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Elliptic Problems in the Upper Half-space with Critical Boundary Conditions

by

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Elliptic Problems in the Upper Half-space with Critical Boundary Conditions

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Dedication

To Guilhermina (in memoriam)

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Abstract

This work is divided in two parts. In the first one, we apply variational methods to study existence and multiplicity of solutions for a class of elliptic nonlinear boundary value problems in the upper half-space. We are mainly interested in the critical growth on the boundary and we exploit the most diverse variations of growth conditions inside the domain. In the second part of the work, we study problems defined in the whole space $\mathbb{R}^N$. Namely, we first concern with multiplicity of solutions for a singular problem and, finally, we obtain an existence result for an indefinite planar equation with critical exponential growth

Keywords: Nonlinear boundary conditions; half-space; self-similar solutions; critical trace problems; sign-changing solutions; concave-convex problems; symmetric functionals; singular problems.

Resumo

Este trabalho está dividido em duas partes. Na primeira, aplicamos métodos variacionais para estudar existência e multiplicidade de soluções para uma classe de problemas elípticos com condição de fronteira não linear no semi-espaço superior. Consideramos, em especial, o crescimento crítico no bordo e exploramos as mais diversas variações de condições de crescimento no interior do domínio. Já na segunda parte do trabalho, estudamos problemas definidos em todo o espaço $\mathbb{R}^N$. A saber, primeiramente estamos interessados em multiplicidade de soluções para um problema singular e, por fim, obtivemos um resultado de existência para uma equação planar indefinida com crescimento crítico exponencial.

Palavras-chave: Condições de fronteira não-linear; semi-espaço; soluções auto-similares; problemas de traço crítico; soluções que trocam de sinal; problemas do tipo côncavo-convexo; funcionais simétricos, problemas singulares.

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Introduction

Let $\mathbb{R}^N_+ = \{(x', x_N) : x' \in \mathbb{R}^{N-1},\, x_N > 0\}$ be the upper half-space. In the first part of this work, we consider nonlinear boundary value problems of type

$$(P) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(x \cdot \nabla u) = a(x) f(u), & x \in \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \eta} = b(x')|u|^{p-2}u, & x' \in \mathbb{R}^{N-1}, \end{cases}$$

where $\frac{\partial}{\partial \eta}$ denotes the outer unit normal derivative, we have identified $\partial\mathbb{R}^N_+ \simeq \mathbb{R}^{N-1}$, $2 < p \le 2_* := 2(N-1)/(N-2)$, the potentials a and b satisfies suitable conditions and the nonlinearity $f : \mathbb{R}^N_+ \to \mathbb{R}$ is a continuous function.

The operator appearing in the right-hand side of the first equation of (P) naturally appears when we look for self-similar solutions, that is, solutions of the special form $w(x,t) = t^{-\lambda}u(t^{-1/2}x)$, for the following nonlinear heat equation

$$w_t - \Delta w = 0,\ x \in \mathbb{R}^N_+ \times (0, +\infty), \qquad \frac{\partial w}{\partial \eta} = |w|^{p-2}w,\ x' \in \mathbb{R}^{N-1} \times (0, +\infty).$$

Among other advantages, this type of solution provides qualitative properties like global existence, blow-up and asymptotic behavior (see e.g. [57, 58, 69]). Moreover, they preserve the PDE scaling and so carry simultaneously information about small and large scale behaviors. The connection with (P) is that, if we put w into the heat equation, we see that the profile u needs to verify the same equation in (P) with $a, b \equiv 1$ and $f(u) = 1/(2(p-2))u$.

It is important to show the connection between problem (P) and the following important class of nonlinear boundary value problems

$$-\Delta v = g(x, v),\ x \in \mathbb{R}^N_+, \qquad \frac{\partial v}{\partial \eta} = h(x', v),\ x' \in \mathbb{R}^{N-1}. \tag{0.1}$$

Its mathematical importance arises, for instance, in the study of conformal deformation of Riemannian manifolds [29, 42, 43, 56], problems of sharp constant in Sobolev trace inequalities [38, 41] and blow-up properties of the solutions of related parabolic equations [47, 60]. This kind of equations also appears in several applied contexts like glaciology [77], population genetics [6], non-Newtonian fluid mechanics [39], nonlinear elasticity [32], among others.

There is a vast literature concerning nonnegative solutions for (0.1). Using the moving plane method, Hu [59] obtained nonexistence of positive solutions when $g \equiv 0$ and $h(v) = v^q$, with $1 < q < N/(N-2)$. Similar results were obtained by Chipot *et al.* in [31] in the case that $g(v) = av^p$ and $h(v) = v^q$ with $1 < p \leq (N+2)/(N-2)$, $1 < q \leq N/(N-2)$, with one the inequalities being strict, and $a > 0$ (see also [94] for existence and multiplicity results in the double subcritical case). In dimension $N = 2$ and $g \equiv 0$, Cabré and Morales [21] presented necessary and sufficient conditions on $h(v)$ for the existence of *layer solutions*, that is, bounded solutions that satisfy some monotonicity properties. When $g \equiv 0$ and $h(v) = (N-2)v^{N/(N-2)}$, existence of positive solution decaying as $|x|^{2-N}$ at infinity was obtained by Escobar [41] using the conformal equivalence between the unit ball in $\mathbb{R}^N$ and the half-space (see also [91]). In the same paper, it was considered the case $g(v) = N(N-2)v^{(N+2)/(N-2)}$ and $h(v) = bv^{N/N(N-2)}$. Later, Chipot *et al.* [30] removed the decay assumption by using the shrinking sphere method to give a complete description of positive solutions when $g(v) = av^{(N+2)/(N-2)}$ and $h(v) = bv^{n/(N-2)}$. Similar results were obtained by Li and Zhu in [66], including a 2-dimensional version with exponential type nonlinearities.

Notice that, if u is a solution of (P), then the function $v = \exp(|x|^2/8)u$ verifies (0.1) for

$$g(x, v) = -\left(\frac{N}{4} + \frac{|x|^2}{16}\right) v + \tilde{a}(x) f(\exp(-|x|^2/8)v), \qquad h(x', v) = \tilde{b}(x')|v|^{2_*-2}v,$$

where $\tilde{a}(x) = a(x)\exp(|x|^2/8)$ and $\tilde{b}(x') = b(x')\exp\left(-\frac{|x'|^2}{4(N-2)}\right)$. Differently from the former cases, this problem is not homogeneous and the nonlinearity g is unbounded in the spatial variable. Hence, the techniques used in the aforementioned works do not apply and we need to perform a different approach to deal with the drift term inside the domain.

In order to overcome this, notice that, if we set

$$K(x) = \exp(|x|^2/4),$$

we have that $2\nabla K = xK$ and the first equation in (P) can be rewritten as

$$-\mathrm{div}(K(x)\nabla u) = K(x)a(x)f(u), \quad \text{in } \mathbb{R}^N_+.$$

Hence, it is natural to look for finite energy solutions belonging to the Sobolev space $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ defined as the closure of $C_c^\infty(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx\right)^{1/2}.$$

This kind of space was first introduced by Escobedo and Kavian [44] who considered a problem in the whole space $\mathbb{R}^N$. The upper half-space case was presented in [47], where it is proved that $\mathcal{D}_K^{1,2}(\mathbb{R}_+^N)$ is compactly embedded into the weighted Lebesgue space

$$L_K^r(\mathbb{R}_+^N) = \left\{ u \in L^r(\mathbb{R}_+^N) \,:\, \|u\|_r = \left(\int_{\mathbb{R}_+^N} K(x)|u|^r dx \right)^{1/r} < +\infty \right\},$$

for any $2 \leq r \leq 2^*$. By taking $r = 2$ in particular, we can solve the linear problem associated with (P), namely

$$(LP) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(x \cdot \nabla u) = \lambda u, & \text{in } \mathbb{R}_+^N, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \mathbb{R}^{N-1}, \end{cases}$$

and use spectral theory to obtain a sequence of eigenvalues $(\lambda_j)_{j \in \mathbb{N}}$ such that

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$$

with $\lim\limits_{j \to \infty} \lambda_j = +\infty$. Moreover, as we will see later, the first eigenvalue is exactly $\lambda_1 = N/2$.

The thesis has five chapters. In the first three, we consider different versions of problem (P), by varying the assumptions on a, b and f. In the two last chapter we consider only the first equation in (P) but in the whole space $\mathbb{R}^{\mathbb{N}}$. Despite having intersections, any chapter can be read independently. In what follows we present the main results of each chapter.

Chapter 1

In the first chapter, we consider our first variation of problem (P), namely

$$(P_1) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(x \cdot \nabla u) = \lambda u, & x \in \mathbb{R}_+^N, \\ \dfrac{\partial u}{\partial \eta} = |u|^{2_*-2}u, & x' \in \mathbb{R}^{N-1}, \end{cases}$$

where $2_* := 2(N-1)/(N-2)$ and $\lambda > 0$ is a parameter.

Problem (P_1) is strongly linked to the classic problem

$$-\Delta u = \lambda u + |u|^{2^*-2}u, \quad u \in H_0^1(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 3$ and $\lambda > 0$ is a parameter. This equation comes from the Yamabe's problem, which deals with the existence of Riemannian metrics with constant scalar curvature. In a seminal paper, Brezis and Nirenberg [19] proved that the existence of positive solution is related to the interaction of the parameter with the first eigenvalue $\lambda_{1,\Omega} > 0$ of the spectrum $\sigma(-\Delta, H_0^1(\Omega))$. Among other things, they showed that the above equation has a positive solution whenever $N \geq 4$ and $\lambda \in (0, \lambda_{1,\Omega})$. This was the starting point of an effusive literature concerning this critical equation. We especially quote here the paper of Capozzi, Fortunato and Palmieri [23], which obtained solution for $\lambda \geq \lambda_{1,\Omega}$ and Cerami, Solimini and Struwe [25], which proved the existence of a sign-changing solution if $0 < \lambda < \lambda_{1,\Omega}$ and $N \geq 6$.

The authors in [47] considered the subcritical version of (P_1), that is, the same problem with 2_* replaced by $p \in (2, 2_*)$. Among other results, they obtained the existence of a positive solution if $\lambda < \lambda_1$. The critical version was recently considered in [48] and the situation turns to be more delicate. After proving the trace embedding $\mathcal{D}_K^{1,2}(\mathbb{R}_+^N) \hookrightarrow L_K^{2_*}(\mathbb{R}^{N-1})$, where

$$L_K^{2_*}(\mathbb{R}^{N-1}) = \left\{ u \in L^{2_*}(\mathbb{R}^{N-1}) \; : \; \pmb{|} u \pmb{|}_{2_*} = \left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*} dx' \right)^{1/2_*} < +\infty \right\},$$

the authors showed that, in the critical case, there is no self-similar solution to the equation. Besides this, they obtained a positive solution whenever $N \geq 7$ and the parameter λ verifies

$$\lambda_N^* = \frac{N}{4} + \frac{N-4}{8} < \lambda < \lambda_1.$$

We notice that, since $\lambda_1 = N/2$, the above range is nonempty.

The first main result completes the above study by considering the case $\lambda > \lambda_1$. Standard arguments show that positive solutions are not expected and therefore we look for sign-changing solutions. More specifically, we prove the following:

Theorem A. *If $N \geq 7$ and $\lambda > \lambda_1$ is not an eigenvalue of (LP), then problem (P_1) has a sign-changing solution.*

In the proof, we apply the Linking Theorem [83] to the energy functional

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{2_*}\pmb{|} u \pmb{|}_{2_*}^{2_*}, \quad u \in \mathcal{D}_K^{1,2}(\mathbb{R}_+^N).$$

Since standard arguments can not be used to verify the linking geometry, we need to perform a detailed study of the structure of solutions of the eigenvalue problem (LP) and prove a trick projection result (see Lemma 1.1.4 and Proposition 1.1.5). The assumption that λ is not an eigenvalue of (LP) is a non-resonant type condition of technical nature and assures that Palais-Smale sequences are bounded. Actually, the arguments used in [23, 83] do not work in unbounded domains and therefore we need to perform a different approach here (see Proposition 1.1.6).

Now, we come back to the range where positive solution exists and ask if it is possible to obtain another solution. In this new setting, we prove the following:

Theorem B. *If $N \geq 7$ and $\lambda \in (\lambda_N^*, \lambda_1)$, then problem (P_1) has a sign-changing solution.*

In order to explain the main steps for the proof, we define $u^+(x) = \max\{u(x), 0\}$ and $u^- = u^+ - u$. After that, inspired by the paper of Cerami, Solimini and Struwe [25], we introduce the Nehari nodal set

$$\mathcal{M}_\lambda = \{u \in \mathcal{D}_K^{1,2}(\mathbb{R}_+^N) \,:\, u^\pm \neq 0,\, I'_\lambda(u^\pm)u^\pm = 0\}$$

and prove that

$$d_\lambda = \inf_{u \in \mathcal{M}_\lambda} I_\lambda(u)$$

is attained by a solution $u \in \mathcal{M}_\lambda$. Since we are dealing with the critical case, the functional I_λ satisfies only a local Palais-Smale condition. So, we need to prove some fine estimates (see Lemmas 1.4.2 and 1.4.3) involving the positive solution obtained in [48] and a slight modification of the *instanton functions* founded independently by Escobar [41] and Beckner [13]. This is essential to guarantee that d_λ belongs to the range where we have compactness. Since $\mathcal{M}_\lambda$ is not a differentiable manifold, it is not easy to construct Palais-Smale sequences on the level d_λ. In order to do this, we adapt some ideas introduced by Tarantello in [90].

Chapter 2

In the second chapter, we deal with the following concave-convex type problem

$$(P_2) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(x \cdot \nabla u) = \lambda a(x)|u|^{q-2}u, & x \in \mathbb{R}_+^N, \\ \dfrac{\partial u}{\partial \eta} = b(x')|u|^{p-2}u, & x' \in \mathbb{R}^{N-1}, \end{cases}$$

where $N \geq 3$, $\lambda > 0$ is a parameter and $1 < q < 2 < p \leq 2_*$. If we denote by $r' = r/(r-1)$ the conjugated exponent of $r > 1$, we can present the basic hypothesis on a, b in the following way:

(a_0) $a \in L_K^{\sigma_q}(\mathbb{R}_+^N) \cap L_{loc}^{N/2}(\mathbb{R}_+^N)$ for some

$$\left(\frac{p}{q}\right)' < \sigma_q \leq \left(\frac{2}{q}\right)';$$

(b_0) $b \in L^\infty(\mathbb{R}^{N-1})$.

Since a and b can change it sign, we may define the sets

$$\Omega_a^+ := \{x \in \mathbb{R}_+^N :\ a(x) > 0\}, \quad \Omega_b^+ := \{x' \in \mathbb{R}^{N-1} :\ b(x') > 0\}.$$

In the first results of the second chapter we obtain existence of two nonnegative solutions when roughly speaking the closure of the set Ω_a^+ intersects Ω_b^+ and the parameter $\lambda > 0$ approaches zero. More specifically, denoting by $B_\delta(0)$ the open ball centered at origin with radii $\delta > 0$, we prove the following:

Theorem C. *Suppose that a, b satisfy (a_0) and (b_0). If $1 < q < 2 < p < 2_*$, then there exists $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, problem (P_2) has at least two nonnegative nonzero solutions provided*

(ab) *there exists $\delta > 0$ such that*

$$(B_\delta(0) \cap \mathbb{R}^N_+) \subset \Omega_a^+, \quad (B_\delta(0) \cap \partial\mathbb{R}^N_+) \subset \Omega_b^+.$$

In the critical case we also obtain two nonnegative solutions, but now we need to add a flatness condition on the potential b:

Theorem D. *Suppose that $N \geq 7$, $p = 2_*$ and the other conditions of Theorem C are verified. Then there exists $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, problem (P_2) has at least two nonnegative nonzero solutions provided*

(b_1) *there exist $M > 0$ and $\sigma > N - 1$ such that*

$$\|b\|_\infty - b(x') \leq M|x'|^\sigma, \quad \text{for a.e. } x' \in B_\delta(0) \cap \partial\mathbb{R}^N_+.$$

The first solution will be obtained with a standard minimization argument while the second one requires finer arguments. This is specially true when $p = 2_*$, since the trace embedding we are going to use fails to be compact. Two points are important to overcome this difficulty: a trick regularization study of the first solution on the boundary and the application of an idea of Brezis and Nirenberg [19], together with fine estimates of a modification of the *instanton functions* founded by Escobar [41] and Beckner [13].

Still in the second chapter, we take advantage of the symmetry to get more and more solutions (with no prescribed sign). Unfortunately, in this case we do not assume that both the potentials are indefinite.

We prove the following:

Theorem E. *Suppose that $1 < q < 2$, $a \geq 0$ and $b \not\equiv 0$ satisfiy (a_0) and (b_0), respectively. Then problem (P_2) has infinitely many solutions in each of the following cases:*

1. *$2 < p < 2_*$ and $\lambda > 0$;*
2. *$p = 2_*$, $b \leq 0$ and $\lambda > 0$;*

3. $p = 2_*$ *and* $\lambda > 0$ *is small.*

Theorem F. *Suppose that* $1 < q < 2 < p < 2_*$*,* $a \not\equiv 0$ *and* $b \geq 0$ *satisfy* (a_0) *and* (b_0)*, respectively. Then, for any* $\lambda > 0$*, problem* (P_2) *has infinitely many solutions.*

The above theorems will be proved as application of suitable versions of the Symmetric Mountain Pass Theorem [3]. These versions were proved by Tonkes in the paper [92] which strongly motivated our second chapter (see also [11, 12] for some earlier results). In the critical case, when $b \leq 0$, the boundary term is related with a semi-norm and therefore we can argue as in the subcritical case. When $p = 2_*$ and b is indefinite in sign, we borrow an argument from [10]. It can be proved that, when $b \leq 0$, the energy of the solutions given by Theorem E are negative and goes to zero. On the other hand, in Theorem F, this energy goes to infinity, the same occurring with the norm of the solutions.

We finally notice that the results of Chapter 2 were recently published in [53].

Chapter 3

The third chapter concerns with existence and multiplicity of solutions for the problem

$$(P_3) \qquad \begin{cases} -\Delta u - \frac{1}{2}(x \cdot \nabla u) = f(u), & x \in \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \eta} = \beta |u|^{2_*-2}u, & x' \in \partial\mathbb{R}^N_+, \end{cases}$$

where $N \geq 3$, $\beta > 0$ is a parameter and $f : \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

(f_0) $f : \mathbb{R} \to \mathbb{R}$ is continuous;

(f_1) there exist $a_1, a_2 > 0$ and $2 < p < 2^* := 2N/(N-2)$ such that

$$|f(s)| \leq a_1 + a_2|s|^{p-1}, \quad \forall\, s \in \mathbb{R};$$

(f_2) there holds

$$\lim_{s \to 0} \frac{f(s)}{s} = 0;$$

(f_3) there exists $2 < \theta < 2_*$ such that

$$0 < \theta F(s) \leq f(s)s, \quad \forall\, s \in \mathbb{R} \setminus \{0\},$$

where $F(s) := \int_0^s f(\tau)\, d\tau$.

The first result of this chapter can be stated as follows:

Theorem G. *Suppose that f is odd and satisfies $(f_0)-(f_3)$. Then, for any given $k \in \mathbb{N}$, there exists $\beta^* = \beta^*(k) > 0$ such that problem (P_3) has at least k pairs of solutions, provided $\beta \in (0, \beta^*)$.*

In the proof, we apply a version of the Symmetric Mountain Pass Theorem. The main task here is the management of Palais-Smale sequences and we follow ideas presented in Silva and Xavier [87]. Since we are dealing with unbounded domains, the former argument does not directly apply and we need to perform a trick adaptation of Bianchi, Chabrowski and Szulkin's ideas [15, 27] and the concentration compactness principle due to Lions [67].

In the second result of the chapter, we do not require symmetry for f and obtained the existence of nonnegative solution. In this case, the parameter β does not play any role and we prove the following:

Theorem H. *Suppose that $N \geq 7$ and f satisfies $(f_0)-(f_3)$. Then problem (P_3) has a nonnegative nonzero solution provided*

$$\lim_{\varepsilon \to 0^+} \varepsilon^{N-2} \int_0^{1/\varepsilon} F\left(\frac{\varepsilon^{-(N-2)/2}}{[s^2+1]^{(N-2)/2}}\right) s^{N-1}\, ds = +\infty. \tag{0.2}$$

In the proof we follow the paper of Brezis and Nirenberg [19]. After obtaining a local compactness condition for the associated functional, we need to prove that it Mountain Pass level belongs to the correct range. At this point we perform some fine estimates and use the technical condition (0.2). It was inspired by a similar one which appeared in [19, Lemma 2.1] and it holds if, for instance, $F(s) \geq \gamma |s|^p$, for some $\gamma > 0$.

Chapter 4

In Chapter 4, we are concerned with positive solutions for the singular equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu h(x) u^{q-1} + \lambda u + u^{2^*-1}, \qquad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\lambda > 0$, $\mu > 0$ is a parameter, $0 < q < 1$ and h has some somability properties. Before presenting the condition on h, we need to say a few words about the variational structure of the problem. After multiplying the equation by $K(x) := \exp(|x|^2/4)$, it can be rewritten as

$$(P_4) \qquad \begin{cases} -\mathrm{div}(K(x)\nabla u) = \mu K(x) h(x) u^{q-1} + \lambda K(x) u + K(x) u^{2^*-1}, & \text{in } \mathbb{R}^N, \\ u > 0, \quad \text{in } \mathbb{R}^N. \end{cases}$$

It is natural to look for solutions in the space $D_K^{1,2}(\mathbb{R}^N)$ defined as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} K(x)|\nabla u|^2\, dx\right)^{1/2}.$$

It was proved in [44] that $D_K^{1,2}(\mathbb{R}^N)$ is a Hilbert space which is continuously embedded into the weighted Lebesgue spaces

$$L_K^p(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) \,:\, \|u\|_p := \left(\int_{\mathbb{R}^N} K(x)|u|^p dx\right)^{1/p} < \infty\right\},$$

for any $p \in [2, 2^*]$.

Due to the difficulties related to the operator and the singular nature of the nonlinearity at the origin, we do not expect to find regular solutions. Hence, as usual in the literature, we call $u \in D_K^{1,2}(\mathbb{R}^N)$ a solution for problem (P_4) if it satisfies $u > 0$ a.e. in $\mathbb{R}^N$ and, for any $\phi \in D_K^{1,2}(\mathbb{R}^N)$, we have that $h(x)u^{q-1}\phi \in L_K^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} K(x)\left[(\nabla u \cdot \nabla \phi) - \mu h(x)u^{q-1}\phi - \lambda u\phi - u^{2^*-1}\phi\right] dx = 0.$$

In our first result we obtain a solution when the paramenter $\mu > 0$ is small. More specifically, we shall prove the following:

Theorem I. *Suppose that $\lambda < N/2$ and $h > 0$ satisfies*

(h) $h \in L_K^1(\mathbb{R}^N) \cap L_K^2(\mathbb{R}^N)$.

Then there exists $\mu^* > 0$ such that problem (P_4) has a solution, whenever $\mu \in (0, \mu^*)$.

In the proof, we apply a minimization argument for a perturbed (nonsingular) problem. We notice that condition $\lambda < N/2$ is necessary for the existence of a solution. Indeed, it is proved in [44] that the linearized version of equation (P_4) has the pair $(\lambda, u) = (N/2, \varphi_1)$ as a solution, where $\varphi_1(x) = \exp(-|x|^2/4) > 0$. So, if $u_0 \in X$ is a solution, we may pick $v = \varphi_1$ in the integral formulation to get

$$\left(\frac{N}{2} - \lambda\right)\int_{\mathbb{R}^N} K(x)u\varphi_1\, dx = \int_{\mathbb{R}^N} K(x)\left[\mu h(x)u^{q-1}\varphi_1 + u^{2^*-1}\varphi_1\right] dx > 0,$$

from which it follows that $\lambda < N/2$.

In our second result concernin (P_4), we obtain another solution under an additional lower bound on the value of λ. More specifically, we prove the following:

Theorem J. *Suppose that* $\max\{1, N/4\} < \lambda < N/2$, $h > 0$ *is continuous and satisfies (h). Then there exists $0 < \mu_* < \mu^*$ such that problem (P_4) has at least two solutions, whenever $\mu \in (0, \mu_*)$*

In order to obtain the second solution, we apply the Mountain Pass Theorem to a perturbed functional, together with a limit process. The extra assumption on λ is related with the range of existence of positive solution for the case $\mu = 0$ (nonsingular) obtained in [44]. It is worth mentioning that the continuity of h may be replaced by the weaker condition that the infimum of h is positive in any ball.

Chapter 5

In Chapter 5, we deal with the following equation

$$(P_5) \qquad -\Delta u + \frac{1}{2}(x \cdot \nabla u) = a(x)f(u), \quad x \in \mathbb{R}^2,$$

where a is a sign-changing potential and the nonlinerity f has an exponential critical growth at infinity.

We follow [2] to impose the assumptions on the indefinite potential a. More specifically, assume that

(a_1) $a : \mathbb{R}^2 \to \mathbb{R}$ is a bounded sign-changing continuous function;

(a_2) if

$$\Omega^+ := \{x \in \mathbb{R}^2; a(x) > 0\}, \quad \Omega^- := \{x \in \mathbb{R}^2; a(x) < 0\},$$

then $\text{dist}(\overline{\Omega^+}, \overline{\Omega^-}) > 0$;

(a_3) there exists $R > 0$ such that $a(x) < 0$ for $|x| \geq R$.

We are interested in the case that f is superlinear both at the origin and at infinity, namely

(f_0) $f \in C(\mathbb{R}, \mathbb{R})$ and there exists $\alpha_0 > 0$ such that

$$\lim_{s \to +\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0, \\ +\infty & \text{if} \quad \alpha < \alpha_0; \end{cases}$$

(f_1) $\lim_{s \to 0} f(s)/s = 0$.

Given $r \geq 2$, it is proved in [50] that $\mathcal{D}^{1,2}_K(\mathbb{R}^N)$ is compactly embedded into the weighted Lebesgue space $L^r_K(\mathbb{R}^2)$. Hence, we can define the constant

$$S_2 := \inf \left\{ \int_{\mathbb{R}^2} K(x)|\nabla u|^2 dx \, : \, \int_{\mathbb{R}^2} K(x)|u|^2 \, dx = 1 \right\}.$$

Since $\overline{\Omega^+}$ is far from $\overline{\Omega^-}$, we can find $\zeta \in C^\infty(\mathbb{R}^2, [0,1])$ such that

$$\zeta \equiv 1, \text{ in } \Omega^+, \qquad \zeta \equiv 0, \text{ in } \Omega^-, \qquad \mathcal{M} := \sup_{\mathbb{R}^2} |\nabla \zeta| < \infty.$$

Our technical assumptions on f can be stated as follows:

(f_2) there exist $\nu > 2$ and $0 < \theta < \nu \left[2(1 + \mathcal{M}S_2^{-1/2})\right]^{-1}$ such that,

$$0 < \frac{\nu}{\theta}F(s) \le f(s)s, \quad \forall\, |s| > 0,$$

where $F(s) := \int_0^s f(\tau)\, d\tau$;

(f_3) there exist K_0, $R_0 > 0$ such that

$$0 < F(s) \le K_0|f(s)|, \quad \forall\, |s| \ge R_0;$$

(f_4) if $x_0 \in \Omega^+$ and $r > 0$ are such that $a(x_0) = \max_{\Omega^+} a$ and $a(x) \ge (\max_{\Omega^+} a)/2$ in $B_r(x_0)$, then

$$\lim_{s\to+\infty} sf(s)e^{-\alpha_0 s^2} \ge \beta_0 > \frac{8}{\alpha_0 r^2 \cdot \max_{\Omega^+} a} \exp\left(\frac{r^2}{8} + \frac{r^4}{512}\right).$$

We prove the following existence result:

Theorem K. *Suppose that $(a_1)-(a_3)$ and $(f_0)-(f_4)$ hold. Then problem (P_5) admits at least a weak nontrivial solution.*

In the proof we apply the Mountain Pass Theorem. Since the potential a changes it sign, it is not so easy to prove that Palais-Smale sequences are bounded. Conditions (a_2) and (f_2) are important in this issue. Condition (f_3) has first appeared in [36] and provides a compactness property for the Palais-Smale sequence. With the aim of overcome the difficulties imposed by the lack of compactness, since we are dealing with the whole space $\mathbb{R}^2$, we invoke a version of the Trudinger-Moser inequality together with assumption (f_4) and the Moser's functions to find the correct localization of the mountain pass level. We notice that (f_4) is weaker than $\lim_{s\to+\infty} f(s)se^{-\alpha_0 s^2} = +\infty$, which have been used in some former papers (see (g_5) in [2] for instance).

We finally notice that the results of Chapter 5 are going to appear in [54].

CHAPTER 1

Sign-changing solutions for an elliptic equation with critical nonlinear boundary condition

Consider the following nonlinear boundary value problem

$$-\Delta v = f(x,v), \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial v}{\partial \eta} = g(x,v), \text{ on } \mathbb{R}^{N-1}, \tag{1.1}$$

where $\mathbb{R}^N_+ = \{(x', x_N) : x' \in \mathbb{R}^{N-1},\ x_N > 0\}$ is the upper half-space, $\frac{\partial}{\partial \eta}$ denotes the outer unit normal derivative and we have identified $\partial\mathbb{R}^N_+ \simeq \mathbb{R}^{N-1}$. Its mathematical importance arises, for instance, in the study of conformal deformation of Riemannian manifolds [29, 42, 43, 56], problems of sharp constant in Sobolev trace inequalities [38, 41] and blow-up properties of the solutions of related parabolic equations [47, 60]. This kind of equations also appears in several applied contexts like glaciology [77], population genetics [6], non-Newtonian fluid mechanics [39], nonlinear elasticity [32], among others.

There is a vast literature concerning nonnegative solutions for (1.1). Using the moving plane method, Hu [59] obtained nonexistence of positive solutions when $f \equiv 0$ and $g(v) = v^q$, with $1 < q < N/(N-2)$. Similar results were obtained by Chipot *et al.* in [31] in the case that $f(v) = av^p$ and $g(v) = v^q$ with $1 < p \leq (N+2)/(N-2)$, $1 < q \leq N/(N-2)$, with one the inequalities being strict, and $a > 0$ (see also [94] for existence and multiplicity results in the double subcritical case). In dimension $N = 2$ and $f \equiv 0$, Cabré and Morales [21] presented necessary and sufficient conditions on $g(v)$ for the existence of *layer solutions*, that is, bounded solutions that satisfy some monotonicity properties. When $f \equiv 0$ and $g(v) = (N-2)v^{N/(N-2)}$, existence of positive solution decaying as $|x|^{2-N}$ at infinity was obtained by Escobar [41] using the conformal equivalence between the unit ball in $\mathbb{R}^N$ and the half-space (see also [91]).

In the same paper, it was considered the case $f(v) = N(N-2)v^{(N+2)/(N-2)}$ and $g(v) = bv^{N/N(N-2)}$. Later, Chipot *et al.* [30] removed the decay assumption by using the shrinking sphere method to give a complete description of positive solutions when $f(v) = av^{(N+2)/(N-2)}$ and $g(v) = bv^{n/(N-2)}$. Similar results were obtained by Li and Zhu in [66], including a 2-dimensional version with exponential type nonlinearities.

In this chapter, we deal with the boundary critical problem

$$(P_1) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(x \cdot \nabla u) = \lambda u, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \eta} = |u|^{2_*-2}u, & \text{on } \mathbb{R}^{N-1}, \end{cases}$$

where $2_* := 2(N-1)/(N-2)$. Notice that, if u is a solution of (P_1), then the function $v = \exp(|x|^2/8)u$ verifies (1.1) for

$$f(x,v) = \left(\lambda - \frac{N}{4} - \frac{|x|^2}{16}\right)v, \qquad g(x,v) = \exp\left(-\frac{|x|^2}{4(N-2)}\right)|v|^{2_*-2}v.$$

Differently from the former cases, this problem is not homogeneous and the nonlinearity f is unbounded in the spatial variable. Hence, the techniques used in the aforementioned works do not apply and we need to perform a different approach to deal with the drift term inside the domain.

Before presenting our result is important to emphasize the similarity of our equation with the classical problem

$$-\Delta u = \lambda u + |u|^{2^*-2}u, \quad u \in H^1_0(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 3$ and $\lambda > 0$ is a parameter. This equation comes from the Yamabe's problem, which deals with the existence of Riemannian metrics with constant scalar curvature. In a seminal paper, Brezis and Nirenberg [19] proved that the existence of positive solution is related to the interaction of the parameter with the first eigenvalue $\lambda_{1,\Omega} > 0$ of the spectrum $\sigma(-\Delta, H^1_0(\Omega))$. Among other things, they showed that the above equation has a positive solution whenever $N \geq 4$ and $\lambda \in (0, \lambda_{1,\Omega})$. This was the starting point of an effusive literature concerning this critical equation. We especially quote here the paper of Capozzi, Fortunato and Palmieri [23], which obtained solution for $\lambda \geq \lambda_{1,\Omega}$ and Cerami, Solimini and Struwe [25], which proved the existence of a sign-changing solution if $0 < \lambda < \lambda_{1,\Omega}$ and $N \geq 6$.

Besides the natural connection with the Brezis and Nirenberg problem, (P_1) is closely related to the nonlinear heat equation

$$w_t - \Delta w = 0, \text{ in } \mathbb{R}^N_+ \times (0, +\infty), \qquad \frac{\partial w}{\partial \eta} = |w|^{p-2}w, \text{ on } \mathbb{R}^{N-1} \times (0, +\infty).$$

A solution with the special form $w(x,t) = t^{-\lambda}u(t^{-1/2}x)$ is called self-similar solution. It is known (see e.g. [57, 58, 69]) that it provides qualitative properties like global existence, blow-up and asymptotic behavior. Moreover, they preserve the PDE scaling and so carry simultaneously information about small and large scale behaviors. The connection with (P_1) is that, if we put w into the heat equation, we see that the profile u needs to verify the same equation in (P_1) with $\lambda = 1/(2(p-2))$ and 2_* replaced by $p \in (2, 2_*]$.

Setting $K(x) = \exp(|x|^2/4)$ and noticing that $2\nabla K = xK$, the first equation in (P_1) can be rewritten as

$$-\mathrm{div}(K(x)\nabla u) = \lambda K(x)u, \quad \text{in } \mathbb{R}^N_+.$$

Hence, it is natural to look for finite energy solutions belonging to the Sobolev space $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ defined as the closure of $C^\infty_c(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx\right)^{1/2}.$$

This kind of space was first introduced by Escobedo and Kavian [44] who considered a problem in the whole space $\mathbb{R}^N$. The upper half-space case was presented in [47], where it is proved that $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ is compactly embedded into the weighted Lebesgue space

$$L^2_K(\mathbb{R}^N_+) = \left\{u \in L^2(\mathbb{R}^N_+) \,:\, \|u\|_2 = \left(\int_{\mathbb{R}^N_+} K(x)u^2 dx\right)^{1/2} < +\infty\right\}.$$

So, we can solve the linear problem associated with (P_1), namely

$$(LP) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(x\cdot\nabla u) = \lambda u, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \eta} = 0, & \text{on } \mathbb{R}^{N-1}, \end{cases}$$

and use spectral theory to obtain a sequence of eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ such that

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \cdots$$

with $\lim_{j\to\infty} \lambda_j = +\infty$. Moreover, as we will see later, the first eigenvalue is exactly $\lambda_1 = N/2$.

The authors in [47] considered the subcritical version of (P_1), that is, the same problem with 2_* replaced by $p \in (2, 2_*)$. Among other results, they obtained the existence of a positive solution if $\lambda < \lambda_1$. As a consequence, self-similar solutions to the associated heat equation exist whenever $2 + (1/N) < p < 2_*$. The critical version was recently considered in [48] and the situation turns to be more delicate. After proving the trace embedding $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+) \hookrightarrow L^{2_*}_K(\mathbb{R}^N_+)$, where

$$L^{2_*}_K(\mathbb{R}^{N-1}) = \left\{u \in L^{2_*}(\mathbb{R}^{N-1}) \,:\, |u|_{2_*} = \left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*} dx'\right)^{1/2_*} < +\infty\right\},$$

the authors showed that, in the critical case, there is no self-similar solution to the equation. Besides this, they obtained a positive solution whenever $N \geq 7$ and the parameter λ verifies

$$\lambda_N^* = \frac{N}{4} + \frac{N-4}{8} < \lambda < \lambda_1.$$

We notice that, since $\lambda_1 = N/2$, the above range is nonempty.

In the first part of this chapter we complete the above study by considering the case $\lambda > \lambda_1$. Standard arguments show that positive solutions are not expected and therefore we look for sign-changing solutions. More specifically, we prove the following:

Theorem A. *If $N \geq 7$ and $\lambda > \lambda_1$ is not an eigenvalue of (LP), then problem (P_1) has a sign-changing solution.*

In the proof, we apply the Linking Theorem [83] to the energy functional

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{2_*}\|u\|_{2_*}^{2_*}, \quad u \in \mathcal{D}_K^{1,2}(\mathbb{R}_+^N).$$

Since standard arguments can not be used to verify the linking geometry, we need to perform a detailed study of the structure of solutions of the eigenvalue problem (LP) and prove a trick projection result (see Lemma 1.1.4 and Proposition 1.1.5). The assumption that λ is not an eigenvalue of (LP) is a non-resonant type condition of technical nature and assures that Palais-Smale sequences are bounded. Actually, the arguments used in [23, 83] do not work in unbounded domains and therefore we need to perform a different approach here (see Proposition 1.1.6).

In the second part of the chapter, we come back to the range where positive solution exists and ask if it is possible to obtain another solution. In this new setting, we prove the following:

Theorem B. *If $N \geq 7$ and $\lambda \in (\lambda_N^*, \lambda_1)$, then problem (P_1) has a sign-changing solution.*

In order to explain the main steps for the proof, we first define $u^+(x) = \max\{u(x), 0\}$ and $u^- = u^+ - u$. After that, inspired by the paper of Cerami, Solimini and Struwe [25], we introduce the Nehari nodal set $\mathcal{M}_\lambda = \{u \in \mathcal{D}_K^{1,2}(\mathbb{R}_+^N) : u^\pm \neq 0,\ I'_\lambda(u^\pm)u^\pm = 0\}$ and prove that

$$d_\lambda = \inf_{u \in \mathcal{M}_\lambda} I_\lambda(u)$$

is attained by a solution $u \in \mathcal{M}_\lambda$. Since we are dealing with the critical case, the functional I_λ satisfies only a local Palais-Smale condition. So, we need to prove some fine estimates (see Lemmas 1.4.2 and 1.4.3) involving the positive solution obtained in [48] and a slight modification of the *instanton functions* founded independently by

Escobar [41] and Beckner [13]. This is essential to guarantee that d_λ belongs to the range where we have compactness. Since $\mathcal{M}_\lambda$ is not a differentiable manifold, it is not easy to construct Palais-Smale sequences on the level d_λ. In order to do this, we adapt some ideas introduced by Tarantello in [90].

The chapter is organized as follows: in the next section, we present the variational framework and some technical results for Theorem A, which is proved after in Section 1.2. In Section 1.3, we establish the minimization scheme for the second case and in the last section, we obtain the solution when $\lambda \in (\lambda_N^*, \lambda_1)$.

1.1 Variational setting and preliminary results

We start this section setting $K(x) := \exp(|x|^2/4)$ and noticing that

$$\operatorname{div}(K(x)\nabla u) = K(x)\left(\Delta u + \frac{1}{2}(x\cdot\nabla u)\right),$$

for any regular function u. Hence, it is natural to define the Banach space $\mathcal{D}^{1,2}_K(\Omega)$ as being the closure of $C_c^\infty(\overline{\Omega})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}_K(\Omega)} := \left(\int_\Omega K(x)|\nabla u|^2\,dx\right)^{\frac{1}{2}},$$

for any open set $\Omega \subset \mathbb{R}^N$. For simplicity, we denote $\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)$ by X and $\|\cdot\|_{\mathcal{D}^{1,2}_K(\mathbb{R}^N_+)}$ by $\|\cdot\|$. We also define, for any $2 \le r \le 2^* := 2N/(N-2)$, the weighted Lebesgue space

$$L^r_K(\mathbb{R}^N_+) := \left\{ u \in L^r(\mathbb{R}^N_+) \,:\, \|u\|_r := \left(\int_{\mathbb{R}^N_+} K(x)|u|^r dx\right)^{1/r} < \infty \right\}.$$

According to [47, Lemma 2.2], the embedding $X \hookrightarrow L^r_K(\mathbb{R}^N_+)$ is continuous for $2 \le r \le 2^*$ and compact for $2 \le r < 2^*$. Moreover, denoting by

$$L^r_K(\mathbb{R}^{N-1}) := \left\{ u \in L^r(\mathbb{R}^{N-1}) \,:\, |u|_r := \left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^r dx'\right)^{1/r} < \infty \right\},$$

it was proved in [47, Lemma 2.4] the compact trace embedding $X \hookrightarrow L^r_K(\mathbb{R}^{N-1})$, for $2 < r < 2_*$. Subsequently, the authors in [48, Theorem 1.1] extended this former result by proving that the embedding is really continuous for $2 \le r \le 2_*$ and compact for $2 \le r < 2_*$. So, the natural range of the trace embedding is covered and we can define the best constant

$$S(K) := \inf_{\varphi \in X\setminus\{0\}} \frac{\|\varphi\|^2}{|\varphi|^2_{2_*}} > 0. \tag{1.2}$$

Actually, it is proved in [48] that the above infimum is achieved and it is equal to the best constant S of the Sobolev trace embedding $\mathcal{D}^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2_*}(\mathbb{R}^{N-1})$.

The energy functional associated with our problem $I_\lambda : X \to \mathbb{R}$ is given by

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{2_*}|\!|\!|u|\!|\!|_{2_*}^{2_*}, \quad \forall\, u \in X.$$

Standard calculations show that $I_\lambda \in C^1(X,\mathbb{R})$ and the weak solutions of (P_1) are precisely the critical points of I_λ.

For proving Theorem A, we shall use the following variant of the Mountain Pass Theorem [83] (see also [93, Theorem 2.12]).

Theorem 1.1.1. *Let $E = V \oplus W$ be a real Banach space with* $\dim V < \infty$. *Suppose $I \in C^1(E,\mathbb{R})$ satisfies*

(I_1) *there exist $\rho, \alpha > 0$ such that $I|_{W\cap\partial B_\rho(0)} \geq \alpha$;*

(I_2) *there exists $e \in W \cap \partial B_1(0)$ and $R > \rho$ such that*

$$I|_{\partial Q} \leq 0,$$

with

$$Q := \left(\overline{B_R(0)} \cap V\right) \oplus \{te : 0 < t < R\}.$$

If

$$c := \inf_{\gamma\in\Gamma} \max_{u\in Q} I(\gamma(u)), \tag{1.3}$$

where $\Gamma := \left\{\gamma \in C(\overline{Q}, E) : \gamma \equiv Id \text{ on } \partial Q\right\}$, then there exists a sequence $(u_n) \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$, as $n \to +\infty$.

We are intending to apply this abstract result with $E = X$ and $I = I_\lambda$. In order to present the decomposition of the space X we consider the linearized problem

$$(LP) \qquad \begin{cases} -\mathrm{div}(K(x)\nabla u) = \lambda K(x) u, & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial \eta} = 0, & \text{on } \mathbb{R}^{N-1}, \\ u \in \mathcal{D}^{1,2}_K(\mathbb{R}^N_+). \end{cases}$$

Thanks to the compact embedding $X \hookrightarrow L^2_K(\mathbb{R}^N_+)$, we can use standard spectral theory to obtain sequence of eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ such that

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$$

with $\lim\limits_{j\to\infty} \lambda_j = +\infty$. A straightforward computation shows that

$$\varphi_1(x) := \exp\left(-|x|^2/4\right)$$

satisfies (LP). Since this function is positive, its associated eigenvalue is the first one. Noticing that $\nabla\varphi_1 = -(x/2)K(x)^{-1}$, we can explicitly compute this first eigenvalue in the following way:

$$\lambda_1 = -\frac{\operatorname{div}(K(x)\nabla\varphi_1)}{K(x)\varphi_1} = \frac{1}{2}\operatorname{div}(x) = \frac{N}{2}.$$

Along all this section we shall assume that $\lambda \in (\lambda_k, \lambda_{k+1})$, for some $k \in \mathbb{N}$. In order to apply Theorem 1.1.1, we set

$$V := \operatorname{span}\{\varphi_1, \dots, \varphi_k\}, \qquad W := V^\perp, \tag{1.4}$$

in such way that $X = V \oplus W$. As it is well known from the variational characterization of the eigenvalue of (LP), we have that

$$\frac{1}{\lambda_k}\|v\|^2 \le \|v\|_2^2, \qquad \|w\|_2^2 \le \frac{1}{\lambda_{k+1}}\|w\|^2, \qquad \forall\, v \in V,\, w \in W. \tag{1.5}$$

The condition (I_1) easily follows from the above inequalities.

Lemma 1.1.2. *The functional I_λ satisfies assumption (I_1) of Theorem 1.1.1 .*

Proof. Using (1.5) and (1.2) we obtain, for any $w \in W$,

$$I_\lambda(w) \ge \frac{1}{2}\left(\frac{\lambda_{k+1}-\lambda}{\lambda_{k+1}}\right)\|w\|^2 - \frac{1}{2_*}|w|_{2_*}^{2_*} \ge \|w\|^2\left(\frac{C_1}{2} - \frac{1}{2_*}S^{-2_*/2}\|w\|^{2_*-2}\right),$$

where $C_1 := (\lambda_{k+1}-\lambda)/\lambda_{k+1} > 0$. Hence,

$$I_\lambda(w) \ge \frac{\rho^2 C_1}{4}, \qquad \forall\, w \in W \cap \partial B_\rho(0),$$

for $\rho := \left[(2_* C_1 S^{2_*/2})/4\right]^{1/(2_*-2)}$. The lemma is proved. □

The proof of (I_2) is more involved and we need to perform a detailed study of the solutions of (LP). We start with an interesting result proved by Escobedo and Kavian [44, Proposition 2.3] via a Fourier Transform approach:

Proposition 1.1.3. *The eigenvalues of the problem*

$$\begin{cases} -\operatorname{div}(K(x)\nabla u) = \mu K(x) u, & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}_K^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.6}$$

are $\mu_k = (N+k-1)/2$, with $k \in \mathbb{N}$. The associated eigenspaces are given by

$$\mathcal{V}_k := \mathit{span}\left\{D^\beta\varphi_1 \,:\, |\beta| = k-1\right\}$$

where $\varphi_1(x) = \exp(-|x|^2/4)$, $\beta \in (\mathbb{N}\cup\{0\})^N$, $|\beta| := \beta_1 + \cdots + \beta_N$ and $D^\beta := \partial^{\beta_1}\cdots\partial^{\beta_N}$. In particular, any eigenfunction can be written as $P(x)\varphi_1(x)$, for some polynomial function P.

As an application of the above result, we can describe the shape of the solutions of the problem (LP). More specifically, we have the following:

Lemma 1.1.4. *If $\varphi \in X$ is an eigenfunction of (LP), then there exists a polynomial $p(x)$ such that $\varphi(x) = p(x)\varphi_1(x)$, for any $x \in \mathbb{R}^N_+$.*

Proof. Suppose that $\varphi \in X$ is an eigenfunction of (LP) and define

$$v(x', x_N) := \begin{cases} \varphi(x', x_N), & \text{if } x_N \geq 0, \\ \varphi(x', -x_N), & \text{if } x_N < 0. \end{cases}$$

Since $\varphi_{x_N}(x', 0) = 0$ in $\mathbb{R}^{N-1}$, we can check that $v \in \mathcal{D}^{1,2}_K(\mathbb{R}^N)$. Moreover, $v_{|_{\mathbb{R}^N_-}} \in \mathcal{D}^{1,2}_K(\mathbb{R}^N_-)$ is a solution of a linear problem analogous to (LP) but with $\mathbb{R}^N_+$ replaced by $\mathbb{R}^N_- := \{(x', x_N) : x' \in \mathbb{R}^{N-1},\, x_N < 0\}$.

Let $\phi \in C^\infty_c(\mathbb{R}^N)$ and denote by $\phi_+ \in C^\infty_c(\overline{\mathbb{R}^N_+})$ the restriction of ϕ to $\mathbb{R}^N_+$. We define ϕ_- in an analogous way and compute

$$\begin{aligned} \int_{\mathbb{R}^N} K(x)(\nabla v \cdot \nabla \phi)\, dx &= \int_{\mathbb{R}^N_+} K(x)(\nabla \varphi \cdot \nabla \phi_+)\, dx \\ &\quad + \int_{\mathbb{R}^N_-} K(x)(\nabla \varphi(x', -x_N) \cdot \nabla \phi_-)\, dx \\ &= \lambda \int_{\mathbb{R}^N_+} K(x) v \phi_+\, dx + \lambda \int_{\mathbb{R}^N_-} K(x) v \phi_-\, dx \\ &= \lambda \int_{\mathbb{R}^N} K(x) v \phi\, dx, \end{aligned}$$

that is, v is an eigenfunction of (1.6). The result follows from Proposition 1.1.3. □

We are ready to prove a technical result which will be useful for verifying the geometric condition (I_2).

Proposition 1.1.5. *Suppose that $\phi \in C^\infty_c(\overline{\mathbb{R}^N_+}) \setminus \{0\}$ is such that $\phi_{|_{\mathbb{R}^{N-1}}} \neq 0$ and its orthogonal projection $\phi^\perp$ over W is nonzero. Then the functional I_λ satisfies assumption (I_2) of Theorem 1.1.1 for $e := \phi^\perp / \|\phi^\perp\|$.*

Proof. Since $\lambda > \lambda_k$, we can use the variational inequality (1.5) to check that $I_\lambda \leq 0$ in V. Thus, since the set Q defined in Theorem 1.1.1 is such that $\partial Q = \{v + te : v \in V,\, \|v\| = R,\ 0 \leq t \leq R\} \cup \{v \in V : \|v\| \leq R\} \cup \{v + Re : v \in V,\, \|v\| \leq R\}$, condition (I_2) holds if we can prove that

$$\lim_{\|z\| \to +\infty,\, z \in V \oplus \mathbb{R}e} I(z) = -\infty. \tag{1.7}$$

In order to prove the above claim, we first notice that there exists a maximal set of indices $L = \{j_1, \ldots, j_l\} \subset \{1, \ldots, k\}$ such that $\mathcal{O} := \{\varphi_{j_1}(x', 0), \ldots, \varphi_{j_l}(x', 0)\}$ is linearly independent and

$$\operatorname{span} \mathcal{O} = \operatorname{span}\{\varphi_1(x', 0), \ldots, \varphi_k(x', 0)\} \tag{1.8}$$

After a rearrangement, we may assume that $L = \{1, 2, \ldots, m\}$, with $m \le k$.

We first show that the function

$$|(b_1, \cdots, b_m, b_{m+1})|_1 := \| b_1\varphi_1 + \cdots + b_m\varphi_m + b_{m+1}\phi^{\perp} \|_{2_*}$$

defines a norm in $\mathbb{R}^{m+1}$. Indeed, suppose that $|(b_1, \cdots, b_m, b_{m+1})|_1 = 0$, in such way that

$$b_1\varphi_1(x', 0) + \cdots + b_m\varphi_m(x', 0) + b_{m+1}\phi^{\perp}(x', 0) = 0, \quad \forall\, x' \in \mathbb{R}^{N-1}.$$

If $b_{m+1} \neq 0$, then $\phi^{\perp}(\cdot, 0)$ is a linear combination of the elements of $\mathcal{O}$. By Lemma 1.1.4, there exists a polynomial q such that $\phi^{\perp}(x', 0) = q(x')\varphi_1(x', 0)$, for any $x' \in \mathbb{R}^{N-1}$. Since $\phi - \phi^{\perp} \in \text{span}\{\varphi_1, \cdots, \varphi_k\}$, it follows again from Lemma 1.1.4 that there exists a polynomial r such that

$$\phi(x', 0) = [(\phi - \phi^{\perp}) + \phi^{\perp}](x', 0) = r(x')\varphi_1(x', 0), \quad \forall\, x' \in \mathbb{R}^{N-1}.$$

But $\phi_{|_{\mathbb{R}^{N-1}}} \neq 0$, $\varphi_1 > 0$ and ϕ has compact support, and therefore we could construct polynomials of type $t \mapsto p(x_1, \ldots, t, \ldots, x_{N-1})$ with infinitely many roots, which is absurd. Thus, we have that $b_{m+1} = 0$ and, since $\mathcal{O}$ is linearly independent, all the others coefficients are also null. The other properties of a norm can be easily verified.

Now we prove that there exist m polynomials $Q_i : \mathbb{R}^k \to \mathbb{R}$ of degree 1, $1 \le i \le m$, and $C_1 > 0$ such that

$$\| a_1\varphi_1 + \cdots + a_k\varphi_k + a_{k+1}\phi^{\perp} \|_{2_*} \ge C_1 \left[\left(\sum_{i=1}^{m} Q_i^2(a_1, \ldots, a_k) \right) + a_{k+1}^2 \right]^{1/2}, \qquad (1.9)$$

for any $a_1, \ldots, a_{k+1} \in \mathbb{R}$. Indeed, since $|\cdot|_1$ is a norm in $\mathbb{R}^{m+1}$, there exists $C_1 > 0$ such that

$$|(b_1, \ldots, b_m, b_{m+1})|_1 \ge C_1 \left(\sum_{i=1}^{m+1} b_i^2 \right)^{1/2}, \qquad (1.10)$$

for any $(b_1, \ldots, b_{m+1}) \in \mathbb{R}^{m+1}$. For each $l = 1, \ldots, k$, we infer from (1.8) that $\varphi_l = \sum_{i=1}^{m} c_i^l \varphi_i$ in $\mathbb{R}^{N-1}$, and consequently

$$\left(\sum_{l=1}^{k} a_l\varphi_l \right) + a_{k+1}\phi^{\perp} = \left(\sum_{i=1}^{m} Q_i(a)\varphi_i \right) + a_{k+1}\phi^{\perp}, \quad \text{in } \mathbb{R}^{N-1},$$

where $Q_i(a) := \sum_{l=1}^{k} a_l c_i^l$ and $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$. Setting $b_i := Q_i(a)$, $1 \le i \le m$, and $b_{m+1} = a_{k+1}$, (1.9) is a direct consequence of the above expression, (1.10) and the definition of $|\cdot|_1$.

We are ready to prove (1.7). Let $z = \left(\sum_{i=1}^{k} a_i\varphi_i \right) + a_{k+1}\phi^{\perp} \in V \oplus \mathbb{R}e$ and notice that, by (LP) and the orthogonality of the eigenfunctions, we have that

$$I_\lambda(z) = -\frac{1}{2} \sum_{i=1}^{k} a_i^2(\lambda - \lambda_i)\|\varphi_i\|_2^2 + \frac{a_{k+1}^2}{2}(\|\phi^{\perp}\|^2 - \lambda\|\phi^{\perp}\|_2^2) - \frac{1}{2_*}\| z \|_{2_*}^{2_*}.$$

Hence, if we set

$$C_2 := \min_{1\le i\le k}(\lambda-\lambda_i)\|\varphi_i\|_2^2 > 0, \qquad C_3 := (\|\phi^\perp\|^2 - \lambda\|\phi^\perp\|_2^2) > 0,$$

it follows from (1.5) and (1.9) that

$$I_\lambda(z) \le -\frac{C_2}{2}\left(\sum_{i=1}^k a_i^2\right) + \frac{C_3}{2}a_{k+1}^2 - \frac{C_1^{2_*}}{2_*}|a_{k+1}|^{2_*}. \tag{1.11}$$

Since $V \oplus \mathbb{R}e$ is finite-dimensional, there exists $C_4 > 0$ such that

$$C_4\|z\|^2 \le \left(\sum_{i=1}^k a_i^2\right) + a_{k+1}^2.$$

So, if $\|z\| \to +\infty$, at least one of the terms on the right-hand side above goes to infinity and therefore (1.7) is a consequence of (1.11). The proposition is proved. $\square$

In the final result of this section, we follow ideas of the celebrated paper of Brezis and Nirenberg [19] to get a local compactness result.

Proposition 1.1.6. *Suppose that $(u_n) \in X$ satisfies*

$$0 \ne \lim_{n\to\infty} I_\lambda(u_n) = d < \frac{1}{2(N-1)}S^{N-1}, \qquad \lim_{n\to\infty} I_\lambda'(u_n) = 0. \tag{1.12}$$

Then (u_n) is bounded and, along a subsequence, (u_n) weakly converges to a nonzero weak solution to (P_λ).

Proof. From (1.12), we obtain

$$\left(\frac{1}{2}-\frac{1}{2_*}\right) |u_n|_{2_*}^{2_*} = I_\lambda(u_n) - \frac{1}{2}I_\lambda'(u_n)u_n \le C_1 + C_1\|u_n\|. \tag{1.13}$$

Using the decomposition $X = V \oplus W$, one can write $u_n = v_n + w_n$, with $v_n \in V$ and $w_n \in W$. Setting

$$J(u) := \frac{1}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*}dx', \quad \forall\, u \in X,$$

we can use (1.12) and (1.5) to get

$$\begin{aligned} C_2 + o_n(1)\|v_n\| &\ge I_\lambda(u_n) - \frac{1}{2}I_\lambda'(u_n)v_n \\ &\ge \frac{1}{2}\|w_n\|^2 - \frac{\lambda}{2}\|w_n\|_2^2 + \frac{1}{2}J'(u_n)v_n - \frac{1}{2_*}|u_n|_{2_*}^{2_*} \\ &\ge \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\|w_n\|^2 + \frac{1}{2}J'(u_n)v_n - \frac{1}{2_*}|u_n|_{2_*}^{2_*}, \end{aligned}$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. If $A_1 := (\lambda_{k+1} - \lambda)/(2\lambda_{k+1}) > 0$, the above expression, (1.13), Holder's inequality and the trace embedding imply that

$$\begin{aligned} A_1\|w_n\|^2 &\leq C_3 + o_n(1)\|v_n\| + C_3\|u_n\| - \frac{1}{2}\int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*-2}u_n v_n\,dx' \\ &\leq C_3 + C_4\|u_n\| + C_5 |u_n|_{2_*}^{2_*-1} |v_n|_{2_*}. \\ &\leq C_3 + C_4\|u_n\| + C_6(C_1 + C_1\|u_n\|)^{(2_*-1)/2_*}\|u_n\| \end{aligned}$$

and therefore

$$A_1\|w_n\|^2 \leq C_3 + C_7\|u_n\| + C_8\|u_n\|^{2-(1/2_*)}. \tag{1.14}$$

On the other hand, from (1.5) we obtain

$$o_n(1)\|v_n\| = I'_\lambda(u_n)v_n \leq \left(1 - \frac{\lambda}{\lambda_k}\right)\|v_n\|^2 - \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*-2}u_n v_n\,dx'.$$

and we can argue as above to get

$$A_2\|v_n\|^2 \leq C_9\|u_n\| + C_{10}\|u_n\|^{2-(1/2_*)},$$

where $A_2 := (\lambda - \lambda_k)/\lambda_k > 0$. Since $\|u_n\|^2 = \|v_n\|^2 + \|w_n\|^2$, the above expression and (1.14) imply that

$$\|u_n\|^2 \leq C_{11} + C_{12}\|u_n\| + C_{13}\|u_n\|^{2-(1/2_*)}.$$

and therefore it follows from $2 - (1/2_*) < 2$ that (u_n) is bounded in X.

Up to a subsequence, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } X, \\ u_n \to u, & \text{strongly in } L^2_K(\mathbb{R}^N_+), \\ u_n \to u, & \text{strongly in } L^s_K(\mathbb{R}^{N-1}), \end{cases}$$

for any $2 \leq s < 2*$ and for some $u \in X$. Given $\phi \in C_c^\infty(\overline{\mathbb{R}^N_+})$, we can use the above convergences, Young's inequality and standard computations to show that

$$0 = \lim_{n\to+\infty} I'_\lambda(u_n)\phi = I'_\lambda(u)\phi,$$

and therefore u is a critical point of I_λ.

We prove now that $u \neq 0$. Suppose, by contradiction, that this is not the case. Then, $u_n \to 0$ in $L^2_K(\mathbb{R}^N_+)$ and we can use $I_\lambda(u_n) \to d$ and $I'_\lambda(u_n)u_n \to 0$ to obtain

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{2_*}|u_n|_{2_*}^{2_*} = d + o_n(1) \tag{1.15}$$

and

$$\|u_n\|^2 - |u_n|_{2_*}^{2_*} = o_n(1).$$

Since we may assume that $\|u_n\|^2 \to l \geq 0$, the above expression shows that $|u_n|_{2_*}^{2_*} \to l$. Thus, it follows from (1.15) that

$$d = \left(\frac{1}{2} - \frac{1}{2_*}\right) l = \frac{1}{2(N-1)} l. \tag{1.16}$$

Recall that the constant $S(K)$ defined in (1.2) is equal to the best constant S of the trace embedding $\mathcal{D}^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2_*}(\mathbb{R}^{N-1})$. So, passing the inequality $S|u_n|_{2_*}^2 \leq \|u_n\|^2$ to the limit we obtain $Sl^{2/2_*} \leq l$. If $l > 0$, we conclude that $l \geq S^{N-1}$. Combining this with (1.16), we obtain $d \geq S^{N-1}/[2(N-1)]$, which is a contradiction. Hence, $l = 0$ and therefore $u_n \to 0$ in X, which implies that $I_\lambda(u_n) \to d = 0$, contrary to the hypothesis. Thus, $u \neq 0$ and we have done. □

1.2 Proof of Theorem A

We devote this section to the proof of Theorem A. For any $\varepsilon > 0$, consider the function

$$U_\varepsilon(x', x_N) := \frac{\varepsilon^{(N-2)/2}}{[|x'|^2 + (x_N + \varepsilon)^2]^{(N-2)/2}}, \qquad (x', x_N) \in \mathbb{R}^N_+.$$

They are the so-called *instantons* which achieves the best constant of the Sobolev trace embedding $\mathcal{D}^{1,2}(\mathbb{R}^N_+) \hookrightarrow L^{2_*}(\mathbb{R}^{N-1})$ (see [41]).

We now fix $R > 0$, pick $\phi \in C^\infty(\overline{\mathbb{R}^N_+}, [0,1])$ such that $\phi \equiv 1$ in $\overline{\mathbb{R}^N_+} \cap B_R(0)$, $\phi \equiv 0$ in $\overline{\mathbb{R}^N_+} \backslash B_{2R}(0)$ and set, for each $\varepsilon > 0$,

$$\psi_\varepsilon(x) := K(x)^{-1/2} \phi(x) U_\varepsilon(x), \quad x \in \mathbb{R}^N_+.$$

This function ψ_ε was extensively exploited in [48], where it was proved that, if $N \geq 7$, then

$$\|\psi_\varepsilon\|^2 = A_N + O(\varepsilon^4) + \varepsilon^2 \gamma_N, \qquad \|\psi_\varepsilon\|_2^2 = O(\varepsilon^{N-2}) + \varepsilon^2 \alpha_N$$

and

$$|\psi_\varepsilon|_{2_*}^{2_*} = B_N^{2_*/2} - \varepsilon^2 D_N + o(\varepsilon^2), \tag{1.17}$$

where the constants A_N, B_N, D_N, α_N, $\gamma_N > 0$ depend only on the dimension N. Moreover, if we set

$$Q_\lambda(u) := \frac{\|u\|^2 - \lambda \|u\|_2^2}{|u|_{2_*}^2}, \quad \forall\, u \in X \setminus \{0\},$$

there exists $E_N > 0$, depending only on N, such that

$$Q_\lambda(\psi_\varepsilon) = S + \varepsilon^2 \left(-E_N + o(1)\right), \tag{1.18}$$

whenever $\lambda > \lambda_N^*$. It is worth mention that, along this section, the notations O and o refers to $\varepsilon \to 0^+$.

Remark 1. *We would like to emphasize that all the constants above can be explicitly computed in terms of the Beta function*

$$B(a,b) := \int_0^\infty \frac{s^{a-1}}{(s+1)^{a+b}} ds, \quad \forall\, a,\, b > 0,$$

the dimension N and the volume σ_{N-2} of the $(N-2)$-dimensional sphere. Actually,

$$A_N := \int_{\mathbb{R}^N_+} |\nabla U_\varepsilon|^2 dx, \qquad B_N := \left(\int_{\mathbb{R}^{N-1}} |U_\varepsilon|^{2_*} dx'\right)^{2/2_*},$$

$$D_N := \frac{\sigma_{N-2}}{8(N-2)} B\left(\frac{N+1}{2}, \frac{N-3}{2}\right), \qquad \alpha_N := \frac{\sigma_{N-2}}{2(N-4)} B\left(\frac{N-1}{2}, \frac{N-3}{2}\right)$$

$$\gamma_N := \frac{\sigma_{N-2}(N-2)}{4(N-4)} \left[B\left(\frac{N+1}{2}, \frac{N-3}{2}\right) + \frac{1}{(N-3)} B\left(\frac{N-1}{2}, \frac{N-1}{2}\right)\right]$$

and

$$E_N := \frac{\lambda\alpha_N - \gamma_N - (2/2_*) A_N B_N^{-2/2_*} D_N}{B_N}.$$

Before stating our next result, we need to introduce some useful notation. For any $u_1,\, u_2 \in X$, we denote

$$(u_1, u_2) := \int_{\mathbb{R}^N_+} K(x)\,(\nabla u_1 \cdot \nabla u_2)\, dx, \qquad (u_1, u_2)_2 := \int_{\mathbb{R}^N_+} K(x) u_1 u_2\, dx. \tag{1.19}$$

Since ψ_ε has compact support, for any $\tau \geq 1$ it is well defined

$$|\!|\psi_\varepsilon|\!|_\tau := \left(\int_{\mathbb{R}^{N-1}} K(x', 0) |\psi_\varepsilon|^\tau dx'\right)^{1/\tau}.$$

Moreover, the following holds:

Lemma 1.2.1. *We have that*

$$|\!|\psi_\varepsilon|\!|_\tau^\tau = O(\varepsilon^{(N-1)-\tau(N-2)/2}), \qquad |\!|\psi_\varepsilon|\!|_1 = O(\varepsilon^{(N-2)/2}), \tag{1.20}$$

$$(v, \psi_\varepsilon) = \|v\|_2 O(\varepsilon^{(N-2)/2}), \qquad (v, \psi_\varepsilon)_2 = \|v\|_2 O(\varepsilon^{(N-2)/2}), \tag{1.21}$$

for any $v \in V$ and $\tau \in \mathbb{R}$ such that $(N-1)/(N-2) < \tau < 2_*$.

Proof. For saving notation, we write only K and ϕ to denote $K(x', 0)$ and $\phi(x', 0)$, respectively. Using the definition of ψ_ε and the change of variable $y' = (x'/\varepsilon)$, we get

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} K|\psi_\varepsilon|^\tau dx' &= \varepsilon^{\tau(N-2)/2} \int_{\mathbb{R}^{N-1}} \frac{K^{(2-\tau)/2}\phi^\tau}{[|x'|^2 + \varepsilon^2]^{\tau(N-2)/2}} dx' \\ &\leq C_1 \varepsilon^{-\tau(N-2)/2} \int_{B_{2R}(0)\cap\mathbb{R}^{N-1}} \frac{1}{[|x'/\varepsilon|^2 + 1]^{\tau(N-2)/2}} dx' \\ &\leq C_1 \varepsilon^{(N-1)-\tau(N-2)/2} \int_{\mathbb{R}^{N-1}} \frac{1}{[|y'|^2 + 1]^{\tau(N-2)/2}} dy'. \end{aligned}$$

Using $\tau > (N-1)/(N-2)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} \frac{1}{[|y'|^2+1]^{\tau(N-2)/2}}dy' &\leq C_2 + \int_{\{|y'|\geq 1\}} \frac{1}{|y'|^{\tau(N-2)}}dy' \\ &\leq C_2 + C_3 \int_1^{+\infty} r^{-\tau(N-2)} r^{N-2} dr < +\infty,\end{aligned}$$

and therefore the first equality in (1.20) holds. For the second one, notice that

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} K|\psi_\varepsilon|\,dx' &= \varepsilon^{(N-2)/2} \int_{\mathbb{R}^{N-1}} \frac{K^{1/2}\phi}{[|x'|^2+\varepsilon^2]^{(N-2)/2}}dx' \\ &\leq C_4\varepsilon^{(N-2)/2} \int_{B_{2R}(0)\cap\mathbb{R}^{N-1}} \frac{1}{[|x'|^2+\varepsilon^2]^{(N-2)/2}}dx' \\ &\leq C_4\varepsilon^{(N-2)/2} \int_{B_{2R}(0)\cap\mathbb{R}^{N-1}} \frac{1}{|x'|^{N-2}}dx'.\end{aligned}$$

Again, the last integral above is finite.

For proving (1.21) we pick $v = \sum_{i=1}^k a_i\varphi_i \in V$ and notice that, since each $\varphi_i \in X$ is a solution to the linear problem (LP) with $\lambda = \lambda_i$, then

$$\begin{aligned}|(v,\psi_\varepsilon)| &= \left|\sum_{i=1}^k \lambda_i a_i(\varphi_i,\psi_\varepsilon)_2\right| \leq \lambda_k \sum_{i=1}^k |a_i||(\varphi_i,\psi_\varepsilon)_2| \\ &\leq \lambda_k \sum_{i=1}^k |a_i|\|\varphi_i\|_{L^\infty(\mathbb{R}^N_+)} \int_{\mathbb{R}^N_+} K(x)|\psi_\varepsilon|\,dx.\end{aligned}$$

Since all the norms in V are equivalent, there exists $C_5 > 0$, independent of v, such that $\sum_{i=1}^k |a_i| \leq C_5\|v\|_2$. Hence, if we set $C_6 := \lambda_k \max\limits_{1\leq i\leq n} \|\varphi_i\|_{L^\infty(\mathbb{R}^N_+)}$, we obtain

$$\begin{aligned}|(v,\psi_\varepsilon)| &\leq C_5C_6\|v\|_2\varepsilon^{(N-2)/2}C_7 \int_{B_{2R}(0)\cap\mathbb{R}^N_+} \frac{1}{|x|^{(N-2)}}\,dx \\ &\leq C_8\|v\|_2\varepsilon^{(N-2)/2},\end{aligned}$$

from which the first equality in (1.21) follows. The second one can be proved along the same lines. □

The following result is the keystone for proving Theorem A.

Proposition 1.2.2. *For any $\varepsilon > 0$ small, there holds*

$$\max_{u\in V\oplus\mathbb{R}\psi_\varepsilon} I_\lambda(u) < \frac{1}{2(N-1)}S^{N-1}.$$

Proof. Given $u \neq 0$, a straightforward computation yields

$$\max_{t\geq 0} I_\lambda(tu) = \frac{1}{2(N-1)} \left(\frac{\|u\|^2 - \lambda\|u\|_2^2}{|\!|u|\!|_{2_*}^2} \right)^{N-1}.$$

Therefore, by homogeneity, we see that it is sufficient to prove that

$$\max_{u\in\Sigma_\varepsilon} \left(\|u\|^2 - \lambda\|u\|_2^2\right) < S, \tag{1.22}$$

where

$$\Sigma_\varepsilon := \{u = v + t\psi_\varepsilon \,:\, v \in V,\, t \in \mathbb{R},\, |\!|u|\!|_{2_*} = 1\}.$$

We first check that, for any $u = v + t\psi_\varepsilon \in \Sigma_\varepsilon$, there holds $t = O(1)$ as $\varepsilon \to 0^+$. Indeed, setting

$$A(u) := |\!|u|\!|_{2_*}^{2_*} - |\!|v|\!|_{2_*}^{2_*} - |\!|t\psi_\varepsilon|\!|_{2_*}^{2_*},$$

integrating the equality

$$\frac{d}{ds}\left(|sv + t\psi_\varepsilon|^{2_*} - |sv|^{2_*}\right) = 2_* \left[|sv + t\psi_\varepsilon|^{2_*-2}(sv + t\psi_\varepsilon) - |sv|^{2_*-2}(sv)\right] v$$

and using the Mean Value Theorem we obtain

$$\begin{aligned}
A(u) &= \int\limits_{\mathbb{R}^{N-1}} K(x',0)\left(|v + t\psi_\varepsilon|^{2_*} - |v|^{2_*} - |t\psi_\varepsilon|^{2_*}\right) dx' \\
&= 2_* \int\limits_{\mathbb{R}^{N-1}} \int_0^1 K(x',0)\left(|sv + t\psi_\varepsilon|^{2_*-2}(sv + t\psi_\varepsilon) - |sv|^{2_*-2}(sv)\right) v\, ds\, dx' \\
&= 2_*(2_*-1) \int\limits_{\mathbb{R}^{N-1}} \int_0^1 K(x',0)(|sv + t\psi_\varepsilon\theta|^{2_*-2} t\psi_\varepsilon v)\, ds\, dx',
\end{aligned}$$

with $\theta(x) \in [0,1]$. Since $s \in [0,1]$, we get

$$\left||sv + t\psi_\varepsilon\theta|^{2_*-2} t\psi_\varepsilon v\right| \leq C_1(|t||v|^{2_*-1}|\psi_\varepsilon| + |t|^{2_*-1}|v||\psi_\varepsilon|^{2_*-1})$$

and therefore it follows from (1.20) with $\tau = 2_* - 1 = N/(N-2)$ that

$$\begin{aligned}
|A(u)| &\leq C_1|t| \int\limits_{\mathbb{R}^{N-1}} K(x',0)|v|^{2_*-1}|\psi_\varepsilon| dx' + C_1|t|^{2_*-1} \int\limits_{\mathbb{R}^{N-1}} K(x',0)|v||\psi_\varepsilon|^{2_*-1} dx' \\
&\leq C_1|t|\|v\|_{L^\infty(\mathbb{R}^{N-1})}^{2_*-1} O(\varepsilon^{(N-2)/2}) + C_1|t|^{2_*-1}\|v\|_{L^\infty(\mathbb{R}^{N-1})} O(\varepsilon^{(N-2)/2}).
\end{aligned}$$

Since V is finite-dimensional and the eigenfunctions φ_i of (LP) are regular up to the boundary (see Lemma 1.1.4), there exists $C_2 > 0$, independent of v, such that $\|v\|_{L^\infty(\mathbb{R}^{N-1})} \leq C_2 |\!|v|\!|_{2_*}$. So, we infer from the above expression that

$$|A(u)| \leq |t| |\!|v|\!|_{2_*}^{2_*-1} O(\varepsilon^{(N-2)/2}) + |t|^{2_*-1} |\!|v|\!|_{2_*} O(\varepsilon^{(N-2)/2}). \tag{1.23}$$

From Young's inequality with exponents $s = 2_*/(2_* - 1)$ and $s' = 2_*$, we get

$$\begin{aligned}
\mathbf{|}v\mathbf{|}_{2_*}^{2_*-1}|t|O(\varepsilon^{(N-2)/2}) &\leq \frac{1}{4}\mathbf{|}v\mathbf{|}_{2_*}^{2_*} + C_3|t|^{2_*}O(\varepsilon^{(N-2)/2})^{2_*} \\
&= \frac{1}{4}\mathbf{|}v\mathbf{|}_{2_*}^{2_*} + C_3|t|^{2_*}O(\varepsilon^{N-1})
\end{aligned}$$

and

$$\mathbf{|}v\mathbf{|}_{2_*}|t|^{2_*-1}O(\varepsilon^{(N-2)/2}) \leq \frac{1}{4}\mathbf{|}v\mathbf{|}_{2_*}^{2_*} + C_4|t|^{2_*}O(\varepsilon^{(N-1)(N-2)/N}).$$

Replacing the above expressions in (1.23) and using $(N-1)(N-2)/N < (N-1)$, we obtain

$$|A(u)| \leq \frac{1}{2}\mathbf{|}v\mathbf{|}_{2_*}^{2_*} + |t|^{2_*}O(\varepsilon^{(N-1)(N-2)/N}).$$

Hence, using (1.17) we get

$$\begin{aligned}
1 &= \mathbf{|}u\mathbf{|}_{2_*}^{2_*} = A(u) + \mathbf{|}v\mathbf{|}_{2_*}^{2_*} + \mathbf{|}t\psi_\varepsilon\mathbf{|}_{2_*}^{2_*} \\
&\geq -\frac{1}{2}\mathbf{|}v\mathbf{|}_{2_*}^{2_*} - |t|^{2_*}O(\varepsilon^{(N-1)(N-2)/N}) + \mathbf{|}v\mathbf{|}_{2_*}^{2_*} + |t|^{2_*}\mathbf{|}\psi_\varepsilon\mathbf{|}_{2_*}^{2_*} \\
&= \frac{1}{2}\mathbf{|}v\mathbf{|}_{2_*}^{2_*} + |t|^{2_*}\left(B_N^{2_*/2} + O(1)\right),
\end{aligned}$$

and therefore $t = O(1)$ as $\varepsilon \to 0^+$.

For any given $u = v + t\psi_\varepsilon \in \Sigma_\varepsilon$, it follows from (1.5), (1.21) and $t = O(1)$ that

$$\begin{aligned}
\|u\|^2 - \lambda\|u\|_2^2 &\leq (\lambda_k - \lambda)\|v\|_2^2 + \|v\|_2 O(\varepsilon^{(N-2)/2}) + \|t\psi_\varepsilon\|^2 - \lambda\|t\psi_\varepsilon\|_2^2 \\
&\leq \frac{1}{4(\lambda-\lambda_k)}O(\varepsilon^{N-2}) + Q_\lambda(t\psi_\varepsilon)\mathbf{|}t\psi_\varepsilon\mathbf{|}_{2_*}^2,
\end{aligned} \tag{1.24}$$

where we have used, in the last inequality, that $as^2 + bs \leq -b^2/(4a)$ for $a < 0$ and $s \in \mathbb{R}$. Since $Q_\lambda(t\psi_\varepsilon) = Q_\lambda(\psi_\varepsilon)$, by (1.18) we obtain that

$$Q_\lambda(t\psi_\varepsilon) = S + \varepsilon^2\left(-E_N + o(1)\right). \tag{1.25}$$

In order to estimate $\mathbf{|}t\psi_\varepsilon\mathbf{|}_{2_*}^2$ we notice that, since the function $s \mapsto |s|^{2_*}$ is convex, we have that

$$\begin{aligned}
1 &= \int_{\mathbb{R}^{N-1}} K|v + t\psi_\varepsilon|^{2_*}dx' \\
&\geq \mathbf{|}t\psi_\varepsilon\mathbf{|}_{2_*}^{2_*} + 2_*\int_{\mathbb{R}^{N-1}} K|t\psi_\varepsilon|^{2^*-2}t\psi_\varepsilon v \\
&\geq \mathbf{|}t\psi_\varepsilon\mathbf{|}_{2_*}^{2_*} - 2_*\|v\|_{L^\infty(\mathbb{R}^{N-1})}|t|^{2_*-1}\mathbf{|}\psi_\varepsilon\mathbf{|}_{2_*-1}^{2_*-1}
\end{aligned}$$

and therefore we infer from (1.20) that

$$\mathbf{|}t\psi_\varepsilon\mathbf{|}_{2_*}^2 \leq \left(1 + \mathbf{|}v\mathbf{|}_{2_*}O(\varepsilon^{(N-2)/2})\right)^{2/2_*} = 1 + O(\varepsilon^{(N-2)/2}).$$

Thus, it follows from (1.25) that

$$Q_\lambda(t\psi_\varepsilon)\mathbf{|}t\psi_\varepsilon\mathbf{|}_{2_*}^2 \leq S + \varepsilon^2\left[-E_N + O(\varepsilon^{(N-6)/2}) + o(1)\right].$$

Using this inequality, $N \geq 7$ and (1.24) we obtain

$$\|u\|^2 - \lambda \|u\|_2^2 \leq S + \varepsilon^2 \left[-E_N + O(\varepsilon^{(N-6)/2}) + O(\varepsilon^{N-4}) + o(1) \right] < S,$$

for any $\varepsilon > 0$ sufficiently small. This establishes (1.22) and concludes the proof. $\square$

We are ready to present the first part of the proof of our main result.

Proof of Theorem A. Consider the decomposition $X = V \oplus W$, with V and W as in (1.4). Let $\varepsilon > 0$ and notice that the function $\phi = \psi_\varepsilon$ verifies all the conditions of Proposition 1.1.5. Hence, we can use Lemma 1.1.2 and Theorem 1.1.1 to obtain $(u_n) \subset X$ such that

$$I_\lambda(u_n) \to c, \qquad I'_\lambda(u_n) \to 0,$$

with the minimax level $c > 0$ defined in (1.3). We can pick $\varepsilon > 0$ so small in such way that Proposition 1.2.2 holds. Since $V \oplus \mathbb{R}e = V \oplus \mathbb{R}\psi_\varepsilon$, this last proposition and (1.3) imply that $c < S^{N-1}/(2(N-1))$. It follows from Proposition 1.1.6 that I_λ has a nonzero critical point $u \in X$. In order to prove that u changes its sign we consider $\varphi_1 > 0$ a first eigenfunction of (LP) and notice that, since $I'_\lambda(u)\varphi_1 = 0$, there holds

$$(\lambda_1 - \lambda) \int_{\mathbb{R}^N_+} K(x) u \varphi_1 dx = \int_{\mathbb{R}^{N-1}} K(x', 0) |u|^{2_*-2} u \varphi_1 dx'.$$

If $u \geq 0$ in $\mathbb{R}^N_+$, it follows from the above expression and $\int_{\mathbb{R}^N_+} K(x) u \varphi_1 dx > 0$ that $\lambda \leq \lambda_1$, which is not true. A similar argument discard $u \leq 0$ and therefore the proof is complete. $\square$

1.3 A Nehary type approach for $\lambda^*_N < \lambda < \lambda_1$

We present in this section some preliminary results for the proof of Theorem B. From now on, we suppose that $\lambda^*_N < \lambda < \lambda_1$. Hence, we can use [48, Theorem 1.5] to obtain a positive solution $u_0 \in X$ of the problem (P_λ). Since $|u_0|_{2_*} \neq 0$, the number $R > 0$ appearing in the definition of the function ψ_ε in the Section 2 can be chosen in such way that

$$\int_{\mathbb{R}^{N-1} \setminus B_{2R}(0)} K(x', 0) u_0^{2_*} dx' > 0. \qquad (1.26)$$

For any given $u \in X$, we define $u^+(x) := \max\{u(x), 0\}$, $u^- := u^+ - u$ and the sets

$$\mathcal{N}_\lambda := \{u \in X \setminus \{0\} \, : \, I'_\lambda(u)u = 0\}, \qquad \mathcal{M}_\lambda := \left\{u \in X \, : \, u^\pm \in \mathcal{N}_\lambda \right\}.$$

Notice that the Nehari manifold $\mathcal{N}_\lambda$ contains all the nonzero critical points of I_λ and $\mathcal{M}_\lambda \subset \mathcal{N}_\lambda$. The idea is to look for a critical point of I_λ which belongs to $\mathcal{M}_\lambda$ and therefore changes sign.

If $u \in \mathcal{N}_\lambda$, we have that

$$\|u\|^2 = \lambda \|u\|_2^2 + \|u\|_{2_*}^{2_*} \leq \frac{\lambda}{\lambda_1}\|u\|^2 + S^{-2_*/2}\|u\|^{2_*},$$

and therefore there exists $\gamma > 0$ such that

$$\|u\| \geq \gamma, \quad \forall\, u \in \mathcal{N}_\lambda. \tag{1.27}$$

Moreover, on $\mathcal{N}_\lambda$ we have that

$$I_\lambda(u) = \left(\frac{1}{2} - \frac{1}{2_*}\right)\|u\|^2 - \lambda\left(\frac{1}{2} - \frac{1}{2_*}\right)\|u\|_2^2 \geq \frac{1}{2(N-1)}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^2,$$

in such way that we can define the positive numbers

$$c_\lambda := \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u), \qquad d_\lambda := \inf_{u \in \mathcal{M}_\lambda} I_\lambda(u).$$

Although $\mathcal{M}_\lambda$ is not a differentiable manifold, we can adapt an argument of [90] for proving the following:

Lemma 1.3.1. *There exists a sequence $(u_n) \subset \mathcal{M}_\lambda$ such that $I_\lambda(u_n) \to d_\lambda$ and $I'_\lambda(u_n) \to 0$, as $n \to +\infty$.*

Proof. Using Ekeland's Variational Principle, we obtain a sequence $(u_n) \subset \mathcal{M}_\lambda$ such that

$$I_\lambda(u_n) \leq d_\lambda + \frac{1}{n}, \qquad I_\lambda(z) \geq I_\lambda(u_n) - \frac{1}{n}\|z - u_n\|, \quad \text{for all } z \in \mathcal{M}_\lambda. \tag{1.28}$$

Using (1.27) and recalling that I_λ is coercive over $\mathcal{M}_\lambda$ we obtain $\mu > \gamma > 0$ such that $\gamma \leq \|u_n^\pm\| \leq \mu$, for all $n \in \mathbb{N}$.

We claim that there exists $K = K(\lambda, \gamma, \mu) > 0$ such that $\|I'_\lambda(u_n)\| \leq K/n$, for all $n \in \mathbb{N}$. If this is true we obtain $I'_\lambda(u_n) \to 0$ and the result follows from (1.28).

In order to prove the claim, we fix $n \in \mathbb{N}$ and $v \in X$ such that $\|v\| \leq 1$ and notice that, since $(u_n - \delta v)^\pm \to u_n^\pm$ as $\delta \to 0$, then

$$\phi_{\delta,n}^\pm := (u_n - \delta v)^\pm \neq 0,$$

for any δ small. For simplicity, we drop the subscript n in what follows. The above expression and a direct computation shows that,

$$z_\delta := t_\delta^+ \phi_\delta^+ - t_\delta^- \phi_\delta^- \in \mathcal{M}_\lambda$$

where $t_\delta^\pm$ are given by

$$t_\delta^\pm = \left(\frac{\|(u - \delta v)^\pm\|^2 - \lambda\|(u - \delta v)^\pm\|_2^2}{\|(u - \delta v)^\pm\|_{2_*}^2}\right)^{1/(2_* - 2)}.$$

Setting $g_\pm(\delta) := t_\delta^\pm$, we obtain from the above expression that $g_\pm(0) = 1$ and

$$(2_* - 2)g'_\pm(0) = \frac{-2(u^\pm, v) + 2\lambda(u^\pm, v)_2 + 2_* \int_{\mathbb{R}^{N-1}} K(x', 0)(u^\pm)^{2_*-1} v \, dx'}{|u^\pm|_{2_*}^{2_*}},$$

where the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_2$ were defined in (1.19). Since $\|u^\pm\| \geq \gamma$, we have that

$$|u^\pm|_{2_*}^{2_*} = \|u^\pm\|^2 - \lambda\|u^\pm\|_2^2 \geq \gamma^2 \left(1 - \frac{\lambda}{\lambda_1}\right)$$

and therefore, using $\|u^\pm\| \leq \mu$ and Hölder's inequality we obtain

$$|g'_\pm(0)| \leq \frac{2\|u^\pm\|\|v\| + 2\lambda\|u^\pm\|_2\|v\|_2 + 2_*|u^\pm|_{2_*}^{2_*-1}|v|_{2_*}}{(2_* - 2)\gamma^2(\lambda_1 - \lambda)/\lambda_1} \leq C_1 \tag{1.29}$$

for

$$C_1 := \frac{2\mu + 2(\lambda/\lambda_1)\mu + 2_* S^{-2_*/2}\mu^{2_*-1}}{(2_* - 2)\gamma^2(\lambda_1 - \lambda)/\lambda_1}.$$

We now notice that

$$z_\delta - u = \left(t_\delta^+ - 1\right)\phi_\delta^+ - \left(t_\delta^- - 1\right)\phi_\delta^- - \delta v, \tag{1.30}$$

and therefore

$$\frac{\|z_\delta - u\|}{\delta} = \|g'_+(0)u^+ - g'_-(0)u^- - v\| + o_\delta(1), \tag{1.31}$$

as $\delta \to 0^+$. Thus, we can use (1.28) to get

$$I'_\lambda(u)(z_\delta - u) + o_\delta(\|z_\delta - u\|) = I_\lambda(z_\delta) - I_\lambda(u) \geq -\frac{1}{n}\|z_\delta - u\|.$$

It follows from (1.30) and $g_\pm(0) = 1$ that

$$\begin{aligned} I'_\lambda(u)v \;\; &\leq \;\; \left(\frac{g_+(\delta) - g_+(0)}{\delta}\right) I'_\lambda(u)\phi_\delta^+ - \left(\frac{g_-(\delta) - g_-(0)}{\delta}\right) I'_\lambda(u)\phi_\delta^- \\ &\quad + \frac{1}{n}\frac{\|z_\delta - u\|}{\delta} + \frac{o_\delta(\|z_\delta - u\|)}{\delta}. \end{aligned}$$

Passing to the limit, recalling that $I'_\lambda(u)\phi_\delta^\pm = I'_\lambda(u)(u^\pm + o_\delta(1)) = o_\delta(1)$, using (1.31), (1.29) and $\|u^\pm\| \leq \mu$ we conclude that

$$I'_\lambda(u)v \leq \frac{1}{n}\|g'_+(0)u^+ - g'_-(0)u^- - v\| \leq \frac{1}{n}\left(2C_1\mu + 1\right), \quad \forall\, v \in X,\ \|v\| \leq 1.$$

and therefore $\|I'_\lambda(u_n)\| \leq K/n$, for $K = 2C_1\mu + 1$. The lemma is proved. □

As in the first case, the energy functional satisfies a local compactness condition.

Proposition 1.3.2. *Suppose that $(u_n) \subset \mathcal{M}_\lambda$ satisfies*

$$\lim_{n\to+\infty} I_\lambda(u_n) = d < c_\lambda + \frac{1}{2(N-1)}S^{N-1}, \qquad \lim_{n\to\infty} I'_\lambda(u_n) = 0.$$

Then (u_n) has a convergent subsequence.

Proof. Since I_λ restricted to $\mathcal{N}_\lambda$ is coercive the sequence (u_n) is bounded in X. So, up to a subsequence, we may assume that $u_n \rightharpoonup u$ weakly X, $u_n^\pm \rightharpoonup u^\pm$ weakly in X and $u_n^\pm \to u^\pm$ strongly in $L^2_K(\mathbb{R}^N_+)$, for some $u \in X$. Arguing as in the proof of Proposition 1.1.6 we obtain $I'_\lambda(u) = 0$. Moreover, since $(u_n) \subset \mathcal{M}_\lambda$, we have that $o_n(1) = I'_\lambda(u_n^+)u_n^+ - I'_\lambda(u)u^+$ and therefore the above convergences imply that

$$\lim_{n\to+\infty} \|u_n^+ - u^+\|^2 = l, \qquad \lim_{n\to+\infty} |u_n^+ - u^+|_{2_*}^{2_*} = l, \tag{1.32}$$

for some $l \geq 0$.

We shall prove that $l = 0$ and therefore $u_n^+ \to u^+$ in X. Suppose, by contradiction, that $l > 0$. Passing the inequality $\|u_n^+ - u^+\|^2 \leq S^{-1}|u_n^+ - u^+|_{2_*}^2$ to the limit we get $l \geq S^{N-1}$. On the other hand, the convergences of (u_n^+) just mentioned and Brezis-Lieb's lemma [18] implies that

$$I_\lambda(u_n^+) = I_\lambda(u_n^+ - u^+) + I_\lambda(u^+) + o_n(1). \tag{1.33}$$

However, by (1.32) and the strong convergence we get

$$I_\lambda(u_n^+ - u^+) = \frac{1}{2}\|u_n^+ - u^+\|^2 - \frac{1}{2_*}|u_n^+ - u^+|_{2_*}^{2_*} + o_n(1) = \frac{1}{2(N-1)}l + o_n(1),$$

and therefore (1.33) implies that

$$I_\lambda(u_n^+) = \frac{1}{2(N-1)}l + I_\lambda(u^+) + o_n(1).$$

Recalling that $u_n^- \in \mathcal{N}_\lambda$, we conclude that $c_\lambda \leq I_\lambda(u_n^-)$. Also, since $I'_\lambda(u)u^+ = 0$, we have that $I_\lambda(u^+) \geq 0$. So, we can use the above inequality and $l \geq S^{N-1}$ to get

$$d + o_n(1) = I_\lambda(u_n) = I_\lambda(u_n^-) + I_\lambda(u_n^+) \geq c_\lambda + \frac{1}{2(N-1)}S^{N-1} + o_n(1).$$

Passing to the limit we obtain a contradiction. Hence $l = 0$ and $u_n^+ \to u^+$ strongly in X. The same argument shows that $u_n^- \to u^-$ strongly in X and the proposition is proved. $\square$

1.4 Proof of Theorem B

Since we already have a Palais-Smale sequence at level d_λ, we need only to show that d_λ belongs to the compactness range of the functional I_λ. We shall use the following intersection property.

Lemma 1.4.1. *There exists α_*, $\beta_* \in \mathbb{R}$ such that $(\alpha_* u_0 + \beta_* \psi_\varepsilon) \in \mathcal{M}_\lambda$.*

Proof. Define

$$J(u) := \frac{|u|_{2_*}^{2_*}}{\|u\|^2 - \lambda\|u\|_2^2}, \quad \forall\, u \in X \setminus \{0\},$$

and $J(0) = 0$. From $\lambda < \lambda_1$ and the continuous embedding $X \hookrightarrow L_K^{2_*}(\mathbb{R}^{N-1})$ we obtain $0 \le J(u) \le C_c\|u\|^{2_*-2}$, and therefore J is continuous.

We now set

$$\sigma(r,s,t) := rt\,[(1-s)u_0 - s\psi_\varepsilon]\,, \quad \forall\, r \ge 0,\ s,\ t \in [0,1].$$

and

$$\Gamma(r) := \inf_{s\in[0,1]} J(\sigma(r,s,1)), \quad \forall\, r > 0.$$

If $\Gamma(1) = 0$, then there exists $s_0 \in [0,1]$ such that $J(\sigma(1,s_0,1)) = 0$, that is, $|(1-s_0)u_0 - s_0\psi_\varepsilon|_{2_*}^{2_*} = 0$. Since ψ_ε is positive in $B_R(0) \cap \mathbb{R}^{N-1}$ and (1.26) holds, we have that $s_0 \in (0,1)$. Thus, recalling that $\psi_\varepsilon \equiv 0$ outside $B_{2R}(0) \cap \mathbb{R}^{N-1}$, we obtain

$$0 = |(1-s_0)u_0 - s_0\psi_\varepsilon|_{2_*}^{2_*} \ge (1-s_0)^{2_*} \int_{\mathbb{R}^{N-1}\setminus B_{2R}(0)} K(x',0)u_0^{2_*}\,dx',$$

which contradicts (1.26). Hence, $\Gamma(1) > 0$ and we infer from $J(\sigma(r,s,1)) = r^{2_*-2}J(\sigma(1,s,1)) \ge r^{2_*-2}\Gamma(1)$ that

$$\lim_{r\to+\infty} \Gamma(r) = +\infty.$$

Let $r_0 > 0$ be such that

$$J(\sigma(r_0,s,1)) \ge \Gamma(r_0) > 2, \quad \forall\, s \in [0,1], \tag{1.34}$$

and define the functions $f,\ g : [0,1]\times[0,1] \to \mathbb{R}$ as

$$f(s,t) := J(\sigma^-(r_0,s,t)) - J(\sigma^+(r_0,s,t))$$

and

$$g(s,t) := J(\sigma^+(r_0,s,t)) + J(\sigma^-(r_0,s,t)) - 2.$$

Since $\sigma(r_0,0,t) = r_0tu_0 \ge 0$ and $\sigma(r_0,1,t) = -r_0t\psi_\varepsilon \le 0$, it follows that

$$f(0,t) = -J(\sigma^-(r_0,1,t)) \le 0, \qquad f(1,t) = J(\sigma^+(r_0,0,t)) \ge 0,$$

for any $t \in [0,1]$. Moreover, for any $s \in [0,1]$,

$$g(s,0) = -2 \le 0, \qquad g(s,1) = J(\sigma^+(r_0,s,1)) + J(\sigma^-(r_0,s,1)) - 2 \ge 0,$$

where we have used $J(u^+) + J(u^-) \ge J(u)$ and (1.34) in the last inequality.

Using the above inequalities and Miranda's Theorem [70] we obtain $s_0,\ t_0 \in [0,1]$ such that $f(s_0,t_0) = 0 = g(s_0,t_0)$ and so

$$J(\sigma^+(r_0,s_0,t_0)) = 1 = J(\sigma^-(r_0,s_0,t_0)).$$

Consequently, $I'_\lambda(\sigma^\pm(r_0,s_0,t_0))\sigma^\pm(r_0,s_0,t_0) = 0$. Since $J(0) = 0$, we also have that $\sigma^\pm(r_0,s_0,t_0) \ne 0$, and therefore the lemma holds for $\alpha_* := r_0t_0(1-s_0)$ and $\beta_* := r_0t_0s_0$. □

The two next results are of technical nature and it will be useful to estimate d_λ.

Lemma 1.4.2. *If τ_1, $\tau_2 > 1$, then there exists $A_1 = A_1(u_0, R, \tau_1, \tau_2) > 0$ such that*

$$\left| \|\alpha u_0 + \beta\psi_\varepsilon\|_{2_*}^{2_*} - \|\alpha u_0\|_{2_*}^{2_*} - \|\beta\psi_\varepsilon\|_{2_*}^{2_*} \right| \leq A_1 \left(|\alpha|^{2_*-1} \|\beta\psi_\varepsilon\|_{\tau_1} + |\alpha| \|\beta\psi_\varepsilon\|_{(2_*-1)\tau_2}^{2_*-1} \right),$$

for any α, $\beta \in \mathbb{R}$.

Proof. For simplicity, we write only K to denote $K(x', 0)$. If we call $\Psi(\alpha, \beta)$ the term into modulus in the inequality above, we have that

$$\begin{aligned} \Psi(\alpha, \beta) &= \int_{\mathbb{R}^{N-1}} K \left(\int_0^1 \frac{d}{ds} \left[|s\alpha u_0 + \beta\psi_\varepsilon|^{2_*} - |s\alpha u_0|^{2_*} \right] ds \right) dx' \\ &= 2_* \int_{\mathbb{R}^{N-1}} K \left(\int_0^1 [g(1) - g(0)]\, \alpha u_0 \, ds \right) dx' \end{aligned}$$

for $g(t) := |s\alpha u_0 + t\beta\psi_\varepsilon|^{2_*-2}(s\alpha u_0 + t\beta\psi_\varepsilon)$. From the Mean Value Theorem we obtain $\theta(x, s) \in (0, 1)$ such that

$$\Psi(\alpha, \beta) = 2_*(2_* - 1) \int_{\mathbb{R}^{N-1}} K \left(\int_0^1 \left[|s\alpha u_0 + \theta\beta\psi_\varepsilon|^{2_*-2} \alpha u_0 \beta\psi_\varepsilon \right] ds \right) dx'.$$

Since s, $\theta \in [0, 1]$, we obtain

$$|\Psi(\alpha, \beta)| \leq C_1 \int_{\mathbb{R}^{N-1}} K |\alpha u_0|^{2_*-1} |\beta\psi_\varepsilon| \, dx' + C_1 \int_{\mathbb{R}^{N-1}} K |\alpha u_0| |\beta\psi_\varepsilon|^{2_*-1} \, dx'. \qquad (1.35)$$

We now notice that the positive solution $u_0 \in X$ of problem (P_λ) given in [48, Theorem 1.5] belongs to $C^2(\mathbb{R}^N_+)$. Although regularity up to the boundary is a more complicated issue, we can adapt the proof of Brezis-Kato's theorem [17] presented by Struwe [88, Lemma B.3] (see also [1, Lemma 4.1] for the normal derivative version) to conclude that $u_0 \in L^\tau_{loc}(\mathbb{R}^{N-1})$, for any $\tau \geq 1$. So, if we set $\Omega := \{x' \in \mathbb{R}^{N-1} : |x'| < 2R\}$ and recall that ψ_ε vanishes outside $B_{2R}(0)$, we can use Hölder's inequality to get

$$\int_{\mathbb{R}^{N-1}} K |\alpha u_0|^{2_*-1} |\beta\psi_\varepsilon| \, dx' \leq |\alpha|^{2_*-1} |\beta| \|u_0\|_{L_K^{(2_*-1)\tau_1'}(\Omega)}^{2_*-1} \|\psi_\varepsilon\|_{\tau_1}$$

and

$$\int_{\mathbb{R}^{N-1}} K |\alpha u_0| |\beta\psi_\varepsilon|^{2_*-1} \, dx' \leq |\alpha| |\beta|^{2_*-1} \|u_0\|_{L_K^{\tau_2'}(\Omega)}^{\tau_2'} \|\psi_\varepsilon\|_{(2_*-1)\tau_2}^{2_*-1},$$

where $\|u_0\|_{L^r_K(\Omega)} := \left(\int_\Omega K(x', 0) |u_0|^r dx' \right)^{1/r}$, for $r > 1$. So, it is sufficient to define

$$A_1 := C_1 \left(\|u_0\|_{L_K^{(2_*-1)\tau_1'}(\Omega)}^{2_*-1} + \|u_0\|_{L_K^{\tau_2'}(\Omega)}^{\tau_2'} \right)$$

and use the two above inequalities together with (1.35). □

Lemma 1.4.3. *If τ_1, $\tau_2 > 1$, then there exists $A_i = A_i(u_0, R, \tau_1, \tau_2, N) > 0$, $i = 2, 3$, such that*

$$\|\alpha u_0 + \beta\psi_\varepsilon\|_{2_*}^{2_*} \geq \frac{1}{3}|\alpha|^{2_*}\|u_0\|_{2_*}^{2_*} + |\beta|^{2_*}\left(\|\psi_\varepsilon\|_{2_*}^{2_*} - A_2\|\psi_\varepsilon\|_{\tau_1}^{2_*} - A_3\|\psi_\varepsilon\|_{(2_*-1)\tau_2}^{2_*}\right),$$

for any α, $\beta \in \mathbb{R}$.

Proof. According to last result, we have that

$$\|\alpha u_0 + \beta\psi_\varepsilon\|_{2_*}^{2_*} \geq \frac{1}{3}|\alpha|^{2_*}\|u_0\|_{2_*}^{2_*} + |\beta|^{2_*}\|\psi_\varepsilon\|_{2_*}^{2_*} + f(|\alpha|) + g(|\alpha|), \tag{1.36}$$

for f, $g : [0, +\infty) \to \mathbb{R}$ given by

$$f(s) := \frac{1}{3}\|u_0\|_{2_*}^{2_*}s^{2_*} - A_1\|\beta\psi_\varepsilon\|_{\tau_1}s^{2_*-1},$$

and

$$g(s) := \frac{1}{3}\|u_0\|_{2_*}^{2_*}s^{2_*} - A_1\|\beta\psi_\varepsilon\|_{(2_*-1)\tau_2}^{2_*-1}s.$$

The function f attains its minimum at the point

$$s_0 := \frac{3(2_* - 1)}{2_*}\frac{A_1}{\|u_0\|_{2_*}^{2_*}}\|\beta\psi_\varepsilon\|_{\tau_1},$$

and therefore

$$f(|\alpha|) \geq f(s_0) = -A_2|\beta|^{2_*}\|\psi_\varepsilon\|_{\tau_1}^{2_*}, \quad \forall\, \alpha \in \mathbb{R},$$

with $A_2 := A_2(u_0, R, \tau_1, \tau_2, N) > 0$. Analogously, there exists $A_3 > 0$ such that

$$g(|\alpha|) \geq -A_3|\beta|^{2_*}\|\psi_\varepsilon\|_{(2_*-1)\tau_2}^{2_*}, \quad \forall\, \alpha \in \mathbb{R}.$$

The lemma follows from the two above inequalities and (1.36). □

We are ready to prove our second main theorem.

Proof of Theorem B. Let $\lambda_N^* < \lambda < \lambda_1$. Invoking Lemma 1.3.1 we obtain $(u_n) \subset \mathcal{M}_\lambda$ such that $I_\lambda(u_n) \to d_\lambda$ and $I'_\lambda(u_n) \to 0$, as $n \to +\infty$. We claim that

$$d_\lambda < c_\lambda + \frac{1}{2(N-1)}S^{N-1}. \tag{1.37}$$

If this is true, it follows from Proposition 1.3.2 that, along a subsequence, $u_n \to u$ strongly in X. Since $\mathcal{M}_\lambda$ is closed, we have that $u \in \mathcal{M}_\lambda$, from which we conclude that $u^\pm \neq 0$. Moreover, recalling that $\mathcal{M}_\lambda \subset \mathcal{N}_\lambda$, we conclude that $I'_\lambda(u) = 0$ and therefore $u \in X$ is a sign-changing solution for (P_λ).

For proving (1.37) we first notice that, according to Lemma 1.4.1, there exists α_*, $\beta_* \in \mathbb{R}$ such that $(\alpha_* u_0 + \beta_*\psi_\varepsilon) \in \mathcal{M}_\lambda$. So, it is sufficient to show that, for some $\varepsilon > 0$,

$$\sup_{\alpha,\beta\in\mathbb{R}} I_\lambda(\alpha u_0 + \beta\psi_\varepsilon) < c_\lambda + \frac{1}{2(N-1)}S^{N-1}.$$

Arguing as in the proof of Lemma 1.4.1, we can check that $W := \text{span}\{u_0, \psi_\varepsilon\}$ is a 2-dimensional subspace. Moreover, using (1.26) and the compact support of ψ_ε, we conclude that $|\alpha u_0 + \beta\psi_\varepsilon|_{2_*} = 0$ if, and only if, $(\alpha, \beta) = (0, 0)$. So, the function $(\alpha, \beta) \mapsto |\alpha u_0 + \beta\psi_\varepsilon|_{2_*}$ defines a norm in W. From the equivalence between norms in finite-dimensional subspaces, we get

$$\lim_{(|\alpha|+|\beta|)\to+\infty} I_\lambda(\alpha u_0 + \beta\psi_\varepsilon) = -\infty,$$

and therefore we can restrict our attention to points $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$|\alpha u_0 + \beta\psi_\varepsilon|_{2_*}^{2_*} \leq C_1,$$

for some $C_1 > 0$ large enough.

Using Lemma 1.4.3, we get

$$C_1 \geq \frac{1}{3}|\alpha|^{2_*}|u_0|_{2_*}^{2_*} + |\beta|^{2_*}\left(|\psi_\varepsilon|_{2_*}^{2_*} - A_2|\psi_\varepsilon|_{\tau_1}^{2_*} - A_3|\psi_\varepsilon|_{(2_*-1)\tau_2}^{2_*}\right), \tag{1.38}$$

If we pick $(N-1)/(N-2) < 1 < \tau_1 < 2(N-1)/(N+2)$, it follows from (1.20) that

$$|\psi_\varepsilon|_{\tau_1} = O(\varepsilon^{2+\nu_1}), \qquad \nu_1 := \frac{2(N-1) - \tau_1(N+2)}{2\tau_1} > 0, \tag{1.39}$$

as $\varepsilon \to 0^+$. Moreover, for

$$\frac{N-1}{N} < 1 < \tau_2 < \frac{2(N-1)}{N+4} < \frac{2(N-1)}{N} < 2_*,$$

we can apply (1.20) with $\tau = (2_* - 1)\tau_2$ to get

$$|\psi_\varepsilon|_{(2_*-1)\tau_2}^{2_*-1} = O(\varepsilon^{2+\nu_2}), \quad \nu_2 := \frac{2(N-1) - \tau_2(N+4)}{2\tau_2} > 0. \tag{1.40}$$

From the above inequalities we conclude that $|\psi_\varepsilon|_{\tau_1}^{2_*} = o(1)$ and $|\psi_\varepsilon|_{(2_*-1)\tau_2}^{2_*} = o(1)$, and therefore it follows from (1.38) and (1.17) that

$$C_1 \geq \frac{1}{3}|\alpha|^{2_*}|u_0|_{2_*}^{2_*} + |\beta|^{2_*}\left(B_N^{2_*/2} + o(1)\right).$$

Since $B_N > 0$, we conclude that $\alpha = O(1)$ and $\beta = O(1)$. It is worth mentioning that the above choices for τ_1 and τ_2 are possible because $N \geq 7$.

Notice that, since $I_\lambda'(u_0)\psi_\varepsilon = 0$, then

$$\int_{\mathbb{R}^N_+} K(x)\left(\nabla u_0 \cdot \nabla\psi_\varepsilon\right)dx - \lambda\int_{\mathbb{R}^N_+} K(x)u_0\psi_\varepsilon dx = \int_{\mathbb{R}^{N-1}} K(x', 0)u_0^{2_*-1}\psi_\varepsilon dx'.$$

Thus,

$$I_\lambda(\alpha u_0 + \beta\psi_\varepsilon) \leq I_\lambda(\alpha u_0) + \frac{\beta^2}{2}\left(\|\psi_\varepsilon\|^2 - \lambda\|\psi_\varepsilon\|_2^2\right) - \frac{|\beta|^{2_*}}{2_*}|\psi_\varepsilon|_{2_*}^{2_*} + \Phi(\varepsilon, \alpha, \beta) \tag{1.41}$$

with

$$\Phi(\varepsilon,\alpha,\beta) := A_1 O(1)\left(|\psi_\varepsilon|_{\tau_1} + |\psi_\varepsilon|^{2_*-1}_{(2_*-1)\tau_2}\right) + \alpha\beta \int_{\mathbb{R}^{N-1}} K(x',0) u_0^{2_*-1}\psi_\varepsilon\, dx',$$

with the number $A_1 > 0$ given by Lemma 1.4.2 and we have used that α and β remain bounded as $\varepsilon \to 0^+$. Arguing as in the proof of Lemma 1.4.2 and recalling that $\alpha = O(1)$ and $\beta = O(1)$ as $\varepsilon \to 0^+$, we get

$$\alpha\beta \int_{\mathbb{R}^{N-1}} K(x',0) u_0^{2_*-1}\psi_\varepsilon\, dx' \le C_2 |u_0|^{2_*-1}_{L_K^{(2_*-1)\tau_1'}(\Omega)} |\psi_\varepsilon|_{\tau_1} = O(\varepsilon^{2+\nu_1}),$$

and therefore we can use (1.39), (1.40) and $\nu_1, \nu_2 > 0$ to conclude that

$$\Phi(\varepsilon,\alpha,\beta) = O(\varepsilon^{2+\nu_1}) + O(\varepsilon^{2+\nu_2}) = o(\varepsilon^2). \tag{1.42}$$

Since $\lambda > \lambda_1$, a straightforward computation shows that the function

$$f(\beta) := \frac{\beta^2}{2}\left(\|\psi_\varepsilon\|^2 - \lambda\|\psi_\varepsilon\|_2^2\right) - \frac{|\beta|^{2_*}}{2_*}|\psi_\varepsilon|^{2_*}_{2_*}, \quad \forall\, \beta \in \mathbb{R},$$

is such that

$$f(\beta) \le \frac{1}{2(N-1)}\left[\frac{\|\psi_\varepsilon\|^2 - \lambda\|\psi_\varepsilon\|_2^2}{|\psi_\varepsilon|^2_{2_*}}\right]^{N-1} = \frac{1}{2(N-1)} Q_\lambda(\psi_\varepsilon)^{N-1},$$

for any $\beta \in \mathbb{R}$. Moreover, using (1.18) and the Mean Value Theorem, we obtain $\theta \in (0,1)$ such that

$$\begin{aligned} Q_\lambda(\psi_\varepsilon)^{N-1} &\le \left[S + \varepsilon^2(-E_N + o(1))\right]^{N-1} \\ &= S^{N-1} + (N-1)\varepsilon^2\left[-E_N + o(1)\right]\left[S + \theta\varepsilon^2(-E_N + o(1))\right]^{N-2} \end{aligned}$$

and therefore

$$f(\beta) \le \frac{1}{2(N-1)} S^{N-1} + \varepsilon^2\left[-\frac{E_N S}{2} + o(1)\right],$$

as $\varepsilon \to 0^+$. Since $I_\lambda(\alpha u_0) \le I_\lambda(u_0) = c_\lambda$, for any $\alpha \in \mathbb{R}$, and $E_N > 0$, we can replace the above inequality and (1.42) in (1.41) to get

$$\begin{aligned} \sup_{\alpha,\beta\in\mathbb{R}} I_\lambda(\alpha u_0 + \beta\psi_\varepsilon) &\le c_\lambda + \frac{1}{2(N-1)} S^{N-1} + \varepsilon^2\left[-\frac{E_N S}{2} + o(1)\right] \\ &< c_\lambda + \frac{1}{2(N-1)} S^{N-1}, \end{aligned}$$

for any $\varepsilon > 0$ small. This finishes the proof of the second case of Theorem B. □

CHAPTER 2

Multiplicity of solutions for a concave-convex type problem

Let $\mathbb{R}^N_+ := \left\{(x', x_N) : x' \in \mathbb{R}^{N-1},\ x_N > 0\right\}$ be the upper half-space and consider the following heat equation with nonlinear boundary condition

$$v_t - \Delta v = 0, \ \text{ in } \mathbb{R}^N_+ \times (0, +\infty), \qquad \frac{\partial v}{\partial \eta} = |v|^{p-2} v, \ \text{ on } \partial\mathbb{R}^N_+ \times (0, +\infty),$$

where $2 < p \leq 2_* := 2(N-1)/(N-2)$ and $\partial u/\partial \eta$ denotes the partial outward normal derivative. Solutions of type

$$v(x,t) = t^{-\lambda} u(t^{-1/2} x),$$

with $\lambda = 1/(2(p-2)) > 0$, are called self-similar solutions. Besides preserve the PDE scaling, they carry simultaneously information about small and large scale behaviors, providing also qualitative properties like global existence, blow-up and asymptotic behavior (see e.g. [57, 58, 69]).

An easy computation shows that the profile u above needs to satisfy

$$-\Delta u - \frac{1}{2}\left(x \cdot \nabla u\right) = \lambda u, \ \text{ in } \mathbb{R}^N_+, \qquad \frac{\partial u}{\partial \eta} = |u|^{p-2} u, \ \text{ on } \partial\mathbb{R}^N_+.$$

Such problem was recently considered in [47, 48], where existence results were presented according to the range of λ. Actually, these papers were strongly motivated by the vast literature concerning the version of the problem for the whole space $\mathbb{R}^N$ with different types of nonlinearities. We could quote [9, 20, 24, 51, 52, 72, 76] and their references for results about existence, nonexistence, multiplicity, decay rate, among other properties of solutions.

Here, we are going to study the effect of replacing the linear term λu in the above equation by a sublinear indefinite function. Our main motivation comes from the problem

$$-\Delta u = \lambda a(x)|u|^{q-2}u + b(x)|u|^{r-2}u, \text{ in } \Omega, \qquad u = 0, \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 < q < 2 < r \leq 2N/(N-2)$ and the potentials a, b satisfy natural regularity conditions. In the celebrated paper [4], Ambrosetti, Brezis and Cerami considered the constant case $a \equiv 1$, $b \equiv 1$ and obtained $\Lambda > 0$ such that the problem admits at least two positive solutions whenever $\lambda \in (0, \Lambda)$, at least one if $\lambda = \Lambda$ and no solution if $\lambda > \Lambda$. Variable and indefinite potentials were considered in [35] (see also [37]). In [55], the authors obtained for

$$-\Delta u + u = |u|^{r-2}u, \text{ in } \Omega, \qquad \frac{\partial u}{\partial \eta} = \lambda |u|^{q-2}u, \text{ on } \partial\Omega,$$

results which are analogous to that of [4]. Some of their results were extended in [94] for the indefinite potential case (see also [82]). All the aforementioned works belong to a huge class of problems which are now called of concave-convex type.

In this chapter, we deal with the concave-convex boundary value problem

$$(P_2) \qquad \begin{cases} -\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda a(x)|u|^{q-2}u, & x \in \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \eta} = b(x')|u|^{p-2}u, & x' \in \mathbb{R}^{N-1}, \end{cases}$$

where $N \geq 3$, $\lambda > 0$ is a parameter, $1 < q < 2 < p \leq 2_*$ and we have identified $\partial\mathbb{R}^N_+ \simeq \mathbb{R}^{N-1}$. For describing the assumptions on the potentials we need first to present the functional space to deal with (P_2). This is done in what follows.

As we shall see, the function $K(x) = \exp(|x|^2/4)$ is closely related with the appropriated space to look for solutions of our problem. In order to present the assumptions on the coefficients, we denote for any $2 \leq r \leq 2^*$ the weighted Lebesgue space

$$L^r_K(\mathbb{R}^N_+) = \left\{ u \in L^r(\mathbb{R}^N_+) \, : \, \|u\|_r = \left(\int_{\mathbb{R}^N_+} K(x)|u|^r dx \right)^{1/r} < \infty \right\}. \tag{2.1}$$

If we denote by $r' = r/(r-1)$ the conjugated exponent of $r > 1$, we can present the basic hypothesis on a, b in the following way:

(a_0) $a \in L^{\sigma_q}_K(\mathbb{R}^N_+) \cap L^{N/2}_{loc}(\mathbb{R}^N_+)$ for some

$$\left(\frac{p}{q}\right)' < \sigma_q \leq \left(\frac{2}{q}\right)';$$

(b_0) $b \in L^\infty(\mathbb{R}^{N-1})$.

Since they can change it sign, we may define the sets

$$\Omega_a^+ := \{x \in \mathbb{R}^N_+ : \; a(x) > 0\}, \quad \Omega_b^+ := \{x' \in \mathbb{R}^{N-1} : \; b(x') > 0\}.$$

In our first results we obtain existence of two nonnegative solutions when roughly speaking the closure of the set Ω_a^+ intersects Ω_b^+ and the parameter $\lambda > 0$ approaches zero. More specifically, denoting by $B_\delta(0)$ the open ball centered at origin with radii $\delta > 0$, we prove the following:

Theorem C. *Suppose that a, b satisfy (a_0) and (b_0). If $1 < q < 2 < p < 2_*$, then there exists $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, problem (P_2) has at least two nonnegative nonzero solutions provided*

(ab) *there exists $\delta > 0$ such that*

$$(B_\delta(0) \cap \mathbb{R}^N_+) \subset \Omega_a^+, \quad (B_\delta(0) \cap \partial\mathbb{R}^N_+) \subset \Omega_b^+.$$

In our second result, we consider the critical case by adding a flatness condition on the potential b:

Theorem D. *Suppose that $N \geq 7$, $p = 2_*$ and the other conditions of Theorem C are verified. Then there exists $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, problem (P_2) has at least two nonnegative nonzero solutions provided*

(b_1) *there exist $M > 0$ and $\sigma > N - 1$ such that*

$$\|b\|_\infty - b(x') \leq M|x'|^\sigma, \quad \text{for a.e. } x' \in B_\delta(0) \cap \partial\mathbb{R}^N_+.$$

The first solution will be obtained with a standard minimization argument while the second one requires finer arguments. This is specially true when $p = 2_*$, since the trace embedding we are going to use fails to be compact. Two points are important to overcome this difficulty: a trick regularization study of the first solution on the boundary and the application of an idea of Brezis and Nirenberg [19], together with fine estimates of a modification of the *instanton functions* founded by Escobar [41] and Beckner [13].

In the second part of the chapter, we take advantage of the symmetry to get more and more solutions (with no prescribed sign). Unfortunately, in this case we do not assume that both the potentials are indefinite.

We prove the following:

Theorem E. *Suppose that $1 < q < 2$, $a \geq 0$ and $b \not\equiv 0$ satisfiy (a_0) and (b_0), respectively. Then problem (P_2) has infinitely many solutions in each of the following cases:*

1. $2 < p < 2_*$ and $\lambda > 0$;
2. $p = 2_*$, $b \leq 0$ and $\lambda > 0$;
3. $p = 2_*$ and $\lambda > 0$ is small.

Theorem F. *Suppose that $1 < q < 2 < p < 2_*$, $a \not\equiv 0$ and $b \geq 0$ satisfy (a_0) and (b_0), respectively. Then, for any $\lambda > 0$, problem (P_2) has infinitely many solutions.*

The above theorems will be proved as application of suitable versions of the Symmetric Mountain Pass Theorem [3]. They were proved by Tonkes in the paper [92] which strongly motivated the second part of our work (see also [11, 12] for some earlier results). In the critical case, when $b \leq 0$, the boundary term is related with a semi-norm and therefore we can argue as in the subcritical case. When $p = 2_*$ and b is indefinite in sign, we borrow an argument from [10]. It can be proved that, when $b \leq 0$, the energy of the solutions given by Theorem E are negative and goes to zero. On the other hand, in Theorem F, this energy goes to infinity, the same occurring with the norm of the solutions.

The chapter is organized as follows: in Section 2.1 we present the variational framework to deal with our problem and obtained the first solution; in Section 2.2 we finish the proof of the first two theorems; Section 2.3 is devoted to the proof of Theorems E and F.

2.1 Variational setting

Throughout the chapter we assume that $1 < q < 2 < p \leq 2_*$ and conditions (a_0), (b_0) hold. Following Escobedo and Kavian [44], we first set

$$K(x) := \exp(|x|^2/4), \quad x \in \mathbb{R}^N_+,$$

and notice that the first equation in (P_2) is equivalent to

$$-\text{div}(K(x)\nabla u) = \lambda K(x)a(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N_+.$$

Hence, it is natural looking for solutions in the space X defined as the closure of $C_c^\infty(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx\right)^{1/2}.$$

Recall the definition of $L^r_K(\mathbb{R}^N_+)$ in (2.1) and define, for each $2 \leq s \leq 2_*$, the space

$$L^s_K(\mathbb{R}^{N-1}) := \left\{ u \in L^s(\mathbb{R}^{N-1}) : |\!|u|\!|_s := \left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^s dx'\right)^{1/s} < \infty \right\}.$$

We collect in the next proposition the abstract results proved in [47, 48].

Proposition 2.1.1. *For any $r \in [2, 2^*)$ and $s \in [2, 2_*)$, the embeddings $X \hookrightarrow L^r_K(\mathbb{R}^N_+)$ and $X \hookrightarrow L^s_K(\mathbb{R}^{N-1})$ are compact. In the critical cases $r = 2^*$ and $s = 2_*$, we have only continuous embeddings.*

Given $2 \leq r \leq 2^*$ and $2 \leq s \leq 2_*$, we can use the above result to define the following embedding constants:

$$S_r := \inf_{u \in X/\{0\}} \frac{\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N_+} K(x)|u|^r dx\right)^{2/r}},$$

$$S_{s,\partial} := \inf_{u \in X/\{0\}} \frac{\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx}{\left(\int_{\mathbb{R}^{N-1}} K(x',0)|u|^s dx'\right)^{2/s}}.$$

By condition (a_0), we have that $2 \leq q\sigma'_q < 2^*$, and therefore we can use Hölder's inequality to get

$$\left|\int_{\mathbb{R}^N_+} K(x)a(x)(u^+)^q dx\right| \leq \|a\|_{\sigma_q} \left(\int_{\mathbb{R}^N_+} K(x)|u|^{q\sigma'_q} dx\right)^{1/\sigma'_q} < +\infty, \qquad (2.2)$$

for any $u \in X$. Hence, condition (b_0) and standard arguments show that the functional

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N_+} K(x)a(x)(u^+)^q\, dx - \frac{1}{p}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')(u^+)^p\, dx'$$

belongs to $C^1(X, \mathbb{R})$. Here and in what follows we will denote $u^+ := \max\{u, 0\}$ and $u^- := u^+ - u$. If $I'_\lambda(u) = 0$, then we can compute $0 = I'_\lambda(u)u^-$ to conclude that $\|u^-\| = 0$, and therefore the critical points of I_λ are nonnegative solutions of problem (P_2).

The first step in the proof of Theorem C is the study of I_λ near origin.

Lemma 2.1.2. *There exist $\rho = \rho(q, p, |b|_\infty) > 0$, $\alpha = \alpha(\rho) > 0$ and $\lambda_* = \lambda_*(q, \rho) > 0$ such that $I_\lambda(u) \geq \alpha > 0$, for any $u \in X$ verifying $\|u\| = \rho$, and $\lambda \in (0, \lambda_*)$.*

Proof. By using (2.2) and Proposition 2.1.1, we get

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\|a\|_{\sigma_q}\|u\|^q_{q\sigma'_q} - \frac{1}{p}|b|_\infty |u|^p_p \\ &= \frac{\|u\|^q}{2}\left[\|u\|^{2-q} - \frac{2}{p}S^{-p/2}_{p,\partial}|b|_\infty\|u\|^{p-q} - \lambda\frac{2}{q}S^{-q/2}_{q\sigma'_q}\|a\|_{\sigma_q}\right]. \end{aligned}$$

The function $g : (0, \infty) \to \mathbb{R}$ given by $g(t) := t^{2-q} - C_1 t^{p-q}$, with $C_1 := 2S^{-p/2}_{p,\partial}|b|_\infty/p$, achieves its maximum value at

$$\rho := \left[\frac{(2-q)}{C_1(p-q)}\right]^{1/(p-2)}.$$

Thus, for any $u \in X$ satisfying $\|u\| = \rho$, there holds

$$I_\lambda(u) \geq \frac{\rho^q}{2}\left(g(\rho) - \lambda \frac{2}{q} S_{q\sigma_q'}^{-q/2}\|a\|_{\sigma_q}\right) \geq \frac{\rho^q}{2}\frac{g(\rho)}{2} = \alpha > 0,$$

whenever

$$\lambda < \lambda_* := \frac{q S_{q\sigma_q'}^{q/2}}{4\|a\|_{\sigma_q}} g(\rho),$$

and the result follows. □

We obtain in the next proposition our first solution.

Proposition 2.1.3. *Let λ_*, $\rho > 0$ be as in the above lemma. For any $\lambda \in (0, \lambda_*)$, we have that*

$$-\infty < c_0 := \inf_{u \in \overline{B_\rho(0)}} I_\lambda(u) < 0$$

and the infimum is attained at $u_0 \in B_\rho(0)$ such that $u_0 \in L^\nu_{loc}(\mathbb{R}^N_+) \cap L^\nu_{loc}(\mathbb{R}^{N-1})$ for any $\nu \geq 1$.

Proof. The inequality $c_0 > -\infty$ is obvious, since I_λ maps bounded sets in bounded sets. Let $\delta > 0$ given by (ab) and consider $\varphi \in C_0^\infty(B_\delta(0))$ such that $\int_{\mathbb{R}^N_+} K(x)a(x)\varphi^q dx > 0$. Then,

$$\frac{I_\lambda(t\varphi)}{t^q} \leq \frac{t^{2-q}}{2}\|\varphi\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N_+} K(x)a(x)\varphi^q dx,$$

and therefore

$$\limsup_{t \to 0^+} \frac{I_\lambda(t\varphi)}{t^q} \leq -\frac{\lambda}{q}\int_{\mathbb{R}^N_+} K(x)a(x)\varphi^q dx < 0,$$

which proves that $I_\lambda(t\varphi) < 0$, for any $t > 0$ small. This implies that $c_0 < 0$.

Let $(u_n) \subset \overline{B_\rho(0)}$ be a minimizing sequence for c_0. We may assume that, for some $u_0 \in X$,

$$\begin{cases} u_n \rightharpoonup u_0 \text{ weakly in } X, \\ u_n \to u_0 \text{ strongly in } L^r_K(\mathbb{R}^N_+), \\ u_n^+(x) \to u_0^+(x),\ |u_n(x)| \leq h_r(x) \text{ for a.e. } x \in \mathbb{R}^N_+, \end{cases} \tag{2.3}$$

for any $2 \leq r < 2^*$ and $h_r \in L^r_K(\mathbb{R}^N_+)$. Moreover, since $I_\lambda \geq \alpha > 0$ on $\partial B_\rho(0)$, we can use $2 \leq q\sigma_q' < 2_*$ and the Ekeland Variational Principle to also assume that

$$\lim_{n \to +\infty} I_\lambda(u_n) = c_0, \qquad \lim_{n \to +\infty} I_\lambda'(u_n) = 0.$$

We claim that $I_\lambda'(u_0) = 0$. Indeed, pick $\phi \in C_0^\infty(\overline{\mathbb{R}^N_+})$ and call Ω its support. Since $\sigma_q > (p/q)' = p/(p-q)$, its possible to choose $p_0 \in (2, p)$ close to p and such that

$$\sigma_q > \frac{p_0}{p_0 - q} > \frac{p_0}{p_0 + 1 - q}.$$

Thus, there exists $t > 1$ satisfying

$$\frac{1}{\sigma_q} + \frac{1}{p_0/(q-1)} + \frac{1}{t} = 1.$$

Using Young's inequality we get

$$|K(x)a(x)(u_n^+)^{q-1}\phi(x)| \leq C_1 \left(|a(x)|^{\sigma_q} + |h_{p_0}|^{p_0} + |\phi(x)|^t\right)$$

for a.e. $x \in \Omega$. It follows from the pointwise convergence in (2.3) and the Lebesgue's Theorem that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N_+} K(x)a(x)(u_n^+)^{q-1}\phi\, dx = \int_{\mathbb{R}^N_+} K(x)a(x)(u_0^+)^{q-1}\phi\, dx.$$

A simpler argument shows that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x')b(x')(u_n^+)^{p-1}\phi\, dx' = \int_{\mathbb{R}^{N-1}} K(x')b(x')(u_0^+)^{p-1}\phi\, dx'.$$

So, the claim follows from the weak convergence of (u_n) and the density of $C_0^\infty(\overline{\mathbb{R}^N_+})$ in X.

From Young's inequality, we obtain

$$|K(x)a(x)(u_n^+)^q| \leq |K(x)| \left(\frac{|a|^{\sigma_q}}{\sigma_q} + \frac{|u_n^+|^{q\sigma_q'}}{\sigma_q'}\right) \leq K(x)\left(|a|^{\sigma_q} + h_{q\sigma_q'}^{q\sigma_q'}(x)\right),$$

for a.e. $x \in \mathbb{R}^N_+$. Since $2 \leq q\sigma_q' < p \leq 2^*$, we can use Hölder's inequality to conclude that this last function is integrable and we infer from Lebesgue's Theorem again that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N_+} K(x)a(x)(u_n^+)^q\, dx = \int_{\mathbb{R}^N_+} K(x)a(x)(u_0^+)^q\, dx.$$

Thus,

$$\begin{aligned}
c_0 &= \liminf_{n\to+\infty} \left[I_\lambda(u_n) - \frac{1}{p} I_\lambda'(u_n)u_n\right] \\
&= \liminf_{n\to+\infty} \left[\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{R}^N_+} K(x)a(x)(u_n^+)^q\, dx\right] \\
&\geq \left[\left(\frac{1}{2} - \frac{1}{p}\right)\|u_0\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{R}^N_+} K(x)a(x)(u_0^+)^q\, dx\right] \\
&= I_\lambda(u_0) - \frac{1}{p} I_\lambda'(u_0)u_0 = I_\lambda(u_0).
\end{aligned}$$

Hence $I(u_0) = c_0 < 0$ and it follows from Lemma 2.1.2 that $u_0 \in B_\rho(0)$.

In order to obtain regularity for the solution, we set $w := \exp(|x|^2/8)u_0 \in W^{1,2}_{loc}(\mathbb{R}^N_+)$ and notice that w weakly solves

$$\begin{cases} -\Delta w = f(x,w), & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial w}{\partial \eta} = g(x',w), & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

where

$$f(x,t) := a(x)\exp((2-q)|x|^2/8)|t|^{q-2}t - \left[(|x|^2+4N)/16\right]t$$

and

$$g(x',t) := b(x')\exp((2-p)|x'|^2/8)|t|^{p-2}t,$$

for $x \in \mathbb{R}^N_+$, $x' \in \mathbb{R}^{N-1}$ and $t \in \mathbb{R}$. It is easy to check that

$$|f(x,t)| \le \Gamma_1(x)(1+|t|), \qquad |g(x',t)| \le \Gamma_2(x')(1+|t|)$$

for the functions

$$\Gamma_1(x) := |a(x)|\exp((2-q)|x|^2/8) + \left[(|x|^2+4N)/16\right], \qquad \Gamma_2(x') := b(x').$$

Using (a_0) and (b_0) we conclude that $\Gamma_1 \in L^{N/2}_{loc}(\mathbb{R}^N_+)$ and $\Gamma_2 \in L^{N-1}_{loc}(\mathbb{R}^{N-1})$. Hence, we can use a version of Brezis-Kato's Theorem [17] (see also [1, Appendix 4]) to conclude that $u_0 \in L^{\nu}_{loc}(\mathbb{R}^N_+) \cap L^{\nu}_{loc}(\mathbb{R}^{N-1})$ for any $\nu \ge 1$. The proposition is proved. □

2.2 Proofs of Theorems C and D

Recall that, if E is a Banach space, $\Phi \in C^1(E,\mathbb{R})$ and $c \in \mathbb{R}$, the functional Φ satisfies the $(PS)_c$ condition if any sequence $(u_n) \subset E$ such that

$$\lim_{n\to+\infty} \Phi(u_n) = c, \qquad \lim_{n\to+\infty} \Phi'(u_n) = 0,$$

has a convergent subsequence. From now on, any such sequence will be called $(PS)_c$-sequence.

Lemma 2.2.1. *If $2 < p < 2_*$, then the functional I_λ satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$.*

Proof. Let $(u_n) \subset X$ be a $(PS)_c$-sequence. Computing $I_\lambda(u_n) - (1/p)I'_\lambda(u_n)u_n$, using (a_0) and Hölders's inequality, we can check that (u_n) is bounded. Then, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in X and $u_n \to u$ strongly in $L^r_K(\mathbb{R}^N_+)$ and $L^s_K(\mathbb{R}^{N-1})$, for any $r \in [2,2^*)$ and $s \in [2,2_*)$, respectively. Setting $q_0 := q\sigma'_q \in [2,p)$ and applying Hölder's inequality with exponents σ_q, $q_0/(q-1)$ and q_0, we get

$$\left|\int_{\mathbb{R}^N_+} K(x)a(x)(u_n^+)^{q-1}(u_n-u)\,dx\right| \le \|a\|_{\sigma_q}\|u_n\|^{q-1}_{q_0}\|u_n-u\|_{q_0} \to 0,$$

as $n \to +\infty$. Analogously,

$$\left|\int_{\mathbb{R}^{N-1}} K(x',0)b(x')(u_n^+)^{p-1}(u_n-u)\,dx'\right| \le |b|_\infty|u_n|^{p-1}_p|u_n-u|_p \to 0.$$

From the two above expressions and the weak convergence we obtain

$$o(1) = I_\lambda'(u_n)(u_n - u) = \|u_n\|^2 - \|u\|^2 + o(1),$$

as $n \to +\infty$. The result is now a consequence of the weak convergence. □

When dealing with the critical case, we need the following local compactness result:

Lemma 2.2.2. *If $p = 2^*$ and the function u_0 given by Proposition 2.1.3 is the only nonzero critical point of I_λ, then I_λ satisfies the Palais-Smale condition at any level*

$$c < \bar{c} := I_\lambda(u_0) + \frac{1}{2(N-1)} \frac{1}{\|b\|_\infty^{N-2}} S_{2_*,\partial}^{N-1}.$$

Proof. Let $(u_n) \subset X$ be a $(PS)_c$-sequence. As in Lemma 2.2.1, we may assume that $u_n \rightharpoonup u$ weakly in X and $u_n \to u$ strongly in $L_K^{q\sigma_q'}(\mathbb{R}_+^N)$. Hence, we infer from the Lebesgue Theorem that, as $n \to +\infty$,

$$\int_{\mathbb{R}_+^N} K(x)a(x)(u_n^+)^q \, dx = \int_{\mathbb{R}_+^N} K(x)a(x)(u^+)^q \, dx + o(1).$$

If $z_n := (u_n - u)$, we can use $I_\lambda'(u_n)u_n = o(1)$ and Brezis-Lieb's lemma [18] to obtain

$$\begin{aligned} o(1) &= \|u_n\|^2 - \lambda \int_{\mathbb{R}_+^N} K(x)a(x)(u_n^+)^q \, dx - \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(u_n^+)^{2_*} dx' \\ &= I_\lambda'(u)u + \|z_n\|^2 - \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(z_n^+)^{2_*} dx' + o(1). \end{aligned}$$

As in the proof of Proposition 2.1.3, we have that $I_\lambda'(u) = 0$. So, passing the above expression to the limit, we obtain $\gamma \geq 0$ such that

$$\lim_{n \to +\infty} \|z_n\|^2 = \gamma = \lim_{n \to +\infty} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(z_n^+)^{2_*} dx'.$$

We need to prove that $\gamma = 0$. In order to do this, we first take the limit in the inequality

$$\int_{\mathbb{R}^{N-1}} K(x',0)b(x')(z_n^+)^{2_*} dx' \leq \|b\|_\infty S_{2_*,\partial}^{-2_*/2} \left(\int_{\mathbb{R}_+^N} K(x)|\nabla z_n|^2 dx \right)^{2_*/2},$$

to obtain $\gamma \leq \|b\|_\infty S_{2_*,\partial}^{-2_*/2} \gamma^{2_*/2}$. Suppose, by contradiction, that $\gamma > 0$. Then

$$\gamma \geq \frac{1}{\|b\|_\infty^{N-2}} S_{2_*,\partial}^{N-1}. \tag{2.4}$$

On the other hand, using Brezis-Lieb again, we obtain

$$c + o(1) = I_\lambda(u_n) = I_\lambda(u) + \frac{1}{2}\|z_n\|^2 - \frac{1}{2_*} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')(z_n^+)^{2_*} \, dx' + o(1).$$

Passing to the limit and using (2.4), we conclude that

$$c = I_\lambda(u) + \frac{1}{2(N-1)}\gamma \geq I_\lambda(u) + \frac{1}{2(N-1)}\frac{1}{|b|_\infty^{N-2}} S_{2_*,\partial}^{N-1}.$$

Recalling that u is a critical point of I_λ, we conclude from the hypotheses that $u = 0$ or $u = u_0$. Since $\max\{I_\lambda(0), I_\lambda(u_0)\} \leq 0$, the above expression contradicts $c < \overline{c}$. So, $\gamma = 0$ and we have done. □

Let $\delta > 0$ be as in assumption (ab) and take $\phi \in C^\infty(\overline{\mathbb{R}^N_+}, [0,1])$ such that $\phi \equiv 1$ in $\overline{\mathbb{R}^N_+} \cap B_{\delta/2}(0)$ and $\phi \equiv 0$ in $\overline{\mathbb{R}^N_+} \backslash B_\delta(0)$. Set, for each $\varepsilon > 0$,

$$u_\varepsilon(x) := K(x)^{-1/2}\phi(x)U_\varepsilon(x), \quad x \in \mathbb{R}^N_+,$$

where

$$U_\varepsilon(x', x_N) := \frac{\varepsilon^{(N-2)/2}}{[|x'|^2 + (x_N+\varepsilon)^2]^{(N-2)/2}}.$$

If $N \geq 7$, it is proved in [48] that,

$$\|u_\varepsilon\|^2 = A_N + O(\varepsilon^2), \qquad |u_\varepsilon|_{2_*}^{2_*} = B_N^{2_*/2} + O(\varepsilon^2), \tag{2.5}$$

as $\varepsilon \to 0^+$. Moreover, the constants A_N, B_N are such that $A_N/B_N = S_{2_*,\partial}$ and the following holds:

Lemma 2.2.3. *If $\psi_\varepsilon := u_\varepsilon/|u_\varepsilon|_{2_*}$ and $(N-1)/(N-2) < \tau < 2_*$, then*

$$\|\psi_\varepsilon\|^{2(N-1)} = S_{2_*,\partial}^{N-1} + O(\varepsilon^2), \qquad |\psi_\varepsilon|_\tau^\tau = O(\varepsilon^{(N-1)-\tau(N-2)/2}), \tag{2.6}$$

as $\varepsilon \to 0^+$.

Proof. Using the Mean Value theorem for $g(r) = r^s$ and a simple computation, we can check that

$$\left[A + O(\varepsilon^t)\right]^s = A^s + O(\varepsilon^t),$$

for any A, s, $t > 0$. Hence, we infer from (2.5) and the defintion of 2_* that

$$\|\psi_\varepsilon\|^{2(N-1)} = \frac{[A_N + O(\varepsilon^2)]^{N-1}}{\left[B_N^{2_*/2} + O(\varepsilon^2)\right]^{N-2}} = \frac{A_N^{N-1} + O(\varepsilon^2)}{B_N^{2_*(N-2)/2} + O(\varepsilon^2)} = \left(\frac{A_N}{B_N}\right)^{N-1} + O(\varepsilon^2).$$

Since $A_N/B_N = S$, we conclude that the first statement in (2.6) holds.

For the second one, we first notice that

$$\begin{aligned} |u_\varepsilon|_\tau^\tau &= \varepsilon^{-\tau(N-2)/2}\int_{\mathbb{R}^{N-1}} \frac{K(x',0)^{-\tau/2}\phi(x',0)^\tau}{[|x'/\varepsilon|^2+1]^{\tau(N-2)/2}}\,dx' \\ &\leq C_1\varepsilon^{-\tau(N-2)/2}\int_{B_\delta(0)\cap\partial\mathbb{R}^N_+} \frac{1}{(|x'/\varepsilon|^2+1)^{\tau(N-2)/2}}\,dx' \\ &\leq C_1\varepsilon^{(N-1)-\tau(N-2)/2}\int_{\mathbb{R}^{N-1}} \frac{1}{(|y'|^2+1)^{\tau(N-2)/2}}\,dy', \end{aligned}$$

where we have used the definition of u_ε, $0 \le \phi \le 1$ and the change of variable $y' = x'/\varepsilon$. But

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} \frac{1}{(|y'|^2+1)^{\tau(N-2)/2}}\,dy' &\le C_2 + \int_{\partial\mathbb{R}^N_+\setminus B_1(0)} \frac{1}{|y'|^{\tau(N-2)}}\,dy' \\ &= C_2 + C_3\int_1^{+\infty} s^{-\tau(N-2)+(N-2)}ds < +\infty,\end{aligned}$$

whenever $\tau > (N-1)/(N-2)$. Since $|u_\varepsilon|^\tau_{2_*} = B_N^{\tau/2} + o(1)$, as $\varepsilon \to 0^+$, the result follows from the above inequalities. □

We are ready to prove our first main results of the chapter.

Proofs of Theorems C and D. According to Lemma 2.1.2 and Proposition 2.1.3, there exists $\lambda_* > 0$ such that, for any $\lambda \in (0, \lambda_*)$, the problem (P_2) has a nonnegative solution $u_0 \in X \setminus \{0\}$ such that $I_\lambda(u_0) < 0$. The second solution will be obtained as an application of the Mountain Pass Theorem.

Recall that $\psi_\varepsilon := u_\varepsilon/|u_\varepsilon|_{2_*}$ and notice that

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')(u_0+t\psi_\varepsilon)^p\,dx' &= O(1) + \int_\Omega K(x',0)b(x')(u_0+t\psi_\varepsilon)^p dx' \\ &\ge O(1) + t^p\int_\Omega K(x',0)b(x')\psi_\varepsilon^p\,dx',\end{aligned}$$

as $t \to +\infty$, where $\Omega := B_\delta(0)\cap\mathbb{R}^{N-1}$. Since a similar argument holds for the integral inside the domain, we get

$$I_\lambda(u_0+t\psi_\varepsilon) \le O(t^2) + O(t^q) - \frac{t^p}{p}\int_\Omega K(x',0)b(x')\psi_\varepsilon^p dx',$$

as $t \to +\infty$. The function in the last integral above is positive, and therefore we can use $1 < q < 2 < p$ to obtain

$$\lim_{t\to+\infty} I_\lambda(u_0+t\psi_\varepsilon) = -\infty. \tag{2.7}$$

Hence, there exists $t_* > 0$ large such that $e := u_0 + t_*\psi_\varepsilon$ satisfies $\|e\| > \rho$ given by Lemma 2.1.2 and $I_\lambda(e) \le 0$. So, it is well defined

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = e\}$. From the Mountain Pass Theorem [3] (see also [93, Theorem 1.15]), we obtain $(u_n) \subset X$ such that

$$\lim_{n\to+\infty} I_\lambda(u_n) = c, \qquad \lim_{n\to+\infty} I'_\lambda(u_n) = 0.$$

If $2 < p < 2^*$, we can use Lemma 2.2.1 to conclude that, along a subsequence, (u_n) converges to a critical point $u_1 \in X$ such that $I_\lambda(u_1) > 0$. Hence, $u_1 \ne u_0$ is the second solution. And we conclude the proof of Theorem C.

The final step in the above argument is more delicate in the critical case $p = 2_*$. Actually, we need to prove that, for $\varepsilon > 0$ small, there holds

$$\max_{t\geq 0} I_\lambda(u_0 + t\psi_\varepsilon) < \bar{c} := I_\lambda(u_0) + \frac{1}{2(N-1)} \frac{1}{|b|_\infty^{N-2}} S_{2_*,\partial}^{N-1}. \tag{2.8}$$

If this is true, we can use Lemma 2.2.2, the Mountain Pass Theorem and a contradiction argument to obtain a nonzero solution $u_1 \neq u_0$.

In order to prove (2.8), we first notice that, since $u_0 \in B_\rho(0)$ is a local minimum of I_λ, we can use (2.7) to obtain $t_\varepsilon > 0$ such that

$$m_\varepsilon := I_\lambda(u_0 + t_\varepsilon\psi_\varepsilon) = \max_{t\geq 0} I_\lambda(u_0 + t\psi_\varepsilon).$$

We claim that $t_\varepsilon = O(1)$, as $\varepsilon \to 0^+$. Indeed, suppose by contradiction that $t_{\varepsilon_n} \to +\infty$, for some sequence $\varepsilon_n \to 0^+$. Recalling that a, $b > 0$ in the support of ψ_ε, we can use $I'_\lambda(u_0 + t_\varepsilon\psi_\varepsilon)\psi_\varepsilon = 0$ and $I'_\lambda(u_0)\psi_\varepsilon = 0$ to get

$$t_\varepsilon^{2_*-1} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')\psi_\varepsilon^{2_*} dx' \leq t_\varepsilon \|\psi_\varepsilon\|^2 + \int_{\mathbb{R}^{N-1}} K(x',0)b(x')u_0^{2_*-1}\psi_\varepsilon \, dx'.$$

Thus, from (2.6), Hölder's inequality and $|\psi_\varepsilon|_{2_*} = 1$, we obtain

$$\int_{\mathbb{R}^{N-1}} K(x',0)b(x')\psi_\varepsilon^{2_*} dx' \leq t_\varepsilon^{2-2_*} \left[S_{2_*,\partial} + O(1)\right] + t_\varepsilon^{1-2_*} |b|_\infty |u_0|_{2_*}^{2_*-1},$$

for all $\varepsilon > 0$. In particular, we can take $\varepsilon = \varepsilon_n$ in the above inequality to conclude that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')\psi_{\varepsilon_n}^{2_*} dx' = 0.$$

On the other hand, using (b_1) and $|\psi_{\varepsilon_n}|_{2_*} = 1$ we obtain

$$o(1) = \int_{\mathbb{R}^{N-1}} K(x',0)b(x')\psi_{\varepsilon_n}^{2_*} dx' \geq |b|_\infty - M \int_{\mathbb{R}^{N-1}} K(x',0)|x'|^\sigma \psi_{\varepsilon_n}^{2_*} \, dx'. \tag{2.9}$$

Moreover, since $|u_{\varepsilon_n}|_{2_*} = B_N^{1/2} + o(1)$,

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} K(x',0)|x'|^\sigma \psi_{\varepsilon_n}^{2_*} \, dx' &\leq C_1 \frac{\varepsilon_n^{N-1}}{|u_\varepsilon|_{2_*}^{2_*}} \int_{B_\delta(0)\cap\mathbb{R}_+^{N-1}} \frac{|x'|^\sigma}{[|x'|^2 + \varepsilon^2]^{N-1}} \, dx' \\ &= O(\varepsilon_n^{N-1}) \int_{B_\delta(0)\cap\mathbb{R}_+^{N-1}} |x'|^{\sigma-2(N-1)} \, dx', \end{aligned}$$

as $n \to +\infty$. Since $\sigma > N-1$, the last integral above is finite and therefore

$$\int_{\mathbb{R}^{N-1}} K(x',0)|x'|^\sigma \psi_{\varepsilon_n}^{2_*} dx' = O(\varepsilon_n^{N-1}), \quad \text{as } n \to +\infty. \tag{2.10}$$

Thus, it follows from (2.9) that $|b|_\infty = 0$, which does not make sense. This proves that (t_ε) is bounded.

We also claim that (t_ε) is far from zero, that is, there exists $M > 0$ such that $t_\varepsilon \geq M$, for all $\varepsilon > 0$. In order to prove the claim let us suppose, by contradiction, the existence of a sequence $(\varepsilon_n) \subset (0, +\infty)$ with $\varepsilon_n \to 0^+$ and $t_{\varepsilon_n} \to 0^+$. As we know, for each $\varepsilon > 0$ we can choose $t_\varepsilon^* > 0$ in such a way that $\|u_0 + t_\varepsilon^* \psi_\varepsilon\| > \rho$ and $I(u_0 + t_\varepsilon^* \psi_\varepsilon) \leq 0$, with $\rho > 0$ as in Lemma 2.1.2. Hence, it is well defined, for each $\varepsilon > 0$,

$$c_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} I_\lambda(\gamma(t)) \geq \alpha > 0,$$

where $\Gamma_\varepsilon := \{\gamma \in C([0,1], X),\ \gamma(0) = u_0 \text{ and } \gamma(1) = u_0 + t_\varepsilon^* \psi_\varepsilon\}$. Consequently,

$$m_\varepsilon \geq c_\varepsilon > 0,$$

for any $\varepsilon > 0$. On the other hand, since

$$\|u_0 - (t_\varepsilon \psi_\varepsilon + u_0)\| = \|u_0 - t_\varepsilon \psi_\varepsilon - u_0\| = |t_\varepsilon| \|\psi_\varepsilon\|,$$

it follows from (2.6) and $t_{\varepsilon_n} \to 0$, that

$$m_{\varepsilon_n} = I_\lambda(u_0 + t_{\varepsilon_n} \psi_{\varepsilon_n}) \to I_\lambda(u_0) < 0,$$

as $n \to +\infty$. This leads to a contradiction and the claim is proved.

Using $I'_\lambda(u_0)\psi_\varepsilon = 0$, we obtain

$$m_\varepsilon = I(u_0) + \frac{t_\varepsilon^2}{2}\|\psi_\varepsilon\|^2 - \frac{\lambda}{q}\Gamma_{1,\varepsilon} - \frac{1}{2_*}\Gamma_{2,\varepsilon}, \tag{2.11}$$

where

$$\Gamma_{1,\varepsilon} := \int_{\mathbb{R}^N_+} K(x)a(x)[(u_0 + t_\varepsilon \psi_\varepsilon)^q - u_0^q - q t_\varepsilon u_0^{q-1} \psi_\varepsilon]\, dx$$

and

$$\Gamma_{2,\varepsilon} := \int_{\mathbb{R}^{N-1}} K(x', 0) b(x')[(u_0 + t_\varepsilon \psi_\varepsilon)^{2_*} - u_0^{2_*} - 2_* t_\varepsilon u_0^{2_*-1} \psi_\varepsilon]\, dx'$$

It follows from the Mean Value Theorem that there exists $\theta(x) \in [0,1]$ such that

$$\begin{aligned} (u_0(x) + t_\varepsilon \psi_\varepsilon(x))^q - u_0(x)^q &= q(u_0(x) + \theta(x) t_\varepsilon \psi_\varepsilon(x))^{q-1} t_\varepsilon \psi_\varepsilon(x) \\ &\geq q t_\varepsilon u_0(x)^{q-1} \psi_\varepsilon(x), \end{aligned}$$

for a.e. $x \in \mathbb{R}^N_+$. Since $a \geq 0$ in the support of ψ_ε we conclude that $\Gamma_{1,\varepsilon} \geq 0$. For estimating $\Gamma_{2,\varepsilon}$ we notice that, given $r,\ s \geq 0$ and $1 < \mu < 2_* - 1$, there holds (see [28])

$$(r + s)^{2_*} \geq r^{2_*} + s^{2_*} + 2_* r^{2_*-1} s + 2_* r s^{2_*-1} - A_\mu r^{2_*-\mu} s^\mu,$$

for some constant $A_\mu > 0$. Picking $r = u_0(x)$ and $s = t_\varepsilon \psi_\varepsilon(x)$, we get

$$\Gamma_{2,\varepsilon} \geq \int_{\mathbb{R}^{N-1}} K(x', 0) b(x') \left[t_\varepsilon^{2_*} \psi_\varepsilon^{2_*} + 2_* t_\varepsilon^{2_*-1} u_0 \psi_\varepsilon^{2_*-1} - A_\mu t_\varepsilon^\mu u_0^{2_*-\mu} \psi_\varepsilon^\mu\right] dx'.$$

Since $\Gamma_{1,\varepsilon} \geq 0$ and $|\psi_\varepsilon|_{2_*} = 1$, we can use the above inequality and (2.11) to obtain

$$m_\varepsilon \leq I(u_0) + \left[\frac{t_\varepsilon^2}{2}\|\psi_\varepsilon\|^2 - \frac{t_\varepsilon^{2_*}}{2_*}|b|_\infty\right] + \Gamma_{2,\varepsilon,1} - \Gamma_{2,\varepsilon,2} + \Gamma_{2,\varepsilon,3},$$

with

$$\Gamma_{2,\varepsilon,1} := \frac{t_\varepsilon^{2_*}}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)[|b|_\infty - b(x')]\psi_\varepsilon^{2_*}dx',$$

$$\Gamma_{2,\varepsilon,2} := t_\varepsilon^{2_*-1}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')u_0\psi_\varepsilon^{2_*-1}dx'$$

and

$$\Gamma_{2,\varepsilon,3} := C_\mu\frac{t_\varepsilon^\mu}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')u_0^{2_*-\mu}\psi_\varepsilon^\mu dx'.$$

As in (2.10), the integral in $\Gamma_{2,\varepsilon,1}$ has order ε^{N-1}, as $\varepsilon \to 0^+$. So, we infer from the boundedness of (t_ε) that $\Gamma_{2,\varepsilon,1} = O(\varepsilon^{N-1})$. Moreover,

$$\max_{t\geq 0}\left\{\frac{t^2}{2}\|\psi_\varepsilon\|^2 - \frac{t^{2_*}}{2_*}|b|_\infty\right\} = \frac{1}{2(N-1)}\frac{\|\psi_\varepsilon\|^{2(N-1)}}{|b|_\infty^{N-2}}$$

and therefore we infer from (2.6) and the above estimate for m_ε that

$$m_\varepsilon \leq \bar{c} + O(\varepsilon^2) - \Gamma_{2,\varepsilon,2} + \Gamma_{2,\varepsilon,3}. \tag{2.12}$$

In order to estimate the last two terms, we recall that $u_0 \in L^\nu_{loc}(\mathbb{R}^N_+) \cap L^\nu_{loc}(\mathbb{R}^{N-1})$ for any $\nu \geq 1$. So, if we denote $\Omega_\partial := B_\delta(0) \cap \mathbb{R}^{N-1}$, we can choose $\tau_1 > 1$ such that

$$\frac{2(N-1)}{(N+4)} < \tau_1 < \frac{2(N-1)}{N}$$

and use Hölder's inequality to get

$$\int_{\mathbb{R}^{N-1}} K(x',0)b(x')u_0\psi_\varepsilon^{2_*-1}dx' \leq |b|_\infty\left(\int_{\Omega_\partial} K(x',0)u_0^{\tau_1'}dx'\right)^{1/\tau_1'}|\psi_\varepsilon|_{(2_*-1)\tau_1}^{2_*-1}.$$

Since $(N-1)/(N-2) < (2_*-1)\tau_1 < 2_*$ and $t_\varepsilon \geq M > 0$, we infer from (2.6) and the choice of τ_1 that

$$\Gamma_{2,\varepsilon,2} \geq O(\varepsilon^{(N-1)/\tau_1-(N/2)}). \tag{2.13}$$

We now set $\mu := (N-1)/(N-2)$, pick $1 < \tau_2 < 2$ and apply Hölder's inequality again to obtain

$$\int_{\mathbb{R}^{N-1}} K(x',0)b(x')u_0^{2_*-\mu}\psi_\varepsilon^\mu\, dx' \leq |b|_\infty\left(\int_{\Omega_\partial} K(x',0)u_0^{(2_*-\mu)\tau_2'}\, dx'\right)^{1/\tau_2'}|\psi_\varepsilon|_{\mu\tau_2}^\mu,$$

from which we conclude that

$$\Gamma_{2,\varepsilon,3} = O(\varepsilon^{(N-1)/\tau_2-(N-1)/2}). \tag{2.14}$$

Since

$$\lim_{\tau\to 2(N-1)/N}\left(\frac{N-1}{\tau}-\frac{N}{2}\right)=0<\frac{N-1}{2}=\lim_{\tau\to 1}\left(\frac{N-1}{\tau}-\frac{N-1}{2}\right),$$

we can choose the numbers τ_1, τ_2 above in such way that

$$\nu_1:=\frac{N-1}{\tau_1}-\frac{N}{2}<2, \qquad \nu_2:=\frac{N-1}{\tau_2}-\frac{N-1}{2}>\nu_1.$$

Replacing (2.13) and (2.14) in (2.12) and using the above inequalities, we obtain

$$m_\varepsilon\le \bar{c}+O(\varepsilon^2)-O(\varepsilon^{\nu_1})+O(\varepsilon^{\nu_2})=\bar{c}+\varepsilon^{\nu_1}\left[O(\varepsilon^{2-\nu_1})-O(1)+O(\varepsilon^{\nu_2-\nu_1})\right],$$

as $\varepsilon\to 0^+$. We conclude that (2.8) holds, for any $\varepsilon>0$ small. The theorem D is proved. □

2.3 Proofs of Theorems E and F

We start this section presenting some definitions and abstract results which will be used to obtain infinitely many solutions for (P_2). Let $E=V\oplus W$ be an infinite dimensional Hilbert space, with $V=\operatorname{span}\{\varphi_1^V,\varphi_2^V,\dots\}$, $W=\operatorname{span}\{\varphi_1^W,\varphi_2^W,\dots\}$ and the basis being orthonormal. For each $n\in\mathbb{N}$, define the subspaces

$$V^n:=\operatorname{span}\{\varphi_1^V,\varphi_2^V,\dots\varphi_n^V\}, \qquad V_n:=\overline{\operatorname{span}\{\varphi_n^V,\varphi_{n+1}^V,\dots\}}.$$

Using the set $\{\varphi_i^W\}_{i\in\mathbb{N}}$ we define W^n and W_n in a similar way.

Given $\Phi\in C^1(E,\mathbb{R})$ and $c\in\mathbb{R}$, we say that Φ satisfies the $(PS)_c^*$-condition (with respect to $V^n\oplus W^n$) if any sequence $(u_n)\subset V^n\oplus W^n$ such that

$$\lim_{n\to+\infty}\Phi(u_n)=c, \qquad \lim_{n\to+\infty}\Phi'_{|V^n\oplus W^n}(u_n)=0,$$

has a subsequence converging to a critical point of Φ. Any such sequence will be called $(PS)_c^*$-sequence.

We are going to obtain infinitely many solutions for (P_2) as applications of the following abstract theorems due to Tonkes [92] (see also [3, 12]):

Theorem 2.3.1. *Let $\Phi\in C^1(E,\mathbb{R})$ be an even functional. Suppose that, for every $n\ge n_0$, there exist $R_n>r_n>0$ such that*

(A_1) $\inf\{\Phi(u)\,:\,u\in V_n\oplus W,\ \|u\|_E=R_n\}\ge 0$;

(A_2) $b_n:=\inf\{\Phi(u)\,:\,u\in V_n\oplus W,\ \|u\|_E\le R_n\}\to 0$, *as* $n\to+\infty$;

(A_3) $d_n:=\sup\{\Phi(u)\,:\,u\in V^n,\ \|u\|_E=r_n\}<0$;

(A_4) Φ *satisfies* $(PS)_c^*$*-condition for all* $c\in[b_{n_0},0)$.

Then Φ has a sequence of critical values $c_n \in [b_n, d_n]$ such that $c_n \to 0$, as $n \to +\infty$.

Theorem 2.3.2. *Let $\Phi \in C^1(E, \mathbb{R})$ be a even functional. Suppose that, for every $n \in \mathbb{N}$, there exist $R_n > r_n > 0$ such that*

$(\widetilde{A_2})$ *$b_n := \inf\{\Phi(u) \,:\, u \in V_n \oplus W,\ \|u\| = r_n\} \to \infty$, as $n \to \infty$;*

$(\widetilde{A_3})$ *$a_n := \max\{\Phi(u) \,:\, u \in V^n,\ \|u\| = R_n\} \leq 0$;*

$(\widetilde{A_4})$ *Φ satisfies $(PS)_c$-condition for all $c > 0$.*

Then Φ has a sequence of critical values $c_n \in (0, +\infty)$ such that $c_n \to +\infty$, as $n \to +\infty$.

Since we are not interested in the sign of the solutions, we redefine the energy function setting

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N_+} K(x)a(x)|u|^q\,dx - \frac{1}{p}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u|^p\,dx'.$$

We are intending to apply Theorem 2.3.1 with $\Phi = I_\lambda$ and $E = X$. In order to define the space decomposition, we recall that $\Omega_a^+ = \left\{x \in \mathbb{R}^{\mathbb{N}}_+ \,:\, a(x) > 0\right\}$ and define

$$W := \left\{u \in X : u(x) = 0 \text{ for a.e. } x \in \operatorname{int}(\Omega_a^+)\right\}.$$

We call V the orthogonal complement of the closed subspace W, in such way that $X = V \oplus W$.

We start with the required compactness properties.

Proposition 2.3.3. *If $2 < p < 2_*$, then I_λ satisfies the $(PS)_c^*$ condition at any level $c \in \mathbb{R}$. The same holds if $p = 2_*$ and $b \leq 0$.*

Proof. Let $(u_n) \subset V^n \oplus W^n$ be a $(PS)_c^*$-sequence. Computing $I_\lambda(u_n) - (1/p)I_\lambda'(u_n)u_n$, using (a_0) and Hölder's inequality, we can check that (u_n) is bounded. Then, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in X. Pick $\phi \in C_0^\infty(\overline{\mathbb{R}^N_+})$ and denote by ϕ^n its projection over the subspace $V^n \oplus W^n$. Since $(I_\lambda'(u_n)) \subset X^*$ is bounded, we have that

$$|I_\lambda'(u_n)(\phi - \phi^n)| \leq \|I_\lambda'(u_n)\|_{X^*}\|\phi - \phi^n\| = o(1),$$

as $n \to +\infty$. Thus, recalling that $I_\lambda'(u_n)\phi^n = o(1)$, we obtain

$$I_\lambda'(u_n)\phi = I_\lambda'(u_n)(\phi - \phi^n) + I_\lambda'(u_n)\phi^n = o(1).$$

Arguing as in the proof of Proposition 2.1.3, we conclude that $I_\lambda'(u)\phi = 0$, for any $\phi \in C_0^\infty(\overline{\mathbb{R}^N_+})$. It follows from a density argument that $I_\lambda'(u) = 0$.

Using Lebesgue Theorem as in the proof of Proposition 2.1.3, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N_+} K(x)a(x)|u_n|^q dx = \int_{\mathbb{R}^N_+} K(x)a(x)|u|^q dx. \tag{2.15}$$

Moreover, in the subcritical case $2 < p < 2_*$, the same kind of convergence holds for the term $\int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u_n|^p dx'$, since the trace embedding is compact. These two convergences and $I'_\lambda(u)u = 0$ provide

$$o(1) = I'_\lambda(u_n)u_n - I'_\lambda(u)u = \|u_n^2\| - \|u\|^2 + o(1),$$

and we infer from the weak convergence that $u_n \to u$ strongly in X.

For the critical case $p = 2_*$, we first use assumption $b \le 0$ to guarantee that $\varphi \mapsto -\int_{\mathbb{R}^{N-1}} K(x',0)b(x')|\varphi|^p\, dx'$ is a seminorm in X. Hence, from the weak lower semicontinuity of a seminorm, we get

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u_n|^{2_*}dx' \le \int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u|^{2_*}dx'.$$

This, (2.15) and $I'_\lambda(u_n)u_n = o(1)$ imply that

$$\limsup_{n\to+\infty}\|u_n\|^2 \le \|u\|^2.$$

On the other hand, the weak convergence provides $\|u\|^2 \le \liminf_{n\to+\infty}\|u_n\|^2$, and therefore the result follows as in the former case. □

If $p = 2_*$ and b changes it sign, we need the following local compactness result.

Proposition 2.3.4. *If $p = 2_*$, then there exists $C_c = C_c(q, N, \|a\|_{\sigma_q}) > 0$ such that the functional I_λ satisfies the $(PS)^*_c$ condition at any level*

$$c < \frac{1}{2(N-1)}\frac{1}{\|b\|_\infty^{N-2}} S_{2_*,\partial}^{N-1} - C_c\lambda^{2/(2-q)}.$$

Proof. Let $(u_n) \subset V^n \oplus W^n$ be a $(PS)^*_c$ sequence. As in the proof of Proposition 2.3.3, we may assume that $u_n \rightharpoonup u$ weakly in X, with $I'_\lambda(u) = 0$. Since $I_\lambda(u) = I_\lambda(u) - (1/2_*)I'_\lambda(u)u$, we obtain

$$I_\lambda(u) = \frac{1}{2(N-1)}\|u\|^2 - \lambda\left(\frac{2_* - q}{2_* q}\right)\int_{\mathbb{R}^N_+} K(x)a(x)|u|^q dx. \tag{2.16}$$

We now set $z_n := (u_n - u)$ and argue as in Proposition 2.2.2 to get

$$\lim_{n\to\infty}\|z_n\|^2 = \gamma = \lim_{n\to\infty}\int_{\mathbb{R}^{N-1}} K(x',0)b(x')|z_n|^{2_*}dx'.$$

for some $\gamma \ge 0$. If $\gamma > 0$, then

$$\gamma \ge \frac{1}{\|b\|_\infty^{N-2}} S_{2_*,\partial}^{N-1}. \tag{2.17}$$

On the other, we infer from Brezis-Lieb's lemma that

$$c+o(1)=I_\lambda(u_n)=\frac{1}{2}\|z_n\|^2-\frac{1}{2_*}\int_{\mathbb{R}^{N-1}}K(x',0)b(x')|z_n|^{2_*}dx'+I_\lambda(u)+o(1).$$

Passing to the limit, using (2.16), Hölder's inequality and (2.17), we conclude that

$$c\geq\frac{1}{2(N-1)}\frac{1}{\|b\|_\infty^{N-2}}S_{2_*,\partial}^{N-1}+g(\|u\|), \tag{2.18}$$

where

$$g(t):=\frac{t^2}{2(N-1)}-\lambda\gamma_q t^q, \qquad t>0,$$

and $\gamma_q:=\|a\|_{\sigma_q}S_{q\sigma_q'}^{-q/2}(2_*-q)/(2_*q)$. Setting

$$C_c:=\left(\frac{2-q}{2q}\right)\frac{1}{(N-1)}\left[(N-1)q\gamma_q\right]^{2/(2-q)},$$

a straightforward computation shows that $g(t)\geq -C_c\lambda^{2/(2-q)}$, for any $t>0$. Hence, we infer from (2.18) that

$$c\geq\frac{S_{2_*,\partial}^{N-1}}{2(N-1)\|b\|_\infty^{N-2}}-C_c\lambda^{2/(2-q)},$$

which does not make sense. This contradiction proves that $\gamma=0$ or, equivalently, $u_n\to u$ strongly in X. □

We finish this section with an important tool for the proof of Theorem E.

Lemma 2.3.5. *Suppose that $a\geq 0$ and set*

$$\mu_n:=\sup_{\{u\in V_n\oplus W\,:\,\|u\|\leq 1\}}\int_{\mathbb{R}^N_+}K(x)a(x)|u|^q\,dx.$$

Then $\mu_n\to 0$, as $n\to\infty$.

Proof. Since $(\mu_n)\subset[0,+\infty)$ is nonincreasing, we have that $\mu_n\to\mu_0\geq 0$, as $n\to\infty$. Let $(u_n)\subset V_n\oplus W$ be such that $\|u_n\|=1$ and

$$\frac{\mu_0}{2}\leq\frac{\mu_n}{2}\leq\int_{\mathbb{R}^N_+}K(x)a(x)|u_n|^q\,dx. \tag{2.19}$$

We may assume that $u_n\rightharpoonup u=v+w$ weakly in X, with $v\in V$ and $w\in W$. The orthogonal decomposition and the definition of V_n imply that $\langle u_n,\varphi_k^V\rangle=0$, for any fixed $k\in\mathbb{N}$ and $n>k$. So,

$$0=\lim_{n\to+\infty}\langle u_n,\varphi_k^V\rangle=\langle u,\varphi_k^V\rangle=\langle v,\varphi_k^V\rangle,$$

and therefore $v=0$ or, equivalently, $u=w$. Using Lebesgue Theorem as in the proof of Proposition 2.1.3, we conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N_+}K(x)a(x)|u_n|^q dx=\int_{\mathbb{R}^N_+}K(x)a(x)|w|^q dx=0,$$

since $w\in W$. The above expression and (2.19) imply that $\mu_0=0$. □

We are ready to prove Theorem E.

Proof of Theorem E. It follows from Lemma 2.3.5 and Proposition 2.1.1 that, for any $u \in V_n \oplus W$, there holds

$$I_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\mu_n\|u\|^q - \frac{1}{p}\|b\|_\infty S_{p,\partial}^{-p/2}\|u\|^p.$$

Hence,

$$I_\lambda(u) \geq \frac{1}{4}\|u\|^2 - \frac{\lambda}{q}\mu_n\|u\|^q, \qquad \forall\, u \in \overline{B_{\rho_1}(0)} \cap (V_n \oplus W), \tag{2.20}$$

where $\rho_1 := \left[pS_{p,\partial}^{p/2}/(4\|b\|_\infty)\right]^{1/(p-2)}$. Since $\mu_n \to 0$, there exists $n_1 \in \mathbb{N}$ such that

$$R_n := \left(\frac{4\lambda\mu_n}{q}\right)^{1/(2-q)} < \rho_1, \qquad \forall\, n \geq n_1.$$

Using (2.20) we can check that $I_\lambda(u) \geq 0$, for any $u \in V_n \oplus W$ such that $\|u\| = R_n$. This proves that (A_1) holds.

In order to verify (A_2) we notice that, for $n \geq n_1$,

$$I_\lambda(u) \geq -\frac{\lambda}{q}\mu_n R_n^q \qquad \forall\, u \in \overline{B_{R_n}(0)} \cap (V_n \oplus W).$$

Thus,

$$0 \geq b_n = \inf\left\{I_\lambda(u) \,:\, u \in \overline{B_{R_n}(0)} \cap (V_n \oplus W)\right\} \geq -\frac{\lambda}{q}\mu_n R_n^q.$$

Since the right-hand side above goes to 0, as $n \to +\infty$, we conclude that (A_2) holds.

Given $u \in V^n$, we have that $\int_{\mathbb{R}^N_+} K(x)a(x)|u|^q dx = 0$ if, and only if, $u = 0$. Hence, this integral defines a norm in the finite dimensional subspace V^n. The equivalence of norms provides $0 < \beta_n < (8\mu_n)/q$, such that

$$\beta_n\|u\|^q \leq \int_{\mathbb{R}^N_+} K(x)a(x)|u|^q\, dx, \qquad \forall\, u \in V^n.$$

Hence, we can argue as above to get

$$I_\lambda(u) \leq \|u\|^2 - \frac{\lambda}{q}\beta_n\|u\|^q, \qquad \forall\, u \in \overline{B_{\rho_2}(0)} \cap V^n, \tag{2.21}$$

where $\rho_2 := \left[pS_{p,\partial}^{p/2}/(2\|b\|_\infty)\right]^{1/(p-2)}$. Since $\beta_n \to 0$, there exists $n_2 \in \mathbb{N}$ such that

$$r_n := \left(\frac{\lambda\beta_n}{2}\right)^{1/(2-q)} < \rho_2, \qquad \forall\, n \geq n_2.$$

A straightforward computation shows that the function $g(t) := t^2 - (\lambda/q)\beta_n t^q$, for $t > 0$, attains its minimum value at $t = r_n$ and

$$d_n := g(r_n) = -\frac{(2-q)}{2q}\lambda\beta_n\left(\frac{\lambda\beta_m}{2}\right)^{q/(2-q)} < 0.$$

Hence, we infer from (2.21) that $I_\lambda(u) \le d_n$, for any $u \in \partial B_{r_n}(0) \cap V^n$ and $n \ge n_2$.

We now define $n_0 := \max\{n_1, n_2\}$. According to the above considerations, I_λ verifies (A_1) and (A_2). Moreover, since $\beta_n < (8\mu_n)/q$, we have that $r_n < R_n$ and therefore (A_3) also holds. It remains to check (A_4). If $2 < p < 2_*$ or $p = 2_*$ and $b \le 0$, condition (A_4) is a direct consequence of Proposition 2.3.3. If $p = 2_*$ but we have no information about the sign of b, we have compactness at any negative level provided $\lambda > 0$ is small in such way that

$$C_c \lambda^{2/(2-q)} < \frac{1}{2(N-1)} \frac{1}{\|b\|_\infty^{N-2}} S_{2_*,\partial}^{N-1},$$

where $C_c > 0$ comes from Proposition 2.3.4. In any case, we may invoke Theorem 2.3.1 to obtain infinitely many critical points for I_λ. □

Remark 2. *Suppose that $b \le 0$ and let $(u_n) \subset X$ be a sequence of solutions given by Theorem E. If we denote by $c_n = I_\lambda(u_n) \in [b_n, 0]$ the energy of the solutions, we can use $I'_\lambda(u_n)u_n = 0$ and an easy computation to get*

$$c_n = \lambda\left(\frac{1}{2} - \frac{1}{q}\right) \int_{\mathbb{R}^N_+} K(x)a(x)|u_n|^q \, dx + \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u_n|^p \, dx'.$$

Hence,

$$0 \le \lambda \int_{\mathbb{R}^N_+} K(x)a(x)|u_n|^q dx \le -\frac{2q}{(2-q)} c_n$$

and

$$\frac{2p}{(p-2)} c_n \le \int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u_n|^p dx' \le 0.$$

Recalling that $b_n \to 0$, the above inequalities and $I'_\lambda(u_n)u_n = 0$ imply that $\|u_n\| \to 0$.

In order to prove Theorem F we recall that $\Omega_b^+ = \left\{x' \in \mathbb{R}^{N-1} : b(x') > 0\right\}$ and redefine the subspace W in the following way:

$$W := \{u \in X : u(x') = 0 \text{ for a.e. } x' \in \operatorname{int}(\Omega_b^+)\}.$$

As before, V is the orthogonal complement of W in X, in such way that $X = V \oplus W$.

Proof of Theorem F. Setting

$$\mu_n := \sup_{\{u \in V_n \oplus W : \ \|u\| \le 1\}} \int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u|^p dx',$$

we can use $2 < p < 2_*$ and the same argument of Lemma 2.3.5 to conclude that $\mu_n \to 0$, as $n \to +\infty$.

If $u \in V_n \oplus W$, then

$$I_\lambda(u) \ge \frac{1}{2}\|u\|^2 - \frac{\lambda}{q} S_{q\sigma'_q}^{-q/2} \|a\|_{\sigma_q} \|u\|^q - \frac{\mu_n}{p}\|u\|^p,$$

and therefore

$$I_\lambda(u) \geq \frac{1}{4}\|u\|^2 - \frac{\mu_n}{p}\|u\|^p, \qquad \forall\, u \in V_n \oplus W, \quad \|u\| \geq \rho_1, \tag{2.22}$$

where $\rho_1 := \left[4\lambda\|a\|_{\sigma_q} S_{q\sigma_q'}^{-q/2}/q\right]^{1/(2-q)}$. Since $\mu_n \to 0$, there exists $n_1 \in \mathbb{N}$ such that

$$r_n := \left(\frac{p}{8\mu_n}\right)^{1/(p-2)} > \rho_1, \qquad \forall\, n \geq n_1.$$

So, using (2.22) we conclude that

$$b_n = \inf\{I_\lambda(u) : \ u \in V_n \oplus W;\ \|u\| = r_n\} \geq \frac{1}{8} r_n^2.$$

It follows from $\mu_n \to 0$ and the definition of r_n that $(\widetilde{A_2})$ holds.

Arguing as in Theorem E, we have that $\int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u|^p dx'$ defines a norm in the finite dimensional subspace V^n. Then, there exists $0 < \beta_n < 8\mu_n$ such that

$$\beta_n \|u\|^p \leq \int_{\mathbb{R}^{N-1}} K(x',0)b(x')|u|^p dx', \qquad \forall\, u \in V_n.$$

Hence,

$$I_\lambda(u) \leq \|u\|^2 - \frac{\beta_n}{p}\|u\|^p \qquad u \in V^n,\ \|u\| \geq \rho_2,$$

where $\rho_2 := \left(2\lambda S_{q\sigma_q'}^{-q/2}\|a\|_{\sigma_q}/q\right)^{1/(2-q)}$. Setting

$$R_n := \max\left\{2\rho_2, \left(\frac{p}{\beta_n}\right)^{1/(p-2)}\right\},$$

a straightforward computation shows that $I_\lambda(u) \leq 0$, whenever $u \in V^n$ satisfies $\|u\| = R_n$. Since $R_n > r_n$, we conclude that requirement $(\widetilde{A_3})$ if fulfilled.

Since $(PS)_c^*$ implies $(PS)_c$ condition, the proof of $(\widetilde{A_4})$ is analogous to that of Proposition 2.3.3. So, we may invoke Theorem 2.3.2 to obtain a sequence of solutions $(u_n) \subset X$ such that $I_\lambda(u_n) = c_n \to +\infty$, as $n \to +\infty$. Since

$$c_n = I_\lambda(u_n) \leq \frac{1}{2}\|u_n\|^2 + \frac{\lambda}{q}\|a\|_{\sigma_q} S_{q\sigma_q'}^{-q/2}\|u_n\|^q + \frac{1}{p}|b|_\infty S_{p,\partial}^{-p/2}\|u_n\|^p,$$

we conclude that $\|u_n\| \to +\infty$. The theorem is proved. □

CHAPTER 3

Multiplicity of solutions for a superlinear problem

This chapter concerns with existence and multiplicity of solutions for the problem

$$(P_3) \qquad \begin{cases} -\Delta u - \frac{1}{2}(x \cdot \nabla u) = f(u), & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \eta} = \beta |u|^{2_*-2} u, & \text{on } \partial \mathbb{R}^N_+, \end{cases}$$

where $\mathbb{R}^N_+ := \{(x', x_N) \in \mathbb{R}^N_+ : x' \in \mathbb{R}^{N-1},\, x_N > 0\}$ is the upper half-space, $\frac{\partial}{\partial \eta}$ is the partial outward normal derivative, $\beta > 0$ is a parameter, f is a superlinear function with subcritical growth and $2_* := 2(N-1)/(N-2)$, for $N \geq 3$.

A physical motivation comes from the nonlinear heat equation

$$v_t - \Delta v = 0, \text{ in } \mathbb{R}^N_+ \times (0, +\infty), \qquad \frac{\partial v}{\partial \eta} = |v|^{p-2} v, \text{ on } \partial \mathbb{R}^N_+ \times (0, +\infty), \tag{3.1}$$

where $x \in \mathbb{R}^N_+$ is the spatial variable and $t > 0$ is time. A solution with the special form $v(x,t) = t^{\mu} u(t^{-1/2} x)$ is called self-similar solution. It preserves the PDE scaling, providing qualitative properties and giving information about large and small scale behaviors. A direct computation shows that the profile $u : \overline{\mathbb{R}^N_+} \to \mathbb{R}$ verifies

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu u, \quad x \in \mathbb{R}^N_+, \qquad \frac{\partial u}{\partial \eta} = |u|^{p-2} u, \quad x' \in \partial \mathbb{R}^N_+,$$

with $\mu = 1/(2(p-2))$.

Problem (3.1) and their variations have been studied in bounded domain, the half-space $\mathbb{R}^N_+$ and even in the whole space in the last decades; see, e.g., [7, 8, 44, 52, 57, 58, 61, 69, 81, 84] and references therein. Different types of results can be found in these works, such as existence, uniqueness of solutions, blow-up or asymptotic behavior results. To the best of our knowledge, Escobedo and Kavian [44] were the first authors to propose a variational approach to nonlinear heat problems. In their paper, they consider the whole space case and settled the abstract Sobolev spaces appropriated to find solution with rapid decay at infinity. This abstract setting was recently extended to the half-space in [47], including the necessary trace embeddings. In these paper, it was considered existence and nonexistence results for some subcritical versions of (P_3). Some extensions for the critical case were recently proved in [48].

In our first result, we study the effect of the parameter $\beta > 0$ on the number of solutions. So, differently from the aforementioned works, we are concerned with the existence of multiple solutions. Our main assumptions on the superlinear nonlinearity f read as:

(f_0) $f : \mathbb{R} \to \mathbb{R}$ is continuous;

(f_1) there exist $a_1, a_2 > 0$ and $2 < p < 2^* := 2N/(N-2)$ such that

$$|f(s)| \leq a_1 + a_2|s|^{p-1}, \quad \forall\, s \in \mathbb{R};$$

(f_2) there holds

$$\lim_{s \to 0} \frac{f(s)}{s} = 0;$$

(f_3) there exists $2 < \theta < 2_*$ such that

$$0 < \theta F(s) \leq f(s)s, \quad \forall\, s \in \mathbb{R} \setminus \{0\},$$

where $F(s) := \int_0^s f(\tau)\, d\tau$.

By taking advantage of the symmetry properties of the problem, we prove the following:

Theorem G. *Suppose that f is odd and satisfies $(f_0) - (f_3)$. Then, for any given $k \in \mathbb{N}$, there exists $\beta^* = \beta^*(k) > 0$ such that problem (P_3) has at least k pairs of solutions, provided $\beta \in (0, \beta^*)$.*

For the proof we apply a version of the Symmetric Mountain Pass Theorem. The main task here is the management of Palais-Smale sequences and we follow ideas presented in Silva and Xavier [87]. Since we are dealing with unbounded domains, the former argument does not directly apply and we need to perform a trick adaptation of Bianchi, Chabrowski and Szulkin's ideas [15, 27] and the concentration compactness principle due to Lions [67].

In our second result, we do not require symmetry for f and obtained the existence of nonnegative solution. In this case, the parameter β does not play any role and we prove the following:

Theorem H. *Suppose that $N \geq 7$ and f satisfies $(f_0)-(f_3)$. Then problem (P_3) has a nonnegative nonzero solution provided*

$$\lim_{\varepsilon\to 0^+} \varepsilon^{N-2} \int_0^{1/\varepsilon} F\left(\frac{\varepsilon^{-(N-2)/2}}{[s^2+1]^{(N-2)/2}}\right) s^{N-1}\,ds = +\infty. \tag{3.2}$$

In the proof we follow the paper of Brezis and Nirenberg [19]. After obtaining a local compactness condition for the associated functional, we need to prove that it Mountain Pass level belongs to the correct range. At this point we perform some fine estimates and use the technical condition (3.2). It was inspired by a similar one which appeared in [19, Lemma 2.1] and it holds if, for instance, $F(s) \geq \gamma|s|^p$, for some $\gamma > 0$ and $2 < p < 2^*$. In order to check that, it is enough to notice that $g(s) = s^{N-1}/(1+s^2)^{p(N-2)/2}$ is increasing in the interval $[0, s_0]$, where $s_0 = [-(N-1)/(N-1-pN+2p)]^{1/2}$. Hence, for $\varepsilon > 0$ enough small,

$$\int_0^{1/\varepsilon} \frac{s^{N-1}}{(1+s^2)^{p(N-2)/2}}\,ds \geq \int_0^{s_0} \frac{s^{N-1}}{(1+s^2)^{p(N-2)/2}}\,ds > 0$$

and therefore

$$\lim_{\varepsilon\to 0^+} \varepsilon^{N-2-p(N-2)/2} \int_0^{1/\varepsilon} \frac{s^{N-1}}{(s^2+1)^{p(N-2)/2}}\,ds = +\infty.$$

It is worth noticing that, in some sense, our work was inspired by the equation

$$-\Delta u = g(u) + |u|^{2^*-2}u, \;\; x \in \Omega, \quad u = 0, \;\; x \in \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, g is a subcritical perturbation of order greater or equal than one. Brezis and Nirenberg [19], after their pioneer work, promoved an intensive study of critical growth problems (see [10, 23, 26, 40, 86, 87] and references therein). We present here a contribution for this huge class of problems. In a more specific way, the main results of this chapter complement that of [48], since we deal here with a superlinear nonlinearity in $\mathbb{R}^N_+$ and, beyond the nonnegative solution, we also obtain multiplicity results.

The chapter is organized as follows. In the next section, we present the variational framework to deal with (P_3) and prove a compactness result. In Section 3.2 we prove Theorem G and, in the final section, we prove Theorem H.

3.1 Variational framework and the Palais-Smale condition

If we define $K(x) := \exp(|x|^2/4)$, a straightforward computation shows that the first equation in (P_3) becomes

$$-\text{div}(K(x)\nabla u) = K(x)f(u), \quad x \in \mathbb{R}^N_+.$$

Hence, it is natural looking for solutions in the space X defined as the closure of $C_c^\infty(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N_+} K(x) |\nabla u|^2 \, dx \right)^{1/2},$$

which is induced by the inner product

$$(u, v) := \int_{\mathbb{R}^N_+} K(x) (\nabla u \cdot \nabla v) \, dx$$

From now on we identify $\partial \mathbb{R}^N_+ \sim \mathbb{R}^{N-1}$. Given $2 \le r \le 2^*$ and $2 \le s \le 2_*$ we consider the weighted Lebesgue spaces

$$L^r_K(\mathbb{R}^N_+) := \left\{ u \in L^r(\mathbb{R}^N_+) \, : \, \|u\|_r := \left(\int_{\mathbb{R}^N_+} K(x) |u|^r dx \right)^{1/r} < \infty \right\},$$

$$L^s_K(\mathbb{R}^{N-1}) := \left\{ u \in L^s(\mathbb{R}^{N-1}) : |u|_s := \left(\int_{\mathbb{R}^{N-1}} K(x', 0) |u|^s dx' \right)^{1/s} < \infty \right\}.$$

and collect in the next proposition the abstract results proved in [47, 48].

Proposition 3.1.1. *For any $r \in [2, 2^*)$ and $s \in [2, 2_*)$, the embeddings $X \hookrightarrow L^r_K(\mathbb{R}^N_+)$ and $X \hookrightarrow L^s_K(\mathbb{R}^{N-1})$ are compact. Moreover, continuous embeddings hold in the critical cases $r = 2^*$ and $s = 2_*$.*

In view of this result we can define, for $r \in [2, 2^*]$ and $s \in [2, 2_*]$, the following constants:

$$S_r := \inf_{u \in X / \{0\}} \frac{\int_{\mathbb{R}^N_+} K(x) |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N_+} K(x) |u|^r dx \right)^{2/r}},$$

and

$$S_{s,\partial} := \inf_{u \in X / \{0\}} \frac{\int_{\mathbb{R}^N_+} K(x) |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^{N-1}} K(x', 0) |u|^s dx' \right)^{2/s}}.$$

Setting $F(s) := \int_0^s f(\tau) \, d\tau$, it follows from $(f_0) - (f_2)$, Proposition 3.1.1 and standard arguments that the functional $I_\beta : X \to \mathbb{R}$ defined as

$$I_\beta(u) := \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N_+} K(x) F(u) \, dx - \frac{\beta}{2_*} |u|^{2_*}_{2_*},$$

is well defined. Actually, $I_\beta \in C^1(X, \mathbb{R})$ and its critical points are precisely the weak solutions of (P_3).

Recall that, if E is a Banach space, $I \in C^1(E,\mathbb{R})$ and $c \in \mathbb{R}$, the functional I is said to satify the $(PS)_c$ condition if any sequence $(u_n) \subset E$ such that

$$\lim_{n\to+\infty} I(u_n) = c, \qquad \lim_{n\to+\infty} I'(u_n) = 0,$$

has a convergent subsequence. From now on, any such sequence will be called $(PS)_c$-sequence.

The main result of this section can be stated as follows:

Proposition 3.1.2. *Suppose that f satisfies (f_0)-(f_3). For any given $M > 0$, the functional I_β satisfies the $(PS)_c$-condition for any $0 < c \le M$, provided $\beta > 0$ satisfies*

$$\beta < \beta^* := \left(\frac{S_{2_*,\partial}^{N-1}}{2(N-1)M} \right)^{1/(N-2)}. \tag{3.3}$$

Proof. Let $M > 0$ and $(u_n) \subset X$ be a $(PS)_c$-sequence for I_β, with $0 < c \le M$. Using (f_3) and a standard argument we can prove that (u_n) is bounded. Hence, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } X, \\ u_n \to u, & \text{strongly in } L_K^r(\mathbb{R}_+^N) \text{ and } L_K^s(\mathbb{R}^{N-1}), \\ u_n(x) \to u(x), & \text{for a.e. } x \in \overline{\mathbb{R}_+^N}, \end{cases} \tag{3.4}$$

for any $r \in [2, 2_*)$ and $s \in [2, 2_*)$. Moreover, we can easily check that $I_\beta'(u) = 0$.

We claim that, if $\beta < \beta^*$, then

$$\lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*}\, dx' = \int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*}\, dx'. \tag{3.5}$$

Since the proof of this convergence is rather long, we postpone it for the end of the section. So, assuming the claim, we can use (3.4) and Lebesgue's theorem to get

$$\begin{aligned} o(1) = I_\beta'(u_n)u_n &= \|u_n\|^2 - \int_{\mathbb{R}_+^N} K(x)f(u_n)u_n\, dx - \beta |u_n|_{2_*}^{2_*} \\ &= \|u_n\|^2 - \int_{\mathbb{R}_+^N} K(x)f(u)u\, dx - \beta |u|_{2_*}^{2_*} + o(1) \\ &= \|u_n\|^2 - \|u\|^2 + I_\beta'(u)u + o(1), \end{aligned}$$

as $n \to +\infty$. The above expression, $I_\beta'(u)u = 0$ and the weak convergence in (3.4) imply that $u_n \to u$ in X. $\square$

We devote the rest of this section to the proof of (3.5). The first step is to apply the Lions' concentration-compactness principle (see [67, Lema 1.2]). In order to do this let us make some definitions. If Ω is a Hausdorff space, we denote by $\mathcal{M}(\Omega)$ the space of finite Radon measures defined on the Borel σ-algebra of Ω. For the convenience of the reader we will prove the following Concentration and Compactness lemma:

Lemma 3.1.3. *If $(u_n) \subset X$ is a bounded sequence, then there exist an at most countable family J, positive numbers $\{\mu_j\}_{j\in J}$, $\{\nu_j\}_{j\in J}$, and points $\{x_j\}_{j\in J} \subset \partial\mathbb{R}^N_+$ such that*

$$\begin{cases} K(x)|\nabla u_n|^2 dx & \rightharpoonup & \mu \geq K(x)|\nabla u|^2 dx + \sum_{j\in J} \mu_j \delta_{x_j}, \\ K(x',0)|u_n|^{2_*} dx' & \rightharpoonup & \nu = K(x',0)|u|^{2_*} dx' + \sum_{j\in J} \nu_j \delta_{x_j}, \\ \mu_j & \geq & S_{2_*,\partial}(\nu_j)^{2/2_*}, \end{cases} \tag{3.6}$$

where the convergences hold in the sense of the measures, $\mu \in \mathcal{M}(\overline{\mathbb{R}^N_+})$, $\nu \in \mathcal{M}(\partial\mathbb{R}^N_+)$ e δ_x is the Dirac mass concentrated on $x \in \mathbb{R}^N$.

Proof. Since (u_n) is bounded in X, we have that $(K|\nabla u_n|^2)$ and $(K(\cdot,0)|u_n|^{2_*})$ are bounded sequences in $L^1(\mathbb{R}^N_+)$ and $L^1(\mathbb{R}^{N-1})$, respectively. Hence, we can suppose that

$$K(x)|\nabla u_n|^2\, dx \rightharpoonup \mu, \qquad K(x',0)|u_n|^{2_*}\, dx' \rightharpoonup \nu,$$

weakly in the sense of measures (see [45, Definition 1.1.2]), where $\mu \in \mathcal{M}(\overline{\mathbb{R}^N_+})$, $\nu \in \mathcal{M}(\partial\mathbb{R}^N_+)$ and we have identified $\partial\mathbb{R}^N_+ \simeq \mathbb{R}^{N-1}$.

Given $\phi \in C_0^\infty(\overline{\mathbb{R}^N_+})$, we have that $(\phi u_n) \subset X$ and hence

$$S_{2_*,\partial}^{1/2} | \phi u_n |_{2_*} \leq \|\phi u_n\| = \|\nabla(\phi u_n)\|_2. \tag{3.7}$$

Notice that

$$| \phi u_n |_{2_*}^{2_*} = \int_{\partial\mathbb{R}^N_+} K(x',0)|\phi|^{2_*}|u_n|^{2_*}\, dx' \to \int_{\partial\mathbb{R}^N_+} |\phi|^{2_*}\, d\nu. \tag{3.8}$$

Moreover,

$$\begin{aligned} \|\nabla(\phi u_n)\|_2^2 &= \int_{\mathbb{R}^N_+} K(x)|\nabla(\phi u_n)|^2\, dx \\ &= \int_{\mathbb{R}^N_+} |\phi|^2 K(x)|\nabla u_n|^2\, dx + \int_{\mathbb{R}^N_+} K(x)|u_n|^2|\nabla\phi|^2\, dx. \end{aligned} \tag{3.9}$$

We may assume that $u_n \rightharpoonup u$ weakly in X and we first consider the case $u = 0$. From the compact embedding $X \hookrightarrow L^2_K(\mathbb{R}^N_+)$, we may also suppose that $u_n \to 0$ strongly in $L^2_K(\mathbb{R}^N_+)$, and thus

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N_+} K(x)|u_n|^2|\nabla\phi|^2\, dx = 0.$$

The above convergence and (3.9) imply that

$$\lim_{n\to+\infty} \|\nabla(\phi u_n)\|_2^2 = \int_{\mathbb{R}^N_+} |\phi|^2\, d\mu.$$

This, (3.7) and (3.8) provide the following inequality

$$\left(\int_{\partial\mathbb{R}^N_+} |\phi|^{2_*}\, d\nu\right)^{1/2_*} \le S_{2_*,\partial}^{-\frac{1}{2}} \left(\int_{\mathbb{R}^N_+} |\phi|^2\, d\mu\right)^{1/2}. \tag{3.10}$$

Denoting by $\mathcal{L}(\mathbb{R}^N)$ the Lebesgue σ-algebra of $\mathbb{R}^N$, let us define the following measures in $\mathcal{L}(\mathbb{R}^N)$:

$$\tilde{\mu}(A) := \mu(A \cap \overline{\mathbb{R}^N_+}), \qquad \tilde{\nu}(A) := \nu(A \cap \partial\mathbb{R}^N_+).$$

Since ν and μ are finite Radon measures and elements of the form $A\cap\partial\mathbb{R}^N_+$ and $A\cap\overline{\mathbb{R}^N_+}$, with $A \in \mathcal{L}(\mathbb{R}^N)$, belong to the Lebesgue σ-algebras of $\partial\mathbb{R}^N_+$ and $\overline{\mathbb{R}^N_+}$, respectively, it follows that $\tilde{\nu}$ are $\tilde{\mu}$ are finite Radon measures . Moreover, from the definitions of $\tilde{\mu}$, $\tilde{\nu}$ and (3.10) we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |\phi|^{2_*}\, d\tilde{\nu}\right)^{\frac{1}{2_*}} &= \left(\int_{\partial\mathbb{R}^N_+} |\phi|^{2_*}\, d\nu\right)^{\frac{1}{2_*}} \\ &\le S_{2_*,\partial}^{-\frac{1}{2}} \left(\int_{\mathbb{R}^N_+} |\phi|^2\, d\mu\right)^{\frac{1}{2}} = S_{2_*,\partial}^{-\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\phi|^2\, d\tilde{\mu}\right)^{\frac{1}{2}},\end{aligned} \tag{3.11}$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$. Hence, it follows from the Lions' concentration-compactness principle [67, Lemma 1.2] for the whole space the existence of a family at most countable J, positive numbers $\{\nu_j\}_{j\in J}$ and points $\{x_j\}_{j\in J} \subset \mathbb{R}^N$ such that

$$\tilde{\nu} = \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \tilde{\mu} \ge S_{2_*,\partial} \sum_{j\in J} \nu_j^{\frac{2}{2_*}} \delta_{x_j}.$$

We claim that $\{x_j\}_{j\in J} \subset \partial\mathbb{R}^N_+$. Indeed, if this is not the case, we can use the above expression and the definition of $\tilde{\nu}$ to get

$$0 < \nu_j = \sum_{j\in J} \nu_j \delta_{x_j}(\{x_j\}) = \tilde{\nu}(\{x_j\}) = \nu(\{x_j\} \cap \partial\mathbb{R}^N_+) = 0,$$

which does not make sense. Hence, the claim holds true.

Since $\tilde{\nu}$ and $\tilde{\mu}$ restricted to $\partial\mathbb{R}^N_+$ and $\overline{\mathbb{R}^N_+}$ coincide with ν and μ, respectively, we have that

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j} \qquad \mu \ge S_{2_*,\partial} \sum_{j\in J} \nu_j^{\frac{2}{2_*}} \delta_{x_j}.$$

In the general case $u \neq 0$, it is sufficient to set $v_n := u_n - u$, notice that $v_n \rightharpoonup 0$ weakly in X, and argue as above with u_n replaced by v_n □

Lemma 3.1.4. *If* (3.3) *holds, then J is empty.*

Proof. Suppose, by contradiction, that $\beta < \beta^*$ and there exists some $j \in J$. We first claim that

$$\nu_j \geq \left(\frac{S_{2_*,\partial}}{\beta}\right)^{N-1}. \tag{3.12}$$

Assuming the claim, we can prove the lemma in the following way. Pick $\psi \in C_0^\infty(B_2(x_j))$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in $B_1(x_j)$. Computing $I_\beta(u_n) - (1/2)I'_\beta(u_n)u_n$ and using (f_3), we obtain

$$\begin{aligned} c + o(1) &\geq \frac{\beta}{2(N-1)} \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*}\, dx' \\ &\geq \frac{\beta}{2(N-1)} \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*}\psi(x',0)\, dx', \end{aligned}$$

where $o(1)$ denotes a quantity approaching zero as $n \to +\infty$. Passing to the limit, using (3.6) and (3.12), we obtain

$$M \geq c \geq \frac{\beta}{2(N-1)} \int_{B_1(x_j)\cap\mathbb{R}^{N-1}} \psi(x',0)\, d\nu \geq \frac{\beta}{2(N-1)}\nu_j \geq \frac{\beta}{2(N-1)} \left(\frac{S_{2_*,\partial}}{\beta}\right)^{N-1},$$

which is equivalent to $\beta \geq \beta^*$, contrary to (3.3). Hence, J is empty.

It remains to prove (3.12). For that, we consider $\phi \in C_0^\infty(B_2(0))$ such that $0 \leq \phi \leq 1$ and $\phi \equiv 1$ in $B_1(0)$ and define

$$\phi_j^\varepsilon(x) := \phi\left(\frac{x - x_j}{\varepsilon}\right), \quad x \in \mathbb{R}^N.$$

Since $I'_\beta(u_n)(u_n\phi_j^\varepsilon) = o(1)$, we obtain

$$\begin{aligned} \left[\int \phi_j^\varepsilon\, d\mu - \beta \int \phi_j^\varepsilon\, d\nu\right] + o(1) &= \int_{\mathbb{R}^N_+} K(x)f(u_n)u_n\phi_j^\varepsilon\, dx \\ &\quad - \int_{\mathbb{R}^N_+} K(x)(\nabla u_n \cdot \nabla\phi_j^\varepsilon)u_n\, dx. \end{aligned} \tag{3.13}$$

We shall estimate each term on the right side above. First notice that, by using $(f_1) - (f_2)$, (3.4) and Lebesgue's theorem we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N_+} K(x)f(u_n)u_n\phi_j^\varepsilon\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^N_+} K(x)f(u)u\phi_j^\varepsilon\, dx.$$

Moreover, since $\mathrm{supp}(\phi_j^\varepsilon) \subset B_{2\varepsilon}(x_j)$, we can use Lebesgue's theorem again to get

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N_+} K(x)f(u_n)u_n\phi_j^\varepsilon\, dx = 0. \tag{3.14}$$

By using Holder's inequality, that (u_n) is bounded and the definition of ϕ_j^ε we get that

$$\begin{aligned} \left|\int_{\mathbb{R}^N_+} K(x)(\nabla u_n \cdot \nabla\phi_j^\varepsilon)u_n\, dx\right| &\leq \|u_n\| \left(\int_{\Omega_\varepsilon^j} K(x)(u_n)^2|\nabla\phi_j^\varepsilon|^2\, dx\right)^{1/2} \\ &= \frac{c_1}{\varepsilon}\left(\int_{\Omega_\varepsilon^j} K(x)(u_n)^2 \left|\nabla\phi\left(\frac{x-x_j}{\varepsilon}\right)\right|^2 dx\right)^{1/2}, \end{aligned}$$

where $\Omega_\varepsilon^j := B_{2\varepsilon}(x_j)\cap\mathbb{R}^N_+$ and $c_1 > 0$ is independent of n. If we call $\Sigma_{n,\varepsilon}$ the left-hand side of the above expression, we can use the change of variable $y = (x - x_j)/\varepsilon$ and the strong convergence $u_n \to u$ in $L^2_K(\mathbb{R}^N_+)$ to get

$$\Sigma_{n,\varepsilon} \le c_2\varepsilon^{(N-2)/2}\left(\int\limits_{B_{2\varepsilon(0)}\cap\mathbb{R}^N_+} K(\varepsilon y + x_j)u^2(\varepsilon y + x_j)\,dy + o(1)\right)^{\frac{1}{2}},$$

where $c_2 = c_1\|\nabla\phi\|_\infty$. It follows that

$$\lim_{\varepsilon\to 0}\lim_{n\to+\infty}\int_{\mathbb{R}^N_+} K(x)(\nabla u_n\cdot\nabla\phi_j^\varepsilon)u_n\,dx = 0.$$

Passing (3.13) to the limit, using the above expression, (3.14) and (3.6), we obtain

$$\beta\nu_j = \beta\lim_{\varepsilon\to 0^+}\int\phi_j^\varepsilon\,d\nu = \lim_{\varepsilon\to 0^+}\int\phi_j^\varepsilon\,d\mu \ge \lim_{\varepsilon\to 0^+}\left[\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2\phi_j^\varepsilon\,dx + \sum_{i\in J}\mu_i\phi_j^\varepsilon(x_i)\right].$$

Now, we can use the Lebesgue's Theorem, the definition of ϕ_j^ε and (3.6) to get that

$$\beta\nu_j \ge \mu_j\phi(0) = \mu_j \ge S_{2_*,\partial}\nu_j^{2/2_*}$$

that is, $\nu_j^{1-(2/2_*)} \ge S_{2_*,\partial}/\beta$, which is equivalent to (3.12). The lemma is proved. □

In the next result we follow an argument due to Bianchi et al. [15].

Lemma 3.1.5. *If*

$$\nu_\infty := \lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{\{x'\in\mathbb{R}^{N-1}:|x'|\ge R\}} K(x',0)|u_n|^{2_*}\,dx',$$

then $\nu_\infty = 0$ *or* $\nu_\infty \ge \left(\frac{S_{2_*,\partial}}{\beta}\right)^{N-1}$.

Proof. Set

$$\mu_\infty := \lim_{R\to+\infty}\limsup_{n\to+\infty}\int_{\mathbb{R}^N_+\setminus B_R(0)} K(x)|\nabla u_n|^2\,dx$$

and consider, for each $R > 1$, a function $\phi_R \in C^\infty(\overline{\mathbb{R}^N})$ such that $\phi_R \equiv 0$ in $B_R(0)$ and $\phi_R \equiv 1$ outside $B_{R+1}(0)$. Since $(u_n\phi_R) \subset X$, we have that $S_{2_*,\partial}\|u_n\phi_R\|^2_{2_*} \le \|u_n\phi_R\|^2$ and we can use (3.4) to obtain

$$\begin{aligned} S_{2_*,\partial}\limsup_{n\to+\infty}\left(\int_{\mathbb{R}^{N-1}} K(x',0)|\phi_R u_n|^{2_*}\,dx'\right)^{\frac{2}{2_*}} \;\le\; & \limsup_{n\to+\infty}\int_{\mathbb{R}^N_+} K(x)|\nabla u_n|^2\phi_R^2\,dx \\ + & \int_{\mathbb{R}^N_+} K(x)|\nabla\phi_R|^2u^2\,dx. \end{aligned}$$

Passing to the limit as $R \to +\infty$, using the definition of ϕ_R and the Lebesgue's theorem, we conclude that

$$S_{2_*,\partial}\nu_\infty^{2/2_*} \le \mu_\infty. \tag{3.15}$$

Using that $I'_\beta(u_n)(u_n\phi_R) = o(1)$, together with (3.4) and Lebesgue's theorem, we get

$$\limsup_{n\to+\infty} B_n \le \int_{\mathbb{R}^N_+} K(x)f(u)u\phi_R\,dx + \beta \limsup_{n\to+\infty} C_n + \limsup_{n\to+\infty} -A_n, \tag{3.16}$$

where

$$A_n := \int_{\mathbb{R}^N_+} K(x)(\nabla u_n \cdot \nabla\phi_R)u_n\,dx, \qquad B_n := \int_{\mathbb{R}^N_+} K(x)|\nabla u_n|^2\phi_R\,dx,$$

and

$$C_n := \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*}\phi_R\,dx'.$$

From Holder's inequality, we obtain

$$-A_n \le \|u_n\|^2 \left(\int_{B_{R+1}(0)\setminus B_R(0)} K(x)|\nabla\phi_R|^2 u_n^2\,dx\right)^{1/2}.$$

The above inequality, Proposition 3.1.1, the definition of ϕ_R and Lebesgue's theorem imply that

$$\lim_{R\to+\infty}\limsup_{n\to+\infty} -A_n \le 0. \tag{3.17}$$

Moreover, as before, it follows from the definition of ϕ_R that

$$\lim_{R\to+\infty}\limsup_{n\to+\infty} B_n = \mu_\infty, \qquad \lim_{R\to+\infty}\limsup_{n\to+\infty} C_n = \nu_\infty.$$

Passing (3.16) to the limit as $R \to +\infty$, using (3.15), (3.17), the above equalities and Lebesgue's theorem, we get $S_{2_*,\partial}\nu_\infty^{2/2_*} \le \mu_\infty \le \beta\nu_\infty$, from which the result follows. □

We are ready to prove that (3.5) holds. First notice that, in view of the pointwise convergence in (3.4) and Fatou's lemma, it is sufficient to check that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*}\,dx' \le \int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*}\,dx'.$$

Since $\beta < \beta^*$, the set J is empty. Hence, the weak convergence in the sense of measure (3.6) imply that (see [46, Theorem 1, Section 1.9]), for each $R > 0$, there holds

$$\begin{aligned}\limsup_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*}\,dx' &= \limsup_{n\to+\infty} \int_{\{x'\in\mathbb{R}^{N-1}:|x'|>R\}} K(x',0)|u_n|^{2_*}\,dx' \\ &+ \int_{\{x'\in\mathbb{R}^{N-1}:|x'|\le R\}} K(x',0)|u|^{2_*}\,dx'.\end{aligned}$$

Passing to the limit as $R \to +\infty$, we obtain

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)|u_n|^{2_*}\,dx' = \nu_\infty + \int_{\mathbb{R}^{N-1}} K(x',0)|u|^{2_*}\,dx',$$

where ν_∞ was defined in Lemma 3.1.5.

It remains to check that $\nu_\infty = 0$. In order to do this, notice that the same argument of the proof of Lemma 3.1.4 provides

$$c + o(1) \geq \frac{\beta}{2(N-1)} \int_{\{x'\in\mathbb{R}^{N-1}:|x'|\geq R\}} K(x',0)|u_n|^{2_*}\,dx',$$

for any $R > 0$. Recalling that $c \leq M$, we obtain

$$M \geq \lim_{R\to+\infty} \limsup_{n\to+\infty} \frac{\beta}{2(N-1)} \int\limits_{\{x'\in\mathbb{R}^{N-1}:|x'|\geq R\}} K(x',0)|u_n|^{2_*}\,dx' = \frac{\beta}{2(N-1)}\nu_\infty.$$

If $\nu_\infty \neq 0$, we can use the above inequality and Lemma 3.1.5 to obtain $\beta \geq \beta^*$, which contradicts (3.3). Hence, $\nu_\infty = 0$ and we conclude that (3.5) is verified.

3.2 Proof of Theorem G

Our first main result will be proved as an application of the following version of the Symmetric Moutain Pass Theorem (see [3]).

Theorem 3.2.1. *Let $E = V \oplus W$, where E is a real Banach space and V is finite dimensional. Suppose $I \in C^1(E,R)$ is an even functional satisfying $I(0) = 0$ and*

- (I^1) *there exist constants $\rho, \alpha > 0$ such that $I\,|_{B_\rho(0)\cap W} \geq \alpha$;*
- (I^2) *there exists a subspace $\tilde{V}$ of E such that $\dim V < \dim \tilde{V} < \infty$ and $\max\limits_{u\in\tilde{V}} I(u) \leq M$, for some constant $M > 0$;*
- (I^3) *I satisfies $(PS)_c$, for any $0 < c < M$.*

Then I has at least $\dim \tilde{V} - \dim V$ pairs of nonzero critical points.

We are intending to apply this abstract result with $E = X$ and $I = I_\beta$. For the required decomposition of the space X we consider the linearized problem

$$(LP) \qquad \begin{cases} -\operatorname{div}(K(x)\nabla u) = \lambda K(x)u, & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial \eta} = 0, & \text{on } \mathbb{R}^{N-1}. \end{cases}$$

Thanks to the compact embedding $X \hookrightarrow L^2_K(\mathbb{R}^N_+)$, we can use standard spectral theory to obtain a sequence of eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ such that

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$$

with $\lim\limits_{j\to\infty} \lambda_j = +\infty$. Moreover, the first eigenvalue is given by

$$\lambda_1 = \left\{ \int_{\mathbb{R}^N_+} K(x)|\nabla u|^2\, dx : \int_{\mathbb{R}^N_+} K(x)u^2\, dx = 1 \right\}.$$

From this, we obtain the following Poincaré inequality

$$\lambda_1 \int_{\mathbb{R}^N_+} K(x)u^2\, dx \leq \int_{\mathbb{R}^N_+} K(x)|\nabla u|^2\, dx, \qquad \forall u \in X. \tag{3.18}$$

We are ready to prove our multiplicity result.

Proof of Theorem G. In order to apply Theorem 3.2.1, we consider $V = \{0\}$ and $W = X$.

Given $\varepsilon > 0$, we can use (f_1) and (f_2) to obtain $c_1 = c_1(\varepsilon) > 0$ such that

$$|F(s)| \leq \frac{\varepsilon}{2}s^2 + c_1|s|^p, \qquad \forall\, s \in \mathbb{R}.$$

Picking $\varepsilon > 0$ such that $\varepsilon < \lambda_1$, we can use (3.18) and Proposition 3.1.1 to get

$$\begin{aligned} I_\beta(w) &\geq \frac{1}{2}\|w\|^2 - \frac{1}{2}\varepsilon\|w\|_2^2 - c_1\|w\|_p^p - \frac{\beta}{2_*}\|w\|_{2_*} \\ &\geq \frac{1}{2}\left[\frac{\lambda_1 - \varepsilon}{\lambda_1}\right]\|w\|^2 - c_1 S_p^{-p/2}\|w\|^p - S_{2_*,\partial}^{-2_*/2}\|w\|^{2_*}, \end{aligned}$$

for any $w \in W$. Since $2 < p < 2^*$, we conclude that

$$I_\beta(w) \geq \frac{\|w\|^2}{2}\left[c_2 + o(\|w\|^2)\right], \quad \text{as } \|w\| \to 0,\ w \in W,$$

with $c_2 = (\lambda_1 - \varepsilon)/(\lambda_1) > 0$. This proves that (I^1) holds.

Given $k \in \mathbb{N}$, we consider $\{\psi_i\}_{i=1}^k \subset C_0^\infty(\overline{\mathbb{R}^N_+})$ smooth functions with disjoint supports and denote

$$\widetilde{V} := \operatorname{span}\{\psi_1, \ldots, \psi_k\}.$$

Then, $\dim \widetilde{V} = k$ and there exists a large ball $B_R(0) \subset \overline{\mathbb{R}^N_+}$ containing the support of all the functions $\psi_1, \ldots, \psi_k$.

Notice that (f_3) provides c_3, $c_4 > 0$ such that

$$F(s) \geq c_3 s^\theta - c_4, \qquad \forall s \in \mathbb{R}, \tag{3.19}$$

with $\theta > 2$. Hence, for any $v \in \widetilde{V}$, the equivalence of norms in $\tilde{V}$ implies that

$$I_\beta(v) \leq \frac{1}{2}\|v\| - c_5\|v\|^\theta - c_4 \text{meas}(B_R(0)) \to -\infty, \qquad \text{as } \|v\| \to +\infty.$$

Since I_β maps bounded sets into bounded sets, it follows from the above expression that $\max_{v\in\widetilde{V}} I_\beta(v) \leq M$, for some constant $M > 0$. This proves (I^2).

We now consider $\beta^* > 0$ as in Proposition 3.1.2 and invoke Theorem 3.2.1 to obtain k pairs of nonzero solution whenever $\beta \in (0, \beta*)$. The theorem is proved. □

3.3 Proof of Theorem H

We prove in this section Theorem H. Since we are looking for nonnegative solutions, we shall assume that $f(s) = 0$, for any $s \le 0$. Moreover, since the parameter $\beta > 0$ does not play any rule in in this case, we assume from now on that $\beta = 1$ and consider the functional

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N_+} K(x)F(u)\,dx - \frac{1}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)(u^+)^{2_*}\,dx',$$

where $u^+(x) := \max\{u(x),0\}$. It is clear that $I \in C^1(X,\mathbb{R})$. Moreover, if u is such that $I'(u) = 0$ and $u^- := u^+ - u$, then $0 = I'(u)u^- = -\|u^-\|^2$. Hence, the critical points of I are nonnegative solutions of our problem.

We start with a local compactness result.

Lemma 3.3.1. *The functional I satisfies the $(PS)_c$-condition for any*

$$c < c^* := \frac{S_{2_*,\partial}^{N-1}}{2(N-1)}.$$

Proof. Let $(u_n) \subset X$ be such that $I(u_n) \to c$ and $I'(u_n) \to 0$. As before, (u_n) is bounded and therefore there exists $u \in X$ such that (3.4) holds. Moreover, from (f_1), (f_2) and the Lebesgue's Theorem, we conclude that $I'(u) = 0$ and

$$\int_{\mathbb{R}^N_+} K(x)F(u_n)\,dx = \int_{\mathbb{R}^N_+} K(x)F(u)\,dx + o(1)$$

and

$$\int_{\mathbb{R}^N_+} K(x)f(u_n)u_n\,dx = \int_{\mathbb{R}^N_+} K(x)f(u)u\,dx + o(1),$$

as $n \to +\infty$.

If $z_n := (u_n - u)$, we can use the above expressions, $I'(u_n)u_n = o(1)$ and Brezis-Lieb's lemma [18] to get

$$\begin{aligned} o(1) &= \|u_n\|^2 - \int_{\mathbb{R}^N_+} K(x)f(u_n)u_n\,dx - \int_{\mathbb{R}^{N-1}} K(x',0)(u_n^+)^{2_*}\,dx' \\ &= I'(u)u + \|z_n\|^2 - \int_{\mathbb{R}^{N-1}} K(x',0)(z_n^+)^{2_*}\,dx' + o(1). \end{aligned}$$

Passing to the limit and using $I'(u) = 0$, we obtain $b \ge 0$ such that

$$\lim_{n\to+\infty} \|z_n\|^2 = b = \lim_{n\to+\infty} \int_{\mathbb{R}^{N-1}} K(x',0)(z_n^+)^{2_*}\,dx'.$$

We claim that $b = 0$. In order to prove this, we first pass to the limit the inequality

$$\int_{\mathbb{R}^{N-1}} K(x',0)(z_n^+)^{2_*}dx' \le S_{2_*,\partial}^{-2_*/2}\left(\int_{\mathbb{R}^N_+} K(x)|\nabla z_n|^2 dx\right)^{2_*/2},$$

to obtain $b \leq S_{2_*,\partial}^{-2_*/2} b^{2_*/2}$. Hence, if $b > 0$, we get

$$b \geq S_{2_*,\partial}^{N-1}. \tag{3.20}$$

On the other hand, using Brezis-Lieb again, we obtain

$$c + o(1) = I(u_n) = I(u) + \frac{1}{2}\|z_n\|^2 - \frac{1}{2_*}\int_{\mathbb{R}^{N-1}} K(x',0)(z_n^+)^{2_*}\,dx' + o(1).$$

Taking the limit and using (3.20), we get that

$$c = I(u) + \frac{\lambda}{2(N-1)} \geq I(u) + \frac{S_{2_*,\partial}^{N-1}}{2(N-1)} = I(u) + c^*.$$

Using (f_3) we obtain $I(u) = I(u) - (1/\theta)I'(u)u \geq 0$, and therefore the above expression implies that $c \geq c_*$, which does not make sense. $\square$

Let us take $\phi \in C^\infty(\overline{\mathbb{R}^N_+}, [0,1])$ such that $\phi \equiv 1$ in $\overline{\mathbb{R}^N_+} \cap B_1(0)$ and $\phi \equiv 0$ in $\overline{\mathbb{R}^N_+} \backslash B_2(0)$. Set, for each $\varepsilon > 0$,

$$u_\varepsilon(x) := K(x)^{-1/2}\phi(x)U_\varepsilon(x), \quad x \in \mathbb{R}^N_+,$$

where

$$U_\varepsilon(x', x_N) := \frac{\varepsilon^{(N-2)/2}}{[|x'|^2 + (x_N+\varepsilon)^2]^{(N-2)/2}}.$$

When $N \geq 7$, it is proved in [48] that, as $\varepsilon \to 0^+$,

$$\|u_\varepsilon\|^2 = A_N + O(\varepsilon^2), \qquad |u_\varepsilon|_{2_*}^{2_*} = B_N^{2_*/2} + O(\varepsilon^2), \tag{3.21}$$

with the constants above being such that $A_N/B_N = S_{2_*,\partial}$. We shall need the following estimates:

Lemma 3.3.2. *If $\psi_\varepsilon := u_\varepsilon/|u_\varepsilon|_{2_*}$ and $N/(N-2) < q < 2N/(N-2)$, then*

$$\|\psi_\varepsilon\|^{2(N-1)} = S_{2_*,\partial}^{N-1} + O(\varepsilon^2), \qquad \|\psi_\varepsilon\|_q^q = O(\varepsilon^{N-q(N-2)/2}), \tag{3.22}$$

as $\varepsilon \to 0^+$.

Proof. Using the Mean Value theorem for $g(r) = r^s$ and a simple computation, we can check that

$$\left[A + O(\varepsilon^t)\right]^s = A^s + O(\varepsilon^t),$$

for any A, s, $t > 0$. Hence, we infer from (3.21) and the defintion of 2_* that

$$\|\psi_\varepsilon\|^{2(N-1)} = \frac{[A_N + O(\varepsilon^2)]^{N-1}}{\left[B_N^{2_*/2} + O(\varepsilon^2)\right]^{N-2}} = \frac{A_N^{N-1} + O(\varepsilon^2)}{B_N^{2_*(N-2)/2} + O(\varepsilon^2)} = \left(\frac{A_N}{B_N}\right)^{N-1} + O(\varepsilon^2).$$

The first statement in (3.22) follows from the above inequality and $A_N/B_N = S_{2_*,\partial}$.

For the second one, we first notice that

$$\begin{aligned}\|u_\varepsilon\|_q^q &= \varepsilon^{-q(N-2)/2}\int_{\mathbb{R}^N_+}\frac{K(x)^{-q/2}\phi(x)^q}{[|x'/\varepsilon|^2+(x_N/\varepsilon+1)^2]^{q(N-2)/2}}\,dx\\ &\leq C_1\varepsilon^{-q(N-2)/2}\int_{B_2(0)\cap\mathbb{R}^N_+}\frac{1}{[|x/\varepsilon|^2+1]^{q(N-2)/2}}\,dx\\ &\leq C_1\varepsilon^{-q(N-2)/2+N}\int_{\mathbb{R}^N_+}\frac{1}{[|y|^2+1]^{q(N-2)/2}}\,dy,\end{aligned}$$

where we have used the definition of u_ε, $0\leq\phi\leq 1$ and the change of variable $y=x/\varepsilon$. But

$$\begin{aligned}\int_{\mathbb{R}^N_+}\frac{1}{[|y|^2+1]^{q(N-2)/2}}\,dy &\leq C_2+\int_{\mathbb{R}^N_+\setminus B_1(0)}\frac{1}{|y|^{q(N-2)}}\,dy\\ &= C_2+C_3\int_1^{+\infty}s^{-q(N-2)+(N-1)}\,ds<+\infty,\end{aligned}$$

whenever $q>N/(N-2)$. Since $|u_\varepsilon|_{2_*}^q=B_N^{q/2}+o(1)$, as $\varepsilon\to 0^+$, the result follows from the above inequalities. □

We are ready to prove our second main result.

Proof of Theorem H. Arguing as in the proof of Theorem A we obtain $\rho,\alpha>0$ such that $I(u)\geq\alpha$, whenever $\|u\|\geq\rho$. Moreover, it follows from (3.19) that

$$\frac{I(t\psi_\varepsilon)}{t^{2_*}}\leq\frac{1}{2t^{2_*-2}}\|\psi_\varepsilon\|^2-\frac{c_3}{t^{2_*-\theta}}\|\psi_\varepsilon\|_\theta^\theta+\frac{c_4}{t^{2_*}}\text{meas}(\text{supp}\psi_\varepsilon)-\frac{1}{2_*}|\psi_\varepsilon|_{2_*}^{2_*},$$

for $t>0$. Thus, there exists $\bar{t}>0$ such that $e=\bar{t}\psi_\varepsilon$ satisfies $I(e)<0$ and $\|e\|>\rho$. If we set

$$c_\varepsilon:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t))\geq\alpha,$$

where $\Gamma:=\{\gamma\in C([0,1],X):\ \gamma(0)=0,\ \gamma(1)=e\}$, we obtain from the Mountain Pass Theorem [3] a sequence $(u_n)\subset X$ such that $I(u_n)\to c_\varepsilon$ and $I'(u_n)\to 0$. If $c_\varepsilon<c^*$, it follows from Lemma 3.3.1 that, along a subsequence, (u_n) strongly converges to a critical point $u\in X$ such that $I(u)=c_\varepsilon\geq\alpha>0$. Thus, $u\geq 0$ is a nonzero solution of the problem.

It remains to check that, for some $\varepsilon>0$ small, there holds $c_\varepsilon<c^*$. In order to do that, we set

$$m_\varepsilon:=\max_{t\geq 0}I(t\psi_\varepsilon)$$

and notice that it is sufficient to prove that $m_\varepsilon<c^*$. Let $t_\varepsilon>0$ be such that $m_\varepsilon=I(t_\varepsilon\psi_\varepsilon)$. Since $I'(t_\varepsilon\psi_\varepsilon)\psi_\varepsilon=0$ and $|\psi_\varepsilon|_{2_*}=1$, we get

$$t_\varepsilon^{2_*-1}=t_\varepsilon\|\psi_\varepsilon\|^2-\int_{\mathbb{R}^N_+}K(x)f(t_\varepsilon\psi_\varepsilon)\psi_\varepsilon\,dx, \tag{3.23}$$

The above identity and (f_3) imply that

$$t_\varepsilon \leq \|\psi_\varepsilon\|^{2/(2_*-2)}.$$

Since the function $g : [0, +\infty) \to \mathbb{R}$ defined by $g(t) := (t^2/2)\|\psi_\varepsilon\|^2 - t^{2_*}/2_*$ is increasing in the interval $[0, \|\psi_\varepsilon\|^{2/(2_*-2)}]$, we can use the above inequality and (3.22) to get

$$\begin{aligned} m_\varepsilon &= g(t_\varepsilon) - \int_{\mathbb{R}^n_+} K(x)F(t_\varepsilon\psi_\varepsilon)\,dx \\ &\leq \frac{\|\psi_\varepsilon\|^{2(N-1)}}{2(N-1)} - \int_{\mathbb{R}^n_+} K(x)F(t_\varepsilon\psi_\varepsilon)\,dx \\ &= \frac{S_{2_*,\partial}^{N-1}}{2(N-1)} + O(\varepsilon^2) - \int_{\mathbb{R}^n_+} K(x)F(t_\varepsilon\psi_\varepsilon)\,dx. \end{aligned}$$

So, it is sufficient to prove that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2}\int_{\mathbb{R}^n_+} K(x)F(t_\varepsilon\psi_\varepsilon)\,dx = +\infty. \tag{3.24}$$

First notice that, by (f_1), (f_2), (3.22) and $p < 2^*$, it follows that

$$\left|\int_{\mathbb{R}^N_+} K(x)f(t_\varepsilon\psi_\varepsilon)\psi_\varepsilon\,dx\right| \leq O(\varepsilon^2) + O(\varepsilon^{N-p(N-2)/2}) = o(1),$$

as $\varepsilon \to 0^+$. This, together with (3.22) and (3.23), implies that $t_\varepsilon \to S_{2_*,\partial}^{(N-2)/2} > 0$, as $\varepsilon \to 0^+$. Thus, since (f_3) implies that F is increasing in $[0, +\infty)$, we can use (3.21), $K \geq 1$ and the definition of ϕ to obtain $C_1 > 0$ such that

$$\int_{\mathbb{R}^N_+} K(x)F(t_\varepsilon\psi_\varepsilon)\,dx \geq \int_{B_1(0)\cap\mathbb{R}^N_+} F\left(C_1\frac{\varepsilon^{(N-2)/2}}{[|x'|^2 + (x_N+\varepsilon)^2]^{(N-2)/2}}\right)dx, \tag{3.25}$$

for any $\varepsilon > 0$ small. If we call Γ_ε the right-hand side above, the change of variables $y = x/\varepsilon$ gives

$$\Gamma_\varepsilon = \varepsilon^N\int_0^{1/\varepsilon}\int_{\partial B_s(0)\cap\mathbb{R}^N_+} F\left(C_1\frac{\varepsilon^{-(N-2)/2}}{[|y'|^2 + (y_N+1)^2]^{(N-2)/2}}\right)d\sigma_y\,ds.$$

Now, using the change of variable $y = sx$, with $x \in \partial B_1(0)$, the monotonicity of F and the inequality $s^2|x'|^2 + (sx_N+1)^2 \leq 4(s^2+1)$, for $x \in \partial B_1(0)$, we obtain

$$\begin{aligned} \Gamma_\varepsilon &\geq \varepsilon^N\int_0^{1/\varepsilon}\int_{\partial B_1(0)\cap\mathbb{R}^N_+} F\left(C_2\frac{\varepsilon^{-(N-2)/2}}{[s^2+1]^{(N-2)/2}}\right)s^{N-1}\,d\sigma_x\,ds \\ &= C_3\varepsilon^N\int_0^{1/\varepsilon} F\left(C_2\frac{\varepsilon^{-(N-2)/2}}{[s^2+1]^{(N-2)/2}}\right)s^{N-1}\,ds \end{aligned}$$

with $C_2 = 4^{-(N-2)/2} C_1 > 0$ and $C_3 = C_3(N)$. After rescaling, we obtain

$$\frac{1}{\varepsilon^2}\Gamma_\varepsilon \geq C_4 \varepsilon^{N-2} \int_0^{C_2^{-2/(N-2)}/\varepsilon} F\left(\frac{\varepsilon^{-(N-2)/2}}{[s^2+1]^{(N-2)/2}}\right) s^{N-1}\, ds.$$

with $C_4 := C_3 C_2^{2N/(N-2)}$. It is easy to see that (3.24) is a consequence of the above expression, (3.25) and hypothesis (3.2). The theorem is proved. □

CHAPTER 4

A singular problem in $\mathbb{R}^N$

Consider the equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = g(x,u), \qquad \text{in } \mathbb{R}^N,$$

with $N \geq 3$. As observed by Escobedo and Kavian in [44], if $g(x,s) = \lambda s + |s|^{p-2}s$ and $2 < p \leq 2^* := 2N/(N-2)$, this equation naturally appears when we deal with the associated heat equation

$$u_t - \Delta u = |u|^{p-2}u, \qquad \text{in } (0,\infty) \times \mathbb{R}^N,$$

and look for solutions with the special form $u_\lambda(t,x) := t^{-\lambda}u(t^{-1/2}x)$, for $\lambda = 1/(p-1)$. We quote the works [9,20,24,52,57,75,76] and references therein for information about existence, nonexistence, decay rate and many other aspects concerning this subject. We emphasize that, in all of those works, the function $g(x,s)$ remains bounded as $s \to 0$. So, it is natural to ask what we can do in the singular case, that is, when $g(x,s) \to +\infty$ as $s \to 0^+$.

This chapter aims to give a first answer to the above question. More specifically, we are concerned with positive solutions for the singular equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu h(x)u^{q-1} + \lambda u + u^{2^*-1}, \qquad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\lambda > 0$, $\mu > 0$ is a parameter, $0 < q < 1$ and h has some somability properties. Before presenting the condition on h, we need to say a few words about the variational structure of the problem. We first notice that, after multiplying the equation by $K(x) := \exp(|x|^2/4)$, it can be rewritten as

$$(P_\mu) \qquad \begin{cases} -\text{div}(K(x)\nabla u) = \mu K(x)h(x)u^{q-1} + \lambda K(x)u + K(x)u^{2^*-1}, & \text{in } \mathbb{R}^N, \\ u > 0, \quad \text{in } \mathbb{R}^N. \end{cases}$$

It is natural to look for solutions in the space X defined as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} K(x)|\nabla u|^2\,dx\right)^{1/2}.$$

It was proved in [44] that X is a Hilbert space which is continuously embedded into the weighted Lebesgue spaces

$$L^p_K(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) \,:\, \|u\|_p := \left(\int_{\mathbb{R}^N} K(x)|u|^p dx\right)^{1/p} < \infty\right\},$$

for any $p \in [2, 2^*]$.

Due to the difficulties related to the operator and the singular nature of the nonlinearity at the origin, we do not expect to find regular solutions. Hence, as usual in the literature, we call $u \in X$ a solution for problem (P_μ) if it satisfies $u > 0$ a.e. in $\mathbb{R}^N$ and, for any $\phi \in X$, we have that $h(x)u^{q-1}\phi \in L^1_K(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} K(x)\left[(\nabla u \cdot \nabla \phi) - \mu h(x)u^{q-1}\phi - \lambda u\phi - u^{2^*-1}\phi\right] dx = 0. \tag{4.1}$$

In our first result we obtain a solution when the parameter $\mu > 0$ is small. More specifically, we shall prove the following:

Theorem I. *Suppose that $\lambda < N/2$ and $h > 0$ satisfies*

(h) $h \in L^1_K(\mathbb{R}^N) \cap L^2_K(\mathbb{R}^N)$.

Then there exists $\mu^* > 0$ such that problem (P_μ) has a solution, whenever $\mu \in (0, \mu^*)$.

In the proof, we apply a minimization argument for a perturbed (nonsingular) problem. We notice that condition $\lambda < N/2$ is necessary for the existence of a solution. Indeed, it is proved in [44] that the linearized version of equation (P_μ) has the pair $(\lambda, u) = (N/2, \varphi_1)$ as a solution, where $\varphi_1(x) = \exp(-|x|^2/4) > 0$. So, if $u_0 \in X$ is a solution, we may pick $v = \varphi_1$ in the integral formulation to get

$$\left(\frac{N}{2} - \lambda\right)\int_{\mathbb{R}^N} K(x)u\varphi_1\,dx = \int_{\mathbb{R}^N} K(x)\left[\mu h(x)u^{q-1}\varphi_1 + u^{2^*-1}\varphi_1\right] dx > 0,$$

from which it follows that $\lambda < N/2$.

In our second result, we obtain another solution under an additional lower bound on the value of λ. More specifically, we prove the following:

Theorem J. *Suppose that* $\max\{1, N/4\} < \lambda < N/2$, $h > 0$ *is continuous and satisfies* (h). *Then there exists $0 < \mu_* < \mu^*$ such that problem (P_μ) has at least two solutions, whenever $\mu \in (0, \mu_*)$*

To obtain the second solution, we apply the Mountain Pass Theorem to a perturbed functional, together with a limit process. The extra assumption on λ is related with the range of existence of positive solution for the nonsingular problem (P_0) obtained in [44]. It is worth mentioning that the continuity of h may be replaced by the weaker condition that the infimum of h is positive in any ball.

We end this introduction with some general comments about the singular problem

$$-\Delta u = g(x,u), \quad \text{in } \Omega, \qquad u > 0, \quad \text{in } \Omega, \qquad u \in H_0^1(\Omega),$$

where $N \geq 3$, $\Omega \subset \mathbb{R}^N$ is a domain and $g(x,s) \to +\infty$, as $s \to 0$. There is a vast literature concerning this kind of problem, mainly due to its applications in boundary layer flow, fluid dynamics, non-Newtonian fluids, reaction–diffusion processes, chemical heterogeneous catalysts, in the theory of heat conduction in electrically conducting materials and in other geophysical and industrial contexts (see for instance [22, 33, 68, 80]).

Although it is impossible to give a complete reference, it seems important to quote the pioneering works of Stuart [89] and Crandall, Rabinowitz and Tartar [34], who considered a general second order operator instead of the laplacian and used some topological arguments to get solutions. Later, Lazer and McKenna [65] proved existence and regularity results for $g(x,s) = h(x)s^{q-1}$, where h is Hölder continuous. Their result was generalized in different ways by Lair and Shaker [63, 64] and Zhang and Cheng [95]. Also in the bounded domain case, we quote the paper of Boccardo and Orsina [16], where the Laplacian is replaced by the operator $u \mapsto \operatorname{div}(M(x)\nabla u)$, with M being a bounded elliptic matrix, $g(x,s) = h(x)s^{q-1}$, with $h \geq 0$ belonging to some Lebesgue space or even being a Radon measure. Some results for quasilinear operators can be found in [5, 78, 79]. For the case of the whole space, we refer the reader to [62, 64, 85], where it is supposed that $g(x,s) = h(x)s^{q-1} + f(x,s)$, h is continuous and f has some mild conditions.

The results presented in this chapter complement the aforementioned works since we deal here with the whole space case and consider a different operator. The rest of the chapter is organized as follows: in the next section we prove Theorem I while the last one is devoted to the proof of Theorem J.

4.1 Proof of Theorem I

Along all this section we write only $\int f$ to denote $\int_{\mathbb{R}^N} f(x)dx$, where $f \in L^1(\mathbb{R}^N)$. For any $s \in \mathbb{R}$, we consider $s^+ := \max\{s, 0\}$ and $s^- := s^+ - s$. Before starting the proofs, we need to say a few words about the linearization of the problem (P_μ), namely

$$-\operatorname{div}(K(x)\nabla u) = \lambda K(x)u, \quad \text{in } \mathbb{R}^N.$$

Its spectrum was completely characterized in [44], where it is proved by a Fourier approach that the first eigenvalue is given by

$$\lambda_1 = \inf \left\{ \int K(x)|\nabla u|^2 \, : \, \int K(x)|u|^2 = 1 \right\} = \frac{N}{2}.$$

From this, we infer the following Poincare type inequality:

$$\lambda_1 \int_{\mathbb{R}^N} K(x)|u|^2 \, dx \leq \int_{\mathbb{R}^N} K(x)|\nabla u|^2 \, dx, \qquad \forall u \in X. \tag{4.2}$$

Since we are going to obtain solutions for small values of the parameter $\mu > 0$, it is important to consider the limit problem

$$-\mathrm{div}(K(x)\nabla u) = \lambda K(x)u + K(x)|u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N, \tag{P_0}$$

and its associated C^1-functional given by

$$I_0(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u^+\|_2^2 - \frac{1}{2^*}\|u^+\|_{2^*}^{2^*}, \quad u \in X.$$

From now on, we assume that $h \in L^1_K(\mathbb{R}^N) \cap L^2_K(\mathbb{R}^N)$. Hence, we can use interpolation to conclude that $h \in L^\theta_K(\mathbb{R}^N)$, where $\theta := 2/(2-q)$. For any $u \in X$, it follows from the Hölder's inequality that

$$\frac{1}{q}\int K(x)h(x)(u^+)^q \leq \frac{1}{q}\|h\|_\theta \|u^+\|_2^q \leq C_1\|u\|^q, \tag{4.3}$$

Thus, we may add the singular term to I_0 and obtain the functional associated with the problem (P_μ), namely

$$I_\mu(u) := I_0(u) - \frac{\mu}{q}\int K(x)h(x)(u^+)^q, \quad u \in X.$$

It is clear that I_μ is a well-defined continuous functional in X. In our first result we study its behavior near the origin.

Lemma 4.1.1. *There exists $\mu^* > 0$ such that, for any $\mu \in (0, \mu^*)$, there holds*

$$I_\mu(u) \geq \rho, \quad \forall\, u \in X \cap \partial B_R(0),$$

with ρ, $R > 0$ independent of μ.

Proof. Given $u \in X$, we can use (4.2) and the embedding $X \hookrightarrow L^{2^*}_K(\mathbb{R}^N)$ to get

$$I_0(u) \geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^2 - C_2\|u\|^{2^*} \geq C_3\|u\|^2, \tag{4.4}$$

if $C_3 = (1 - \lambda/\lambda_1)/4$ and

$$\|u\| \leq R := \left(\frac{C_3}{C_2}\right)^{1/(2^*-2)}.$$

This and (4.3) imply that

$$I_\mu(u) \geq \|u\|^q \left(C_3\|u\|^{2-q} - \mu C_1\right) \geq \rho := \frac{C_3}{2}R^q,$$

whenever $\|u\| = R$ and

$$0 < \mu < \mu^* := \frac{C_3}{2C_1}R^{2-q}.$$

The lemma is proved. □

Let μ^*, $R > 0$ as in Lemma 4.1.1 and $\mu \in (0, \mu^*)$. By picking a nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$, we get

$$\lim_{t\to 0^+} \frac{I_\mu(t\varphi)}{t^q} = -\frac{\mu}{q}\int K(x)h(x)\varphi^q < 0,$$

and therefore there exists $t_0 > 0$ small in such a way that $\|t_0\varphi\| \leq R$ and $I_\mu(t_0\varphi) < 0$. This shows that

$$m_\mu := \inf_{\|u\|\leq R} I_\mu(u) < 0.$$

Since I_μ maps bounded sets onto bounded sets, we have that $m_\mu > -\infty$.

Even if we prove that m_μ is attained in $B_R(0)$, the singular term of the equation gives rise to a difficulty. Actually, since $0 < q < 1$, the term $\int K(x)h(x)(u^+)^q$ is continuous but not differentiable, and therefore it is not clear that minimizers are solutions of our problem. However, using a direct calculation, we may prove that this hold, as we can see from the next result.

Lemma 4.1.2. *If $u \in B_R(0)$ is such that $I_\mu(u) = m_\mu$, then u is a solution for problem (P_μ).*

Proof. Let $\psi \in X$ be a nonnegative function. Since $\|u\| < R$, we have that $\|u+t\psi\| < R$, for any $t > 0$ small. If we divide the inequality $I_\mu(u) \leq I_\mu(u + t\psi)$ by $t > 0$ and take the limit as $t \to 0^+$, we obtain

$$\frac{\mu}{q}\liminf_{t\to 0^+}\int \frac{K(x)h(x)[((u+t\psi)^+)^q - (u^+)^q]}{t} \leq I_0'(u)\psi.$$

Since $\psi \geq 0$, we have that $(u + t\psi)^+ \geq u^+$. Then, we can use the Fatou's lemma to obtain

$$I_0'(u)\psi - \mu\int K(x)h(x)(u^+)^{q-1}\psi \geq 0, \quad \forall\, \psi \in X,\ \psi \geq 0. \tag{4.5}$$

By setting $t_0 := (R/\|u\|) - 1 > 0$, a straightforward computation shows that $\|(1 + t)u\| < R$, whenever $t \in (-1, t_0)$. Hence, the function

$$\gamma(t) := I_\mu((1 + t)u), \quad t \in (-1, t_0),$$

attains its minimum value at $t = 0$. Thus,

$$\gamma'(0) = I_0'(u)u - \mu \int K(x)h(x)(u^+)^q = 0. \tag{4.6}$$

Pick $\varepsilon > 0$, $\phi \in X$ and define $\Omega_\varepsilon^+ := [u^+ + \varepsilon\phi < 0]$. By using (4.5) with $\psi = (u^+ + \varepsilon\phi)^+$ we get, after some computations,

$$\begin{aligned} 0 \ \leq \ & -\|u^-\|^2 + I_0'(u)u - \mu \int K(x)h(x)(u^+)^q \\ & + \varepsilon I_0'(u)\phi - \varepsilon\mu \int K(x)h(x)(u^+)^{q-1}\phi - \int_{\Omega_\varepsilon^+} K(x)[\nabla u \cdot \nabla(u^+ + \varepsilon\phi)]\,dx \\ & + \int_{\Omega_\varepsilon^+} K(x)(u^+ + \varepsilon\phi)\left[\lambda u^+ + \mu h(x)(u^+)^{q-1} + (u^+)^{2^*-1}\right] dx. \end{aligned}$$

Hence, it follows from (4.6) that

$$\begin{aligned} 0 \ \leq \ & \varepsilon I_0'(u)\phi - \varepsilon\mu \int K(x)h(x)(u^+)^{q-1}\phi - \int_{\Omega_\varepsilon^+} K(x)[\nabla u \cdot \nabla(u^+ + \varepsilon\phi)]\,dx \\ \leq \ & \varepsilon\left[I_0'(u)\phi - \mu \int K(x)h(x)(u^+)^{q-1}\phi - \int_{\Omega_\varepsilon^+} K(x)[\nabla u \cdot \nabla\phi]\,dx\right] \end{aligned}$$

If we divide the previous expression by $\varepsilon > 0$, take the limit as $\varepsilon \to 0^+$ and notice that

$$\lim_{\varepsilon \to 0^+} \mathbf{1}_{\Omega_\varepsilon^+}(x) = 0, \qquad \text{a.e. in } \mathbb{R}^N,$$

where $\mathbf{1}_{\Omega_\varepsilon^+}$ stands for the characteristic function of the set Ω_ε^+, we can use Lebesgue theorem to conclude that

$$I_0'(u)\phi - \mu \int K(x)h(x)(u^+)^{q-1}\phi \geq 0, \qquad \forall\, \phi \in X.$$

Since this inequality also holds with write $-\phi$ instead of ϕ, we conclude that $u \in X$ satisfies the integral equation (4.1). Moreover, by picking $\phi = u$ in the above equality, we obtain $\|u^-\| = 0$, which shows that $u \geq 0$ a.e. $\mathbb{R}^N$.

It remains to be proved that $u > 0$ a.e. $\mathbb{R}^N$. In order to do that, we consider $\Omega \subset \mathbb{R}^N$ an open bounded set and $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\phi \geq 0$ in $\mathbb{R}^N$ and $\phi \equiv 1$ in Ω. Since $K(x)h(x)u^{q-1}\phi \geq K(x)h(x)u^{q-1} \geq 0$ for a.e. $x \in \Omega$ and

$$\int_\Omega K(x)h(x)(u^+)^{q-1}dx < +\infty,$$

we conclude that $K(x)h(x)u^{q-1}$ is finite a.e. in Ω, from which it follows that $u > 0$ a.e. in Ω. Since Ω is arbitrary, the lemma is proved. □

We now notice that I_μ is not of class C^1, and therefore we cannot perform standard minimization arguments. So, instead of a direct approach, we are going to consider the following perturbation process: for each $k \in \mathbb{N}$, we define $\mathcal{X}_k : \mathbb{R} \to \mathbb{R}$ as

$$\mathcal{X}_k(s) := \int_0^s \left(t^+ + \frac{1}{k}\right)^{q-1} dt = \frac{1}{q}\left[\left(s^+ + \frac{1}{k}\right)^q - \left(\frac{1}{k}\right)^q\right] + \left(\frac{1}{k}\right)^{q-1} s^-, \tag{4.7}$$

and the functional

$$I_{\mu,k}(u) := I_0(u) - \mu \int K(x)h(x)\mathcal{X}_k(u), \quad u \in X.$$

Since

$$\mathcal{X}_k'(s) = \left(s^+ + \frac{1}{k}\right)^{q-1}, \quad s \in \mathbb{R}, \tag{4.8}$$

it is clear that $I_{\mu,k} \in C^1(X, \mathbb{R})$.

We are going to show that $I_{\mu,k}$ attains its minimum at $u_k \in B_R(0)$ and the desired solution will be obtained passing to the limit as $k \to +\infty$. The details can be found in the next proposition.

Proposition 4.1.3. *Let μ^*, $R > 0$ be given by Lemma 4.1.1. For any $\mu \in (0, \mu^*)$ there exists $u \in X$ such that $\|u\| < R$ and $I_\mu(u) = m_\mu$. In particular, the problem (P_μ) has a solution with negative energy.*

Proof. Since $\mathcal{X}_k(s) \leq \int_0^s (t^+)^{q-1} dt$, we have that $I_{\mu,k}(u) \geq I_\mu(u)$, for any $u \in X$ and $k \in \mathbb{N}$. It follows from Lemma 4.1.1 that $I_{\mu,k} \geq \rho$ on $\partial B_R(0)$. Thus, since $I_{\mu,k}(0) = 0$, we can define

$$m_{\mu,k} := \inf_{\|u\| \leq R} I_{\mu,k}(u),$$

and use the Ekeland Variational Principle to obtain a sequence $(u_{n,k})_{n \in \mathbb{N}} \subset B_R(0)$ such that

$$\lim_{n \to +\infty} I_{\mu,k}(u_{n,k}) = m_{\mu,k}, \quad \lim_{n \to +\infty} I'_{\mu,k}(u_{n,k}) = 0.$$

Up to a subsequence, we have that, as $n \to +\infty$,

$$\begin{cases} u_{n,k} \rightharpoonup u_k, & \text{wealy in } X, \\ u_{n,k} \to u_k, & \text{stronlgy in } L^s_K(\mathbb{R}^N), \\ u^\pm_{n,k}(x) \to u^\pm_k(x), & \text{a.e. in } \mathbb{R}^N, \\ |u_{n,k}(x)| \leq g_s(x), & \text{a.e. in } \mathbb{R}^N, \end{cases} \tag{4.9}$$

for any $s \in [2, 2^*)$ and some $g_s \in L^s_K(\mathbb{R}^N)$. By noticing that

$$|\mathcal{X}_k(s)| \leq \int_0^{|s|} k^{q-1}\, dt = k^{q-1}|s|, \quad s \in \mathbb{R}, \tag{4.10}$$

we infer from (4.9) that

$$|K(x)h(x)\mathcal{X}_k(u_{n,k})| \leq \left(\frac{1}{k}\right)^{1-q} K(x)h(x)g_2(x),$$

a.e. in $\mathbb{R}^N$. Since the right-hand side above belongs to $L^1(\mathbb{R}^N)$, we can use the Lebesgue Theorem to obtain

$$\lim_{n \to +\infty} \int K(x)h(x)\mathcal{X}_k(u_{n,k}) = \int K(x)h(x)\mathcal{X}_k(u_k). \tag{4.11}$$

Setting $v_{n,k} := u_{n,k} - u_k$, we can use the above inequality, (4.9) and the Brezis-Lieb lemma [18] to get

$$\begin{aligned} m_{\mu,k} &= I_{n,k}(u_{n,k}) + o_n(1) \\ &= \frac{1}{2}\|v_{n,k}\|^2 + \frac{1}{2}\|u_k\|^2 - \frac{\lambda}{2}\|u_k^+\|_2^2 - \mu \int K(x)h(x)\mathcal{X}_k(u_k) \\ &\quad -\frac{1}{2^*}\int K(x)(v_{n,k}^+)^{2^*} - \frac{1}{2^*}\int K(x)(u_k^+)^{2^*}\,dx + o_n(1), \end{aligned} \tag{4.12}$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. Recalling that $\|u_{n,k}\| < R$ and using the weak convergence, we obtain

$$\begin{aligned} \limsup_{n\to+\infty} \|v_{n,k}\|^2 &< R^2 + \|u_k\|^2 - 2\lim_{n\to+\infty}\int K(x)(\nabla u_{n,k}\cdot\nabla u_k) \\ &= R^2 - \|u_k\|^2 \le R^2. \end{aligned}$$

This shows that $\|v_{n,k}\| \le R$, whenever $n \ge n_0(k)$. Hence, it follows from (4.4) that

$$\frac{1}{2}\|v_{n,k}\|^2 - \frac{1}{2^*}\int K(x)(v_{n,k}^+)^{2^*} \ge 0, \quad \forall\, n \ge n_0(k),$$

which combined with (4.12) imply that

$$m_{\mu,k} \ge I_{\mu,k}(u_k) + o_n(1).$$

Passing to the limit as $n \to +\infty$ we conclude that $m_{\mu,k} = I_{\mu,k}(u_k)$. Moreover, since $m_{\mu,k} \le I_{\mu,k}(0) = 0$ and $I_{\mu,k} \ge \rho > 0$ on $\partial B_R(0)$, we have that $\|u_k\| < R$.

For any $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have that

$$\left| K(x)h(x)\left(u_{n,k}^+(x) + \frac{1}{k}\right)^{q-1}\varphi(x)\right| \le k^{1-q}K(x)h(x)|\varphi(x)|, \quad \text{a.e. in } \mathbb{R}^N.$$

By using the pointwise convergence and Lebesgue's theorem, we obtain

$$\lim_{n\to+\infty}\int K(x)h(x)\left(u_{n,k}^+ + \frac{1}{k}\right)^{q-1}\varphi = \int K(x)h(x)\left(u_k^+ + \frac{1}{k}\right)^{q-1}\varphi.$$

This inequality, (4.9) and a standard density argument imply that $I'_{\mu,k}(u_k) = 0$.

We are going to show that

$$\lim_{k\to+\infty} I_{\mu,k}(u_k) = m_\mu. \tag{4.13}$$

Since $I_{\mu,k}(u_k) \ge I_\mu(u_k) \ge m_\mu$, it is sufficient to verify that

$$\limsup_{k\to+\infty} I_{\mu,k}(u_k) \le m_\mu. \tag{4.14}$$

In order to do this, let $(w_n) \subset B_R(0)$ be such that $I_\mu(w_n) \to m_\mu$, as $n \to +\infty$. Then

$$\begin{aligned} I_\mu(w_n) &= I_{\mu,k}(w_n) + \mu \int K(x)h(x)\mathcal{X}_k(w_n) - \frac{\mu}{q}\int K(x)h(x)(w_n^+)^q \\ &\geq m_{\mu,k} + \mu \int K(x)h(x)\mathcal{X}_k(w_n) - \frac{\mu}{q}\int K(x)h(x)(w_n^+)^q. \end{aligned} \tag{4.15}$$

Fixed $n \in \mathbb{N}$, we can use that $\mathcal{X}_k(w_n)(x) \to w_n^+(x)^q/q$ for a.e. $x \in \mathbb{R}^N$, as $k \to +\infty$, and (4.10), to obtain

$$\begin{aligned} \int K(x)h(x)\mathcal{X}_k(w_n)\,dx &= \int K(x)h(x)\frac{(w_n^+ + \frac{1}{k})^q - (\frac{1}{k})^q}{q} \\ &\quad + k^{1-q}\int_{\mathbb{R}^N} K(x)h(x)w_n^-\,dx \\ &= \frac{1}{q}\int K(x)h(x)(w_n^+)^q + o_k(1). \end{aligned}$$

By combining this expression with (4.15) and taking the limsup as $k \to +\infty$, we obtain

$$I_\mu(w_n) \geq \limsup_{k\to+\infty} m_{\mu,k} = \limsup_{k\to+\infty} I_{\mu,k}(u_k).$$

Once again, passing to the limit as $n \to +\infty$, we immediately obtain (4.14).

We are now able to prove that m_μ is attained. Since (u_k) is bounded, along a subsequence $u_k \rightharpoonup u$ weakly in X. As before, we can prove that

$$\lim_{k\to+\infty} \int K(x)h(x)\mathcal{X}_k(u_k) = \frac{1}{q}\int K(x)h(x)(u^+)^q.$$

Hence, we can use (4.13) and the same argument used to prove that $I_{\mu,k}(u_k) = m_{\mu,k}$ (but now considering the limits in the index k) to conclude that $I_\mu(u) = m_\mu$. We omit the details. □

4.2 Proof of Theorem J

Now we have obtained a first solution, we are going to apply the Mountain Pass Theorem for the perturbed functional and obtain a second solution as a limit process. First, we present some important facts about the problem (P_0) stated in the beginning of the previous section. In order to describe some results proved in [44], we redefine the associated energy functional as

$$I_0(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{2^*}\|u^+\|_{2^*}^{2^*}, \quad u \in X.$$

The least energy level of (P_0) is defined as

$$c_0 := \inf_{u\in\mathcal{N}_0} I_0(u),$$

where $\mathcal{N}_0 := \{u \in X \setminus \{0\} : I_0'(u)u = 0\}$ is the Nehari manifold. In [44], the authors obtained ground state solution for (P_0) using the minimization problem

$$S_\lambda(K) := \inf_{u \in X \setminus \{0\}} \frac{\|u\|^2 - \lambda \|u\|_2^2}{\|u^+\|_{2^*}^2}.$$

They proved that $0 < S_\lambda(K) < S_{2^*}$, when $\max\{1, N/4\} < \lambda < N/2$. As a consequence, the above infimum is attained by a positive function $u_0 \in X \cap C^2(\mathbb{R}^N)$. Since the problem is homogeneous, a scaling argument provides $\tau > 0$ such that the function $\omega_0 := \tau^{2^*-2} u_0$ is a solution for (P_0) with $I_0(\omega_0) = c_0$. We finally mention that, since $u_0 = \tau^{1/(2-2^*)}\omega_0$ and $\omega_0 \in \mathcal{N}_0$, we have that

$$S_{2^*} > S_\lambda(K) = \frac{\|u_0\|^2 - \lambda \|u_0\|_2^2}{\|u_0^+\|_{2^*}^2} = \frac{\|\omega_0\|^2 - \lambda \|\omega_0\|_2^2}{\|\omega_0^+\|_{2^*}^2} = \left(\|\omega_0\|^2 - \lambda \|\omega_0\|_2^2\right)^{2/N},$$

which leads to the following useful inequality

$$\frac{1}{N} S_{2^*}^{N/2} > \frac{1}{N}\left(\|\omega_0\|^2 - \lambda \|\omega_0\|_2^2\right) = I_0(\omega_0) = c_0. \tag{4.16}$$

From now on, we are going to look for a second solution for problem (P_μ) as a positive energy critical point of

$$I_{\mu,k}(u) := I_0(u) - \mu \int_{\mathbb{R}^N} K(x) h(x) \mathcal{X}_k(u)\, dx, \quad u \in X,$$

with I_0 redefined as before. It is clear that its critical points are weak solutions for the (nonsingular) problem

$$-\mathrm{div}(K(x)\nabla u) = K(x)\frac{h(x)}{(u + 1/k)^{1-q}} + \lambda K(x) u + K(x)|u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N. \tag{$P_{\mu,k}$}$$

In our next result we prove that such solutions are, indeed, zero or positive in $\mathbb{R}^N$.

Lemma 4.2.1. *If $u_k \in X$ is a nonzero critical point of $I_{\mu,k}$, then it is a positive weak solution for $(P_{\mu,k})$.*

Proof. It is clear that u_k weakly solves the problem. Moreover, computing

$$0 = I_{\mu,k}'(u_k)u_k^- = \|u_k^-\|^2 - \lambda \|u_k^-\|_2^2 - \mu k^{1-q} \int K(x) h(x) u_k^-,$$

we conclude that $u_k \geq 0$ a.e. in $\mathbb{R}^N$. In order to prove that $u_k > 0$, we consider $\Sigma \subset B_R(0)$ a compact subset of $\mathbb{R}^N$. Using that $K, \lambda \geq 1$ and the relation

$$(s + s^{2^*-1}) + \frac{a}{(s+1)^{1-q}} \geq \min\left\{1, \frac{a}{2^{1-q}}\right\}, \quad \forall\, a > 0,\ s \geq 0,$$

we conclude that, for a.e. $x \in B_R(0)$, there holds

$$\begin{aligned} -\mathrm{div}(K(x)\nabla u(x)) &= \lambda K(x) u(x) + \mu \frac{K(x) h(x)}{(u(x) + 1/k)^{1-q}} + K(x) u(x)^{2^*-1} \\ &\geq (u(x) + u(x)^{2^*-1}) + \mu \frac{h(x)}{(u(x)+1)^{1-q}} \\ &\geq C_R, \end{aligned}$$

where
$$C_R := \min\left\{1, \mu\frac{\min_{x\in B_R(0)} h(x)}{2^{1-q}}\right\} > 0.$$
On the other hand, using the Lax-Milgram theorem, we obtain a nonnegative $v \in H_0^1(B_R(0))$ such that
$$-\mathrm{div}(K(x)\nabla v) = C_R, \text{ in } B_R(0).$$
Following the ideas developed in [44, Theorem 3.12], we can prove that $v \in C^2(B_R(0) \cap C(\overline{B_R(0)})$ and therefore the Strong Maximum Principle ensures that $v > 0$ in $B_R(0)$. Thus, there exists a constant $C_\Sigma > 0$ such that $v(x) \geq C_\Sigma$, for any $x \in \Sigma$.

Since $\mathrm{div}(K(x)\nabla u) \leq \mathrm{div}(K(x)\nabla v)$ in $B_R(0)$, we have that
$$\int_{B_R(0)} K(x)(\nabla u \cdot \nabla\varphi)\, dx \geq \int_{B_R(0)} K(x)(\nabla v \cdot \nabla v)\, dx, \quad \forall\, \varphi \in H_0^1(B_R(0)).$$
If we $\varphi := \max\{v-u, 0\}$ and use $K \geq 1$ again, we obtain
$$\|\varphi\|^2_{H_0^1(B_R(0))} \leq \int_{[v\geq u]} K(x)|\nabla\varphi|^2 dx \leq \int_{B_R(0)} K(x)(\nabla(v-u)\cdot\nabla\varphi)\, dx \leq 0,$$
from which we conclude that $\varphi = 0$ or, equivalently, $u \geq v$ a.e. in $B_R(0)$. Hence, $u \geq v \geq C_\Sigma > 0$ in the (arbitrary) set Σ and the lemma is proved. □

Remark 3. *In the above proof, we have used the continuity of h to guarantee that $C_R > 0$. So, it is clear that the same result is true if we just assume that, for any $R > 0$, there holds*
$$\inf_{x\in B_R(0)} h(x) > 0.$$

If $d \in \mathbb{R}$, we say that $(u_n) \subset X$ is a $(PS)_d$ sequence for $I_{\mu,k}$ if
$$\lim_{n\to+\infty} I_{\mu,k}(u_n) = d, \quad \lim_{n\to+\infty} I'_{\mu,k}(u_n) = 0.$$
The functional $I_{\mu,k}$ satisfies the Palais-Smale condition at level d if any such sequence has a convergent subsequence. In what follows, we prove that our functional satisfies this compactness condition in an appropriated subset of $\mathbb{R}$.

Lemma 4.2.2. *There exists $M_1 = M_1(q, \lambda, N, \|h\|_\theta) > 0$ and $M_2 = M_2(q, \|h\|_1) > 0$ such that, for any $\mu > 0$ and $k \in \mathbb{N}$, the functional $I_{\mu,k}$ satisfies the Palais-Smale condition at any level*
$$d < \frac{1}{N} S_{2^*}^{N/2} - M_1\mu^\theta - \frac{M_2}{k^q}\mu.$$

Proof. Let $(u_n) \subset X$ be a $(PS)_d$ sequence for $I_{\mu,k}$. In order to verify that it is a bounded sequence, we set
$$\alpha_0 := \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^*}\right)\left(\frac{\lambda_1 - \lambda}{\lambda_1}\right) \tag{4.17}$$

and use (4.2) to get

$$\begin{aligned} d + o_n(1)(1 + \|u_n\|) &= I_{\mu,k}(u_n) - \frac{1}{2^*} I'_{\mu,k}(u_n)u_n \\ &\geq 2\alpha_0\|u_n\|^2 - \mu \int K(x)h(x)\mathcal{X}_k(u_n) \\ &\quad + \frac{\mu}{2^*}\int K(x)h(x)\mathcal{X}'_k(u_n)u_n. \end{aligned}$$

It follows from the above expression and (4.7)-(4.8) that

$$d + o_n(1)(1 + \|u_n\|) \geq 2\alpha_0\|u_n\|^2 - \frac{\mu}{q}\int K(x)h(x)\left(u_n^+ + \frac{1}{k}\right)^q. \tag{4.18}$$

Since $(a+b)^q \leq C_q(a^q + b^q)$, for some $C_q > 0$ and any a, $b \geq 0$, we can use the Young's inequality to obtain, for each $\varepsilon > 0$, a constant $C_{\varepsilon,q} > 0$ such that

$$\begin{aligned} \frac{\mu}{q}K(x)h(x)\left(u_n^+(x) + \frac{1}{k}\right)^q &\leq C_q\frac{\mu}{q}K(x)h(x)\left[(u_n^+)^q(x) + k^{-q}\right] \\ &\leq \varepsilon K(x)u_n(x)^2 + C_{\varepsilon,q}K(x)h(x)^\theta + C_q\frac{\mu}{q}K(x)h(x), \end{aligned}$$

for a.e. $x \in \mathbb{R}^N$. Picking $\varepsilon = \alpha_0\lambda_1$, we can use the above expression, (4.18) and (4.2), to obtain

$$d + o_n(1)(1 + \|u_n\|) \geq \alpha_0\|u_n\|^2 - M_1\mu^\theta - M_2\frac{\mu}{k^q},$$

where

$$M_1 := C_{\varepsilon,q}\|h\|_\theta^\theta, \quad M_2 := \frac{C_q}{q}\|h\|_1.$$

Thus, $(u_n) \subset X$ is bounded.

Up to a subsequence, we may assume that $u_n \rightharpoonup u$ weakly in X and an analogous of (4.9) holds. Arguing as in the proof of Lemma 4.1.3, we can prove that $I'_{\mu,k}(u) = 0$ and

$$\lim_{n\to+\infty}\int K(x)h(x)\mathcal{X}'_k(u_n)u_n = \int K(x)h(x)\mathcal{X}'_k(u)u.$$

Moreover, the former computations provide

$$I_{\mu,k}(u) = I_{\mu,k}(u) - \frac{1}{2^*}I'_{\mu,k}(u)u \geq \alpha_0\|u\|^2 - M_1\mu^\theta - M_2\frac{\mu}{k^q}. \tag{4.19}$$

Hence, if we set $v_n := (u_n - u)$, we can use (4.9) and the Brezis-Lieb lemma to get

$$o_n(1) = I'_{\mu,k}(u_n)u_n = \|v_n\|^2 - \|v_n\|_{2^*}^{2^*} - I'_{\mu,k}(u)u + o_n(1),$$

from which it follows that

$$\lim_{n\to+\infty}\|v_n\|^2 = l = \lim_{n\to+\infty}\|v_n\|_{2^*}^{2^*},$$

for some $l \geq 0$.

Suppose, by contradiction, that $l > 0$. Then we can use the definition of S_{2^*} to conclude that $l \geq S_{2^*}^{N/2}$. On the other hand, using Brezis-Lieb lemma again, (4.11) and (4.19) we obtain

$$\begin{aligned} d + o_n(1) &= I_{\mu,k}(u_n) = \frac{1}{2}\|v_n\|^2 - \frac{1}{2^*}\|v_n\|_{2^*}^{2^*} + I_{\mu,k}(u) + o_n(1) \\ &\geq \frac{1}{2}\|v_n\|^2 - \frac{1}{2^*}\|v_n\|_{2^*}^{2^*} + \alpha_0\|u\|^2 - M_1\mu^\theta - M_2\frac{\mu}{k^q} + o_n(1). \end{aligned}$$

Taking the limit as $n \to +\infty$ and recalling that $l \geq S_{2^*}^{N/2}$, we obtain

$$d \geq \frac{1}{N}S_{2^*}^{N/2} - M_1\mu^\theta - M_2\frac{\mu}{k^q},$$

which contradicts the hypotheses. Hence, $l = 0$ or, equivalently, $u_n \to u$ strongly in X. □

We solve in the sequel the modified problem.

Proposition 4.2.3. *Let μ^*, $\rho > 0$ be given by Lemma 4.1.1. Then, there exists $k_* = k_*(q,h) > 0$ and $\mu_* = \mu_*(q,N,h) < \mu^*$ such that, for any $k \geq k_*$ and $\mu \in (0,\mu_*)$, the functional $I_{\mu,k}$ has a positive critical point $u_k \in X$ verifying $I_{\mu,k}(u_k) \geq \rho > 0$.*

Proof. Let M_1, M_2 be given by Lemma 4.2.2. Recalling that the function ω_0 obtained in the beginning of the section is positive, we obtain $I_{\mu,k}(t\omega_0) \leq I_0(t\omega_0)$, for any $t \geq 0$. Since $I_0(t\omega_0) \to 0$, as $t \to 0^+$, we can find $t_* > 0$, independent of μ and k, such that

$$\max_{0 \leq t \leq t_*} I_{\mu,k}(t\omega_0) < \frac{c_0}{2} < c_0 - M_1\mu^\theta - M_2\frac{\mu}{k^q}, \tag{4.20}$$

whenever

$$\mu < \min\left\{1, \frac{c_0}{2(M_1+M_2)}\right\}.$$

Moreover, since the function $t \mapsto t\omega_0(x)\,[t\omega_0(x)+1]^{q-1}$ is increasing in $[0,+\infty)$, a change of variables provides

$$\begin{aligned} \mathcal{X}_k(t\omega_0(x)) = \int_{1/k}^{t\omega_0(x)+1/k} \frac{1}{\tau^{1-q}}\,d\tau &\geq \frac{t\omega_0(x)}{[s\omega_0(x)+1/k]^{1-q}} \\ &\geq \frac{t\omega_0(x)}{[t\omega_0(x)+1]^{1-q}} \\ &\geq \frac{t_*\omega_0(x)}{[t_*\omega_0(x)+1]^{1-q}}, \end{aligned} \tag{4.21}$$

for any $x \in \mathbb{R}^N$ and $t \geq t_*$. Hence, if we define

$$C_{h,q} := t_* \int K(x)h(x)\frac{\omega_0}{(t_*\omega_0+1)^{1-q}},$$

we can use (4.21) and that $I_0(\omega_0) = \max_{t\geq 0} I_0(t\omega_0)$ to obtain

$$I_{\mu,k}(t\omega_0) = I_0(t\omega_0) - \mu \int K(x)h(x)\mathcal{X}_k(t\omega_0) \leq c_0 - C_{h,q}\mu, \quad t \geq t_*. \tag{4.22}$$

We now notice that, if

$$k \geq k_* := \left(\frac{2M_2}{C_{h,q}}\right)^{1/q}, \quad \mu^{\theta-1} < \frac{C_{h,q}}{2M_1},$$

then

$$M_1\mu^\theta + M_2\frac{\mu}{k^q} < \frac{C_{h,q}}{2}\mu + \frac{C_{h,q}}{2}\mu = \mu C_{h,q}.$$

This inequality, together with (4.22) and (4.20), imply that

$$\sup_{t\geq 0} I_{\mu,k}(t\omega_0) < c_0 - M_1\mu^\theta - M_2\frac{\mu}{k^q}, \tag{4.23}$$

whenever $k \geq k_*$ and

$$0 < \mu < \mu_* := \min\left\{1, \frac{c_0}{2(M_1+M_2)}, \left(\frac{C_{h,q}}{2M_1}\right)^{1/(\theta-1)}\right\}.$$

Since

$$\lim_{t\to+\infty} \frac{I_{\mu,k}(t\omega_0)}{t^{2^*}} \leq -\frac{1}{2^*}\|\omega_0\|_{2^*}^{2^*} < 0,$$

there exists $T > 0$, independent of μ and k, such that $\|T\omega_0\| > \rho$ and $I_{\mu,k}(t\omega_0) < 0$, for any $t \geq T$. Thus, we can use Lemma 4.1.1 to define the Moutain Pass level

$$c_{\mu,k} := \inf_{\gamma\in\Gamma} \sup_{0\leq t\leq 1} I_{\mu,k}(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], X) : \ \gamma(0) = 0, \gamma(1) = T\omega_0\}$.

The definition of Γ and (4.23) imply that

$$c_{\mu,k} < c_0 - M_1\mu^\theta - M_2\frac{\mu}{k^q}, \tag{4.24}$$

whenever $k \geq k_*$ and $\mu \in (0, \mu_*)$. Using Lemma 4.2.2 and the Mountain Pass Theorem we obtain a critical point $u_k \in X$ such that $I_{\mu,k}(u_k) \geq \rho$. By Lemma 4.2.1 this solution is positive and the proposition is proved. □

We are ready to prove our final main result.

Proof of Theorem J.. Let $\mu^* > 0$ be given by Proposition 4.2.3 and $0 < \mu < \mu^*$. Using Proposition 4.1.3, we obtain a first positive solution with negative energy. In order to obtain the second one, we denote by $(u_k)_{k\geq k^*} \subset X$ the sequence of positive solutions given by Proposition 4.2.3. As in Lemma 4.2.2, we can prove that

$$c_{\mu,k} = I_{\mu,k}(u_k) - \frac{1}{2^*}I'_{\mu,k}(u_k)u_k \geq \alpha_0\|u_k\|^2 - M_1\mu^\theta - M_2\frac{\mu}{k^q},$$

where $\alpha_0 > 0$ was defined in (4.17) and M_1, M_2 come from Lemma 4.2.2. The above inequality and (4.24) imply that $(u_k) \subset X$ is bounded.

Up to a subsequence, we may assume that $u_k \rightharpoonup u$ weakly in X, as $k \to +\infty$, and an analogous of (4.9) holds. Arguing as in the proof of Lemma 4.1.3, we can prove that $I'_\mu(u) = 0$. Moreover, for each compact set $\Sigma \subset \mathbb{R}^N$, it follows from the proof of Lemma 4.2.1 that $u_k(x) \geq C_\Sigma$, for some $C_\Sigma > 0$ independent of k. Thus, we infer from the pointwise convergence of (u_k) that $u \geq C_\Sigma > 0$ a.e. in the (arbitrary) set Σ, and therefore u is a solution for (P_μ).

In order to guarantee that u is different from the first solution, we shall prove that $I_\mu(u) > 0$. We first notice that, arguing as in Lemma 4.2.2 and using $u \neq 0$, we get

$$I_\mu(u) \geq \alpha_0 \|u\|^2 - M_1\mu^\theta > -M_1\mu^\theta. \tag{4.25}$$

By setting $v_k := u_k - u$, using Brezis-Lieb lemma, $I_\mu(u_k)u_k = 0$ and repeating the calculations of Lemma 4.2.2, we obtain

$$o_k(1) = \|v_k\|^2 - \|v_k\|_{2^*}^{2^*} + I'_\mu(u)u + o_k(1),$$

and therefore, for some $l \geq 0$, there holds

$$\lim_{k\to+\infty} \|v_k\|^2 = l = \lim_{k\to+\infty} \|v_k\|_{2^*}^{2^*}.$$

Thus, we can use (4.24) and the same argument employed in the proof of Lemma 4.2.2 to obtain

$$c_0 - M_1\mu^\theta - M_2\frac{\mu}{k^q} > I_{\mu,k}(u_k) = \frac{1}{N}l + I_\mu(u) + o_k(1).$$

If $l > 0$, then $l \geq S_{2^*}^{N/2}$ and we can pass to the limit as $k \to +\infty$, use (4.16) and (4.25) to obtain

$$c_0 - M_1\mu^\theta \geq \frac{1}{N}S_{2^*}^{2/N} + I_\mu(u) > c_0 - M_1\mu^\theta,$$

which does not make sense. Hence, $l = 0$ and therefore $u_k \to u$ strongly in X. This implies that

$$\rho \leq I_{\mu,k}(u_k) = I_\mu(u) + o_k(1),$$

and therefore $I_\mu(u) \geq \rho > 0$. The theorem is proved. □

CHAPTER 5

Indefinite problem with exponential critical growth in $\mathbb{R}^2$

We are concerned with the equation

$$(P_5) \qquad -\Delta u + \frac{1}{2}(x \cdot \nabla u) = a(x)f(u), \quad x \in \mathbb{R}^2,$$

where a is a sign-changing potential and the nonlinerity f has an exponential critical growth at infinity. The operator in (P_5) naturally appears when we look for self-similar solutions for homogeneous heat equations, namely solutions of the form $\omega(t,x) = t^{-1/(p-2)}u(t^{-1/2}x)$ for the evolution equation

$$\omega_t - \Delta\omega = |\omega|^{p-2}\omega, \text{ in } (0,+\infty) \times \mathbb{R}^N.$$

More specifically, ω is a solution for the above equation if, and only if, the profile u is a solution for the elliptic equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \lambda u + |u|^{p-2}u, \quad x \in \mathbb{R}^N.$$

There is a vast literature concerning the above problem with several types of nonlinearities for bounded domains, the whole space $\mathbb{R}^N$ and even the upper half-space $\mathbb{R}^N_+$. Without intention to present a complete list of references, we could cite [9, 20, 24, 44, 52, 57, 72, 75, 76] and references therein. In these works the authors find results about existence, nonexistence, multiplicity, decay rate, among other properties of solutions via ODE techniques or variational methods. As far as we know, Escobedo and Kavian [44] were the first to treat this operator in a variational way and particularly inspired works as [49, 51], that considered problem (P_5) with sign-changing nonlinearity having a concave-convex prototype.

In this chapter, we deal with an indefinite potential a. More specifically, we follow [2] and assume that

(a_1) $a : \mathbb{R}^2 \to \mathbb{R}$ is a bounded sign-changing continuous function;

(a_2) if

$$\Omega^+ := \{x \in \mathbb{R}^2; a(x) > 0\}, \quad \Omega^- := \{x \in \mathbb{R}^2; a(x) < 0\},$$

then $\mathrm{dist}(\overline{\Omega^+}, \overline{\Omega^-}) > 0$;

(a_3) there exists $R > 0$ such that $a(x) < 0$ for $|x| \geq R$.

We are interested in the case that f is superlinear both at the origin and at infinity, namely

(f_0) $f \in C(\mathbb{R}, \mathbb{R})$ and there exists $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0, \\ +\infty & \text{if} \quad \alpha < \alpha_0; \end{cases}$$

(f_1) $\lim_{s\to 0} f(s)/s = 0.$

In order to present the other conditions on f we need to say some words about our functional space. So, we set $K(x) := \exp(|x|^2/4)$ and notice that $\mathrm{div}(K(x)\nabla u) = K(x)\left[\Delta u + (1/2)(x \cdot \nabla u)\right]$, in such way that we can use a variational approach and look for solutions in the space X defined as the closure of $C_c^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^2} K(x)|\nabla u|^2\, dx\right)^{1/2}.$$

Given $s \geq 2$, it is proved in [50] that X is compactly embedded into the weighted Lebesgue space $L_K^s := L^s(\mathbb{R}^2, K(x))$. Hence, we can define the constant

$$S_2 := \inf\left\{\int_{\mathbb{R}^2} K(x)|\nabla u|^2 dx \,:\, \int_{\mathbb{R}^2} K(x)|u|^2\, dx = 1\right\}.$$

Since $\overline{\Omega^+}$ is far from $\overline{\Omega^-}$, we can find $\zeta \in C^\infty(\mathbb{R}^2, [0,1])$ such that

$$\zeta \equiv 1, \text{ in } \Omega^+, \qquad \zeta \equiv 0, \text{ in } \Omega^-, \qquad \mathcal{M} := \sup_{\mathbb{R}^2} |\nabla \zeta| < \infty.$$

Our technical assumptions on f can be stated as follows:

(f_2) there exist $\nu > 2$ and $0 < \theta < \nu\left[2(1 + \mathcal{M}S_2^{-1/2})\right]^{-1}$ such that, for $F(s) := \int_0^s f(\tau)\, d\tau$, there holds

$$0 < \frac{\nu}{\theta}F(s) \leq f(s)s, \quad \forall\, |s| > 0;$$

(f_3) there exist K_0, $R_0 > 0$ such that

$$0 < F(s) \leq K_0|f(s)|, \quad \forall\, |s| \geq R_0;$$

(f_4) if $x_0 \in \Omega^+$ and $r > 0$ are such that $a(x_0) = \max_{\Omega^+} a$ and $a(x) \geq (\max_{\Omega^+} a)/2$ in $B_r(x_0)$, then

$$\lim_{s \to +\infty} sf(s)e^{-\alpha_0 s^2} \geq \beta_0 > \frac{8}{\alpha_0 r^2 \cdot \max_{\Omega^+} a} \exp\left(\frac{r^2}{8} + \frac{r^4}{512}\right).$$

We prove the following existence result:

Theorem K. *Suppose that* $(a_1)-(a_3)$ *and* $(f_0)-(f_4)$ *hold. Then problem* (P_5) *admits at least a weak nontrivial solution.*

In the proof we apply the Mountain Pass Theorem. Since the potential a changes it sign, it is not so easy to prove that Palais-Smale sequences are bounded. Conditions (a_2) and (f_2) are important in this issue. Condition (f_3) has first appeared in [36] and provides a compactness property for the Palais-Smale sequence. With the aim of overcome the difficulties imposed by the lack of compactness, since we are dealing with the whole space $\mathbb{R}^2$, we invoke a version of the Trudinger-Moser inequality together with assumption (f_4) and the Moser's functions to find the correct localization of the mountain pass level. We notice that (f_4) is weaker than $\lim_{s\to+\infty} f(s)se^{-\alpha_0 s^2} = +\infty$, which have been used in some former papers (see (g_5) in [2] for instance). It is not difficult to see that, if we pick $q > \nu/\theta$, then the function

$$f(s) = (q|s|^{q-2}s + 2\alpha_0|s|^q s)e^{\alpha_0|s|^q}$$

satisfies all the conditions $(f_0) - (f_4)$ above.

We finish this introduction quoting the paper [14], where the authors considered

$$-\Delta u + u = a(x)f(u), \text{ in } \Omega \qquad Bu = 0, \text{ on } \partial\Omega,$$

in a bounded domain, $Bu = \partial u/\partial\nu$ or $Bu = u$, $a \in C(\Omega, \mathbb{R})$ is a sign-changing potential and f is a power type subcritical nonlinearity. The N-laplacian case is considered in [2] for an exterior domain Ω, Dirichlet boundary conditions and f having exponential critical growth. Theorem A is a complement of these papers since we deal with the whole space case and a different operator.

The chapter contains two more sections. In the first one, we present the variational framework to deal with (P_5) and some auxiliary results. Theorem K is proved in Section 5.2.

5.1 Variational framework and technical results

We start by quoting a Trudinger-Moser type inequality proved in [50].

Theorem 5.1.1 (Trudinger-Moser)**.** *If $u \in X$, $\beta > 0$ and $p \geq 0$ then $K(x)|u|^{2+p}(e^{\beta u^2} - 1) \in L^1(\mathbb{R}^2)$. Moreover, if $\|u\| \leq M$, with $\beta M^2 < 4\pi$, then there exists a constant $C = C(\beta, M, p) > 0$ such that*

$$\int_{\mathbb{R}^2} K(x)|u|^{2+p}(e^{\beta u^2} - 1)\, dx \leq C\|u\|^{2+p}.$$

Let $\alpha > \alpha_0$ and $q \geq 1$. It follows from (f_0) that

$$\lim_{|s| \to +\infty} \frac{f(s)}{|s|^{1-q}(e^{\alpha s^2} - 1)} = 0.$$

Hence, we can use (f_1) to obtain, for any given $\varepsilon > 0$, a constant $C_\varepsilon > 0$ such that

$$\max\{|f(s)s|, |F(s)|\} \leq \varepsilon s^2 + C_\varepsilon |s|^q (e^{\alpha s^2} - 1), \tag{5.1}$$

for any $s \in \mathbb{R}$. Since $a \in L^\infty(\mathbb{R}^2)$, we can use the above estimates and Theorem 5.1.1 to show that the functional $I : X \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} K(x)a(x)F(u)\, dx$$

is well-defined, it belongs to $C^1(\mathbb{R}^2, \mathbb{R})$ and its critical points are weak solutions for problem (P_5).

Let $x_0 \in \Omega^+$ and $r > 0$ be given by condition (f_4). We define a slight adaptation of the Green's function considered by Moser in [71], namely

$$\widetilde{M}_n(x) := \frac{1}{\sqrt{2\pi}} \cdot \begin{cases} K(r/n)^{-1/2}(\log n)^{1/2}, & \text{if} \quad |x - x_0| \leq r/n, \\ K(x)^{-1/2} \dfrac{\log(r/|x - x_0|)}{(\log n)^{1/2}}, & \text{if} \quad r/n \leq |x - x_0| < r, \\ 0, & \text{if} \quad |x - x_0| \geq r. \end{cases}$$

As we shall see, the location of $x_0 \in \mathbb{R}^2$ does not play any role in our next calculations. So, we assume with no loss of generality that $x_0 = 0$. We have that $\widetilde{M}_n \in H^1(\mathbb{R}^2)$ and $\operatorname{supp}(\widetilde{M}_n) = \overline{B}_r(0)$. Moreover, it is proved in [50, Lemma 4.6] that there exists a sequence $(d_n) \subset \mathbb{R}$ such that

$$\|\widetilde{M}_n\|^2 = 1 + \frac{1}{\log n}\left(\frac{r^2}{8} + \frac{r^4}{512}\right) - d_n, \qquad \lim_{n \to +\infty} d_n \log n = 0. \tag{5.2}$$

In particular, $\|\widetilde{M}_n\|^2 \to 1$, as $n \to +\infty$.

Lemma 5.1.2. *Suppose that $(a_1) - (a_3)$, (f_2) and (f_4) hold. If $M_n := \widetilde{M}_n / \|\widetilde{M}_n\|$, then there exists $n \in \mathbb{N}$ such that*

$$\max_{s \geq 0} I(sM_n) = \max\left\{\frac{s^2}{2} - \int_{\mathbb{R}^2} K(x)a(x)F(sM_n)\, dx\right\} < \frac{2\pi}{\alpha_0}.$$

Proof. For each $n \in \mathbb{N}$, consider the function $g_n(s) := I(sM_n)$, for $s \geq 0$. From (f_2), we obtain $C_1, C_2 > 0$ such that $F(s) \geq C_1|s|^{\nu/\theta} - C_2$, for any $s \in \mathbb{R}$. Thus, since $\mathrm{supp}(M_n) \subset \Omega^+$, we have that

$$g_n(s) \leq \frac{s^2}{2} - C_1 s^{\nu/\theta} \int_{\Omega^+} K(x)a(x)M_n^{\nu/\theta}\,dx + C_2 \int_{\Omega^+} K(x)a(x)\,dx.$$

Recalling that $\nu/\theta > 2$, we obtain $g_n(s) \to -\infty$, as $s \to +\infty$. Hence, g_n attains its global maximum at $s_n > 0$ which satisfies $0 = g_n'(s_n)$ or, equivalently,

$$s_n^2 = \int_{B_r(0)} K(x)a(x)f(s_nM_n)s_nM_n\,dx. \tag{5.3}$$

Suppose, by contradiction, that the result of the lemma is false. Then $g_n(s_n) \geq (2\pi)/\alpha_0$ and we can use the definition of g_n, $\mathrm{supp}(M_n) \subset \Omega^+$ and $F \geq 0$, to get

$$s_n^2 \geq \frac{4\pi}{\alpha_0}. \tag{5.4}$$

Let $\beta_0 > 0$ be given by (f_4). If $0 < \varepsilon < \beta_0$, there exists $R_\varepsilon > 0$ such that

$$sf(s) \geq (\beta_0 - \varepsilon)e^{\alpha_0 s^2}, \quad \forall\, |s| \geq R_\varepsilon. \tag{5.5}$$

Using the definition of M_n, (5.4) and $\|\tilde{M}_n\| \to 1$, as $n \to +\infty$, we conclude that

$$s_nM_n(x) = s_n\frac{\tilde{M}_n}{\|\widetilde{M}_n\|} \geq \frac{e^{-r^2/(8n^2)}}{\|\widetilde{M}_n\|}\sqrt{\frac{4\pi\log n}{\alpha_0}} \geq R_\varepsilon,$$

for any $|x| < r/n$ and n large. Hence, it follows from (5.3), (5.5), $K \geq 1$, the choice of $r > 0$ in (f_4), the previous inequality and the definition of M_n that

$$\begin{aligned}
s_n^2 &\geq \int_{B_{r/n}(0)} K(x)a(x)f(s_nM_n)s_nM_n\,dx \\
&\geq c_0(\beta_0-\varepsilon)\int_{B_{r/n}(0)} \exp(\alpha_0(s_nM_n)^2)\,dx \\
&= c_0(\beta_0-\varepsilon)\int_{B_{r/n}(0)} \exp\left(\alpha_0 s_n^2\frac{e^{-r^2/(4n^2)}\log n}{2\pi\|\widetilde{M}_n\|^2}\right)dx \\
&= c_0(\beta_0-\varepsilon)\frac{\pi r^2}{n^2}\exp\left(\alpha_0 s_n^2\frac{e^{-r^2/(4n^2)}\log n}{2\pi\|\widetilde{M}_n\|^2}\right),
\end{aligned}$$

where $c_0 := (\max_{\Omega^+} a)/2$. Using that $1/n^2 = \exp(-2\log n)$, we obtain

$$s_n^2 \geq c_0(\beta_0-\varepsilon)\pi r^2 \exp\left(2\left[\frac{e^{-r^2/(4n^2)}}{\|\widetilde{M}_n\|^2}\frac{\alpha_0}{4\pi}s_n^2 - 1\right]\log n\right), \tag{5.6}$$

and hence, recalling that $\exp(s) \geq s$, we get that

$$s_n^2 \geq 2c_0(\beta_0-\varepsilon)\pi r^2\left[\frac{e^{-r^2/(4n^2)}}{\|\widetilde{M}_n\|^2}\frac{\alpha_0}{4\pi}s_n^2 - 1\right]\log n. \tag{5.7}$$

Since $e^{-r^2/(4n^2)}\|\widetilde{M}_n\|^{-2} \to 1$, we conclude from the above inequality that (s_n) is bounded. Hence, up to a subsequence, $s_n^2 \to \gamma \geq 4\pi/\alpha_0$. If $\gamma > 4\pi/\alpha_0$, we obtain a contradiction after passing (5.7) to the limit. Thus, $\gamma = 4\pi/\alpha_0$. Combining inequalities (5.4), (5.6) and Lemma 5.2, we obtain

$$s_n^2 \geq c_0(\beta_0 - \varepsilon)\pi r^2 \exp\left\{\frac{-2}{\|\widetilde{M}_n\|^2}(\|\widetilde{M}_n\|^2 - e^{-r^2/(4n^2)})\log n\right\}.$$

Passing to the limit in n, using (5.2) and a straightforward computation, we obtain

$$\frac{4\pi}{\alpha_0} \geq c_0(\beta_0 - \varepsilon)\pi r^2 \exp\left(-2\left(\frac{r^2}{8} + \frac{r^4}{512}\right)\right).$$

Letting $\varepsilon \to 0$ and recalling that $c_0 = (\max_{\Omega^+} a)/2$, we finally conclude that

$$\beta_0 \leq \frac{8}{\alpha_0 r^2 \cdot \max_{\Omega^+} a} \exp\left(\frac{r^2}{4} + \frac{r^4}{256}\right),$$

which contradicts assumption (f_4). The result is proved. □

We prove in the sequel that I has the Mountain Pass geometry.

Lemma 5.1.3. *Suppose that $(a_1) - (a_3)$ and $(f_0) - (f_2)$ hold. If $n \in \mathbb{N}$ is given by Lemma 5.1.2, we have that*

(i) there exist ξ, $\rho > 0$ such that $I(u) \geq \xi$, for any $u \in X$, $\|u\| = \rho$.

(ii) there exists $s_0 > 0$ such that $\|s_0 M_n\| > \rho$ and $I(s_0 M_n) < 0$.

Proof. Given $\alpha > \alpha_0$ and $\varepsilon > 0$, it follows from (5.1) (with $q = 3$) that

$$\begin{aligned}\int_{\mathbb{R}^2} K(x)a(x)F(u)\,dx &\leq \int_{\Omega^+} K(x)a(x)F(u)\,dx \leq \varepsilon\|a\|_{L^\infty(\Omega^+)}\|u\|_2^2 \\ &+ \|a\|_{L^\infty(\Omega^+)}C_\varepsilon \int_{\mathbb{R}^N} K(x)|u|^3(e^{\alpha u^2} - 1)\,dx.\end{aligned}$$

If $0 < M < 1$ is such that $\alpha M^2 < 4\pi$, we can use Theorem 5.1.1 to obtain $C_1 = C_1(M, \alpha) > 0$ such that

$$\int_{\mathbb{R}^2} K(x)a(x)F(u)\,dx \leq \varepsilon\|a\|_{L^\infty(\Omega^+)}S_2^{-1}\|u\|_2 + C_1\|u\|^3,$$

whenever $\|u\| \leq M$. Hence, picking $\varepsilon > 0$ in such a way that $(1 - 2\varepsilon\|a\|_{L^\infty(\Omega^+)}S_2^{-1}) = C_2 > 0$, we get that

$$I(u) \geq \frac{1}{2}(1 - 2\varepsilon\|a\|_{L^\infty(\Omega^+)}S_2^{-1})\|u\|^2 - C_1\|u\|^3 = \|u\|^2\left(\frac{C_2}{2} - C_1\|u\|\right),$$

and item (i) clearly holds for $\rho := C_2/(4C_1)$ and $\xi := \rho^2 C_2/4$. The second statement is a direct consequence of the proof of the last lemma, where we have that $I(sM_n) \to -\infty$, as $s \to +\infty$. □

The above result ensures the existence of a Palais-Smale sequence at the mountain pass level [3] (see also [93, Theorem 1.15]), that is, a sequence $(u_n) \subset X$ such that

$$\lim_{n\to+\infty} I'(u_n) = 0, \qquad \lim_{n\to+\infty} I(u_n) = c_{MP}$$

where

$$c_{MP} := \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} I(\gamma(s)) \in \left(0, \frac{2\pi}{\alpha_0}\right),$$

and $\Gamma := \{\gamma \in C([0,1], X); \gamma(0) = 0, \gamma(1) = e\}$, with $e := s_0 M_n \in X$ given by Lemma 5.1.3. Notice that the path $\gamma(s) := s s_0 M_n$ belongs to Γ and therefore we really have that $c_M < 2\pi/\alpha_0$.

Lemma 5.1.4. *There exists $u_0 \in X$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ weakly in X.*

Proof. It is sufficient to prove that (u_n) is bounded in X. Computing $I(u_n) - (\theta/\nu)I'(u_n)(\zeta u_n)$ and using the properties of the function ζ we get that

$$\begin{aligned} c + o_n(1) + o_n(1)\|u_n\| &= \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^2} K(x)a(x)F(u_n)\,dx \\ &- \frac{\theta}{\nu}\int_{\mathbb{R}^2} K(x)\left[\nabla u_n \nabla(\zeta u_n) - a(x)f(u_n)\zeta u_n\right]dx \\ &\geq \left(\frac{1}{2} - \frac{\theta}{\nu}\right)\|u_n\|^2 - \frac{\theta\mathcal{M}}{\nu}\int_{\mathbb{R}^2} K(x)|\nabla u_n||u_n|\,dx \\ &+ \int_{\Omega^+} K(x)a(x)\left[\frac{\theta}{\nu}f(u_n)u_n - F(u_n)\right]dx \end{aligned}$$

and therefore we can use (f_2) to obtain

$$c + o_n(1) + o_n(1)\|u_n\| \geq \left(\frac{1}{2} - \frac{\theta}{\nu}\right)\|u_n\|^2 - \frac{\theta\mathcal{M}}{\nu}\int_{\mathbb{R}^2} K(x)|\nabla u_n||u_n|\,dx. \tag{5.8}$$

It follows from Hölder's inequality and the continuous embedding that

$$\frac{\theta\mathcal{M}}{\nu}\int_{\mathbb{R}^2} K(x)|\nabla u_n||u_n|\,dx \leq \frac{\theta\mathcal{M}S_2^{-1/2}}{\nu}\|u_n\|^2,$$

which together with (5.8) lead to

$$c + o_n(1) + o_n(1)\|u_n\| \geq \left(\frac{1}{2} - \frac{\theta}{\nu} - \frac{\theta\mathcal{M}S_2^{-1/2}}{\nu}\right)\|u_n\|^2.$$

By (f_2), the term into parenthesis above is positive, which implies that (u_n) is bounded in X.

□

Since X is compactly embedded in $L^s_K(\mathbb{R}^2)$, it follows from the above lemma that

$$\begin{cases} u_n & \to & u_0 \text{ strongly in } L^s(\mathbb{R}^2), \\ u_n(x) & \to & u_0(x) \text{ a.e. in } \mathbb{R}^2, \\ |u_n(x)| & \leq & h_s(x) \text{ a.e. in } \mathbb{R}^2, \end{cases} \tag{5.9}$$

for any $s \geq 2$ and some $h_s \in L^s_K(\mathbb{R}^2)$.

Lemma 5.1.5. *Suppose that* $(a_1)-(a_3)$ *and* $(f_0)-(f_4)$ *hold. If* $a^{\pm}(x) := \max\{\pm a(x), 0\}$ *and* $u_0 \in X$ *is given by Lemma 5.1.4, then* $K(x)a^{\pm}(x)f(u_n) \to K(x)a^{\pm}(x)f(u_0)$ *in* $L^1_{loc}(\mathbb{R}^2)$.

Proof. Fixed $\sigma > 0$, we can compute $I(u_n) - (\sigma/\nu)I'(u_n)(\zeta u_n)$ and argue as in Lemma 5.1.4 to obtain

$$\begin{aligned} c + o_n(1) + o_n(1)\|u_n\| & \geq \left(\frac{1}{2} - \frac{\sigma}{\nu} - \frac{\sigma \mathcal{M} S_2^{-1/2}}{\nu}\right)\|u_n\|^2 \\ & + \left(\frac{\sigma}{\nu} - \frac{\theta}{\nu}\right)\int_{\Omega^+} K(x)a(x)f(u_n)u_n\, dx. \end{aligned}$$

Choosing $\sigma > \nu\left[2(1 + \mathcal{M}S_2^{-1/2})\right]^{-1} > \theta$ and recalling that (u_n) is bounded, we obtain

$$\int_{\Omega^+} K(x)a(x)f(u_n)u_n\, dx \leq C_1.$$

Moreover, since $I'(u_n)u_n = 0$, we have that

$$\int_{\Omega^-} K(x)a(x)f(u_n)u_n\, dx \leq \int_{\mathbb{R}^N} K(x)a(x)f(u_n)u_n\, dx = \|u_n\| + o_n(1) \leq C_2.$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded set. Given $\varepsilon > 0$, is is clear that

$$|f(s)| \leq \varepsilon f(s)s, \quad \forall\, |s| \geq R_\varepsilon := 1/\varepsilon.$$

Consequently,

$$\int_{[|u_n| \geq R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)|f(u_n)|\, dx \leq \varepsilon \int_{[|u_n| \geq R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)f(u_n)u_n\, dx \leq \varepsilon C_3, \tag{5.10}$$

with $C_3 := (C_1 + C_2)$. Thus, from the pointwise convergence and Fatou's lemma, we obtain

$$\int_{[|u_0| \geq R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)|f(u_0)|\, dx \leq \varepsilon C_3. \tag{5.11}$$

On the other hand,

$$\begin{aligned} \int_{\Omega} K(x)a^{\pm}(x)|f(u_n) - f(u_0)|\, dx & \leq \int_{[|u_n| \geq R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)|f(u_0)|\, dx \\ & + \int_{[|u_n| \geq R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)|f(u_n)|\, dx \\ & + \int_{[|u_n| < R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)|f(u_n) - f(u_0)|\, dx. \end{aligned}$$

Thus, we infer from (5.10) and (5.11) that

$$\begin{aligned}\int_{\Omega} K(x)a^{\pm}(x)|f(u_n)-f(u_0)|\,dx &\leq 2\varepsilon C_3+\int_{\Sigma_{n,\varepsilon}\cap\Omega} K(x)a^{\pm}(x)|f(u_0)|\,dx\\ &\quad+\int_{[|u_n|<R_\varepsilon]\cap\Omega} K(x)a^{\pm}(x)|f(u_n)-f(u_0)|\,dx,\end{aligned}$$

with $\Sigma_{n,\varepsilon} := [|u_0| < R_\varepsilon] \cap [|u_n| \geq R_\varepsilon]$. Passing the above inequality to the limit as $n \to +\infty$, using that Ω is bounded, Lebesgue's theorem and the arbitrariness of $\varepsilon > 0$, we obtain

$$\lim_{n\to+\infty}\int_{\Omega} K(x)a^{\pm}(x)f(u_n)\,dx=\int_{\Omega} K(x)a^{\pm}(x)f(u_0)\,dx,$$

and the lemma is proved. □

5.2 Proof of Theorem K

We prove in this section the main theorem of the chapter. The idea is proving that the weak limit u_0 given by Lemma 5.1.4 is a nonzero solution of (P_5). First notice that, since $I'(u_n) \to 0$, as $n \to +\infty$, we can use Lemmas 5.1.4 and 5.1.5 to conclude that $I'(u_0)\varphi = 0$, for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. A density argument shows that u_0 is a critical point of I.

Suppose, by contradiction, that $u_0 = 0$. Using condition (f_3), the continuity of f and that Ω^+ is bounded, we obtain $C_1 > 0$ such that

$$K(x)a(x)F(u_n) \leq C_1 + K_0K(x)a(x)|f(u_n)|, \quad \text{for a.e. } x \in \Omega^+.$$

As a byproduct of the proof of Lemma 5.1.5, we see that the right hand side above goes to zero. So, we can use the pointwise convergence and Lebesgue's theorem to conclude that $\int_{\Omega^+} K(x)a(x)F(u_n)\,dx \to 0$. Hence,

$$\begin{aligned}c_{MP}+o_n(1)=I(u_n) &= \frac{1}{2}\|u_n\|^2-\int_{\mathbb{R}^2} K(x)a(x)F(u_n)\,dx\\ &\geq \frac{1}{2}\|u_n\|^2-\int_{\Omega^+} K(x)a(x)F(u_n)\,dx=\frac{1}{2}\|u_n\|^2+o_n(1),\end{aligned}$$

from which we conclude that $\limsup_{n\to+\infty}\|u_n\|^2 \leq 2c_{MP} < 4\pi/\alpha_0$. This provides m, $n_0 > 0$ be such that

$$\|u_n\|^2 < m < \frac{4\pi}{\alpha_0}, \quad \forall\, n \geq n_0.$$

We now claim that $\int_{\mathbb{R}^2} K(x)a(x)f(u_n)u_n = o_n(1)$. If this is true, we can use $I'(u_n)u_n = o_n(1)$ and (5.1) to get

$$\|u_n\|^2=\int_{\mathbb{R}^2} K(x)a(x)f(u_n)u_n\,dx+o_n(1)=o_n(1),$$

which implies that $I(u_n) \to 0$. But this is impossible because $I(u_n) \to c_{MP} > 0$. Then, $u_0 \neq 0$ is the desired solution.

In order to prove the claim, we pick $\alpha > \alpha_0$, $q > 2$ and $s > 1$ to be chosen later, and apply (5.1) together with Hölder's inequality to write

$$\begin{aligned}\int_{\mathbb{R}^2} K(x)a(x)f(u_n)u_n\,dx &\leq C_2\|u_n\|^2_{L^2_K} + C_3\int_{\mathbb{R}^2} K(x)|u_n|^{2q}(e^{\alpha u_n^2}-1)dx \\ &\leq C_2\|u_n\|^2_{L^2_K} \\ &+ C_3\|u_n\|^q_{L^{qs'}_K}\left[\int_{\mathbb{R}^2} K(x)|u_n|^{qs}\left(e^{\alpha u_n^2}-1\right)^s\right]^{1/s},\end{aligned}$$

Using the inequality $(1+a)^s \geq 1 + a^s$ with $a = e^t - 1$, we get $(e^t-1)^s \leq e^{ts} - 1$. So, setting $v_n := u_n/\|u_n\|$ and noticing that $\alpha s u_n^2 = \alpha s\|u_n\|^2|v_n|^2 \leq \alpha s m|v_n|^2$, for $n \geq n_0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} K(x)a(x)f(u_n)u_n\,dx &\leq C_2\|u_n\|^2_{L^2_K} \\ &+ C_4\|u_n\|^q_{L^{qs'}_K}\left[\int_{\mathbb{R}^2} K(x)|v_n|^{qs}\left(e^{\alpha s m|v_n|^2}-1\right)\right]^{1/s}.\end{aligned}$$

Since $\alpha s m \to \alpha_0 m < 4\pi$, as $\alpha \to \alpha_0$ and $s \to 1^+$, we can choose α, s, q close to the numbers α_0, 1, 2, respectively, and use Theorem 5.1.1 to guarantee that the term into brackets above is uniformly bounded. It is sufficient now to recall that $u_n \to 0$ strongly in the weighted Lebesgue spaces to obtain

$$\int_{\mathbb{R}^2} K(x)a(x)f(u_n)u_n \leq C_1\|u_n\|^2_{L^2_K} + C_5\|u_n\|^q_{L^{qs'}_K} = o_n(1),$$

and we have done.

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