde{A}} = -\frac{1}{2}\sum_{\tilde{D}=1}^{n+p}\frac{\partial \tilde{g}_{00}}{\partial x_{\tilde{D}}}\tilde{g}^{\tilde{D}\tilde{A}} \\
&= -\frac{1}{2}\sum_{\tilde{D}=1}^{n+p}\left[\left(2e^{2f}\frac{\partial f}{\partial x_{\tilde{D}}}\right)\tilde{g}^{\tilde{D}\tilde{A}}\right] \\
&= -e^{2f}\sum_{\tilde{D}=1}^{n+p}\frac{\partial f}{\partial x_{\tilde{D}}}\tilde{g}^{\tilde{D}\tilde{A}} = -e^{2f}\sum_{D=1}^{n+p}\frac{\partial f}{\partial x_D}g^{DA}.
\end{aligned}$$

Rewriting in the Einstein's sum convention, we obtain the expression desired for $\tilde{\overline{\Gamma}}^{A}_{00}$.

We obtain relations for the curvature tensor analyzing each case, considering the definition of the Riemannian tensor curvature in the beginning of the section 1.2 and keeping in mind the local expressions of the connection and the Lie Brackets (see Proposition 1.2 for a reference).

1. $A \neq 0, B \neq 0, C \neq 0$:

   Since the Levi-Civita connection with respect to the orthonormal basis $\{e_i\}_{i=0}^{n}$ depends only of the Christoffel symbols and $\tilde{\overline{\Gamma}}^{\tilde{A}}_{\tilde{B}\tilde{C}} = \overline{\Gamma}^{A}_{BC}$, we have that Levi-Civita connection on N and on $\tilde{N}$ coincide. Furthermore,

$$\begin{aligned}
\tilde{R}_{ABCD} = &\quad \tilde{g}\left(\overline{\nabla}_{\tilde{e}_A}\overline{\nabla}_{\tilde{e}_B}\tilde{e}_C - \overline{\nabla}_{\tilde{e}_B}\overline{\nabla}_{\tilde{e}_A}\tilde{e}_C - \overline{\nabla}_{[\tilde{e}_A,\tilde{e}_B]}\tilde{e}_C, \tilde{e}_D\right) \\
= &\quad \begin{pmatrix}\overline{\nabla}_{\tilde{e}_A}\overline{\nabla}_{\tilde{e}_B}\tilde{e}_C - \overline{\nabla}_{\tilde{e}_B}\overline{\nabla}_{\tilde{e}_A}\tilde{e}_C - \overline{\nabla}_{[\tilde{e}_A,\tilde{e}_B]}\tilde{e}_C & 0\end{pmatrix}\begin{pmatrix}e^{2f} & 0 \\ 0 & g_{AB}\end{pmatrix}\begin{pmatrix}0 \\ e_D\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\overline{\nabla}_{\tilde{e}_A}\overline{\nabla}_{\tilde{e}_B}\tilde{e}_C - \overline{\nabla}_{\tilde{e}_B}\overline{\nabla}_{\tilde{e}_A}\tilde{e}_C - \overline{\nabla}_{[\tilde{e}_A,\tilde{e}_B]}\tilde{e}_C \quad 0\right)\begin{pmatrix} 0 \\ (g_{AB})e_D \end{pmatrix} \\
&= \left(\overline{\nabla}_{\tilde{e}_A}\overline{\nabla}_{\tilde{e}_B}(0,e_C) - \overline{\nabla}_{\tilde{e}_B}\overline{\nabla}_{\tilde{e}_A}(0,e_C) - \overline{\nabla}_{[\tilde{e}_A,\tilde{e}_B]}(0,e_C) \quad 0\right)\begin{pmatrix} 0 \\ (g_{AB})e_D \end{pmatrix} \\
&= \left(\overline{\nabla}_{(0,e_A)}\overline{\nabla}_{(0,e_B)}e_C - \overline{\nabla}_{(0,e_B)}\overline{\nabla}_{(0,e_A)}e_C - \overline{\nabla}_{[(0,e_A),(0,e_B)]}e_C \quad 0\right)\begin{pmatrix} 0 \\ (g_{AB})e_D \end{pmatrix} \\
&= \left(0 \quad \overline{\nabla}_{e_A}\overline{\nabla}_{e_B}e_C - \overline{\nabla}_{e_B}\overline{\nabla}_{e_A}e_C - \overline{\nabla}_{[e_A,e_B]}e_C\right)\begin{pmatrix} 0 \\ (g_{AB})e_D \end{pmatrix} \\
&= \left(\overline{\nabla}_{e_A}\overline{\nabla}_{e_B}e_C - \overline{\nabla}_{e_B}\overline{\nabla}_{e_A}e_C - \overline{\nabla}_{[e_A,e_B]}e_C\right)(g_{AB})e_D \\
&= g\left(\overline{\nabla}_{e_A}\overline{\nabla}_{e_B}e_C - \overline{\nabla}_{e_B}\overline{\nabla}_{e_A}e_C - \overline{\nabla}_{[e_A,e_B]}e_C, e_D\right) \\
&= R_{ABCD},
\end{aligned}
$$

where the fifth and sixth equalities are true by the local representations of the connection and of the Lie bracket.

2. $A = 0, B \neq 0, C \neq 0, D \neq 0$:

   Observe that $\overline{\nabla}_{\tilde{e}_B}\tilde{e}_C$ and $\tilde{e}_C$ do not depend of s, furthermore, $[e^{-f}\tilde{e}_0, \tilde{e}_B] = 0$ by its local representation, therefore $\overline{\nabla}_{e^{-f}\tilde{e}_0}\overline{\nabla}_{\tilde{e}_B}\tilde{e}_C - \overline{\nabla}_{\tilde{e}_B}\overline{\nabla}_{e^{-f}\tilde{e}_0}\tilde{e}_C - \overline{\nabla}_{[e^{-f}\tilde{e}_0,\tilde{e}_B]}\tilde{e}_C = 0$. From this, we have

$$\tilde{R}_{0BCD} = \tilde{g}\left(\overline{\nabla}_{e^{-f}\tilde{e}_0}\overline{\nabla}_{\tilde{e}_B}\tilde{e}_C - \overline{\nabla}_{\tilde{e}_B}\overline{\nabla}_{e^{-f}\tilde{e}_0}\tilde{e}_C - \overline{\nabla}_{[e^{-f}\tilde{e}_0,\tilde{e}_B]}\tilde{e}_C, \tilde{e}_D\right) = \tilde{g}\left(0, \tilde{e}_D\right) = 0.$$

3. $A = C = 0, B \neq 0, D \neq 0$:

   Considering geodesic normal coordinates at a point of N and the Christoffel symbols computed previously, we get

$$
\begin{aligned}
\tilde{R}_{0B0D} &= \tilde{g}\left(\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}\tilde{\overline{\nabla}}_{\tilde{e}_B}(e^{-f}\tilde{e}_0) - \tilde{\overline{\nabla}}_{\tilde{e}_B}\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(e^{-f}\tilde{e}_0) - \tilde{\overline{\nabla}}_{[e^{-f}\tilde{e}_0,\tilde{e}_B]}(e^{-f}\tilde{e}_0), \tilde{e}_D\right)\\
&= \tilde{g}\left(\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(\tilde{\overline{\Gamma}}^{\tilde{K}}_{B0}\tilde{e}_{\tilde{K}}) - \tilde{\overline{\nabla}}_{\tilde{e}_B}(\tilde{\overline{\Gamma}}^{\tilde{L}}_{00}\tilde{e}_{\tilde{L}}), \tilde{e}_D\right)\\
&= \tilde{g}\left(\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(\tilde{\overline{\Gamma}}^{0}_{B0}e^{-f}\tilde{e}_0 + \tilde{\overline{\Gamma}}^{K}_{B0}\tilde{e}_K) - \tilde{\overline{\nabla}}_{\tilde{e}_B}\tilde{\overline{\Gamma}}^{\tilde{L}}_{00}\tilde{e}_{\tilde{L}} - \tilde{\overline{\Gamma}}^{\tilde{L}}_{00}\tilde{\overline{\nabla}}_{\tilde{e}_B}\tilde{e}_{\tilde{L}}, \tilde{e}_D\right)\\
&= \tilde{g}\left(\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(\tilde{\overline{\Gamma}}^{0}_{B0}e^{-f}\tilde{e}_0 + \tilde{\overline{\Gamma}}^{K}_{B0}\tilde{e}_K) - \tilde{\overline{\nabla}}_{\tilde{e}_B}\tilde{\overline{\Gamma}}^{\tilde{L}}_{00}\tilde{e}_{\tilde{L}} - \tilde{\overline{\Gamma}}^{\tilde{L}}_{00}\tilde{\overline{\Gamma}}^{\tilde{P}}_{B\tilde{L}}\tilde{e}_{\tilde{P}}, \tilde{e}_D\right)\\
&= \tilde{g}\left(\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(\tilde{\overline{\Gamma}}^{0}_{B0}e^{-f}\tilde{e}_0 + \tilde{\overline{\Gamma}}^{K}_{B0}\tilde{e}_K) - \tilde{\overline{\nabla}}_{\tilde{e}_B}\tilde{\overline{\Gamma}}^{D}_{00}\tilde{e}_D - \tilde{\overline{\Gamma}}^{\tilde{L}}_{00}\tilde{\overline{\Gamma}}^{D}_{B\tilde{L}}\tilde{e}_D, \tilde{e}_D\right)\\
&= \tilde{g}\left(\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(\tilde{\overline{\Gamma}}^{0}_{B0}e^{-f}\tilde{e}_0 + \tilde{\overline{\Gamma}}^{K}_{B0}\tilde{e}_K) - \tilde{\overline{\nabla}}_{\tilde{e}_B}\tilde{\overline{\Gamma}}^{D}_{00}\tilde{e}_D - \tilde{\overline{\Gamma}}^{\tilde{L}}_{00}\tilde{\overline{\Gamma}}^{D}_{B\tilde{L}}\tilde{e}_D, \tilde{e}_D\right)\\
&= \tilde{g}\left(\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(\tilde{\overline{\Gamma}}^{0}_{B0}e^{-f}\tilde{e}_0 + \tilde{\overline{\Gamma}}^{K}_{B0}\tilde{e}_K) - \tilde{\overline{\nabla}}_{\tilde{e}_B}\tilde{\overline{\Gamma}}^{D}_{00}\tilde{e}_D - \tilde{\overline{\Gamma}}^{0}_{00}\tilde{\overline{\Gamma}}^{D}_{B0}\tilde{e}_D - \tilde{\overline{\Gamma}}^{L}_{00}\tilde{\overline{\Gamma}}^{D}_{BL}\tilde{e}_D, \tilde{e}_D\right)\\
&= \tilde{g}\Big(\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}\tilde{\overline{\Gamma}}^{0}_{B0}e^{-f}\tilde{e}_0 + \tilde{\overline{\Gamma}}^{0}_{B0}\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(e^{-f}\tilde{e}_0) + \tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}\tilde{\overline{\Gamma}}^{K}_{B0}\tilde{e}_K + \tilde{\overline{\Gamma}}^{K}_{B0}\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}\tilde{e}_K\\
&\quad - \tilde{\overline{\nabla}}_{\tilde{e}_B}\tilde{\overline{\Gamma}}^{D}_{00}\tilde{e}_D, \tilde{e}_D\Big)\\
&= \tilde{g}\left(\tilde{\overline{\Gamma}}^{0}_{B0}\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(e^{-f}\tilde{e}_0) - \tilde{\overline{\nabla}}_{\tilde{e}_B}\tilde{\overline{\Gamma}}^{D}_{00}\tilde{e}_D, \tilde{e}_D\right)\\
&= \tilde{g}\left(\tilde{\overline{\Gamma}}^{0}_{B0}\tilde{\overline{\Gamma}}^{\tilde{P}}_{00}\tilde{e}_{\tilde{P}} - \tilde{\overline{\nabla}}_{\tilde{e}_B}\tilde{\overline{\Gamma}}^{D}_{00}\tilde{e}_D, \tilde{e}_D\right)\\
&= \tilde{\overline{\Gamma}}^{0}_{B0}\tilde{\overline{\Gamma}}^{D}_{00} - \tilde{\overline{\nabla}}_{\tilde{e}_B}\tilde{\overline{\Gamma}}^{D}_{00}\\
&= \overline{\nabla}_{e_B}f(-e^{2f}\overline{\nabla}_{e_D}f) - \tilde{\overline{\nabla}}_{\tilde{e}_B}(-e^{2f}\overline{\nabla}_{e_D}f)\\
&= \overline{\nabla}_{e_B}f(-e^{2f}\overline{\nabla}_{e_D}f) + e^{2f}2\tilde{\overline{\nabla}}_{\tilde{e}_B}f\overline{\nabla}_{e_D}f + e^{2f}\tilde{\overline{\nabla}}_{\tilde{e}_B}\overline{\nabla}_{e_D}f\\
&= \overline{\nabla}_{e_B}f(-e^{2f}\overline{\nabla}_{e_D}f) + e^{2f}2\overline{\nabla}_{e_B}f\overline{\nabla}_{e_D}f + e^{2f}\overline{\nabla}_{e_B}\overline{\nabla}_{e_D}f\\
&= e^{2f}\left(\overline{\nabla}_{e_B}\overline{\nabla}_{e_D}f + \overline{\nabla}_{e_B}f\overline{\nabla}_{e_D}f\right),
\end{aligned}
$$

where $\tilde{\overline{\Gamma}}^{D}_{BL}$ in the seventh equality vanishes by 2.6 and because we are considering geodesic normal coordinates at a point of N.

Now we compute the Ricci tensor curvature.

$$
\tilde{R}_{00} = \tilde{g}^{\tilde{A}\tilde{B}}\tilde{R}_{\tilde{A}00\tilde{B}} = -\tilde{g}^{\tilde{A}\tilde{B}}e^{2f}(\tilde{\overline{\nabla}}_{\tilde{A}}\tilde{\overline{\nabla}}_{\tilde{B}}f + \tilde{\overline{\nabla}}_{\tilde{A}}f\tilde{\overline{\nabla}}_{\tilde{B}}f) = -e^{2f}(\tilde{\overline{\Delta}}f + |\tilde{\overline{\nabla}}f|^2).
$$

$$
\begin{aligned}
\tilde{R}_{\tilde{A}\tilde{B}} &= \tilde{g}^{\tilde{C}\tilde{D}}\tilde{R}_{\tilde{C}\tilde{A}\tilde{B}\tilde{D}} = \tilde{g}^{00}\tilde{R}_{0\tilde{A}\tilde{B}0} + \tilde{g}^{\tilde{C}\tilde{D}}\tilde{R}_{\tilde{C}\tilde{A}\tilde{B}\tilde{D}} \\
&= e^{-2f}(-e^{2f}(\tilde{\overline{\nabla}}_{\tilde{A}}\tilde{\overline{\nabla}}_{\tilde{B}}f + \tilde{\overline{\nabla}}_{\tilde{A}}f\tilde{\overline{\nabla}}_{\tilde{B}}f)) + \tilde{g}^{\tilde{C}\tilde{D}}\tilde{R}_{\tilde{C}\tilde{A}\tilde{B}\tilde{D}} \\
&= R_{AB} - \tilde{\overline{\nabla}}_{\tilde{A}}\tilde{\overline{\nabla}}_{\tilde{B}}f - \tilde{\overline{\nabla}}_{\tilde{A}}f\tilde{\overline{\nabla}}_{\tilde{B}}f,
\end{aligned}
$$

$$
\tilde{R}_{\tilde{A}0} = \tilde{g}^{\tilde{B}\tilde{C}}\tilde{R}_{\tilde{B}\tilde{A}0\tilde{C}} = \tilde{g}^{\tilde{B}\tilde{C}}\tilde{R}_{0\tilde{C}\tilde{B}\tilde{A}} = \tilde{g}^{\tilde{B}\tilde{C}}0 = 0.
$$

$\tilde{R}_{0\tilde{B}} = 0$ is obtained similarly.

Finally, we compute the mean curvature vector of the fiber $[x]$. Considering (2.4), the orthonormal frame of $\tilde{N}$ in the beginning of this section and the computation of the Christoffel symbols done previously,

$$
\begin{aligned}
\mathbf{H}_{[x]} &= \sum_{\alpha=n+1}^{n+p}\langle\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(e^{-f}\tilde{e}_0),\tilde{\nu}_\alpha\rangle\tilde{\nu}_\alpha + \sum_{i=1}^{n}\langle\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(e^{-f}\tilde{e}_0),\tilde{e}_i\rangle\tilde{e}_i \\
&= e^{-f}\left(\sum_{\alpha=n+1}^{n+p}\langle\tilde{\overline{\nabla}}_{\tilde{e}_0}(e^{-f}\tilde{e}_0),\tilde{\nu}_\alpha\rangle\tilde{\nu}_\alpha + \sum_{i=1}^{n}\langle\tilde{\overline{\nabla}}_{\tilde{e}_0}(e^{-f}\tilde{e}_0),\tilde{e}_i\rangle\tilde{e}_i\right) \\
&= e^{-f}\left(\sum_{\alpha=n+1}^{n+p}\left\langle e^{-f}\left(-\tilde{\overline{\nabla}}_{\tilde{e}_0}f\tilde{e}_0 + \tilde{\overline{\nabla}}_{\tilde{e}_0}\tilde{e}_0\right),\tilde{\nu}_\alpha\right\rangle\tilde{\nu}_\alpha + \sum_{i=1}^{n}\left\langle e^{-f}\left(-\tilde{\overline{\nabla}}_{\tilde{e}_0}f\tilde{e}_0 + \tilde{\overline{\nabla}}_{\tilde{e}_0}\tilde{e}_0\right),\tilde{e}_i\right\rangle\tilde{e}_i\right) \\
&= e^{-2f}\left(\sum_{\alpha=n+1}^{n+p}\left\langle\tilde{\overline{\nabla}}_{\tilde{e}_0}\tilde{e}_0,\tilde{\nu}_\alpha\right\rangle\tilde{\nu}_\alpha + \sum_{i=1}^{n}\left\langle\tilde{\overline{\nabla}}_{\tilde{e}_0}\tilde{e}_0,\tilde{e}_i\right\rangle\tilde{e}_i\right) \\
&= e^{-2f}\left(\sum_{\alpha=n+1}^{n+p}\tilde{\overline{\Gamma}}_{00}^{\alpha}\tilde{\nu}_\alpha + \sum_{i=1}^{n}\tilde{\overline{\Gamma}}_{00}^{i}\tilde{e}_i\right) \\
&= e^{-2f}\left(\sum_{\alpha=n+1}^{n+p}\left(-e^{2f}\frac{\partial f}{\partial x_\alpha}\right)\tilde{\nu}_\alpha + \sum_{i=1}^{n}\left(-e^{2f}\frac{\partial f}{\partial x_i}\right)\tilde{e}_i\right) \\
&= \sum_{\alpha=n+1}^{n+p}\left(0,-\frac{\partial f}{\partial x_\alpha}\nu_\alpha\right) + \sum_{i=1}^{n}\left(0,-\frac{\partial f}{\partial x_i}e_i\right) = (0,-\overline{\nabla}f).
\end{aligned}
$$

□

Lemma 2.1. The mean curvature vector $\mathbf{H}_{\tilde{M}}$ of $\tilde{M}$ on $\tilde{N}$ at (s,x) is given by

$$
\mathbf{H}_{\tilde{M}}(s,x) = (0,\mathbf{H}_M(x)) + \mathbf{H}_{[x]}^{\perp},
$$

where $\mathbf{H}_M(x)$ denotes the mean curvature of M on N at x and $\perp$ denotes the projection on the normal bundle of $\tilde{M}$ on $\tilde{N}$.

Proof. By the definition of the mean curvature vector and our choice of frame,

$$\mathbf{H}_{\tilde{M}}(s,x) = \sum_{\alpha=n+1}^{n+p} \langle \tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(e^{-f}\tilde{e}_0), \tilde{\nu}_\alpha\rangle \tilde{\nu}_\alpha + \sum_{\alpha=n+1}^{n+p}\sum_{i=1}^{n} \langle \tilde{\overline{\nabla}}_{\tilde{e}_i}\tilde{e}_i, \tilde{\nu}_\alpha\rangle \tilde{\nu}_\alpha, \tag{2.11}$$

$$\mathbf{H}_M(x) = \sum_{\alpha=n+1}^{n+p}\sum_{i=1}^{n} \langle \overline{\nabla}_{e_i} e_i, \nu_\alpha\rangle \nu_\alpha \tag{2.12}$$

$$\mathbf{H}_{[x]} = \sum_{\alpha=n+1}^{n+p} \langle \tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(e^{-f}\tilde{e}_0), \tilde{\nu}_\alpha\rangle \tilde{\nu}_\alpha + \sum_{i=1}^{n} \langle \tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(e^{-f}\tilde{e}_0), \tilde{e}_i\rangle \tilde{e}_i. \tag{2.13}$$

By (2.3), (2.4), the local expression of the connection in local coordinates and the Christoffel symbols computed previously,

$$\begin{aligned}
\mathbf{H}_{\tilde{M}}(s,x) - \mathbf{H}_{[x]}^{\perp} &= \sum_{\alpha=n+1}^{n+p}\sum_{i=1}^{n} \langle \tilde{\overline{\nabla}}_{\tilde{e}_i}\tilde{e}_i, \tilde{\nu}_\alpha\rangle \tilde{\nu}_\alpha \\
&= \sum_{\alpha=n+1}^{n+p}\sum_{i=1}^{n} \langle (0, \overline{\nabla}_{e_i} e_i), (0, \nu_\alpha)\rangle (0, \nu_\alpha) \\
&= \sum_{\alpha=n+1}^{n+p}\sum_{i=1}^{n} \Big(0, \langle \overline{\nabla}_{e_i} e_i, \nu_\alpha\rangle \nu_\alpha\Big) = (0, \mathbf{H}_M(x)).
\end{aligned}$$

□

As a direct consequence, we have

Corollary 2.1. The mean curvature vector of $\tilde{M}$ at (s,x) is given by

$$\mathbf{H}_{\tilde{M}}(s,x) = (0, \mathbf{H}_M(x) - (\overline{\nabla} f)^{\perp}).$$

Proof. By Proposition 2.1, we have

$$\mathbf{H}_{\tilde{M}}(s,x) = (0, \mathbf{H}_M(x)) + \mathbf{H}_{[x]}^{\perp} = (0, \mathbf{H}_M(x)) + (0, (-\overline{\nabla} f)^{\perp}).$$

□

Now, we are able to prove a correspondence between conformal solitons and minimal submanifolds, first proved by Smoczyk for hypersurfaces.

Theorem 2.1. Assume that $\mathbf{X}$ is a conformal vector field on a simply connected Riemannian manifold (N^{n+p}, g) satisfying (2.2). Then there exists a warped product metric $\tilde{g}$ on $\tilde{N} = \mathbb{R} \times N^{n+p}$ such that a submanifold $M^n \subset N^{n+p}$ satisfies the soliton equation (2.1) if and only if the associated submanifold $\tilde{M} = \mathbb{R} \times M^n \subset \tilde{N}$ is a minimal submanifold in $(\tilde{N}, \tilde{g})$.

Proof. Let $\omega = \sum_{i=1}^{n} X_i dx^i$ be a 1-form dual to the vector $\mathbf{X}$ defined on N. From the hypothesis done in the beginning of this chapter, $\overline{\nabla}_j X_i = \overline{\nabla}_i X_j$ for each $i, j = 1, \cdots n$. This hypothesis, the symmetry of the connection $\overline{\nabla}$ and as it was seen in the proof of the Proposition 1.1 provide

$$\begin{aligned}
\overline{\nabla}_j X_i = \overline{\nabla}_i X_j \iff & \quad (\nabla_{\partial_j}\omega)(\partial_i) = (\nabla_{\partial_i}\omega)(\partial_j) \\
\iff & \quad \frac{\partial}{\partial x_j}(\omega(\partial_i)) - \omega(\nabla_{\partial_j}\partial_i) = \frac{\partial}{\partial x_i}(\omega(\partial_j)) - \omega(\nabla_{\partial_i}\partial_j) \\
\iff & \quad \frac{\partial}{\partial x_j}(\omega(\partial_i)) - \omega(\nabla_{\partial_j}\partial_i - \nabla_{\partial_i}\partial_j) = \frac{\partial}{\partial x_i}(\omega(\partial_j)) \\
\iff & \quad \frac{\partial}{\partial x_j}(\omega(\partial_i)) - \omega([\partial_j, \partial_i]) = \frac{\partial}{\partial x_i}(\omega(\partial_j)) \\
\iff & \quad \frac{\partial}{\partial x_j}(\omega(\partial_i)) = \frac{\partial}{\partial x_i}(\omega(\partial_j)) \\
\iff & \quad \frac{\partial X_i}{\partial x_j} = \frac{\partial X_j}{\partial x_i}.
\end{aligned}$$

Thus,

$$\begin{aligned}
d\omega &= d\left(\sum_{i=1}^{n} X_i dx^i\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j} dx^i \wedge dx^j \\
&= \sum_{1 \le i < j \le n} \left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}\right) dx^i \wedge dx^j = 0.
\end{aligned}$$

Since ω is defined on a simply connected Riemannian manifold N and $d\omega = 0$, it follows from the Theorem 1.5 the existence of a smooth map $f : N \longrightarrow \mathbb{R}$ such that $\omega = df$. By the duality between vector fields and 1-forms, $\mathbf{X} = \overline{\nabla} f$. Defining a warped metric by $\tilde{g}(s,x) = e^{2f(x)}ds^2 + g(x)$, the result follows from the previous corollary.

□

2.2 Minimality and stability in arbitrary codimension

Lemma 2.2. The following equality holds on a hypersurface $\tilde{M} \subset \tilde{N}$:

$$|\tilde{A}_\alpha|^2 + \widetilde{Ric}(\widetilde{\nu}_\alpha, \widetilde{\nu}_\alpha) = |A_\alpha|^2 + Ric(\nu_\alpha, \nu_\alpha) - \nabla_{\nu_\alpha}\nabla_{\nu_\alpha} f.$$

Proof. Recalling that $\widetilde{\nu}_\alpha = (0, \nu_\alpha)$, (2.10) implies

$$\widetilde{Ric}(\widetilde{\nu}_\alpha, \widetilde{\nu}_\alpha) = Ric(\nu_\alpha, \nu_\alpha) - \nabla_{\nu_\alpha}\nabla_{\nu_\alpha} f - \langle \nu_\alpha, \nabla f\rangle^2,$$

for each $\alpha \in \{n+1, \cdots, n+p\}$ and where $\langle \cdot, \cdot \rangle$ is the inner product on TN. The squared norm of the second fundamental form of $\tilde{N}$ with respect to the normal $\widetilde{\nu}_\alpha$ is

$$\begin{aligned}
|\tilde{A}_\alpha|^2 &= \tilde{g}^{\tilde{P}\tilde{Q}}\tilde{g}^{\tilde{S}\tilde{T}}\tilde{h}^\alpha_{\tilde{P}\tilde{S}}\tilde{h}^\alpha_{\tilde{Q}\tilde{T}} \\
&= g^{PQ}g^{ST}h^\alpha_{PS}h^\alpha_{QT} + \tilde{g}^{00}\tilde{g}^{00}\tilde{h}_{00}\tilde{h}_{00} \\
&= |A_\alpha|^2 + (e^{-2f})(e^{-2f})\left(\langle \overline{\nabla}_{\tilde{e}_0}\tilde{e}_0, \tilde{\nu}_\alpha\rangle\right)\left(\langle \overline{\nabla}_{\tilde{e}_0}\tilde{e}_0, \tilde{\nu}_\alpha\rangle\right) \\
&= |A_\alpha|^2 + \left(\langle \overline{\nabla}_{e^{-f}\tilde{e}_0}(e^{-f}\tilde{e}_0), \tilde{\nu}_\alpha\rangle\right)\left(\langle \overline{\nabla}_{e^{-f}\tilde{e}_0}(e^{-f}\tilde{e}_0), \tilde{\nu}_\alpha\rangle\right) \\
&= |A_\alpha|^2 + \left(\langle \mathbf{H}_{[x]}, \tilde{\nu}_\alpha\rangle\right)^2 \\
&= |A_\alpha|^2 + \left(\langle -\overline{\nabla} f, \nu_\alpha\rangle\right)^2.
\end{aligned}$$

Combining this identity and the previous identity, we proved the lemma. □

Definition 2.1. If $\tilde{M}$ is the submanifold associated to M, then we call a deformation of $\tilde{M}$ symmetric, if it is constant along the fiber directions $[x]$, i.e., $|\tilde{\nabla}^\perp_{e^{-f}\tilde{e}_0}\tilde{\mathbf{S}}|^2 = 0$. A minimal submanifold $\tilde{M}$ in $\tilde{N}$ that is associated to a submanifold $M \subset N$ is called symmetric stable if

$$\int_{[0,1]\times M} \left(\sum_{i,j=1}^{n} \langle \tilde{\overline{\nabla}}_{\tilde{e}_i} \tilde{\mathbf{S}}, \tilde{e}_j \rangle^2 + \langle \tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0} \tilde{\mathbf{S}}, e^{-f}\tilde{e}_0 \rangle^2 + \sum_{i=1}^{n} \tilde{R}(\tilde{e}_i, \tilde{\mathbf{S}}, \tilde{\mathbf{S}}, \tilde{e}_i) + \tilde{R}(e^{-f}\tilde{e}_0, \tilde{\mathbf{S}}, \tilde{\mathbf{S}}, e^{-f}\tilde{e}_0) \right) d\tilde{\mu}$$
$$\leq \int_{[0,1]\times M} \left(\sum_{i=1}^{n} |\tilde{\nabla}^{\perp}_{\tilde{e}_i} \tilde{\mathbf{S}}|^2 + |\tilde{\nabla}^{\perp}_{e^{-f}\tilde{e}_0} \tilde{\mathbf{S}}|^2 \right) d\tilde{\mu}$$

for each $\tilde{\mathbf{S}} \in \Gamma^{\infty}_{0,sym}(\nu([0,1]\times M)) := \{\tilde{\mathbf{S}} \in \Gamma(\nu([0,1]\times M))\ ;\ \tilde{\mathbf{S}}(s,x) = \tilde{\mathbf{S}}(0,x)$ for all $s \in [0,1]$ and $\mathbf{S}(x) := \tilde{\mathbf{S}}(0,x)$ is a compactly supported smooth normal vector field on M in $N\}$. $\Gamma(\nu([0,1]\times M))$ is the normal bundle of $[0,1]\times M$ in $\tilde{N}$.

In the next lemma, we consider the following set of functions:

$$C^{\infty}_{0,sym}([0,1]\times M) := \{\tilde{u}\ ;\ \tilde{u}(s,x) = \tilde{u}(0,x) \text{ for all } s \in [0,1] \text{ and } u(x) := \tilde{u}(0,x) \in C^{\infty}_c(M)\}.$$

Lemma 2.3. Assume that $\mathbf{X}$ is a conformal vector field satisfying (2.2) on a Riemannian manifold simply connected (N,g). Further assume that $M \subset N$ is a hypersurface that solves the soliton equation (2.1). Then there exists a smooth function on N with $\overline{\nabla} f = \mathbf{X}$ (unique up to adding a constant) such that the associated minimal hypersurface $\tilde{M} \subset (\tilde{N}, e^{2f}ds^2 + g)$ is stable under symmetric deformations if and only if

$$\int_M (|A|^2 + Ric(\nu,\nu) - \lambda) u^2 e^f d\mu \leq \int_M |\nabla u|^2 e^f d\mu$$

for each test function $u \in C^{\infty}_c(M)$.

Proof. From the Theorem 2.1, there exists a smooth function on N with $\overline{\nabla} f = \mathbf{X}$ and it is clear that such f is unique up to adding a constant because if f_1 and f_2 are two functions with the same property and such that $f_1 - f_2$ is a constant, then $\overline{\nabla} f_1 = \overline{\nabla} f_2 = \mathbf{X}$. A hypersurface $\tilde{M} \subset \tilde{N}$ is symmetric stable if and only if

$$\int_{[0,1]\times M} (|\tilde{A}|^2 + \widetilde{Ric}(\tilde{\nu},\tilde{\nu}) - \lambda)\tilde{u}^2 d\tilde{\mu}(s,x) \leq \int_{[0,1]\times M} |\nabla \tilde{u}|^2 d\tilde{\mu}(s,x)$$

for each $\tilde{u} \in C^{\infty}_{0,sym}([0,1]\times M)$.

Recalling that $\nabla f = \mathbf{X}$, the proof of the Proposition 1.1 and considering an orthonormal frame $\{e_\alpha\}_{\alpha=n+1}^{n+p}$,

$$\begin{aligned}\overline{\nabla}_\beta X_\alpha &= (\nabla_{e_\beta}\omega)(e_\alpha)\\ &= \frac{\partial}{\partial x_\beta}(\omega(e_\alpha)) - \omega(\nabla_{e_\beta}e_\alpha)\\ &= \frac{\partial}{\partial x_\beta}(g(\mathbf{X}, e_\alpha))\\ &= \frac{\partial}{\partial x_\beta}(g(\nabla f, e_\alpha))\\ &= \frac{\partial}{\partial x_\beta}(\nabla_\alpha f)\\ &= \nabla_\beta\nabla_\alpha f\end{aligned}$$

This and hypothesis (2.2) done in the beginning of this chapter give $\nabla_\nu\nabla_\nu f = \lambda g(\nu,\nu) = \lambda$. Therefore the previous lemma implies

$$\begin{aligned}\int_{[0,1]\times M}(|\tilde{A}|^2 + \widetilde{Ric}(\widetilde{\nu},\widetilde{\nu}))\tilde{u}^2 - |\tilde{\nabla}\tilde{u}|^2 d\tilde{\mu}(s,x) &= \int_{[0,1]\times M}(|A|^2 + Ric(\nu,\nu) - \lambda)\tilde{u}^2 - |\tilde{\nabla}\tilde{u}|^2 d\tilde{\mu}(s,x)\\ &= \int_{[0,1]\times M}(|A|^2 + Ric(\nu,\nu) - \lambda)\tilde{u}^2 - |\nabla\tilde{u}|^2 d\tilde{\mu}(s,x),\end{aligned}$$

where the last equality follows from the fact that the connection $\tilde{\nabla}$ on $\tilde{M}$ is induced by the connection ∇ on M.

Defining $\tilde{u} := u$ with $u \in C^\infty_{0,sym}(M)$ and observing that $d\tilde{\mu} = ds\sqrt{\det\tilde{g}} = dse^f\sqrt{\det g} = dse^f d\mu$ by (2.5), we obtain the result.

$\square$

The following lemma is necessary to prove the next lemma.

Lemma 2.4.

$$\sum_{i,j=1}^{n}\langle\tilde{\overline{\nabla}}_{\tilde{e}_i}\tilde{\mathbf{S}},\tilde{e}_j\rangle^2+\langle\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}\tilde{\mathbf{S}},e^{-f}\tilde{e}_0\rangle^2+\sum_{i=1}^{n}\tilde{R}(\tilde{e}_i,\tilde{\mathbf{S}},\tilde{\mathbf{S}},\tilde{e}_i)+\tilde{R}(e^{-f}\tilde{e}_0,\tilde{\mathbf{S}},\tilde{\mathbf{S}},e^{-f}\tilde{e}_0)$$
$$=\sum_{i,j=1}^{n}\langle\overline{\nabla}_{e_i}\mathbf{S},\tilde{e}_j\rangle^2+\sum_{i=1}^{n}R(e_i,\mathbf{S},\mathbf{S},e_i)-\sum_{\alpha,\beta=n+1}^{n+p}S^\alpha S^\beta\overline{\nabla}_\alpha\overline{\nabla}_\beta f,$$

where $\tilde{\mathbf{S}}=S^\alpha\tilde{\nu}_\alpha=(0,S^\alpha\nu_\alpha)=(0,\mathbf{S})$.

Proof. For each $1\leq i,j\leq n$, we have by (2.6) that

$$\langle\tilde{\overline{\nabla}}_{\tilde{e}_i}\tilde{\mathbf{S}},\tilde{e}_j\rangle=\sum_{\alpha=n+1}^{n+p}\langle\tilde{\overline{\nabla}}_{\tilde{e}_i}(S^\alpha\tilde{\nu}_\alpha),\tilde{e}_j\rangle=\sum_{\alpha=n+1}^{n+p}S^\alpha\langle\tilde{\overline{\nabla}}_{\tilde{e}_i}\tilde{\nu}_\alpha,\tilde{e}_j\rangle$$
$$=\sum_{\alpha=n+1}^{n+p}S^\alpha(-\tilde{\overline{\Gamma}}_{ij}^\alpha)=\sum_{\alpha=n+1}^{n+p}S^\alpha(-\overline{\Gamma}_{ij}^\alpha)=\sum_{\alpha=n+1}^{n+p}S^\alpha\langle\overline{\nabla}_{e_i}\nu_\alpha,e_j\rangle=\langle\overline{\nabla}_{e_i}\mathbf{S},e_j\rangle.$$

We also have

$$\langle\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}\tilde{\mathbf{S}},e^{-f}\tilde{e}_0\rangle=e^{-2f}\sum_{\alpha=n+1}^{n+p}S^\alpha\langle\tilde{\overline{\nabla}}_{\tilde{e}_0}\tilde{\nu}_\alpha,\tilde{e}_0\rangle=e^{-2f}\sum_{\alpha=n+1}^{n+p}S^\alpha\tilde{h}_{00}^\alpha,$$

where $\tilde{h}_{00}^\alpha$ is the second fundamental form of $\tilde{M}$ in $\tilde{N}$. Using (2.7) and (2.9), we have, for each $1\leq i\leq n$,

$$\tilde{R}(\tilde{e}_i,\tilde{\mathbf{S}},\tilde{\mathbf{S}},\tilde{e}_i)=\sum_{\alpha,\beta=n+1}^{n+p}S^\alpha S^\beta\tilde{R}(\tilde{e}_i,\tilde{\nu}_\alpha,\tilde{\nu}_\beta,\tilde{e}_i)=\sum_{\alpha,\beta=n+1}^{n+p}S^\alpha S^\beta\tilde{R}_{i\alpha\beta i}$$
$$=\sum_{\alpha,\beta=n+1}^{n+p}S^\alpha S^\beta R_{i\alpha\beta i}=\sum_{\alpha,\beta=n+1}^{n+p}S^\alpha S^\beta R(e_i,\nu_\alpha,\nu_\beta,e_i)=R(e_i,\mathbf{S},\mathbf{S},e_i)$$

and

$$\tilde{R}(e^{-f}\tilde{e}_0,\tilde{\mathbf{S}},\tilde{\mathbf{S}},e^{-f}\tilde{e}_0)=\sum_{\alpha,\beta=n+1}^{n+p}e^{-2f}S^\alpha S^\beta\tilde{R}_{0\alpha\beta0}=-\sum_{\alpha,\beta=n+1}^{n+p}e^{-2f}S^\alpha S^\beta\tilde{R}_{0\alpha0\beta}$$
$$=-\sum_{\alpha,\beta=n+1}^{n+p}S^\alpha S^\beta\left(\overline{\nabla}_\alpha\overline{\nabla}_\beta f+\overline{\nabla}_\alpha f\overline{\nabla}_\beta f\right).$$

Thus,

$$\langle \tilde{\overline{\nabla}}_{\tilde{e}_i}\tilde{\mathbf{S}},\tilde{e}_j\rangle^2+\langle \tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}\tilde{\mathbf{S}},e^{-f}\tilde{e}_0\rangle^2+\sum_{i=1}^{n}\tilde{R}(\tilde{e}_i,\tilde{\mathbf{S}},\tilde{\mathbf{S}},\tilde{e}_i)+\tilde{R}(e^{-f}\tilde{e}_0,\tilde{\mathbf{S}},\tilde{\mathbf{S}},e^{-f}\tilde{e}_0) \tag{2.14}$$
$$=\langle \overline{\nabla}_{e_i}\mathbf{S},e_j\rangle^2+\sum_{i=1}^{n}R(e_i,\mathbf{S},\mathbf{S},e_i)-\sum_{\alpha,\beta=n+1}^{n+p}S^\alpha S^\beta\overline{\nabla}_\alpha\overline{\nabla}_\beta f+e^{-4f}\left(\sum_{\alpha=n+1}^{n+p}S^\alpha\tilde{h}^\alpha_{00}\right)^2$$
$$-\sum_{\alpha,\beta=n+1}^{n+p}S^\alpha S^\beta\overline{\nabla}_\alpha f\overline{\nabla}_\beta f.$$

Now, we compute the last two terms. As f is independent of s, remains, by (2.13), that

$$\mathbf{H}_{[x]}=-\sum_{\alpha=n+1}^{n+p}e^{-2f}\tilde{h}^\alpha_{00}\tilde{\mathbf{v}}_\alpha+\sum_{i=1}^{n}\langle\tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}(e^{-f}\tilde{e}_0),\tilde{e}_i\rangle\tilde{e}_i,$$

then

$$\langle\mathbf{H}_{[x]},\tilde{\mathbf{S}}\rangle=-e^{-2f}\sum_{\alpha=n+1}^{n+p}S^\alpha\tilde{h}^\alpha_{00}.$$

On the other hand,

$$\langle\mathbf{H}_{[x]},\tilde{\mathbf{S}}\rangle=\langle(0,-\overline{\nabla}f),(0,\mathbf{S})\rangle=-\langle\overline{\nabla}f,\mathbf{S}\rangle=-\sum_{\alpha=n+1}^{n+p}S^\alpha\overline{\nabla}_\alpha f.$$

These two equalities provide

$$\sum_{\alpha,\beta=n+1}^{n+p}S^\alpha S^\beta\overline{\nabla}_\alpha f\overline{\nabla}_\beta f=\left(\sum_{\alpha=n+1}^{n+p}S^\alpha\overline{\nabla}_\alpha f\right)^2=e^{-4f}\left(\sum_{\alpha=n+1}^{n+p}S^\alpha\tilde{h}^\alpha_{00}\right)^2.$$

Substituting this last equality in (2.14), we have the result. □

Lemma 2.5. Assume that $\mathbf{X}$ is a conformal vector field on a simply connected Riemannian manifold (N,g) such that $\overline{\nabla}_iX_j=\lambda g_{ij}$ for a smooth function λ. Further, assume that $M\subset N$ is a submanifold which satisfies $\mathbf{H}=\mathbf{X}^\perp$, then there exists a smooth function on N with $\overline{\nabla}f=\mathbf{X}$, which is unique up to adding constant, such that the associated minimal submanifold $\tilde{M}\subset(\tilde{N},e^{2f}ds^2+g)$ is stable under symmetric deformations if and only if

$$\int_M \left(\sum_{i,j=1}^{n} \langle \overline{\nabla}_{e_i}\mathbf{S}, e_j\rangle^2 + R(e_i,\mathbf{S},\mathbf{S},e_i) - \lambda|\mathbf{S}|^2 \right) e^f d\mu \le \int_M \sum_{i=1}^{n} |\nabla^{\perp}_{e_i}\mathbf{S}|^2 e^f d\mu \tag{2.15}$$

for every normal vector field $\mathbf{S}$ with compact support on M, where $\nabla^{\perp}$ is the induced normal connection on the normal bundle of M.

Proof. By the Theorem 2.1, there exists a smooth function on N with $\overline{\nabla} f = \mathbf{X}$ and it is clear that such f is unique up to adding a constant because if f_1 and f_2 are two functions with the same property, then $\overline{\nabla}(f_1 - f_2) = \overline{\nabla} f_1 - \overline{\nabla} f_2 = X - X = 0$, which implies that $f_1 - f_2$ is a constant. By the warped product metric (2.3) which we defined, it remains that $d\tilde{\mu}(s,x) = e^{f(x)} ds d\mu(x)$. Arguing analogously the Lemma 2.3, $\overline{\nabla}_\alpha \overline{\nabla}_\beta f = \lambda g_{\alpha\beta}$, therefore

$$\sum_{\alpha,\beta=n+1}^{n+p} S^\alpha S^\beta \overline{\nabla}_\alpha \overline{\nabla}_\beta f = \sum_{\alpha,\beta=n+1}^{n+p} S^\alpha S^\beta \lambda g_{\alpha\beta} = \lambda |\mathbf{S}|^2.$$

As $\tilde{\mathbf{S}}$ independent of s, $\tilde{\nabla}^{\perp}_{e^{-f}\tilde{e}_0}\tilde{\mathbf{S}} = 0$. Now, observe that is suficient prove that $|\tilde{\nabla}^{\perp}_{\tilde{e}_i}\tilde{\mathbf{S}}|^2 = |\nabla^{\perp}_{e_i}\mathbf{S}|^2$ for each $1 \le i \le n$. Indeed, if this holds, then the beginning of the demonstration, the previous Lemma and the definition (2.1) imply that

$$\begin{aligned}
&\int_M \left(\sum_{i,j=1}^{n} \langle \overline{\nabla}_{e_i}\mathbf{S}, e_j\rangle^2 + \sum_{i=1}^{n} R(e_i,\mathbf{S},\mathbf{S},e_i) - \lambda|\mathbf{S}|^2 \right) e^f d\mu \\
&= \int_{[0,1]\times M} \left(\sum_{i,j=1}^{n} \langle \overline{\nabla}_{e_i}\mathbf{S}, e_j\rangle^2 + \sum_{i=1}^{n} R(e_i,\mathbf{S},\mathbf{S},e_i) - \lambda|\mathbf{S}|^2 \right) e^f ds d\mu \\
&= \int_{[0,1]\times M} \left(\sum_{i,j=1}^{n} \langle \overline{\nabla}_{e_i}\mathbf{S}, e_j\rangle^2 + \sum_{i=1}^{n} R(e_i,\mathbf{S},\mathbf{S},e_i) - \lambda|\mathbf{S}|^2 \right) d\tilde{\mu} \\
&= \int_{[0,1]\times M} \left(\sum_{i,j=1}^{n} \langle \overline{\nabla}_{e_i}\mathbf{S}, e_j\rangle^2 + \sum_{i=1}^{n} R(\mathbf{S},e_i,\mathbf{S},e_i) - \sum_{\alpha,\beta=n+1}^{n+p} S^\alpha S^\beta \overline{\nabla}_\alpha \overline{\nabla}_\beta f \right) d\tilde{\mu} \\
&= \int_{[0,1]\times M} \left(\sum_{i,j=1}^{n} \langle \tilde{\overline{\nabla}}_{\tilde{e}_i}\tilde{\mathbf{S}}, \tilde{e}_j\rangle^2 + \langle \tilde{\overline{\nabla}}_{e^{-f}\tilde{e}_0}\tilde{\mathbf{S}}, e^{-f}\tilde{e}_0\rangle^2 + \sum_{i=1}^{n} \tilde{R}(\tilde{e}_i,\tilde{\mathbf{S}},\tilde{\mathbf{S}},\tilde{e}_i) + \tilde{R}(e^{-f}\tilde{e}_0,\tilde{\mathbf{S}},\tilde{\mathbf{S}},e^{-f}\tilde{e}_0) \right) d\tilde{\mu} \\
&\le \int_{[0,1]\times M} \left(\sum_{i=1}^{n} |\tilde{\nabla}^{\perp}_{\tilde{e}_i}\tilde{\mathbf{S}}|^2 + |\tilde{\nabla}^{\perp}_{e^{-f}\tilde{e}_0}\tilde{\mathbf{S}}|^2 \right) d\tilde{\mu}
\end{aligned}$$

$$= \int_{[0,1]\times M} \left(\sum_{i=1}^{n} |\nabla^{\perp}_{e_i}\mathbf{S}|^2 \right) e^f ds d\mu$$
$$= \int_{M} \left(\sum_{i=1}^{n} |\nabla^{\perp}_{e_i}\mathbf{S}|^2 \right) e^f d\mu,$$

where the penultimate equality holds because the deformation is symmetric. This proves the first part. By an analogous reasoning, the converse also holds, therefore the Theorem will be proved. Thus, we will show that $|\tilde{\nabla}^{\perp}_{\tilde{e}_i}\tilde{\mathbf{S}}|^2 = |\nabla^{\perp}_{e_i}\mathbf{S}|^2$ for each $1 \leq i \leq n$. By (2.6),

$$\begin{aligned}
\tilde{\nabla}^{\perp}_{\tilde{e}_i}\tilde{\mathbf{S}} &= \tilde{\nabla}^{\perp}_{\tilde{e}_i}\left(\sum_{\alpha=n+1}^{n+p} S^{\alpha}\tilde{\nu}_{\alpha} \right) = \sum_{\alpha=n+1}^{n+p} \tilde{e}_i(S^{\alpha})\tilde{\nu}_{\alpha} + S^{\alpha}\tilde{\nabla}^{\perp}_{\tilde{e}_i}\tilde{\nu}_{\alpha} \\
&= \sum_{\alpha=n+1}^{n+p} (0,e_i)(S^{\alpha})\tilde{\nu}_{\alpha} + S^{\alpha}\tilde{\nabla}^{\perp}_{\tilde{e}_i}\tilde{\nu}_{\alpha} = \sum_{\alpha=n+1}^{n+p} e_i(S^{\alpha})\tilde{\nu}_{\alpha} + S^{\alpha}\tilde{\nabla}^{\perp}_{\tilde{e}_i}\tilde{\nu}_{\alpha} \\
&= \sum_{\alpha=n+1}^{n+p} \left(e_i(S^{\alpha})\tilde{\nu}_{\alpha} + S^{\alpha}\tilde{\overline{\Gamma}}^{\beta}_{i\alpha}\tilde{\nu}_{\beta} \right) = \sum_{\alpha=n+1}^{n+p} \left(e_i(S^{\alpha})(0,\nu_{\alpha}) + S^{\alpha}\overline{\Gamma}^{\beta}_{i\alpha}(0,\nu_{\beta}) \right) \\
&= \left(0, \sum_{\alpha=n+1}^{n+p} e_i(S^{\alpha})\nu_{\alpha} + S^{\alpha}\overline{\Gamma}^{\beta}_{i\alpha}\nu_{\beta} \right) = \left(0, \nabla^{\perp}_{e_i}\left(\sum_{\alpha=n+1}^{n+p} S^{\alpha}\nu_{\alpha} \right) \right) \\
&= (0, \nabla^{\perp}_{e_i}\mathbf{S}),
\end{aligned}$$

therefore $|\tilde{\nabla}^{\perp}_{\tilde{e}_i}\tilde{\mathbf{S}}|^2 = |\nabla^{\perp}_{e_i}\mathbf{S}|^2$ for each $1 \leq i \leq n$. □

Definition 2.2. A conformal soliton M^n on N^{n+p} is stable if it satisfies (2.15) for any compactly supported normal vector field $\mathbf{S}$ on M.

2.3 Totally geodesic submanifolds.

As observed by Arezzo and Sun, the Theorem 2.1 indicates that it is natural find for special minimal submanifolds in $\tilde{N}$.

Proposition 2.2. Assume that $\mathbf{X}$ is a conformal vector field on a simply connected Riemannian manifold (N^{n+p}, g) satisfying (2.2). Then there exists a warped product metric $\tilde{g}$ on $\tilde{N} =$

$\mathbb{R} \times N^{n+p}$ such that a submanifold $M^n \subset N^{n+p}$ is a totally geodesic submanifold in (N,g) if and only if the associated submanifold $\tilde{M} = \mathbb{R} \times M^n \subset \tilde{N}$ is a totally geodesic submanifold in $(\tilde{N},\tilde{g})$.

Proof. Defining $\tilde{g} = e^{2f(x)}ds^2 + g$ with f satisfying $\overline{\nabla} f = \mathbf{X}$. We will denote by $\tilde{h}^{\alpha}_{ij}$ of $\tilde{M}$ in $(\tilde{N},\tilde{g})$ and h^{α}_{ij} the second fundamental form of M in (N,g). The hypothesis that $F : M \longrightarrow N$ is an immersion and the Inverse Function Theorem allow us write M locally as

$$\begin{aligned} F : U \subset M &\longrightarrow N \\ x &\mapsto F(x), \end{aligned}$$

then $\tilde{M}$ is given by

$$\begin{aligned} \tilde{F} : \mathbb{R} \times U &\longrightarrow \tilde{N} \\ (s,x) &\mapsto (s,F(x)). \end{aligned}$$

By the Gauss' equation,

$$\frac{\partial^2 F^A}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial F^A}{\partial x^k} + \overline{\Gamma}^A_{DE}\frac{\partial F^D}{\partial x^i}\frac{\partial F^E}{\partial x^j} = -h^{\beta}_{ij}\nu^A_{\beta} \tag{2.16}$$

and

$$\frac{\partial^2 \tilde{F}^{\tilde{A}}}{\partial x^i \partial x^j} - \tilde{\Gamma}^c_{ab}\frac{\partial \tilde{F}^{\tilde{A}}}{\partial x^c} + \tilde{\overline{\Gamma}}^{\tilde{A}}_{\tilde{D}\tilde{E}}\frac{\partial \tilde{F}^{\tilde{D}}}{\partial x^a}\frac{\partial \tilde{F}^{\tilde{E}}}{\partial x^b} = -\tilde{h}^{\beta}_{ab}\tilde{\nu}^{\tilde{A}}_{\beta}. \tag{2.17}$$

From the Gauss' equation for F, we have

$$-h^{\beta}_{ij} = g_{AC}\nu^C_{\beta}\frac{\partial^2 F^A}{\partial x^i \partial x^j} + g_{AC}\nu^C_{\beta}\overline{\Gamma}^A_{DE}\frac{\partial F^D}{\partial x^i}\frac{\partial F^E}{\partial x^j}. \tag{2.18}$$

From the Gauss' equation for $\tilde{F}$, we have

$$-\tilde{h}^{\beta}_{ab} = \tilde{g}_{\tilde{A}\tilde{C}}\tilde{\nu}^{\tilde{C}}_{\beta}\frac{\partial^2 \tilde{F}^{\tilde{A}}}{\partial x^i \partial x^j} + \tilde{g}_{\tilde{A}\tilde{C}}\tilde{\nu}^{\tilde{C}}_{\beta}\tilde{\overline{\Gamma}}^{\tilde{A}}_{\tilde{D}\tilde{E}}\frac{\partial \tilde{F}^{\tilde{D}}}{\partial x^a}\frac{\partial \tilde{F}^{\tilde{E}}}{\partial x^b}. \tag{2.19}$$

By the definition (2.4) of $\tilde{v}_\alpha$, we have

$$\tilde{v}^A_\beta = v^A_\beta$$

for $A \geq 1$ and $\tilde{v}^0_\beta = 0$. By (2.5), (2.6), (2.16) and the local expression of $\tilde{F}$, we have, for $i, j \geq 1$,

$$-\tilde{h}^\beta_{ij} = \tilde{g}_{\tilde{A}\tilde{C}}\tilde{v}^{\tilde{C}}_\beta \frac{\partial^2 \tilde{F}^{\tilde{A}}}{\partial x^i \partial x^j} + \tilde{g}_{\tilde{A}\tilde{C}}\tilde{v}^{\tilde{C}}_\beta \tilde{\overline{\Gamma}}^{\tilde{A}}_{\tilde{D}\tilde{E}} \frac{\partial \tilde{F}^{\tilde{D}}}{\partial x^a}\frac{\partial \tilde{F}^{\tilde{E}}}{\partial x^b} \tag{2.20}$$

$$= g_{AC}v^C_\beta \frac{\partial^2 F^A}{\partial x^i \partial x^j} + g_{AC}v^C_\beta \overline{\Gamma}^A_{DE}\frac{\partial F^D}{\partial x^i}\frac{\partial F^E}{\partial x^j} = h^\beta_{ij}, \tag{2.21}$$

i.e.,

$$\tilde{h}^\beta_{ij} = h^\beta_{ij}, 1 \leq i, j \leq n, n+1 \leq \beta \leq n+p. \tag{2.22}$$

Similarly, we have, for $j \geq 1$,

$$-\tilde{h}^\beta_{0j} = \tilde{g}_{\tilde{A}\tilde{C}}\tilde{v}^{\tilde{C}}_\beta \frac{\partial^2 \tilde{F}^{\tilde{A}}}{\partial s \partial x^j} + \tilde{g}_{\tilde{A}\tilde{C}}\tilde{v}^{\tilde{C}}_\beta \tilde{\overline{\Gamma}}^{\tilde{A}}_{\tilde{D}\tilde{E}}\frac{\partial \tilde{F}^{\tilde{D}}}{\partial s}\frac{\partial \tilde{F}^{\tilde{E}}}{\partial x^j} = \tilde{g}_{\tilde{A}\tilde{C}}\tilde{v}^{\tilde{C}}_\beta \tilde{\overline{\Gamma}}^{\tilde{A}}_{0\tilde{E}}\frac{\partial \tilde{F}^{\tilde{E}}}{\partial x^j} = 0,$$

i.e.,

$$\tilde{h}^\beta_{0j} = 0, 1 \leq j \leq n, n+1 \leq \beta \leq n+p. \tag{2.23}$$

Finally, we have

$$\begin{aligned}-\tilde{h}^\beta_{00} &= \tilde{g}_{\tilde{A}\tilde{C}}\tilde{v}^{\tilde{C}}_\beta \frac{\partial^2 \tilde{F}^{\tilde{A}}}{\partial s^2} + \tilde{g}_{\tilde{A}\tilde{C}}\tilde{v}^{\tilde{C}}_\beta \tilde{\overline{\Gamma}}^{\tilde{A}}_{\tilde{D}\tilde{E}}\frac{\partial \tilde{F}^{\tilde{D}}}{\partial s}\frac{\partial \tilde{F}^{\tilde{E}}}{\partial s} = g_{AC}v^C_\beta \tilde{\overline{\Gamma}}^{\tilde{A}}_{00} \\ &= -g_{AC}v^C_\beta e^{2f}g^{AB}\frac{\partial f}{\partial x^B} = -e^{2f}v^B_\beta \frac{\partial f}{\partial x^B} = -e^{2f}\langle \overline{\nabla} f, v_\beta \rangle,\end{aligned}$$

i.e.,

$$\tilde{h}^\beta_{00} = e^{2f}\langle \overline{\nabla} f, v_\beta \rangle, \; n+1 \leq \beta \leq n+p. \tag{2.24}$$

If $\tilde{M}$ is totally geodesic in $(\tilde{N},\tilde{g})$, then the second fundamental form of $\tilde{M}$ vanishes, i.e., $\tilde{h}_{ij}^{\beta}=0$ for $1\le i,j\le n$ and $n+1\le\beta\le n+p$. From this and (2.22), follows that the second fundamental form of M vanishes, which implies that M is totally geodesic in (N,g).

If M is totally geodesic in (N,g), then the second fundamental form of M vanishes and, consequentely, M is a minimal submanifold. Thus, follows from the equation for the conformal soliton that $\mathbf{X}^{\perp}=\mathbf{H}\equiv 0$. As seen in the proof of the Theorem 2.1, it must exists a potential function f so that $\mathbf{X}=\overline{\nabla}f$, therefore $(\overline{\nabla}f)^{\perp}\equiv 0$ and, by (2.24),

$$\tilde{h}_{00}^{\beta}=e^{2f}\langle\overline{\nabla}f,\nu_{\beta}\rangle\equiv 0,\ n+1\le\beta\le n+p.$$

This, (2.22) and (2.23), imply that the second fundamental form of $\tilde{M}$ vanishes, therefore $\tilde{M}$ is totally geodesic in $(\tilde{N},\tilde{g})$. □

Corollary 2.2. A conformal soliton M in $(\mathbb{R}^{n+p},\delta)$ satisfying (2.2) is a linear subspace if and only if its associated submanifold $\tilde{M}$ is totally geodesic in $(\mathbb{R}^{n+p+1},\tilde{g})$.

Proof. The previous Theorem combined with the Theorem 1.1 and the observation (1.3) gives the result. □

Remark 2.1. Roughly speaking, a linear subspace in the corollary is understood as a linear subspace, its translation or a submanifold which is a linear subspace or its translation in each connected component of the submanifold.

Chapter 3

Variational principle applied to conformal solitons

The first and second variation's formulas of a weighted functional are computed to show that the conformal solitons to the mean curvature flow are the only critical points for such functional. The second variation's formula of the weighted functional gives a stability notion for conformal solitons, which coincides with the stability derived in the previous chapter. Also, we present some examples of hypersurfaces and submanifolds which are stable and a proof that conformal solitons are related with singularities of the mean curvature flow. Finally, we present a proof that compact self-shrinkers in $\mathbb{R}^{n+1}$ are non stable as well as a proof that "grim reaper" in $\mathbb{R}^2$ and the "grim reaper" cylinder in $\mathbb{R}^{n+1}$ are stable.

3.1 A variational principle

Suppose that $\mathbf{X}$ is an arbitrary conformal vector field on a simply connected Riemannian manifold (N^{n+p}, g) such that $\overline{\nabla}_i X_j = \lambda g_{ij}$ for a smooth function λ, then there exists a smooth function f on N such that $\overline{\nabla} f = \mathbf{X}$ as we did see. Define the weighted volume functional $\mathcal{G}$ on a submanifold M of N by

$$\mathcal{G}(M) := \int_M e^f d\mu,$$

where $d\mu$ is the volume element induced on M.

Let $F : M \times (-\varepsilon, \varepsilon) \longrightarrow N$ be a variation with compact support, that is, $F = Id$ outside of some compact set and $F(x,0) = x$.

Let F_s restrict to M be the variational vector field. Let $\{x_i\}_{i=1}^n$ be local coordinates on M, then the induced metric on $F(M,s)$ is given by

$$g_{ij}(s) = g\left(F_{x_i}, F_{x_j}\right).$$

Denote by $\overline{\nabla}$ and ∇ the Levi-Civita's connection of N and M, respectively. Define

$$\nu(s) = e^f \frac{\sqrt{\det(g_{ij}(s))}}{\sqrt{\det(g_{ij}(0))}}.$$

The function ν is well-defined and independent of the choice of the coordinate system. Furthermore,

$$\mathcal{G}(M) = \int_M \nu(s)\sqrt{\det(g_{ij}(0))}.$$

Lemma 3.1.

$$\frac{d}{ds}\nu(s)\bigg|_{s=0} = \langle \mathbf{X}^\perp - \mathbf{H}, F_s\rangle e^f + \operatorname{div}_M(e^f F_s^T).$$

Proof.

$$\begin{aligned}
\frac{d\nu}{ds}(s) = d\nu(F_s) &= df(F_s)e^f \frac{\sqrt{\det(g_{ij}(s))}}{\sqrt{\det(g_{ij}(0))}} + e^f \frac{1}{\sqrt{\det(g_{ij}(0))}} \frac{1}{2\sqrt{\det(g_{ij}(s))}} \sum_{i=1}^n \operatorname{adj}(g_{ij}(s)) dg_{ij}(F_s) \\
&= \langle \overline{\nabla} f, F_s\rangle e^f \frac{\sqrt{\det(g_{ij}(s))}}{\sqrt{\det(g_{ij}(0))}} + e^f \frac{1}{\sqrt{\det(g_{ij}(0))}} \frac{1}{2\sqrt{\det(g_{ij}(s))}} \sum_{i=1}^n \operatorname{adj}(g_{ij}(s)) g'_{ij}(s) \\
&= \langle \overline{\nabla} f, F_s\rangle e^f \frac{\sqrt{\det(g_{ij}(s))}}{\sqrt{\det(g_{ij}(0))}} + e^f \frac{1}{\sqrt{\det(g_{ij}(0))}} \frac{1}{2\sqrt{\det(g_{ij}(s))}} \sum_{i=1}^n \det(g_{ij}(s)) g^{ij}(s) g'_{ij}(s) \\
&= \langle \overline{\nabla} f, F_s\rangle e^f \frac{\sqrt{\det(g_{ij}(s))}}{\sqrt{\det(g_{ij}(0))}} + e^f \frac{1}{\sqrt{\det(g_{ij}(0))}} \frac{\sqrt{\det(g_{ij}(s))}}{2} \sum_{i=1}^n g^{ij}(s) g'_{ij}(s) \\
&= \left(\langle \overline{\nabla} f, F_s\rangle + \frac{1}{2}\sum_{i=1}^n g^{ij}(s) g'_{ij}(s) \right) \nu(s),
\end{aligned}$$

where adj denotes the adjoint matrix. Thus,

$$\frac{d\nu}{ds}(s) = \left(\langle \overline{\nabla} f, F_s \rangle + \frac{1}{2}\sum_{i=1}^{n} g^{ij}(s) g'_{ij}(s) \right) \nu(s), \tag{3.1}$$

where

$$g'_{ij}(s) = \frac{d}{ds}\langle F_{x_i}, F_{x_j}\rangle = \langle \overline{\nabla}_{F_s} F_{x_i}, F_{x_j}\rangle + \langle F_{x_i}, \overline{\nabla}_{F_s} F_{x_j}\rangle = \langle F_{x_i s}, F_{x_j}\rangle + \langle F_{x_i}, F_{x_j s}\rangle. \tag{3.2}$$

Fixed $x \in M$, we can compute pointwise $\left.\frac{d\nu}{ds}(s)\right|_{s=0}$ and choose a coordinate system at x so that $\{F_{x_i}(0)\}_{i=1}^{n}$ is an orthonormal basis of $T_x M$ with a induced metric $g(0)$. Using the fact that $\overline{\nabla}_{F_s} F_{x_i} - \overline{\nabla}_{F_{x_i}} F_s = [F_s, F_{x_i}] = 0$, we have at x that

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{n} g^{ij}(0) g'_{ij}(0) = \frac{1}{2}\sum_{i=1}^{n} g'_{ii}(0) = \sum_{i=1}^{n}\langle \overline{\nabla}_{F_s} F_{x_i}, F_{x_i}\rangle &= \sum_{i=1}^{n}\langle \overline{\nabla}_{F_{x_i}} F_s, F_{x_i}\rangle \\
&= \sum_{i=1}^{n}\langle \overline{\nabla}_{F_{x_i}} F_s^{\perp}, F_{x_i}\rangle + \sum_{i=1}^{n}\langle \overline{\nabla}_{F_{x_i}} F_s^{T}, F_{x_i}\rangle \\
&= -\sum_{i=1}^{n}\langle F_s^{\perp}, \overline{\nabla}_{F_{x_i}} F_{x_i}\rangle + \mathrm{div}_M F_s^T \\
&= -\langle \mathbf{H}, F_s\rangle + \mathrm{div}_M F_s^T.
\end{aligned}$$

Substituting this and $\mathbf{X} = \overline{\nabla} f$ on (3.1), we have

$$\begin{aligned}
\left.\frac{d}{ds}\nu(s)\right|_{s=0} &= \left(\langle \mathbf{X}, F_s\rangle - \langle \mathbf{H}, F_s\rangle + \mathrm{div}_M F_s^T\right) e^f \\
&= \left(\langle \mathbf{X}^{\perp} - \mathbf{H}, F_s\rangle + \langle \mathbf{X}^T, F_s\rangle + \mathrm{div}_M F_s^T\right) e^f \\
&= \langle \mathbf{X}^{\perp} - \mathbf{H}, F_s\rangle e^f + \langle \mathbf{X}^T, F_s\rangle e^f + \mathrm{div}_M (F_s^T) e^f \\
&= \langle \mathbf{X}^{\perp} - \mathbf{H}, F_s\rangle e^f + \langle \mathbf{X}, F_s^T\rangle e^f + e^f \sum_{i=1}^{n}\langle \overline{\nabla}_{F_{x_i}} F_s^T, F_{x_i}\rangle \\
&= \langle \mathbf{X}^{\perp} - \mathbf{H}, F_s\rangle e^f + \langle \overline{\nabla} f, F_s^T\rangle e^f + e^f \sum_{i=1}^{n}\langle \overline{\nabla}_{F_{x_i}} F_s^T, F_{x_i}\rangle \\
&= \langle \mathbf{X}^{\perp} - \mathbf{H}, F_s\rangle e^f + \left\langle \sum_{i=1}^{n}\langle \overline{\nabla} f, F_{x_i}\rangle F_{x_i}, F_s^T \right\rangle e^f + e^f \sum_{i=1}^{n}\langle \overline{\nabla}_{F_{x_i}} F_s^T, F_{x_i}\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle \mathbf{X}^{\perp} - \mathbf{H}, F_s \rangle e^f + e^f \sum_{i=1}^{n} \left\langle F_{x_i}, \langle \overline{\nabla} f, F_{x_i} \rangle F_s^T \right\rangle e^f + e^f \sum_{i=1}^{n} \langle \overline{\nabla}_{F_{x_i}} F_s^T, F_{x_i} \rangle \\
&= \langle \mathbf{X}^{\perp} - \mathbf{H}, F_s \rangle e^f + \sum_{i=1}^{n} \left\langle F_{x_i}, df(F_{x_i}) e^f F_s^T \right\rangle + e^f \sum_{i=1}^{n} \langle \overline{\nabla}_{F_{x_i}} F_s^T, F_{x_i} \rangle \\
&= \langle \mathbf{X}^{\perp} - \mathbf{H}, F_s \rangle e^f + \sum_{i=1}^{n} \left\langle F_{x_i}, \overline{\nabla}_{F_{x_i}} f e^f F_s^T \right\rangle + \sum_{i=1}^{n} \langle e^f \overline{\nabla}_{F_{x_i}} F_s^T, F_{x_i} \rangle \\
&= \langle \mathbf{X}^{\perp} - \mathbf{H}, F_s \rangle e^f + \sum_{i=1}^{n} \langle \overline{\nabla}_{F_{x_i}} (e^f F_s^T), F_{x_i} \rangle \\
&= \langle \mathbf{X}^{\perp} - \mathbf{H}, F_s \rangle e^f + \mathrm{div}_M (e^f F_s^T).
\end{aligned}
$$

□

Theorem 3.1 (First Variation Formula of the weighted volume functional)**.**

$$
\frac{d}{ds} \mathcal{G}(F(M,s)) \Big|_{s=0} = \int_M \langle \mathbf{X}^{\perp} - \mathbf{H}, F_s \rangle e^f d\mu .
$$

Proof. Observe that

$$
\frac{d}{ds} \mathcal{G}(F(M,s)) \Big|_{s=0} = \int_M \frac{d}{ds} \nu(s) \Big|_{s=0} \sqrt{\det(g_{ij}(0))} \tag{3.3}
$$

From the Lemma 3.1,

$$
\int_M \frac{d}{ds} \nu(s) \Big|_{s=0} \sqrt{\det(g_{ij}(0))} = \int_M \left(\langle \mathbf{X}^{\perp} - \mathbf{H}, F_s \rangle e^f + \mathrm{div}_M (e^f F_s^T) \right) \sqrt{\det(g_{ij}(0))}
$$

By the Divergence Theorem,

$$
\int_M \mathrm{div}_M (e^f F_s^T) d\mu = 0.
$$

Thus,

$$
\frac{d}{ds} \mathcal{G}(F(M,s)) \Big|_{s=0} = \int_M \langle \mathbf{X}^{\perp} - \mathbf{H}, F_s \rangle e^f d\mu . \tag{3.4}
$$

□

Corollary 3.1. *M is a critical point for the $\mathcal{G}$-functional if and only if $\mathbf{X}^\perp = \mathbf{H}$ on M, i.e., M is a conformal soliton for the mean curvature flow with conformal vector field $\mathbf{X}$.*

Proof. From (3.4), it is clear that every conformal soliton M is a critical point of the $\mathcal{G}$-functional. Reciprocally, suppose that M is a critical point of the $\mathcal{G}$-functional. Once that the normal variation F is arbitrary, we can choose F such that $F_s = u(\mathbf{X}^\perp - \mathbf{H})$, where u is a positive function with compact support on M, then M is a conformal soliton from this, (3.4) and the hypothesis that M is a critical point of the $\mathcal{G}$-functional. $\square$

Now, suppose that $M^n \subset N^{n+p}$ is a conformal soliton, i.e., $\mathbf{X}^\perp = \mathbf{H}$. We will compute the second variation of the $\mathcal{G}$-functional for the normal variational F of M ($F_s^T \equiv 0$) with compact support.

As before, we will compute $\left.\dfrac{d^2}{ds^2}\nu(s)\right|_{s=0}$ pointwise so that, for a point x fixed, we have an orthonormal coordinate system at x. Before that, we will need the following claim.

Claim 3.1.

$$\sum_{i,j=1}^{n} (g^{ij})' g'_{ij} = -\sum_{i,j=1}^{n} (g'_{ij})^2.$$

Proof. We will omit the point 0 for simplicity and we will consider normal coordinates. Observe that

$$(g^{ip} g_{pj})' = (\delta^i_j)' \Longrightarrow (g^{ij})' = -g'_{ij}$$

and

$$g'_{ij} = \langle F_{x_i s}, F_{x_j}\rangle + \langle F_{x_i}, F_{x_j s}\rangle = -2\langle A(F_{x_i}, F_{x_j}), F_s\rangle.$$

The last equality implies that (g'_{ij}) is diagonalizable. This and the first equality imply that $((g^{ij})')$ is diagonalizable. If D is the diagonal matrix of (g'_{ij}), then

$$\sum_{i,j=1}^{n} (g^{ij})' g'_{ij} = \operatorname{tr}\,((G^{-1})' G')$$

$$
\begin{aligned}
&= \mathrm{tr}\,((Q^{-1}(-D)Q)(Q^{-1}DQ))\\
&= \mathrm{tr}\,(Q^{-1}(-D)DQ)\\
&= \mathrm{tr}\,((-D)DQQ^{-1})\\
&= -\mathrm{tr}\,(D^2)\\
&= -\mathrm{tr}\,((G')^2)\\
&= -\sum_{i,j=1}^{n}(g'_{ij})^2,
\end{aligned}
$$

where tr denotes the trace of a matrix. □

We return to compute $\left.\frac{d^2}{ds^2}\nu(s)\right|_{s=0}$.

Lemma 3.2.

$$
\begin{aligned}
\left.\frac{d^2}{ds^2}\nu(s)\right|_{s=0} = \Bigg(&|\overline{\nabla}^{\perp}F_s|^2 - \sum_{i,j=1}^{n}\langle B(F_{x_i},F_{x_j}),F_s\rangle^2 + \lambda\langle F_s,F_s\rangle\\
&-\sum_{i=1}^{n}\langle R_N(F_s,E_i)E_i,F_s\rangle\Bigg)e^f + (\mathrm{div}_M F_{ss} + \langle\overline{\nabla}f,F_{ss}\rangle)e^f.
\end{aligned}
$$

Proof. By (3.1), the previous claim and considering normal coordinates,

$$
\begin{aligned}
\left.\frac{d^2}{ds^2}\nu(s)\right|_{s=0} &= \left(\langle\overline{\nabla}f,F_s\rangle' + \frac{1}{2}\sum_{i,j=1}^{n}\left((g^{ij})'(0)g'_{ij}(0) + g^{ij}(0)g''_{ij}(0)\right)\right)\nu(0)\\
&+\left(\langle\overline{\nabla}f,F_s\rangle + \frac{1}{2}\sum_{i,j=1}^{n}g^{ij}(0)g'_{ij}(0)\right)\left.\frac{d\nu}{ds}(s)\right|_{s=0}\\
&= \left(\langle\overline{\nabla}f,F_s\rangle' + \frac{1}{2}\sum_{i,j=1}^{n}\left((g^{ij})'(0)g'_{ij}(0) + g^{ij}(0)g''_{ij}(0)\right)\right)\nu(0)\\
&+\left(\langle\overline{\nabla}f,F_s\rangle + \frac{1}{2}\sum_{i,j=1}^{n}g^{ij}(0)g'_{ij}(0)\right)^2\nu(0)
\end{aligned}
$$

$$= \left(\langle \overline{\nabla} f, F_s \rangle' + \frac{1}{2} \sum_{i,j=1}^{n} \left((g^{ij})'(0) g'_{ij}(0) + g^{ij}(0) g''_{ij}(0) \right) \right) e^f$$
$$+ \left(\langle \overline{\nabla} f, F_s \rangle + \frac{1}{2} \sum_{i,j=1}^{n} g^{ij}(0) g'_{ij}(0) \right)^2 e^f$$
$$= \left(\langle \overline{\nabla} f, F_s \rangle' - \frac{1}{2} \sum_{i,j=1}^{n} (g'_{ij}(0))^2 + \frac{1}{2} \sum_{i=1}^{n} g''_{ii}(0) \right) e^f$$
$$+ \left(\langle \overline{\nabla} f, F_s \rangle + \frac{1}{2} \sum_{i,j=1}^{n} (g^{ij} g'_{ij})(0) \right)^2 e^f,$$

i.e.,

$$\frac{d^2}{ds^2} \nu(s) \Big|_{s=0} = \left(\langle \overline{\nabla} f, F_s \rangle' - \frac{1}{2} \sum_{i,j=1}^{n} (g'_{ij}(0))^2 + \frac{1}{2} \sum_{i=1}^{n} g''_{ii}(0) \right) e^f + \left(\langle \overline{\nabla} f, F_s \rangle + \frac{1}{2} \sum_{i,j=1}^{n} (g^{ij} g'_{ij})(0) \right)^2 e^f \tag{3.5}$$

From the fact that M is a conformal soliton, that $\frac{1}{2} \sum_{i=1}^{n} g^{ij}(0) g'_{ij}(0) = -\langle \mathbf{H}, F_s \rangle + \mathrm{div}_M F_s^T$ and that $F_s^T \equiv 0$, follows that

$$\frac{d^2}{ds^2} \nu(s) \Big|_{s=0} = \left(\langle \overline{\nabla} f, F_s \rangle' - \frac{1}{2} \sum_{i,j=1}^{n} (g'_{ij}(0))^2 + \frac{1}{2} \sum_{i=1}^{n} g''_{ii}(0) \right) e^f. \tag{3.6}$$

We will compute the three terms of the right side of the equality above separately. Recalling that $\overline{\nabla}_i X_j = \lambda g_{ij}$, we see that $\overline{\nabla}^2 f = \overline{\nabla} \mathbf{X} = \lambda g$, therefore

$$\langle \overline{\nabla} f, F_s \rangle' = \overline{\nabla}^2 f(F_s, F_s) + \langle \overline{\nabla} f, F_{ss} \rangle = \lambda \langle F_s, F_s \rangle + \langle \overline{\nabla} f, F_{ss} \rangle. \tag{3.7}$$

Defining $E_i := F_{x_i}(0)$ for each $i = 1, \cdots, n$ and considering (3.2),

$$g'_{ij}(0) = \langle F_{x_i s}, F_{x_j} \rangle + \langle F_{x_i}, F_{x_j s} \rangle = -2 \langle B(E_i, E_j), F_s \rangle. \tag{3.8}$$

Recalling that $\{F_{x_i}(0)\}_{i=1}^n$ is an orthonormal basis of T_xM with a induced metric $g(0)$,

$$\begin{aligned}
\sum_{i=1}^n g_{ii}''(0) &= \sum_{i=1}^n (2\langle F_{x_i s}, F_{x_i}\rangle)' \\
&= \sum_{i=1}^n (2\langle F_{x_i ss}, F_{x_i}\rangle + 2\langle F_{x_i s}, F_{x_i s}\rangle) \\
&= \sum_{i=1}^n \Big(2\langle \overline{\nabla}_{F_s}\overline{\nabla}_{F_s} F_{x_i}, F_{x_i}\rangle + 2\langle \overline{\nabla}_{F_s} F_{x_i}, \overline{\nabla}_{F_s} F_{x_i}\rangle\Big) \\
&= \sum_{i=1}^n \Big(2\langle \overline{\nabla}_{F_s}\overline{\nabla}_{F_{x_i}} F_s, F_{x_i}\rangle + 2\langle \overline{\nabla}_{F_{x_i}} F_s, \overline{\nabla}_{F_{x_i}} F_s\rangle\Big) \\
&= \sum_{i=1}^n \Big(2\langle R_N(F_s, F_{x_i}, F_s) + 2\overline{\nabla}_{F_{x_i}}\overline{\nabla}_{F_s} F_s, F_{x_i}\rangle + 2\langle \overline{\nabla}_{F_{x_i}} F_s, \overline{\nabla}_{F_{x_i}} F_s\rangle\Big) \\
&= 2\mathrm{div}_M(F_{ss}) + \sum_{i=1}^n 2\langle R_N(F_s, F_{x_i}, F_s), F_{x_i}\rangle + 2\langle \overline{\nabla}_{F_{x_i}} F_s, \overline{\nabla}_{F_{x_i}} F_s\rangle \\
&= 2\mathrm{div}_M(F_{ss}) + \sum_{i=1}^n 2\,\langle R_N(F_s, F_{x_i}, F_s), F_{x_i}\rangle \\
&+ \sum_{i=1}^n 2\left\langle \left(\overline{\nabla}_{F_{x_i}} F_s\right)^T + \left(\overline{\nabla}_{F_{x_i}} F_s\right)^\perp, \left(\overline{\nabla}_{F_{x_i}} F_s\right)^T + \left(\overline{\nabla}_{F_{x_i}} F_s\right)^\perp \right\rangle \\
&= 2\mathrm{div}_M(F_{ss}) + \sum_{i=1}^n 2\langle R_N(F_s, F_{x_i}, F_s), F_{x_i}\rangle \\
&+ \sum_{i=1}^n 2\left\langle \left(\overline{\nabla}_{F_{x_i}} F_s\right)^T, \left(\overline{\nabla}_{F_{x_i}} F_s\right)^T \right\rangle + 2\left\langle \left(\overline{\nabla}_{F_{x_i}} F_s\right)^\perp, \left(\overline{\nabla}_{F_{x_i}} F_s\right)^\perp \right\rangle \\
&= 2\mathrm{div}_M(F_{ss}) + \sum_{i=1}^n 2\langle R_N(F_s, F_{x_i}, F_s), F_{x_i}\rangle \\
&+ \sum_{i=1}^n 2g^{ii}\left(\left\langle \left(\overline{\nabla}_{F_{x_i}} F_s\right)^T, \left(\overline{\nabla}_{F_{x_i}} F_s\right)^T \right\rangle + \left\langle \left(\overline{\nabla}_{F_{x_i}} F_s\right)^\perp, \left(\overline{\nabla}_{F_{x_i}} F_s\right)^\perp \right\rangle\right),
\end{aligned}$$

where $g^{ii} = 1$ for each $i = 1, \cdots, n$ because $\{F_{x_i}(0)\}_{i=1}^n$ is an orthonormal basis of T_xM with an induced metric $g(0)$.

Observe that, on a normal coordinate system,

$$\left(\overline{\nabla}_{F_{x_i}} F_s\right)^T = \sum_{j=1}^n \left\langle \left(\overline{\nabla}_{F_{x_i}} F_s\right)^T, F_{x_j} \right\rangle F_{x_j} = \sum_{j=1}^n \langle F_{x_i s}, F_{x_j}\rangle F_{x_j} = \sum_{j=1}^n -\langle B(F_{x_i}, F_{x_j}), F_s\rangle F_{x_j},$$

where the last equality follows from the fact that

$$
\begin{aligned}
F_{x_i}\langle F_s, F_{x_j}\rangle = 0 &\Longrightarrow \langle F_{sx_i}, F_{x_j}\rangle + \langle F_s, F_{x_jx_i}\rangle = 0\\
&\Longrightarrow \langle F_{sx_i}, F_{x_j}\rangle = -\langle F_s, F_{x_jx_i}\rangle\\
&\Longrightarrow \langle F_{sx_i}, F_{x_j}\rangle = -\langle F_s, \overline{\nabla}_{F_{x_j}} F_{x_i}\rangle\\
&\Longrightarrow \langle F_{sx_i}, F_{x_j}\rangle = -\left\langle F_s, \left(\overline{\nabla}_{F_{x_j}} F_{x_i}\right)^T + \left(\overline{\nabla}_{F_{x_j}} F_{x_i}\right)^\perp\right\rangle\\
&\Longrightarrow \langle F_{sx_i}, F_{x_j}\rangle = -\left\langle F_s, \left(\overline{\nabla}_{F_{x_j}} F_{x_i}\right)^\perp\right\rangle\\
&\Longrightarrow \langle F_{sx_i}, F_{x_j}\rangle = -\langle F_s, B(F_{x_i}, F_{x_j})\rangle.
\end{aligned}
$$

Thus,

$$
\left\langle \left(\overline{\nabla}_{F_{x_i}} F_s\right)^T, \left(\overline{\nabla}_{F_{x_i}} F_s\right)^T \right\rangle = \sum_{j=1}^n \langle B(F_{x_i}, F_{x_j}), F_s\rangle^2,
$$

which provide us

$$
\begin{aligned}
\sum_{i=1}^n g_{ii}''(0) = {}& 2\mathrm{div}_M(F_{ss}) + \sum_{i=1}^n 2\langle R_N(F_s, F_{x_i}, F_s), F_{x_i}\rangle\\
&+ 2g^{ii}\left(\left\langle \left(\overline{\nabla}_{F_{x_i}} F_s\right)^T, \left(\overline{\nabla}_{F_{x_i}} F_s\right)^T\right\rangle + \left\langle \left(\overline{\nabla}_{F_{x_i}} F_s\right)^\perp, \left(\overline{\nabla}_{F_{x_i}} F_s\right)^\perp\right\rangle\right)\\
= {}& 2\mathrm{div}_M(F_{ss}) + \sum_{i=1}^n 2\langle R_N(F_s, F_{x_i}, F_s), F_{x_i}\rangle + 2\sum_{i,j=1}^n \langle B(F_{x_i}, F_{x_j}), F_s\rangle^2 + 2|\overline{\nabla}^\perp F_s|^2,
\end{aligned}
\tag{3.9}
$$

Recalling that (3.6), (3.7), (3.8) and (3.9), we obtain, at x, that

$$
\frac{d^2}{ds^2}\nu(s) = \left(\langle \overline{\nabla} f, F_s\rangle' - \frac{1}{2}\sum_{i,j=1}^n (g_{ij}'(0))^2 + \frac{1}{2}\sum_{i=1}^n g_{ii}''(0)\right) e^f
$$

$$
\begin{aligned}
&= \left(\lambda \langle F_s, F_s \rangle + \langle \overline{\nabla} f, F_{ss} \rangle - \sum_{i,j=1}^{n} \langle B(F_{x_i}, F_{x_j}), F_s \rangle^2 \right. \\
&\left. + |\overline{\nabla}^{\perp} F_s|^2 - \sum_{i=1}^{n} \langle R_N(F_s, E_i)E_i, F_s \rangle + \mathrm{div}_M F_{ss} \right) e^f \\
&= \left(|\overline{\nabla}^{\perp} F_s|^2 - \sum_{i,j=1}^{n} \langle B(F_{x_i}, F_{x_j}), F_s \rangle^2 + \lambda \langle F_s, F_s \rangle \right. \\
&\left. - \sum_{i=1}^{n} \langle R_N(F_s, E_i)E_i, F_s \rangle \right) e^f + (\mathrm{div}_M F_{ss} + \langle \overline{\nabla} f, F_{ss} \rangle) e^f.
\end{aligned}
$$

□

Lemma 3.3.

$$\int_M (\mathrm{div}_M F_{ss} + \langle \overline{\nabla} f, F_{ss} \rangle) = 0.$$

Proof. As $\mathbf{H} = \sum_{i=1}^{n} \overline{\nabla}_{E_i} E_i$, we get

$$
\begin{aligned}
\mathrm{div}_M F_{ss} &= \sum_{i=1}^{n} \langle \overline{\nabla}_{E_i} F_{ss}, E_i \rangle \\
&= \sum_{i=1}^{n} \langle \overline{\nabla}_{E_i} (F_{ss})^T, E_i \rangle + \sum_{i=1}^{n} \langle \overline{\nabla}_{E_i} (F_{ss})^{\perp}, E_i \rangle \\
&= \mathrm{div}_M F_{ss}^T - \sum_{i=1}^{n} \langle (F_{ss})^{\perp}, \overline{\nabla}_{E_i} E_i \rangle \\
&= \mathrm{div}_M F_{ss}^T - \langle (F_{ss})^{\perp}, \mathbf{H} \rangle \\
&= \mathrm{div}_M F_{ss}^T - \langle F_{ss}, \mathbf{H} \rangle.
\end{aligned}
$$

Recalling (2.1), we get

$$
\begin{aligned}
(\mathrm{div}_M F_{ss} + \langle \overline{\nabla} f, F_{ss} \rangle) e^f &= (\mathrm{div}_M F_{ss}^T + \langle \mathbf{X} - \mathbf{H}, F_{ss} \rangle) e^f \\
&= (\mathrm{div}_M F_{ss}^T + \langle \mathbf{X}^T, F_{ss} \rangle) e^f \\
&= (\mathrm{div}_M F_{ss}^T + \langle \nabla f, F_{ss}^T \rangle) e^f
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{i=1}^{n}\langle\overline{\nabla}_{E_i}F_{ss}^T, E_i\rangle + \left\langle \sum_{i=1}^{n}\langle\nabla f, F_{x_i}\rangle F_{x_i}, F_{ss}^T\right\rangle\right)e^f \\
&= \left(\sum_{i=1}^{n}\langle\overline{\nabla}_{E_i}F_{ss}^T, E_i\rangle + \left\langle \sum_{i=1}^{n} F_{x_i}, \langle\nabla f, F_{x_i}\rangle F_{ss}^T\right\rangle\right)e^f \\
&= \left(\sum_{i=1}^{n}\langle\overline{\nabla}_{E_i}F_{ss}^T, E_i\rangle + \sum_{i=1}^{n}\left\langle F_{x_i}, df(F_{x_i})F_{ss}^T\right\rangle\right)e^f \\
&= \left(\sum_{i=1}^{n}\langle\overline{\nabla}_{E_i}F_{ss}^T, E_i\rangle + \sum_{i=1}^{n}\left\langle E_i, df(E_i)F_{ss}^T\right\rangle\right)e^f \\
&= \left(\sum_{i=1}^{n}\langle\overline{\nabla}_{E_i}F_{ss}^T, E_i\rangle + \sum_{i=1}^{n}\left\langle E_i, \overline{\nabla}_{E_i}fF_{ss}^T\right\rangle\right)e^f \\
&= \mathrm{div}_M(e^f F_{ss}^T).
\end{aligned}
$$

By the Divergence Theorem,

$$\int_M \mathrm{div}_M(e^f F_{ss}^T) = 0.$$

□

Lemma 3.4.

$$\int_M |\nabla_M^\perp S|^2 e^f d\mu = -\int_M \left(\langle\Delta_M^\perp S, S\rangle + S^\alpha\langle\nabla_M S^\alpha, \nabla_M f\rangle\right)e^f d\mu, \tag{3.10}$$

where $\nabla_M^\perp S = E_i(S^\alpha)E_i \otimes \nu_\alpha$ and

$$\Delta_M^\perp S = \sum_{i=1}^{n}\left(\nabla_{E_i}(\nabla_{E_i}S)^\perp\right)^\perp - \sum_{i=1}^{n}\left(\nabla_{\nabla_{E_i}E_i}S\right)^\perp \tag{3.11}$$

is the Laplacian on a normal bundle.

Proof.

Claim 3.2.

$$\langle\Delta_M^\perp S, S\rangle = \frac{1}{2}\Delta_M|S|^2 - |\nabla_M^\perp S|^2.$$

Proof. Indeed, using geodesic normal coordinates, we get

$$
\begin{aligned}
\Delta_M |S|^2 &= \sum_{i=1}^{n} \nabla_{E_i} \nabla_{E_i} |S|^2 \\
&= 2 \sum_{i=1}^{n} \nabla_{E_i} \langle \nabla_{E_i} S, S \rangle \\
&= 2 \sum_{i=1}^{n} \nabla_{E_i} \langle (\nabla_{E_i} S)^{\perp}, S \rangle \\
&= 2 \sum_{i=1}^{n} \langle \nabla_{E_i} (\nabla_{E_i} S)^{\perp}, S \rangle + \langle (\nabla_{E_i} S)^{\perp}, \nabla_{E_i} S \rangle \\
&= 2 \sum_{i=1}^{n} \langle \left(\nabla_{E_i} (\nabla_{E_i} S)^{\perp} \right)^{\perp}, S \rangle + \langle (\nabla_{E_i} S)^{\perp}, (\nabla_{E_i} S)^{\perp} \rangle \\
&= 2 \langle \Delta_M^{\perp} S, S \rangle + 2 \left| (\nabla_M S)^{\perp} \right|^2 .
\end{aligned}
$$

□

If we use geodesic normal coordinates and the previous claim, we obtain

$$
\begin{aligned}
&\left(|\nabla_M^{\perp} S|^2 + \langle \Delta_M^{\perp} S, S \rangle + S^{\alpha} \langle \nabla_M S^{\alpha}, \nabla_M f \rangle \right) e^f \\
&= \frac{1}{2} \left(\Delta_M |S|^2 + \langle \nabla_M |S|^2, \nabla_M f \rangle \right) e^f \\
&= \frac{1}{2} \left(\operatorname{div}(\nabla_M |S|^2) + \langle \nabla_M |S|^2, \nabla_M f \rangle \right) e^f \\
&= \frac{1}{2} \left(\sum_{i=1}^{n} \langle \nabla_{E_i} (\nabla_{E_i} |S|^2), E_i \rangle + \sum_{i=1}^{n} \langle \nabla_M |S|^2, E_i \rangle \langle \nabla_M f, E_i \rangle \right) e^f \\
&= \frac{1}{2} \left(\sum_{i=1}^{n} \langle \nabla_{E_i} (\nabla_{E_i} |S|^2), E_i \rangle + \sum_{i=1}^{n} \langle \nabla_{E_i} |S|^2, E_i \rangle df(E_i) \right) e^f \\
&= \frac{1}{2} \left(\sum_{i=1}^{n} \langle \nabla_{E_i} (\nabla_{E_i} |S|^2), E_i \rangle e^f + \sum_{i=1}^{n} \langle \nabla_{E_i} |S|^2, E_i \rangle df(E_i) e^f \right) \\
&= \frac{1}{2} \left(\sum_{i=1}^{n} \langle \nabla_{E_i} (\nabla_{E_i} |S|^2), E_i \rangle e^f + \sum_{i=1}^{n} \langle \nabla_{E_i} |S|^2, E_i \rangle \nabla_{E_i} e^f \right) \\
&= \frac{1}{2} \left(\sum_{i=1}^{n} \langle e^f \nabla_{E_i} (\nabla_{E_i} |S|^2), E_i \rangle + \sum_{i=1}^{n} \langle \nabla_{E_i} e^f \nabla_{E_i} |S|^2, E_i \rangle \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\sum_{i=1}^{n}\langle \nabla_{E_i}(e^f \nabla_{E_i}|S|^2), E_i\rangle\right)\\
&= \frac{1}{2}\mathrm{div}(e^f \nabla_M |S|^2).
\end{aligned}
$$

Integrating it and using the Divergence Theorem, we get the result. □

Theorem 3.2 (Second variation Formula of the weighted volume functional)**.**

$$
\begin{aligned}
\frac{d^2}{ds^2}\mathcal{G}(F(M,s))\bigg|_{s=0} = \int_M \Bigg(& |\overline{\nabla}^{\perp} F_s|^2 - \sum_{i,j=1}^{n}\langle B(F_{x_i},F_{x_j}),F_s\rangle^2 + \lambda\langle F_s,F_s\rangle \\
& - \sum_{i=1}^{n}\langle R_N(F_s,E_i)E_i,F_s\rangle\Bigg) e^f d\mu \\
= & -\int_M \langle F_s, \tilde{L}F_s\rangle e^f d\mu,
\end{aligned}
$$

where the stability operator $\tilde{L}$ is defined on a normal vector field S on M by

$$
\tilde{L}S = \Delta_M^{\perp} S + \langle \nabla_M^{\perp} S, \mathbf{X}\rangle + \sum_{i=1}^{n}(R_N(S,E_i)E_i)^{\perp} + \tilde{B}(S) - \lambda S, \tag{3.12}
$$

and $\tilde{B}$ is the Simons' operator defined by

$$
\tilde{B}(S) = \sum_{i,j=1}^{n}\langle B(E_i,E_j),S\rangle B(E_i,E_j)
$$

with $S = S^{\alpha}\nu_{\alpha}$.

Proof. The second variation Formula of the weighted volume functional is

$$
\frac{d^2}{ds^2}\mathcal{G}(F(M,s))\bigg|_{s=0} = \int_M \frac{d^2}{ds^2}\nu(s)\bigg|_{s=0}\sqrt{\det(g_{ij}(0))} \tag{3.13}
$$

Using lemmas 3.2, 3.3 and 3.4 and (3.12), we get the formula desired. □

Observe that, in local coordinates, we have

$$\begin{aligned}
\sum_{i,j=1}^{n}\langle B(E_i,E_j),S\rangle^2 &= \sum_{i,j=1}^{n}(\langle B(E_i,E_j),S\rangle)(\langle B(E_i,E_j),S\rangle)\\
&= \sum_{i,j=1}^{n}\left(\left\langle B(E_i,E_j),\sum_{\alpha=n+1}^{n+p}S^{\alpha}\nu_{\alpha}\right\rangle\right)\left(\left\langle B(E_i,E_j),\sum_{\beta=n+1}^{n+p}S^{\beta}\nu_{\beta}\right\rangle\right)\\
&= \sum_{i,j=1}^{n}\sum_{\alpha,\beta=n+1}^{n+p}\langle B(E_i,E_j),S^{\alpha}\nu_{\alpha}\rangle\langle B(E_i,E_j),S^{\beta}\nu_{\beta}\rangle\\
&= \sum_{i,j=1}^{n}\sum_{\alpha,\beta=n+1}^{n+p}S^{\alpha}S^{\beta}\langle B(E_i,E_j),\nu_{\alpha}\rangle\langle B(E_i,E_j),\nu_{\beta}\rangle\\
&= \sum_{i,j=1}^{n}\sum_{\alpha,\beta=n+1}^{n+p}S^{\alpha}S^{\beta}h_{ij}^{\alpha}h_{ij}^{\beta} = \sum_{i,j=1}^{n}\langle\overline{\nabla}_{e_i}S,e_j\rangle^2.
\end{aligned}$$

Based on the second variation of the $\mathcal{G}$-functional, we define the stability of conformal solitons

Definition 3.1. A conformal soliton M^n on N^{n+p} it is said $\mathcal{G}$-stable if for every normal vector field S with compact support on M,

$$\int_M\left(\sum_{i,j=1}^{n}\langle\overline{\nabla}_{e_i}S,e_j\rangle^2+\sum_{i=1}^{n}R_N(e_i,S,S,e_i)-\lambda|S|^2\right)e^f d\mu \le \int_M\sum_{i=1}^{n}|\nabla_{e_i}^{\perp}S|^2e^f d\mu \tag{3.14}$$

or, equivalently,

$$-\int_M\langle S,\tilde{L}S\rangle e^f d\mu \ge 0, \tag{3.15}$$

where $\tilde{L}$ is defined by (3.12).

If M is a hypersurface on N, then (3.14) is simply

$$\int_M(|A|^2+\mathrm{Ric}_N(\nu,\nu)-\lambda)u^2e^f d\mu \le \int_M|\nabla u|^2e^f d\mu. \tag{3.16}$$

Indeed, $S = u\nu$ when M is a hypersurface in N. Thus,

$$\int_M \left(\sum_{i,j=1}^n \langle \overline{\nabla}_{e_i} S, e_j \rangle^2 + \sum_{i=1}^n R_N(e_i, S, S, e_i) - \lambda |S|^2 \right) e^f d\mu \leq \int_M \sum_{i=1}^n |\nabla^\perp_{e_i} S|^2 e^f d\mu$$

$$\Longleftrightarrow \int_M \left(\sum_{i,j=1}^n \langle \overline{\nabla}_{e_i} (u\nu), e_j \rangle^2 + \sum_{i=1}^n R_N(e_i, u\nu, u\nu, e_i) - \lambda |u\nu|^2 \right) e^f d\mu \leq \int_M \sum_{i=1}^n |\nabla^\perp_{e_i} (u\nu)|^2 e^f d\mu$$

$$\Longleftrightarrow \int_M \left(u^2 \sum_{i,j=1}^n \langle \overline{\nabla}_{e_i} \nu, e_j \rangle^2 + \sum_{i=1}^n u^2 R_N(e_i, \nu\nu, e_i) - \lambda u^2 \right) e^f d\mu \leq \int_M \sum_{i=1}^n |\nabla_{e_i} u|^2 e^f d\mu$$

$$\Longleftrightarrow \int_M \left(|A|^2 + \mathrm{Ric}_N(\nu, \nu) - \lambda \right) u^2 e^f d\mu \leq \int_M |\nabla u|^2 e^f d\mu.$$

$\mathcal{G}$-stability is the same notion that was proved in the Lemma 2.3 or the inequality (2.15) for the case of hypersurfaces.

We need the two identities below to deduce the stability's $\mathcal{G}$-operator.

$$\mathscr{L}u := \Delta u + \langle \nabla u, \nabla f \rangle = e^{-f} \mathrm{div}\,(e^f \nabla u)$$

and

Proposition 3.1. If $M \subset N$ is a hypersurface, u is a C^1 function with compact support and v is a C^2 function, then

$$\int_M u(\mathscr{L}v) e^f d\mu = -\int_M \langle \nabla u, \nabla v \rangle e^f d\mu$$

Proof. The proposition follows immediately from the Stokes' Theorem and the identity above. □

The two identities above, the inequality (3.16) and the observation that $\mathbf{X} = \nabla f$ done in the beginning of the section 3.1 provide us the operator of stability for conformal solitons (the stability's $\mathcal{G}$-operator)

$$\tilde{L}u = \Delta_M u + \langle \nabla u, \mathbf{X} \rangle + |A|^2 u + \mathrm{Ric}(\nu, \nu) u - \lambda u. \tag{3.17}$$

We will see some examples.

Example 3.1 (Self-shrinkers in $\mathbb{R}^{n+1}$)**.** Let $(N,g) = (\mathbb{R}^{n+1}, \delta)$ be the Euclidean space and assume that $\mathbf{X} = -\frac{1}{2}x$, where x is the position vector on $\mathbb{R}^{n+1}$, then X is a conformal vector field with $\lambda = -\frac{1}{2}$, because $\overline{\nabla}_i X_j = -\frac{1}{2}g_{ij}$. The potential function is $f = -\frac{|x|^2}{4}$ once that $\mathbf{X} = \overline{\nabla} f$ as we saw in the proof of the Theorem 2.1. In this case, the conformal soliton for $\mathbf{X}$ is the self-shrinker for the mean curvature flow which satisfies

$$\mathbf{H} = -\frac{1}{2}x^{\perp}. \tag{3.18}$$

Observe that $\mathrm{Ric}(\nu,\nu) = 0$ for the Euclidean metric, then follows from 2.3 that a self-shrinker M^n in $\mathbb{R}^{n+1}$ is stable if and only if

$$\int_M \left(|A|^2 + \frac{1}{2}\right) u^2 e^{-\frac{|x|^2}{4}} d\mu \leq \int_M |\nabla u|^2 e^{-\frac{|x|^2}{4}} d\mu \tag{3.19}$$

for every test function $u \in C_c^{\infty}(M)$. The $\mathcal{G}$-functional is

$$\int_M e^{-\frac{|x|^2}{4}} d\mu \tag{3.20}$$

for this case. The operator of stability for conformal solitons $\tilde{L}$ is

$$\tilde{L}u = \Delta_M u - \frac{1}{2}\langle \nabla u, x\rangle + |A|^2 u + \frac{1}{2}u. \tag{3.21}$$

Example 3.2 (Self-shrinkers in $\mathbb{R}^{n+p}$)**.** Let $(N,g) = (\mathbb{R}^{n+p}, \delta)$ be is the Euclidean space and assume that $\mathbf{X} = -\frac{1}{2}x$, where x is the position vector in $\mathbb{R}^{n+p}$. Analogous to the previous example, $\mathbf{X}$ is the conformal vector field in $\mathbb{R}^{n+p}$ with $\lambda = -\frac{1}{2}$ and the potential function $f = -\frac{|x|^2}{4}$ once that $\mathbf{X} = \overline{\nabla} f$ as we saw in the proof of the Theorem 2.1. In this case, the conformal soliton for the $\mathbf{X}$ is a self-shrinker for the mean curvature flow in $\mathbb{R}^{n+p}$ satisfying (3.18). Observe that $R(e_i, S, S, e_i) = 0$ for the Euclidean metric. In this case, the operator of stability for conformal solitons $\tilde{L}$ is

$$\tilde{L}S = \Delta_M^{\perp} S - \frac{1}{2}\langle \nabla_M^{\perp} S, x\rangle + \tilde{A}(S) + \frac{1}{2}S. \tag{3.22}$$

Example 3.3 (Translating solitons in $\mathbb{R}^{n+1}$)**.** Let $(N,g)=(\mathbb{R},\delta)$ be the Euclidean space and assume that $\mathbf{X}=T$ is a constant vector field on $\mathbb{R}^{n+1}$, then $\mathbf{X}$ is a conformal vector field with $\lambda=0$ and the potential function is $f=\langle T,x\rangle$ once that $\mathbf{X}=\overline{\nabla}f$ as we saw in the proof of the Theorem 2.1. The conformal soliton for $\mathbf{X}$ is the translating soliton for the mean curvature flow that satisfies

$$\mathbf{H}=T^{\perp} \tag{3.23}$$

in this case. Suppose that the tangential part of T is V so that

$$T=V+\mathbf{H}. \tag{3.24}$$

Observe that $\mathrm{Ric}(\nu,\nu)=0$, because the metric is Euclidean, then follows from 2.3 that a translating soliton M^n in $\mathbb{R}^{n+1}$ is stable if and only if

$$\int_M |A|^2u^2e^{\langle T,x\rangle}d\mu \leq \int_M |\nabla u|^2e^{\langle T,x\rangle}d\mu \tag{3.25}$$

for every test function $u\in C_c^{\infty}(M)$.

The operator of stability for conformal solitons $\tilde{L}$ is

$$\tilde{L}u=\Delta_M+\langle V,\nabla u\rangle+|A|^2u. \tag{3.26}$$

Example 3.4 (Self-expanders in $\mathbb{R}^{n+1}$)**.** Let $(N,g)=(\mathbb{R}^{n+1},\delta)$ be is the Euclidean space and assume that $\mathbf{X}=\frac{1}{2}x$, where x is the position vector in $\mathbb{R}^{n+1}$, then $\mathbf{X}$ is a conformal vector field with $\lambda=\frac{1}{2}$, because $\overline{\nabla}_iX_j=-\frac{1}{2}g_{ij}$. The potential function is $f=\frac{|x|^2}{4}$ once that $\mathbf{X}=\overline{\nabla}f$ as we saw in the proof of the Theorem 2.1. The conformal soliton for $\mathbf{X}$ is a self-shrinker for the mean curvature flow that satisfies

$$\mathbf{H}=\frac{1}{2}x^{\perp} \tag{3.27}$$

in this case. Observe that $\mathrm{Ric}(\nu,\nu)=0$ for the Euclidean metric, then follows from 2.3 that a self-shrinker M^n in $\mathbb{R}^{n+1}$ is stable if and only if

$$\int_M \left(|A|^2+\frac{1}{2}\right)u^2 e^{-\frac{|x|^2}{4}}d\mu \leq \int_M |\nabla u|^2 e^{-\frac{|x|^2}{4}}d\mu \tag{3.28}$$

for every test function $u\in C_c^\infty(M)$. The operator of stability for conformal solitons $\tilde{L}$ is

$$\tilde{L}u=\Delta_M u+\frac{1}{2}\langle x,\nabla u\rangle+|A|^2u-\frac{1}{2}u. \tag{3.29}$$

3.2 On the stability of conformal solitons in $\mathbb{R}^{n+1}$.

In this section, we provide a description of conformal solitons satisfying $\overline{\nabla}_j X_i=\lambda g_{ij}$ for some smooth function λ. We saw in the previous section that the self-similar solutions and the translating solitons are conformal solitons on an Euclidean space. We show that theses solitons are the unique conformal solitons in $\mathbb{R}^{n+p}$. We also consider in this section the stability of self-shrinkers and translating solitons.

Proposition 3.2. Every conformal soliton in $\mathbb{R}^{n+p}$ satisfying $\overline{\nabla}_j X_i=\lambda g_{ij}$ for some smooth function λ must be a self-shrinker, self-expander or translating soliton.

Proof. Suppose that $\mathbf{X}=\sum_{i=1}^{n+p}X^A e_A$, where $\{e_A\}_{1\leq A\leq n+p}$ is the canonical frame in $\mathbb{R}^{n+p}$, then

$$\frac{\partial X^A}{\partial x_B}=\lambda\delta_{AB}$$

by hypothesis which implies

$$X^A=\lambda x_A+\mu_A.$$

Thus, $\mathbf{X}=\lambda x+\mu$ on what $\mu=(\mu_1,\cdots,\mu_{n+p})\in\mathbb{R}^{n+p}$ is a fixed vector.

If $\lambda=0$, then $\mathbf{X}=\mu$ is a fixed vector in $\mathbb{R}^{n+p}$. As the conformal soliton satisfies the equation $\mathbf{X}^\perp=\mathbf{H}$ and μ is fixed, the soliton must be a translating soliton.

If $\lambda \neq 0$, then $\mathbf{X} = \lambda\left(x + \frac{\mu}{\lambda}\right)$. The equation of the conformal soliton is

$$\mathbf{H} = \lambda\left(x + \frac{\mu}{\lambda}\right)^{\perp}.$$

This is an equation for a self-shrinker if $\lambda < 0$ or for a self-expander if $\lambda > 0$ centered in $-\frac{\mu}{\lambda}$. □

Theorem 3.3. Every compact self-shrinker M in $\mathbb{R}^{n+1}$ is unstable.

Proof. We know from 3.1 that M is stable if and only if

$$\int_M \left(|A|^2 + \frac{1}{2}\right) u^2 e^{-\frac{|x|^2}{4}} d\mu \leq \int_M |\nabla u|^2 e^{-\frac{|x|^2}{4}} d\mu$$

for every test function $u \in C_c^{\infty}(M)$.

Suppose by contradiction that M is stable. Observe that constant functions defined on M are test functions on M once that M is compact, but these functions do not satisfy the inequality above, otherwise,

$$0 < \int_M \left(|A|^2 + \frac{1}{2}\right) u^2 e^{-\frac{|x|^2}{4}} d\mu \leq \int_M |\nabla u|^2 e^{-\frac{|x|^2}{4}} d\mu = 0,$$

which is a contradiction, therefore M is unstable. □

3.2.1 Stable translating solitons in $\mathbb{R}^{n+1}$

The "grim reaper" in $\mathbb{R}^2$ is defined by

$$\begin{aligned} F : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) &\longrightarrow \mathbb{R}^2 \\ x &\mapsto (x, -\log(\cos x)). \end{aligned}$$

Lemma 3.5. The "grim reaper" is the only translating soliton of the mean curvature flow in $\mathbb{R}^2$.

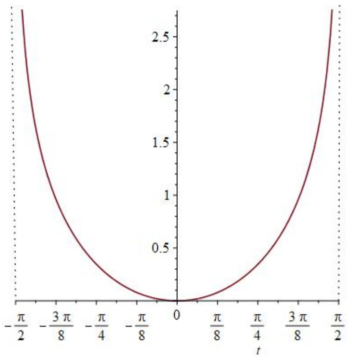

Fig. 3.1 "Grim reaper" in $\mathbb{R}^2$.

Proof. Let $I \subset \mathbb{R}$ be an open set and $f : I \longrightarrow \mathbb{R}^2$ a curve. The curvature of the curve is

$$\kappa(x) = \frac{f''(x)}{(1+(f'(x))^2)^{\frac{3}{2}}}$$

and its normal is

$$\nu(x) = \frac{(-f'(x),1)}{(1+(f'(x))^2)^{\frac{1}{2}}}.$$

Thus, if such curve is a translating soliton of the mean curvature flow, then it must satisfies

$$\frac{f''(x)}{(1+(f'(x))^2)^{\frac{3}{2}}} = \left\langle (u_1,u_2), \frac{(-f'(x),1)}{(1+(f'(x))^2)^{\frac{1}{2}}} \right\rangle,$$

where $u = (u_1,u_2) \in \mathbb{R}^2$ is a fixed vector.

Choosing $u = (1,0)$,

$$\frac{f''(x)}{1+(f'(x))^2} = -f'(x)$$

and integrating it, we get

$$\arctan(f'(x)) = -f(x),$$

therefore

$$f'(x) = \tan(-f(x)) = -\tan(f(x)) \iff \frac{\cos(f(x))}{\sin(f(x))} f'(x) = -1.$$

Integrating it,

$$\log(\cos(f(x))) = -x,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Apply a rotation of $\dfrac{\pi}{2}$ to the graph of f, followed by the change of variables $x \mapsto -x$ and the fact that $\log(\cos x)$ is an even function to get the "grim reaper".

The uniqueness of the "grim reaper" as a translating soliton of the mean curvature flow in $\mathbb{R}^2$ follows from the Theorem of existence and uniqueness of ODEs. $\square$

Theorem 3.4. The "grim reaper" is a stable translating soliton in $\mathbb{R}^2$.

Proof. Let $T = e_2$ be the direction of the translation. Keeping in mind the example 3.3, $f(F(x)) = \langle T, F(x)\rangle = \langle (0,1), (x, -\log(\cos x))\rangle = -\log(\cos x)$.

We do some computations in the sequence. The tangent vector is $F_x = (1, \tan x)$, the induced metric is $g_{xx} = \langle F_x, F_x\rangle = 1 + \tan^2 x = \dfrac{1}{\cos^2 x}$ and the induced volume form is $d\mu = \dfrac{1}{\cos x}dx$. If $\nu = (\sin x, -\cos x)$, then $h_{xx} = -\langle F_{xx}, \nu\rangle = -\dfrac{1}{\cos x}$ and $|A|^2 = H^2 = (g^{xx}h_{xx})^2 = \cos^2 x$. As observed previously, $f(F(x)) = -\log(\cos x)$, i.e., $e^{f(F(x))} = \dfrac{1}{\cos x}$, therefore

$$\int_M |A|^2 u^2 e^{\langle T,x\rangle} d\mu = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 x) u^2 \frac{1}{\cos x}\frac{1}{\cos x} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u^2 dx.$$

Observe that

$$|\nabla u|^2 = g^{xx}\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} = \cos^2 x (u'(x))^2,$$

therefore

$$\int_M |\nabla u|^2 e^{\langle T,x\rangle} d\mu = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x (u'(x))^2 \frac{1}{\cos x}\frac{1}{\cos x} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (u'(x))^2 dx.$$

Thus, given $u \in C_c^\infty\left(\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)\right)$,

$$\int_M |A|^2 u^2 e^{\langle T,x\rangle} d\mu = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (u(x))^2 dx \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (u'(x))^2 dx = \int_M |\nabla u|^2 e^{\langle T,x\rangle} d\mu,$$

where the inequality follows from the Proposition 1.4. This shows (3.25), which is the stability condition for translating solitons in $\mathbb{R}^{n+1}$. $\square$

The "grim reaper" cylinder is given by

$$F:\mathbb{R}^{n-1}\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\longrightarrow\mathbb{R}^{n+1}$$
$$(x_1,\cdots,x_n)\mapsto(x_1,\cdots,x_{n-1},x_n,-\log(\cos x_n)).$$

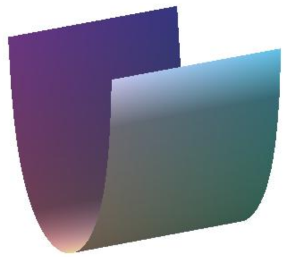

Fig. 3.2 "Grim reaper" cylinder in $\mathbb{R}^{n+1}$.

Lemma 3.6. The "grim reaper" cylinder $\mathbb{R}^{n-1}\times\Gamma$ is a translating soliton of the mean curvature flow in $\mathbb{R}^{n+1}$, where Γ is the "grim reaper" in $\mathbb{R}^2$.

Proof. Let $I\subset\mathbb{R}$ be an open set and $f:I\longrightarrow\mathbb{R}^2$ a curve. We will show that the parametrization given by

$$F:\mathbb{R}^{n-1}\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\longrightarrow\mathbb{R}^{n+1}$$
$$(x_1,\cdots,x_n)\mapsto(x_1,\cdots,x_{n-1},x_n,f(x_n)).$$

is a "grim reaper" cylinder which is a translating soliton. Indeed, The normal of the hypersurface at a point $x=(x_1,\cdots,x_n)$ is

$$\nu(x)=\frac{1}{(1+(f'(x_n))^2)^{\frac{1}{2}}}(0,\cdots,0,0,-f'(x_n),1).$$

We can see that

$$\begin{cases} g_{ij}(x) & = 0, \ 1 \le i < j \le n \\ g_{ij}(x) & = 1, \ 1 \le i = j < n \\ g_{ij}(x) & = 1 + (f'(x_n))^2, \ i = j = n \end{cases}$$

and

$$\begin{cases} \left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}(x), \nu(x) \right\rangle & = 0, \ 1 \le i \le j \le n-1 \\ \left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}(x), \nu(x) \right\rangle & = 0, \ 1 \le i < j \le n-1 \\ \left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}(x), \nu(x) \right\rangle & = 0, \ 1 \le i < j \le n \\ \left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}(x), \nu(x) \right\rangle & = \frac{f''(x_n)}{(1 + (f'(x_n))^2)^{\frac{1}{2}}}, \ i = j = n. \end{cases}$$

Recalling that $h_{ij} = -\left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}(x), \nu(x) \right\rangle$, we get

$$H(x) = g^{nn} h_{nn} = -\frac{f''(x_n)}{(1 + (f'(x_n))^2)^{\frac{3}{2}}}.$$

Now, the proof follows from the previous lemma. □

Next we consider the stability of the grim-reaper cylinder. The proof that this hypersurface is a stable translating soliton (which was omitted in [2]) follows the previous results, with some adaptations.

Theorem 3.5. The "grim reaper" cylinder $\mathbb{R}^{n-1} \times \Gamma$ is a stable translating soliton in $\mathbb{R}^{n+1}$, where Γ is the "grim reaper" in $\mathbb{R}^2$.

Proof. Let $x = (x_1, \cdots, x_n)$ and $u \in C_c^\infty\left(\mathbb{R}^{n-1} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$. As u is a test function, we can choose a $(n-1)$-cube Q which size has length l depending on u such that

$$\operatorname{supp} u \subset Q \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{3.30}$$

We do some computations for the "grim reaper" cylinder.

$$\begin{cases} g_{ij}(x) & = 0,\ 1 \le i < j \le n \\ g_{ij}(x) & = 1,\ 1 \le i = j < n \\ g_{ij}(x) & = 1 + \tan^2(x_n) = \sec^2(x_n),\ i = j = n, \end{cases}$$

$$\nu(x) = \frac{1}{\sec(x_n)}\left(0, \cdots, 0, 0, -\tan(x_n), 1\right).$$

and

$$\begin{cases} \left\langle \dfrac{\partial^2 F}{\partial x_i \partial x_j}(x), \nu(x) \right\rangle & = 0,\ 1 \le i \le j \le n-1 \\ \left\langle \dfrac{\partial^2 F}{\partial x_i \partial x_j}(x), \nu(x) \right\rangle & = 0,\ 1 \le i < j \le n-1 \\ \left\langle \dfrac{\partial^2 F}{\partial x_i \partial x_j}(x), \nu(x) \right\rangle & = 0,\ 1 \le i < j \le n \\ \left\langle \dfrac{\partial^2 F}{\partial x_i \partial x_j}(x), \nu(x) \right\rangle & = \sec(x_n),\ i = j = n \end{cases}$$

Recalling that $h_{ij} = -\left\langle \dfrac{\partial^2 F}{\partial x_i \partial x_j}(x), \nu(x) \right\rangle$, we get

$$|A|^2 = (g^{iq} g^{pj} h_{iq} h_{pj})^2 = (g^{nn} h_{nn})^2 = \cos^2(x_n).$$

Let $T = e_{n+1}$ be the direction of the translation. Keeping in mind the example 3.3,

$$f(F(x)) = \langle T, F(x) \rangle = \langle (0, \cdots, 0, 0, 1), (x_1, \cdots, x_{n-1}, x_n, -\log(\cos x_n)) \rangle = -\log(\cos x_n).$$

Arguing similarly to the previous theorem,

$$|A|^2 u^2 e^{\langle T, F(x) \rangle} d\mu = \left(\cos^2(x_n)\right) u^2 \left(\frac{1}{\cos x_n}\right) \left(\frac{1}{\cos x_n} dx_1 \cdots dx_n\right) = u^2 dx_1 \cdots dx_n$$

and

$$
\begin{aligned}
|\nabla u|^2 e^{\langle T,F(x)\rangle} d\mu &= \left(\sum_{i=1}^{n-1}\left(\frac{\partial u}{\partial x_i}\right)^2 + \left(\cos(x_n)\frac{\partial u}{\partial x_n}\right)^2\right)\left(\frac{1}{\cos x_n}\right)\left(\frac{1}{\cos x_n} dx_1\cdots dx_n\right)\\
&= \left(\sum_{i=1}^{n-1}\left(\sec(x_n)\frac{\partial u}{\partial x_i}\right)^2 + \left(\frac{\partial u}{\partial x_n}\right)^2\right) dx_1\cdots dx_n.
\end{aligned}
$$

Proposition 1.4 gives

$$
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u^2 dx_n \le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\partial u}{\partial x_n}\right)^2 dx_n.
$$

Recalling (3.30) and integrating the inequality above,

$$
\begin{aligned}
\int_{\mathbb{R}^{n-1}\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)} |A|^2 u^2 e^{\langle T,F(x)\rangle} d\mu &= \int_Q \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u^2 dx_n dx_1\cdots dx_{n-1}\\
&\le \int_Q \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\partial u}{\partial x_n}\right)^2 dx_n dx_1\cdots dx_{n-1}\\
&\le \int_Q \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sum_{i=1}^{n-1}\left(\frac{\partial u}{\partial x_i}\right)^2 + \left(\frac{\partial u}{\partial x_n}\right)^2\right) dx_n dx_1\cdots dx_{n-1}\\
&\le \int_Q \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sum_{i=1}^{n-1}\left(\sec(x_n)\frac{\partial u}{\partial x_i}\right)^2 + \left(\frac{\partial u}{\partial x_n}\right)^2\right) dx_n dx_1\cdots dx_{n-1}\\
&= \int_{\mathbb{R}^{n-1}\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)} |\nabla u|^2 e^{\langle T,F(x)\rangle} d\mu.
\end{aligned}
$$

This shows inequality (3.25), which is the stability condition for translating solitons in $\mathbb{R}^{n+1}$.

$\square$

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