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Massera's theorem for generalized ODEs and applications

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Resumo

Nesse trabalho, o objetivo principal é provar uma versão do Teorema de Massera para as equações diferenciais ordinárias generalizadas (EDOs generalizadas). Tal teorema fornece condições para garantir a existência de soluções periódicas para equações diferenciais quando há uma solução limitada. Além de estudar esse resultado para EDOs generalizadas, usamos as correspondências entre essas equações e as equações diferenciais em medida, equações diferenciais com impulso e equações dinâmicas em escalas temporais para obter versões do Teorema de Massara para cada uma dessas equações. Esses resultados são novos na literatura e podem ser encontrados em [14].

Abstract

In this work, the main objective is to prove a version of Massera's Theorem for generalized ordinary differential equations (generalized ODEs). Such theorem provides conditions to guarantee the existence of periodic solutions for differential equations when there is a bounded solution. Besides studying this result for generalized ODEs, we use the correspondences between these equations and the measure differential equations, impulse differential equations and dynamic equations on time scales to obtain versions of Massara's Theorem for each of those equations. All these results are new in the literature and they can be found in [14].

Contents

Introduction

The pendulum has been an object of study of many physicists and mathematicians over the last few centuries. It is quite often an example of equation describing its behavior in many ODE books, as in [15]. It is a known fact that its movement can be described by the equation

$$\ddot{\theta} = \frac{g}{L}\sin(\theta),$$

where $g \in (0,\infty)$ represents the acceleration of the gravity, $L \in (0,\infty)$ is the length of the pendulum and $\theta : \mathbb{R} \to \mathbb{R}$ is the angle which the pendulum makes with the vertical at each instant of time t.

Throughout the years, many other types of pendulums have been studied. One of them is the Kapitza pendulum, studied originally by Stephenson Andrew in [38]. This object had not only a mass moving around the support, but also the support itself was oscillating at a high frequency $\omega \in (0,\infty)$. Figure 1, presented below, represents such object.

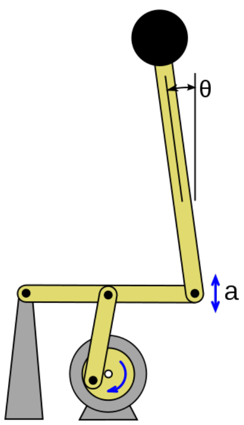

Figure 1 Kapitza pendulum, from [10].

A very interesting property of the Kapitza pendulum concerns its equilibrium points. As a consequence of the very high oscillation ω, the point where the mass is exactly above the support, that would be normally an unstable equilibrium position in a classical pendulum, turns to be a stable equilibrium position. Only in 1951 that the first proper explanation for this phenomenon was given by the Nobel laureate physicist Pyotr Kapitza in [20] and [21]. He was able to describe this movement using the following equation:

$$\ddot{\theta} = \frac{g}{L}\sin(\theta) - \frac{a\omega^2 \sin(\omega t)}{L}\sin(\theta), \tag{1}$$

where $g \in (0,\infty)$ is the acceleration of the gravity, $L \in (0,\infty)$ is the length of the pendulum, $a \in (0,\infty)$ is the amplitude that the support is vibrating and $\theta \colon \mathbb{R} \to \mathbb{R}$ is the angle which the pendulum makes with the vertical when the mass is placed upwards.

Equation (1) presents some interesting challenges for the mathematicians. Due to the high oscillation ω, the solution of this problem could not be found using the Riemann or even Lebesgue integral. Jaroslav Kurzweil, motivated by this type of equation, constructed a new type of integral in 1957 ([22]). In his latest book [23], Kurzweil also presents the Kapitza pendulum from the mathematics perspective. Independently, Ralph Henstock also arrived in an equivalent formulation of this integral in 1961 ([17]). Due to the contribution of both mathematicians, this integral is now known as the Henstock–Kurzweil integral.

The new integral attracted many mathematicians not only due to its possibility to integrate more functions, but also because of the simplicity involved in its definition. The Lebesgue integral, for example, requires a very robust measure theory to be well–defined, as done in [35]. On the other hand, the Henstock–Kurzweil integral uses the same idea of partition applied in the Riemann integral. The main difference, as shown in [3], is that the first type of integral uses an auxiliary function, called **gauge**, to control the size of the partition that is used to integrate.

The Henstock–Kurzweil integral was then further generalized in what is called the Kurzweil integral, presented in [37]. The new integral uses the same idea of controlling the partition with a gauge, but it also includes more integrals, such as the Stieltjes–type of integral. Using this more general integral, it is possible to introduce a class of equations called **generalized ODEs**, which are integral equations. This equation was originally constructed in [22] and is presented in [37] and in Chapter 1 of this work.

Motivated by this type of integral, we study another interesting result, the Massera's Theorem. Before we talk more about the relation between this problem and the Henstock–Kurzweil integral, let us present the classical Massera's Theorem for ODEs. In 1950, Massera published results concerning the existence of periodic solutions of ODEs (see [26]). He proved that the existence of a bounded solution also implies the existence of a periodic solution in an 1–dimensional system $x' = f(x,t)$, where $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $x\colon \mathbb{R} \to \mathbb{R}$.

Furthermore, he also showed that such bounded solution gets closer to the periodic solution as the time increases.

On the other hand, Massera also proved, in the same article, that the result was no longer valid for systems in higher dimensions. Nevertheless, he showed that the existence of a bounded solution for linear systems of order n, $n \in \mathbb{N}$, also implies the existence of a periodic solution. He also presented an example to prove that the bounded solution does not need to get closer to the periodic solution. In the last chapter of this work, we present such example, as well as the theorems that were originally obtained.

Since then, the problem has been further generalized due to its relevance (see [4] and [25], for example). In this work, we study this problem and obtain its extensions for the generalized ODEs. All these results are new in the literature and were published in [14] and there are no similar theorems for this class of equations. Besides that, the proof of such results are not analogous to the original proof. This is because the solutions of a generalized ODE, unlike the solutions of an ODE, can be discontinuous. This leads us to some interesting mathematical challenges to be overcome.

Furthermore, [14] also shows another reason why the generalized ODEs are so attractive. Once we have this result in hands, we can also obtain the Massera's Theorem for other types of differential equations. Since the generalized ODEs are very general, they include other types of equations such as measure DEs ([37]), dynamic equations on time scales ([13]) and impulsive differential equations ([30]). This is also shown in Chapter 3 of this dissertation.

Therefore, we also proved versions of Massera's Theorem for each of this differential equations (see [14] and also Chapter 4 of this work). There were no versions of Massera's Theorem for measure DEs in the literature and, thus, the results presented in [14] are completely new for these equations.

Regarding the impulsive DEs, there are a few results about the Massera's Theorem in this class of equations (see [1] and [19]). On the other hand, since our results are derived from the generalized ODEs, our results have weaker conditions and also, due to the fact that we are dealing with Henstock–Kurzweil integrals, our functions can be highly oscillating. Lastly, we also obtained version of Massera's Theorem for the dynamic equations on time scales. Before we can talk more about that, let us explain what are the time scales and the dynamic equations on time scales.

In 1988, Stefan Hilger introduced the concept of a time scale in his PhD thesis [18]. Hilger's idea was to construct a theory that encompasses the discrete and continuous analysis. For that, he defined a time scale, which is any nonempty closed subset of the real numbers. One of the advantages of this theory is the unification and extension of the results from the continuous and discrete analysis. Therefore, it prevents the results from being proved twice and also, it allows us to obtain results for ″hybrid″ cases. Such theory is presented in [5], [6] and also in Chapter 2 of this dissertation.

With this theory, Hilger introduced the so called dynamic equations on time scales, which are differential equations using the time scales as a domain. Concerning the classical differential equations, the continuous differential equations would be the ODEs while the discrete differential equations are the difference equations. As the ″hybrid″ case, it can be used in many applications such as to describe insect populations that may stay dormant during the winter (discrete case) while the model can be continuous outside that period. Such applications are presented in [5].

Since 1988, the theory of time scales aroused the interest of many mathematicians, due to its aplications (see [1, 4–7, 13, 25, 34]). Thus, many interesting problems were proved for this area in all of these articles and books. One of these problems is the Massera's Theorem, which has versions for the dynamic equations on time scales, such as [4] and [25].

In our work, we also obtained new versions of the Massera's Theorem for dynamic equations on time scales, presented in [14]. However, we deal with periodic time scales for this problem and, therefore, it is very different from the time scales that appear in [4], which presents a version of Massera's Theorem for a specific type of time scale, that is, $q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, $q > 1$, the quantum scale. On the other hand, in [25], although the main focus be on the periodic time scales, the results in [14] have weaker assumptions about the regularity of the involved functions, due to the fact that it was generated from the generalized ODEs.

Therefore, this dissertation shows some very interesting and general results about the Massera's Theorem using original techniques. It also makes clear why the generalized ODEs are so relevant, since it capable of working with such general results.

Chapter 1

Generalized Ordinary Differential Equations

The concept of an integral has been developed by many mathematicians over the last few centuries. One of these types of integrals aroused from the necessity to integrate functions with high oscillations and it was first studied by Jaroslav Kurzweil ([22]) and Ralph Henstock ([17]). It attracted many mathematicians due to its simplicity and very interesting properties. Such integral is known as Henstock–Kurzweil integral. This integral was later generalized into what is presented here as the Kurzweil integral, in order to include order types of integrals (see [37]).

In [22], Kurzweil also showed why an ordinary differential equation or even a measure differential equation using the Lebesgue integral was not sufficient to solve problems involving rapidly oscillating external forces. Because of that, he introduced the concept of a **generalized ordinary differential equation**. Such concept is also presented here.

One of the mathematicians that also became very important in the area was Štefan Schwabik. In addition to the various articles published in the area, Schwabik also wrote one of the main books about the Kurzweil integral, [37]. This book is the main reference of this chapter and also, for the ones who are interested to learn more about this theory. Here, we start by defining and showing some basic properties of the so called **Kurzweil integral**. Next, we present the definition of the generalized ODEs. Finally, we show some results concerning the existence and uniqueness of solutions of such differential equations. All of these results and definitions will be used throughout Chapters 3 and 4 and can be found in [37]. Also, in this chapter, we bring the results in a didactic way in order to provide a good text for the ones who work in the area.

1.1 Kurzweil Integral

1.1.1 Definition of the integral

A pair (τ, J) of a point $\tau \in \mathbb{R}$ and a compact interval $J \subset \mathbb{R}$ is called a **tagged interval** and τ is called a **tag** of J. Consider an interval $[a,b] \subset \mathbb{R}$ such that $-\infty < a < b < +\infty$. The finite collection of tagged intervals $\Delta = \{(\tau_i, J_i), i = 1, \dots, k\}$ is called a **partition** of $[a,b]$ if the following conditions are satisfied:

1. $\tau_i \in J_i \subset [a,b]$ for every $i = 1, \dots, k$;

2. $Int(J_i) \cap Int(J_j) = \emptyset$ for every $i \neq j$ and $i, j \in \{1, \dots, k\}$;

3. $\bigcup_{i=1}^{k} J_i = [a,b]$.

A **gauge** on $[a,b]$ is any positive function $\delta\colon\ [a,b] \to (0, +\infty)$. Considering such a function, a partition $\Delta = \{(\tau_i, J_i), i = 1, \dots, k\}$ of $[a,b]$ is called **δ–fine** if

$$J_i \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$$

for every $i = 1, \dots, k$.

With the definitions above, it is possible to characterize the Kurzweil integrable functions.

Definition 1.1 ([37, Definition 1.2n])**.** *A function $U\colon\ [a,b] \times [a,b] \to \mathbb{R}^n$ is called **Kurzweil integrable** over $[a,b]$ if there exists an $I \in \mathbb{R}^n$ such that for every $\varepsilon > 0$, there exists a gauge $\delta(\varepsilon)\colon\ [a,b] \to (0, +\infty)$ such that*

$$\left\| \sum_{i=1}^{k} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})] - I \right\| < \varepsilon$$

for every δ–fine partition

$$D = \{(\tau_i, [\alpha_{i-1}, \alpha_i]), i = 1, \dots, k\}.$$

We usually denote $\delta(\varepsilon)$ as δ and the sum $\sum_{i=1}^{k} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})]$ as $S(U, D)$ when there is no risk of confusion. It is also common to write the partition as

$$D = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}.$$

In the definition above, $I \in \mathbb{R}^n$ is called the **Kurzweil integral of U** over the interval $[a,b]$ and will be denoted as $\int_a^b DU(\tau,t)$. If such integral exists, we also define $\int_b^a DU(\tau,t) = -\int_a^b DU(\tau,t)$ and $\int_a^b DU(\tau,t) = 0$ if $a = b$. We are also going to denote as $\mathcal{K}([a,b],\mathbb{R}^n)$, or simply $\mathcal{K}([a,b])$ when the codomain is clear, for the class of Kurzweil integrable functions on $[a,b]$.

A problem the reader may be concerned when reading the definition of the Kurzweil integrable functions is about the existence of a δ–fine partition given a gauge δ. For instance, one may think that the definition could be satisfied only by the empty set and this could result in other problems. The next lemma shows that this is not the case.

Lemma 1.2 (Cousin Lemma - [27])**.** *Given a gauge δ on $[a,b]$, there is a δ–fine partition of $[a,b]$.*

Proof. Define $E \subset [a,b]$ as the set of all points $a < x \leqslant b$ such that there exists a δ–fine partition of $[a,x]$. Note first that $E \neq \emptyset$ because for any $x \in (a, a+\delta(a))$, it is possible to set the partition $\Delta = \{(a,[a,x])\}$ on the interval $[a,x]$, proving that $(a, a+\delta(a)) \subset E$.

Define $u = \sup E$. Our goal is to show that $u \in E$ and $u = b$ to conclude the result. For the first part, observe that by the definition of supremum, there exists an $y \in E$ such that $u - \delta(u) < y \leqslant u$. By definition of E, there is a partition of $[a,y]$ and it is possible to add to such a partition the set $\{(u,[y,u])\}$, converting it into a partition of $[a,u]$ and, thus, proving that $u \in E$.

Now suppose, by contradiction, that $u < b$. Let $p \in (u, u+\delta(u)) \cap (u,b)$ be any point in that interval. It is possible to construct a partition of $[a,p]$ considering a partition of $[a,u]$ and then, adding $\{(u,[u,p])\}$. Therefore, $p \in E$ and $u < p$, which is a contradiction, since u is the supremum of E. $\square$

By Lemma 1.2, the Kurzweil integral is well–defined. One of the first questions that may arise now about this integral, is how it relates to other types of integrals. Notice first that if $\delta(t) = \delta$ is a constant and $U(\tau,t) = f(\tau)t$, then the definition above corresponds exactly to the Riemann integral. Also, note that, in this case, we have

$$S(U,D) = \sum_{i=1}^{k} [U(\tau_i,\alpha_i) - U(\tau_i,\alpha_{i-1})] = \sum_{i=1}^{k} [f(\tau_i)(\alpha_i - \alpha_{i-1})]$$

for any δ–fine partition D. Thus, it is clear that any Riemann integrable function is also Kurzweil integrable. Since the case $U(\tau,t) = f(\tau)t$ may appear often in this dissertation, we fix the following notation:

$$\int_a^b DU(\tau,t) = \int_a^b D[f(\tau)t] = \int_a^b f(t)\mathrm{d}t.$$

Similarly, the Henstock–Kurzweil–Stieltjes integral $U(\tau,t) = f(\tau)g(t)$ is also included in the notion of Kurzweil integrable functions. In this case, we are going to use the notation:

$$\int_a^b DU(\tau,t) = \int_a^b D[f(\tau)g(t)] = \int_a^b f(t)\mathrm{d}g(t).$$

A follow–up question that can appear from the latest paragraphs is if any Kurzweil integrable function is also Riemann integrable. The example below shows that this is not the case.

Example 1.3 ([3, Example 2.3])**.** *The Dirichlet function* $f\colon [0,1] \to [0,1]$ *given by*

$$f(t) = \begin{cases} 1, & \text{if } t \in [0,1] \cap \mathbb{Q}, \\ 0, & \text{otherwise}, \end{cases}$$

is not Riemann integrable, but it is Kurzweil integrable.

Proof. We can see that the Dirichlet function is not Riemann integrable by applying the well–known fact that a function f is Riemann integrable if, and only if, f is continuous almost everywhere.

Now, we show that f is Kurzweil integrable. First, we enumerate all rationals in $[0,1]$ in some order $\{r_n\colon n \in \mathbb{N}\}$. Then, define the gauge $\delta\colon [0,1] \to (0,+\infty)$ as

$$\delta(t) = \begin{cases} \dfrac{\varepsilon}{2^{n+1}}, & \text{if } t = r_n, \\ 1, & \text{otherwise}, \end{cases}$$

for any given $\varepsilon > 0$.

Consider an arbitrary δ–fine partition $D = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$. Notice that any terms of the sum $S(f,D)$ that have irrational tags τ_i are irrelevant, because $f(\tau_i) = 0$ in this case. Let us denote by τ_{i_r} the tags of D that are rational. Therefore,

$$[\alpha_{i_r-1}, \alpha_{i_r}] \subset (\tau_{i_r} - \delta(\tau_{i_r}), \tau_{i_r} + \delta(\tau_{i_r}))$$

implies that

$$(\alpha_{i_r} - \alpha_{i_r-1}) < 2\delta(\tau_{i_r}) = \frac{\varepsilon}{2^{i_r}}.$$

It could happen that $\tau_{i_r} = \tau_{i_r+1}$ for some i_r. But in this case, we would have

$$(\alpha_{i_r+1} - \alpha_{i_r-1}) < 2\delta(\tau_{i_r}) = \frac{\varepsilon}{2^{i_r}}.$$

Putting all the information above together, we conclude that

$$\|S(f,D)-0\| = \left\| \sum_{i_r=1}^{k} f(\tau_{i_r})(\alpha_{i_r}-\alpha_{i_r-1}) \right\| < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$$

Thus,

$$\int_0^1 D[f(\tau)t] = \int_0^1 f(t)\mathrm{d}t = 0,$$

and the result follows. □

We do not show in details the relation between the Kurzweil and Lebesgue integral in this dissertation, but the reader can find in [3] that any Lebesgue integrable function is also Kurzweil integrable, however the reciprocal is not true.

1.1.2 Basic Properties

In this subsection, we present some basic properties of the Kurzweil integral. Many of them are expected, since it appears in other types of integrals. The first one is the linearity of the integral.

Theorem 1.4 (Linearity - [37, Theorem 1.9]). *Consider two functions $U,V \in \mathcal{K}([a,b])$ and $c_1, c_2 \in \mathbb{R}$. Then $c_1U + c_2V \in \mathcal{K}([a,b])$ and*

$$\int_a^b D[c_1U(\tau,t)+c_2V(\tau,t)] = c_1\int_a^b DU(\tau,t)+c_2\int_a^b DV(\tau,t).$$

Proof. Given $\varepsilon > 0$, there is a gauge $\delta_U\colon [a,b] \to (0,+\infty)$ such that for every δ_U–fine partition D_U of $[a,b]$, we obtain

$$\left\| S(U,D_U) - \int_a^b DU(\tau,t) \right\| < \frac{\varepsilon}{2}.$$

Similarly, there is a gauge $\delta_V\colon [a,b] \to (0,+\infty)$ such that for every δ_V–fine partition D_V of $[a,b]$, the inequality below holds:

$$\left\| S(V,D_V) - \int_a^b DV(\tau,t) \right\| < \frac{\varepsilon}{2}.$$

It is easy to see that $S(c_1U + c_2V, D) = c_1S(U,D) + c_2S(V,D)$ for any partition D of $[a,b]$. Now, consider $\delta\colon [a,b] \to (0,+\infty)$ such that $0 < \delta(t) \leqslant \min\{\delta_U(t), \delta_V(t)\}$ for every

$t \in [a,b]$. For any δ–fine partition D, we get

$$\begin{aligned}
&\left\| S(c_1U + c_2V, D) - c_1 \int_a^b DU(\tau,t) - c_2 \int_a^b DV(\tau,t) \right\| \\
&\leqslant \left\| c_1 S(U,D) - c_1 \int_a^b DU(\tau,t) \right\| + \left\| c_2 S(V,D) - c_2 \int_a^b DV(\tau,t) \right\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

and the proof is complete. □

The theorem below shows that instead of working with $U\colon [a,b]\times[a,b] \to \mathbb{R}^n$, we could consider only $U\colon [a,b]\times[a,b] \to \mathbb{R}$.

Theorem 1.5 ([37, Theorem 1.6]). *A function $U\colon [a,b]\times[a,b] \to \mathbb{R}^n$ denoted as $U = (U_1, U_2, \ldots, U_n)$ is Kurzweil integrable if and only if each component U_j, $j = 1,2,\ldots,n$ is Kurzweil integrable.*

Proof. Suppose first that $\int_a^b DU(\tau,t) = I = (I_1, I_2, \ldots, I_n)$ exists. Then, given $\varepsilon > 0$, there is a gauge $\delta\colon [a,b] \to (0,+\infty)$ such that for every δ–fine partition D of $[a,b]$, we have

$$\|S(U,D) - I\| < \varepsilon.$$

It is easy to see that if we define as D_m the projection of D in the m-th coordinate, then

$$|S(U_m, D_m) - I_m| \leqslant \|S(U,D) - I\| < \varepsilon.$$

Since m is chosen arbitrarily, we can conclude one of the implications of the theorem.

To show the other implication of the theorem, consider first an arbitrary $\varepsilon > 0$. We know that there is a gauge $\delta_m\colon [a,b] \to (0,+\infty)$ such that for every δ_m–fine partition D_m, we get

$$|S(U_m, D_m) - I_m| < \frac{\varepsilon}{n^{1/2}},$$

where n is the dimension of the codomain of U. Define now the gauge $\delta\colon [a,b] \to (0,+\infty)$ as

$$\delta(t) = \min\{\delta_m(t)\colon m = 1,2,\ldots,n\}.$$

It is immediate to see that for any δ–fine partition D of $[a,b]$, we obtain

$$\|S(U,D) - I\| = \left(\sum_{m=1}^{n} (S(U_m, D_m) - I_m)^2 \right)^{1/2} < \left(\sum_{m=1}^{n} \frac{\varepsilon^2}{n} \right)^{1/2} = \varepsilon$$

and the proof is complete. $\square$

Although we just proved that we can consider only $U\colon [a,b]\times[a,b]\to\mathbb{R}$, we will consider the more general case $U\colon [a,b]\times[a,b]\to\mathbb{R}^n$ throughout the theorems of this section. Nonetheless, the above result will be relevant for some proofs that will appear later.

An important property that the Kurzweil integral also satisfies is the Bolzano–Cauchy Condition.

Theorem 1.6 (Bolzano–Cauchy Condition - [37, Theorem 1.7])**.** *The function $U\colon [a,b]\times[a,b]\to\mathbb{R}^n$ is Kurzweil integrable on the interval $[a,b]$ if and only if for every $\varepsilon>0$ there is a gauge δ on $[a,b]$ such that for every δ–fine partitions D_1 and D_2, we get*

$$\|S(U,D_1)-S(U,D_2)\|<\varepsilon.$$

Proof. Assume that the Bolzano–Cauchy Condition holds. Then, for every gauge δ on $[a,b]$, define the set

$$C_\delta=\{S(U,D)\colon D \text{ is a } \delta\text{–fine partition of } [a,b]\}.$$

Observe first that $\delta_1\leqslant\delta_2$ implies $C_{\delta_1}\subset C_{\delta_2}$ because if D is δ_1–fine, it is also δ_2–fine. Moreover, if δ is the gauge corresponding to the Bolzano–Cauchy Condition for $\varepsilon>0$, we obtain

$$diam\,(C_\delta)\leqslant\varepsilon. \tag{1.1}$$

Now, consider a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of real positive numbers such that $\varepsilon_n\to 0$. For each ε_n, let δ_n be the gauge corresponding to the Bolzano–Cauchy Condition. We can assume that $\delta_{n+1}\leqslant\delta_n$ because if that is not true, we can change δ_{n+1} by $\min\{\delta_n,\delta_{n+1}\}$. As a consequence of (1.1), we know that $diam\left(C_{\delta_n}\right)\to 0$. Using the Cantor's Intersection Theorem (see [2, Theorem 3.25]), there exists an $I\in\mathbb{R}^n$ such that $\{I\}=\bigcap_n C_{\delta_n}$. Thus, for every $\varepsilon>0$, there is an $\varepsilon_n<\varepsilon$ and for every δ_n–fine partition D, we have

$$\|S(U,D)-I\|<\varepsilon$$

and, therefore, we conclude that U is Kurzweil integrable.

Conversely, consider that U is a Kurzweil integrable function on the interval $[a,b]$. Given $\varepsilon>0$, there exists a gauge δ such that for every δ–fine partition D, it follows that

$$\left\|S(U,D)-\int_a^b DU(\tau,t)\right\|<\frac{\varepsilon}{2}.$$

Let D_1, D_2 be two δ–fine partitions. Then

$$\|S(U,D_1)-S(U,D_2)\| \leqslant \left\|S(U,D_1)-\int_a^b DU(\tau,t)\right\| + \left\|\int_a^b DU(\tau,t)-S(U,D_2)\right\| < \varepsilon,$$

proving the desired result. □

As one may already expect for this integral, it has the additive property. The result below describes this property.

Theorem 1.7 (Additive Property - [37, Theorem 1.11])**.** *Let* $U\colon [a,b]\times[a,b]\to\mathbb{R}^n$ *be such that* $U\in\mathcal{K}([a,c])$ *and* $U\in\mathcal{K}([c,b])$ *for* $c\in(a,b)$*. Then* $U\in\mathcal{K}([a,b])$ *and also*

$$\int_a^c DU(\tau,t)+\int_c^b DU(\tau,t)=\int_a^b DU(\tau,t).$$

Proof. Consider $\varepsilon>0$. By the definition of the integral, there is a gauge δ_A on the interval $[a,c]$ such that for every δ_A–fine partition D_A of $[a,c]$, we obtain

$$\left\|S(U,D_A)-\int_a^c DU(\tau,t)\right\| < \frac{\varepsilon}{2}.$$

Similarly, there is also a gauge δ_B on the interval $[c,b]$ such that for every δ_B–fine partition D_B of $[c,b]$, we also get

$$\left\|S(U,D_B)-\int_c^b DU(\tau,t)\right\| < \frac{\varepsilon}{2}.$$

The idea of the proof now is to use the functions δ_A and δ_B to construct a gauge δ on $[a,b]$ that can be applied in the definition of the integral in that interval. For instance, consider first the function $\tilde{\delta}\colon [a,b]\to(0,+\infty)$ defined as

$$\tilde{\delta}(t)=\begin{cases}\delta_A(t), & \text{if } t\in[a,c),\\ \min\{\delta_A(c),\delta_B(c)\}, & \text{if } t=c,\\ \delta_B(t), & \text{if } t\in(c,b].\end{cases}$$

Define now the gauge $\delta\colon [a,b]\to(0,+\infty)$ as

$$\delta(t)=\begin{cases}\min\left\{\tilde{\delta}(t),|t-c|\right\}, & \text{if } t\neq c,\\ \delta(c), & \text{if } t=c.\end{cases}$$

Consider now a δ–fine partition $D=\{\alpha_0,\tau_1,\alpha_1,\ldots,\alpha_{k-1},\tau_k,\alpha_k\}$ of the interval $[a,b]$. **Claim 1**: c is a tag of some interval of the partition. To prove the claim, suppose that $c\in[\alpha_{j-1},\alpha_j]$ for some $j\in\mathbb{N}$ and consider the tag τ_j of that interval. By the definition of δ–fine partition, we know that

$$c\in[\alpha_{j-1},\alpha_j]\subset(\tau_j-\delta(\tau_j),\tau_j+\delta(\tau_j)).$$

On the other hand, $\delta(t)<|t-c|$ if $t\neq c$. Thus, if $c\neq\tau_j$, then $c\notin(\tau_j-\delta(\tau_j),\tau_j+\delta(\tau_j))$ and this is not possible. Therefore, $c=\tau_j$.

Using this tag c, it is possible to divide the sum $S(U,D)$ in two sums, one on the interval $[a,c]$ and another one on the interval $[c,b]$. Such a division is presented below:

$$\begin{aligned}S(U,D)&=\sum_{i=1}^{j-1}[U(\tau_i,\alpha_i)-U(\tau_i,\alpha_{i-1})]+U(c,\alpha_j)-U(c,\alpha_{j-1})\\&\quad+\sum_{i=j+1}^{k}[U(\tau_i,\alpha_i)-U(\tau_i,\alpha_{i-1})]\\&=\sum_{i=1}^{j-1}[U(\tau_i,\alpha_i)-U(\tau_i,\alpha_{i-1})]+U(c,c)-U(c,\alpha_{j-1})\\&\quad+U(c,\alpha_j)-U(c,c)+\sum_{i=j+1}^{k}[U(\tau_i,\alpha_i)-U(\tau_i,\alpha_{i-1})]=S(U,D_{\tilde{A}})+S(U,D_{\tilde{B}}),\end{aligned}$$

where $D_{\tilde{A}}$ is a δ_A–fine partition of $[a,c]$, since $\delta(t)<\delta_A(t)$ for every $t\in[a,c]$. Similarly, $D_{\tilde{B}}$ is a δ_B–fine partition of $[c,b]$. Hence,

$$\begin{aligned}&\left\|S(U,D)-\int_a^c DU(\tau,t)-\int_c^b DU(\tau,t)\right\|\\&=\left\|S(U,D_{\tilde{A}})+S(U,D_{\tilde{B}})-\int_a^c DU(\tau,t)-\int_c^b DU(\tau,t)\right\|\\&\leqslant\left\|S(U,D_{\tilde{A}})-\int_a^c DU(\tau,t)\right\|+\left\|S(U,D_{\tilde{B}})-\int_c^b DU(\tau,t)\right\|\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Thus, by the definition of the Kurzweil integral, $\int_a^b DU(\tau,t)$ exists and

$$\int_a^b DU(\tau,t)=\int_a^c DU(\tau,t)+\int_c^b DU(\tau,t),$$

proving the result. □

It is also important to have a result ensuring the existence of the integral of a function $U\colon [a,b]\times[a,b]\to\mathbb{R}^n$ in sub-intervals of $[a,b]$. This is the content of the next result.

Theorem 1.8 ([37, Theorem 1.10])**.** *Assume that $U\colon [a,b]\times[a,b]\to\mathbb{R}^n$ is such that $U\in\mathcal{K}([a,b])$. Then for every $[c,d]\subset[a,b]$, $U\in\mathcal{K}([c,d])$.*

Proof. Let $\varepsilon>0$ be given. By the Bolzano–Cauchy Condition (Theorem 1.6), there is a gauge δ on $[a,b]$ such that

$$\|S(U,D_1)-S(U,D_2)\|<\varepsilon,$$

for every δ–fine partitions D_1 and D_2 of $[a,b]$.

Suppose now that $a<c<d<b$ and the other cases can be proved in a similar way. Let $\widetilde{D_1}$ and $\widetilde{D_2}$ be two δ–fine partitions of $[c,d]$. Consider also that D_A is a δ–fine partition of $[a,c]$ and D_B is a δ–fine partition of $[d,b]$. Note that the existence of all these partitions are guaranteed by the Cousin Lemma (Lemma 1.2). Furthermore, we use the following notations

$$\begin{aligned}\widetilde{D_1}&=\{\alpha_0,\tau_1,\alpha_1,\ldots,\alpha_{k-1},\tau_k,\alpha_k\},\\ D_A&=\{\alpha_0^A,\tau_1^A,\alpha_1^A,\ldots,\alpha_{l-1}^A,\tau_l^A,\alpha_l^A\},\\ D_B&=\{\alpha_0^B,\tau_1^B,\alpha_1^B,\ldots,\alpha_{m-1}^B,\tau_m^B,\alpha_m^B\},\end{aligned}$$

where $\alpha_0=a$, $\alpha_k=\alpha_0^A=c$, $\alpha_l^A=\alpha_0^B=d$ and $\alpha_m^B=b$. Putting the three sets above together, we can create a new partition of $[a,b]$ as represented below:

$$D_1=\{\alpha_0,\tau_1,\alpha_1,\ldots,\alpha_{k-1},\tau_k,\alpha_k,\tau_1^A,\alpha_1^A,\ldots,\alpha_{l-1}^A,\tau_l^A,\alpha_l^A,\tau_1^B,\alpha_1^B,\ldots,\alpha_{m-1}^B,\tau_m^B,\alpha_m^B\}.$$

It is immediate to see that D_1 is δ–fine. Similarly, we can put together $\widetilde{D_2}$, D_A and D_B to obtain a δ–fine partition of $[a,b]$ that we denote by D_2. It is easy to see that

$$\left\|S(U,\widetilde{D_1})-S(U,\widetilde{D_2})\right\|=\|S(U,D_1)-S(U,D_2)\|<\varepsilon$$

and we can use the Bolzano–Cauchy Condition (Theorem 1.6) to obtain the desired result. □

The following lemma was originally formulated by Stanislaw Saks and it was generalized by Ralph Henstock (see [37]). It is a useful tool and we use this result to prove more elaborated theorems later in this chapter.

Lemma 1.9 (Saks–Henstock Lemma - [37, Lemma 1.13])**.** *Let $U\colon [a,b]\times[a,b]\to\mathbb{R}^n$ be such that $U\in\mathcal{K}([a,b])$ and $\varepsilon>0$. Consider a gauge δ on $[a,b]$ such that for every δ–fine* *partition* $D = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}$ *of* $[a,b]$*, we get*

$$\left\| \sum_{i=1}^{k} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})] - \int_a^b DU(\tau,t) \right\| < \varepsilon.$$

If $\{(\xi_i, [\beta_i, \gamma_i]), i = 1, \dots, m\}$ *represents a* δ*–fine system, that is,*

$$\xi_i \in [\beta_i, \gamma_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)), \quad \textit{for } i = 1, \dots, m,$$
$$(\beta_i, \gamma_i) \cap (\beta_j, \gamma_j) = \emptyset, \quad \textit{whenever } i \neq j,$$

then

$$\left\| \sum_{i=1}^{m} \left[U(\xi_i, \gamma_i) - U(\xi_i, \beta_i) - \int_{\beta_i}^{\gamma_i} DU(\tau,t) \right] \right\| \leqslant \varepsilon.$$

Proof. First, we can assume without loss of generality that $\beta_i < \gamma_i$ for $i = 1, \dots, m$, since $\int_{\beta_i}^{\gamma_i} DU(\tau,t) = 0$ when $\beta_i = \gamma_i$. Moreover, the integral $\int_{\beta_i}^{\gamma_i} DU(\tau,t)$ exists by Theorem 1.8.

Define $\gamma_0 = a$ and $\beta_{m+1} = b$. If $\gamma_j < \beta_{j+1}$ for $j = 0, \dots, m$, then for every $\theta > 0$, there is a gauge δ_j on the interval $[\gamma_j, \beta_{j+1}]$ such that for every δ_j–fine partition D^j of $[\gamma_j, \beta_{j+1}]$, we have

$$\left\| S(U, D^j) - \int_{\gamma_j}^{\beta_{j+1}} DU(\tau,t) \right\| < \frac{\theta}{m+1}.$$

Without loss of generality, it is possible to assume that $\delta_j(\tau) < \delta(\tau)$ for $\tau \in [\gamma_j, \beta_{j+1}]$ because if that is not true, we can swap $\delta_j(\tau)$ by $\min\{\delta_j(\tau), \delta(\tau)\}$. As a consequence, it follows that

$$S(U,D) = \sum_{i=1}^{m} [U(\xi_i, \gamma_i) - U(\xi_i, \beta_i)] + \sum_{j=0}^{m} S(U, D^j)$$

represents an integral sum where D is a δ–fine partition. Using the additive property (Theorem 1.7), we get

$$\begin{aligned}
&\left\| \sum_{i=1}^{m} \left[U(\xi_i, \gamma_i) - U(\xi_i, \beta_i) - \int_{\beta_i}^{\gamma_i} DU(\tau,t) \right] \right\| \\
&\leqslant \left\| \sum_{i=1}^{m} \left[U(\xi_i, \gamma_i) - U(\xi_i, \beta_i) - \int_{\beta_i}^{\gamma_i} DU(\tau,t) \right] + \sum_{j=0}^{m} \left[S(U, D^j) - \int_{\gamma_j}^{\beta_{j+1}} DU(\tau,t) \right] \right\| \\
&\quad + \sum_{j=0}^{m} \left\| S(U, D^j) - \int_{\gamma_j}^{\beta_{j+1}} DU(\tau,t) \right\| \\
&< \left\| S(U,D) - \int_a^b DU(\tau,t) \right\| + \sum_{j=0}^{m} \frac{\theta}{m+1} < \varepsilon + \theta.
\end{aligned}$$

Since the choice of θ is arbitrary, the result follows. $\square$

Remark 1.10. *Although the inequality in the above lemma is not strict, it is possible to change ε by $\varepsilon/2$ in the statement of the theorem to obtain that if*

$$\left\|\sum_{i=1}^{k}[U(\tau_i,\alpha_i)-U(\tau_i,\alpha_{i-1})]-\int_a^b DU(\tau,t)\right\|<\frac{\varepsilon}{2}$$

then

$$\left\|\sum_{i=1}^{m}\left[U(\xi_i,\gamma_i)-U(\xi_i,\beta_i)-\int_{\beta_i}^{\gamma_i} DU(\tau,t)\right]\right\|<\varepsilon.$$

The next two theorems give us important information about the behavior of the integral. The results below directly imply that the Kurzweil integral does not have to be continuous.

Theorem 1.11 ([37, Theorem 1.16])**.** *Consider a function $U\colon [a,b]\times[a,b]\to\mathbb{R}^n$ such that $U\in\mathcal{K}([a,b])$. For any $c\in[a,b]$,*

$$\lim_{x\to c}\left[\int_a^x DU(\tau,t)-U(c,x)+U(c,c)\right]=\int_a^c DU(\tau,t). \tag{1.2}$$

Proof. Given $\varepsilon>0$, consider a gauge δ on $[a,b]$ such that for every δ–fine partition D of $[a,b]$, we obtain

$$\left\|S(U,D)-\int_a^b DU(\tau,t)\right\|<\frac{\varepsilon}{2}.$$

If $s\in[c-\delta(c),c+\delta(c)]\cap[a,b]$, then the Saks–Henstock Lemma (Lemma 1.9) implies that

$$\left\|U(c,s)-U(c,c)-\int_s^c DU(\tau,t)\right\|<\varepsilon.$$

Using the additive property of the integral (Theorem 1.7), we obtain

$$\left\|\int_a^s DU(\tau,t)-U(c,s)+U(c,c)-\int_a^c DU(\tau,t)\right\|=\left\|U(c,s)-U(c,c)-\int_s^c DU(\tau,t)\right\|<\varepsilon$$

and the result follows. $\square$

Notice that a necessary and sufficient condition for the map

$$c\in[a,b]\mapsto\int_a^c DU(\tau,t)$$

to be continuous is that the function $U(c,\cdot)$ is continuous at c.

Theorem 1.12 ([37, Theorem 1.14]). *Assume that* $U\colon [a,b]\times[a,b]\to\mathbb{R}^n$ *is such that* $U\in\mathcal{K}([a,c])$ *for every* $c\in[a,b)$ *and that*

$$\left\|\lim_{c\to b-}\left[\int_a^c DU(\tau,t)-U(b,c)+U(b,b)\right]\right\|<+\infty.$$

Then $U\in\mathcal{K}([a,b])$ *and*

$$\lim_{c\to b-}\left[\int_a^c DU(\tau,t)-U(b,c)+U(b,b)\right]=\int_a^b DU(\tau,t)$$

holds.

Similarly, if $U\colon [a,b]\times[a,b]\to\mathbb{R}^n$ *is such that* $U\in\mathcal{K}([c,b])$ *for every* $c\in(a,b]$ *and satisfies*

$$\left\|\lim_{c\to a+}\left[\int_c^b DU(\tau,t)-U(a,c)+U(a,a)\right]\right\|<+\infty,$$

then $U\in\mathcal{K}([a,b])$ *and*

$$\lim_{c\to a+}\left[\int_c^b DU(\tau,t)-U(a,c)+U(a,a)\right]=\int_a^b DU(\tau,t).$$

Proof. Only the proof of the first part of the theorem will be presented below. The second part follows in a similar way.

By the hypotheses of the theorem, there is an $I\in\mathbb{R}^n$ such that for any $\varepsilon>0$ given, there is a number $\beta\in[a,b]$ for which

$$\left\|\left[\int_a^c DU(\tau,t)-U(b,c)+U(b,b)-I\right]\right\|<\frac{\varepsilon}{2}$$

holds for every $c\in[\beta,b)$.

Now consider an increasing sequence $\{c_n\}_{n\in\mathbb{N}}$ such that $c_0=a$ and $\lim\limits_{k\to+\infty} c_k=b$. Also by the assumptions, $U\in\mathcal{K}([a,c_j])$ for every $j\in\mathbb{N}$. Thus, there is a gauge $\delta_j\colon [a,c_j]\to(0,+\infty)$ such that for every δ_j–fine partition D_j the inequality

$$\left\|S(U,D_j)-\int_a^{c_j} DU(\tau,t)\right\|<\frac{\varepsilon}{2^{j+3}} \tag{1.3}$$

holds.

Before constructing the gauge on $[a,b]$, let us build an auxiliary function $\tilde{\delta}$ on $[a,b)$. In order to do that, first observe that for any $t\in[a,b)$, there is only one number $j=j(t)$ such that $t\in[c_{j(t)-1},c_{j(t)})$. Consider then a function $\tilde{\delta}\colon [a,b)\to(0,+\infty)$ such that for every $t \in [c_{j(t)-1}, c_{j(t)})$ we have $\tilde{\delta}(t) < \delta_j(t)$ and $[t - \tilde{\delta}(t), t + \tilde{\delta}(t)] \cap [a,b) \subset [a, c_{j(t)}]$. Let $c \in [a,b)$ and consider a $\tilde{\delta}$–fine partition of $[a,c]$

$$\tilde{D} = \{a = \alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-2}, \tau_{k-1}, \alpha_{k-1} = c\}.$$

It is possible to control the sum

$$\sum_{i=1}^{k-1} \left[U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1}) - \int_{\alpha_{i-1}}^{\alpha_i} DU(\tau, t) \right]$$

in terms of $j(t) \in \mathbb{N}$. In other words, fix $j(t)$ and consider all intervals $[\alpha_{i-1}, \alpha_i]$ such that $\tau_i \in [c_{j(t)-1}, c_{j(t)})$. It follows from the definition of $\tilde{\delta}$ that $[\alpha_{i-1}, \alpha_i] \subset [\tau_i - \tilde{\delta}(\tau_i), \tau_i + \tilde{\delta}(\tau_i)] \subset [a, c_{j(t)}]$ and also, $[\alpha_{i-1}, \alpha_i] \subset [\tau_i - \delta_{j(t)}(\tau_i), \tau_i + \delta_{j(t)}(\tau_i)]$. Let us denote these points of the partition as:

$$\left\{ \alpha_{0,j(t)}, \tau_{1,j(t)}, \alpha_{1,j(t)}, \ldots, \alpha_{l-1,j(t)}, \tau_{l,j(t)}, \alpha_{l,j(t)} \right\}.$$

Thus, the sum

$$\sum_{i=1}^{l} \left[U(\tau_{i,j(t)}, \alpha_{i,j(t)}) - U(\tau_{i,j(t)}, \alpha_{i-1,j(t)}) - \int_{\alpha_{i-1,j(t)}}^{\alpha_{i,j(t)}} DU(\tau, t) \right]$$

represents only the tags that are in $[c_{j(t)-1}, c_{j(t)})$. By the Saks–Henstock Lemma (Lemma 1.9), we have

$$\left\| \sum_{i=1}^{l} \left[U(\tau_{i,j(t)}, \alpha_{i,j(t)}) - U(\tau_{i,j(t)}, \alpha_{i-1,j(t)}) - \int_{\alpha_{i-1,j(t)}}^{\alpha_{i,j(t)}} DU(\tau, t) \right] \right\| < \frac{\varepsilon}{2^{j+2}}.$$

Therefore,

$$\begin{aligned}
&\left\| \sum_{i=1}^{k-1} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})] - \int_a^c DU(\tau, t) \right\| \\
&= \left\| \sum_{i=1}^{k-1} \left[U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1}) - \int_{\alpha_{i-1}}^{\alpha_i} DU(\tau, t) \right] \right\| \\
&\leqslant \sum_{j(t)=1}^{\infty} \left\| \sum_{i=1}^{l} \left[U(\tau_{i,j(t)}, \alpha_{i,j(t)}) - U(\tau_{i,j(t)}, \alpha_{i-1,j(t)}) - \int_{\alpha_{i-1,j(t)}}^{\alpha_{i,j(t)}} DU(\tau, t) \right] \right\| \\
&< \sum_{j(t)=1}^{\infty} \frac{\varepsilon}{2^{j(t)+2}} = \frac{\varepsilon}{2},
\end{aligned}$$

where the first inequality above is valid because the sum originally would have finite terms (from $j(t)=1$ until $j(c)$) but we can add more positive terms to major the sum.

Now, define the gauge $\delta\colon [a,b]\to(0,+\infty)$ as

$$0<\delta(t)<\min\{b-t,\tilde{\delta}(t)\}$$

for $t\in[a,b)$ and

$$0<\delta(b)<b-\beta.$$

Let $D=\{\alpha_0,\tau_1,\alpha_1,\dots,\alpha_{k-1},\tau_k,\alpha_k\}$ be a δ–fine partition on $[a,b]$. By the definition of gauge, we must have $\alpha_k=\tau_k=b$ and $\alpha_{k-1}\in(\beta,b)$. Thus,

$$\begin{aligned}
\|S(U,D)-I\| &= \left\|\sum_{i=1}^{k-1}[U(\tau_i,\alpha_i)-U(\tau_i,\alpha_{i-1})]+U(b,b)-U(b,\alpha_{k-1})-I\right\| \\
&= \left\|\sum_{i=1}^{k-1}[U(\tau_i,\alpha_i)-U(\tau_i,\alpha_{i-1})]-\int_a^{\alpha_{k-1}}DU(\tau,t)\right\| \\
&\quad+\left\|\int_a^{\alpha_{k-1}}DU(\tau,t)+U(b,b)-U(b,\alpha_{k-1})-I\right\| \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}$$

and we conclude the proof. □

The Kurzweil integral also satisfies a version of the substitution theorem.

Theorem 1.13 (Substitution - [37, Theorem 1.18])**.** *Assume that $\phi\colon [a,b]\to\mathbb{R}$ is a continuous strictly monotone function on $[a,b]$ and consider another function $U\colon [\phi(a),\phi(b)]\times[\phi(a),\phi(b)]\to\mathbb{R}^n$. If one of the integrals*

$$\int_{\phi(a)}^{\phi(b)}DU(\tau,t),\qquad \int_a^b DU(\phi(\sigma),\phi(s))$$

exists then the other integral also exists and they have the same value.

Proof. Assume that $\phi\colon [a,b]\to\mathbb{R}$ is increasing and that $\int_a^b DU(\phi(\sigma),\phi(s))$ exists. By the existence of the integral on $[a,b]$, given $\varepsilon>0$, there is a gauge $\delta\colon [a,b]\to(0,+\infty)$ such that for every δ–fine partition

$$D=\{\beta_0,\sigma_1,\beta_1,\dots,\beta_{k-1},\sigma_k,\beta_k\}$$

of $[a,b]$, we get

$$\left\| \sum_{i=1}^{k} [U(\phi(\sigma_i),\phi(\beta_i)) - U(\phi(\sigma_i),\phi(\beta_{i-1}))] - \int_a^b DU(\phi(\sigma),\phi(s)) \right\| < \varepsilon.$$

Since ϕ is continuous and strictly increasing, the inverse $\phi^{-1}\colon [\phi(a),\phi(b)] \to [a,b]$ exists and is also a continuous and strictly increasing function on $[\phi(a),\phi(b)]$. Thus, we can associate for each $\tau \in [\phi(a),\phi(b)]$ exactly one point $\sigma = \phi^{-1}(\tau) \in [a,b]$. Using this relation, it is possible to construct a gauge $\omega\colon [\phi(a),\phi(b)] \to (0,+\infty)$ such that

$$[\tau - \omega(\tau), \tau + \omega(\tau)] \cap [\phi(a),\phi(b)] \subset \phi([\sigma - \delta(\sigma), \sigma + \delta(\sigma)] \cap [a,b]). \tag{1.4}$$

The inclusion above is important to show that an ω–fine partition of $[\phi(a),\phi(b)]$ can be transformed in a δ–fine partition of $[a,b]$ using the function ϕ^{-1}. To see that, consider an arbitrary ω–fine partition of $[\phi(a),\phi(b)]$

$$\{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}.$$

Define $\beta_j = \phi^{-1}(\alpha_j)$ for $j = 0,\dots,k$ and $\sigma_j = \phi^{-1}(\tau_j)$ for $j = 1,\dots,k$. It is easy to see that $\{\beta_0, \sigma_1, \beta_1, \dots, \beta_{k-1}, \sigma_k, \beta_k\}$ is a partition of $[a,b]$. Furthermore, we obtain

$$\tau_j - \omega(\tau_j) \leqslant \alpha_{j-1} \leqslant \tau_j \leqslant \alpha_j \leqslant \tau_j + \omega(\tau_j),$$

since the partition is ω–fine. In addition, inclusion (1.4) implies that

$$\phi(\sigma - \delta(\sigma)) \leqslant \tau_j - \omega(\tau_j) < \tau_j + \omega(\tau_j) \leqslant \phi(\sigma + \delta(\sigma)).$$

Therefore, we conclude

$$\begin{aligned} \sigma - \delta(\sigma) = \phi^{-1}(\phi(\sigma - \delta(\sigma))) \leqslant \phi^{-1}(\alpha_{j-1}) = \beta_{j-1} \\ \leqslant \beta_j = \phi^{-1}(\alpha_j) \leqslant \phi^{-1}(\phi(\sigma + \delta(\sigma))) = \sigma + \delta(\sigma) \end{aligned}$$

and it follows that $\{\beta_0, \sigma_1, \beta_1, \dots, \beta_{k-1}, \sigma_k, \beta_k\}$ is δ–fine. Thus,

$$\begin{aligned} &\left\| \sum_{i=1}^{k} [U(\tau_i,\alpha_i) - U(\tau,\alpha_{i-1})] - \int_a^b DU(\phi(\sigma),\phi(s)) \right\| \\ &= \left\| \sum_{i=1}^{k} [U(\phi(\sigma_i),\phi(\beta_i)) - U(\phi(\sigma_i),\phi(\beta_{i-1}))] - \int_a^b DU(\phi(\sigma),\phi(s)) \right\| < \varepsilon. \end{aligned}$$

By definition, $\int_{\phi(a)}^{\phi(b)} DU(\tau,t)$ exists and it is equal to $\int_a^b DU(\phi(\sigma),\phi(s))$.

The proof that $\int_{\phi(a)}^{\phi(b)} DU(\tau,t)$ implies the existence of $\int_a^b DU(\phi(\sigma),\phi(s))$ is similar and it will be omitted. We also omit the case when ϕ is a decreasing function, since it follows in a similar way with obvious adaptations. $\square$

Remark 1.14. *It is also possible to change the theorem above to consider when $\phi\colon [a,b]\to\mathbb{R}$ is a continuous and monotone function. In this case, there may not be a bijection between $[a,b]$ and $[\phi(a),\phi(b)]$, but the proof above can be adapted defining*

$$\phi^{-1}(t) = \{s\in[a,b]\colon \phi(t)=s\}.$$

1.1.3 Convergence theorems

This subsection presents theorems and properties involving sequence of functions. Although the results are more technical than the ones in the previous subsection, they are all needed to prove the existence and uniqueness of solutions of generalized ODEs using the Kurzweil integral that will be presented later.

Theorem 1.15 ([37, Theorem 1.25])**.** *Suppose that $U, U_m\colon [a,b]\times[a,b]\to\mathbb{R}^n$, $m\in\mathbb{N}$, are such that $U_m\in\mathcal{K}([a,b])$. Furthermore, assume that there is a gauge $\widetilde{\delta}$ on $[a,b]$ satisfying*

$$\lim_{m\to+\infty}[U_m(\tau,t_2)-U_m(\tau,t_1)] = U(\tau,t_2)-(\tau,t_1) \tag{1.5}$$

for every $\tau\in[a,b]$ and $t_1,t_2\in\mathbb{R}$ in a way that $[t_1,t_2]\subset[\tau-\widetilde{\delta}(\tau),\tau+\widetilde{\delta}(\tau)]$. Finally, suppose that given $\xi>0$, there is a gauge δ on $[a,b]$ such that for every δ–fine partition D and every $m\in\mathbb{N}$, the following inequality holds:

$$\left\| S(U_m,D) - \int_a^b DU_m(\tau,t)\right\| < \xi. \tag{1.6}$$

Then $U\in\mathcal{K}([a,b])$ and also

$$\lim_{m\to+\infty}\int_a^b DU_m(\tau,t) = \int_a^b DU(\tau,t).$$

Proof. Given $\varepsilon>0$, there is a gauge δ on $[a,b]$ such that for every δ–fine partition $D=\{\alpha_0,\tau_1,\alpha_1,\dots,\alpha_{k-1},\tau_k,\alpha_k\}$, we have

$$\left\| S(U_m,D) - \int_a^b DU_m(\tau,t)\right\| < \frac{\varepsilon}{4}.$$

Without loss of generality, it is possible to assume that $\delta(t) \leqslant \widetilde{\delta}(t)$ for every $t \in [a,b]$. This follows from the fact that the function $\overline{\delta}(t) = \min\{\delta(t), \widetilde{\delta}(t)\}$ is also a gauge. With that information in hands, we can use (1.5) to guarantee that there exists an integer $m_0 > 0$ such that for every $m > m_0$ the inequality

$$\|S(U_m,D) - S(U,D)\| = \left\| \sum_{i=1}^{k} [U_m(\tau_i,\alpha_i) - U_m(\tau_i,\alpha_{i-1}) - U(\tau_i,\alpha_i) + U(\tau_i,\alpha_{i-1})] \right\| < \frac{\varepsilon}{4}$$

holds. Combining the inequalities above, we obtain

$$\lim_{m\to+\infty} S(U_m,D) = S(U,D)$$

and

$$\left\| S(U,D) - \int_a^b DU_m(\tau,t) \right\| < \frac{\varepsilon}{2} \tag{1.7}$$

for $m > m_0$.

On the other hand, (1.7) also implies that for every $l, m > m_0$, we get

$$\left\| \int_a^b DU_l(\tau,t) - \int_a^b DU_m(\tau,t) \right\| < \varepsilon.$$

Consequently, $\{\int_a^b DU_m(\tau,t)\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}^n$ and there is a vector $I \in \mathbb{R}^n$ such that

$$\lim_{m\to+\infty} \int_a^b DU_m(\tau,t) = I.$$

Lastly, we obtain

$$\|S(U,D) - I\| \leqslant \left\| S(U,D) - \int_a^b DU_m(\tau,t) \right\| + \left\| \int_a^b DU_m(\tau,t) - I \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and the result follows. □

The condition in Theorem 1.15 may seem very strong at a first look, but we will show a very useful example satisfying these hypotheses in Corollary 1.21. Besides that, such conditions will be very important later when we talk about the uniqueness of solutions of generalized ODEs using the Kurzweil integral. Because of that, we give a special name for the sequences satisfying these conditions. This name is presented in the definition below.

Definition 1.16 ([37, Definition 1.26]). *A sequence of functions $U_m\colon [a,b] \times [a,b] \to \mathbb{R}^n$, $m \in \mathbb{N}$ such that $U_m \in \mathcal{K}([a,b])$ is **equi–integrable** if the condition* (1.6) *of Theorem 1.15 is satisfied.*

The next theorem is very technical and has an extensive proof. Because of that, its proof will be omitted, but it can be found in the cited reference.

Theorem 1.17 ([37, Theorem 1.28]). *Consider* $U, U_m \colon [a,b] \times [a,b] \to \mathbb{R}$, $m \in \mathbb{N}$, *are such that* $U_m \in \mathcal{K}([a,b])$ *for* $m \in \mathbb{N}$ *and also:*

i. there is a gauge $\gamma \colon [a,b] \to (0,+\infty)$ *such that for every* $\varepsilon > 0$, *there is a function* $p \colon [a,b] \to \mathbb{N}$ *and a superadditive set function* Φ *defined from any closed interval* $J = [\alpha, \beta] \subset [a,b]$ *to a real number* $\Phi(J) \in (0,+\infty)$ *with* $\Phi([a,b]) < \varepsilon$ *such that for every* $\tau \in [a,b]$, *we get*

$$|U_m(\tau,\beta) - U_m(\tau,\alpha) - U(\tau,\beta) + U(\tau,\alpha)| < \Phi([\alpha,\beta])$$

provided $m > p(\tau)$ *and* $\tau \in [\alpha,\beta] \subset [\tau - \gamma(\tau), \tau + \gamma(\tau)]$;

ii. there is a gauge $\omega \colon [a,b] \to (0,+\infty)$ *such that for any function* $m \colon [a,b] \to \mathbb{N}$ *and any* ω*–fine partition* $D = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ *the inequalities*

$$B \leqslant \sum_{i=1}^{k} U_{m(\tau_i)}(\tau_i, \alpha_i) - U_{m(\tau_i)}(\tau_i, \alpha_{i-1}) \leqslant C$$

hold for some constants $B, C \in \mathbb{R}$.

Then the sequence $(U_m)_{m=1}^{\infty}$ *is equi-integrable.*

The next result is a direct consequence of the Theorem 1.17.

Corollary 1.18 ([37, Theorem 1.29]). *Suppose that* $U, U_m \colon [a,b] \times [a,b] \to \mathbb{R}$, $m \in \mathbb{N}$ *are functions satisfying all conditions of Theorem 1.17. Then* $U \in \mathcal{K}([a,b])$ *and*

$$\lim_{m \to +\infty} \int_a^b DU_m(\tau,t) = \int_a^b DU(\tau,t).$$

Proof. To prove this result, we apply Theorem 1.15. Notice first that condition (1.6) is already satisfied by Theorem 1.17. On the other hand, condition (1.5) of Theorem 1.15 is also satisfied directly by the item (i) of the hypotheses of Theorem 1.17 that we are also assuming here. All the conditions of Theorem 1.15 are valid and the proof is complete. □

The corollary below is a type of Dominated Convergence Theorem for Kurzweil integrable functions.

Corollary 1.19 ([37, Corollary 1.31]). *Let $U, U_m\colon [a,b]\times[a,b]\to\mathbb{R}$, $m\in\mathbb{N}$, be such that $U_m\in\mathcal{K}([a,b])$ and condition (i) of Theorem 1.17 is satisfied. Assume also that there are functions $V, W\colon [a,b]\times[a,b]\to\mathbb{R}$, $V,W\in\mathcal{K}([a,b])$, and there is a gauge ω such that for any ω–fine partition $D=\{\alpha_0,\tau_1,\alpha_1,\ldots,\alpha_{k-1},\tau_k,\alpha_k\}$ and every $m\in\mathbb{N}$, we obtain*

$$V(\tau_i,\alpha_i)-V(\tau_i,\alpha_{i-1})\leqslant U_m(\tau_i,\alpha_i)-U_m(\tau_i,\alpha_{i-1})\leqslant W(\tau_i,\alpha_i)-W(\tau_i,\alpha_{i-1}) \tag{1.8}$$

for $i=1,2,\ldots,k$.

Then $U\in\mathcal{K}([a,b])$ and

$$\lim_{m\to+\infty}\int_a^b DU_m(\tau,t)=\int_a^b DU(\tau,t).$$

Proof. Using the fact that $V,W\in\mathcal{K}([a,b])$, there is a gauge δ of $[a,b]$ such that for every δ–fine partition D, it follows that

$$\left|S(V,D)-\int_a^b DV(\tau,t)\right|<1,\quad \left|S(W,D)-\int_a^b DW(\tau,t)\right|<1.$$

It is also possible to assume that $\delta(t)<\omega(t)$ for every $t\in[a,b]$ (if this is not true, change $\delta(t)$ for $\min\{\delta(t),\omega(t)\}$). Applying (1.8), we conclude that

$$\begin{aligned}\int_a^b DV(\tau,t)-1<S(V,D)&\leqslant\sum_{i=1}^{k}U_{m(\tau_i)}(\tau_i,\alpha_i)-U_{m(\tau_i)}(\tau_i,\alpha_{i-1})\\&\leqslant S(W,D)<\int_a^b DW(\tau,t)+1.\end{aligned}$$

Thus, condition (ii) of Theorem 1.17 is satisfied with

$$B=\int_a^b DV(\tau,t)-1$$

and

$$C=\int_a^b DW(\tau,t)+1$$

and the result follows. □

Before we present the last consequence of Theorem 1.17, we define the space of bounded variation functions from $[\alpha,\beta]\subset\mathbb{R}$ to $\mathbb{R}^n$.

Definition 1.20. *Given a function* $x\colon [\alpha,\beta]\to\mathbb{R}^n$*, we define the* **variation of** x *in* $[\alpha,\beta]$ *as*

$$var_{\alpha}^{\beta}(x)=\sup\left\{\sum_{i=1}^{k}\|x(s_i)-x(s_{i-1})\| : P=\{s_0,s_1,\ldots,s_k\} \text{ is a partition of } [\alpha,\beta]\right\}.$$

If $var_{\alpha}^{\beta}(x)<+\infty$*, then* x *is called a* **function of bounded variation**. *We denote the space of functions with such property as* $BV([\alpha,\beta],\mathbb{R}^n)$ *or simply as* BV *when there is no risk of confusion. We can also use the following norm for this space:*

$$\|x\|_{BV}=\|x(\alpha)\|+var_{\alpha}^{\beta}(x).$$

For more details about this space, see [33].

We present a particular case of Corollary 1.19 to conclude this subsection. It is also a concrete example of a sequence of functions satisfying all the conditions of Theorem 1.15.

Corollary 1.21 ([37, Corollary 1.32])**.** *Let* $g\colon [a,b]\to\mathbb{R}$ *be a nondecreasing function and* $f_m\colon [a,b]\to\mathbb{R}$*,* $m\in\mathbb{N}$*, is a sequence of functions such that* $\int_a^b f_m(s)\mathrm{d}g(s)$ *exists for every* $m\in\mathbb{N}$*. Assume also that for every* $t\in[a,b]$*,*

$$\lim_{m\to+\infty} f_m(t)=f(t)$$

and there exist two functions $v,w\colon [a,b]\to\mathbb{R}$ *such that the integrals* $\int_a^b v(s)\mathrm{d}g(s)$ *and* $\int_a^b w(s)\mathrm{d}g(s)$ *exist and*

$$v(t)\leqslant f_m(t)\leqslant w(t)$$

holds for every $t\in[a,b]$ *and* $m\in\mathbb{N}$*. Then* $\int_a^b f(s)\mathrm{d}g(s)$ *exists and*

$$\lim_{m\to+\infty}\int_a^b f_m(s)\mathrm{d}g(s)=\int_a^b f(s)\mathrm{d}g(s).$$

Proof. We show that both conditions of Theorem 1.17 are satisfied and conclude the result. Firstly, we can prove that the condition (ii) is satisfied in a similar way as done in Corollary 1.19. The only difference is that in this case we have to define $U_m(\tau,t)=f_m(\tau)g(t)$, $U(\tau,t)=f(\tau)g(t)$, $V(\tau,t)=v(\tau)g(t)$ and $W(\tau,t)=w(\tau)g(t)$ for $\tau,t\in[a,b]$.

Remember now that the condition (i) of Theorem 1.17 is: there is a gauge $\gamma\colon [a,b]\to(0,+\infty)$ such that for every $\varepsilon>0$ there are a function $p\colon [a,b]\to\mathbb{N}$ and a superadditive set function Φ defined from any closed interval $J=[\alpha,\beta]\subset[a,b]$ to a real number $\Phi(J)\in$

$(0,+\infty)$ with $\Phi([a,b]) < \varepsilon$ such that for every $\tau \in [a,b]$, we obtain

$$|U_m(\tau,\beta) - U_m(\tau,\alpha) - U(\tau,\beta) + U(\tau,\alpha)| < \Phi([\alpha,\beta])$$

provided $m > p(\tau)$ and $\tau \in [\alpha,\beta] \subset [\tau - \gamma(\tau), \tau + \gamma(\tau)]$.

Define now the function

$$\Phi([\alpha,\beta]) = \frac{\varepsilon var_{\alpha}^{\beta}(g)}{2var_a^b(g)+1}$$

for any closed interval $[\alpha,\beta] \subset [a,b]$. Notice that $\Phi([a,b]) < \varepsilon/2 < \varepsilon$ and that Φ is a superadditive set function.

Using $U_m(\tau,t) = f_m(\tau)g(t)$, $U(\tau,t) = f(\tau)g(t)$ and the fact that $\lim\limits_{m\to+\infty} f_m(t) = f(t)$, we can guarantee that given $\tau \in [a,b]$, there is an $m_0 \in \mathbb{N}$ such that for every $m > m_0$, we have

$$|U_m(\tau,\beta) - U_m(\tau,\alpha) - U(\tau,\beta) + U(\tau,\alpha)| = |(f_m(\tau) - f(\tau))(g(\beta) - g(\alpha))| < \Phi([\alpha,\beta]).$$

We can conclude that for any gauge ω of $[a,b]$, the desired conditions are satisfied and the proof follows from Theorem 1.17. $\square$

1.1.4 Inequalities

In this subsection, we present two inequalities involving the Kurzweil integral. The first inequality will be used in the next section of this chapter, while the second one will be used to prove the uniqueness of solutions of a generalized ODE using the Kurzweil integral.

Theorem 1.22 ([37, Theorem 1.35])**.** *Suppose that $U\colon [a,b]\times[a,b] \to \mathbb{R}^n$ is Kurzweil integrable. If $V\colon [a,b]\times[a,b] \to \mathbb{R}$ is a function for which the integral $\int_a^b DV(\tau,t)$ exists and there is a gauge ω on $[a,b]$ such that*

$$|t-\tau|\,\|U(\tau,t) - U(\tau,\tau)\| \leqslant (t-\tau)(V(\tau,t) - V(\tau,t)) \tag{1.9}$$

for every $t \in [\tau - \omega(\tau), \tau + \omega(\tau)]$. Then

$$\left\| \int_a^b DU(\tau,t) \right\| \leqslant \int_a^b DV(\tau,t).$$

Proof. Given $\varepsilon > 0$, there is a gauge δ such that $\delta(s) \leqslant \omega(s)$ for every $s \in [a,b]$ and for every δ–fine partition $D = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$, we obtain

$$\left\| S(U,D) - \int_a^b DU(\tau,t) \right\| < \frac{\varepsilon}{2}, \quad \left| S(V,D) - \int_a^b DV(\tau,t) \right| < \frac{\varepsilon}{2}.$$

By (1.9),

$$\|U(\tau_i,\alpha_i) - U(\tau_i,\tau_i)\| \leqslant (V(\tau_i,\alpha_i) - V(\tau_i,\tau_i))$$

when $\alpha_i > \tau_i$. Otherwise,

$$\|U(\tau_i,\alpha_i) - U(\tau_i,\tau_i)\| \leqslant (V(\tau_i,\tau_i) - V(\tau_i,\alpha_i)).$$

It follows that

$$\begin{aligned}\|U(\tau_i,\alpha_i) - U(\tau_i,\alpha_{i-1})\| &\leqslant \|U(\tau_i,\alpha_i) - U(\tau_i,\tau_i)\| + \|U(\tau_i,\tau_i) - U(\tau_i,\alpha_{i-1})\| \\ &\leqslant V(\tau_i,\alpha_i) - V(\tau_i,\alpha_{i-1}).\end{aligned}$$

The inequalities above imply that $\|S(U,D)\| \leqslant S(V,D)$. Thus, we get

$$\begin{aligned}\left\| \int_a^b DU(\tau,t) \right\| &\leqslant \left\| S(U,D) - \int_a^b DU(\tau,t) \right\| + \|S(U,D)\| \\ &< \frac{\varepsilon}{2} + S(V,D) - \int_a^b DV(\tau,t) + \int_a^b DV(\tau,t) < \varepsilon + \int_a^b DV(\tau,t)\end{aligned}$$

and the proof is complete. □

The next theorem is a nonlinear version of the Gronwall Inequality for the Henstock–Kurzweil–Stieltjes integral. Due to its extensive proof, it will be omitted, but one can find it in the cited reference. Besides being used to prove the uniqueness of solutions of generalized ODEs later in this chapter, it will also be used to prove some results later in the Chapter 4.

Theorem 1.23 ([37, Theorem 1.40])**.** *Consider $\psi\colon [a,b] \to [0,+\infty)$, $h\colon [a,b] \to \mathbb{R}$ and $\omega\colon [0,+\infty) \to [0,+\infty)$ are such that ψ is bounded, h is continuous and nondecreasing and ω is continuous, nondecreasing, $\omega(0) = 0$ and $\omega(t) > 0$ for every $t > 0$.*

Define the function $\Omega\colon (0,+\infty) \to \mathbb{R}$ by

$$\Omega(s) = \int_{s_0}^{s} \frac{1}{\omega(r)} \mathrm{d}r, \quad s \in (0,+\infty)$$

for some $s_0 > 0$. Assume that Ω is increasing on $(0,+\infty)$ such that $\Omega(s_0) = 0$, $\lim_{s\to 0+} \Omega(s) = \alpha \geqslant -\infty$ and $\lim_{s\to+\infty} \Omega(s) = \beta \leqslant +\infty$.

If there is a $k > 0$ such that for every $\xi \in [a,b]$, the inequalities

$$\psi(\xi) \leqslant k + \int_a^{\xi} \omega(\psi(\tau))\mathrm{d}h(\tau)$$

and

$$\Omega(k) + h(b) - h(a) < \beta$$

hold, then

$$\psi(\xi) \leqslant \Omega^{-1}(\Omega(k) + h(\xi) - h(a)).$$

1.2 Generalized Ordinary Differential Equations

In this section, we present the definition of a generalized ODE. We also show some properties of this type of equation.

Throughout this section, suppose that $O \subset \mathbb{R}^n \times \mathbb{R}$ is an open set. Furthermore, consider $F\colon O \to \mathbb{R}^n$.

Definition 1.24 ([37, Definition 3.1])**.** *A function $x\colon [\alpha,\beta] \to \mathbb{R}^n$ is a* ***solution of the generalized ordinary differential equation***

$$\frac{dx}{d\tau} = DF(x,t) \tag{1.10}$$

on the interval $[\alpha,\beta]$ if $(x(t),t) \in O$ whenever $t \in [\alpha,\beta]$ and

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau),t) \tag{1.11}$$

is true for every $s_1, s_2 \in [\alpha,\beta]$.

The integral in (1.11) must be interpreted as the Kurzweil integral defined in the previous section.

It is also important to note that the symbols $dx/d\tau$ and D in (1.10) are only notations and they do not mean any kind of derivative. The example below shows a case where the involved functions are not even differentiable.

Example 1.25 ([37])**.** *Consider a function $f\colon A \subset \mathbb{R} \to \mathbb{R}$ that is continuous but not differentiable, such as the Weierstrass function (see [16]). Define $F(x,t) = f(t)$ and suppose that $s_1, s_2 \in A$ are real numbers such that $s_1 < s_2$. In this case, it is immediate that*

$$\int_{s_1}^{s_2} DF(x(\tau),t) = \int_{s_1}^{s_2} Df(t) = f(s_2) - f(s_1).$$

Thus, the solution of the generalized ODE

$$\frac{dx}{d\tau} = DF(x,t) = Df(t)$$

is $x(t) = f(t)$*, although* x *is clearly not differentiable.*

The following proposition brings a property one may already expect from the generalized ODEs.

Proposition 1.26 ([37, Proposition 3.5])**.** *Let* $x\colon [\alpha,\beta] \to \mathbb{R}^n$ *be a solution of the generalized ODE* (1.10) *on* $[\alpha,\beta]$*. Then for every* $\gamma \in [\alpha,\beta]$ *fixed, the equality*

$$x(s) = x(\gamma) + \int_{\gamma}^{s} DF(x(\tau),t), \quad s \in [\alpha,\beta] \tag{1.12}$$

is valid. On the other hand, if $x\colon [\alpha,\beta] \to \mathbb{R}^n$ *satisfies* (1.12) *for some* $\gamma \in [\alpha,\beta]$ *and* $(x(t),t) \in O$ *for every* $t \in [\alpha,\beta]$*, then* x *is a solution of the generalized ODE* (1.10) *on* $[\alpha,\beta]$*.*

Proof. It is enough to apply the definition of a solution of the generalized ODE using $s_1 = \gamma$ and $s_2 = s$ to conclude the first statement of the proposition.

For the second part, consider a function $x\colon [\alpha,\beta] \to \mathbb{R}^n$ satisfying (1.12) and $(x(t),t) \in O$ for every $t \in [\alpha,\beta]$. Given $s_1, s_2 \in [\alpha,\beta]$, we can use the additive property of the integral (Theorem 1.7) to obtain that

$$x(s_2) - x(s_1) = x(\gamma) + \int_{\gamma}^{s_2} DF(x(\tau),t) - x(\gamma) - \int_{\gamma}^{s_1} DF(x(\tau),t) = \int_{s_1}^{s_2} DF(x(\tau),t).$$

It follows that x is a solution of the generalized ODE (1.10). $\square$

With Proposition 1.26 at hands, we are able to define the solution of the initial value problem (see [37]).

Definition 1.27. *A function* $x\colon [\alpha,\beta] \to \mathbb{R}^n$ *is a* ***solution of the generalized ODE*** (1.10) ***with the initial condition*** $x(t_0) = x_0$ *if*

$$x(s) = x_0 + \int_{t_0}^{s} DF(x(\tau),t)$$

is satisfied for every $s \in [\alpha,\beta]$*.*

The next proposition shows that the solution of a generalized ODE does not need to be continuous. More precisely, x is continuous at t if $F(x(t),\cdot)$ is continuous.

Proposition 1.28 ([37, Proposition 3.6])**.** *Let* $x\colon [\alpha,\beta] \to \mathbb{R}^n$ *be a solution of* (1.10). *Then the following property holds for all* $\sigma \in [\alpha,\beta]$

$$\lim_{s\to\sigma}[x(s) - F(x(\sigma),s) + F(x(\sigma),\sigma)] = x(\sigma).$$

Proof. Given $\sigma \in [\alpha,\beta]$, we know by Proposition 1.26 that

$$x(s) - \int_{\sigma}^{s} DF(x(\tau),t) = x(\sigma).$$

Besides that, Theorem 1.11 implies

$$\lim_{s\to\sigma}\left[\int_{\sigma}^{s} DF(x(\tau),t) - F(x(\sigma),s) + F(x(\sigma),\sigma)\right] = 0.$$

On the other hand, we get

$$\begin{aligned} x(s) - F(x(\sigma),s) + F(x(\sigma),\sigma) &= x(s) - \int_{\sigma}^{s} DF(x(\tau),t) - x(\sigma) \\ &\quad - F(x(\sigma),s) + F(x(\sigma),\sigma) + \int_{\sigma}^{s} DF(x(\tau),t) + x(\sigma) \end{aligned}$$

and the result follows by applying the limit when s goes to σ in the equation above. $\square$

Example 1.29 ([37])**.** *Consider a nondecreasing function* $j\colon [\alpha,\beta] \to \mathbb{R}$ *such that* $(j(t),t) \in O$*, for every* $t \in [\alpha,\beta]$*. If we define* $F(x,t) = j(t)$*, then the solution of the generalized ODE*

$$\frac{dx}{d\tau} = DF(x,t) = Dj(t)$$

is

$$\int_{s_1}^{s_2} DF(x(\tau),t) = \int_{s_1}^{s_2} Dj(t) = j(s_2) - j(s_1)$$

for every $s_1, s_2 \in [\alpha,\beta]$*. Thus, the solution* x *inherits the same properties of* j*, including its continuity.*

Proposition 1.28 shows that the solutions of a generalized ODE can be difficult to control in some sense. Because of that, we define a class of functions $\mathcal{F}(G,h,\omega)$ with special properties that will be used later to prove the existence and uniqueness of solutions of the generalized ODE (1.10).

Before the next definition, we need to fix some notation. Given $c > 0$, we define

$$B_c = \{x \in \mathbb{R}^n \colon \ \|x\| < c\}.$$

Define also

$$G = B_c \times (a,b)$$

where $-\infty < a < b < +\infty$. Finally, suppose that $h\colon [a,b] \to \mathbb{R}$ is a nondecreasing function on $[a,b]$ and $\omega\colon [0,+\infty) \to [0,+\infty)$ is a continuous, increasing function with $\omega(0) = 0$.

Definition 1.30 ([37, Definition 3.8]). *$F\colon G \to \mathbb{R}^n$ belongs to the* ***class*** *$\mathcal{F}(G,h,\omega)$ if:*

1. *for every $(x,t_1),(x,t_2) \in G$, we get*

$$\|F(x,t_2) - F(x,t_1)\| \leqslant |h(t_2) - h(t_1)|; \tag{1.13}$$

2. *for all $(x,t_1),(x,t_2),(y,t_1),(y,t_2) \in G$, we obtain*

$$\|F(x,t_2) - F(x,t_1) - F(y,t_2) + F(y,t_1)\| \leqslant |h(t_2) - h(t_1)|\omega(\|x-y\|). \tag{1.14}$$

The first property of the functions in $\mathcal{F}(G,h,\omega)$ is a type of Carathéodory condition, while the second one is a type of Osgood condition.

The next result shows how it is possible to control the integral of a function in $\mathcal{F}(G,h,\omega)$ using h. Corollary 1.33, for example, shows that we can control the variation of a solution of a generalized ODE using the function h when $F \in \mathcal{F}(G,h,\omega)$.

Theorem 1.31 ([37, Lemma 3.9]). *Suppose that $F\colon G \to \mathbb{R}^n$ satisfies* (1.13) *of Definition 1.30. Further, consider that $[\alpha,\beta] \subset (a,b)$, $x\colon [\alpha,\beta] \to \mathbb{R}^n$ is such that $(x(t),t) \in G$ for every $t \in [\alpha,\beta]$ and the integral $\int_\alpha^\beta DF(x(\tau),t)$ exists. Then*

$$\left\| \int_{s_1}^{s_2} DF(x(\tau),t) \right\| \leqslant |h(s_2) - h(s_1)|$$

for $s_1,s_2 \in [\alpha,\beta]$.

Proof. Combining (1.13) with the fact that h is a nondecreasing function, we obtain

$$|\tau - t|\,\|F(x,\tau) - F(x,t)\| \leqslant (\tau - t)(h(\tau) - h(t)).$$

As done in Example 1.29,

$$\int_{s_1}^{s_2} Dh(t) = h(s_2) - h(s_1)$$

for $s_1,s_2 \in [\alpha,\beta]$. Then,

$$\left\| \int_{s_1}^{s_2} DF(x(\tau),t) \right\| \leqslant |h(s_2) - h(s_1)|$$

follows from Theorem 1.22. □

As a direct consequence, we are able to control the solution of a generalized ODE using the function h.

Corollary 1.32 ([37, Lemma 3.10])**.** *Suppose that $F\colon G \to \mathbb{R}^n$ satisfies* (1.13) *of Definition 1.30 and that $x\colon [\alpha,\beta] \to \mathbb{R}^n$ is a solution of (1.10) where $[\alpha,\beta] \subset (a,b)$. Then*

$$\|x(s_2)-x(s_1)\| \leqslant |h(s_2)-h(s_1)|$$

for every $s_1,s_2 \in [\alpha,\beta]$.

Proof. The result follows immediately from Theorem 1.31 and the fact that for every $s_1,s_2 \in [\alpha,\beta]$, we have

$$x(s_2)-x(s_1) = \int_{s_1}^{s_2} DF(x(\tau),t)$$

by the definition of a solution of the generalized ODE (1.10). □

The next corollary is a direct consequence of the above result.

Corollary 1.33 ([37, Corollary 3.11])**.** *Suppose that $F\colon G \to \mathbb{R}^n$ satisfies* (1.13) *of Definition 1.30. If $x\colon [\alpha,\beta] \to \mathbb{R}^n$ is a solution of* (1.10) *where $[\alpha,\beta] \subset (a,b)$, then x is of bounded variation and*

$$var_{\alpha}^{\beta}(x) \leqslant h(\beta)-h(\alpha).$$

Proof. Consider $\alpha = t_0 < t_1 < \ldots < t_k = \beta$. Corollary 1.32 implies that

$$\sum_{i=1}^{k} \|x(t_i)-x(t_{i-1})\| \leqslant \sum_{i=1}^{k} h(t_i)-h(t_{i-1}) = h(\beta)-h(\alpha).$$

Since the choice of the points $t_0,t_1,\ldots,t_k$ is arbitrary, the result follows. □

Theorem 1.34 ([28, Lemma 5])**.** *Suppose that $F\colon G \to \mathbb{R}^n$ satisfies* (1.14) *of Definition 1.30. Then*

$$\left\| \int_{s_1}^{s_2} D[F(x(\tau),t)-F(y(\tau),t)] \right\| \leqslant \int_{s_1}^{s_2} \omega(\|x(t)-y(t)\|)\mathrm{d}h, \quad s_1,s_2 \in (a,b).$$

Proof. Given an arbitrary partition of $[s_1,s_2]$

$$D = \{(\tau_i,[\alpha_{i-1},\alpha_i]), i = 1,\ldots,k\},$$

we obtain from (1.14) that

$$\left\| \sum_{i=1}^{k} (F(x(\tau_i),\alpha_i) - F(x(\tau_i),\alpha_{i-1})) - (F(y(\tau_i),\alpha_i) - F(y(\tau_i),\alpha_{i-1})) \right\| \leqslant \sum_{i=1}^{k} \omega(\|x(\tau_i) - y(\tau_i)\|)(h(\alpha_i) - h(\alpha_{i-1})).$$

Since the function F is Kurzweil integrable and $\omega(\|x(t) - y(t)\|)$ is Henstock–Kurzweil integrable with respect to the nondecreasing function h, we also know that given $\varepsilon > 0$, there is a δ–fine partition D of $[a,b]$, denoted in the same way, such that

$$\left\| \int_{s_1}^{s_2} D[F(x(\tau),t) - F(y(\tau),t)] \right\| < \left\| \sum_{i=1}^{k} (F(x(\tau_i),\alpha_i) - F(x(\tau_i),\alpha_{i-1})) - (F(y(\tau_i),\alpha_i) - F(y(\tau_i),\alpha_{i-1})) \right\| + \frac{\varepsilon}{2}$$

and

$$\sum_{i=1}^{k} \omega(\|x(\tau_i) - y(\tau_i)\|)(h(\alpha_i) - h(\alpha_{i-1})) < \left\| \int_{s_1}^{s_2} \omega(\|x(t) - y(t)\|)\mathrm{d}h(t) \right\| + \frac{\varepsilon}{2}$$

Combining all the information above, we get

$$\begin{aligned} \left\| \int_{s_1}^{s_2} D[F(x(\tau),t) - F(y(\tau),t)] \right\| &< \left\| \sum_{i=1}^{k} (F(x(\tau_i),\alpha_i) - F(x(\tau_i),\alpha_{i-1})) - (F(y(\tau_i),\alpha_i) - F(y(\tau_i),\alpha_{i-1})) \right\| + \frac{\varepsilon}{2} \\ &\leqslant \sum_{i=1}^{k} \omega(\|x(\tau_i) - y(\tau_i)\|)(h(\alpha_i) - h(\alpha_{i-1})) + \frac{\varepsilon}{2} \\ &< \left\| \int_{s_1}^{s_2} \omega(\|x(t) - y(t)\|)\mathrm{d}h(t) \right\| + \varepsilon \end{aligned}$$

and the proof is complete. □

Lemma 1.35 ([37, Lemma 3.12]). *Assume that* $x\colon [\alpha,\beta] \to \mathbb{R}^n$*, where* $[\alpha,\beta] \subset (a,b)$*, is a solution of* (1.10) *and that* $F\colon G \to \mathbb{R}^n$ *satisfies* (1.13)*. Then,*

$$\lim_{t\to s+} x(t) - x(s) = \lim_{t\to s+} F(x(s),t) - F(x(s),s) \tag{1.15}$$

for $s \in [\alpha, \beta)$ *and*

$$x(s) - \lim_{t \to s-} x(t) = F(x(s), s) - \lim_{t \to s-} F(x(s), t) \tag{1.16}$$

for $s \in (\alpha, \beta]$.

Proof. We will prove that (1.15) holds and the other equation can be proved in a similar way. Observe first that $\lim_{t \to s+} h(t)$ exists, because h is a nondecreasing function. Thus, using (1.13), we can guarantee that $\lim_{t \to s+} F(x(s), t)$ also exists.

By the definition of solution of (1.10), we have

$$\lim_{t \to s+} x(t) - x(s) = \lim_{t \to s+} \int_s^t DF(x(\tau), t).$$

On the other hand, by Theorem 1.11, we get

$$\lim_{t \to s+} \int_s^t DF(x(\tau), t) = \lim_{t \to s+} [F(x(s), t) - F(x(s), s)]$$

and it is enough to combine the equalities above to obtain the result. □

So far we have presented some basic properties of the generalized ODE and also, the properties of the functions in the class $\mathcal{F}(G, h, \omega)$. To conclude this section, we show some important properties to prove the existence of solutions in the next section.

Lemma 1.36 ([37, Corollary 3.15])**.** *Suppose that* $F \in \mathcal{F}(G, h, \omega)$ *and* $x\colon [\alpha, \beta] \to \mathbb{R}^n$*, where* $[\alpha, \beta] \subset (a, b)$*, is a step function. Then* $\int_\alpha^\beta DF(x(\tau), t)$ *exists.*

Proof. By hypothesis, x is a step function. Thus, there are constants $\alpha = c_0 < c_1 < \ldots < c_k = \beta$ and vectors $C_0, C_1, \ldots, C_k \in \mathbb{R}^n$ such that $x(t) = C_j$ for $t \in (c_{j-1}, c_j]$ and $x(\alpha) = C_0$.

First, we prove that the integral $\int_{c_{j-1}}^{c_j} DF(x(\tau), t)$ exists for $j = 1, 2, \ldots, k$. Observe that for any $c_{j_1} < \sigma_1 < \sigma_2 < c_j$ and an arbitrary gauge $\delta\colon [\sigma_1, \sigma_2] \to (0, +\infty)$, we have that for any δ–fine partition $D = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{l-1}, \tau_l, \alpha_l\}$, the following equality holds:

$$\begin{aligned} S(F, D) &= \sum_{i=1}^{l} F(x(\tau_i), \alpha_i) - F(x(\tau_i), \alpha_{i-1}) \\ &= \sum_{i=1}^{l} F(C_j, \alpha_i) - F(C_j, \alpha_{i-1}) = F(C_j, \sigma_2) - F(C_j, \sigma_1). \end{aligned}$$

Thus,

$$\int_{\sigma_1}^{\sigma_2} DF(x(\tau), t) = F(C_j, \sigma_2) - F(C_j, \sigma_1).$$

On the other hand, Theorem 1.15 implies that

$$\begin{aligned}\int_{s_{j-1}}^{\sigma_2} DF(x(\tau),t) &= \lim_{t\to s_{j-1}} \left[\int_t^{\sigma_2} DF(x(\tau),t) + F(x(s_{j-1}),t) - F(x(s_{j-1}),s_{j-1})\right] \\ &= \lim_{t\to s_{j-1}} \left[F(C_j,\sigma_2) - F(C_j,t) + F(x(s_{j-1}),t) - F(x(s_{j-1}),s_{j-1})\right] \\ &= F(C_j,\sigma_2) - \lim_{t\to s_{j-1}} F(C_j,t) + \lim_{t\to s_{j-1}} F(C_{j-1},t) - F(C_{j-1},s_{j-1}).\end{aligned}$$

Similarly,

$$\int_{\sigma_1}^{s_j} DF(x(\tau),t) = F(C_j,\sigma_1) - \lim_{t\to s_j} F(C_j,t) + \lim_{t\to s_j} F(C_j,t) - F(C_j,s_j).$$

It is immediate that $\int_{s_{j-1}}^{s_j} DF(x(\tau),t)$ exists. From the additive property of the integral (Theorem 1.7), we also have that $\int_\alpha^\beta DF(x(\tau),t)$ exists and the proof is complete. □

The next theorem tells us about the existence of the integral $\int_\alpha^\beta DF(x(\tau),t)$, when x is the limit of a sequence of functions $(x_k)_{k\in\mathbb{N}}$.

Theorem 1.37 ([37, Corollary 3.14]). *Assume that $F \in \mathcal{F}(G,h,\omega)$ and $x\colon [\alpha,\beta] \to \mathbb{R}^n$ is the pointwise limit of a sequence of functions $(x_k)_{k\in\mathbb{N}}$ where $x_k\colon [\alpha,\beta] \to \mathbb{R}^n$. Suppose also that $(x_k(s),s),(x(s),s) \in G$ for every $k \in \mathbb{N}$ and $s \in [\alpha,\beta]$ and that $\int_\alpha^\beta DF(x_k(\tau),t)$ exists. Then $\int_\alpha^\beta DF(x(\tau),t)$ exists and*

$$\int_\alpha^\beta DF(x(\tau),t) = \lim_{k\to+\infty} \int_\alpha^\beta DF(x_k(\tau),t).$$

Proof. Without loss of generality, it is possible to assume that F is a real valued function. This assumption is valid because of Theorem 1.5.

To prove this result, we are going to use Corollary 1.19. For that, first we need to show that there is a gauge $\gamma\colon [\alpha,\beta] \to (0,+\infty)$ such that for every $\varepsilon > 0$ there are a function $p\colon [\alpha,\beta] \to \mathbb{N}$ and a superadditive set function Φ defined from any closed interval $J = [t_1,t_2] \subset [\alpha,\beta]$ mapping to a real number $\Phi(J) \in (0,+\infty)$ with $\Phi([\alpha,\beta]) < \varepsilon$ such that for every $\tau \in [\alpha,\beta]$, the inequality

$$|F(x(\tau),t_2) - F(x(\tau),t_1) - F(x_k(\tau),t_2) + F(x_k(\tau),t_1)| < \Phi([\alpha,\beta])$$

holds provided $m > p(\tau)$ and $\tau \in [t_1,t_2] \subset [\tau - \gamma(\tau), \tau + \gamma(\tau)]$.

Using the fact that $F \in \mathcal{F}(G,h,\omega)$, we know by (1.14) that

$$|F(x(\tau),t_2) - F(x(\tau),t_1) - F(x_k(\tau),t_2) + F(x_k(\tau),t_1)| \leqslant (h(t_2) - h(t_1))\omega(\|x_k(\tau) - x(\tau)\|)$$

where $\alpha \leqslant t_1 \leqslant \tau \leqslant t_2 \leqslant \beta$. Define $\mu\colon [\alpha,\beta] \to \mathbb{R}$ by

$$\mu(t) = \frac{\varepsilon}{h(\beta) - h(\alpha) + 1} h(t).$$

It is easy to see that μ is a nondecreasing function because h is nondecreasing. Furthermore,

$$\mu(\beta) - \mu(\alpha) = \frac{h(\beta) - h(\alpha)}{h(\beta) - h(\alpha) + 1} \varepsilon < \varepsilon.$$

Besides that, there is a $p(\tau) \in \mathbb{N}$ such that for every $k > p(\tau)$, we get

$$\omega(\|x_k(\tau) - x(\tau)\|) < \frac{\varepsilon}{h(\beta) - h(\alpha) + 1}.$$

Thus, we obtain

$$|F(x(\tau,t_2) - F(x(\tau,t_1) - F(x_k(\tau,t_2) + F(x_k(\tau,t_1))| \leqslant \mu(t_2) - \mu(t_1)$$

and the first condition of Corollary 1.19 is satisfied with $\Phi([t_1,t_2]) = \mu(t_2) - \mu(t_1)$.

To conclude the proof, we need to show that there are two functions $V, W\colon [\alpha,\beta] \to \mathbb{R}$, $V, W \in \mathcal{K}([\alpha,\beta])$ and a gauge $\delta\colon [\alpha,\beta] \to (0,+\infty)$ such that for any δ–fine partition $D = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ and every $k \in \mathbb{N}$, we have

$$V(\tau_i, \alpha_i) - V(\tau_i, \alpha_{i-1}) \leqslant F(x_k(\tau_i), \alpha_i) - F(x_k(\tau_i), \alpha_{i-1}) \leqslant W(\tau_i, \alpha_i) - W(\tau_i, \alpha_{i-1}).$$

But that condition holds because h is a Henstock–Kurzweil integrable function (see Example 1.29) and, by (1.13), we get

$$-(h(\alpha_{i+i}) - h(\alpha_i)) \leqslant F(x_k(\tau_i), \alpha_{i+i}) - F(x_k(\tau_i), \alpha_i) \leqslant h(\alpha_{i+1}) - h(\alpha_i).$$

Thus, all conditions of Corollary 1.19 are satisfied and the proof is complete. □

Combining Lemma 1.36 and Theorem 1.37, we obtain the following important result.

Corollary 1.38 ([37, Corollary 3.16])**.** *Suppose that $F \in \mathcal{F}(G,h,\omega)$ and $x\colon [\alpha,\beta] \to \mathbb{R}^n$, $[\alpha,\beta] \subset (a,b)$ is a regulated function. Then $\int_\alpha^\beta DF(x(\tau),t)$ exists.*

Proof. The result follows immediately from Lemma 1.36, Theorem 1.37 and the fact that every regulated function is the uniform limit of step functions. □

1.3 Existence and Uniqueness of Solutions of Generalized ODEs

Consider the generalized ODE

$$\frac{dx}{d\tau} = DF(x,t) \tag{1.17}$$

as in Definition 1.24 and also, suppose that $F\colon G \to \mathbb{R}^n$ belongs to $\mathcal{F}(G,h,\omega)$. We assume, as done in the previous section, that $G = \{x \in \mathbb{R}^n\colon \|x\| < c\} \times (a,b)$ and that $\omega\colon [0,+\infty) \to [0,+\infty)$ is continuous, increasing and $\omega(0) = 0$. The only difference in this section is that we are going to suppose that $h\colon [a,b] \to \mathbb{R}$ is a nondecreasing and also, left–continuous function.

We already know that the solution of (1.17) does not need to be continuous. A problem that could occur is that $(x_0,t_0) \in G$, but since the solution may jump at this point and may be no longer in G. On the other hand, we know by Lemma 1.35 that

$$\lim_{t\to t_0+} x(t) = x_0 + \lim_{t\to t_0+} F(x_0,t) - F(x_0,t_0) \tag{1.18}$$

for every $(x_0,t_0) \in G$. Thus, we use the following notation

$$x_0+ := x_0 + \lim_{t\to t_0+} F(x_0,t) - F(x_0,t_0) \tag{1.19}$$

and suppose that $x_0+ \in B_c = \{x \in \mathbb{R}^n\colon \|x\| < c\}$.

Remark 1.39. *By Corollary 1.32, we already have that $x \in BV$. It is also important to note that any function of bounded variation is also a regulated function and we can conclude that Corollary 1.38 is valid for x.*

The next two theorems are well–known results and their proofs will be omitted here. One can find the complete proof of these results in the cited references of each theorem.

Theorem 1.40 (Helly's Choice Theorem - [33])**.** *Let A be a family of functions defined on $[a,b]$. If all functions of A are such that for every $t \in [a,b]$, we have a constant $K > 0$ for which*

$$\|z(t)\| \leqslant K, \quad var_a^b(z) \leqslant K, \quad \forall z \in A.$$

Then there exists a sequence $(z_k)_{k\in\mathbb{N}} \subset A$ that converges pointwise on $[a,b]$ to a function of bounded variation ϕ.

Theorem 1.41 (Schauder–Tychonoff Fixed Point Theorem - [12, Theorem 3.2 of Appendix One])**.** *Let A be a closed convex set in a Banach space and $f\colon A \to A$ be a continuous map. If $\overline{f(A)}$ is compact, then f has a fixed point.*

As the reader may already suspect, we apply the Schauder–Tychonoff Fixed Point Theorem to prove the existence of solutions of generalized ODE.

Theorem 1.42 ([37, Theorem 4.2]). *Suppose that $F\colon G \to \mathbb{R}^n$ belongs to $\mathcal{F}(G,h,\omega)$, $(x_0,t_0) \in G$ and $x_0+ \in B_c$. Then, there are $\Delta^-,\Delta^+ > 0$ such that $x\colon [t_0-\Delta^-,t_0+\Delta^+] \to \mathbb{R}^n$ is a solution of the generalized ODE*

$$\frac{dx}{d\tau} = DF(x,t)$$

with $x(t_0) = x_0$.

Proof. The main idea of the proof is to apply the Schauder–Tychonoff Theorem (Theorem 1.41) to the map

$$T\colon \mathcal{A} \to T(\mathcal{A})$$
$$Tz(s) = x_0 + \int_{t_0}^{s} DF(z(\tau),t)$$

where $\mathcal{A}$ is a subset of BV precisely constructed to satisfy the hypotheses of such theorem.

Part I: constructing $\mathcal{A}$.

Firstly, remember that $B_c = \{x \in \mathbb{R}^n\colon \|x\| < c\}$ is an open set and we are assuming that $h\colon [a,b] \to \mathbb{R}$ is a nondecreasing and left–continuous function. Thus, there exists $\Delta^- > 0$ such that for every $t \in [t_0-\Delta^-,t_0]$, it implies that $\|x(t)-x_0\| \leqslant (h(t_0)-h(t))$ and also, $(x(t),t) \in G = B_c \times (a,b)$. Similarly, since h is a regulated function, there exists $\Delta^+ > 0$ such that for every $t \in (t_0,t_0+\Delta^+]$, we obtain $\|x(t)-x_0+\| \leqslant (h(t) - \lim\limits_{s\to t_0+} h(s))$ and $(x(t),t) \in G = B_c \times (a,b)$.

Define $\mathcal{A}$ as the set of all functions $z\colon [t_0-\Delta^-,t_0+\Delta^+] \to \mathbb{R}^n$ such that $z \in BV[t_0-\Delta^-,t_0+\Delta^+]$, $\|z(t)-x_0\| \leqslant (h(t_0)-h(t))$ if $t \in [t_0-\Delta^-,t_0]$ and $\|z(t)-x_0+\| \leqslant (h(t) - \lim\limits_{s\to t_0+} h(s)))$ if $t \in (t_0,t_0+\Delta^+]$. It is immediate that $z(t_0) = x_0$ and $\lim\limits_{s\to t_0+} z(s) = x_0+$ for every $z \in \mathcal{A}$.

Claim I: $\mathcal{A}$ is a closed and convex.

To see that $\mathcal{A}$ is convex, let $z_1,z_2 \in \mathcal{A}$ and $\alpha \in [0,1]$. Then for $t \in [t_0-\Delta^-,t_0]$, we get

$$\begin{aligned}\|\alpha z_1(t)+(1-\alpha)z_2(t)-x_0\| &\leqslant \|\alpha z_1(t)-\alpha x_0\| + \|(1-\alpha)z_2(t)-(1-\alpha)x_0\| \\ &\leqslant \alpha(h(t_0)-h(t)) + (1-\alpha)(h(t_0)-h(t)) \\ &= (h(t_0)-h(t)).\end{aligned}$$

Similarly, for every $t \in (t_0,t_0+\Delta^+]$, we obtain

$$\|\alpha z_1(t)+(1-\alpha)z_2(t)-x_0+\| \leqslant (h(t) - \lim_{s\to t_0+} h(s))$$

and $\alpha z_1 + (1-\alpha)z_2 \in \mathcal{A}$. Therefore, $\mathcal{A}$ is convex.

Now, let us prove that $\mathcal{A}$ is closed. For that, suppose that $(z_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{A}$, which converges to a function $z \in BV$. For every $t \in [t_0-\Delta^-, t_0+\Delta^+]$, we have

$$\|z_k(t) - z(t)\| \leqslant \|z_k(t) - z(t)\|_{BV}.$$

As a consequence,

$$\lim_{k\to+\infty} \|z_k(t) - z(t)\| = 0$$

converges uniformly for $t \in [t_0-\Delta^-, t_0+\Delta^+]$. Thus, for every $\varepsilon > 0$ and $t \in [t_0-\Delta^-, t_0]$, there is $N \in \mathbb{N}$ such that for every $k > N$, we get

$$\|z(t) - x_0\| \leqslant \|z(t) - z_k(t)\| + \|z_k(t) - x_0\| \leqslant \varepsilon + (h(t_0) - h(t)).$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\|z(t) - x_0\| \leqslant (h(t_0) - h(t))$. Similarly, we can show that

$$\|z(t) - x_0+\| \leqslant (h(t) - \lim_{s\to t_0+} h(s)))$$

for $t \in (t_0, t_0+\Delta^+]$ and we conclude that $z \in \mathcal{A}$.

Part II: defining $T\colon \mathcal{A} \to \mathcal{A}$.

Define the function

$$T\colon \mathcal{A} \to \mathcal{A}$$
$$Tz(s) = x_0 + \int_{t_0}^{s} DF(z(\tau), t), \quad s \in [t_0-\Delta^-, t_0+\Delta^+]. \tag{1.20}$$

Notice first that, by Corollary 1.38, the integral $\int_{t_0}^{s} DF(z(\tau), t)$ exists and the right–hand side of (1.20) makes sense.

Let us show now that $T(\mathcal{A}) \subset \mathcal{A}$. For every $s \in (t_0, t_0+\Delta^+]$, the following equalities hold

$$\begin{aligned}
\|Tz(s) - x_0+\| &= \left\| x_0 + \int_{t_0}^{s} DF(z(\tau), t) - x_0+ \right\| \\
&= \left\| \int_{t_0}^{s} DF(z(\tau), t) - \left[\lim_{r\to t_0+} F(x_0, r) - F(x_0, t_0) \right] \right\| \\
&= \left\| \int_{t_0}^{s} DF(z(\tau), t) - \left[\lim_{r\to t_0+} F(z(t_0), r) - F(z(t_0), t_0) \right] \right\| \\
&= \lim_{r\to t_0+} \left\| \int_{r}^{s} DF(z(\tau), t) \right\|,
\end{aligned} \tag{1.21}$$

where the second and the fourth equalities hold because of Theorem 1.12.

By Theorem 1.31, we get

$$\left\| \int_r^s DF(z(\tau),t) \right\| \leqslant |h(s)-h(r)|. \tag{1.22}$$

Combining (1.21) and (1.22), we conclude

$$\|Tz(s)-x_0+\| \leqslant (h(s)-\lim_{r\to t_0+} h(r)).$$

Similarly, we can show that for every $s \in [t_0-\Delta^-, t_0]$, we have

$$\|Tz(s)-x_0\| \leqslant (h(t_0)-h(s)).$$

Therefore, $Tz \in \mathcal{A}$ for every $z \in \mathcal{A}$.

Claim II: T is a continuous map.

Consider a sequence of functions $(z_k)_{k\in\mathbb{N}} \subset \mathcal{A}$ converging to a function $z \in BV$. This means that

$$\lim_{k\to+\infty} \|z_k - z\|_{BV} = 0.$$

From Claim I of Part I, $z \in \mathcal{A}$. We will show now that $\lim\limits_{k\to+\infty} \|Tz_k - Tz\|_{BV} = 0$.

Note first that by hypothesis, $F \in \mathcal{F}(G,h,\omega)$. From the linearity of the integral (Theorem 1.4), it follows that for any $x,y \in \mathcal{A}$ and $t_1,t_2 \in [t_0-\Delta^-, t_0+\Delta^+]$, the inequality holds:

$$\begin{aligned}\|Tx(t_2)-Tx(t_1)-Ty(t_2)+Ty(t_1)\| &= \left\| \int_{t_1}^{t_2} D[F(x(\tau),t)-F(y(\tau),t)] \right\| \\ &\leqslant \int_{t_1}^{t_2} \omega(\|x(t)-y(t)\|)\mathrm{d}h(t)\end{aligned}$$

where the last inequality follows from Theorem 1.34. Thus, we conclude that

$$var_{t_0-\Delta^-}^{t_0+\Delta^+}(Tx-Ty) \leqslant \int_{t_0-\Delta^-}^{t_0+\Delta^+} \omega(\|x(t)-y(t)\|)\mathrm{d}h(t).$$

Therefore,

$$\begin{aligned}\|Tz_k - Tz\|_{BV} &= \left\|Tz_k(t_0-\Delta^-) - Tz(t_0-\Delta^-)\right\| + var_{t_0-\Delta^-}^{t_0+\Delta^+}(Tz_k - Tz) \\ &\leqslant \int_{t_0-\Delta^-}^{t_0+\Delta^+} \omega(\|z_k(t)-z(t)\|)\mathrm{d}h(t).\end{aligned}$$

By Theorem 1.37, we get

$$\lim_{k\to+\infty}\int_{t_0-\Delta^-}^{t_0+\Delta^+}\omega(\|z_k(t)-z(t)\|)\mathrm{d}h(t)=0$$

and we conclude that $\lim\limits_{k\to+\infty}\|Tz_k-Tz\|=0.$

Claim III: $\overline{T(\mathcal{A})}$ **is compact.**

Since BV is a metrizable space, the notion of compactness is equivalent to sequentially compactness (see [32, Theorem 28.2]). Therefore, we are going to show that $T(\mathcal{A})$ is sequentially compact to conclude the desired property.

Consider then an arbitrary sequence $(Tz_k)_{k\in\mathbb{N}}\subset T(\mathcal{A})$. As done in Claim I, we can find a bound for the value of the preimage $z_k(t)$ for every $t\in[\alpha,\beta]\subset(a,b)$ using the function h and the variation of these functions can also be controlled by the same function h. Thus, we can apply the Helly's Choice Theorem (Theorem 1.40) to conclude that $(z_k)_{k\in\mathbb{N}}$ has a subsequence $(z_{k_n})_{k_n\in\mathbb{N}}$ converging to ϕ. Since $\mathcal{A}$ is closed, we have that $\phi\in\mathcal{A}$ and thus, $T\phi\in T(\mathcal{A})$. Besides that, the fact that T is continuous implies that $(Tz_{k_n})_{k_n\in\mathbb{N}}$ converges to $T\phi$ and it is clear that $T(\mathcal{A})$ is sequentially compact.

Therefore, by the Schauder–Tychonoff Theorem (Theorem 1.41), T has a fixed point, which implies that

$$x(s)=Tx(s)=x_0+\int_{t_0}^{s}DF(x(\tau),t)$$

and the proof is complete. □

We will not talk about maximal solutions of the generalized ODE (1.17) in this work, but the next lemma gives some information about prolongation of solutions.

Lemma 1.43 ([37, Lemma 4.4])**.** *Suppose that* $x\colon[\alpha,\beta]\to\mathbb{R}^n$ *and* $y\colon[\beta,\gamma]\to\mathbb{R}^n$ *are solutions of* (1.17) *where* $a<\alpha\leqslant\beta\leqslant\gamma<b$ *and* $x(\beta)=y(\beta)$. *Then* $z\colon[\alpha,\gamma]\to\mathbb{R}^n$, *defined by* $z(s)=x(s)$ *for* $s\in[\alpha,\beta]$ *and* $z(s)=y(s)$ *for* $s\in[\beta,\gamma]$, *is also a solution of (1.17).*

Proof. Suppose that $\alpha\leqslant s_1<\beta<s_2\leqslant\gamma$. By the additive property of the integral (Theorem 1.7), we obtain

$$\begin{aligned}z(s_2)-z(s_1)&=z(s_2)-z(\beta)+z(\beta)-z(s_1)=y(s_2)-y(\beta)+x(\beta)-x(s_1)\\&=\int_{\beta}^{s_2}DF(y(\tau),t)+\int_{s_1}^{\beta}DF(x(\tau),t)=\int_{s_1}^{s_2}DF(z(\tau),t)\end{aligned}$$

and the proof is complete. □

Next, the definition of unique solution of the generalized ODE (1.17) in the future is presented.

Definition 1.44 ([37, Definition 4.7])**.** *Let* $x\colon [\alpha_1,\beta_1]\to\mathbb{R}^n$ *be a solution of the generalized ODE* (1.17) *x is called an* ***unique solution in the future*** *if for any other solution* $y\colon [\alpha_2,\beta_2]\to\mathbb{R}^n$ *of* (1.17) *with* $x(t_0)=y(t_0)$ *for some* $t_0\in[\alpha_1,\beta_1]\cap[\alpha_2,\beta_2]$*, we have* $x(t)=y(t)$ *for every* $t\in[t_0,\beta_1]\cap[t_0,\beta_2]$.

We just defined a unique solution in the future. To finish this section, we present a result about the uniqueness of solutions in the future.

Theorem 1.45 ([37, Theorem 4.8])**.** *Suppose that* $h\colon [a,b]\to\mathbb{R}$ *is a nondecreasing and left–continuous function and* $F\in\mathcal{F}(G,h,\omega)$*. Assume also that* $\omega\colon [0,+\infty)\to[0,+\infty)$ *is continuous, nondecreasing,* $\omega(0)=0$*,* $\omega(t)>0$ *for every* $t>0$ *and for every* $s>0$*, we get*

$$\lim_{r\to 0+}\int_r^s \frac{1}{\omega(t)}\mathrm{d}t=+\infty.$$

Then every solution $x\colon [t_0,t_0+\kappa]\to\mathbb{R}^n$ *of* (1.17) *is unique in the future.*

Proof. Consider that $x,y\colon [t_0,t_0+\kappa]\to\mathbb{R}^n$ are two solutions of (1.17) with $x(t_0)=y(t_0)=x_0$. We will show that $x(t)=y(t)$ for every $t\in[t_0,t_0+\kappa]$. First, from $F\in\mathcal{F}(G,h,\omega)$, we have

$$\|F(x(\tau),t_2)-F(x(\tau),t_1)+F(y(\tau),t_2)-F(y(\tau),t_1)\|\leqslant|h(t_2)-h(t_1)|\omega(\|x(\tau)-y(\tau)\|)$$

for every $t_0\leqslant t_1\leqslant\tau\leqslant t_2\leqslant t_0+\kappa$. Therefore, by Theorem 1.34, we obtain

$$\begin{aligned}\|x(s)-y(s)\| &= \left\|\int_{t_0}^s D[F(x(\tau),t)-F(y(\tau),t)]\right\|\\ &\leqslant \int_{t_0}^s \omega(\|x(\tau)-y(\tau)\|)\mathrm{d}h(\tau)\\ &= \int_{t_0}^{t_0+\delta}\omega(\|x(\tau)-y(\tau)\|)\mathrm{d}h(\tau)+\int_{t_0+\delta}^s \omega(\|x(\tau)-y(\tau)\|)\mathrm{d}h(\tau),\end{aligned}$$

for $0<\delta<s-t_0$. From Theorem 1.12, we have

$$\begin{aligned}&\int_{t_0}^{t_0+\delta}\omega(\|x(\tau)-y(\tau)\|)\mathrm{d}h(\tau)\\ &=\omega(\|x(t_0)-y(t_0)\|)\left(\lim_{r\to t_0+}h(r)-h(t_0)\right)+\lim_{r\to t_0+}\int_r^{t_0+\delta}\omega(\|x(\tau)-y(\tau)\|)\mathrm{d}h(\tau)\\ &\leqslant \sup\{\omega(\|x(\tau)-y(\tau)\|)\colon \tau\in(t_0,t_0+\delta]\}(h(t_0+\delta)-\lim_{r\to t_0+}h(r)) =: A(\delta).\end{aligned}$$

As a consequence,

$$\|x(s)-y(s)\| \leqslant A(\delta)+\int_{t_0+\delta}^{s} \omega(\|x(\tau)-y(\tau)\|)\mathrm{d}h(\tau)$$

holds for $\delta \leqslant s \leqslant \kappa$.

Consider now $u_0>0$ and define the function $\Omega\colon (0,+\infty)\to\mathbb{R}$ as

$$\Omega(u)=\int_{u_0}^{u}\frac{1}{\omega(t)}\mathrm{d}t.$$

By Theorem 1.23, we get

$$\|x(s)-y(s)\| \leqslant \Omega^{-1}(\Omega(A(\delta))+h(s)-h(t_0+\delta))$$

for $\delta \leqslant s \leqslant \kappa$ whenever $\Omega(A(\delta))+h(t_0+\kappa)-h(t_0+\delta) \leqslant \lim\limits_{u\to+\infty}\Omega(u) \leqslant +\infty$.

Notice that $\lim\limits_{\delta\to 0+} A(\delta)=0$, since $\lim\limits_{\delta\to 0+} h(t_0+\delta)=\lim\limits_{r\to t_0+} h(r)$. Using the hypothesis, we get

$$\lim_{r\to 0+}\int_{r}^{s}\frac{1}{\omega(t)}\mathrm{d}t=+\infty$$

for every $s>0$. Furthermore,

$$\begin{aligned}\lim_{\delta\to 0+}\Omega(A(\delta))+h(t_0+\kappa)-h(t_0+\delta) &\leqslant \lim_{\beta\to 0+}\Omega(A(\beta))+h(t_0+\kappa)-h(t_0+\delta)\\ &=\lim_{\delta\to 0+}\Omega(\delta)+h(t_0+\kappa)-h(t_0+\delta)=-\infty.\end{aligned}$$

Thus, there is a $\delta_0>0$ such that for every $0<\delta<\delta_0$, the inequalities

$$\Omega(A(\delta))+h(t_0+\kappa)-h(t_0+\delta) \leqslant \lim_{u\to+\infty}\Omega(u) \leqslant +\infty,$$

hold. Therefore,

$$\|x(s)-y(s)\| \leqslant \Omega^{-1}(\Omega(A(\delta))+h(s)-h(t_0+\delta)),$$

which implies

$$\Omega(\|x(s)-y(s)\|) \leqslant \Omega(A(\delta))+h(s)-h(t_0+\delta).$$

Hence,

$$\Omega(\|x(s)-y(s)\|)-\Omega(A(\delta)) \leqslant h(s)-\lim_{\delta\to 0+} h(t_0+\delta).$$

As a consequence, we obtain

$$\int_{A(\delta)}^{\|x(s)-y(s)\|} \frac{1}{\omega(r)} dr \leqslant h(s) - \lim_{\delta\to 0+} h(t_0+\delta).$$

By contradiction, suppose that $\|x(s_0)-y(s_0)\| = B > 0$ for some $s_0 \in (t_0, t_0+\kappa]$. Then

$$\int_{A(\delta)}^{B} \frac{1}{\omega(r)} dr \leqslant h(s) - \lim_{\beta\to 0+} h(t_0+\beta) < +\infty, \quad \forall \delta \in (0, \delta_0).$$

On the other hand, by hypothesis,

$$\lim_{\delta\to 0+} \int_{A(\delta)}^{B} \frac{1}{\omega(r)} dr = +\infty.$$

Clearly, we have a contradiction. Therefore, $\|x(s_0)-y(s_0)\| = 0$ for every $s \in [t_0, t_0+\kappa]$. $\square$

1.4 Generalized Linear Differential Equations

In this section, we present the theory of **generalized linear differential equations** or simply generalized linear ODEs. Such equations are of the form

$$\frac{dx}{d\tau} = D[A(t)x + g(t)] \tag{1.23}$$

where $A\colon J \subset \mathbb{R} \to L(\mathbb{R}^n)$ is an $n \times n$ real valued matrix on the interval J and $g\colon J \subset \mathbb{R} \to \mathbb{R}^n$ is a function. Notice that (1.23) is a particular case of the generalized ODE (1.17) with $F(x,t) = A(t)x + g(t)$.

It is usual for this part of the theory to consider that A and g are **locally of bounded variation**. In other words, for every compact interval $[a,b] \subset J$, we suppose that A and g are of bounded variation. This will be important to guarantee the existence and uniqueness of solutions of this type of equation. Because of that, throughout this section, we will assume that A and g are locally of bounded variation without mentioning it in every result.

The next result is presented in the beginning of Chapter 6 in [37].

Lemma 1.46 ([37])**.** *The function $F\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defined as $F(x,t) = A(t)x + g(t)$ belongs to the class $\mathcal{F}(G,h,\omega)$. In other words, there are a nondecreasing function $h\colon [a,b] \to \mathbb{R}$, a continuous function $\omega\colon [0,+\infty) \to [0,+\infty)$ with $\omega(0) = 0$ and a set $G = B_c \times (a,b)$ such that:*

1. *for every* $(x,t_1),(x,t_2)\in G$, *we get*

$$\|F(x,t_2)-F(x,t_1)\|\leqslant|h(t_2)-h(t_1)|; \tag{1.24}$$

2. *for all* $(x,t_1),(x,t_2),(y,t_1),(y,t_2)\in G$, *we obtain*

$$\|F(x,t_2)-F(x,t_1)-F(y,t_2)+F(y,t_1)\|\leqslant|h(t_2)-h(t_1)|\omega(\|x-y\|). \tag{1.25}$$

Proof. For the first item, consider an arbitrary point $(x_0,t_0)\in\mathbb{R}^n\times(a,b)$, $c\leqslant 1$ and define

$$G=\{x\in\mathbb{R}^n:\ \|x-x_0\|\leqslant c\}\times(a,b).$$

Let $h\colon[a,b]\to\mathbb{R}$ be defined by $h(t)=(c+\|x_0\|)(\mathrm{var}_a^t A)+(\mathrm{var}_a^t g)$. Thus,

$$\|F(x,t_2)-F(x,t_1)\|\leqslant\|A(t_2)-A(t_1)\|\,\|x\|+\|g(t_2)-g(t_1)\|\leqslant|h(t_2)-h(t_1)|$$

for every $(x,t_1),(x,t_2)\in G$.

The proof of the second item follows in a similar way and it will be omitted here. □

From the definition of a generalized ODE (see Definition 1.24), it is easy to see that the solution $x\colon[\alpha,\beta]\to\mathbb{R}^n$ of a generalized linear ODE satisfies

$$x(s_2)-x(s_1)=\int_{s_1}^{s_2}d[A(t)x(\tau)]+g(s_2)-g(s_1),\quad s_1,s_2\in[\alpha,\beta].$$

It is more common to denote the equation above as

$$x(s_2)-x(s_1)=\int_{s_1}^{s_2}d[A(\tau)]x(\tau)+g(s_2)-g(s_1).$$

The next lemma is important to prove the uniqueness of solutions of a generalized linear ODE. Due to its extensive proof, we only state the result.

Lemma 1.47 ([37, Proposition 6.2]). *If* $x\in BV([a,b],\mathbb{R}^n)$, *then* $Tx\colon[a,b]\to\mathbb{R}^n$ *given by*

$$Tx(t)=\int_{t_0}^{t}d[A(\tau)]x(\tau),\quad t,t_0\in[a,b] \tag{1.26}$$

is such that $Tx\in BV([a,b],\mathbb{R}^n)$.

The next result is about the existence of solutions of a generalized linear ODE. It also gives us a hint about the uniqueness of solutions.

Theorem 1.48 ([37, Proposition 6.3])**.** *Consider $t_0, t \in [a,b]$. Then either:*

1. *the equation*
$$x(t) = \int_{t_0}^{t} d[A(\tau)]x(\tau) + f(t) \tag{1.27}$$
has a unique solution that belongs to $BV([a,b],\mathbb{R}^n)$ for any $f \in BV([a,b],\mathbb{R}^n)$;

   or

2. *the equation*
$$x(t) = \int_{t_0}^{t} d[A(\tau)]x(\tau) \tag{1.28}$$
admits a nontrivial solution in $BV([a,b],\mathbb{R}^n)$.

Proof. Using the same notation of (1.26), we can rewrite (1.27) and (1.28) as

$$x(t) - Tx(t) = f, \quad x(t) - Tx(t) = 0.$$

From Lemma 1.47 and the fact that the sum of functions of bounded variation is also a function of bounded variation, it follows that $T\colon BV([a,b],\mathbb{R}^n) \to BV([a,b],\mathbb{R}^n)$. Since T is a linear transformation, the result follows immediately from the Fredholm alternative (see [36, Theorem 4.12]). □

The theorem below gives us a way to verify if the second item of Theorem 1.48 occurs or not. In other words, it brings a condition to ensure the uniqueness of solutions of the generalized linear ODE.

Before presenting the next result, we fix some notations: I is the $n \times n$ identity matrix, $\Delta^- A, \Delta^+ A$ are defined as follows:

$$\Delta^- A := A(t) - \lim_{s \to t-} A(s),$$

$$\Delta^+ A := \lim_{s \to t+} A(s) - A(t).$$

Theorem 1.49 ([37, Proposition 6.4])**.** *Suppose that $t_0 \in [a,b]$. Then the equation*

$$x(t) = x(t_0) + \int_{t_0}^{t} d[A(\tau)]x(\tau) \tag{1.29}$$

admits only the trivial solution if and only if $I - \Delta^- A(t)$ is invertible for $t \in (t_0, b]$ and $I - \Delta^+ A(t)$ is invertible for $t \in [a, t_0)$.

Proof. Suppose that

$$I-\Delta^{-}A(t),\quad I-\Delta^{+}A(s)$$

are invertible matrices for $t\in(t_0,b]$ and $s\in[a,t_0)$, respectively. By Theorem 1.48, we already know that the solution of (1.29) exists. To prove one of the implications, it remains to show that only the trivial solution is in $BV([a,b],\mathbb{R}^n)$.

For that, define $\xi\colon [a,b]\to\mathbb{R}$ as

$$\xi(t)=\mathrm{var}_{t_0}^{t}A.$$

It is easy to see that $\xi(t_0)=0$ and that ξ is a nondecreasing function. Without loss of generality, suppose that $t_0<b$ and consider $c\in(t_0,b]$ for which $\xi(c)-\xi(t_0)<\dfrac{1}{2}$.

Let x be a solution of (1.29) in $BV([a,b],\mathbb{R}^n)$. For any $s_1,s_2\in[t_0,c]$, $s_1<s_2$, we have

$$\|x(s_2)-x(s_1)\|=\left\|\int_{s_1}^{s_2}d[A(\tau)]x(\tau)\right\|\leqslant\int_{s_1}^{s_2}\|x(s)\|\,\mathrm{dA(s)}\leqslant\int_{\mathrm{s}_1}^{\mathrm{s}_2}\|x(s)\|\,\mathrm{d}\xi(\mathrm{s}).$$

From the calculation above, it follows that

$$\mathrm{var}_{t_0}^{c}(x)\leqslant\int_{t_0}^{c}\|x(s)\|\,\mathrm{d}\xi(\mathrm{s}).$$

Applying Theorem 1.12, we get

$$\begin{aligned}\int_{t_0}^{c}\|x(s)\|\,\mathrm{d}\xi(\mathrm{s})&=\|x(t_0)\|\left[\lim_{s\to t_0+}\xi(s)-\xi(t_0)\right]+\lim_{\delta\to0+}\int_{t_0+\delta}^{c}\|x(s)\|\,\mathrm{d}\xi(\mathrm{s})\\&\leqslant\|x\|_{BV([t_0,c])}\left[\lim_{s\to t_0+}\xi(s)-\xi(t_0)\right]+\lim_{\delta\to0+}\int_{t_0+\delta}^{c}\|x\|_{BV([t_0,c])}\,\mathrm{d}\xi(s)\\&\leqslant\|x\|_{BV([t_0,c])}\left[\xi(c)-\lim_{\delta\to0+}\xi(t_0+\delta)\right].\end{aligned}$$

On the other hand, we choose c in a way that $\xi(c)-\xi(t_0)<1/2$. Combining this with the calculation above, we obtain

$$\|x\|_{BV([t_0,c])}\leqslant\|x\|_{BV([t_0,c])}\frac{1}{2}.$$

Therefore, $x(t)=0$ for every $t\in[t_0,c]$.

The next step of the proof is to show that $x(t)=0$ for every $t\in[t_0,b]$. For that, define

$$M=\sup\{t\in[t_0,b]\colon x(s)=0,\ \forall s\in[t_0,t]\}.$$

Applying Theorem 1.11, we have

$$\begin{aligned} x(M) - \lim_{s \to M-} x(s) &= \lim_{\delta \to 0+} \int_{M-\delta}^{M} d[A(\tau)]x(\tau) \\ &= \left[A(M) - \lim_{\delta \to 0+} A(M-\delta)\right] x(M) = \Delta^{-}A(M)x(M). \end{aligned}$$

Therefore, using the fact that $x(s) = 0$ for every $s \in [t_0, M)$, we obtain

$$0 = \lim_{s \to M-} x(s) = \Delta^{-}A(M)x(M) - x(M) = \left[\Delta^{-}A(M) - I\right] x(M).$$

Since $\Delta^{-}A(M) - I$ is invertible, we conclude that $x(M) = 0$.

Suppose now that $M < b$. It is possible to use the same argument we already did previously in this proof to show that there is a $c \in (M, b]$ such that $x(s) = 0$ for all $s \in [M, c]$. This is a contradiction with the definition of M and we conclude that $M = b$. Thus, $x(t) = 0$ for all $t \in [t_0, b]$. In a similar way, it is possible to show that $x(t) = 0$ for all $t \in [a, t_0]$ and we have one of the implications of the theorem.

The proof of the other implication of the theorem is omitted here, but it can be found in [37]. □

Remark 1.50. *It is easy to see from the proof above that if we are interested in the uniqueness of solutions going forward on time, we can only consider that $I - \Delta^{-}A(t)$ is invertible. This fact will be useful later in Chapter 4.*

Chapter 2

Theory of Time Scales

The theory of time scales was first introduced in 1988 by Stefan Hilger in his PhD thesis [18]. His main idea was to construct a theory that could unify discrete and continuous analysis. Since then, this concept has gained popularity and many books and articles about the subject have been published, such as [4–7, 13, 25, 34].

This chapter is based on [5] and [6]. The idea here is to present basic definitions and results from the theory of time scales, such as the concept of derivative and integrals on a time scale. We also present the definition of the exponential function in this setting.

Some of the results presented here will have their proof omitted, due to its very technical and extensive nature, but the reader can always find such proofs in the cited references presented throughout the chapter.

2.1 Definition and basic properties

We start this section by presenting the definition of a time scale.

Definition 2.1 ([5]). *A* ***time scale*** *is any closed nonempty subset of* $\mathbb{R}$. *It is usual to denote a time scale as* $\mathbb{T}$.

There are some classical examples of time scales such as when $\mathbb{T}$ is $\mathbb{R}$ or $\mathbb{Z}$, which are known as the continuous and discrete cases, respectively. There are also other important time scales such as $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, which is used on the called **quantum calculus** or q**–calculus** (see [11]). Given $q > 1$, we define this time scale as

$$q^{\mathbb{N}_0} = \{q^n \colon n \in \mathbb{N}_0\}.$$

Of course, there are many other examples of time scales and some may even combine the discrete and continuous cases to obtain hybrid types of time scales. We also assume throughout this chapter that $\mathbb{T}$ has the topology induced from $\mathbb{R}$ with the standard topology.

We present now the backward and forward operators and also, the graininess function. All these functions are going to be used later to define a derivative and an integral of a function on a time scale.

Definition 2.2 ([5, Definition 1.1]). *Consider a time scale $\mathbb{T}$ and suppose $t \in \mathbb{T}$. The **forward jump operator** $\sigma\colon \mathbb{T} \to \mathbb{T}$ is defined as*

$$\sigma(t) = \inf\{s \in \mathbb{T}\colon s > t\}.$$

*Similarly, the **backward jump operator** $\rho\colon \mathbb{T} \to \mathbb{T}$ is defined as*

$$\rho(t) = \sup\{s \in \mathbb{T}\colon s < t\}.$$

*Finally, the **graininess function** $\mu\colon \mathbb{T} \to [0, +\infty)$ is defined as*

$$\mu(t) = \sigma(t) - t.$$

Here, we use the convention that $\inf \emptyset = \sup \mathbb{T}$ *and* $\sup \emptyset = \inf \mathbb{T}$. *Thus, $\sigma(t) = t$ if $t = \sup \mathbb{T}$ and $\rho(t) = t$ if $t = \inf \mathbb{T}$.*

Note first that $\rho(t), \sigma(t) \in \mathbb{T}$ whenever $t \in \mathbb{T}$ because $\mathbb{T}$ is a closed subset of the real line. We can also classify the points of the time scale using the functions of the previous definition. The table below describes these classifications and can be found in [5].

Table 2.1 Classification of points, [5, Table 1.1].

Classification of t	Property
right–scattered	$t < \sigma(t)$
right–dense	$t = \sigma(t)$
left–scattered	$\rho(t) < t$
left–dense	$\rho(t) = t$
dense	$\rho(t) = t = \sigma(t)$
isolated	$\rho(t) < t < \sigma(t)$

The figure below illustrates each case of the classification from the table above. In the figure, t_1 is right–dense, left–dense and also, dense. t_2 is left–dense and right–scattered. t_3 is left–scattered and right dense. t_4 is right–scattered, left–scattered and also, isolated.

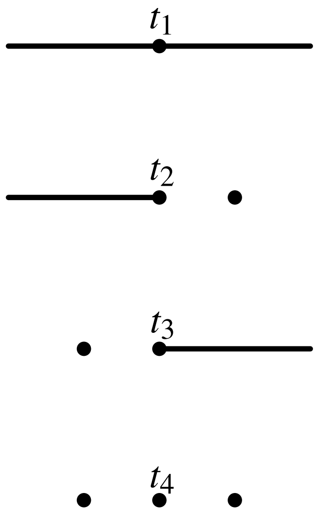

Figure 2.1 Classification of points [5, Figure 1.1].

Before we continue with some results, we present some examples of values of forward, backward operators and graininess function for some time scales.

Table 2.2 Examples of forward, backward and graininess functions.

Time scale	$\sigma(t)$	$\rho(t)$	$\mu(t)$
$\mathbb{R}$	t	t	0
$\mathbb{Z}$	$t+1$	$t-1$	1
$q^{\mathbb{N}_0}$	qt	$\frac{t}{q}$	$(q-1)t$

The idea now is to define a derivative of a function on a time scale, the so–called **delta derivative**. For that, we need to establish a definition and notation for a specific subset in the time scale. The motivation behind of this definition is to exclude a specific point that may be problematic for the delta derivative. This fact is explained just after the definition of a delta derivative of a function.

Definition 2.3 ([5, Definition 1.10]). *We define $\mathbb{T}^\kappa$ as*

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < +\infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = +\infty. \end{cases}$$

Now, we present the delta derivative.

Definition 2.4 ([5, Definition 1.10]). *Suppose that $f \colon \mathbb{T} \to \mathbb{R}^n$ and $t \in \mathbb{T}^\kappa$. The **delta derivative** of f at t, if it exists, is the vector $f^\Delta(t)$ with the following property: given $\varepsilon > 0$, there is a number $\delta > 0$ such that*

$$\left\| [f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s] \right\| \leqslant \varepsilon |\sigma(t) - s|, \quad \forall s \in (t - \delta, t + \delta) \cap \mathbb{T}.$$

*The function f is **delta differentiable** on $\mathbb{T}^\kappa$ if $f^\Delta(t)$ exists for every $t \in \mathbb{T}^\kappa$. In this case, we say that $f^\Delta \colon \mathbb{T}^\kappa \to \mathbb{R}^n$ is the **delta derivative** of f in $\mathbb{T}^\kappa$.*

Remark 2.5. *To see that $f^\Delta \colon \mathbb{T}^\kappa \to \mathbb{R}^n$ is well–defined, we need to prove that $f^\Delta(t)$ is unique for every $t \in \mathbb{T}^\kappa$. Therefore, consider $\alpha, \beta \in \mathbb{R}^n$ satisfying the delta derivative property of f at t. Hence, given an arbitrary $\varepsilon > 0$, there is a $\delta > 0$ such that for every $s \in (t - \delta, t + \delta) \cap \mathbb{T}$, we have*

$$\begin{aligned} \|(\alpha - \beta)[\sigma(t) - s]\| &\leqslant \|[f(\sigma(t)) - f(s)] - \alpha[\sigma(t) - s]\| + \|[f(\sigma(t)) - f(s)] - \beta[\sigma(t) - s\| \\ &\leqslant 2\varepsilon \, |\sigma(t) - s| \, . \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\sigma(t) = s$ or $\alpha = \beta$. The only way that $\sigma(t) = s$ for every $s \in (t - \delta, t + \delta)$ is when $t = \sup \mathbb{T}$ and t is a left–scattered point. This case is excluded because $t \in \mathbb{T}^\kappa$ and it follows that $\alpha = \beta$. See [5].

The remark above also shows why it is important to exclude points in $\mathbb{T} \setminus \mathbb{T}^\kappa$. If we did not exclude these points, it would be possible to have multiple values of delta derivative for them.

The next result presents some important characterization of the delta derivative.

Theorem 2.6 ([5, Theorem 1.16]). *Consider a function $f \colon \mathbb{T} \to \mathbb{R}^n$ and a point $t \in \mathbb{T}^\kappa$. The following statements are true:*

1. *if f is delta differentiable at t, then f is also continuous at t;*

2. *if t is right–scattered and f is continuous at t, then f^Δ exists and*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)};$$

3. *if t is right–dense, then $f^{\Delta}(t)$ exists if and only if*

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

also exists. In this case, we obtain

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s};$$

4. *if f is differentiable at t, then*

$$f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t).$$

Proof. **Item 1.** By hypothesis, f is delta differentiable at t. Given $0<\varepsilon<1$, let

$$\varepsilon_0=\frac{\varepsilon}{1+\left\|f^{\Delta}(t)\right\|+2\mu(t)}.$$

By definition, there is a $\delta>0$ such that

$$\left\|f(\sigma(t))-f(s)-f^{\Delta}(t)[\sigma(t)-s]\right\|\leqslant\varepsilon_0\left|\sigma(t)-s\right|,\qquad\forall s\in(t-\delta,t+\delta)\cap\mathbb{T}.$$

Therefore, for every $s\in(t-\delta,t+\delta)\cap(t-\varepsilon_0,t+\varepsilon_0)\cap\mathbb{T}$, we obtain

$$\begin{aligned}\|f(t)-f(s)\|&\leqslant\left\|-f(\sigma(t))+f(t)+f^{\Delta}(t)[\sigma(t)-s]\right\|+\left\|f^{\Delta}(t)(t-s)\right\|\\&\quad+\left\|f(\sigma(t))-f(s)-f^{\Delta}(t)[\sigma(t)-t]\right\|\\&<\varepsilon_0\left|\sigma(t)-t\right|+\varepsilon_0\left|\sigma(t)-s\right|+\varepsilon_0\left\|f^{\Delta}(t)\right\|\\&<\varepsilon_0\mu(t)+\varepsilon_0\left|\sigma(t)-t+t-s\right|+\varepsilon_0\left\|f^{\Delta}(t)\right\|\\&\leqslant\varepsilon_0\mu(t)+\varepsilon_0\left|t-s\right|+\varepsilon_0\mu(t)+\varepsilon_0\left\|f^{\Delta}(t)\right\|\\&<\varepsilon_0\left[2\mu(t)+1+\left\|f^{\Delta}(t)\right\|\right]=\varepsilon.\end{aligned}$$

The last inequality above holds because $s\in(t-\varepsilon_0,t+\varepsilon_0)$ and $0<\varepsilon_0<1$.

Item 2. Assume now that f is continuous at t and t is right–scattered. Since f is continuous at t, we get

$$\lim_{s\to t}\frac{f(\sigma(t))-f(s)}{\sigma(t)-s}=\frac{f(\sigma(t))-f(t)}{\mu(t)}.$$

Using the definition of limit, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left\| \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - \frac{f(\sigma(t)) - f(t)}{\mu(t)} \right\| < \varepsilon, \quad \forall s \in (t-\delta, t+\delta).$$

Therefore,

$$\left\| f(\sigma(t)) - f(s) - \frac{f(\sigma(t)) - f(t)}{\mu(t)} [\sigma(t) - s] \right\| \leqslant \varepsilon \left| \sigma(t) - s \right|, \quad \forall s \in (t-\delta, t+\delta),$$

and the result follows.

Item 3. Suppose that t is right–dense and

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s} = \alpha.$$

Since t is right–dense, $\sigma(t) = t$. By definition, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left\| \frac{f(t) - f(s)}{t - s} - \alpha \right\| \leqslant \varepsilon, \quad \forall s \in (t-\delta, t+\delta) \cap \mathbb{T} \text{ and } s \neq t.$$

Then, for every $s \in (t-\delta, t+\delta) \cap \mathbb{T}$, we get

$$\| [f(\sigma(t)) - f(s)] - \alpha [\sigma(t) - s] \| = \| [f(t) - f(s)] - \alpha [t - s] \| \leqslant \varepsilon \left| t - s \right| = \varepsilon \left| \sigma(t) - s \right|.$$

The reciprocal of this result can be proved in a similar way and, because of that, its proof will be omitted.

Item 4. Assume that f is delta differentiable at t. If t is right–dense, then $\sigma(t) = t$ and it is immediate that

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t),$$

because $\mu(t) = \sigma(t) - t = 0$ in this case.

Suppose now that t is right–scattered. Combining the first and second items of this theorem, we obtain

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$$

and the proof is complete. □

The item 4 from Theorem 1.17 is a very useful characterization of the delta derivative and it will be used to prove many results in this chapter.

The next theorem presents some other properties of the delta derivative. As the reader may notice, they are very similar to the rules from the classical derivative.

Theorem 2.7 ([5, Theorem 1.20]). *Suppose that $f, g\colon \mathbb{T} \to \mathbb{R}^n$ and $h, j\colon \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^\kappa$. Then:*

1. *for any constants $\alpha, \beta \in \mathbb{R}$, we have*
$$(\alpha f + \beta g)^\Delta(t) = \alpha f^\Delta(t) + \beta g^\Delta(t);$$
2. *the product $fg\colon \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*
$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f^\Delta(t)g(\sigma(t)) + f(t)g^\Delta(t);$$
3. *suppose that $h(t)h(\sigma(t)) \neq 0$. Then,*
$$\left(\frac{1}{h}\right)^\Delta(t) = -\frac{h^\Delta(t)}{h(t)h(\sigma(t))};$$
4. *$h(t)h(\sigma(t)) \neq 0$ implies that*
$$\left(\frac{j}{h}\right)^\Delta(t) = \frac{j^\Delta(t)h(t) - j(t)h^\Delta(t)}{h(t)h(\sigma(t))}.$$

Proof. **Item 1.** We are going to assume that $\alpha, \beta > 0$ and the other cases follow in a similar way. Given $\varepsilon > 0$, there are $\delta_1, \delta_2 > 0$ such that
$$\left\| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right\| \leqslant \frac{\varepsilon}{2\alpha} |\sigma(t) - s|, \quad \forall s \in (t - \delta_1, t + \delta_1) \cap \mathbb{T},$$
and
$$\left\| g(\sigma(t)) - g(s) - g^\Delta(t)[\sigma(t) - s] \right\| \leqslant \frac{\varepsilon}{2\beta} |\sigma(t) - s|, \quad \forall s \in (t - \delta_2, t + \delta_2) \cap \mathbb{T}.$$
Therefore, for every $s \in (t - \delta_1, t + \delta_1) \cap (t - \delta_2, t + \delta_2) \cap \mathbb{T}$, we get
$$\begin{aligned} &\left\| (\alpha f + \beta g)(\sigma(t)) - (\alpha f + \beta g)(s) - \left[\alpha f^\Delta(t) + \beta g^\Delta(t)\right] [\sigma(t) - s] \right\| \\ &\leqslant \left\| \alpha f(\sigma(t)) - \alpha f(s) - \alpha f^\Delta(t)[\sigma(t) - s] \right\| + \left\| \beta g(\sigma(t)) - \beta g(s) - \beta g^\Delta(t)[\sigma(t) - s] \right\| \\ &\leqslant \alpha \frac{\varepsilon}{2\alpha} |\sigma(t) - s| + \beta \frac{\varepsilon}{2\beta} |\sigma(t) - s| = \varepsilon |\sigma(t) - s|. \end{aligned}$$

Item 2. Now, we will prove that

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t).$$

The other equality can be obtained simply by interchanging f and g. Given $0 < \varepsilon < 1$, define

$$\varepsilon_0 = \left[1 + \|f(t)\| + \left\|g^{\Delta}(t)\right\| + \|g(\sigma(t))\|\right]^{-1}\varepsilon.$$

Thus, there are $\delta_1, \delta_2, \delta_3 > 0$ for which

$$\left\|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\right\| \leqslant \varepsilon_0\,|\sigma(t) - s|, \quad \forall s \in (t - \delta_1, t + \delta_1) \cap \mathbb{T},$$

$$\left\|g(\sigma(t)) - g(s) - g^{\Delta}(t)[\sigma(t) - s]\right\| \leqslant \varepsilon_0\,|\sigma(t) - s|, \quad \forall s \in (t - \delta_2, t + \delta_2) \cap \mathbb{T},$$

$$\|f(t) - f(s)\| \leqslant \varepsilon_0, \quad \forall s \in (t - \delta_3, t + \delta_3) \cap \mathbb{T}$$

where the last inequality holds, since f is continuous at t.

Suppose now that $s \in (t - \delta_1, t + \delta_1) \cap (t - \delta_2, t + \delta_2) \cap (t - \delta_3, t + \delta_3) \cap \mathbb{T}$. Then, adding and subtracting the correct terms, we have

$$\begin{aligned}
&\left\|(fg)(\sigma(t)) - (fg)(s) - \left[f^{\Delta}(t)g(\sigma(t)) + f(t)g^{\Delta}(t)\right](\sigma(t) - s)\right\| \\
&\leqslant \left\|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\right\| \|g(\sigma(t))\| + \left\|g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t) - s)\right\| \|f(t)\| \\
&\quad + \left\|g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t) - s)\right\| \|f(t) - f(s)\| + \left\|(\sigma(t) - s)g^{\Delta}(t)[f(s) - f(t)]\right\| \\
&\quad \leqslant \varepsilon_0 \|g(\sigma(t))\|\,|\sigma(t) - s| + \varepsilon_0 \|f(t)\|\,|\sigma(t) - s| + \varepsilon_0^2\,|\sigma(t) - s| + \varepsilon_0 \left\|g^{\Delta}(t)\right\| |\sigma(t) - s| \\
&\quad < \varepsilon_0 \|g(\sigma(t))\|\,|\sigma(t) - s| + \varepsilon_0 \|f(t)\|\,|\sigma(t) - s| + \varepsilon_0\,|\sigma(t) - s| + \varepsilon_0 \left\|g^{\Delta}(t)\right\| |\sigma(t) - s| \\
&\quad = |\sigma(t) - s|\,\varepsilon_0 \left[\|g(\sigma(t))\| + \|f(t)\| + 1 + \left\|g^{\Delta}(t)\right\|\right] \\
&\quad = |\sigma(t) - s|\,\varepsilon.
\end{aligned}$$

Item 3. Assume that $h(t)h(\sigma(t)) \neq 0$. Suppose first that t is right–dense. Applying Theorem 2.6, we get

$$h^{\Delta}(t) = \lim_{s \to t} \frac{h(t) - h(s)}{t - s}.$$

Once more, by Theorem 2.6, we obtain

$$\left(\frac{1}{h}\right)^{\Delta}(t) = \lim_{s \to t} \frac{\frac{1}{h}(t) - \frac{1}{h}(s)}{t - s} = \lim_{s \to t} \frac{h(s) - h(t)}{h(t)h(s)} \frac{1}{t - s} = \frac{-h^{\Delta}(t)}{h(t)h(t)}$$

and the result follows from the fact that $t = \sigma(t)$ when t is right–dense.

Suppose now that t is right–scattered. By Theorem 2.6,

$$h^{\Delta}(t) = \frac{h(\sigma(t)) - h(t)}{\mu(t)}.$$

Thus, it follows that

$$\begin{aligned}
\left(\frac{1}{h}\right)^{\Delta}(t) &= \frac{\frac{1}{h}(\sigma(t)) - \frac{1}{h}(t)}{\mu(t)} \\
&= \frac{\frac{1}{h(\sigma(t))} - \frac{1}{h(t)}}{\sigma(t) - t} \\
&= \frac{\frac{h(t) - h(\sigma(t))}{h(\sigma(t))h(t)}}{\sigma(t) - t} \\
&= \frac{-(h(\sigma(t)) - h(t))}{\sigma(t) - t}\frac{1}{h(\sigma(t))h(t)} = \frac{-h^{\Delta}(t)}{h(\sigma(t))h(t)}.
\end{aligned}$$

Item 4. It follows directly from items 2 and 3. □

To finish this section, we present the definition and some important properties of a regulated and rd–continuous function. Such definitions will appear later in the end of this chapter and also, in Chapters 3 and 4.

Definition 2.8 ([5, Definition 1.57]). *A function $f\colon \mathbb{T} \to \mathbb{R}^n$ is called* ***regulated*** *if its right-sided limit exists for every right–dense point in $\mathbb{T}$ and its left–sided limit exists for every left–dense point in $\mathbb{T}$.*

Definition 2.9 ([5, Definition 1.58]). *A function $f\colon \mathbb{T} \to \mathbb{R}^n$ is called* ***rd–continuous*** *if it is continuous at every right–dense point in $\mathbb{T}$ and its left–sided limit exists for every left–dense point in $\mathbb{T}$. We denote as*

$$C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$$

the set of all rd–continuous functions.

The following theorem shows how the definitions above are connected.

Theorem 2.10 ([5, Theorem 1.60]). *Suppose that $f\colon \mathbb{T} \to \mathbb{R}^n$ is a function. Then:*

1. *if f is continuous, then f is also rd–continuous;*
2. *if f is rd–continuous, then f is also regulated;*

3. *the operator* $\sigma\colon \mathbb{T}\to\mathbb{T}$ *is rd–continuous;*

4. *if* f *is regulated or rd–continuous, then* $f\circ\sigma\colon \mathbb{T}\to\mathbb{R}^n$ *has the same property;*

5. *suppose that* f *is continuous and* $g\colon \mathbb{T}\to\mathbb{T}$ *is regulated or rd–continuous, then* $f\circ g$ *has the same property.*

Proof. Items 1 and 2 follow directly from the definition of continuous, regulated and rd–continuous functions.

For item 3, remember first that the jump operator is defined as follows:

$$\sigma(t)=\inf\{s\in\mathbb{T}\colon\, s>t\}.$$

If $t\in\mathbb{T}$ is right–dense, then it is easy to see that $\sigma(t)=t$. It is also immediate to see that $\lim\limits_{s\to t}\sigma(s)=t$ in this case, regardless of whether t is left–dense or left–scattered and we obtain one of the conditions for σ to be rd–continuous.

Suppose now that $t\in\mathbb{T}$ is left–dense. Hence, it is easy to see that there is a neighborhood U of t where $\sigma(s)=s$ for every $s<t$. Thus, we conclude that $\lim\limits_{s\to t-}\sigma(s)=t$ and, therefore, σ is rd–continuous.

For item 4, suppose that f is a regulated function. Hence, for every right–dense point $t\in\mathbb{T}$, $\lim\limits_{s\to t+}f(s)$ and $\lim\limits_{s\to t+}\sigma(s)$ exist. Thus, it follows that $\lim\limits_{s\to t+}f(\sigma(s))$ also exists. We can prove in a similar way that the left–sided limit $\lim\limits_{s\to t-}f(\sigma(s))$ exists for every left–dense point $t\in\mathbb{T}$. Therefore, $f\circ\sigma$ is regulated. The other case of this item follows analogously.

Item 5 can be proved in a similar way of item 4 and its proof will be omitted. $\square$

2.2 Integration

Throughout this chapter, we based our definitions and results in [5] and [6]. However, in this section, we will define an integral which is not presented in that book. Since this dissertation involves the Henstock–Kurzweil–Stieltjes integral, we present here the **Henstock–Kurzweil** $\boldsymbol{\Delta}$**–integral** (see [34]) instead of the Δ–integral which is in [5] and the Δ–integral in the sense of Riemann or Lebesgue which is in [6]. We will comment more about the relation between these different integrals on time scales after we define the Henstock–Kurzweil Δ–integral.

Before we start with the definition, let us fix a useful notation. We denote the interval of points between a and b that are in the time scale as

$$[a,b]_{\mathbb{T}}=[a,b]\cap\mathbb{T}.$$

Now, we present the definition of a tagged partition, a Δ–gauge and also a δ–fine partition of an interval $[a,b]_{\mathbb{T}}$. All of these concepts are presented in [34].

Definition 2.11 ([34]). *A* ***tagged partition*** *of $[a,b]_{\mathbb{T}}$ is a collection of points $a = s_0 < s_1 < \ldots < s_k = b$ where $k \in \mathbb{N}$ and also, tags $\tau_1, \tau_2, \ldots, \tau_k \in [a,b]_{\mathbb{T}}$ such that $\tau_i \in [s_{i-1}, s_i]_{\mathbb{T}}$ for $i = 1, 2 \ldots, k$. We often denote the partition as*

$$D = \{s_0, \tau_1, s_1, \ldots, s_{k-1}, \tau_k, s_k\}.$$

We also define a ***Δ–gauge*** *as a function $\delta\colon [a,b]_{\mathbb{T}} \to (0,+\infty) \times (0,+\infty)$ given by*

$$\delta(t) = (\delta_L(t), \delta_R(t)),$$

where $\delta_R(t) \geqslant \mu(t)$, $\delta_R(t) > 0$ and $\delta_L(t) > 0$ for every $t \in [a,b]_{\mathbb{T}}$.

Given a Δ–gauge $\delta\colon [a,b]_{\mathbb{T}} \to (0,+\infty) \times (0,+\infty)$, the partition is called ***δ–fine*** *if*

$$[s_{i-1}, s_i] \subset [\tau_i - \delta_L(\tau_i), \tau_i + \delta_R(\tau_i)]$$

for $i = 1, 2, \ldots, k$.

With the definitions above, we can define the Henstock–Kurzweil Δ–integral. It is defined in [34].

Definition 2.12 ([34]). *A function $f\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is* ***Henstock–Kurzweil Δ–integrable*** *if, given $\varepsilon > 0$, there is an $I \in \mathbb{R}^n$ and a Δ–gauge such that for every δ–fine partition*

$$D = \{s_0, \tau_1, s_1, \ldots, s_{k-1}, \tau_k, s_k\}$$

we obtain

$$\left\| \sum_{i=1}^{k} f(\tau_i)(s_i - s_{i-1}) - I \right\| < \varepsilon.$$

In this case, I is called the ***Henstock–Kurzweil Δ–integral*** *and it is denoted $I = \int_a^b f(t)\Delta t$. As done in the last chapter, we can also denote the sum above as $S(f,D)$.*

Remark 2.13. *It is also possible to consider a Stieltjes–type integral. For that, given a function $g\colon [a,b]_{\mathbb{T}} \to \mathbb{R}$, we only need to change the sum in the definition of the Henstock–Kurzweil Δ–integral to*

$$\sum_{i=1}^{k} f(\tau_i)(g(s_i) - g(s_{i-1})).$$

In this case, the integral is denoted as $\int_a^b f(t)\Delta g(t)$. *All results presented in this chapter can be changed for this more general type of integral. For more details about that, see [31].*

Remark 2.14. *The Henstock–Kurzweil* Δ*–integral generalized the* Δ*–integral (see [5]), the Riemann* Δ*–integral and the Lebesgue* Δ*–integral (the last two integrals are defined in [6]). The* Δ*–integral uses the idea of an anti–derivative of a function (see [5]) and it is a particular case of the Riemann* Δ*–integral (see [6]).*

The relations between all these integrals on time scales are analogous to the relations between the Henstock–Kurzweil integral, the Riemann integral and the Lebesgue integral, shown in [3], i.e., any function that is Riemann integrable (or Lebesgue integrable) is also Henstock–Kurzweil integrable, but the reciprocal is not true.

Before we can show some properties of the Henstock–Kurzweil Δ–integral, let us prove a version of the Cousin Lemma (Lemma 1.2) for time scales, so we can guarantee that the Henstock–Kurzweil Δ–integral is well–defined.

Lemma 2.15 ([34, Lemma 1.9])**.** *Given a* Δ*–gauge* δ *of* $[a,b]_{\mathbb{T}}$*, there exists a* δ*–fine partition of* $[a,b]_{\mathbb{T}}$.

Proof. Suppose, by contradiction, that there is no δ–fine partition of $[a,b]_{\mathbb{T}}$. Define

$$c = \sup\left\{t \in [a,b]_{\mathbb{T}} \colon t \leqslant \frac{b-a}{2}\right\}, \quad d = \inf\left\{t \in [a,b]_{\mathbb{T}} \colon t \geqslant \frac{b-a}{2}\right\}.$$

By hypothesis, either $[a,c]_{\mathbb{T}}$ or $[d,b]_{\mathbb{T}}$ have no δ–fine partition. Let $[a_1,b_1]_{\mathbb{T}}$ be one of these intervals that has no δ–fine partition. Analogously, we define

$$c_1 = \sup\left\{t \in [a_1,b_1]_{\mathbb{T}} \colon t \leqslant \frac{b_1-a_1}{2}\right\}, \quad d_1 = \inf\left\{t \in [a_1,b_1]_{\mathbb{T}} \colon t \geqslant \frac{b_1-a_1}{2}\right\}.$$

Either $[a_1,c_1]_{\mathbb{T}}$ or $[d_1,b_1]_{\mathbb{T}}$ have no δ–fine partition, because there is no δ–fine partition of $[a_1,b_1]$. Now, we define $[a_2,b_2]$ as one of the intervals without any δ–fine partition.

Proceeding that way, we obtain a sequence of nested intervals $[a_n,b_n]_{\mathbb{T}}$, $n \in \mathbb{N}$, such that $(b_n - a_n) \leqslant (b-a)/2^n$. Consider

$$\tau = \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n.$$

Hence, there is $m \in \mathbb{N}$ sufficiently large such that

$$\tau - \delta_L(\tau) \leqslant a_m \leqslant b_m \leqslant \tau + \delta_R(\tau).$$

Therefore, $D = \{a_m, \tau, b_m\}$ is a δ–fine partition of $[a_m,b_m]_{\mathbb{T}}$, which is a contradiction. $\square$

We show now some basic properties of the Henstock–Kurzweil Δ–integral.

Theorem 2.16 ([34, Theorem 2.12]). *Consider $f\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ and $c \in [a,b]_{\mathbb{T}}$. f is Henstock–Kurzweil Δ–integrable on $[a,b]_{\mathbb{T}}$ if and only if it is Henstock–Kurzweil Δ–integrable on $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$. It also follows that*

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t.$$

Proof. Suppose that f is Henstock–Kurzweil Δ–integrable on $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$. Therefore, given $\varepsilon > 0$, there is a Δ–gauge $\delta^A = (\delta_L^A, \delta_R^A)$ of $[a,c]_{\mathbb{T}}$ and a Δ–gauge δ^B of $[c,b]_{\mathbb{T}}$ such that for every $\delta^B = (\delta_L^B, \delta_R^B)$–partition D_A and every δ^B–partition D_B, we have

$$\|S(f,D_A) - I_A\| < \frac{\varepsilon}{2}, \quad \|S(f,D_B) - I_B\| < \frac{\varepsilon}{2}.$$

Define the Δ–gauge $\delta = (\delta_L, \delta_R)$ on the interval $[a,b]_{\mathbb{T}}$ as: $\delta_L(t) = \delta_L^A$ if $t \in [a,c)$, $\delta_L(t) = \min\{\delta_L^B, (t-c)/2\}$ if $t \in (c,b]_{\mathbb{T}}$ and

$$\delta_L(c) = \begin{cases} \delta_L^A(c), & \text{if } c \text{ is left–dense, i.e., } c = \rho(c), \\ \min\left\{\delta_L^A(c), \dfrac{c-\rho(c)}{2}\right\}, & \text{otherwise.} \end{cases}$$

Analogously, define $\delta_R(t) = \delta_R^B(t)$ if $t \in [c,b]_{\mathbb{T}}$ and

$$\delta_R(t) = \min\left\{\delta_R^B, \max\left\{\mu(t), \frac{c-t}{2}\right\}\right\}, \quad t \in [a,c)_{\mathbb{T}}.$$

Now, consider a δ–fine partition of $[a,b]_{\mathbb{T}}$

$$D = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

Let $[\alpha_{m-1}, \alpha_m]_{\mathbb{T}}$ be an interval that contains c. Then, no point in $[\alpha_{m-1}, c)_{\mathbb{T}}$ can be a tag because $\delta_R(t) \leqslant (t-c)/2 < (t-c)$ in that interval. Analogously, no point in $(c, \alpha_m]_{\mathbb{T}}$ can be a tag. Therefore, c has to be a tag of $[\alpha_{m-1}, \alpha_m]_{\mathbb{T}}$ and $\alpha_{m-1} = c$ if c is left–scattered, because of the definition of $\delta_L(c)$. Therefore, if c is left–dense, we obtain

$$f(c)(\alpha_m - \alpha_{m-1}) = f(c)(\alpha_m - c) + f(c)(c - \alpha_{m-1}).$$

Thus, we get

$$\begin{aligned}
&\left\|\sum_{i=1}^{k} f(\tau_i)(\alpha_i-\alpha_{i-1})-I_A-I_B\right\| \\
&=\left\|\sum_{i=1}^{m-1} f(\tau_i)(\alpha_i-\alpha_{i-1})+f(c)(\alpha_m-\alpha_{m-1})+\sum_{i=m+1}^{k} f(\tau_i)(\alpha_i-\alpha_{i-1})-I_A-I_B\right\| \\
&\leqslant\left\|\sum_{i=1}^{m-1} f(\tau_i)(\alpha_i-\alpha_{i-1})+f(c)(c-\alpha_{m-1})-I_A\right\| \\
&\quad+\left\|f(c)(\alpha_m-c)+\sum_{i=m+1}^{k} f(\tau_i)(\alpha_i-\alpha_{i-1})-I_B\right\| \\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}$$

The case where c is left–scattered is analogous. Because of that, it will be omitted here. □

As the reader may have noticed, the proof of Theorem 2.16 is similar to the proof of Theorem 1.7. In fact, most of the properties of this integral can be obtained doing some adaptations in the proofs of the properties of the Kurzweil integral, presented in Chapter 1. Because of that, we show other properties of the Henstock–Kurzweil Δ–integral without presenting its proof.

Theorem 2.17 ([34]). *Let $f,g\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ be Henstock–Kurzweil Δ–integrable on $[a,b]_{\mathbb{T}}$. Then:*

1. *given $\alpha,\beta\in\mathbb{R}$, the function $\alpha f+\beta g$ is also Henstock–Kurzweil Δ–integrable on $[a,b]_{\mathbb{T}}$ and*
$$\int_a^b (\alpha f+\beta g)\Delta t=\alpha\int_a^b f\Delta t+\beta\int_a^b g\Delta t;$$
2. *if n=1 and $f(t)\leqslant g(t)$ a.e. on $[a,b)_{\mathbb{T}}$, then*
$$\int_a^b f(t)\Delta t\leqslant\int_a^b g(t)\Delta t.$$

2.3 Exponential Function

In this section, we define and show some properties of the exponential function on time scales. However, instead of presenting the 1–dimensional exponential and then, showing the n–dimensional exponential, we study here only the n–dimensional case and the other one is a particular case.

Let us start then by presenting the definition of rd–continuous and some properties of the delta derivative for higher dimensions.

Definition 2.18 ([5, Definition 5.1]). *A matrix $A\colon \mathbb{T} \to \mathbb{R}^{m\times n}$ is called* ***rd–continuous*** *if each of its entries are rd–continuous on $\mathbb{T}$. In this case, we also denote that A belongs to*

$$C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{m\times n}).$$

We also say that A is ***delta differentiable*** *on $\mathbb{T}^\kappa$ if each entry of A is differentiable on $\mathbb{T}^\kappa$. In this case, we define*

$$A^\Delta \colon = (a_{ij}^\Delta)_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}, \qquad \text{where } A = (a_{ij})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}.$$

Theorem 2.19 ([5, Theorems 5.2 and 5.3]). *Suppose that $A, B\colon \mathbb{T} \to \mathbb{R}^{n\times n}$ are delta differentiable and that $\alpha, \beta \in \mathbb{R}$. Then the following statements are true:*

1. $A \circ \sigma = A + \mu A^\Delta$.
2. $(\alpha A + \beta B)^\Delta = \alpha A^\Delta + \beta B^\Delta$;
3. $(AB)^\Delta = (A\circ\sigma)B^\Delta + A^\Delta B = AB^\Delta + A^\Delta(B\circ\sigma)$;
4. $(A^{-1})^\Delta = -(A\circ\sigma)^{-1}A^\Delta A^{-1} = -A^{-1}A^\Delta(A\circ\sigma)^{-1}$ *if A and $A\circ\sigma$ are invertible;*
5. $(AB^{-1})^\Delta = (A^\Delta - AB^{-1}B^\Delta)(B\circ\sigma)^{-1} = (A^\Delta - [(AB^{-1})\circ\sigma]B^\Delta)B^{-1}$ *if B and $B\circ\sigma$ are invertible;*
6. $(A^*)^\Delta = (A^\Delta)^*$, *where A^* is the conjugate transpose of A.*

Proof. **Item 1.** By Theorem 2.6, $f\circ\sigma = f + \mu f^\Delta$ for any delta differentiable function f. Thus, it follows that

$$A\circ\sigma = (a_{ij}\circ\sigma) = (a_{ij} + \mu a_{ij}^\Delta) = A + \mu A^\Delta.$$

Items 2, 3 and 6. They can be proved in a similar way of item 1. Because of that, their proof will be omitted.

Item 4. It is a direct consequence of item 3 applied to $I = AA^{-1}$. Differentiating this formula, we get

$$0 = A^\Delta A^{-1} + (A\circ\sigma)(A^{-1})^\Delta = A^\Delta[(A^{-1})\circ\sigma] + A(A^{-1})^\Delta.$$

Rewriting the above equations, we obtain

$$(A^{-1})^{\Delta} = -(A \circ \sigma)^{-1} A^{\Delta} A^{-1} = -A^{-1} A^{\Delta} (A \circ \sigma)^{-1}.$$

Item 5 is a consequence of items 3 and 4 and its proof will be omitted. □

We consider the linear system

$$y^{\Delta}(t) = A(t)y(t) \tag{2.1}$$

where $A\colon \mathbb{T} \to \mathbb{R}^{n\times n}$. A function $y\colon \mathbb{T} \to \mathbb{R}^n$ is called a **solution of** (2.1) if it satisfies $y^{\Delta}(t) = A(t)y(t)$ for every $t \in \mathbb{T}^{\kappa}$. Before we can present a result about the existence of solutions of this system, we are going to define a regressive matrix and regressive system.

Definition 2.20 ([5, Definition 5.5]). *A function $A\colon \mathbb{T} \to \mathbb{R}^{n\times n}$ is **regressive** if $I + \mu(t)\, A(t)$ is invertible for all $t \in \mathbb{T}^{\kappa}$. The class of all regressive and rd–continuous functions is denoted by*

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n}).$$

*We also say that the system (2.1) is **regressive** if $A \in \mathcal{R}$.*

The next theorem is a result about the existence and uniqueness of solutions of a more general kind of system than the one we presented previously due to the perturbation that is also presented here. The proof will be omitted due to its size, but it can be found in [5].

Theorem 2.21 ([5, Theorem 5.8]). *Suppose that $A \in \mathcal{R}$ is an $n \times n$ matrix-valued function and that $f\colon \mathbb{T} \to \mathbb{R}^n$ is rd–continuous function. Consider also $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$\begin{cases} y^{\Delta} = A(t)y + f(t), \\ y(t_0) = y_0, \end{cases}$$

has a unique solution.

Remark 2.22 ([5]). *As a consequence of the above theorem, the matrix initial value problem with the constant $n \times 1$ matrix y_0*

$$\begin{cases} y^{\Delta} = A(t)y, \\ y(t_0) = y_0, \end{cases}$$

also has a unique solution.

With the existence and uniqueness of a regressive linear system, we can define the exponential function in the following way.

Definition 2.23 ([5, Definition 5.18])**.** *Suppose that $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ and $t_0 \in \mathbb{T}$. The unique matrix–valued solution of the initial value problem*

$$\begin{cases} y^{\Delta} = A(t)y, \\ y(t_0) = I, \end{cases}$$

*is called **matrix exponential function**. We denote such function as $e_A(\cdot, t_0)$.*

Before we present some properties of the exponential function, let us define and show some properties of two operations between regressive functions.

Definition 2.24 ([5, Definitions 5.10 and 5.12])**.** *Suppose that $A, B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$. The operation $A \oplus B$ is defined by*

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t), \quad \forall t \in \mathbb{T}^{\kappa}.$$

We also define $\ominus A$ as

$$(\ominus A)(t) = -\left[I + \mu(t)A(t)\right]^{-1} A(t), \quad \forall t \in \mathbb{T}^{\kappa}.$$

It is natural to define $A \ominus B$ by

$$(A \ominus B)(t) = (A \oplus (\ominus B))(t), \quad \forall t \in \mathbb{T}^{\kappa}.$$

Theorem 2.25 ([5, Lemma 5.12])**.** *Suppose that $A, B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ and $t \in \mathbb{T}^{\kappa}$. Then:*

1. $(\ominus A)(t) = -A(t)\left[I + \mu(t)A(t)\right]^{-1}$;
2. $A \oplus B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$;
3. $\ominus A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$;
4. $(A \ominus A)(t) = 0$.

Proof. **Item 1.** By definition, we have

$$(\ominus A)(t) = -\left[I + \mu(t)A(t)\right]^{-1} A(t).$$

It is immediate to see that $I+\mu(t)A(t)$ and $A(t)$ commute. It is also a well–known fact that if $AB = BA$, then $AB^{-1} = B^{-1}A$ for any $n\times n$ matrices A and B. As a consequence, $[I+\mu(t)A(t)]^{-1}$ and $A(t)$ also commute and the result follows.

Item 2. To prove this item, we need to show that $I+\mu(t)(A\oplus B)(t)$ is invertible. First, notice that

$$\begin{aligned} I+\mu(t)(A\oplus B)(t) &= I+\mu(t)(A(t)+B(t)+\mu(t)A(t)B(t)) \\ &= I+\mu(t)A(t)+\mu(t)B(t)+\mu^2(t)A(t)B(t) \\ &= [I+\mu(t)A(t)]\,[I+\mu(t)B(t)]\,. \end{aligned}$$

Since $A,B\in\mathcal{R}$, it follows that $I+\mu(t)A(t)$ and $I+\mu(t)B(t)$ are invertible and the result holds.

The proof of item 3 is similar to the proof of item 2. Because of that, it will be omitted.

Item 4. By definition of $\ominus$, we have

$$\begin{aligned} (A\ominus A)(t) &= (A\oplus(\ominus A))\,(t) \\ &= A(t)+(\ominus A(t))+\mu(t)A(t)(\ominus A(t)) \\ &= A(t)+(I+\mu(t)A(t))(\ominus A(t)) \\ &= A(t)+(-A(t)(\ominus A(t))^{-1})(\ominus A(t)) \\ &= A(t)-A(t)=0. \end{aligned}$$

Hence, the proof is complete. □

With the operations that we have just defined, it is possible to show some interesting properties of the exponential function.

Theorem 2.26 ([5, Theorem 5.21])**.** *Suppose that $A,B\in\mathcal{R}(\mathbb{T},\mathbb{R}^{n\times n})$. Then:*

1. $e_0(t,s)=I$ *and* $e_A(t,t)=I$;
2. $e_A(\sigma(t),s)=(I+\mu(t)A(t))e_A(t,s)$;
3. $e_A^{-1}(t,s)=e^*_{\ominus A^*}(t,s)$, *where* A^* *is the conjugate transpose of* A.;
4. $e_A(t,s)=e_A^{-1}(s,t)$;
5. $e_A(t,s)e_A(s,r)=e_A(t,r)$;
6. $e_A(t,s)e_B(t,s)=e_{A\oplus B}(t,s)$.

Proof. **Item 1.** $e_0(t,s)$ is the solution of the initial value problem

$$\begin{cases} y^{\Delta} = 0, \\ y(s) = I. \end{cases}$$

It is easy to see that I is a solution of that problem. Theorem 2.21 guarantees the uniqueness of the solution and we conclude that $e_0(t,s) = I$. The other part of this item follows directly from the definition of the exponential on time scales.

Item 2. Applying the first item of Theorem 2.19, we obtain

$$\begin{aligned} e_A(\sigma(t),s) &= e_A(t,s) + \mu(t)e_A^{\Delta}(t,s) \\ &= e_A(t,s) + \mu(t)A(t)e_A(t,s) \\ &= (I + \mu(t)A(t))e_A(t,s). \end{aligned}$$

Item 3. First, we are going to show that $(e_A^{-1}(t,s))^*$ is a solution of

$$\begin{cases} y^{\Delta} = (\ominus A^*)(t)y, \\ y(s) = I. \end{cases}$$

Define $y(t) = (e_A^{-1}(t,s))^*$. Using the items 4 and 6 of Theorem 2.19, we get

$$y^{\Delta}(t) = ((e_A^{-1}(t,s))^*)^{\Delta} = ((e_A^{-1}(t,s))^{\Delta})^* = -(e_A^{-1}(\sigma(t),s)e_A^{\Delta}(t,s)e_A^{-1}(t,s))^*.$$

Applying the definition of e_A and the item 2 of this proof, we obtain

$$\begin{aligned} -(e_A^{-1}(\sigma(t),s)e_A^{\Delta}(t,s)e_A^{-1}(t,s))^* &= -(e_A^{-1}(\sigma(t),s)A(t)e_A(t,s)e_A^{-1}(t,s))^* \\ &= -(e_A^{-1}(t,s)(I + \mu(t)A(t))^{-1}A(t))^*. \end{aligned}$$

Combining the equations above, we have

$$\begin{aligned} y^{\Delta}(t) &= -(e_A^{-1}(t,s)(I + \mu(t)A(t))^{-1}A(t))^* \\ &= (e_A^{-1}(t,s)(\ominus A)(t))^* = (\ominus A)^*(t)(e_A^{-1}(t,s))^* = (\ominus A)^*(t)y(t). \end{aligned}$$

From item 1, $y(s) = I$. As a consequence, $(e_A^{-1}(t,s))^* = e_{\ominus A^*}(t,s)$. We conclude that $e_A^{-1}(t,s) = e_{\ominus A^*}^*(t,s)$.

Items 4 and 5. For this part, we wil use a different notation. Denote as $y(t,t_0,y_0)$ the solution of the initial problem

$$\begin{cases} y^{\Delta} = A(t)y, \\ y(t_0) = y_0. \end{cases}$$

It is immediate that $y(t,s,I) = e_A(t,s)$.

Since y_0 is a constant, $(y(t,s,I)y_0)^{\Delta} = y^{\Delta}(t,s,I)y_0$. This fact implies that $y(t,s,y_0) = y(t,s,I)y_0$ because of the uniqueness of solutions of the initial value problem. As a consequence, it follows that

$$y(t,s,I) = y(t,r,y(r,s,I)) \tag{2.2}$$

because if we apply $t = r$ in the equation above, we get $y(r,s,I)$ on the left–hand side and $y(r,r,y(r,s,I)) = y(r,r,I)y(r,s,I) = y(r,s,I)$ on the right–hand side.

From (2.2), we obtain $y(t,s,I) = y(t,r,I)y(r,s,I)$. Using the same notation of the statement of theorem, we get $e_A(t,s) = e_A(t,r)e_A(r,s)$.

The other item follows immediately from the property above, because $y(t,s,I)y(s,t,I) = y(t,t,y(s,s,I)) = I$.

Item 6. The proof of this item is similar to the one done in item 3. The main difference here is that we define $y(t) = e_A(t,s)e_B(t,s)$ and the rest follows similarly. □

Next, we have some other properties of the exponential function. Its proof will be omitted due to its size, but it can be found in the cited reference.

Theorem 2.27 ([5, Theorem 5.23])**.** *Suppose that $A \in \mathcal{R}(\mathbb{T},\mathbb{R}^{n\times n})$ and $a,b,c \in \mathbb{T}$. Then*

$$[e_A(c,\cdot)]^{\Delta} = -[e_A(c,\sigma(\cdot))]A$$

and

$$\int_a^b e_A(c,\sigma(t))A(t)\Delta t = e_A(c,a) - e_A(c,b).$$

To conclude this chapter, we enunciate and prove a type of Variation of Constants Theorem for time scales.

Theorem 2.28 (Variation of Constants - [5, Theorem 5.23])**.** *Consider $A \in \mathcal{R}(\mathbb{T},\mathbb{R}^{n\times n})$, $f \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$, $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$\begin{cases} y^{\Delta} = A(t)y + f(t), \\ y(t_0) = y_0, \end{cases}$$

has a unique solution $y\colon \mathbb{T} \to \mathbb{R}^n$ *defined as*

$$y(t) = e_A(t,t_0)y_0 + \int_{t_0}^{t} e_A(t,\sigma(\tau))f(\tau)\Delta\tau. \tag{2.3}$$

Proof. It is easy to see that y as in (2.3) is well–defined. Using the properties of the exponential, we can also write y as

$$y(t) = e_A(t,t_0)\left[y_0 + \int_{t_0}^{t} e_A(t_0,\sigma(\tau))f(\tau)\Delta\tau\right].$$

It is immediate that $y(t_0) = y_0$. We are going to prove that $y^\Delta = A(t)y + f(t)$. Applying Theorem 2.19, we can differentiate the equation above in the following way:

$$\begin{aligned} y^\Delta(t) &= A(t)e_A(t,t_0)\left[y_0 + \int_{t_0}^{t} e_A(t_0,\sigma(\tau))f(\tau)\Delta\tau\right] + e_A(\sigma(t),t_0)e_A(t_0,\sigma(t))f(t) \\ &= A(t)y(t) + f(t). \end{aligned}$$

It remains only to prove the uniqueness of the solution. Hence, suppose that $z\colon \mathbb{T} \to \mathbb{R}^n$ is another solution of the initial value problem. Define the auxiliary function $x(t) = e_A(t_0,t)z(t)$. Then, we get

$$\begin{aligned} A(t)e_A(t,t_0)x(t) + f(t) &= A(t)z(t) + f(t) \\ &= z^\Delta(t) \\ &= A(t)e_A(t,t_0)x(t) + e_A(\sigma(t),t_0)x^\Delta(t). \end{aligned}$$

This implies that

$$x^\Delta(t) = e_A(t_0,\sigma(t))f(t).$$

As a consequence, we get

$$x(t) = y_0 + \int_{t_0}^{t} e_A(t_0,\sigma(\tau))f(\tau)\Delta\tau.$$

From the above equation, $z(t) = y(t)$. Thus, the solution of the problem is unique and the proof is complete. $\square$

Chapter 3

Correspondences Between Generalized ODEs and Other Types of Equations

In this chapter, we present one of the reasons why the generalized ODE is a focus of study of many mathematicians. This type of equation includes many other differential equations such as measure differential equations (see [7] and [37]), dynamic equations on time scales (see [7], [13] and [31]) and impulsive differential equations (see [7], [13] and [30]).

Here, we will show the correspondence between the generalized ODEs and each one of the other cited differential equations. A big advantage of having these relations is that one may find results for the generalized ODEs and simply use these correspondences to obtain results for all the other types of equations mentioned before, as done in Chapter 4.

3.1 Measure Differential Equations

The measure DEs are problems of the type

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \gamma], \tag{3.1}$$

where $\gamma > 0$, $B \subset \mathbb{R}^n$ is an open set, $f\colon B \times [t_0, t_0 + \gamma] \to \mathbb{R}^n$ is a function and $g\colon [t_0, t_0 + \gamma] \to \mathbb{R}$ is a nondecreasing function. Besides that, the integral on the right–hand side is in the sense of Henstock–Kurzweil–Stieltjes.

Here, we only consider the integral form of the measure DEs. However, under certain circumstances, it is possible to obtain an equivalent differential form

$$\begin{cases} Dx = f(x,t)Dg, \\ x(t_0) = x_0, \end{cases} \tag{3.2}$$

where Dx and Dg denote the distributional derivatives of x and g in the sense of L. Schwartz (see [8] for more details about the distributional derivative). As shown in [29], the measure DE (3.2) has an equivalent integral form (3.1) when g is a regulated function, $t \mapsto f(x(t),t)$ is a function of bounded variation and we are considering $n = 1$. The equivalence between both forms in a more general case is still an open problem of the area.

Therefore, a measure DE, here, refers to the integral equation (3.1). Now, we define the concept of a solution of that problem.

Definition 3.1 ([7, Definition 3.2]). *The function $x\colon [t_0, t_0+\gamma] \to \mathbb{R}^n$ is a* ***solution of the measure differential equation*** (3.1)*, with initial condition $x(t_0) = x_0$, if:*

1. *$x(t_0) = x_0 \in B$, where $B \subset \mathbb{R}^n$ is an open set;*
2. *x is a regulated function and $(x(t),t) \in B \times [t_0, t_0+\gamma]$;*
3. *the Henstock–Kurzweil–Stieltjes integral $\int_{t_0}^{t_0+\gamma} f(x(s),s)\mathrm{d}g(s)$ exists;*
4. $x(t) = x_0 + \displaystyle\int_{t_0}^{t} f(x(s),s)\mathrm{d}g(s), \quad \forall t \in [t_0, t_0+\gamma].$

The next definition is an adaptation of [37, Definition 5.1]. In that reference, the author is dealing with the Lebesgue integral. Here, we use the concept of the Henstock–Kurzweil–Stieltjes integral and, because of that, our conditions are more general. In Remark 3.3, we give more details about the difference of these assumptions.

Definition 3.2. *$f\colon B \times [t_0, t_0+\gamma] \to \mathbb{R}^n$ belongs to the class $\mathcal{D}(B \times [t_0, t_0+\gamma], g)$ if*

1. *$\int_{t_0}^{t_0+\gamma} f(x(s),s)\mathrm{d}g(s)$ exists in the sense of Henstock–Kurzweil–Stieltjes;*
2. *there exists a Henstock–Kurzweil–Stieltjes function $m\colon [t_0, t_0+\gamma] \to \mathbb{R}$ such that for every $s_1, s_2 \in [t_0, t_0+\gamma]$ with $s_1 \leqslant s_2$, we obtain*
$$\left\| \int_{s_1}^{s_2} f(x(s),s)\mathrm{d}g(s) \right\| \leqslant \int_{s_1}^{s_2} m(s)\mathrm{d}g(s); \tag{3.3}$$
3. *there exists a regulated function $L\colon [t_0, t_0+\gamma] \to \mathbb{R}_+$ such that*
$$\left\| \int_{s_1}^{s_2} f(x(s),s) - f(y(s),s)\mathrm{d}g(s) \right\| \leqslant \int_{s_1}^{s_2} L(s)\left\|x(s) - y(s)\right\|\mathrm{d}g(\mathrm{s}), \tag{3.4}$$
where $s_1, s_2 \in [t_0, t_0+\gamma]$ are such that $s_1 \leqslant s_2$ and $x, y \in B$.

Remark 3.3. *In [37],* (3.3) *is replaced by an assumption of the type* $\|f(x(s),s)\| \leqslant m(s)$ *and, later in [37, Lemma 5.3], Schwabik shows that a condition of the type* (3.3) *is satisfied. In other words, his assumption implies in* (3.3)*. However, as shown in [7, Example 3.10], the converse may not be valid.*

The next result is a generalization of [37, Proposition 5.5], changing the hypothesis of f, as done in Definition 3.2.

Lemma 3.4. *Suppose that* $f \in \mathcal{D}((B \times [t_0, t_0+\gamma], g)$*. Then the function* $F\colon B \times [t_0, t_0+\gamma] \to \mathbb{R}^n$ *defined as*

$$F(x,t) = \int_{t_0}^{t} f(x(s),s)\,\mathrm{d}g(s)$$

is in the class $\mathcal{F}(B \times [t_0, t_0+\gamma], h, \omega)$*, i.e., there exists a function* $h\colon [t_0, t_0+\gamma] \to \mathbb{R}$ *that is nondecreasing on* $[t_0, t_0+\gamma]$ *and a function* $\omega\colon [0,+\infty) \to [0,+\infty)$ *that is a continuous, increasing function with* $\omega(0) = 0$*, such that:*

1. *for every* $(x,t_1),(x,t_2) \in B \times [t_0, t_0+\gamma]$*, we get*

$$\|F(x,t_2) - F(x,t_1)\| \leqslant |h(t_2) - h(t_1)|; \tag{3.5}$$

2. *for all* $(x,t_1),(x,t_2),(y,t_1),(y,t_2) \in B \times [t_0, t_0+\gamma]$*, we obtain*

$$\|F(x,t_2) - F(x,t_1) - F(y,t_2) + F(y,t_1)\| \leqslant |h(t_2) - h(t_1)|\omega(\|x-y\|). \tag{3.6}$$

Proof. By hypothesis, $f \in \mathcal{D}(B \times [t_0, t_0+\gamma], g)$. Using the functions $m\colon [t_0, t_0+\gamma] \to \mathbb{R}$ and $L\colon [t_0, t_0+\gamma] \to \mathbb{R}$ that appear in that class, define $h\colon [t_0, t_0+\gamma] \to \mathbb{R}$ as

$$h(t) = \int_{t_0}^{t} (m(s) + L(s))\,\mathrm{dg(s)}, \quad \mathrm{t} \in [\mathrm{t_0}, \mathrm{t_0} + \gamma]. \tag{3.7}$$

Let us prove, first, that h defined as (3.7) is a nondecreasing function. Indeed, given $t_1 \leqslant t_2$, we get

$$0 \leqslant \left\| \int_{t_1}^{t_2} f(x(s),s)\mathrm{d}g(s) \right\| \leqslant \int_{t_1}^{t_2} m(s)\mathrm{d}g(s).$$

Therefore,

$$\int_{t_0}^{t_2} m(s)\mathrm{d}g(s) = \int_{t_0}^{t_1} m(s)\mathrm{d}g(s) + \int_{t_1}^{t_2} m(s)\mathrm{d}g(s) \geqslant \int_{t_0}^{t_1} m(s)\mathrm{d}g(s).$$

Hence, $\int_{t_0}^{t} m(s)\mathrm{d}g(s)$ is a nondecreasing function.

On the other hand, since L is a positive function, it is clear that

$$\int_{t_0}^{t} L(s)\mathrm{d}g(s)$$

is a nondecreasing function. Thus,

$$h(t) = \int_{t_0}^{t} (m(s) + L(s))\,\mathrm{d}g(\mathrm{s})$$

is a nondecreasing function.

To prove item 1, suppose that $x \in B$ and $s_1, s_2 \in [t_0, t_0 + \gamma]$. We can use property 2 of Definition 3.2 to obtain:

$$\begin{aligned}
\|F(x,s_2) - F(x,s_1)\| &= \left\| \int_{s_1}^{s_2} f(x(s),s)\,\mathrm{d}g(s) \right\| \\
&\leqslant \left| \int_{s_1}^{s_2} m(s)\mathrm{d}g(s) \right| \\
&\leqslant |h(s_2) - h(s_1)|.
\end{aligned}$$

For item 2, define $\omega\colon [0,+\infty) \to [0,+\infty)$ as $\omega(t) = t$. Therefore, we have

$$\begin{aligned}
\|F(x,s_2) - F(x,s_1) - F(y,s_2) + F(y,s_1)\| &= \left\| \int_{s_1}^{s_2} f(x(s),s) - f(y(s),s)\,\mathrm{d}g(s) \right\| \\
&\leqslant \left| \int_{s_1}^{s_2} L(s)\,\|x-y\|\,\mathrm{d}g(\mathrm{s}) \right| \\
&\leqslant |h(s_2) - h(s_1)|\omega(\|x-y\|),
\end{aligned}$$

and the proof is complete. □

Now we can present a result that links the generalized ODEs and the measure DEs. The next result is a version of [37, Proposition 5.12].

Theorem 3.5. *Consider $f \in \mathcal{D}(B \times [t_0, t_0+\gamma], g)$. Define the function $F\colon B \times [t_0, t_0+\gamma] \to \mathbb{R}^n$ as*

$$F(x,t) = \int_{t_0}^{t} f(x,s)\,\mathrm{d}g(s).$$

If $x\colon [\alpha,\beta] \subset [t_0, t_0+\gamma] \to B$ is a regulated function, then $\int_{\alpha}^{\beta} DF(x(\tau),t)$ exists and we have

$$\int_{\alpha}^{\beta} DF(x(\tau),t) = \int_{\alpha}^{\beta} f(x(s),s)\,\mathrm{d}g(s).$$

Proof. First, Theorem 1.12 implies that F is a regulated function. Therefore, we can apply Lemma 3.4 and also Corollary 1.38 to obtain that $\int_\alpha^\beta DF(x(\tau),t)$ exists. The equality between both integrals is immediate from the definition and the result follows. $\square$

3.2 Dynamic Equations on Time Scales

In this section, we will present a relation between the measure DEs and the dynamic equations on time scales. Before doing that, let us start this section by remembering some notations and definitions from Chapter 2.

A time scale $\mathbb{T}$ is any closed nonempty subset of the real line. The forward jump operator $\sigma\colon \mathbb{T}\to\mathbb{T}$ is defined as

$$\sigma(t)=\inf\{s\in\mathbb{T}\colon s>t\}$$

and we are assuming that $\inf\emptyset=\sup\mathbb{T}$. We define the **extension of a time scale**, denoted by $\mathbb{T}^*$, as

$$\mathbb{T}^*=\begin{cases}(-\infty,\sup\mathbb{T}] & \text{if } \sup\mathbb{T}<+\infty,\\ (-\infty,+\infty) & \text{if } \sup\mathbb{T}=+\infty.\end{cases}$$

In this set, we define the operator $*\colon \mathbb{T}^*\to\mathbb{T}$ as

$$t^*=\inf\{s\in\mathbb{T}\colon s\geqslant t\}.$$

Given a function $f\colon \mathbb{T}\to\mathbb{R}^n$, we also denote as $f^*\colon \mathbb{T}^*\to\mathbb{R}^n$ for the composition $f^*(t)=f(t^*)$. Similarly, given $f\colon \mathbb{R}^n\times\mathbb{T}\to\mathbb{R}^n$, we define $f^*\colon \mathbb{R}^n\times\mathbb{T}^*\to\mathbb{R}^n$ as $f^*(x,t)=f(x,t^*)$.

Also, we will simply denote the interval of points between a and b that are in the time scale as

$$[a,b]_{\mathbb{T}}=[a,b]\cap\mathbb{T}.$$

Now, we can present the definition a dynamic equation on time scales.

Definition 3.6 ([7]). *Consider a time scale $\mathbb{T}$ where $t_0,t_0+\gamma\in\mathbb{T}$ for a given $\gamma>0$. Besides that, let $B\subset\mathbb{R}^n$ be an open set and a function $f\colon B\times[t_0,t_0+\gamma]_{\mathbb{T}}\to\mathbb{R}^n$ that is Henstock–Kurzweil Δ–integrable. A* ***dynamic equation on time scales*** *is given by*

$$x^\Delta(s)=f(x(s),s),\quad s\in[t_0,t_0+\gamma]_{\mathbb{T}}.$$

Integrating both sides, we obtain the equivalent integral form:

$$x(t)=x(t_0)+\int_{t_0}^t f(x(s),s)\Delta s,\quad t\in[t_0,t_0+\gamma]_{\mathbb{T}}. \tag{3.8}$$

Next, we define a solution of a dynamic equation on time scales (3.8).

Definition 3.7 ([7, Definition 3.27]). *A function $x\colon [t_0, t_0+\gamma]_{\mathbb{T}} \to \mathbb{R}^n$ is a* ***solution of a dynamic equation on time scales*** (3.8)*, with initial condition $x(t_0) = x_0$ if:*

1. $x(t_0) = x_0 \in B$;

2. *x is a regulated function (see Definition 2.8) and $(x(t),t) \in B \times [t_0, t_0+\gamma]_{\mathbb{T}}$;*

3. *$\int_{t_0}^{t_0+\gamma} f(x(s),s)\Delta s$ exists in the sense of Henstock–Kurzweil Δ–integral;*

4. $x(t) = x_0 + \displaystyle\int_{t_0}^{t} f(x(s),s)\Delta s, \quad t \in [t_0, t_0+\gamma]_{\mathbb{T}}.$

Before we continue with the theory, let us present an example of a dynamic equation using the concepts and results from Chapter 2.

Example 3.8 ([5, Example 2.55]). *Consider the time scale*

$$\mathbb{T} = q^{\mathbb{N}_0} = \{q^n \colon n \in \mathbb{N}_0 \text{ and } q > 1\}.$$

Suppose also that the function $p\colon \mathbb{T} \to \mathbb{R}$ is regressive, i.e., $1 + \mu(t)p(t) \neq 0$ for every $t \in \mathbb{T}^\kappa$ and the dynamic equation

$$\begin{cases} y^\Delta(t) = p(t)y(t), \\ y(1) = 1. \end{cases}$$

In Chapter 2, we already calculated that $\sigma(t) = qt$ and $\mu(t) = (q-1)t$ for $t \in q^{\mathbb{N}_0}$. To find the solution of this system, we can use Theorem 2.6, which ensures that if f is differentiable, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

Applying this theorem for our system, we get

$$y(qt) = y(\sigma(t)) = y(t) + \mu(t)y^\Delta(t) = [1 + (q-1)tp(t)]y(t).$$

Therefore, it is easy to see that

$$y(t) = \prod_{s \in \mathbb{T} \cap (1,t)} [1 + (q-1)sp(s)]y(1).$$

Since $y(1) = 1$, we can simplify the equation above. Besides that, we defined the exponential as the solution of this system, so we can use the following notation:

$$e_p(t,1) = y(t) = \prod_{s \in \mathbb{T} \cap (0,t)} (1 + (q-1)sp(s)).$$

Now, we present results that will be used to establish the relation between a measure DE and a dynamic equation on time scales. The next theorem can be found in [13], but it is also contained in [31, Theorem 8.6.8].

Theorem 3.9 ([13, Theorem 4.2])**.** *Consider $a \in \mathbb{T}$, a function $f\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ and define $g(t) = t^*$ for $t \in [a,b]$. The Henstock–Kurzweil Δ–integral $\int_a^b f(t)\Delta t$ exists if and only if the Henstock–Kurzweil-Stieltjes integral $\int_a^b f^*(t)\,\mathrm{d}g(t)$ exists. In this case,*

$$\int_a^b f(t)\Delta t = \int_a^b f^*(t)\,\mathrm{d}g(t).$$

Proof. Suppose first that $\int_a^b f(t)\Delta t$ exists. Thus, given $\varepsilon > 0$, there is a Δ–gauge $\delta(t) = (\delta_L(t), \delta_R(t))$ on the interval $[a,b]_{\mathbb{T}}$ such that, if

$$D = \{s_0, \tau_1, s_1, \ldots, s_{l-1}, \tau_l, s_l\}$$

is a δ–fine tagged partition, then

$$\left\| \sum_{i=1}^{l} f(\tau_i)(s_i - s_{i-1}) - \int_a^b f(t)\Delta t \right\| < \varepsilon.$$

Define a gauge $\overline{\delta}\colon [a,b] \to (0,+\infty)$ in the following way:

$$\overline{\delta}(t) = \begin{cases} \min(\delta_L(t), \sup\{d\colon t+d \in [a,b]_{\mathbb{T}} \text{ and } d \leqslant \delta_R(t)\}), & \text{if } t \in (a,b) \cap \mathbb{T}; \\ \sup\{d\colon d+a \in [a,b]_{\mathbb{T}} \text{ and } d \leqslant \delta_R(a)\}, & \text{if } t = a; \\ \delta_L(b), & \text{if } t = b \text{ and } b \in \mathbb{T}; \\ \dfrac{1}{2}\inf\{|t-s| : s \in \mathbb{T}\}, & \text{if } t \in [a,b] \setminus \mathbb{T}. \end{cases}$$

Notice that the supremum in the definition above is always greater than zero, because $\delta_R(t) \geqslant \mu(t)$ and this implies that there is a point $x \in (t, t+\delta_R(t)] \cap \mathbb{T}$. Therefore, $\overline{\delta}(t) > 0$ for every $t \in [a,b]$.

The idea now is to show that the gauge $\overline{\delta}$ satisfies the definition of the Henstock–Kurzweil–Stieltjes integral for the function $f^*(\tau)g(t)$. From now on, we denote the sum in terms of an arbitrary partition D as

$$S(D) = \sum_{i=1}^{k} f^*(\tau_i)(g(s_i) - g(s_{i-1})) = \sum_{i=1}^{k} f(\tau_i^*)(s_i^* - s_{i-1}^*).$$

We will show that if

$$\overline{D} = \{\overline{s_0}, \overline{\tau_1}, \overline{s_1}, \dots, \overline{s_{k-1}}, \overline{\tau_k}, \overline{s_k}\}$$

is a $\overline{\delta}$–fine partition, it is possible to find another δ–fine partition D' such that every point of the new partition is in the time scale and $S(D') = S(\overline{D})$.

First, observe that for the partition $\overline{D}$, we either have that $\overline{\tau_i} \in \mathbb{T}$ or $[\overline{s_{i-1}}, \overline{s_i}] \cap \mathbb{T} = \emptyset$. This is true because for every point $t \in [a,b] \setminus \mathbb{T}$, we get $\overline{\delta}(t) = (1/2)\inf\{|t-s| : s \in \mathbb{T}\}$.

Let us construct $D' \subset \mathbb{T}$ using the following finite induction: by hypothesis, $\overline{s_0} = a \in \mathbb{T}$. Now, consider an interval $[\overline{s_{i-1}}, \overline{s_i}]$ such that $\overline{s_{i-1}} \in \mathbb{T}$. From the last paragraph, we already know that $\overline{\tau_i} \in \mathbb{T}$. If $\overline{s_i} \notin \mathbb{T}$, change $\overline{s_i}$ by $\overline{s_i}^*$ in the partition and exclude all points and tags of $\overline{D}$ that are in $(\overline{s_i}, \overline{s_i}^*)$.

We claim that $S(D') = S(\overline{D})$. To prove this, note first that

$$f^*(\overline{\tau_i})(g(\overline{s_i}) - g(\overline{s_{i-1}})) = f^*(\overline{\tau_i})(\overline{s_i}^* - \overline{s_{i-1}}^*) = f^*(\overline{\tau_i})(g(\overline{s_i}) - g(\overline{s_{i-1}}^*)),$$

proving that the intervals $[\overline{s_{i-1}}, \overline{s_i}]$ and $[\overline{s_{i-1}}, \overline{s_i}^*]$ have the same contribution in the sum $S(D')$. Besides that, any other interval $[\overline{s_{j-1}}, \overline{s_j}] \subset (\overline{s_i}, \overline{s_i}^*)$ are such that

$$g(\overline{s_{j-1}}) = \overline{s_{j-1}}^* = \overline{s_i}^* = \overline{s_j}^* = g(\overline{s_j})$$

and these points do not contribute to the sum $S(\overline{D})$.

To conclude this part of the proof, we need to show that the new division D' is δ–fine. For that, notice first that for any interval $[\overline{s_{i-1}}, \overline{s_i}]$ that remained unchanged from the original $\overline{D}$ is δ–fine because

$$[\overline{s_{i-1}}, \overline{s_i}] \subset (\tau_i - \overline{\delta}(\tau_i), \tau_i + \overline{\delta}(\tau_i)) \subset [\tau_i - \delta_L(\tau_i), \tau_i + \delta_{\mathbb{R}}(\tau_i)].$$

Suppose now that $[\overline{s_{i-1}}, \overline{s_i}]$ was changed, i.e, that $\overline{s_i}$ is one of the points of $\overline{D}$ that was excluded but $\overline{s_{i-1}}$ was not. Hence, the new interval in the partition D' is $[\overline{s_{i-1}}, \overline{s_i}^*]$. Define $M = \sup\{[a, \tau_i + \delta_R(\tau_i)] \cap \mathbb{T}\}$. We have that $M \in [a,b]_{\mathbb{T}}$ and also

$$\overline{s_i} \leqslant \overline{\tau_i} + \overline{\delta}(\overline{\tau_i}) \leqslant \overline{\tau_i} + \sup\{d \colon d + \overline{\tau_i} \in [a,b]_{\mathbb{T}} \text{ and } d \leqslant \delta_R(\overline{\tau_i})\} = M.$$

The inequalities above hold because the original partition $\overline{D}$ is $\overline{\delta}$–fine. Thus, if $\overline{s_i}$ is one of the points of $\overline{D}$ that we excluded, $\overline{s_i} \notin \mathbb{T}$. The fact that $M \in \mathbb{T}$ implies that $\overline{s_i}^* \leqslant M$. As a result,

$$\overline{s_i}^* \leqslant M \leqslant \overline{\tau_i} + \delta_R(\overline{\tau_i})$$

and it follows that D' is δ–fine.

Since $S(D') = S(\overline{D})$ and D' is δ–fine,

$$\left\| S(\overline{D}) - \int_a^b f(t)\Delta t \right\| = \left\| S(D') - \int_a^b f(t)\Delta t \right\| < \varepsilon$$

and we obtain that $\int_a^b f(t)\Delta t = \int_a^b f^*(t)\,\mathrm{d}g(t)$.

Reciprocally, assume that $\int_a^b f^*(t)\,\mathrm{d}g(t)$ exists. Hence, given $\varepsilon > 0$, there is a gauge $\overline{\delta}\colon [a,b] \to (0,+\infty)$ such that for every $\overline{\delta}$–fine partition, we get

$$\left\| \sum_{i=1}^{k} f(\tau_i^*)(s_i^* - s_{i-1}^*) - \int_a^b f^*(t)\,\mathrm{d}g(t) \right\| < \varepsilon.$$

Define now the Δ–gauge $\delta(t) = (\delta_L(t), \delta_R(t))$ where $t \in [a,b]_T$, $\delta_L(t) = \overline{\delta}(t)$ and also $\delta_R(t) = \max\{\overline{\delta}(t), \mu(t)\}$.

Consider a δ–fine partition on $[a,b]_{\mathbb{T}}$ denoted as

$$D = \{s_0, \tau_1, s_1, \ldots, s_{k-1}, \tau_k, s_k\}.$$

As done in the other part of the proof, we will construct another partition D' of $[a,b]$ such that the new partition is $\overline{\delta}$–fine and also $S(D) = S(D')$.

Define the partition D' by replacing the points s_i by $\tau_i + \overline{\delta}(\tau_i)$ and keeping the same tag τ_i for the new interval $[s_{i-1}, \tau_i + \overline{\delta}(\tau_i)]$. For the interval $[\tau_i + \overline{\delta}(\tau_i), s_i]$, use any $\overline{\delta}$–fine partition. It is immediate to see that D' is $\overline{\delta}$–fine. The equality $S(D) = S(D')$ holds because $g(t) = t^*$ is constant on the interval $(\tau_i, s_i]$. Therefore, we conclude that

$$\left\| S(\overline{D}) - \int_a^b f^*(t)\,\mathrm{d}g(t) \right\| = \left\| S(D') - \int_a^b f^*(t)\,\mathrm{d}g(t) \right\| < \varepsilon$$

and the proof is complete. □

The next result is a direct consequence of Theorem 3.9.

Corollary 3.10 ([31, Corollary 8.6.9]). *Given $f\colon [a,b] \to \mathbb{R}^n$, consider $\tilde{f}\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ another function such that $f(t) = \tilde{f}(t)$ for every $t \in [a,b]_{\mathbb{T}}$. If we define $g(t) = t^*$ for every $t \in [a,b]$, then the Henstock–Kurzweil Δ–integral $\int_a^b f(t)\Delta t$ exists if and only if the Henstock–Kurzweil–Stieltjes integral $\int_a^b \tilde{f}(t)\,\mathrm{d}g(t)$ exists and they have the same value.*

Proof. This result is a direct consequence of Theorem 3.9. Just notice that for any partition D of $[a,b]$, we get the sum

$$S(D) = \sum_{i=1}^{k} \tilde{f}(\tau_i)(g(s_i) - g(s_{i-1})).$$

If $\tau_i \notin \mathbb{T}$, then there is an interval $J = [\alpha, \beta] \subset [a,b]$ such that $J \cap \mathbb{T} = \emptyset$. This is true because $[a,b] \setminus \mathbb{T}$ is an open set. Thus, it is possible to refine the partition D by adding the points α, β and the tag $\tau_i \notin \mathbb{T}$. This interval does not contribute to the sum $S(D)$ because

$$\tilde{f}(\tau_i)(g(\beta) - g(\alpha)) = 0$$

and the result follows. □

The next proposition will be useful in Chapter 4. It shows how we can change the extremes of integration from points outside of the time scale to points in the time scale without changing the value of the integral.

Proposition 3.11 ([13, Lemma 4.4]). *Suppose that $a, b \in \mathbb{T}$ and define $g(t) = t^*$ for $t \in [a,b]$. If $f\colon [a,b] \to \mathbb{R}^n$ is a function such that the integral $\int_a^b f(t)\,\mathrm{d}g(t)$ exists, then for every $c, d \in [a,b]$, we have*

$$\int_c^d f(t)\,\mathrm{d}g(t) = \int_{c^*}^{d^*} f(t)\,\mathrm{d}g(t).$$

Proof. Since g is constant on the intervals $[c, c^*]$ and $[d, d^*]$, it is immediate to see that $\int_c^{c^*} f(t)\,\mathrm{d}g(t) = \int_d^{d^*} f(t)\,\mathrm{d}g(t) = 0$. Therefore, using the additive property of the integral,

$$\int_c^d f(t)\,\mathrm{d}g(t) = \int_c^{c^*} f(t)\,\mathrm{d}g(t) + \int_c^d f(t)\,\mathrm{d}g(t) + \int_d^{d^*} f(t)\,\mathrm{d}g(t) = \int_{c^*}^{d^*} f(t)\,\mathrm{d}g(t)$$

and we obtain the result. □

The next theorem presents a link between the dynamic equations on time scales and the measure DEs. Since we already established a connection between the measure DEs and the generalized ODEs in the last section, we also get the desired relation between dynamic equations on time scales and generalized ODEs.

Theorem 3.12 ([31, Theorem 8.7.1]). *Consider $a, b, t_0 \in \mathbb{T}$ such that $a \leqslant t_0 \leqslant b$ and $g(t) = t^*$. Consider also two functions $f\colon B \times [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ and $\tilde{f}\colon B \times [a,b] \to \mathbb{R}^n$ where $B \subset \mathbb{R}^n$ and $f(x,t) = \tilde{f}(x,t)$ for every $(x,t) \in B \times [a,b]_{\mathbb{T}}$.*

If a function $x\colon [a,b]_{\mathbb{T}} \to B$ *satisfies*

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s),s)\,\Delta s, \quad t \in [a,b]_{\mathbb{T}}, \tag{3.9}$$

then the function $y\colon [a,b] \to B$ *given by* $y = x^*$ *satisfies*

$$y(t) = y(t_0) + \int_{t_0}^{t} \tilde{f}(y(s),s)\mathrm{d}g(s), \quad t \in [a,b]. \tag{3.10}$$

Conversely, each function $y\colon [a,b] \to B$ *satisfying* (3.10) *has the form* $y = x^*$, *where* $x\colon [a,b]_{\mathbb{T}} \to B$ *satisfies* (3.9).

Proof. Suppose that $x\colon [a,b]_{\mathbb{T}} \to B$ satisfies (3.9) and $y(t) = x^*(t) = x(t^*)$. Using Corollary 3.10 and Proposition 3.11, we get

$$\begin{aligned}
y(t) = x(t^*) = x(t_0) + \int_{t_0}^{t^*} f(x(s),s)\Delta s \\
= x(t_0^*) + \int_{t_0}^{t^*} \tilde{f}(x^*(s),s)\,\mathrm{d}g(s) \\
= y(t_0) + \int_{t_0}^{t^*} \tilde{f}(y(s),s)\,\mathrm{d}g(s) \\
= y(t_0) + \int_{t_0}^{t} \tilde{f}(y(s),s)\,\mathrm{d}g(s).
\end{aligned}$$

The reciprocal follows analogously and its proof will be omitted. □

3.3 Impulsive Differential Equations

In this section, we study the impulsive differential equations of the form

$$\begin{cases} x'(t) = f(x(t),t), \\ \Delta^+ x(\tau_j) = I_j(x(\tau_j)), \quad j \in \mathbb{Z}, \end{cases} \tag{3.11}$$

where $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $I_j\colon \mathbb{R}^n \to \mathbb{R}^n$ for each $j \in \mathbb{Z}$, $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ is an increasing sequence and $\Delta^+ x(\tau_j) = \lim\limits_{s\to\tau_j+} x(s) - x(\tau_j)$. We assume that the first equality in (3.11) holds almost everywhere, the solution x is left–continuous and regulated on each interval $(\tau_{j-1},\tau_j]$, $j \in \mathbb{Z}$.

Then, the corresponding integral form is

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), s)\, \mathrm{d}s + \sum_{\substack{j \in \mathbb{Z}, \\ t_0 \leqslant \tau_j < t}} I_j(x(\tau_j)), \tag{3.12}$$

where the integral on the right–hand side is in the sense of Henstock–Kurzweil. For more details about this type of equations, see [24].

Definition 3.13 ([7, Definition 3.4]). *A function $x\colon [a,b] \to \mathbb{R}^n$ is a* ***solution of the impulsive DE*** (3.12) *with initial condition $x(t_0) = x_0$ if it satisfies:*

1. $x(t_0) = x_0$;
2. *x is a regulated function and left–continuous;*
3. *$\int_{t_0}^{t_0+\gamma} f(x(s), s)\mathrm{d}s$ exists in the sense of Henstock–Kurzweil;*
4. (3.12) *holds for every $t \in [a,b]$.*

Before we present more results, let us show an example of an impulsive DE.

Example 3.14 ([24, Example 1.1.2]). *Consider the impulsive DE*

$$\begin{cases} x'(t) = x^2(t) + 1, & \text{if } 0 \leqslant t \text{ and } t \neq \dfrac{k\pi}{4},\ k \in \mathbb{N}; \\ \Delta^+ x(t) = -1, & \text{if } t = \dfrac{k\pi}{4},\ k \in \mathbb{N}; \\ x(0) = 0. \end{cases} \tag{3.13}$$

We will show that the solution of this problem is the function

$$x(t) = \begin{cases} \tan(t), & \text{if } t \in \left[0, \dfrac{\pi}{4}\right], \\ \tan\left(t - \dfrac{k\pi}{4}\right), & \text{if } t \in \left(\dfrac{k\pi}{4}, \dfrac{(k+1)\pi}{4}\right],\ k \in \mathbb{N}. \end{cases}$$

It is easy to see that $x'(t) = x^2(t) + 1$ on the points where the derivative is defined. On the other points, notice that if $t = k\pi/4, k \in \mathbb{N}$, then

$$\Delta^+ x(t) = \lim_{s \to k\pi/4+} \tan\left(s - \frac{(k+1)\pi}{4}\right) - \tan\left(\frac{k\pi}{4} - \frac{k\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right) = -1.$$

Thereby, x is a solution of the impulsive DE. Notice that the solution is defined on $[0, +\infty)$. The graph below represents $x(t)$.

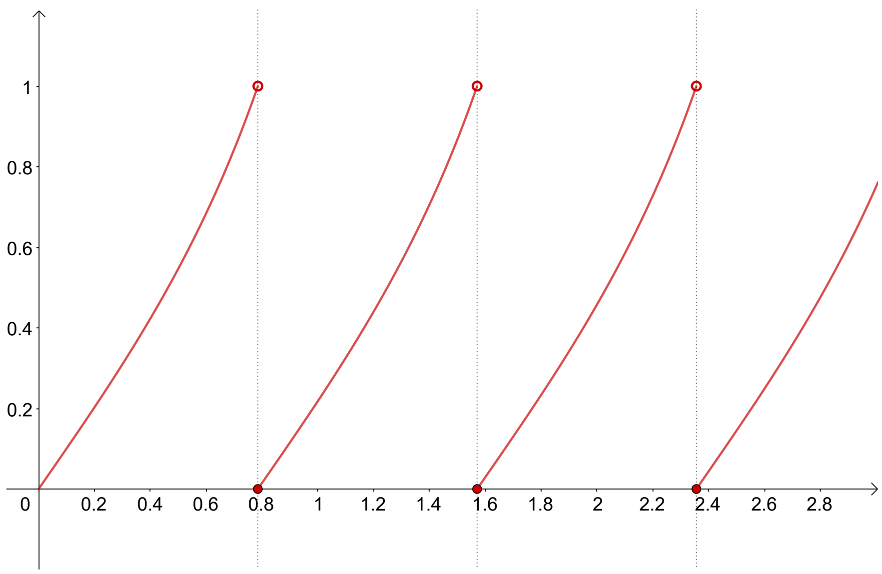

Figure 3.1 Solution of the impulsive DE (3.13).

The following result is about the Henstock–Kurzweil–Stieltjes integral. It will be important later to establish the relation between the impulsive DEs and the measure DEs.

Theorem 3.15 ([13, Lemma 2.4]). *Consider points $a \leqslant t_1 < t_2 < \ldots < t_m \leqslant b$ where $m \in \mathbb{N}$ and also two functions $f\colon [a,b] \to \mathbb{R}^n$ and $g\colon [a,b] \to \mathbb{R}$ such that g is regulated, left–continuous on $[a,b]$ and continuous on the points $t_1, t_2, \ldots, t_m$. Suppose also that there is a pair of functions $\tilde{f}\colon [a,b] \to \mathbb{R}^n$ and $\tilde{g}\colon [a,b] \to \mathbb{R}$ such that $f(t) = \tilde{f}(t)$ for every $t \in [a,b] \setminus \{t_1, t_2, \ldots, t_m\}$ and $\tilde{g} - g$ is constant for each interval $[a,t_1], (t_1,t_2], \ldots, (t_m,b]$.*

Then, $\int_a^b \tilde{f}(s)\,\mathrm{d}\tilde{g}(s)$ exists if and only if $\int_a^b f(s)\,\mathrm{d}g(s)$ exists. In this case, we obtain

$$\int_a^b \tilde{f}(s)\,\mathrm{d}\tilde{g}(s) = \int_a^b f(s)\,\mathrm{d}g(s) + \sum_{\substack{k \in \{1,2,\ldots,m\},\\ t_k < b}} \tilde{f}(t_k)\Delta^+ \tilde{g}(t_k).$$

Proof. First, we will calculate the value of the integral $\int_a^b \tilde{f}(s)\,\mathrm{d}(\tilde{g}-g)(s)$. Since $\tilde{g}-g$ is constant on the interval $[a,t_1]$, we get

$$\int_a^{t_1} \tilde{f}(s)\,\mathrm{d}(\tilde{g}-g)(s) = 0.$$

Next, we calculate the value of the integral in $[t_{k-1},t_k]$ for $k \in \{2,\ldots,m\}$. For this, notice that $\tilde{g}-g$ is also constant on the interval $(t_{k-1},t_k]$. Therefore, applying Theorem 1.11, we obtain

$$\begin{aligned}\int_{t_{k-1}}^{t_k} \tilde{f}(s)\,\mathrm{d}(\tilde{g}-g)(s) &= \lim_{s\to t_{k-1}+}\left[\int_s^{t_k} \tilde{f}(s)\,\mathrm{d}(\tilde{g}-g)(s) - \tilde{f}(t_k)[(\tilde{g}-g)(s)-(\tilde{g}-g)(t_k)]\right]\\ &= \tilde{f}(t_k)\Delta^+\tilde{g}(t_k).\end{aligned} \tag{3.14}$$

For the integral $\int_{t_m}^b \tilde{f}(s)\,\mathrm{d}(\tilde{g}-g)(s)$, we either have that it is zero if $t_m = b$ or we can do a similar calculation to what is done in (3.14) to obtain

$$\int_{t_m}^b \tilde{f}(s)\,\mathrm{d}(\tilde{g}-g)(s) = \tilde{f}(t_m)\Delta^+\tilde{g}(t_m)$$

As a result, we have

$$\int_a^b \tilde{f}(s)\,\mathrm{d}(\tilde{g}-g)(s) = \sum_{\substack{k\in\{1,2,\ldots,m\},\\ t_k<b}} \tilde{f}(t_k)\Delta^+\tilde{g}(t_k).$$

On the other hand, we can use Theorem 1.12 to do the following calculations:

$$\int_a^{t_1} \tilde{f}(s)\,\mathrm{d}g(s) = \lim_{x\to t_1-}\int_a^x \tilde{f}(s)\,\mathrm{d}g(s) = \lim_{x\to t_1-}\int_a^x f(s)\,\mathrm{d}g(s) = \int_a^{t_1} f(s)\,\mathrm{d}g(s).$$

For $k\in\{2,\ldots,m\}$, we also get

$$\begin{aligned}\int_{t_{k-1}}^{t_k} \tilde{f}(s)\,\mathrm{d}g(s) &= \lim_{x\to t_{k-1}}\left(\lim_{r\to t_k-}\int_x^r \tilde{f}(s)\,\mathrm{d}g(s)\right)\\ &= \lim_{x\to t_{k-1}}\left(\lim_{r\to t_k-}\int_x^r f(s)\,\mathrm{d}g(s)\right) = \int_{t_{k-1}}^{t_k} f(s)\,\mathrm{d}g(s).\end{aligned}$$

Analogously, we obtain

$$\int_{t_k}^b \tilde{f}(s)\,\mathrm{d}g(s) = \int_{t_k}^b f(s)\,\mathrm{d}g(s).$$

Combining the above information, we conclude that $\int_a^b \tilde{f}(s)\,\mathrm{d}g(s)$ exists if and only if $\int_a^b f(s)\,\mathrm{d}g(s)$ exists and both integrals have the same value. Thus,

$$\begin{aligned}\int_a^b \tilde{f}(s)\,\mathrm{d}\tilde{g}(s) &= \int_a^b \tilde{f}(s)\,\mathrm{d}g(s) + \int_a^b \tilde{f}(s)\,\mathrm{d}(\tilde{g}-g)(s)\\ &= \int_a^b f(s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,2,\ldots,m\},\\ t_k<b}} \tilde{f}(t_k)\Delta^+\tilde{g}(t_k)\end{aligned}$$

and the proof is complete. □

Notice that any function g satisfying the assumptions of Theorem 3.15 has the form

$$g(s) = g(a) + (s-a) + \sum_{j=1}^{k} \chi_{(t_j,+\infty)}(s), \quad s \in [a,b], \tag{3.15}$$

and is, therefore, unique up to an additive constant.

The next result is also about some relations between the Henstock–Kurzweil integral and the Henstock–Kurzweil–Stieltjes integral.

Corollary 3.16 ([30, Lemma 5.2]). *Let $a \leqslant t_1 < t_2 < \cdots t_k < b$. Suppose that there are two functions $f\colon [a,b] \to \mathbb{R}^n$ and $\tilde{f}\colon [a,b] \to \mathbb{R}^n$ such that $\tilde{f}(s) = f(s)$ for every $s \in [a,b) \setminus \{t_1,\dots,t_k\}$. Let $g\colon [a,b] \to \mathbb{R}$ be a left–continuous function with $\Delta^+ g(t_j) = 1$ for each $j \in \{1,\dots,k\}$, and $g(t) - g(u) = t - u$ whenever $[u,t] \cap \{t_1,\dots,t_k\} = \emptyset$.*

Then the Henstock–Kurzweil–Stieltjes integral $\int_a^b \tilde{f}(s)\,\mathrm{d}g(s)$ exists if and only if the Henstock–Kurzweil–Stieltjes integral $\int_a^b f(s)\,\mathrm{d}s$ exists. In this case,

$$\int_a^b \tilde{f}(s)\,\mathrm{d}g(s) = \int_a^b f(s)\,\mathrm{d}s + \sum_{j=1}^{k} \tilde{f}(t_j).$$

Proof. It is a direct consequence of Theorem 3.15. □

The next result connects the impulsive DEs and the measure DEs.

Theorem 3.17 ([30, Theorem 5.3]). *Consider $a \leqslant t_1 < t_2, \ldots < t_m < b$ where $m \in \mathbb{N}$, $y_0 \in \mathbb{R}^n$, $f\colon \mathbb{R}^n \times [a,b] \to \mathbb{R}^n$ and $I_1,\dots,I_m\colon \mathbb{R}^n \to \mathbb{R}^n$. Then, $y\colon [a,b] \to \mathbb{R}^n$ is a solution of the impulsive DE*

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y(s),s)\,\mathrm{d}s + \sum_{k;\ t_k<t} I_k(y(t_k)), \quad t \in [a,b], \tag{3.16}$$

if and only if it is a solution of the measure DE

$$y(t) = y(t_0) + \int_{t_0}^{t} \tilde{f}(y(s),s)\,\mathrm{d}g(s), \tag{3.17}$$

where $g\colon [a,b] \to \mathbb{R}$ is given by (3.15) and

$$\tilde{f}(z,t) = \begin{cases} f(z,t), & \text{if } t \in [a,b] \setminus \{t_1,t_2,\dots,t_m\}; \\ I_k(z), & \text{if } t \in \{t_1,t_2,\dots,t_m\}. \end{cases}$$

Proof. From Corollary 3.16, Equation (3.17) is equivalent to

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y(s),s)\,\mathrm{d}s + \sum_{k;\ t_k<t} \tilde{f}(y(t_k),t_k), \quad t \in [a,b].$$

By definition, $\tilde{f}(y(t_k),t_k) = I_k(y(t_k))$ and the proof is complete. $\square$

Chapter 4

Massera's Theorem

In 1950, Massera published important results about the existence of periodic solutions of ODEs (see [26]). He showed that for an 1–dimensional system $x' = F(x,t)$, the existence of a bounded solution implies the existence of a periodic solution. He also proved that this did not happen on higher dimensions in general, but a similar result was presented if one considered only linear systems.

The idea of this chapter is to present briefly some of Massera's results. Besides that, we also extend the results to generalized ODEs, measure DEs, dynamic equations on time scales and impulsive DEs.

All the results for these equations are new in the literature, and they can be found in [14], which is the main reference here. Also, the results for dynamic equations on time scales and impulsive DEs presented in this chapter are more general than the ones found in the literature. Such results are presented in [14].

4.1 Massera's Theorem for ODEs

Consider the system

$$\begin{cases} x'(t) = F(x,t), \\ x(t_0) = x_0, \end{cases} \tag{4.1}$$

where x' represents the derivative of x with respect to time and $F : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ is a continuous function that is also locally Lipschitz with respect to x. Let us denote by $x(t,t_0,x_0)$ the solution of (4.1).

Since the focus of this chapter is to present results in classes of equations that are more general than the ODEs, all the results involving this type of equations will have their proof omitted. The next lemma is a useful tool to determine whether a solution is periodic or not.

Lemma 4.1 ([9, Lemma 4.1.3])**.** *Suppose that* (4.1) *is such that* $F(x,t) = F(x,t+T)$ *for some* $T > 0$*. Then:*

1. *if* $x(t)$ *is a solution of* (4.1)*,* $x(t+T)$ *is also a solution of the same system;*

2. (4.1) *has a periodic solution if and only if there is a solution* $x(t,t_0,x_0)$ *such that* $x(t,t_0,x_0) = x(t+T,t_0,x_0)$*.*

We now present the definition of an asymptotic solution to another one.

Definition 4.2. *A solution* $x\colon [t_0,+\infty) \to \mathbb{R}^n$ *of a differential equation* (4.1) *is said to be* ***asymptotic to another solution*** $y\colon [t_0,+\infty) \to \mathbb{R}^n$ *if*

$$\lim_{t\to+\infty} (x(t) - y(t)) = 0.$$

The next theorem was first presented by Massera in [26], but it can also be found in [9, Theorem 4.1.10].

Theorem 4.3 ([26, Theorem 1])**.** *Consider the system* (4.1)*, where* $F\colon \mathbb{R}\times[0,+\infty) \to \mathbb{R}$ *is T–perioric in the second variable, i.e.,* $F(x,t) = F(x,t+T)$ *for some* $T > 0$ *and for all* $t \in [0,+\infty)$*. Then, the existence of a bounded solution* $x\colon [t_0,+\infty) \to \mathbb{R}$ *implies the existence of a T–periodic solution* $y\colon [t_0,+\infty) \to \mathbb{R}$*. Besides that, x is asymptotic to y.*

For systems of order higher than one, the bounded solution may not be asymptotic to a periodic solution, even if we consider a linear system. The next example illustrates this case and it is presented in [14].

Example 4.4. *Consider the system*

$$\begin{cases} x'(t) = -y(t), \\ y'(t) = x(t). \end{cases}$$

To better understand the properties of the function $F\colon \mathbb{R}^2\times\mathbb{R} \to \mathbb{R}^2$*, we rewrite the system above as*

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Thus, $F(x,t)$ *is T–periodic on the second variable for any arbitrary* $T > 0$*, because all the coefficients are constants. Moreover, it is immediate to see that* $(x(t),y(t)) = (0,0)$ *is a T–periodic solution of our system. However, given* $(x_0,y_0) \in \mathbb{R}^2$*, a solution with initial condition* $(x(0),y(0)) = (x_0,y_0)$ *has the form* $(x(t),y(t)) = (x_0\cos t - y_0\sin t, x_0\sin t + y_0\cos t)$ *and,* *although they are bounded and 2π–periodic, they are not asymptotic to the zero solution that is T–periodic.*

Although the bounded solutions of the linear system are not necessarily asymptotic to a periodic solution, it is still possible to guarantee the existence of a periodic solution if there is a bounded solution. Before such theorem is presented, let us start by defining a linear system.

Suppose that $A\colon [t_0,+\infty) \to L(\mathbb{R}^n)$ and $b\colon [t_0,+\infty) \to \mathbb{R}^n$ are continuous functions and $x_0 \in \mathbb{R}^n$. The system of equations

$$\begin{cases} x' = A(t)x + b(t), \\ x(0) = x_0 \end{cases} \tag{4.2}$$

is called a **linear system of differential equations** or, simply, **linear system**.

The next theorem is a verson of Massera's theorem for linear systems.

Theorem 4.5 ([26, Theorem 4])**.** *Consider $A\colon [t_0,+\infty) \to L(\mathbb{R}^n)$ and $b\colon [t_0,+\infty) \to \mathbb{R}^n$ such that $A(t+T) = A(t)$ and $b(t+T) = b(t)$ for some $T > 0$ and every $t \in [t_0,+\infty)$. If the linear system* (4.2) *has a bounded solution, it also has a T–periodic solution.*

4.2 Massera's Theorem for Generalized ODEs

In this section, we state versions of the Massera's Theorem for generalized ODEs. This section contains only new results, which can be found in [14].

In order to do that, consider an open set $O \subset \mathbb{R}^n \times \mathbb{R}$ and a function $F\colon O \to \mathbb{R}^n$. In Chapter 1, we defined a generalized ODE (see Definition 1.24), denoted by

$$\frac{dx}{d\tau} = DF(x,t), \tag{4.3}$$

on the interval $[\alpha,\beta]$ if $(x(t),t) \in O$ whenever $t \in [\alpha,\beta]$ and

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau),t), \quad \forall s_1,s_2 \in [\alpha,\beta].$$

The next result is a useful tool to connect two distinct solutions.

Lemma 4.6 ([14, Lemma 2.5])**.** *Suppose that $F\colon \mathbb{R}^n \times [a,b] \to \mathbb{R}^n$ is regulated on the second variable and $x,y\colon \mathbb{R}^n \to \mathbb{R}^n$ are solutions of* (4.3).

1. *If* $c \in [a,b)$ *and* $\lim_{s \to c+} x(s) = \lim_{s \to c+} y(s)$*, then* $z\colon [a,b] \to \mathbb{R}^n$ *defined as*

$$z(t) = \begin{cases} x(t) & \text{if } t \in [a,c], \\ y(t) & \text{if } t \in (c,b], \end{cases}$$

is also a solution of (4.3)*;*

2. *if* $c \in [a,b)$ *and* $\lim_{s \to c-} x(s) = \lim_{s \to c-} y(s)$*, then* $w\colon [a,b] \to \mathbb{R}^n$ *defined as*

$$w(t) = \begin{cases} x(t) & \text{if } t \in [a,c), \\ y(t) & \text{if } t \in [c,b], \end{cases}$$

is a solution of (4.3).

Proof. Let us start by proving item 1. First, notice that

$$z(s_2) - z(s_1) = \int_{s_1}^{s_2} DF(z(\tau),t),$$

for every $s_1, s_2 \in [a,c]$ or $s_1, s_2 \in (c,b]$. We only need to proof that the same relation still holds if $s_1 \in [a,c]$ and $s_2 \in (c,b]$.

Applying Proposition 1.28 and the fact that F is regulated, we get

$$\lim_{s \to c+} \left[\int_{s}^{s_2} DF(z(\tau),t) - F(z(c),s) + F(z(c),c) \right] = \int_{c}^{s_2} DF(z(\tau),t).$$

Therefore, we can do the following calculation:

$$\begin{aligned}
\int_{s_1}^{s_2} DF(z(\tau),t) &= \int_{s_1}^{c} DF(z(\tau),t) + \int_{c}^{s_2} DF(z(\tau),t) \\
&= \int_{s_1}^{c} DF(z(\tau),t) + \lim_{s \to c+} \left[\int_{s}^{s_2} DF(z(\tau),t) - F(z(c),s) + F(z(c),c) \right] \\
&= x(c) - x(s_1) + \lim_{s \to c+} [y(s_2) - y(s) - F(x(c),s) + F(x(c),c)] \\
&= y(s_2) - x(s_1) + x(c) + \lim_{s \to c+} [-x(s) - F(x(c),s) + F(x(c),c)] \\
&= z(s_2) - z(s_1)
\end{aligned}$$

and we obtain the result.

The second item follows analogously and its proof will be omitted here. □

The next lemma is a version of Lemma 4.1 for generalized ODEs. It is a way to characterize the periodic solutions.

Lemma 4.7 ([14, Lemma 2.6]). *Consider* $F\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *and* $M\colon \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$F(x,t+T) - F(x,t) = M(x)$$

for a $T > 0$ *and every* $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. *Then:*

1. $x\colon [a+T, b+T] \to \mathbb{R}^n$ *is a solution of* (4.3) *if and only if* $y\colon [a,b] \to \mathbb{R}^n$ *defined as* $y(t) = x(t+T)$ *is also a solution of* (4.3)*;*
2. (4.3) *has a* T*–periodic solution if and only if there is solution* $x\colon [t_0, t_0+T] \to \mathbb{R}^n$ *such that* $x(t_0) = x(t_0+T)$.

Proof. We will show only one of the implications of item 1 and the other one follows analogously. First, notice that $F(x,t+T) - F(x,t) = M(x)$ implies that

$$\sum_{i=1}^{k} F(x(\tau_i+T), s_i+T) - F(x(\tau_i+T), s_{i-1}+T) = \sum_{i=1}^{k} F(x(\tau_i+T), s_i) - F(x(\tau_i+T), s_{i-1})$$

for any tagged partition D.

On the other hand, we can use the Substitution Theorem (Theorem 1.13) to obtain that

$$y(s_2) - y(s_1) = x(s_1+T) - x(s_2+T) = \int_{s_1+T}^{s_2+T} DF(x(\tau),t) = \int_{s_1}^{s_2} DF(x(\tau+T),t+T)$$

for any $s_1, s_2 \in [a,b]$.

Using both information above, we obtain

$$y(s_2) - y(s_1) = \int_{s_1}^{s_2} DF(x(\tau+T),t+T) = \int_{s_1}^{s_2} DF(y(\tau),t)$$

and the first part of this proof is complete.

For the second item, we can combine Lemma 1.43 and the first part of this proof to create a solution $z\colon [t_0, +\infty) \to \mathbb{R}^n$ such that $z(t) = x(t)$ for every $t \in [t_0, t_0+T]$ and $z(t+nT) = x(t)$ for any $n \in \mathbb{N}$. It is immediate to see that z is T–periodic. □

In Chapter 1, we defined the class $\mathcal{F}(G,h,\omega)$ (see Definition 1.30) and showed that if F belongs to that class, then (4.3) has a unique solution with $x(t_0) = x_0$ (Theorem 1.45). Here, we consider stronger conditions than those in class $\mathcal{F}(G,h,\omega)$. The extra conditions are important in order to apply the nonlinear Gronwall's Inequality (Theorem 1.23).

Consider a function $F\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. Then, we define the conditions (C1) and (C2) as:

(C1) For every bounded set $B \subset \mathbb{R}^n$, there is a function $h\colon \mathbb{R} \to \mathbb{R}$ is a nondecreasing left–continuous function such that

$$\|F(x,t_2) - F(x,t_1)\| \leqslant |h(t_2) - h(t_1)|$$

for every $x \in B$ and $t_1, t_2 \in \mathbb{R}$;

(C2) For every bounded set $B \subset \mathbb{R}^n$, there exists a function $\omega\colon [0,+\infty) \to [0,+\infty)$ such that $\omega(0) = 0$, $\lim\limits_{u\to 0+} \int_u^v 1/\omega(r)\mathrm{d}r = +\infty$ for a certain $v > 0$ and a nondecreasing left–continuous function $h\colon \mathbb{R} \to \mathbb{R}$ such that for every for all $x, y \in B$ and $t_1, t_2 \in \mathbb{R}$, we obtain

$$\|F(x,t_2) - F(x,t_1) - F(y,t_2) + F(y,t_1)\| \leqslant |h(t_2) - h(t_1)|\omega(\|x-y\|). \tag{4.4}$$

It is clear that if F satisfies the conditions above, $F \in \mathcal{F}(G,h,\omega)$.

Although the next lemma is more technical, it will be important to prove the Massera's theorem for generalized ODEs.

Lemma 4.8 ([14, Lemma 2.7]). *Suppose that $F\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is such that conditions* (C1) *and* (C2) *are satisfied and there exists a $M\colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$F(x,t+T) - F(x,t) = M(x)$$

for a $T > 0$ and every $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Then the function $h\colon \mathbb{R} \to \mathbb{R}$ satisfying (C1) *and* (C2) *can be chosen in a way that $t \mapsto h(t+T) - h(t)$ is a constant function for every $t \in \mathbb{R}$.*

Proof. By hypothesis, we know that given $B \subset \mathbb{R}^n$, there is a function h satisfying (C1) and (C2). Denote $C = h(T) - h(0)$ and define $\overline{h}\colon \mathbb{R} \to \mathbb{R}$ as

$$\overline{h}(t) = \begin{cases} h(t), & \text{if } t \in [0,T), \\ h(t-nT) + nC, & \text{if } t \in [nT,(n+1)T) \text{ and } n \in \mathbb{Z}\setminus\{0\}. \end{cases}$$

Since h is nondecreasing and left–continuous, it is easy to see that $\overline{h}$ has the same properties in each interval $[nT,(n+1)T)$. To see that this is still valid for all $\mathbb{R}$, notice that

$$\begin{aligned} \lim_{t\to(n+1)T-} \overline{h}(t) &= \lim_{t\to T-} h(t) + nC = h(T) + nC = h(0) + h(T) - h(0) + nC \\ &= h(0) + (n+1)C = \overline{h}((n+1)T). \end{aligned}$$

It remains to show that $\overline{h}$ also satisfies (C1) and (C2). For that, let us suppose, without loss of generality, that $t_1 < t_2$. Consider also the unique numbers $a, b \in \mathbb{Z}$ satisfying $aT \leqslant t_1 \leqslant (a+1)T$ and $bT \leqslant t_2 \leqslant (b+1)T$. By hypothesis,

$$F(x,t) - F(x,t+nT) = nM(x)$$

for any $x \in B$ and $n \in \mathbb{Z}$.

Suppose first that $a = b$. Then, (C1) follows from the below calculation:

$$\begin{aligned}
\|F(x,t_2) - F(x,t_1)\| &= \|F(x,t_2 - aT) + aM(x) - F(x,t_1 - aT) - aM(x)\| \\
&\leqslant h(t_2 - aT) - h(t_1 - aT) \\
&= h(t_2 - aT) + aT - h(t_1 - aT) - aT \\
&= \overline{h}(t_2) - \overline{h}(t_1).
\end{aligned}$$

For (C2), we have

$$\begin{aligned}
&\|F(x,t_2) - F(x,t_1) - F(y,t_2) + F(y,t_1)\| \\
&= \|F(x,t_2 - aT) + aM(x) - F(x,t_1 - aT) - aM(x) \\
&\quad - F(y,t_2 - aT) - aM(y) + F(y,t_1 - aT) + aM(y)\| \\
&\leqslant \omega(\|x - y\|)(h(t_2 - aT) - h(t_1 - aT)) \\
&= \omega(\|x - y\|)(\overline{h}(t_2) - \overline{h}(t_1)
\end{aligned}$$

for any $x, y \in O$.

Suppose now that $a < b$. Before we can do the calculation for this case, notice first that

$$\|M(x)\| = \|F(x,T) - F(x,0)\| \leqslant h(T) - h(0) = C.$$

Using this information, we get

$$\begin{aligned}
\|F(x,t_2) - F(x,t_1)\| &\leqslant \|F(x,t_2) - F(x,bT)\| + \|F(x,bT) - F(x,(a+1)T)\| \\
&\quad + \|F(x,(a+1)T) - F(x,t_1)\| \\
&\leqslant \|F(x,t_2 - bT) - F(x,0)\| + (b - a - 1)\,\|M(x)\| \\
&\quad + \|F(x,T) - F(x,t_1 - aT)\| \\
&\leqslant h(t_2 - bT) - h(0) + (b - a - 1)C + h(T) - h(t_1 - aT) \\
&= h(t_2 - bT) + bT - h(t_1 - aT) - aT - C + C \\
&= \overline{h}(t_2) - \overline{h}(t_1).
\end{aligned}$$

The calculation above shows that $\overline{h}$ satisfies (C1). Condition (C2) can be proved in a similar way and, because of that, its proof will be omitted. With that, the proof is complete. $\square$

Our goal is to first present an 1–dimensional version of Massera's Theorem (see Theorem 4.3) and then, afterward, to present the result for higher dimensions. As the next lemma may already indicate, we use a relation of order between solutions to obtain the 1–dimensional result. Of course, we need to use a different argument for the higher dimension case.

The next lemma presents a condition to guarantee a certain relation of order between two solutions. Observe that such lemma is not necessary for ODEs because of the uniqueness and continuity of solutions, but here, our solutions do not need to be continuous.

Lemma 4.9 ([14, Lemma 3.1])**.** *Suppose that* $F\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *satisfies conditions* (C1)*,* (C2) *and also*

(C3) *if* $z,w\in\mathbb{R}$ *are such that* $z<w$*, then*

$$z+\lim_{s\to t+}F(z,s)-F(z,t)\leqslant w+\lim_{s\to t+}F(w,s)-F(w,t)$$

for all $t\in\mathbb{R}$*.*

If $x,y\colon [a,b]\to\mathbb{R}$ *are solutions of* (4.3) *with* $x(a)\leqslant y(a)$*, then* $x(t)\leqslant y(t)$ *for all* $t\in[a,b]$*.*

Proof. By contradiction, suppose that $y(t)<x(t)$ for some $t\in(a,b]$. Define

$$S=\inf\{t\in(a,b]\colon y(t)<x(t)\}.$$

Since h is left–continuous, Corollary 1.32 implies that the solutions are also left–continuous. As a consequence, we can conclude that $x(S)\leqslant y(S)$. Therefore,

$$S\notin\{t\in(a,b]\colon y(t)<x(t)\}$$

and we also get that $S\neq b$ (from the left continuity of the solution).

Suppose now that $x(S)=y(S)$. In this case, we would have two solutions x and y that coincide in a point S but are different in $[S,b]$. This is a contradiction because conditions (C1) and (C2) imply the uniqueness of solutions in the future (Theorem 1.45).

Therefore, the only remain option is that $x(S)<y(S)$. From condition (C3), we have

$$\lim_{t\to S+}x(t)\leqslant\lim_{t\to S+}y(t).$$

On the other hand, the definition of infimum implies that the only possibility is that

$$\lim_{t \to S+} x(t) = \lim_{t \to S+} y(t).$$

We can now apply Lemma 4.6 and construct the solution

$$z(t) = \begin{cases} y(t) & \text{if } t \in [a, t_0], \\ x(t) & \text{if } t \in (t_0, b]. \end{cases}$$

But z and y are two different solutions with $z(a) = y(a)$ and this is also a contradiction. Thereby, the result follows. □

The next result is a version of Massera's Theorem (Theorem 4.3) for generalized ODEs. It is a new result and it can be found in [14].

Theorem 4.10 ([14, Theorem 3.1])**.** *Consider that* $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (C1)-(C3) *and suppose that there is a function* $M\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$F(x, t+T) - F(x, t) = M(x)$$

for a $T > 0$ *and every* $(x, t) \in \mathbb{R} \times \mathbb{R}$*. Then the existence of a bounded solution of* (4.3) *also implies the existence of a* T*–periodic solution of* (4.3)*. Furthermore, each bounded solution of* (4.3) *is asymptotic to a* T*–periodic solution of* (4.3)*.*

Proof. Suppose that $x_0\colon [t_0, +\infty) \to \mathbb{R}$ is a bounded solution of (4.3). Hence, there is a constant $P > 0$ such that $|x(t)| \leqslant P$ for all $t \in [t_0, +\infty)$. For each $n \in \mathbb{N}$, define the function

$$\begin{aligned} x_n\colon [t_0, +\infty) &\to \mathbb{R}, \\ t &\mapsto x(t + nT). \end{aligned}$$

From Lemma 4.7, x_n is also a solution of (4.3).

We will suppose that $x(t_0) \geqslant x_1(t_0)$ and the other case follows analogously. Applying Lemma 4.9, $x(t) \geqslant x_1(t)$ for every $t \in [t_0, +\infty)$. This also implies that

$$x_n(t) = x(t + nT) \geqslant x_1(t + nT) = x_{n+1}(t).$$

Therefore, $\{x_n\}_{n \in \mathbb{N}}$ is a nonincreasing bounded sequence of functions. Thus, there exists a function $y\colon [t_0, +\infty) \to \mathbb{R}$ that is the pointwise limit of the sequence $\{x_n\}_{n \in \mathbb{N}}$.

It is possible to apply Corollary 1.18 because of conditions (C1) and (C2). As a consequence,

$$\lim_{n\to+\infty}\int_a^b DF(x_n(\tau),t) = \int_a^b DF(y(\tau),t).$$

From the calculation below,

$$y(t) = \lim_{n\to+\infty} x_n(t) = \lim_{n\to+\infty}\left(x_n(t_0) + \int_a^b DF(x_n(\tau),t)\right) = y(t_0) + \int_a^b DF(y(\tau),t),$$

it follows that y is also a solution of (4.3). On the other hand,

$$y(t_0+T) = \lim_{n\to+\infty} x_n(t_0+T) = \lim_{n\to+\infty} x_{n+1}(t_0) = y(t_0)$$

and we conclude that y is a T–periodic solution of (4.3).

It remains to show that x is asymptotic to y. Condition (C2) guarantees that there is a $v>0$ such that

$$\Omega(u) = \int_v^u \frac{\mathrm{d}r}{\omega(r)}, \quad u \in (0,+\infty)$$

is a continuous increasing function with $\lim\limits_{u\to 0+}\Omega(u) = -\infty$ and $\lim\limits_{u\to+\infty}\Omega(u) = \beta \leqslant +\infty$. As a consequence, Ω^{-1} can be defined in $(-\infty,\beta)$ and it is also an increasing function.

Therefore, given $\varepsilon > 0$, choose an $\eta > 0$ such that $\Omega(\eta) + h(t_0+T) - h(t_0) < \beta$ and $\Omega^{-1}(\Omega(\eta) + h(t_0+T) - h(t_0)) < \varepsilon$. By Lemma 4.8, we can assume that $h(t+T) - h(t)$ has the same value for every t that h is defined.

Define now the function $\varphi(t) = |x(t) - y(t)|$ for $t \in [t_0,+\infty)$. From the first part of this proof, we know that y is T–periodic and, therefore,

$$\lim_{m\to+\infty} x(t_0+mT) = \lim_{m\to+\infty} x_m(t_0) = y(t_0) = y(t_0+mT).$$

Thus, there is $m_0 \in \mathbb{N}$ such that $\varphi(t_0+mT) = |x(t_0+mT) - y(t_0+mT)| < \eta$ for every $m \geqslant m_0$.

Consider now $t \geqslant t_0 + m_0T$. There is a unique $m \geqslant m_0$ such that $t_0 + mT \leqslant t \leqslant t_0 + m(T+1)$. Applying the remarks above and condition (C2), we obtain

$$\begin{aligned}
\varphi(t) &= \left| x(t_0+mT) - \int_{t_0+mT}^t DF(x(\tau),s)) - y(t_0+mT) - \int_{t_0+mT}^t DF(y(\tau),s))\right| \\
&\leqslant |x(t_0+mT) - y(t_0+mT)| + \left|\int_{t_0+mT}^t D[F(x(\tau),s)) - F(y(\tau),s]\right| \\
&< \eta + \int_{t_0+mT}^t \omega(|x(s)-y(s)|)\mathrm{d}h(s).
\end{aligned}$$

From the nonlinear version of the Gronwall's Inequality (Theorem 1.23), it follows that

$$\begin{aligned}\varphi(t) &\leqslant \Omega^{-1}(\Omega(\eta)+h(t)-h(t_0+mT)) \leqslant \Omega^{-1}(\Omega(\eta)+h(t_0+(m+1)T)-h(t_0+mT)) \\ &= \Omega^{-1}(\Omega(\eta)+h(t_0+T)-h(t_0)) < \varepsilon.\end{aligned}$$

Thereby, $\lim_{t\to+\infty} \psi(t) = 0$ and the proof is complete. □

Remark 4.11. *Notice that Theorem 4.10 does not require the function $F(x,t)$ to be periodic on the second variable, as done in Theorem 4.3. Our assumptions are weaker, without loosing any properties in the result.*

Besides that, another difference between both results is that here, we need condition (C3) *and Lemma 4.9 to guarantee that the periodic solution of* (4.3) *exists. Such auxiliary result is only needed because the solution of a generalized ODE can be discontinuous and, otherwise, we would not be able get a monotone sequence of solutions of* (4.3)*. On the other hand, such result is not valid for a classical ODE, which the solution is usually assumed to be continuous.*

Lastly, here we applied a nonlinear version of the Gronwall's Inequality (Theorem 1.23) to guarantee that the bounded solution of 4.3 is asymptotic to the periodic solution of 4.3. On the original case, we can use a simpler way applying the Arzelà-Ascoli's Theorem, as done in [9, Theorem 4.1.10].

The next example shows a function that satisfies all conditions of Theorem 4.10.

Example 4.12 ([14, Example 3.4])**.** *Let*

$$\frac{dx}{d\tau} = DF(x,t),$$

where $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is given by $F(x,t) = (i+1-t)x$ for all $t \in (i,i+1]$, $i \in \mathbb{Z}$.

One of the hypotheses of Theorem 4.10 is satisfied because $F(x,t+1) = F(x,t)$ for all $x,t \in \mathbb{R}$. In this case, $T = 1$. To see that (C1) *and* (C2) *are also fulfilled, suppose that $O \subset [-K,K] \subset \mathbb{R}$ is a bounded set and $K \geqslant 1$.*

Consider $\omega : [0,+\infty) \to [0,+\infty)$ defined as $\omega(r) = r$ and $h : \mathbb{R} \to \mathbb{R}$ by $h(s) = K(s+i)$ for all $s \in (i,i+1]$, $i \in \mathbb{Z}$. Let $s_1, s_2 \in \mathbb{R}$ be such that $s_1 \leqslant s_2$. There are unique $m,n \in \mathbb{Z}$ such that $s_1 \in (m,m+1]$, $s_2 \in (n,n+1]$. For all $x,y \in O$, we obtain

$$\begin{aligned}|F(x,s_2)-F(x,s_1)| &= |(n+1-s_2)x-(m+1-s_1)x| \leqslant K|(n-m)-(s_2-s_1)| \\ &\leqslant K|(n-m)+(s_2-s_1)| = |h(s_2)-h(s_1)|.\end{aligned}$$

Similarly,

$$|F(x,s_2) - F(x,s_1) - F(y,s_2) + F(y,s_1)| \leqslant |h(s_2) - h(s_1)|\omega(|x-y|).$$

It remains to show that condition (C3) *holds. Observe that such condition is always valid where* $t \mapsto F(x,t)$ *is continuous. Suppose now that* $x < y$ *and* $t \in \mathbb{Z}$*. As a consequence,*

$$x + F(x,t+) - F(x,t) = 2x \leqslant 2y = y + F(y,t+) - F(y,t)).$$

Therefore, all conditions of Theorem 4.10 are satisfied. If we can show that there is a bounded solution, the theorem implies that it is asymptotic to an 1*–periodic solution.*

Since we are able to write F *as* $F(x(\tau),t) = x(\tau)g(t)$*, our integral* $\int_a^b DF(x(\tau),s)$ *can be seen simply as* $\int_a^b x(s)\,\mathrm{d}g(s)$*. Furthermore, if* $[s_1,s_2] \subset (i,i+1]$ *with* $i \in \mathbb{Z}$*, then* $g(s_2) - g(s_1) = s_1 - s_2$*. Therefore,* $\int_{s_1}^{s_2} x(s)\,\mathrm{d}g(s) = -\int_{s_1}^{s_2} x(s)\,\mathrm{d}s$*.*

In other words, the generalized ODE reduces to the ODE $x'(t) = -x(t)$ *in each interval* $(i,i+1]$*. Thus, we only need to see what happens to the solution at* $i \in \mathbb{Z}$*, because we already know how it behaves in each interval* $(i,i+1)$*. For that, Lemma 1.35 implies the following equality*

$$x(i+) = x(i) + F(x(i),i+) - F(x(i),i) = 2x(i).$$

Combining all the information above, we obtain $x(t) = 2x(i)e^{-(t-i)}$ *for all* $t \in (i,i+1]$*. Therefore, every solution is bounded and, by Theorem 4.10, they are all asymptotic to the* 1*–periodic solution. It is also easy to see that the only periodic solution is the constant zero. Figure 4.1 illustrates some solutions of this problem.*

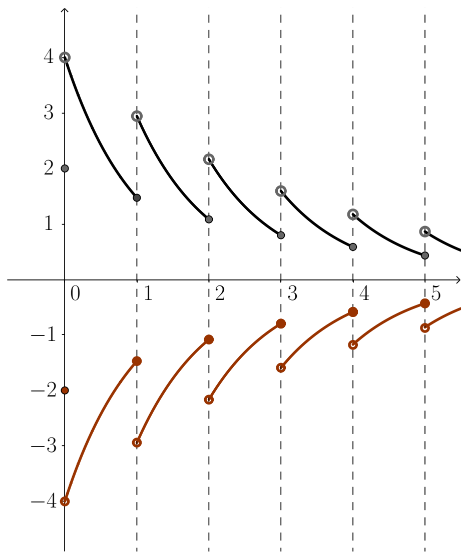

Figure 4.1 Solutions with $x(0) = 2$ and $x(0) = -2$, [14], Figure 1.

Before we present a version of Massera's Theorem for higher dimensions, let us remember what is a system of linear generalized ODEs.

Consider a function $A\colon \mathbb{R} \to L(\mathbb{R}^n)$ and $f\colon \mathbb{R} \to \mathbb{R}^n$. An n–dimensional system of linear generalized ODEs is represented by

$$\frac{dx}{dt} = D[A(t)x + f(t)], \tag{4.5}$$

where x takes values in $\mathbb{R}^n$. (4.5) is a special case of the generalized ODE with $F(x,t) = A(t)x + f(t)$ and is equivalent to the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{d}[A(s)]x(s) + f(t) - f(t_0).$$

We introduce the following conditions:

(D1) $A\colon \mathbb{R} \to L(\mathbb{R}^n)$ has locally bounded variation and $I - \Delta^- A(t)$ is invertible for each $t \in \mathbb{R}$.

(D2) $f\colon \mathbb{R} \to \mathbb{R}^n$ is regulated.

In Chapter 1, we saw that combining condition (D1) the fact that $f\colon \mathbb{R} \to \mathbb{R}^n$ is a function of bounded variation, we obtained the existence of a unique solution defined on $[t_0, +\infty)$ and satisfying $x(t_0) = x_0$. It is possible to assume (D1) and (D2) instead (see [31, Theorem 7.4.5]) and obtain the same result. Let us denote it by $x(\cdot, t_0, x_0)$ the solution with initial condition $x(t_0) = x_0$.

To prove the version of Massera's Theorem for higher dimensions, the Brouwer Fixed Point Theorem will be applied. Such theorem is presented below and it can be found in [12].

Theorem 4.13 (Brouwer Fixed Point Theorem)**.** *Suppose that $P\colon K \to K$ is a continuous map and K is a convex compact subset of an Euclidean space. Then, P has a fixed point.*

The next result is a version of Massera's Theorem (4.5) for generalized linear systems. It is new in the literature and it can be found in [14].

Theorem 4.14 ([14, Theorem 4.1])**.** *Let $A\colon \mathbb{R} \to L(\mathbb{R}^n)$ and $f\colon \mathbb{R} \to \mathbb{R}^n$ be such that* (D1) *and* (D2) *hold. Furthermore, suppose that there are $T > 0$, $C \in L(\mathbb{R}^n)$ and $D \in \mathbb{R}^n$ such that*

$$A(t+T) - A(t) = C \quad \textit{and} \quad f(t+T) - f(t) = D \quad \textit{for every } t \in \mathbb{R}.$$

If (4.5) *has a bounded solution on $[t_0, +\infty)$, then it has a T–periodic solution.*

Proof. Suppose that $x(t,t_0,x_0)$ is a bounded solution of (4.5) on $[t_0,+\infty)$. Hence, there is a constant $B>0$ such that $\|x(t,t_0,x_0)\| \leqslant B$ for every $t\in[t_0,+\infty)$. The idea of the proof now is to apply the Brouwer Fixed Point Theorem on the Poincaré map $P\colon \mathbb{R}^n \to \mathbb{R}^n$

$$P(y)=x(t_0+T,t_0,y)=y+\int_{t_0}^{t_0+T} d\,[A(s)]\,x(s,t_0,y)+f(t)-f(t_0).$$

It is immediate to see that P is a continuous map on $\mathbb{R}^n$.

Let us consider its restriction to the set

$$\Lambda=\{y\in\mathbb{R}^n\colon\ \|x(t,t_0,y)\|\leqslant M \ \text{ for every } t\geqslant t_0\}\,.$$

Since $x_0\in\Lambda$, we already know that the set is not empty. To see that Λ is also convex, consider $z_0,y_0\in\Lambda$ and $\alpha\in(0,1)$, then

$$\begin{aligned}
&\alpha x(t,t_0,z_0)+(1-\alpha)x(t,t_0,y_0)\\
&=\alpha\left(z_0+\int_{t_0}^{t_0+T} d\,[A(s)]\,x(s,t_0,z_0)+f(t)-f(t_0)\right)\\
&+(1-\alpha)\left(y_0+\int_{t_0}^{t_0+T} d\,[A(s)]\,x(s,t_0,y_0)+f(t)-f(t_0)\right)\\
&=\alpha x_0+(1-\alpha)y_0+\int_{t_0}^{t_0+T} d\,[A(s)]\,(\alpha x(s,t_0,x_0)+(1-\alpha)x(s,t_0,y_0))+f(t)-f(t_0).
\end{aligned}$$

Therefore, the function $\alpha x(\cdot,t_0,x_0)+(1-\alpha)x(\cdot,t_0,y_0)$ is also a solution of (4.5). Notice that it coincides with the solution $x(\cdot,t_0,\alpha x_0+(1-\alpha)y_0)$ at t_0 and, by the uniqueness of solutions, we have

$$\|x(t,t_0,\alpha x_0+(1-\alpha)y_0)\|=\|\alpha x(t,t_0,x_0)+(1-\alpha)x(t,t_0,y_0)\|\leqslant \alpha M+(1-\alpha)M=M$$

for each $t\geqslant t_0$. From the calculation above, we conclude that $\alpha x_0+(1-\alpha)y_0\in\Lambda$.

It is clear that Λ is bounded. It remains to show that Λ is closed. For that, consider a sequence $\{y_n\}_{n\in\mathbb{N}}\subset\Lambda$ such that $y_n\to y$. Then $\|x(t,t_0,y_n)\|\leqslant M$ for every $t\geqslant t_0$ and, thus, $\|x(t,t_0,y)\|\leqslant M$ for every $t\geqslant t_0$. Hence, $y\in\Lambda$ and we obtain that Λ is a bounded closed subset of $\mathbb{R}^n$.

Notice now that we can apply Lemma 4.8 for $F(x,t)=A(t)x+g(t)$. Thus, for every $y\in\mathbb{R}^n$, the map $t\mapsto x(t+T,t_0,y)$ is a solution of (4.5). At the point t_0, it coincides with $x(t,t_0,P(y))$. Therefore, if $y\in\Lambda$, we get

$$\|x(t,t_0,P(y))\|=\|x(t+T,t_0,y)\|\leqslant M.$$

This implies that $P\colon \Lambda \to \Lambda$. By the Brouwer Fixed Point Theorem, we conclude that P has a fixed point $\tilde{x}_0 \in \Lambda$. Therefore, $x(t_0,t_0,\tilde{x}_0) = \tilde{x}_0 = P(\tilde{x}_0) = x(t_0+T,t_0,\tilde{x}_0)$. Using Lemma 4.8, $x(t,t_0,\tilde{x}_0)$ is a T–periodic solution of (4.5). □

4.3 Massera's Theorem for Other Types of Equations

Once we obtain the results for the generalized ODEs, it is easy to get similar results for different types of differential equations with discontinuous solutions. It is only necessary to apply the relations presented in Chapter 3.

Let us start by presenting versions of Massera's Theorem for measure DEs. Such type of differential equations have the following equivalent integral form:

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s),s)\,\mathrm{d}g(s). \tag{4.6}$$

The integral above is in the Henstock–Kurzweil–Stieltjes sense.

To see more details about this type of equations, see Chapter 3. We begin with the new result considering only scalar measure DEs, found in [14].

Theorem 4.15 ([14, Theorem 5.1])**.** *Suppose that $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R}\to\mathbb{R}$ are such that:*

- *g is left–continuous, nondecreasing and there are real numbers $K,T > 0$ such that $g(t+T)-g(t) = K$ for each $t \in \mathbb{R}$;*
- *f is continuous in the first variable and T–periodic in the second;*
- *for each $s_1,s_2 \in \mathbb{R}$ and every $x \in \mathbb{R}$, the Henstock–Kurzweil–Stieltjes integral $\int_{s_1}^{s_2} f(x,s)\,\mathrm{d}g(s)$ exists;*
- *for every bounded set $B \subset \mathbb{R}$, there exists a function $m\colon \mathbb{R}\to\mathbb{R}$ for which $\int_{s_1}^{s_2} m(s)\,\mathrm{d}g(s)$ exists for every $s_1,s_2 \in \mathbb{R}$ and there is a continuous increasing function $\omega\colon [0,+\infty) \to [0,+\infty)$ with $\omega(0) = 0$ and $\lim\limits_{u\to 0+} \int_u^v 1/\omega(r)\mathrm{d}r = +\infty$ for a certain $v > 0$ such that:*

$$\left| \int_{s_1}^{s_2} f(x,s)\,\mathrm{d}g(s) \right| \leqslant \int_{s_1}^{s_2} m(s)\,\mathrm{d}g(s), \tag{4.7}$$

$$\left| \int_{s_1}^{s_2} (f(x,s)-f(y,s))\,\mathrm{d}g(s) \right| \leqslant \omega(|x-y|) \int_{s_1}^{s_2} m(s)\,\mathrm{d}g(s) \tag{4.8}$$

for every $x,y \in B$ and $s_1 \leqslant s_2$;

- *if* $x, y \in \mathbb{R}$ *with* $x < y$*, then* $x + f(x,t)\Delta^+ g(t) \leqslant y + f(y,t)\Delta^+ g(t)$ *for every* $t \in \mathbb{R}$*.*

Then the existence of a bounded solution of (4.6) *also implies the existence of a T–periodic solution of* (4.6)*. Besides that, each bounded solution of* (4.6) *is asymptotic to a T–periodic solution of* (4.6)*.*

Proof. Define $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$F(x,t) = \int_{t_0}^{t} f(x,s)\,\mathrm{d}g(s)$$

where the integral above is in the Henstock–Kurzweil–Stieltjes sense. We will see that all the conditions of Theorem 4.10 are satisfied and then, we can use the correspondence between the differential equations to obtain the result.

The hypotheses (4.7) and (4.8) imply that F satisfies conditions (C1) and (C2) with $h(t) = \int_{t_0}^{t} m(s)\,\mathrm{d}g(s)$, $t \in \mathbb{R}$. Notice that h is left–continuous, as it was supposed in these conditions, because g is left–continuous and we can apply Theorem 1.11.

On the other hand, (C3) follows directly from the hypotheses and the calculation:

$$x + \lim_{s \to t+} F(x,s) - F(x,t) = x + f(x,t)\Delta^+ g(t) \leqslant y + f(y,t)\Delta^+ g(t) = y + \lim_{s \to t+} F(y,s) - F(y,t)$$

for all $t \in \mathbb{R}$ and every $x, y \in \mathbb{R}$ such that $x \leqslant y$.

Since $f(x,t+T) = f(x,t)$ and $g(t+T) - g(t) = K$ for every $t \in \mathbb{R}$, we also obtain that

$$F(x,t+T) - F(x,t) = \int_{t}^{t+T} f(x,s)\,\mathrm{d}g(s) = \int_{t_0}^{t_0+T} f(x,s)\,\mathrm{d}g(s).$$

Therefore, F satisfies all the conditions of Theorem 4.10 with $M(x) = \int_{t_0}^{t_0+T} f(x,s)\,\mathrm{d}g(s)$. We can now apply Theorem 3.5 to obtain the result for measure DEs and the proof is complete. $\square$

A linear measure DE is an integral equation of the form

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s) + q(s))\,\mathrm{d}g(s), \tag{4.9}$$

where $t_0 \in \mathbb{R}$, $p\colon \mathbb{R} \to L(\mathbb{R}^n)$, $q\colon \mathbb{R} \to \mathbb{R}^n$ and $g\colon \mathbb{R} \to \mathbb{R}$. In other words, (4.9) is a particular case of (4.6) with $f(x,t) = p(t)x + q(t)$.

Next, we present Massera's Theorem for this type of linear equations. It is a new result and it can be found in [14].

Theorem 4.16 ([14, Theorem 5.2]). *Consider $g\colon \mathbb{R} \to \mathbb{R}$, $p\colon \mathbb{R} \to L(\mathbb{R}^n)$ and $q\colon \mathbb{R} \to \mathbb{R}^n$ with the following properties:*

- *g is left–continuous, nondecreasing and there exist $K, T > 0$ such that $g(t+T) - g(t) = K$ for every $t \in \mathbb{R}$;*
- *p, q are T–periodic functions and are integrable with respect to g on every interval $[s_1, s_2] \subset \mathbb{R}$;*
- *there exists a function $m\colon \mathbb{R} \to \mathbb{R}$ that is integrable on every interval $[s_1, s_2] \subset \mathbb{R}$ and also*

$$\left\| \int_{s_1}^{s_2} p(s)\,\mathrm{d}g(s) \right\| \leqslant \int_{s_1}^{s_2} m(s)\,\mathrm{d}g(s). \tag{4.10}$$

Then the existence of a bounded solution of (4.9) *implies in the existence of a T–periodic solution.*

Proof. Define $A\colon \mathbb{R} \to L(\mathbb{R}^n)$ and $f\colon \mathbb{R} \to \mathbb{R}^n$ as

$$A(t) = \int_{t_0}^{t} p(s)\,\mathrm{d}g(s), \quad f(t) = \int_{t_0}^{t} q(s)\,\mathrm{d}g(s)$$

and the integrals above are in the Henstock–Kurzweil–Stieltjes sense.

Since g is regulated and left–continuous, Theorem 1.11 guarantees that A and f are also regulated and left–continuous. Besides that, it follows directly from (4.10) that

$$\mathrm{var}_{s_1}^{s_2} A \leqslant \int_{s_1}^{s_2} m(s)\,\mathrm{d}g(s)$$

for each $[s_1, s_2] \subset \mathbb{R}$.

As a consequence of p and q being T–periodic, we get

$$\begin{aligned} A(t+T) - A(t) &= \int_{t}^{t+T} p(s)\,\mathrm{d}g(s) = \int_{t_0}^{t_0+T} p(s)\,\mathrm{d}g(s) = A(t_0+T) - A(t_0), \\ f(t+T) - f(t) &= \int_{t}^{t+T} q(s)\,\mathrm{d}g(s) = \int_{t_0}^{t_0+T} q(s)\,\mathrm{d}g(s) = f(t_0+T) - f(t_0). \end{aligned}$$

From the correspondences, we obtain $\int_{t_0}^{t} \mathrm{d}[A(s)]x(s) = \int_{t_0}^{t} p(s)x(s)\,\mathrm{d}g(s)$. Therefore, the measure DE (4.9) is equivalent to the generalized linear ODE

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{d}[A(s)]x(s) + f(t) - f(t_0),$$

and the result follows from Theorem 4.14. $\square$

We will consider now a time scale $\mathbb{T}$ and a nonlinear dynamic equation on time scale as

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), s)\Delta s, \quad t \in \mathbb{T}, \tag{4.11}$$

where $t_0 \in \mathbb{T}$ and $f\colon \mathbb{R} \times \mathbb{T} \to \mathbb{R}$. For more details about this type of equation, see Chapters 2 and 3. Since we study periodic problems in this Chapter, we also need the definition of a periodic time scale.

Definition 4.17 ([14, Definition 6.1]). *Given $T > 0$, a time scale $\mathbb{T}$ is called T–periodic if $t - T, t + T \in \mathbb{T}$ for every $t \in \mathbb{T}$.*

We now prove Massera's Theorem for this integral equations.

Theorem 4.18 ([14, Theorem 5.2]). *Let $\mathbb{T}$ be a T–periodic time scale and suppose that $f\colon \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ satisfies:*

- *f is continuous in the first variable and T–periodic in the second variable;*
- *$\int_{s_1}^{s_2} f(x,s)\,\Delta s$ exists in the sense of Henstock–Kurzweil–Stieltjes for every $[s_1, s_2]_{\mathbb{T}}$ and $x \in \mathbb{R}$;*
- *for every bounded set $B \subset \mathbb{R}$, there exist a function $m\colon \mathbb{T} \to \mathbb{R}$ that is Δ-integrable on every interval $[s_1, s_2]_{\mathbb{T}}$ and a continuous increasing function $\omega\colon [0, +\infty) \to [0, +\infty)$ with $\omega(0) = 0$ and $\lim\limits_{u \to 0+} \int_u^v 1/\omega(r)\mathrm{d}r = +\infty$ for a certain $v > 0$ such that*

  $$\left| \int_{s_1}^{s_2} f(x,s)\,\Delta s \right| \leqslant \int_{s_1}^{s_2} m(s)\,\Delta s,$$
  $$\left| \int_{s_1}^{s_2} (f(x,s) - f(y,s))\,\Delta s \right| \leqslant \omega(|x-y|) \int_{s_1}^{s_2} m(s)\,\Delta s$$

  for every $x, y \in B$ and $s_1 \leqslant s_2$;
- *if $x, y \in \mathbb{R}$ with $x < y$, then $x + f(x,t)\mu(t) \leqslant y + f(y,t)\mu(t)$ for every $t \in \mathbb{T}$.*

Then the existence of a bounded solution of (4.11) *also implies the existence of a T–periodic solution of* (4.11)*. Moreover, each bounded solution of* (4.11) *is asymptotic to a T–periodic solution of* (4.11)*.*

Proof. On Chapter 2, we defined the operator $*\colon \mathbb{T}^* \to \mathbb{T}$ as $t^* = \inf\{s \in \mathbb{T}\colon s \geqslant t\}$, where

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < +\infty, \\ (-\infty, +\infty) & \text{if } \sup \mathbb{T} = +\infty. \end{cases}$$

Notice that in this case, since $\mathbb{T}$ is T–periodic, $\sup\mathbb{T} = +\infty$. Therefore, $\mathbb{T}^* = \mathbb{R}$. Besides that, for any function $h\colon \mathbb{T} \to \mathbb{R}^n$, we also denoted as h^* for the composition $h^*(t) = h(t^*)$.

Given $f\colon \mathbb{R}\times\mathbb{T} \to \mathbb{R}$, define $f^*\colon \mathbb{R}\times\mathbb{T}^* \to \mathbb{R}$ as $f^*(x,t) = f(x,t^*)$. Consider also $g\colon \mathbb{T}^* \to \mathbb{T}$ defined as $g(t) = t^*$. It is easy to see that $g\colon \mathbb{T}^* \to \mathbb{T}$ is a nondecreasing and left–continuous function. Furthermore, $g(t+T) - g(t) = T$ because $\mathbb{T}$ is T–periodic time scale.

Besides that, for any $t \in \mathbb{T}$, we get

$$\Delta^+ g(t) = \lim_{x\to t+} \inf\{s\in\mathbb{T}\colon s \geqslant x\} - \inf\{s\in\mathbb{T}\colon s\geqslant t\} = \inf\{s\in\mathbb{T}\colon s>t\} - t = \mu(t).$$

Analogously, we obtain $\Delta^+ g(t) = 0$ for $t \in \mathbb{R}\setminus\mathbb{T}$. It follows that, if $x,y\in\mathbb{R}$ are such that $x<y$, then

$$x + f^*(x,t)\Delta^+ g(t) \leqslant y + f^*(y,t)\Delta^+ g(t), \quad \forall t \in \mathbb{T}^* = \mathbb{R}.$$

Theorem 3.9 and Proposition 3.11 imply that

$$\int_{s_1^*}^{s_2^*} f(x,s)\Delta s = \int_{s_1^*}^{s_2^*} f^*(x,s)\,\mathrm{d}g(s) = \int_{s_1}^{s_2} f^*(x,s)\,\mathrm{d}g(s)$$

for every $[s_1,s_2]\subset\mathbb{R}$ and $x\in\mathbb{R}$. If we consider $x \in B\subset\mathbb{R}$, we obtain

$$\left|\int_{s_1}^{s_2} f^*(x,s)\,\mathrm{d}g(s)\right| = \left|\int_{s_1^*}^{s_2^*} f(x,s)\,\Delta s\right| \leqslant \int_{s_1^*}^{s_2^*} m(s)\,\Delta s = \int_{s_1}^{s_2} m^*(s)\,\mathrm{d}g(s).$$

Similarly,

$$\left|\int_{s_1}^{s_2} (f^*(x,s)-f^*(y,s))\,\mathrm{d}g(s)\right| \leqslant \omega(|x-y|)\int_{s_1}^{s_2} m^*(s)\,\mathrm{d}g(s).$$

This shows that the functions f^*, g satisfy all the assumptions of Theorem 4.15 and we can apply Theorem 3.12 to obtain the desired result. $\square$

Next, we consider a system of linear dynamic equation on time scales

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s)+q(s))\,\Delta s, \quad t\in\mathbb{T}, \tag{4.12}$$

where $t_0\in\mathbb{T}$, $p\colon \mathbb{T}\to L(\mathbb{R}^n)$ and $q\colon\mathbb{T}\to\mathbb{R}^n$. (4.12) has an equivalent form as

$$x^\Delta(t) = p(t)x + q(t), \quad t\in\mathbb{T}.$$

A version of Massera's Theorem for this type of equations is presented below. It generalizes the ones found in the literature and more details about that will be given after the result.

Theorem 4.19 ([14, Theorem 6.6]). *Let $\mathbb{T}$ be a T–periodic time scale. Suppose that $p\colon \mathbb{T} \to L(\mathbb{R}^n)$ and $q\colon \mathbb{T} \to \mathbb{R}^n$ satisfy:*

- *p and q are T–periodic and Henstock–Kurzweil Δ-integrable on every interval $[s_1, s_2]_{\mathbb{T}}$;*
- *there exists $m\colon \mathbb{T} \to \mathbb{R}$ that is Henstock–Kurzweil Δ-integrable on every interval $[s_1, s_2]_{\mathbb{T}}$ and also*

$$\left\| \int_{s_1}^{s_2} p(s)\,\Delta s \right\| \leqslant \int_{s_1}^{s_2} m(s)\,\Delta s.$$

If (4.12) *has a bounded solution, then it has a T–periodic solution on $\mathbb{T}$.*

Proof. Since $\mathbb{T}$ is T–periodic, $\mathbb{T}^* = \mathbb{R}$. Define $g\colon \mathbb{R} \to \mathbb{T}$ as $g(t) = t^*$ for every $t \in \mathbb{R}$. As commented in the previous theorem, we already have that g is nondecreasing, left–continuous and $g(t+T) - g(t) = T$ for every $t \in \mathbb{R}$. It is also easy to see that p^* and q^* are T–periodic, because p, q are T–periodic and the time scale is T–periodic.

Applying Theorem 3.9 and Proposition 3.11, p^* is Henstock–Kurzweil–Stieltjes integrable on an arbitrary interval $[s_1, s_2]$ and, also,

$$\left\| \int_{s_1}^{s_2} p^*(s)\,\mathrm{d}g(s) \right\| = \left\| \int_{s_1^*}^{s_2^*} p(s)\,\Delta s \right\| \leqslant \int_{s_1^*}^{s_2^*} m(s)\,\Delta s = \int_{s_1}^{s_2} m^*(s)\,\mathrm{d}g(s).$$

Theorem 3.9 also implies that q^* is integrable on $[s_1, s_2]$. Therefore, all the assumptions of Theorem 4.16 are satisfied. It is enough to apply Theorem 3.12 for $y(t) = y(t_0) + \int_{t_0}^{t} (p^*(s)y(s) + q^*(s))\,\mathrm{d}g(s)$ to obtain the desired result. $\square$

Remark 4.20. *There are other results in the literature involving the Massera's Theorem in time scales, such as [4] and [25]. However, [4] deals with the case $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, which is not a periodic time scale and, therefore, there is no intersection between the results in that article and in [14].*

In [25], the authors deal with periodic time scales and, although our definition is different, it can be easily adapted to the same type of condition presented there. Besides that, their version of Massera's Theorem for linear equations ([25, Theorem 3.1]) has stronger hypotheses (f is required to be rd–continuous). There are no results for the 1–dimension case and no results involving the asymptotic behavior of solutions.

Moreover, we could have gotten even more general results if we considered the Henstock–Kurzweil–Stieltjes Δ–integral, instead of the Henstock–Kurzweil Δ–integral. We commented about difference between both equations in Chapter 2 and one may find more details in [31].

We can also consider impulsive DE

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s),s)\,\mathrm{d}s + \sum_{\substack{j\in\mathbb{Z},\\ t_0\leqslant\tau_j<t}} I_j(x(\tau_j)), \tag{4.13}$$

where $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $I_j\colon \mathbb{R}^n \to \mathbb{R}^n$ for each $j \in \mathbb{Z}$, $\{\tau_j\}_{j\in\mathbb{Z}}$ is an increasing real sequence. Such equations can be treated as particular cases of the generalized ODEs. With this correspondence, we are able to obtain the Massera's Theorem for nonlinear scalar impulsive differential equations.

Theorem 4.21 ([14, Theorem 7.3]). *Consider a sequence $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ and functions $F\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $I_j\colon \mathbb{R} \to \mathbb{R}$, $j \in \mathbb{Z}$, such that $\tau_j < \tau_{j+1}$ and $T > 0$, $m \in \mathbb{N}$ for which $\tau_j = \tau_{j-m} + T$ and $I_j = I_{j-m}$ for each $j \in \mathbb{Z}$. Besides that, suppose:*

- *f is continuous in the first variable and T–periodic in the second;*
- *The Henstock–Kurzweil integral $\int_{s_1}^{s_2} f(x,s)\,\mathrm{d}s$ exists for each interval $[s_1,s_2] \subset \mathbb{R}$ and every $x \in \mathbb{R}$;*
- *for every bounded set $B \subset \mathbb{R}$, there exist a function $m\colon \mathbb{R} \to \mathbb{R}$ that is Henstock–Kurzweil integrable in every interval $[s_1,s_2] \subset [0,+\infty)$ and a continuous increasing function $\omega\colon [0,+\infty) \to [0,+\infty)$ with $\omega(0) = 0$, $\lim\limits_{u\to 0+} \int_u^v 1/\omega(r)\mathrm{d}r = +\infty$ for a certain $v > 0$, and*

$$\left|\int_{s_1}^{s_2} f(x,s)\,\mathrm{d}s\right| \leqslant \int_{s_1}^{s_2} m(s)\,\mathrm{d}s,$$
$$\left|\int_{s_1}^{s_2} (f(x,s) - f(y,s))\,\mathrm{d}s\right| \leqslant \omega(|x-y|)\int_{s_1}^{s_2} m(s)\,\mathrm{d}s$$

  for all $x,y \in B$ and $s_1 \leqslant s_2$. Assume also that for each $j \in \{1,\dots,m\}$, there exists a constant $m_j \geqslant 0$ such that

$$|I_j(x)| \leqslant m_j \quad \textit{and} \quad |I_j(x) - I_j(y)| \leqslant \omega(|x-y|)m_j$$

  for all $x,y \in B$;
- *if $z,y \in \mathbb{R}$ with $z < y$, then $z + I_j(z) \leqslant y + I_j(y)$ for each $j \in \{1,\dots,m\}$.*

Then the existence of a bounded solution of (4.13) *also implies the existence of a T–periodic solution of* (4.13)*. Moreover, each bounded solution of* (4.13) *is asymptotic to a T–periodic solution of* (4.13).

Proof. Define $\tilde{f}\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ as done in Theorem 3.17, that is,

$$\tilde{f}(z,t)=\begin{cases} f(z,t), & \text{if } t\in\mathbb{R}\setminus\{\tau_j : j\in\mathbb{Z}\};\\ I_k(z), & \text{if } t\in\{\tau_j : j\in\mathbb{Z}\}.\end{cases}$$

Also, define $g\colon\mathbb{R}\to\mathbb{R}$ as

$$g(t)=t+j,\quad t\in(\tau_j,\tau_{j+1}] \text{ and } j\in\mathbb{Z}.$$

Hence, by Theorem 3.17, (4.13) is equivalent to the measure DE

$$x(t)=x(t_0)+\int_{t_0}^{t}\tilde{f}(x(s),s)\,\mathrm{d}g(s). \tag{4.14}$$

Now, we want to prove that $\tilde{f}$ and g satisfy all assumptions of Theorem 4.15. First, it is easy to see that g is a nondecreasing, left-continuous and $g(t+T)-g(t)=m$ for every $t\in\mathbb{R}$.

On the other hand, consider $z,y\in\mathbb{R}$ such that $z<y$. If $t\in\mathbb{R}\setminus\{\tau_j : j\in\mathbb{Z}\}$, it is immediate that $\Delta^+g(t)=0$. Therefore,

$$z+\tilde{f}(z,t)\Delta^+g(t)=z<y=y+\tilde{f}(y,t)\Delta^+g(t).$$

If $t\in\{\tau_j : j\in\mathbb{Z}\}$, then $\Delta^+g(t)=1$ and we obtain

$$z+\tilde{f}(z,t)\Delta^+g(t)=u+I_j(z)\leqslant y+I_j(y)=y+\tilde{f}(y,t)\Delta^+g(t),$$

for some $j\in\mathbb{Z}$.

From the definition of $\tilde{f}$, it is easy to see that this function is continuous on the first variable and T–periodic on the second. Besides that, the Henstock–Kurzweil integral $\int_{s_1}^{s_2}f(x,s)\,\mathrm{d}s$ exists for each interval $[s_1,s_2]\subset\mathbb{R}$ by hypothesis. Therefore, applying Corollary 3.16, the Henstock–Kurzweil-Stieltjes integral $\int_{s_1}^{s_2}f(x,s)\,\mathrm{d}g(s)$ also exists.

It remains only to show that (4.7) and (4.8) are satisfied. By hypothesis, given a bounded set $O\subset\mathbb{R}$, there is a Henstock–Kurzweil integrable function $m\colon\mathbb{R}\to\mathbb{R}$ such that

$$\left|\int_{s_1}^{s_2}f(x,s)\,\mathrm{d}s\right|\leqslant\int_{s_1}^{s_2}m(s)\,\mathrm{d}s,\quad \forall s_1\leqslant s_2.$$

Besides that, there are integers m_j, $j\in\{1,\ldots,m\}$ such that $\left|I_j\right|\leqslant m_j$. Define the sequence $\{m_i\}_{i\in\mathbb{Z}}$ such that $m_{j+zT}=m_j$ where $j\in\{1,\ldots,m\}$ and $z\in\mathbb{Z}$. Finally, define $\tilde{m}\colon\mathbb{R}\to\mathbb{R}$ as

$$\tilde{m}(t) = \begin{cases} m(t), & \text{if } t \in \mathbb{R} \setminus \{\tau_i : i \in \mathbb{Z}\}, \\ m_j, & \text{if } t = \tau_i \text{ for some } i \in \mathbb{Z}. \end{cases}$$

Applying Corollary 3.16, we get

$$\begin{aligned} \left| \int_{s_1}^{s_2} \tilde{f}(x,s)\, \mathrm{d}g(s) \right| &= \left| \int_{s_1}^{s_2} f(x,s)\, \mathrm{d}s + \sum_{\substack{j \in \mathbb{Z}, \\ s_1 \leqslant \tau_j < s_2}} I_j(x) \right| \\ &\leqslant \int_{s_1}^{s_2} m(s)\, \mathrm{d}s + \sum_{\substack{j \in \mathbb{Z}, \\ s_1 \leqslant \tau_j < s_2}} m_j = \int_{s_1}^{s_2} \tilde{m}(s)\, \mathrm{d}g(s), \end{aligned}$$

for every $x \in O$ and $s_1 \leqslant s_2$, proving that (4.7) is satisfied. Analogously, we can show that (4.8) is satisfied. Therefore, by Theorem 4.15 and the equivalence between (4.14) and (4.13) (Corollary 3.16), we obtain the desired result. □

Finally, we present the result for linear impulsive DE:

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s) + q(s))\, \mathrm{d}s + \sum_{\substack{j \in \mathbb{Z}, \\ t_0 \leqslant \tau_j < t}} (A_j x(\tau_j) + b_j), \tag{4.15}$$

with $p\colon \mathbb{R} \to L(\mathbb{R}^n)$, $q\colon \mathbb{R} \to \mathbb{R}^n$, $A_j \in L(\mathbb{R}^n)$, and $b_j \in \mathbb{R}^n$ for every $j \in \mathbb{Z}$. Notice that (4.15) represents a special case of (4.13) with $f(x,t) = p(t)x + q(t)$ and $I_j(x) = A_j x + b_j$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $j \in \mathbb{Z}$.

Theorem 4.22 ([14, Theorem 7.4])**.** *Consider a sequence $\{\tau_j\}_{j \in \mathbb{Z}} \subset \mathbb{R}$, $A_j \in L(\mathbb{R}^n)$ and $b_j \in \mathbb{R}^n$, $j \in \mathbb{Z}$, such that $\tau_j < \tau_{j+1}$ and there exist $T > 0$ and $m \in \mathbb{N}$ for which $\tau_j = \tau_{j-m} + T$, $A_j = A_{j-m}$ and $b_j = b_{j-m}$ for $j \in \mathbb{Z}$.*

Besides that, suppose that $p\colon \mathbb{R} \to L(\mathbb{R}^n)$ and $q\colon \mathbb{R} \to \mathbb{R}^n$ satisfy:

- *p, q are T–periodic and integrable on every interval $[s_1, s_2] \subset \mathbb{R}$;*
- *there exists $m\colon \mathbb{R} \to \mathbb{R}$ such that m is integrable in the sense of Henstock–Kurzweil–Stieltjes on every interval $[s_1, s_2] \subset \mathbb{R}$ and also*

$$\left\| \int_{s_1}^{s_2} p(s)\, \mathrm{d}s \right\| \leqslant \int_{s_1}^{s_2} m(s)\, \mathrm{d}s.$$

If (4.15) *has a bounded solution, then it also has a T–periodic solution.*

Proof. Define $g\colon \mathbb{R} \to \mathbb{R}$ as

$$g(t) = t + j, \quad t \in (\tau_j, \tau_{j+1}] \text{ and } j \in \mathbb{Z}.$$

Also define $\tilde{p}\colon \mathbb{R} \to L(\mathbb{R}^n)$ and $\tilde{q}\colon \mathbb{R} \to \mathbb{R}^n$ as

$$\tilde{p}(t) = \begin{cases} p(t), & \text{if } t \in \mathbb{R} \setminus \{\tau_j : j \in \mathbb{Z}\}, \\ A_j, & \text{if } t = \tau_j \text{ for some } j \in \mathbb{Z}, \end{cases}$$

$$\tilde{q}(t) = \begin{cases} q(t), & \text{if } t \in \mathbb{R} \setminus \{\tau_j : j \in \mathbb{Z}\}, \\ b_j, & \text{if } t = \tau_j \text{ for some } j \in \mathbb{Z}. \end{cases}$$

By Theorem 3.17, (4.14) is equivalent to

$$x(t) = x(t_0) + \int_{t_0}^{t} (\tilde{p}(s)x(s) + \tilde{q}(s))\,\mathrm{d}g(s).$$

It remains only to show that $\tilde{p}$, $\tilde{q}$ and g satisfy all the hypotheses of Theorem 4.16.

It is immediate to see that g is left–continuous, nondecreasing and $g(t+T) - g(t) = m$, for every $t \in \mathbb{R}$. Besides that, it is also easy to see that $\tilde{p}$ and $\tilde{q}$ are T–periodic and Corollary 3.16 guarantees that both functions are Henstock–Kurzweil–Stieltjes integrable.

Lastly, define $\tilde{m}\colon \mathbb{R} \to \mathbb{R}$ as

$$\tilde{m}(t) = \begin{cases} m(t), & \text{if } t \in \mathbb{R} \setminus \{\tau_j : j \in \mathbb{Z}\}, \\ \|A_j\|, & \text{if } t = \tau_j \text{ for some } j \in \mathbb{Z}. \end{cases}$$

Applying Corollary 3.16 once again, we get

$$\begin{aligned} \left\| \int_{s_1}^{s_2} \tilde{p}(s)\,\mathrm{d}g(s) \right\| &= \left\| \int_{s_1}^{s_2} p(s)\,\mathrm{d}s + \sum_{\substack{j \in \mathbb{Z}, \\ s_1 \leqslant \tau_j < s_2}} A_j \right\| \\ &\leqslant \int_{s_1}^{s_2} m(s)\,\mathrm{d}s + \sum_{\substack{j \in \mathbb{Z}, \\ s_1 \leqslant \tau_j < s_2}} \|A_j\| = \int_{s_1}^{s_2} \tilde{m}(s)\,\mathrm{d}g(s), \end{aligned}$$

for every $s_1 \leqslant s_2$. Thus, applying Theorems 3.17 and 4.16, we conclude the proof. □

Remark 4.23. *There are other versions of Massera's Theorem for impulsive DEs in the literature, such as [1] and [19]. Nevertheless, the results obtained here are more general* *because they derive from the Henstock–Kurzweil–Stieltjes integral. Therefore, unlike what is done in the other articles, the Riemann or even Lebesgue integrals of the functions we study here do not necessarily exist.*

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