Universidade de Brasília

A type of Rabinowitz Theorem for compact operators on a strip and an application to a quasilinear problem

Vinicius Kobayashi Ramos

Orientador: Dr. Carlos Alberto Pereira dos Santos

Departamento de Matemática
Universidade de Brasília

Dissertação apresentada como requisito parcial para obtenção do grau de *Mestre em Matemática*

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por

Vinicius Kobayashi Ramos *

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Comissão Examinadora:

____

Prof. Dr. Carlos Alberto Pereira dos Santos- MAT/UnB (Orientador)

Willian Cintra

____

Prof. Dr. Willian Cintra– MAT/UnB (Membro)

____

Prof. Dr. Antonio Suárez Fernández– Universidad de Sevilla (Membro)

Lais Moreira dos Santos

____

Profa. Dra. Laís Moreira dos Santos – Universidade de Viçosa(Suplente)

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Resumo

O Teorema da Alternativa Global de Bifurcação de Rabinowitz foi explorado por muitos autores com o objetivo de estabelecer resultados de existência de bifurcação. Nesse trabalho, nos engajamos nesse sentido propondo um teorema que garante a existência de alternativa global de bifurcação numa faixa, isto é, uma generalização do resultado original de Rabinowitz cuja formulação envolve problemas do tipo $u = K(\lambda, u)$ em que $K : I \times E \to E$ é um operador compacto e I um intervalo fechado (possivelmente ilimitado) com interior não vazio. Então, como uma aplicação deste, nós provamos um resultado de bifurcação no infinito semelhante ao obtido por Arcoya, Carmona e Pellaci em 2001 [4].

Abstract

The Global Bifurcation Alternative of Rabinowitz Theorem was explored by many authors in order to establish bifurcation existence results. In this work we engage in this sense on proposing a theorem that guarantee global bifurcation alternative in a strip that is a generalization of the original one which is formulated for problems in the form $u = K(\lambda, u)$, where $K : I \times E \to E$ is a compact operator and I is a closed interval (possibly unbounded) with nonempty interior. Then we apply it in order to obtain a bifurcation at infinity existence result similar to that given by Arcoya, Carmona and Pellaci in 2001 [4].

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Chapter 1

Introduction

Consider the problem of solving an equation

$$\Phi(\lambda, u) = 0 \tag{1.0.0.1}$$

for $(\lambda, u) \in \mathbb{R} \times E$ where E is a real Banach space, $\Phi : \mathbb{R} \times E \to E$ and assume that $u = 0$ trivially satisfies (1.0.0.1) for every λ, that is, the set of solutions of (1.0.0.1) contains the curve

$$\{(\lambda, 0);\ \lambda \in \mathbb{R}\}.$$

This happens, for example in a Dirichlet problem of the form

$$\begin{cases} -\Delta u = \lambda u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of the euclidean space $\mathbb{R}^N$. Indeed, from the literature we know that, under some conditions, this problem can be formulated as

$$\Phi(\lambda, u) = 0,$$

where $\Phi(\lambda, u) : \mathbb{R} \times E \to E$ is a compact perturbation of the identity defined as $\Phi(\lambda, u) = u - S(\lambda u^p)$, $S : E \to E$ is the solution operator of the problem

$$\begin{cases} -\Delta u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and E is some appropriated Banach space.

Commonly, the solutions of interest are the nontrivial ones, that is, the solutions (λ, u) with $u \neq 0$. The bifurcation theory is concerned to the problem of finding nontrivial solutions for (1.0.0.1) by taking advantage, somehow, of the well known curve of trivial solutions of (1.0.0.1). We say that $(\lambda_0, 0)$ is a bifurcation point of the equation (1.0.0.1) from the curve of trivial solutions $\{(\lambda, 0); \lambda \in \mathbb{R}\}$ if any neighbourhood of $(\lambda_0, 0)$ contains at least one nontrivial solution of (1.0.0.1).

Let us investigate the possibility of existence of nontrivial solutions near some point $(\lambda_0, 0)$. Consider (a, b) be an interval (possibly unbounded) containing λ_0, U be a bounded open subset of E containing $u = 0$ and just for a moment suppose that $\Phi : (a, b) \times U \to E$ satisfies the hypotheses of Implicit Function Theorem, that is,

i) Φ is continuous;

ii) Φ_u exists and is continuous in V;

iii) $\Phi_u(\lambda_0, 0)$ is invertible with continuous inverse.

In this context the Implicit Function Theorem states that in a neighbourhood U of $(\lambda_0, 0)$, the solutions of $\Phi(\lambda, u) = 0$ are constituted by a unique curve $(\lambda, \varphi(\lambda))$. But we already know that $\lambda \mapsto (\lambda, 0)$ is a curve of solutions of $\Phi(\lambda, u) = 0$ and so we conclude that there is no nontrivial solutions in U. This analysis leads us to a necessary condition for a point $(\lambda_0, 0)$ to be a bifurcation point of $\Phi(\lambda, u) = 0$ from the curve of trivial solutions $\{(\lambda, 0);\ \lambda \in \mathbb{R}\}$ when Φ satisfies i) and ii), which is the following. If $\Phi : U \to E$ satisfies i), ii) and $(\lambda_0, 0)$ is a bifurcation point from the curve of trivial solutions of $\Phi(\lambda, u) = 0$, then item iii) is not verified. If we require less in ii), then we deduce a similar necessary condition given by Lemma B.

Now, let us discard the hypothesis ii) over Φ and compensate it by assuming that (1.0.0.1) is actually a fixed point problem, that is, $\Phi : \mathbb{R} \times \overline{U} \to E$ is a continuous operator of the form

$$\Phi(\lambda, u) = u - K(\lambda, u),$$

where U is a bounded open subset of some real Banach space E. By imposing that K is a compact operator, the Degree Theory gives us a range of tools that allow us to study the existence of bifurcation points for the problem (1.0.0.1). So in Chapter 2, we introduce two objects of the Degree Theory: the Brouwer degree and the Leray-Schauder degree. The first one is a function which associates an integer to each ordered triple

$$(f, U, v) \mapsto deg(f, U, v) \in \mathbb{Z}, \tag{1.0.0.2}$$

where $f : \overline{U} \to \mathbb{R}^N$ is a continuous function, U is a bounded open subset of $\mathbb{R}^N$ and $v \notin f(\partial U)$. The number $\deg(f, U, v)$ gives information about the existence of solutions of $f(u) = v$. The Leray-Schauder degree is an extension of the Brouwer degree for abstract Banach spaces E and was introduced by Leray and Schauder in [28] (1934). Such degree is a map as (1.0.0.2), where now f is substituted by a compact perturbation of the identity $\Phi = I - K : \overline{U} \to E$, U is a bounded open subset of a real Banach space E and $v \notin \Phi(\partial U)$. As well as the Brouwer degree, the number $\deg(\Phi, U, v)$ gives information about the existence of solutions of $\Phi(u) = v$.

The power of this theory lies in the fact that the degree is a topological invariant, that is, it obeys some properties of invariance with respect to the topology considered in E. The most useful among them is the homotopy invariance of the Leray-Schauder degree, which can be stated as follows.

Theorem 1.0.1. *If $H : [0, 1] \times \overline{U} \to E$ is a function of the form*

$$H(t, u) = u - T(t, u),$$

where $T : [0, 1] \times \overline{U} \to E$ is a compact operator, then

$$deg(H(t, \cdot), U, v) = \text{constant for all } t \in [0, 1], \tag{1.0.0.3}$$

for each $v \notin H(\partial U)$.

This property is powerful because it allows us to determine the degree of a function Φ, whose the degree is difficult to calculate, by finding an homotopy between Φ and another compact perturbation of the identity Ψ for each the degree is known.

Moreover, there is a variation of this property that states that (1.0.0.3) holds not only for t varying, but also for U varying. This is the content of the next result.

Theorem 1.0.2. *Assume that* $K : \overline{U} \to E$ *is a compact operator, where* U *is a bounded open subset of* $[a,b] \times E$*, and that the equation*

$$\Phi(\lambda, u) := u - K(\lambda, u) = 0$$

does not admit solutions on ∂U*. Then*

$$deg(\Phi_\lambda, U_\lambda, 0) \text{ is constant in } \lambda \in [a,b],$$

where $U_\lambda := \{u \in E; (\lambda, u) \in U\}$ *and* $\Phi_\lambda := \Phi(\lambda, \cdot)$.

For the case where u is an isolated solution of $\Phi(u) = v$, there is a limit version of the Leray-Schauder degree, which is called "index of an isolated solution" and is defined as

$$i(\Phi, u) = \lim_{\varepsilon \to 0} \deg(\Phi, B_\varepsilon(u), v). \tag{1.0.0.4}$$

The index of an isolated solution $u = 0$ of $\Phi(u) = 0$ satisfies the following identity

$$i(\Phi, 0) = (-1)^\beta, \tag{1.0.0.5}$$

where β is the sum of the algebraic multiplicities of the characteristic values of $K'(0)$ contained in $(0, 1)$. The identity (1.0.0.5) is usually called in the literature as "Leray-Schauder Formula" and it holds for operators $\Phi = I - K$ satisfying some apropriated conditions (see Chapter 2).

Once presented the Leray-Schauder degree and its properties in Chapter 2, we arrive at Chapter 3, where a brief Bifurcation Theory overview is presented. Throughout this Chapter, except for the two last sections 3.3 and 3.4, we will assume the following hypotheses and definitions.

H1) $I \subset \mathbb{R}$ *is a closed interval with non-empty interior*.

H2)

$$\begin{aligned} K : I \times E &\to E \\ (\lambda, u) &\mapsto K(\lambda, u) \end{aligned}$$

is a compact operator.

H3) K *has the form*

$$K(\lambda, u) = L(\lambda)u + H(\lambda, u), \tag{1.0.0.6}$$

where

$$\begin{aligned} L : I &\to \mathcal{H}_c(E) \\ \lambda &\mapsto L(\lambda) \end{aligned}$$

is a continuous operator,

$$\mathcal{H}_c(E) := \{T : E \to E;\ T \text{ is a homogeneous compact operator of degree } 1\}$$

and

$$\begin{aligned} H : I \times E &\to E \\ (\lambda, u) &\mapsto H(\lambda, u) \end{aligned}$$

is a compact operator such that

$$H(\lambda, u) = o(\|u\|) \text{ near } u = 0, \text{ uniformly on each compact interval of } \lambda \text{ contained in } I. \tag{1.0.0.7}$$

Definition 1.0.1. *Let $L : \mathbb{R} \to H_c(E)$ be a continuous operator. Then we define $L_0 := L(1)$ and the set $r(L_0)$ of all characteristic values of L_0 as*

$$r(L_0) := \{\lambda \in \mathbb{R};\ \text{there exists some } 0 \neq u \in E,\ \text{such that } u = L(\lambda)u\}.$$

N) *We will denote*

$$\Phi(\lambda, u) := u - K(\lambda, u).$$

P) *It will be studied the problem* $\Phi(\lambda, u) = 0$.

HI) *In the case of Krasnosel'skii and Rabinowitz theorems, we assume that* $I = \mathbb{R}$. *In the results involving the set* $r(L_0)$, we assume that the interior of I contains the value $\lambda = 1$.

These general assumptions are inspired by the hypotheses of Theorem 1 of [12], which is a generalization of the Unilateral Bifurcation Theorem of Rabinowitz (Theorem 1.25 of [33], which corresponds to Theorem 3.3.2) and our motivations to adopt them are the following.

$\mathbf{M_1}$) These assumptions generalizes the hypotheses of two results that are studied in this work: the Global Bifurcation Alternative of Rabinowitz (Theorem 1.3 of [33], which corresponds to Theorem 3.2.2) and consequently also generalizes the hypotheses of the first well known bifurcation existence result of Krasnosel'skii (Theorem 2.1 of [24] which corresponds to Theorem 3.2.1).

$\mathbf{M_2}$) The hypothesis about the continuity of the operator $\lambda \mapsto L(\lambda)$ in Dai and Feng [12] (2019) is not clear because it requires the assumption that the image of L lives in some topological space and such space is not evidenced by the authors. Motivated by this issue, we construct a norm in the space $H_c(E)$ with which the space is Banach, as we prove by combining the two below lemmas.

Lemma $\mathbf{A_0}$. *The space*

$$H_f(E) := \left\{ H \in H(E);\ \sup_{E \setminus \{0\}} \left\| \frac{H(u)}{\|u\|} \right\| < \infty \right\}$$

is a Banach subspace of $H(E)$.

Lemma A. $H_c(E)$ *is a closed subspace of* $H_f(E)$.

$\mathbf{M_3}$) As a corollary of Theorem A, which is our main result, we prove

Theorem B (A type of Dai's Theorem on a strip). *Assume that $\lambda_0 \in r(L_0)$ is isolated and satisfies*

$$i(\Phi(\lambda_0 - \eta, \cdot), 0) \neq i(\Phi(\lambda_0 + \xi, \cdot), 0) \tag{1.0.0.8}$$

for sufficiently small positive numbers ξ and η. Then there exists a continuum (connected and closed subset of $\mathcal{S}$) $\mathcal{C}_{\lambda_0}$ of

$$\mathcal{S} := \overline{\{(\lambda, u) \in I \times (E \setminus \{0\});\ \Phi(\lambda, u) = 0\}}$$

containing $(\mu, 0)$ such that $\mathcal{C}_{\lambda_0}$ satisfies, at least, one of the following (non-excluding) alternatives:

i) $\mathcal{C}_{\lambda_0}$ *is unbounded;*

ii) $\mathcal{C}_0$ *intercepts some* $(d,u) \in I \times E$*, where* d *is an extremity of the interval* I *(if* I *possesses some extremity) or (not exclusive) intercepts some* $(\lambda_1, 0)$ *with* $\lambda_0 \neq \lambda_1 \in I$*.*

Theorem B is a Dai's correspondent type of the Global Bifurcation Alternative of Rabinowitz (Theorem 1.3 of [33], which corresponds to Theorem 3.2.2) and, as a consequence of the Leray-Schauder formula (1.0.0.5), we also prove

Corollary C. *The Global Bifurcation Alternative of Rabinowitz (Theorem 3.2.2) is a corollary of Theorem B.*

$\mathbf{M}_4$) The indexes in the hypothesis (1.0.0.8) are well defined as a consequence of

Lemma D. *If* $(\lambda_0, 0)$ *is a bifurcation point from the curve of trivial solutions of*

$$(\lambda, u) = 0,$$

then $\lambda_0 \in r(L_0)$*.*

Lemma D is motivated by the statement "[...] the Leray-Schauder degree, $\deg(I - L(\lambda), B_r, 0)$, is well defined for arbitrary r ball B_r and $\lambda \notin r(L_0)$[...]" made in the introduction of [12] which leads us to infer that the necessary condition of Krasnosel'skii for a value λ_0 to be a bifurcation point (Theorem 3.2.3) holds not only for the case when $L(\lambda) = \lambda L$, with L linear, as the necessary condition of Krasnosel'skii (Theorem 3.2.3) does, but also when $\lambda \mapsto L(\lambda)$ is a continuous function and $L(\lambda) \in H_c(E)$. However, we do not find such a result in the literature and so we prove it.

Once motivated our choice for general assumptions of Chapter 3, let us present some of the principal results of this chapter.

After the construction of the real Banach space $H_c(E)$ and an exposition of the evolution of the bifurcation point definition from Krasnosel'skii's to the modern one, we enunciate the first well known bifurcation existence result of Krasnosel'skii [24] (1964).

Theorem 1.0.3 (Krasnosel'skii's Theorem). *Let* $A : E \to E$ *be a compact operator with Fréchet derivative* $A'(0)$ *such that* $A(0) = 0$*. Then, each characteristic value* λ_0 *of odd algebraic multiplicity of the linear operator* $A'(0)$ *is a bifurcation point of* $(\lambda, u) = 0$*, where we are identifying the operators* L *and* H *from hypothesis H3) as* $L(\lambda) = \lambda A'(0)$*,* $H(\lambda, u) = \lambda H(u)$ *and* $H(u) = A(u) - A'(0)u$ *with* $H(u) = o(\|u\|)$*.*

Moveover, associated with λ_0 *there exists a continuous branch of eigenvectors of the operator* A*.*

For a continuous branch of eigenvectors we mean that every neighbourhood $U \subset E$ of 0 contained in a small ball B with $0 \in B$ is such that ∂U contains at least one eigenvector of A associated to some characteristic value λ near λ_0. Krasnosel'skii's Theorem was improved, seven years later, by P.H. Rabinowitz in [33] and his remarkable result became known in the literature as "The Rabinowitz Global Bifurcation Alternative" and whose statement is the following.

Theorem 1.0.4 (Global Bifurcation Alternative of Rabinowitz). *Suppose that the operators* L *and* H *of hypothesis H3) are such that* $L(\lambda) = \lambda L$*, where* L *is a compact linear operator and* $H : \mathbb{R} \times E \to E$ *is a compact operator. If* λ_0 *is a characteristic value of* L *of odd algebraic multiplicity, then there exists a maximal continuum* $\mathcal{C}_0$ *of* $\mathcal{S}$ *containing* $(\lambda_0, 0)$ *such that* $\mathcal{C}_0$ *satisfies at least one of the following (non-excluding) alternatives:*

i) $\mathcal{C}_{\lambda_0}$ *is unbounded,*

ii) $\mathcal{C}_{\lambda_0} \cap \{(\lambda_1, 0)\} \neq \emptyset$ *for some* $\lambda_1 \neq \lambda_0$,

where

$$\mathcal{S} := \overline{\{(\lambda, u);\ \Phi(\lambda, u) = 0,\ u \neq 0\}}$$

With Lemma 2.1 of [24], Krasnosel'skii proves a result that is, in particular[1], a necessary condition for some $(\lambda_0, 0)$ being a bifurcation point of $\Phi(\lambda, u) = 0$, by assuming the hypotheses of Theorem 1.0.3. In the proof of Theorem 1.0.4 (which corresponds to Theorem 1.3 of [33]), Rabinowitz uses this result, although the hypotheses of Theorem 1.0.4 are slightly more general than the bifurcation problem studied by Krasnosel'skii in [24] (see Theorem 3.2.2). However, the author does not justify why the same result holds with more general assumptions. Motivated by this lack, we propose the following lemma, which proves not only that the necessary condition given by Lemma 2.1 of [24] also holds with the assumptions of Theorem 1.0.4 of Rabinowitz, but also a more general result that applies to more general operators.

Lemma B (Generalization of Krasnosel'skii's necessary condition for homogeneous $L(\lambda)$). *Suppose that the operator* $I - L(\lambda_0) : E \to E$ *admits an inverse operator*

$$(I - L(\lambda_0))^{-1} \in H_f(E).$$

Then there exists a ball $B \subset E$ *centered at* 0*, such that*

$$\Phi(\lambda, u) = 0$$

does not admit any nontrivial solution in B *for* λ *lying in a interval* $(\lambda_0 - \xi, \lambda_0 + \xi)$ *where* ξ *is a positive number depending only on* L *and* λ_0*. In particular,* $(\lambda_0, 0)$ *is not a bifurcation point from the curve of trivial solutions of* $\Phi(\lambda, u) = 0$*.*

Lemma B is a generalization of the necessary condition given by Krasnosel'skii in [24]. We point out the following similarity between Lemma B and the necessary condition given in the context of Implicit Function Theorem which was mentioned above. Note that by (1.0.0.7), it follows that $I - L(\lambda_0)$ behaves like a differential $\Phi_u(\lambda_0, 0)$ despite not being necessarily a linear operator. Thus, Lemma B states that if $(\lambda_0, 0)$ is a bifurcation point from the curve of trivial solutions of $\Phi(\lambda, u) = 0$, then the operator $I - L(\lambda_0)$, which behaves like a differential $\Phi_u(\lambda_0, 0)$, does not admits an inverse operator $(I - L(\lambda_0))^{-1} \in H_f(E)$. While the necessary condition given in the context of Implicit Function Theorem states that $\Phi_u(\lambda_0, 0)$ does not admits a continuous inverse.

Besides being a well known fact, we did not find in the literature such a result that proves that Krasnosel'skii's Theorem (Theorem 1.0.3) is a corollary of Rabinowitz Theorem (Theorem 1.0.4) and so we prove this fact.

Corollary B. *Krasnosel'skii's Theorem is a corollary of Rabinowitz Global Bifurcation Alternative.*

The argument of the proof of Corollary B is based on supposing that λ_0 does not satisfies the conclusion of Krasnosel'skii's Theorem and then obtain a contradiction with the connectedness of $\mathcal{C}_{\lambda_0}$.

[1]the result is actually stronger then just a necessary condition.

The idea of the proof of Rabinowitz Theorem (Theorem 1.0.4) is based on the fact that the oddness of the multiplicity implies, by using the Leray-Schauder formula, that the index changes its sign when λ crosses λ_0. This argument of the "index sign change" in the proof of Rabinowitz Global Bifurcation Alternative was explored by Ambrosetti and Hess [1] (1980) and in order to obtain a global bifurcation result for the asymptotic linear elliptic eigenvalue problem

$$\begin{cases} Lu = \lambda f(u) & \text{in } \ , \\ u = 0 & \text{on } \partial \ , \end{cases} \qquad (P\ )$$

where

$$Lu = \quad a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + a_i \frac{\partial u}{\partial x_i} + au$$

is a uniformly elliptic operator with symmetric coefficients $a_{ij} = a_{ji}$ and a 0, $f : \mathbb{R}^+ \to \mathbb{R}$ is a C^1 function satisfying $f(0)$ 0 and asymptotically linear in the sense that there exists a positive number m_∞, a function g and a constant C such that

$$f(s) = m_\infty s + g(s), \ |g(s)| \quad C, \ \forall\, s \in \mathbb{R}^+.$$

By using the compactness of the operator $L^{\ 1}$, they handled the equation in $(P\ )$ to formulate the problem as

$$u = K(\lambda, u),$$

where K is a compact operator. However, to guarantee the index sign change it was not required K to be as in Rabinowitz Theorem (Theorem 1.0.4), instead they used homotopy invariance of the Leray Schauder degree to calculate the index and show that it changes its value (not necessarily the sign) when the parameter crosses the first positive eigenvalue λ_∞ of the problem

$$\begin{cases} Lu = \lambda u & \text{in } \ , \\ u = 0 & \text{on } \partial \ . \end{cases}$$

From this, it was deduced the existence of an unbounded continuum of

$$cl\left(\left\{(\lambda, z) \in [0, +\infty) \quad C_0(\bar{\ }); \frac{z}{\|z\|_0^2} \text{ is a positive solution of } (P\ )\right\}\right).$$

emanating from $(\lambda_\infty, 0)$, where $C_0(\bar{\ })$ stands for the space of all continuous functions $u : \bar{\ } \to \mathbb{R}$ that vanishes at ∂ (see Proposition 3.5 of [1]). In the proof, the authors just mentioned that they were using an adaptation of the proof of Rabinowitz Global Bifurcation Alternative.

Later, David Arcoya, José Carmona and Benedetta Pellacci [4] (2001), by following the ideas from [1], studied bifurcation for the quasilinear elliptic problem

$$\begin{cases} \operatorname{div}(A(x,u)\nabla u) = f(\lambda, x, u) & \text{in } \ , \\ u = 0 & \text{on } \partial \ , \end{cases} \qquad (P\ )$$

where is a bounded open subset of $\mathbb{R}^N$ with smooth boundary and f is a Carathéodory function satisfying, besides some hypotheses about its signal and integrability, the property that f is asymptotically linear at $s = +\infty$, more precisely,

$$\lim_{s \to +\infty} \frac{f(\lambda, x, s)}{s} = \lambda f'_\infty(x), \qquad (f_2)$$

where f_1^0 is a nonzero $L^r(\Omega)$ function with $r \geq N/2$. Also, the matrix A satisfies uniformly elliptic conditions and

$$\exists \lim_{s \to +\infty} A(x,s) = A(x,+\infty), \text{ a.e. in } \Omega.$$

By De Figueiredo's result [16] (1982), the problem

$$\begin{cases} -\mathrm{div}(A(x,+\infty)\nabla u) = \lambda f_1^0(x)u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

admits a first positive eigenvalue λ_1. Under these conditions, the two first statements of Theorem 3.4 of [4] are the following.

Theorem 1.0.5 (First two statements of Theorem 3.4 of [4]). *It emanates from $(\lambda_1, 0)$ a continuum $\mathcal{C}_1$ of*

$$\Sigma_1 := cl\left(\left\{(\lambda, z) \in \mathbb{R} \times C_0(\overline{\Omega});\ \frac{z}{\|z\|^2} \text{ is a positive solution of } (P)\right\}\right)^2$$

where cl( ) denotes the $C_0(\overline{\Omega})$-closure, $\|\cdot\| := \|\nabla(\cdot)\|$ and by positive we mean non trivial and non negative. Moreover, under the additional hypothesis

$$f(0,x,s) \neq 0 \ \forall x \in \Omega, s \geq 0,$$

the continuum $\mathcal{C}_1$ is unbounded.

Our goal in Chapter 4 is to study Theorem 1.0.5 from the point of view of application of Theorem A.

Based on the ideas of [4], we propose the following two alternative results. To state them, let A be a matrix and f a function satisfying, besides the hypotheses required in [4], the following two additional hypotheses.

i) *There exists $s_0 > 0$ and a function $C_0 \in L^r(\Omega)$ such that*

$$f(0,x,s) \leq C_0(x) \text{ a.e. in } \Omega \text{ for every } s \geq s_0. \qquad (f_0)$$

ii) *The function $f_1^0 \in L^r(\Omega)$ is bounded away from zero a.e. in Ω.*

Then it holds the following two theorems.

Theorem C. *It emanates from $(\lambda_1, 0)$ a continuum $\mathcal{C}_1$ of*

$$cl\left(\left\{(\lambda, z) \in [0,+\infty) \times H_0^1(\Omega);\ \frac{z}{\|z\|^2} \text{ is a positive solution of } (P)\right\}\right).$$

Moreover, if

$$f(0,x,s) = 0 \ \forall x \in \Omega, \forall s \geq 0, \qquad (1.0.0.9)$$

then $\mathcal{C}_1$ is unbounded.

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[2] the authors did not specified whether the continuum lies in $\mathbb{R} \times C_0(\overline{\Omega})$ or in $[0,+\infty) \times C_0(\overline{\Omega})$, so in order to not make adulterated reference, we choose the neutral option $\mathbb{R} \times C_0(\overline{\Omega})$. However, in all the graphs presented by them, the continuum lies in the $[0,+\infty) \times C_0(\overline{\Omega})$.

Theorem D. *It emanates from* $(\lambda_1, 0)$ *a continuum* $\mathcal{C}_1$ *of*

$$cl\left(\left\{(\lambda,z)\in[0,+\infty)\times C_0(\overline{\Omega});\ \frac{z}{\|z\|_0^2} \text{ is a positive solution of } (P)\right\}\right).$$

Moreover, if

$$f(0,x,s)=0 \quad \forall x\in\Omega,\ \forall s\geq 0, \tag{1.0.0.10}$$

then $\mathcal{C}_1$ *is unbounded.*

Let us point out the main ideas used to prove these results. By applying the existence result by Leray and Lions [27] (1965) and the uniqueness result by Artola [6], we construct a solution operator $S=Q^{-1}$ of the problem

$$Q(u) = -\mathrm{div}(A(x,u)\nabla u) = h,$$

for each $h\in H^{-1}(\Omega)$, whence the problem (P) can be formulated as

$$u = S(f(\lambda,x,u)).$$

In order to study bifurcation at infinity, the change of variable $u = z/\|z\|_E^2$ (where $E=H_0^1(\Omega)$ in the case of Theorem C and $E=C_0(\overline{\Omega})$ in the case of Theorem D) is applied in the above equation, in ways that the object of study are the solutions of

$$\Phi(\lambda,u)=0,$$

where $\Phi:\mathbb{R}\times E\to E$ is the operator defined by

$$\Phi(\lambda,z)=\begin{cases} z-\|z\|_E^2 S\left(\tau f\left(\lambda,x,\dfrac{z}{\|z\|_E^2}\right)\right), & \text{if } z\neq 0,\\ 0, & \text{if } z=0,\end{cases}$$

The argument to deduce the existence of $\mathcal{C}_1$, is to prove that the index of Φ changes when λ crosses λ_1. In order to prove that the index is 1 for $\lambda<\lambda_1$, we adopt the homotopy H_1 that maps $[0,1]\times\overline{B_{R^{-1}}(0)}$ into E, R is a certain positive number and

$$H_1(\tau,z)=\begin{cases} z-\|z\|_E^2 S\left(\tau f\left(\lambda,x,\dfrac{z}{\|z\|_E^2}\right)\right) & \text{if } z\neq 0,\\ 0 & \text{if } z=0.\end{cases}$$

While to prove that the index is 0 for $\lambda>\lambda_1$, we adopt the homotopy H_2 that maps $[0,1]\times\overline{B_{R_0^{-1}}(0)}$ into E, R_0 is a certain positive number and

$$H_2(t,z)=\begin{cases} z-\|z\|_E^2 S\left(f\left(\lambda,x,\dfrac{z}{\|z\|_E^2}\right)+\dfrac{t\phi}{\|z\|_E^2}\right) & \text{if } z\neq 0,\\ -\psi_t & \text{if } z=0,\end{cases}$$

where $\psi_t:\overline{\Omega}\to\mathbb{R}$ is the unique weak solution of

$$\begin{cases} -\mathrm{div}(A(x,\infty)\nabla u)=t\phi \text{ in } \Omega\\ u=0 \text{ on } \partial\Omega.\end{cases}$$

Observe that the operator $: \mathbb{R}\times E \to E$ defined in (1) is a perturbation of the identity $z - K(\lambda, z)$, where $K : \mathbb{R}\times E \to E$ is the operator defined by

$$K(\lambda, z) = \begin{cases} \|z\|_E^2 S\left(\tau f\left(\lambda, x, \dfrac{z}{\|z\|_E^2}\right)\right), & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases} \tag{1.0.0.11}$$

As in [1], the proof of Theorem 1.0.5 of [4] does not needed the operator K to have the structure required in the hypotheses of Rabinowitz Theorem (Theorem 1.0.4) to ensure the index sign change, instead the authors used the invariance under homotopy property and showed that the index changes value when λ crosses λ_∞. So they mentioned that the conclusion follows from the argument used in Proposition 3.5 of [1]. As we mentioned before, this argument is that the result follows from a combination of the index sign change with an adaptation of the proof of Rabinowitz theorem. From this arises a demand for a new formulation of the bifurcation point theorem involving a more general compact perturbation of the identity than that considered in Theorem 1.0.4 and which can be applied to deduce bifurcation existence results for problems like those studied in [1] and [4]. Motivated by this, we began to study how could we do this. In this sense, consider the following considerations. Since the operator $K : \mathbb{R}\times E \to E$ of Theorem 1.0.4 is compact, then it arise the following questions:

Q_1) is the operator K, defined in (1.0.0.11), compact in $\mathbb{R}\times E$? If the answer is no, one can ask: there exists some subset of $\mathbb{R}\times E$ in which this operator is compact?

Q_2) once answered Q_1, how did we can adapt the Rabinowitz Global Bifurcation Alternative (Theorem 1.0.4) for problems like $u = K(\lambda, u)$, where K is an arbitrary compact operator?

The answer to Q_1 is no. The operator K, defined in (1.0.0.11), is not necessarily compact in $\mathbb{R}\times E$, since to prove the compactness of K at points (λ, z) with $\lambda < 0$, it is necessary to have information about the behaviour of $f(\lambda, x, s)$ for $\lambda < 0$ and $s \to +\infty$, what we do not have in [4]. Although, the hypotheses given in [4], allow us to deduce that this operator is compact in $[0, +\infty)\times E$. This fact motivated us to propose some version of the Rabinowitz theorem that applies to compact operators $K : I\times E \to E$ where I is a closed interval, with non empty interior and E is a real Banach space. While the answer to Q_2 is given by our main result:

Theorem A (A type of Rabinowitz Theorem on a strip). *Let E be a Banach space, I a closed interval (not necessarily bounded) with non empty interior, $K : I\times E \to E$ an abstract operator and $\lambda_0 \in int(I)$ satisfying the following hypothesis:*

1) *K is a compact operator;*

2) *there exists an interval $(a, b) \subset I$ such that*

$$[((a, b)\times\{0\}) \setminus \{(\lambda_0, 0)\}] \cap S = \emptyset; \tag{1.0.0.12}$$

3) *one holds*

$$i(I - K(\lambda_0 - \eta, \cdot), 0) \neq i(I - K(\lambda_0 + \xi, \cdot), 0) \tag{1.0.0.13}$$

for η and ξ positive numbers small enough.

Then there exists a continuum $\mathcal{C}_0$ *of*

$$\mathcal{S} := \overline{\{(\lambda, u) \in I \times (E \setminus \{0\}); u = K(\lambda, u)\}}$$

containing $(\lambda_0, 0)$ *such that* $\mathcal{C}_0$ *satisfies at least one of the following (non-excluding) alternatives:*

i) $\mathcal{C}_0$ *is unbounded,*

ii) $\mathcal{C}_0$ *intercepts some* $(d, u) \in I \times E$ *where* d *is an extremity of the interval* I *(if* I *possesses some extremity) for some* $u \in E$ *or (not exclusive) intercepts some* $(\lambda_1, 0)$ *with* $\lambda_0 \neq \lambda_1 \in I$*.*

The proof of Theorem A is an adaptation of the one of Theorem 1.0.4 and a generalization of Theorem 11 of [3], since the first is formulated for operators K defined in a closed interval I (possibly unbounded), instead of in $\mathbb{R}$ as done in Theorem 11 of [3].

As proposed, we deduce from Theorem A the existence of the continuum $\mathcal{C}_1$ satisfying the statements of Theorem C and D.

Now, we will comment some parallel exposition and results. In Section 3.3, we expose some results on bifurcation theory that behave like variations from the idea of the "index sign change" argument in the proof of Rabinowitz theorem and so we made a pun by saying that these results "bifurcates" from Rabinowitz theorem. These are developed by Dancer, Lopez Gomez, Fleckinger, Dai and Feng, et al. We attribute the rank of parallel exposition for this section motivated by the fact that all these authors, except of Fleckinger, established some bifurcation existence results for problems involving an operator $G : \mathbb{R} \times E \to E$ (not necessarily compact) of the form

$$G(\lambda, u) = L(\lambda)u + H(\lambda, u), \tag{1.0.0.14}$$

where

$$H(\lambda, u) = o(\|u\|) \text{ near } u = 0, \text{ uniformly on each compact interval of } \lambda, \tag{1.0.0.15}$$

and the operator L is well behaved in some sense. And so, it runs away from the main objective of our work which is to obtain (as we did on presenting Theorem A) a generalization of the Global Bifurcation Alternative of Rabinowitz that applies to general compact perturbations of the identity $u - K(\lambda, u)$ defined in a strip.

In order to case the reader's consultation of some conventions and notations that we adopt in the text, we will compile them in this section.

Results nomenclature.

We adopt the following naming pattern for the results (Lemmas, Theorems, Corollaries and Propositions). Results with numeric naming (for example Theorem 1.0.5) are already existent in the literature, with possible changes in their formulations in relation to the original ones. Results with alphabetical naming (for example, Theorem A) fit in the following cases.

i) An apparently unpublished variation of some result of the literature (this is the case of Theorem A, B, C and D).

ii) Well known fact of the literature, although, whose demonstration was not found in widely known textbooks that compile results in bifurcation theorem (this is the case of Corollary B, for example).

iii) A fact that was mentioned, without proof, in some paper (for example, Corollary A is mentioned by Rabinowitz in [33] and Lemma D is mentioned by Dai and Feng in [12]).

iv) Technical result that was not found in widely known textbooks and which is necessary for application in some result of the text (Lemma A_0 and Lemma A, for example).

General notations

i) For each measurable set $A \subset \mathbb{R}^N$, the notation $m(A)$ stands for the Lebesgue's measure of A.

ii) Unless otherwise specified (as in the case of Section 4.3 and 4.4), E is an arbitrary real Banach space.

iii) Given $p \geq 1$, we denote by $\|\cdot\|_p$ the usual norm of the space $L^p(\Omega)$.

Chapter 2.
Section 2.1.

i) U is a bounded open subset of $\mathbb{R}^N$.

ii) For each $v \in \mathbb{R}^N$ and $A \subset \mathbb{R}^N$, we denote $\rho(v, A) := \text{dist}(v, A)$.

Section 2.2.

i) U is a bounded open subset of E.

ii) For each $v \in E$ and $A \subset E$, we denote $\rho(v, A) := \text{dist}(v, A)$.

iii) $C(\overline{U}; E)$ is the space of all continuous operators $T : \overline{U} \to E$.

iv) $Q(\overline{U}; E)$ is the family of all compact operators $K : \overline{U} \to E$. That is, $K \in C(\overline{U}; E)$ and maps subsets of $\overline{U}$ to relatively compact sets of E.

v) For each $K \in Q(\overline{U}; E)$ we will consider the norm

$$\|K\|_\infty := \sup_{u \in \overline{U}} \|K(u)\|.$$

Chapter 3.
Except for Section 3.3 and 3.4.

i) We assume the hypotheses H1), H2), H3) and HI).

ii) We consider the notation given in N).

iii) We study the problem defined in P).

Chapter 4.

i) Ω is a bounded open subset of $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$.

ii) Since Ω is bounded, Poincaré's inequality ensures that we can adopt, in $H_0^1(\Omega)$ the norm $\|u\| := \|\nabla u\|_2$, $u \in H_0^1(\Omega)$.

iii) $C_0(\overline{\Omega})$ is the space of the continuous functions $u : \overline{\Omega} \to \mathbb{R}$ that vanishes at $\partial\Omega$. In this space, we will adopt the usual norm $\|u\|_0 := \sup_{\Omega} |u|$.

iv)) $H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$.

Section 4.3

i) $E = H_0^1(\Omega)$.

Section 4.4

i) $E = C_0(\overline{\Omega})$.

Chapter 2

Degree Theory

Let us consider the problem of solving

$$f(u) = v, \tag{2.0.0.1}$$

where $f : \overline{\Omega} \to E$ is a continuous function, Ω is an open bounded subset of a Banach space E and $v \in E$ is given. When $E = \mathbb{R}^N$, by using some results of Complex Analysis about the index of a plane curve[1], it is possible to construct a topological invariant associated to the function $f : \overline{\Omega} \to \mathbb{R}^N$, the point v and the domain Ω, which is an integer number called the Brouwer Topological Degree of f in the point v, relative to Ω and denoted by $deg(f, \Omega, v)$. This invariant can give some information about existence and uniqueness of solutions of (2.0.0.1).

For a more general Banach space E, that is, when E is not necessarily a finite dimensional space, we have an extension of the Brouwer degree which is called the Leray-Schauder degree.

Our approach is based mainly on the book of Kesavan [23], but with more thorough proofs. Morever, based on the proof of Theorem 4.1 of [2], we prove a certain homotopic invariance at the section of properties of the Leray-Schauder degree.

Both Brouwer and Leray-Schauder degree satisfy a common list of properties that carry the power of the degree theory. We highlight the homotopy invariance property, that is a powerful tool to solve equations. Also, in Leray-Schauder Degree section, we enunciate a very important theorem, sometimes called the Leray-Schauder Formula, which is a key result in applying degree theory to obtain bifurcation results.

In each of the two sections of the chapter, we illustrate the respective concept through an example of application in its conclusion.

2.1 The Brouwer degree

The Brower topological degree will be defined, at first, for $C^1(\overline{U})$-functions $f : \overline{U} \to \mathbb{R}^N$ and regular values $v \in \mathbb{R}^N \setminus f(\partial U)$ of f. This concept is extended for v not necessarily regular values and for continuous functions by using the Sard's Lemma and the density of the space $C^2(\overline{U})$ in $C(\overline{U})$, respectively.

After a list of properties satisfied by the degree is presented and proved. We define the limit version of the Brouwer degree, called "index of an isolated solution" and finally, by using this concept and the homotopy invariance property we present an application

[1] see Cho, Qing and Chen 2006 [10].

where we show an existence result for a non linear system of equations.

Before presenting the mathematical definition of this object, let us recall some basic notions about the Jacobian of a differentiable function.

Let $f : \overline{U} \subset \mathbb{R}^N \to \mathbb{R}^N$ and $v \in \mathbb{R}^N$. If $f \in C^1(\overline{U}, \mathbb{R}^N)$, then we define the Jacobian of f at $u \in U$ by $J_f(u) = det(f'(u))$, where

$$f'(u) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial u_1} & \cdots & \frac{\partial f_N}{\partial u_N} \end{bmatrix}.$$

We say that $v \in f(\overline{U})$ is a regular value of f, if $J_f(u) \neq 0$ for all $u \in f^{-1}(v) \cap U$. If $J_f(u) = 0$, we call u a critical point of f. We denote the set of critical points of f in U by S_f (or S).

Lemma 2.1.1. *Let $f \in C^1(\overline{U}, \mathbb{R}^N)$ and $v \notin f(S) \cup f(\partial U)$. Then, $f^{-1}(v)$ is a finite set.*

Proof. For each $u \in f^{-1}(v)$ we have $J_f(u) \neq 0$ and so $f'(u) : \mathbb{R}^N \to \mathbb{R}^N$ is a isomorphism. By Inverse Function Theorem, f is invertible in a neighborhood V_u of u. Now, since $f^{-1}(v)$ is a closed subset of the compact $\overline{U}$ (since it is the inverse image of the closed set $\{v\}$), it follows that $f^{-1}(v)$ is compact.

Suppose by contradiction that $f^{-1}(v)$ contains an amount of infinite points

$$\{p_1, p_2, \ldots, p_n, \ldots\} \subset f^{-1}(v).$$

By Bolzano Weierstrass's Theorem we have $p_n \to p_0$ up to a subsequence for some $p_0 \in \overline{U}$. Now, the continuity of f implies $f(p_0) = v$. Since $v \notin f(\partial U)$, it follows that $p_0 \in U$ and since v is a regular value of f, we get $J_f(p_0) \neq 0$. Then, the Inverse Function Theorem guarantees the existence of a neighborhood V_{p_0} of p_0 where f is invertible, but since $p_m \to p_0$, it follows that there exists $p_k \in V_{p_0} \setminus \{p_0\}$. But $p_k \in f^{-1}(v)$ and this contradicts the injectivity of f in V_{p_0}. Then, $f^{-1}(v)$ is a finite set. □

Definition 2.1.1. *Let $f \in C^1(\overline{U}, \mathbb{R}^N)$ and $v \notin f(S) \cup f(\partial U)$. We define de Brouwer degree of f, in U, relative to point v by*

$$deg(f, U, v) = \begin{cases} 0, & \text{if } f^{-1}(v) = \emptyset, \\ \displaystyle\sum_{u \in f^{-1}(v)} sgn(J_f(u)), & \text{if } f^{-1}(v) \neq \emptyset. \end{cases}$$

Note that the degree is well defined, since $f^{-1}(v)$ is finite, $f \in C^1(\overline{U}; \mathbb{R}^N)$ and $J_f(u) \neq 0$ for all $u \in f^{-1}(v)$.

Lemma 2.1.2. *Under the the conditions of the Definition 2.1.1, we have*

$$deg(f, U, v) = deg(f - v, U, 0).$$

Proof. It is sufficient to note that $J_f = J_{f-v}$ and $f^{-1}(v) = (f - v)^{-1}(0)$. □

Observe that the above definition requires regularity $C^1(\overline{U})$ of the function and also that $v \notin f(S)$. We want to extend this definition to a broader sense. Consider the following lemma.

Lemma 2.1.3. *Let v be as in Definition 2.1.1. Then, there exists $\varepsilon_0 > 0$ such that*

$$deg(f,U,v) = \int_U \varphi_\varepsilon(f(u)-v)J_f(u)du, \text{ for all } 0<\varepsilon<\varepsilon_0, \tag{2.1.0.1}$$

where $\varphi_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ is such that $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^N)$, $supp\varphi_\varepsilon \subset B_\varepsilon(0)$ and

$$\int_{\mathbb{R}^N} \varphi_\varepsilon(u)du = 1.$$

Proof. Let us suppose initially that $f^{-1}(v) = \emptyset$. In this case, take $\varepsilon_0 = \rho(v, f(\overline{U}))/2$. By our assumption, it follows that $\varepsilon_0 > 0$. Now, by the definition of ε_0, we have that $\|f(u)-v\| \geq \varepsilon_0$ for all $u \in \overline{U}$ and combining this fact with $supp\varphi_\varepsilon \subset B_\varepsilon(0)$, we have $\varphi_\varepsilon(f(u)-v) = 0$ in U for each $0<\varepsilon<\varepsilon_0$ and so the right hand side of (2.1.0.1) is zero. Then, (2.1.0.1) holds.

Now, assume $f^{-1}(v) \neq \emptyset$. Then, it follows from Lemma 2.1.1 that $f^{-1}(v) = \{p_1,\ldots,p_m\}$. Moreover, for each $1 \leq i \leq m$, we have $J_f(p_i) \neq 0$, thereby the Inverse Function Theorem ensures the existence of a neighborhood V_i of p_i and U_i of v such that $V_i \cap V_j = \emptyset$ for all $i \neq j$ and $f|_{V_i}: V_i \to U_i$ is an homeomorphism. Moreover, since $f \in C^1$ we obtain $sgnJ_f(u) = sgnJ_f(p_i)$, for all $u \in V_i$. Let $\varepsilon_0 > 0$ be such that $B_{\varepsilon_0}(v) \subset \cap_{i=1}^m U_i$ and define $W_i = f^{-1}(B_{\varepsilon_0}(v)) \cap V_i$. We claim that for each $0<\varepsilon<\varepsilon_0$, it holds

$$\|f(u)-v\| \geq \varepsilon, \text{ in } \overline{U}\setminus W_i, \ \forall\, i = 1,2,\ldots,m.$$

In fact, on the contrary, there would exist $0<\varepsilon<\varepsilon_0$, $k \in \{1,2,\ldots,m\}$ and $u \in \overline{U}\setminus W_k$ such that

$$\|f(u)-v\| < \varepsilon.$$

So $f(u) \in B_{\varepsilon_0}(v) \subset \cap_{i=1}^m U_i$. Since $f|_{V_i}$ is a bijective function, it follows that $u \in V_i$ for all $i \in \{1,2,\ldots\}$, which contradicts $V_i \cap V_j = \emptyset$ for all $i \neq j$. Thus, the claim is proved and by the fact that supp$\varphi_\varepsilon \subset B_\varepsilon(0)$, it follows that for each $0<\varepsilon<\varepsilon_0$ one has $\varphi_\varepsilon(f(u)-v) = 0$ in $\overline{U}\setminus W_i$, for all $i \in \{1,2,\ldots,m\}$. Consequently

$$\begin{aligned}
\int_U \varphi_\varepsilon(f(u)-v)J_f(u)du &= \sum_{i=1}^m \int_{W_i} \varphi_\varepsilon(f(u)-v)|J_f(u)|sgn(J_f(u))du \\
&= \sum_{i=1}^m sgn(J_f(p_i)) \int_{W_i} \varphi_\varepsilon(f(u)-v)|J_f(u)|du \\
&\overset{(*)}{=} \sum_{u\in f^{-1}(v)} sgn(J_f(u)) \int_{B_\varepsilon(0)} \varphi_\varepsilon(w)dw \\
&= deg(f,U,v).
\end{aligned}$$

To obtain $(*)$, we use the change of variable $f(u)-v = w$ and the following facts

- $J_f = J_{f-v}$ and
- $f(W_i) - v = B_\varepsilon(0)$

□

Based on this result, we get the next result.

Lemma 2.1.4. *Let $f \in C^2(\overline{U},\mathbb{R}^N)$, $v \notin f(\partial U)$, $\rho_0 = \rho(v, f(\partial U)) > 0$ and $v^1, v^2 \in B_{\rho_0}(v)$. Assume $v^i \notin f(S) \cup f(\partial U), i = 1,2$, then*

$$d(f,U,v^1) = d(f,U,v^2).$$

Proof. Let us suppose $v \neq v^i$, $i \in \{1,2\}$, and take $0 < \delta < \rho_0 - \max\{|v - v^1|, |v - v^2|\}$ (note that $\rho_0 - \max\{|v - v^1|, |v - v^2|\} > 0$, because $v^1, v^2 \in B_{\rho_0}(v)$). Like this, we have

$$\delta < \rho_0 - |v - v^i|, \quad \forall\, i \in \{1,2\}.$$

Since $v^i \notin f(S) \cup f(\partial U)$, it follows from Lemma (2.1.3) that there exists $\varepsilon < \delta$ such that

$$d(f,U,v^i) = \int_U \varphi_\varepsilon(f(u) - v^i) J_f(u)\,du, \quad i = 1,2.$$

Let

$$w(u) = (v^1 - v^2)\int_0^1 \varphi_\varepsilon((u - v^1) + t(v^1 - v^2))\,dt$$

and note that

$$\begin{aligned}
\frac{\partial w_i}{\partial u_i} &= \int_0^1 \langle (v^1 - v^2), e_i\rangle \frac{\partial}{\partial u_i}\varphi_\varepsilon((u - v^1) + t(v^1 - v^2))\,dt \\
&= \int_0^1 \langle \nabla(\varphi_\varepsilon((u - v^1) + t(v^1 - v^2))), e_i\rangle\langle v^1 - v^2, e_i\rangle\,dt
\end{aligned}$$

whence

$$\begin{aligned}
\operatorname{div}(w(u)) &= \sum_{i=1}^{N}\int_0^1 \langle \nabla(\varphi_\varepsilon((u - v^1) + t(v^1 - v^2))), e_i\rangle\langle v^1 - v^2, e_i\rangle\,dt \\
&= \int_0^1 \langle \nabla(\varphi_\varepsilon((u - v^1) + t(v^1 - v^2))), (v^1 - v^2)\rangle\,dt \\
&= \int_0^1 \frac{d}{dt}(\varphi_\varepsilon(u - v^1) + t(v^1 - v^2))\,dt \\
&= \varphi_\varepsilon(u - v^2) - \varphi_\varepsilon(u - v^1)
\end{aligned}$$

Now, if $u \in f(\partial U)$, then

$$\begin{aligned}
|u - (1-t)v^1 - tv^2| &= |(u - v) + (1-t)(v - v^1) + t(v - v^2)| \\
&\geq |u - v| - (1-t)|v - v^1| - t|v - v^2| \\
&\geq \rho_0 - (1-t)(\rho_0 - \delta) - t(\rho_0 - \delta) \\
&= \delta > \varepsilon,
\end{aligned}$$

that is, $\varphi_\varepsilon(u - v^1 + t(v^1 - v^2)) = 0$ for all $t \in [0,1]$. Thus, $\operatorname{supp} w \cap f(\partial U) = \emptyset$. Let us define

$$v_i(u) = \begin{cases} \displaystyle\sum_{j=1}^{N} w_j(f(u))A_{ij}(u) & u \in \overline{U}, \\ 0 & \mathbb{R}^N \setminus \overline{U}, \end{cases}$$

where

$$A_{ij}(u) := ij - \text{cofactor of } J_f(u).$$

So $v_j(u) = 0$, for all $u \in \partial U$, and by the product rule we obtain

$$\frac{\partial v_i}{\partial u_i}(u) = \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial w_j}{\partial u_k}(f(u))\frac{\partial f_k}{\partial u_i}A_{ij}(u) + \sum_{j=1}^{N} w_j(f(u))\frac{\partial}{\partial u_i}A_{ij}(u), \ \forall\, u \in U,$$

which implies

$$\text{div}(v(u)) = \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial w_j}{\partial u_k}(f(u))\sum_{i=1}^{N}\frac{\partial f_k}{\partial u_i}A_{ij}(u) + \sum_{j=1}^{N} w_j(f(u))\sum_{i=1}^{N}\frac{\partial}{\partial u_i}A_{ij}(u), \ \forall\, u \in U.$$

Now, we need the following auxiliar claim, whose proof can be found in Kesavan [23], page 39.

Claim 2.1.1. *Let $f \in C^2(\overline{U}, \mathbb{R}^N)$ and $A_{ij}(u)$ as defined above. Then,*

$$\sum_{i=1}^{N}\frac{\partial}{\partial u_i}A_{ij}(u) = 0, \ \textit{for } 1 \leq j \leq N.$$

It follows from this claim, the cofactor definition and the Laplace formula for the calculation of the determinant that

$$\sum_{i=1}^{N}\frac{\partial}{\partial u_i}A_{ij}(u) = 0 \quad \text{and} \quad \sum_{i=1}^{N}\frac{\partial f_k}{\partial u_i}(u)A_{ij}(u) = \delta_{jk}J_f(u).$$

So,

$$\text{div}(v(u)) = \sum_{k=1}^{N}\frac{\partial w_j}{\partial u_k}(f(u))J_f(u) = \text{div}(w(f(u)))J_f(u).$$

Therefore, by using that $v = 0$ on ∂U and Divergence Theorem, we get

$$\begin{aligned} d(f,U,v^1) - d(f,U,v^2) &= \int_U (\varphi_\varepsilon(f(u) - v^1) - \varphi_\varepsilon(f(u) - v^2))J_f(u)du \\ &= \int_U \text{div}(w(f(u)))J_f(u)du \\ &= \int_U \text{div}(v(u))du = \int_{\partial U} v(u)\eta(u)dS_u \\ &= 0, \end{aligned}$$

where η stands for the outward unit normal vector around ∂U. So the lemma is proved. □

Finally, the last needed result to guarantee that we can define the degree at points that are not necessarily regular is the following.

Lemma 2.1.5 (Sard's Lemma). *If $f \in C^1(U)$, then $f(S)$ has null Lebesgue measure.*

Proof. See Theorem 1.3.4 of [23]. □

Definition 2.1.2. *Let $f \in C^2(\overline{U}, \mathbb{R}^N)$, $v \notin f(\partial U)$ and $\rho_0 = \rho(v, f(\partial U))$. The degree of f in U relative to point v is defined by*

$$d(f,U,v) = d(f,U,v'),$$

where v' is a regular point of f in $B_{\rho_0}(v)$.

Observe that, by Sard's Lemma, there exists such $v' \notin f(S)$. Moreover, Lemma 2.1.4 ensures that the definition above does not depend on the choice of v'. Note that Definition 2.1.2 gives us a concept of degree where the condition $v \notin f(S)$ is no longer needed.

Another important feature of the degree is its invariance in small neighbourhoods of regular functions. Before showing it, consider the following lemma.

Lemma 2.1.6. *Let $f \in C^2(\overline{U}, \mathbb{R}^N)$ and $v \notin f(\partial U)$. Then, for $g \in C^2(\overline{U}, \mathbb{R}^N)$, there exists $\varepsilon = \varepsilon(f, g, v) > 0$ such that,*

$$\mathit{deg}(f + tg, U, v) = \mathit{deg}(f, U, v), \ \text{for any } 0 < |t| < \varepsilon.$$

Proof. Let us divide the proof in three cases:

First Case: $f^{-1}(v) = \emptyset$. By definition, we have $deg(f, U, v) = 0$ and $\rho_0 = \rho(v, f(\overline{U})) > 0$. Then,

$$|v - (f(u) + tg(u))| \geq |f(u) - v| - |t||g(u)| \geq \rho_0 - |t|M, \ \ \forall\, u \in \overline{U},$$

where $M = \max_{\overline{U}} |g(u)|$. Taking $\varepsilon = \rho_0/(2M)$, we have that $v \notin (f + tg)(\overline{U})$ for $|t| < \varepsilon$ and so

$$deg(f + tg, U, v) = 0 = deg(f, U, v).$$

Second Case: $v \notin f(S)$ and $f^{-1}(v) \neq \emptyset$. By Lemma 2.1.1, we have

$$f^{-1}(v) = \{p_1, \ldots, p_m\}, \ \text{where } J_f(p_i) \neq 0, \ 1 \leq i \leq m.$$

Let $f_t = f + tg$ and $h(t, u) = f_t(u) - v$, hence

- $h(0, p_i) = 0, \ \ 1 \leq i \leq m,$
- $J_{h(0,\cdot)}(p_i) = J_f(p_i) \neq 0.$

Thus, the Implicit Function Theorem ensures the existence of a neighborhood $(-\delta_i, \delta_i)$ of 0 and U_i of p_i in U such that the graph of the functions $\varphi_i : (-\delta_i, \delta_i) \to U_i$ are formed just by the solutions of $h(t, u) = 0$ in $(-\delta_i, \delta_i) \times U_i$, with $\varphi_i(0) = p_i$. Combining this fact with the C^2 regularity of φ, guaranteed by the implicit function theorem, we can find $\delta > 0$ small enough such that $\varphi_i : (-\delta, \delta) \to U_i$ satisfies $\cap_{i=1}^m U_i = \emptyset$ and $\operatorname{sgn} J_f(u) = \operatorname{sgn} J_f(p_i)$ in U_i, by reducing δ if it is necessary.

Denoting $\mathcal{U} = \cup_{i=1}^m U_i$, we have $f_t^{-1}(v) \cap \mathcal{U} = \{\varphi_1(t), \ldots, \varphi_m(t)\}$, for $t \in (-\delta, \delta)$. By the uniqueness of $(t, \varphi_i(t))$ as solution of $h(t, u) = 0$ in $(-\delta, \delta) \times U_i$ and the compactness of $\overline{U} \setminus \mathcal{U}$, we conclude that $\rho_0 := \rho(v, f(\overline{U} \setminus \mathcal{U})) > 0$. Now, if $|t| \leq \rho_0/(2M)$, we have

$$\begin{aligned} |v - f_t(u)| &= |v - (f(u) + tg(u))| \\ &\geq |f(u) - v| - |t||g(u)| \\ &\geq \rho_0 - |t|M \\ &\geq \frac{\rho_0}{2}, \ \forall\, u \in \overline{U} \setminus \mathcal{U}. \end{aligned}$$

Moreover, if $|t| < \delta$, then $\varphi_i(t) \in \mathcal{U}$ and so

$$f_t^{-1}(v) = \{\varphi_1(t), \ldots, \varphi_m(t)\}, \ |t| < \min\left\{\delta, \frac{\rho_0}{2M}\right\}.$$

Let $i \in \{1, 2, \ldots, m\}$. By the continuity of the functions $u \mapsto J_f(u)$ and $t \mapsto \varphi_i(t)$, we have

$$\mathrm{sgn} J_f(p_i) = \mathrm{sgn} J_f(\varphi_i(t)) \quad \text{for small } t. \tag{2.1.0.2}$$

Furthermore, since $g \in C^2(\overline{U})$ and $\overline{U}$ is a compact set, we have for each $k \in \{1, 2, \ldots, N\}$,

$$\begin{aligned} \left| \frac{\partial f_t}{\partial u_k}(\varphi_i(t)) - \frac{\partial f}{\partial u_k}(\varphi_i(t)) \right| &= \left| \frac{\partial (f + tg)}{\partial u_k}(\varphi_i(t)) - \frac{\partial f}{\partial u_k}(u) \right| \\ &= |t| \left| \frac{\partial g}{\partial u_k}(\varphi_i(t)) \right| \to 0 \text{ as } |t| \to 0. \end{aligned}$$

This implies

$$\mathrm{sgn} J_f(\varphi_i(t)) = \mathrm{sgn} J_{f_t}(\varphi_i(t)), \quad \text{for small } t. \tag{2.1.0.3}$$

Finally, by combining (2.1.0.2) and (2.1.0.3), we get

$$\mathrm{sgn} J_f(u_i) = \mathrm{sgn} J_{f_t}(\varphi_i(t)), \quad \text{for small } t$$

and consequently $\deg(f + tg, U, v) = \deg(f, U, v)$, for small t.

Third Case: $v \in f(S)$.

Let $\rho_0 = \rho(v, f(\partial U))$. By Sard's lemma, we can find $v_0 \in B_{\frac{\rho_0}{3}}(v) \setminus f(S)$, so proceding as in Second Case we obtain

$$\deg(f + tg, U, v_0) = \deg(f, U, v_0) = \deg(f, U, v), \ \forall\, |t| < \varepsilon_0,$$

where the last equality comes from Definition 2.1.2. For $u \in \partial U$, we have

$$|f(u) - v_0| \geq |f(u) - v| - |v - v_0| \geq \rho_0 - \frac{\rho_0}{3} = \frac{2\rho_0}{3}.$$

So, if $u \in \partial U$ and $|t| < \rho_0/(3M)$, then

$$|f(u) + tg(u) - v_0| \geq |f(u) - v_0| - |t||g(u)| > \frac{2\rho_0}{3} - \frac{\rho_0}{3} > \frac{\rho_0}{3},$$

that is, $v_0 \notin (f + tg)(\partial U)$. Thus, by Definition 2.1.2,

$$\deg(f + tg, U, v) = \deg(f + tg, U, v_0), \quad |t| < \frac{\rho_0}{3M}$$

By taking $\varepsilon = \min\left\{\frac{\rho_0}{3M}, \varepsilon_0\right\}$, we conclude that

$$\deg(f + tg, U, v) = \deg(f, U, v), \quad \text{for } |t| < \varepsilon.$$

This ends the proof. $\square$

Corollary 2.1.1. *Let $f \in C(\overline{U}, \mathbb{R}^N)$, $v \notin f(\partial U)$, $\rho_0 = \rho(v, f(\overline{U}))$ and $g \in C^2(\overline{U}, \mathbb{R}^N)$ such that $\|g - f\|_1 < \frac{\rho_0}{2}$. Then, the function $\varphi : [0, 1] \to \mathbb{Z}$ defined by*

$$\varphi(t) = \deg(f + tg, U, v)$$

is locally constant.

Proof. It is sufficient to prove that for each fixed $t_0 \in [0,1]$, we have

$$\deg(f + (t_0 - t)g, U, v) = \deg(f + t_0 g, U, v),$$

for $|t| < \varepsilon$, with $\varepsilon > 0$ small enough.

Observe that $f + (t_0 - t)g = [f + t_0 g] - tg$ and if $v \notin f(\partial U)$, then

$$\begin{aligned}
|f(u) + t_0 g(u) - v| &= |f(u) + t_0 f(u) - t_0 f(u) + t_0 g(u) - v| \\
&= |(1 + t_0) f(u) - v + t_0 (g(u) - f(u))| \\
&\geq (1 + t_0)|f(u) - v| - t_0 |g(u) - f(u)| \\
&\geq (1 + t_0)\rho_0 - t_0 |g(u) - f(u)| \\
&\geq (1 + t_0)\rho_0 - t_0 \frac{\rho_0}{2} \\
&\geq \rho_0 - \left(t_0 - \frac{t_0}{2}\right)\rho_0 \\
&= \rho_0 - \frac{1}{2} t_0 \rho_0 > 0,
\end{aligned}$$

whence $v \notin (f + t_0 g)(\partial U)$ and so we can apply Lemma 2.1.6 to $\tilde{f} - tg$, where $\tilde{f} := f + t_0 g$. □

Corollary 2.1.2. *The Brouwer degree is constant for functions C^2 sufficiently close to a continuous function with respect to the norm of the supremum $\|\cdot\|_\infty$.*

Proof. Let $f \in C(\overline{U}, \mathbb{R}^N)$, $v \notin f(\partial U)$ and $\rho_0 = \rho(v, f(\overline{U}))$ and $g \in C^2(\overline{U}, \mathbb{R}^N)$ such that $\|g - f\|_\infty < \frac{\rho_0}{2}$ [2] So, $v \notin g(\partial U)$, $i = 1, 2$. Furthermore, by defining $\tilde{g} = g - f$, we have for $u \in \partial U$ and $t \in [0,1]$

$$\begin{aligned}
|v - (f(u) + t\tilde{g}(u))| &= |(v - f(u)) - t(g(u) - f(u))| \\
&\geq |v - f(u)| - |t||g(u) - f(u)| \\
&\geq |v - f(u)| - |g(u) - f(u)| \\
&\geq \rho_0 - \rho_0/2 > 0,
\end{aligned}$$

that is, the function $h(t,u) = f(u) + t\tilde{g}(u)$ is such that $v \notin h(t, \partial U)$, for any $t \in [0,1]$ and so $\varphi(t) = \deg(h(t, \cdot), U, v)$ is well defined. Thus, by Lemma 2.1.1 we know that $\varphi(t)$ is locally constant and so continuous. Hence, since $[0,1]$ is a connected set, and φ is continuous, we conclude that $\varphi([0,1])$ is a connected set. As $\varphi([0,1]) \subset \mathbb{Z}$, the connectedness of $\varphi([0,1])$ implies that it is a singleton, that is, $\varphi(t)$ is constant in $[0,1]$. Therefore,

$$\deg(f, U, v) = \varphi(0) = \varphi(1) = \deg(g, U, v).$$

□

Motivated by Corollary 2.1.2, we can extend the concept of degree to continuous functions as follows:

Definition 2.1.3. *Let $f \in C(\overline{U}, \mathbb{R}^N)$, $v \notin f(\partial U)$ and $\rho_0 = \rho(v, f(\partial U))$. We define the Brouwer degree of f in U with respect to v by*

$$\deg(f, U, v) = \deg(g, U, v),$$

for any $g \in C^2()$ satisfying $\|f - g\|_\infty < \frac{\rho_0}{2}$.

[2] see Deimling [17], page 6 for a proof of existence of this function.

2.1.1 Properties of the Brouwer degree

The Brouwer degree satisfies some properties that makes it very useful, specially in the bifurcation theory. One of them deserves to be highlighted: the invariance under homotopy. This property acts as the principal argument in many proofs of the existence of bifurcation points. In the following proposition, we list and prove the main properties of the degree.

Proposition 2.1.1 (Properties of the Brouwer degree). *Let $f \in C(\overline{U};\mathbb{R}^N)$, $v \notin f(\partial U)$, $H \in C(\overline{U} \times [0,1],\mathbb{R}^N)$ be such that $v \notin H(\partial\overline{U} \times [0,1])$ and $Z \subset \overline{U}$ a compact set such that $v \notin f(Z) \cup f(\partial U)$. Then the following properties of the Brouwer degree hold.*

- *P1) (Normalization): $\deg(I,U,v) = \begin{cases} 1 & \text{if } v \in U, \\ 0 & \text{if } v \notin U. \end{cases}$*
- *P2) (Continuity in f): There exists a neighborhood $\mathcal{U}$ of f in the topology $(C(\overline{U},\mathbb{R}^N),\|\cdot\|_\infty)$, such that*
$$\deg(f,U,v) = \deg(g,U,v),\ \forall\, g \in \mathcal{U}.$$
- *P3) (Invariance under Homotopy): The number $\deg(H(\cdot,t),U,v))$ is independent of $t \in [0,1]$.*
- *P4) (Constant over connected components of $\mathbb{R}^N \setminus f(\partial U)$): $\deg(f,U,\cdot)$ is constant in each connected component of $\mathbb{R}^N \setminus f(\partial U)$.*
- *P5) (Additivity): If U_1 and U_2 are bounded open subsets of U such that $U_1 \cap U_2 = \emptyset$, $U = U_{12}$ and $v \notin f(\partial U_1) \cup f(\partial U_2)$, then*
$$\deg(f,U,v) = \deg(f,U_1,v) + \deg(f,U_2,v).$$
- *P6) (Existence of Solution): Assume $\deg(f,U,v) \neq 0$. Then, there exists $u \in U$ such that $f(u) = v$.*
- *P7) (Excision): The following equality holds.*
$$\deg(f,U,v) = \deg(g,U \setminus Z,v).$$
- *P8) (Boundary Dependence): $\deg(g,U,v) = \deg(f,U,v)$, whenever $g|_{\partial U} = f|_{\partial U}$ with $g \in C(\overline{U},\mathbb{R}^N)$.*

Proof. P1): It follows from the facts that $I^{-1}(v) = \{v\}$ and $\operatorname{sgn}(J_I(v)) = \operatorname{sgn}(1) = 1$.

P2): Define $\mathcal{U} = \{g \in C(\overline{U},\mathbb{R}^N);\ \|f - g\|_\infty < \rho_0/4\}$, where $\rho_0 = \rho(v,f(\partial U))$. Thus, $v \notin g(\partial U)$ and the degree $\deg(g,U,v)$ is well defined. Let $h \in C^2(\overline{U},\mathbb{R}^N)$ be such that $\|f - h\|_\infty < \frac{\rho_0}{8}$. So,

$$\|g-h\|_\infty \leq \|g-f\|_\infty + \|f-h\|_\infty < \frac{\rho_0}{4} + \frac{\rho_0}{8} = \frac{3\rho_0}{8} = \frac{3\rho_0}{4}\frac{1}{2} < \frac{1}{2}\rho(v,f(\partial U))$$

Finally, by Definition 2.1.3,

$$\deg(g,U,v) = \operatorname{def}(h,U,v) = \deg(f,U,v).$$

P3): Let $t_0 \in [0,1]$ and $\mathcal{U}$ be the $(C(\overline{U},\mathbb{R}^N),\|\cdot\|_\infty)$-neighborhood of $H(\cdot,t_0)$ given in the proof of P2). Since $H \in C(\overline{U}\times[0,1],\mathbb{R}^N)$, we obtain that $H(\cdot,t) \in \mathcal{U}$ for t sufficiently close to t_0, so $\deg(H(\cdot,t),U,v)$ is constant for t sufficiently close to t_0. In other words, $\deg(H(\cdot,t),U,v)$ is locally constant and so continuous in $[0,1]$. By arguing as done in Corollary 2.1.2, we get the result.

P4): Since $\mathbb{R}^N$ is locally connected[3], it follows from Theorem 5.1.2, in Topology Section in Appendix, that any connected component of the open set $\mathbb{R}^N \setminus f(\partial U)$ is an open subset of $\mathbb{R}^N$, so by Theorem 5.1.1 it is path-connected. Thus, given a connected component C of $\mathbb{R}^N \setminus f(\partial U)$ and $v^1, v^2 \in C$, there exists a curve $\gamma : [0,1] \to C$ with $\gamma(0) = v^1$ and $\gamma(1) = v^2$. Now, by defining the homotopy $H(u,t) = f(u) - \gamma(t)$ and using the fact $\deg(f,U,v) = \deg(f-v,U,0)$, the result follows from P3).

P5): Let $\rho_0 = \rho(v, f(\partial U))$ and $g \in C^2(\overline{U},\mathbb{R}^N)$ such that $\|f-g\|_\infty < \rho_0/2$, so it follows from P2) that

$$\deg(f,U,v) = \deg(g,U,v) \text{ and } \deg(f,U_i,v) = \deg(g,U_i,v).$$

By the definition of ρ_0 and the fact $U_i \subset U$, for $i = 1,2$, we have that $B = B_{\rho_0/2}(v) \subset (\mathbb{R}^N \setminus g(\partial U)) \cap (\mathbb{R}^N \setminus g(\partial U_i))$, for $i = 1,2$. Moreover, B is a connected set, consequently it is contained in one of each connected component of the sets $\mathbb{R}^N \setminus g(\partial U), \mathbb{R}^N \setminus g(\partial U_1)$ and $\mathbb{R}^N \setminus g(\partial U_2)$. By using Sard's Lemma, we can take $w \in B$ a regular point near v and use P4) to conclude that

$$\deg(g,U,v) = \deg(g,U,w) \text{ and } \deg(g,U_i,v) = \deg(g,U_i,w), \; i = 1,2.$$

Therefore,

$$\begin{aligned}
\deg(f,U,v) &= \deg(g,U,v)\\
&= \deg(g,U,w)\\
&= \sum_{p_i \in f^{-1}(w)\cap U_1} \operatorname{sgn}(J_f(p_i)) + \sum_{z_i \in f^{-1}(w)\cap U_2} \operatorname{sgn}(J_f(z_i))\\
&= \deg(g,U_1,w) + \deg(g,U_2,w)\\
&= \deg(f,U_1,v) + \deg(f,U_2,v).
\end{aligned}$$

This ends the proof.

P6): We will prove the contrapositive by contradiction. Suppose that there is no $u \in U$ such that $f(u) = v$. Let $\rho_0 = \rho(v, f(\overline{U}))$ and take $g \in C^2(\overline{U},\mathbb{R}^N)$, with $\|f-g\|_\infty < \rho_0/2$, such that

$$\deg(f,U,v) = \deg(g,U,v).$$

By the definition of ρ_0, it follows that $v \notin g(\overline{U})$ and, in particular, v is a regular value of g. Suppose by contradiction that $\deg(f,U,v) \neq 0$, then $\deg(g,U,v) \neq 0$. By Lemma 2.1.3, there exists $\varepsilon_0 > 0$ such that

$$0 \neq \deg(g,U,v) = \int_{\{u\in U;\ v\in B_\varepsilon(g(u))\}} \varphi_\varepsilon(g(u)-v)J_f(u)du, \quad \forall \varepsilon < \varepsilon_0.$$

[3] see Definition 5.1.1 and Example 5.1.1.

So,

$$\{u \in U;\ v \in B_\varepsilon(g(u))\} \neq \emptyset,$$

in particular, there exists a sequence (p_n) in U such that $|f(p_n) - v| < 1/n.$ The compactness of $\overline{U}$ and the continuity of f provides the existence of $p_0 \in U$ that $p_n \to p_0$, for some $p_0 \in \overline{U}$, and $g(p_0) = \lim_{n\to\infty} g(p_n) = v$. Since $v \notin g(\partial U)$, it follows that $p_0 \in U$ and $g(p_0) = v$, which is a contradiction.

P7): Let $\rho_0 = \rho(v, f(\overline{U}))$, $\rho_1 = \rho(v, f(\partial(U \setminus Z)))$ $g \in C^2(\overline{U}, \mathbb{R}^N)$, with $\|f - g\|_\infty < \delta < \min\{\rho_0/2, \rho_1/2\}$, for $\delta > 0$ sufficiently small such that $\rho_0' := \rho(g(Z), v) > 0$, C be a connected component of $\mathbb{R}^N \setminus g(\partial(U \setminus Z)) \subset \mathbb{R}^N \setminus g(\partial U)$ and v' a regular value of g near v such that $v' \in C$ and

$$\deg(f, U, v) = \deg(g, U, v').$$

By Lemma 2.1.3, there exists $\varepsilon_0 > 0$ such that

$$\deg(g, U, v') = \int_U \varphi_\varepsilon(g(u) - v) J_f(u)du,$$

for each $\varepsilon < \varepsilon_0$. In particular, if $\varepsilon < \rho_0'$, then

$$\begin{aligned}\int_U \varphi_\varepsilon(g(u) - v) J_f(u)du &= \int_{U\setminus Z} \varphi_\varepsilon(g(u) - v) J_f(u)du \\ &= \deg(g, U \setminus Z, v')\end{aligned}$$

and so

$$\deg(f, U, v) = \deg(g, U, v') = \deg(g, U \setminus Z, v').$$

Moreover, since $v' \in C$, it follows by property P4) that

$$\deg(g, U \setminus Z, v') = \deg(g, U \setminus Z, v)$$

whence

$$\begin{aligned}\deg(f, U, v) &= \deg(g, U \setminus Z, v) \\ &= \deg(f, U \setminus Z, v),\end{aligned}$$

as we wish.

P8): Let us define $H(t, u) = tf(u) + (1 - t)g(u),\ \ t \in [0, 1]$. Observe that if $u \in \partial U$, then $H(t, u) = tf(u) + (1 - t)g(u) = tf(u) + (1 - t)f(u) = f(u) \neq v$. Then, the result follows from P3). □

There are many interesting applications of the Brouwer degree, as Theorems of Borsuk Ulam, Hedgehog, Open Mapping, Fixed Point and Surjective Mapping. But exploring them is not the goal of this text. See Deimling [17] or Kesavan [23] for more details.

2.1.2 The index of isolated solutions

For isolated solutions, there is the concept of index of an isolated solution, which is the limit of the degree in a certain sense that will become clear soon.

Let $f : \overline{U} \to \mathbb{R}^N$ be a continuous function and assume that u is an isolated solution of $f(u) = v$, that is, there exists $r > 0$ such that u is the unique solution of $f(u) = v$ in $B_r(u) \subset U$. Then by applying the excision property for the compact set $\overline{(B_r(u) \setminus B_\varepsilon(u))}$, one gets

$$\begin{aligned} \deg(f, B_r(u), v) &= \deg(f, B_r(u) \setminus \overline{(B_r(u) \setminus B_\varepsilon(u))}, v) \\ &= \deg(f, B_\varepsilon(u), v), \ \forall\, 0 < \varepsilon < r. \end{aligned} \tag{2.1.2.1}$$

Thus, we define the index of an isolated solution of f in u relative to the point v, by

$$i(f, u, v) = \lim_{\varepsilon \to 0} \deg(f, B_\varepsilon(u), v).$$

Remark 2.1.1. *The equality (2.1.2.1) implies that*

$$\lim_{\varepsilon \to 0} \deg(f, B_\varepsilon(u), v) = \deg(f, B_r(u), v),$$

and so, by the excision property we have

$$\begin{aligned} \deg(f, B_r(u), v) &= \deg(f, B_r(u) \setminus \overline{(B_r(u) \setminus B_\varepsilon(u))}, v) \\ &= \deg(f, B_\varepsilon(u), v), \ \forall\, 0 < \varepsilon < r. \end{aligned}$$

Moreover, by the excision property we have

$$\deg(f, U, v) = \deg(f, U \setminus \partial B_r(u), v),$$

for each $r < \varepsilon$. On the other hand, by the additivity property we have

$$\begin{aligned} \deg(f, U \setminus \partial B_r(u), v) &= \deg(f, U \setminus \overline{B}_\varepsilon(u), v) + \\ &+ \deg\left(f, U \setminus \overline{U \setminus \overline{B}_\varepsilon(u)}, v\right) \\ &= \deg(f, U \setminus \overline{B}_\varepsilon(u), v) + \\ &+ \deg(f, B_\varepsilon(u), v) \\ &= \deg(f, U \setminus \overline{B}_\varepsilon(u), v) + \\ &+ i(f, u, v) \end{aligned}$$

and so

$$\deg(f, U, v) = \deg(f, U \setminus \overline{B}_\varepsilon(u), v) + i(f, u, v), \tag{2.1.2.2}$$

for each $0 < \varepsilon < r$.

2.1.3 Application

In this section, we present a application of the construction of Brouwer degree to solve equations of the type $f(u) = v$, where $f : \overline{U} \to \mathbb{R}^N$ is a continuous function and U is a bounded open subset of $\mathbb{R}^N$, for the case $N = 2$.

Example 3.5.1 of [23]. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^1-function such that $|\varphi(x)| \leq 1$ and $\varphi'(x) > 0$ for all $x \in \mathbb{R}$. Assume $\varphi(1) = 0$. Then, the system

$$\left.\begin{aligned} x^3 - 3xy^2 + \varphi(x) &= 1 \\ -y^3 + 3x^2y &= 0 \end{aligned}\right\} \tag{2.1.3.1}$$

has, at least, three solutions in $B_2(0) = \{(x,y) \in \mathbb{R}^2;\ x^2+y^2 < 4\}$. We will use the invariance under homotopy property to prove it. Define

$$\begin{aligned} H : \overline{B}_2(0) \times [0,1] &\to \mathbb{R}^2 \\ (x,y,t) &\mapsto (x^3 - 3xy^2 + t\varphi(x),\ -y^3 + 3x^2y). \end{aligned}$$

First let us verify that H is an admissible homotopy, that is, $H(x,y,t) \neq (1,0)$ for all $(x,y,t) \in \partial B_2(0) \times [0,1]$. Indeed, let $(x,y,t) \in \overline{B}_2(0) \times [0,1]$ be such that $H(x,y,t) = (1,0)$. The second equation in (2.1.3.1) gives

$$y(-y^2 + 3x^2) = 0 \Rightarrow y = 0 \text{ or } y^2 = 3x^2.$$

Note that $\partial B_2(0) = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 = 4\}$, so if

$$\begin{aligned} (x,y) \in \partial B_2(0) \text{ and } y = 0 &\Rightarrow x^2 + 0^2 = 4 \\ &\Rightarrow x \in \{-2, 2\} \\ &\Rightarrow (x,y) \in \{(-2,0),(2,0)\}. \end{aligned}$$

If

$$\begin{aligned} (x,y) \in \partial B_2(0) \text{ and } y^2 = 3x^2 &\Rightarrow x^2 + 3x^2 = 4 \\ &\Rightarrow x \in \{-1, 1\} \\ &\Rightarrow 1 + y^2 = 4 \\ &\Rightarrow y \in \{-\sqrt{3}, \sqrt{3}\} \\ &\Rightarrow (x,y) \in \left\{\left(1,\sqrt{3}\right), \left(1,-\sqrt{3}\right), \left(-1,\sqrt{3}\right), \left(-1,-\sqrt{3}\right)\right\}. \end{aligned}$$

Moreover, the fact $H(x,y,t) = (1,0)$ implies in

$$t\varphi(x) = 1 - x^3 + 3xy^2,$$

whence

$$1 \geq |t\varphi(x)| = |1 - x^3 + 3xy^2|. \tag{2.1.3.2}$$

On the other hand,

$$|1 - x^3 + 3xy^2|\Big|_{(1,-\sqrt{3})} = |1 - x^3 + 3xy^2|\Big|_{(1,\sqrt{3})} = |1 - 1 + 3.3| = 9 > 1$$

$$|1 - x^3 + 3xy^2|\Big|_{(-1,-\sqrt{3})} = |1 - x^3 + 3xy^2|\Big|_{(-1,\sqrt{3})} = |1 + 1 + 3.3| = 11 > 1$$

and also

$$|1 - x^3 + 3xy^2|\Big|_{(-2,0)} = |1 + 8 + 0| = 9 > 1$$

$$|1 - x^3 + 3xy^2|\Big|_{(2,0)} = |1 - 8 + 0| = 7 > 1.$$

Hence, by combining the above calculations with (2.1.3.2), we conclude that there are no solutions of $H(x,y,t) = (1,0)$ in $\partial B_2(0) \times [0,1]$. So by the invariance under homotopy property, we obtain that $\deg(H(\cdot,t), B_2(0), (1,0))$ is constant in $[0,1]$.

Consider the system $H(x,y,0) = (1,0)$, that is

$$\begin{cases} x^3 - 3xy^2 = 1, & (2.1.3.3) \\ y^3 + 3x^2y = 0. & (2.1.3.4) \end{cases}$$

Let us find the solutions of it. From (2.1.3.4) we can note that there are only possibilities:

$$\begin{aligned} y &= 0, \\ \text{or } y^2 &= -3x^2. \end{aligned}$$

In the first case, the equation (2.1.3.3) implies that $x = 1$. Thus, the unique solution in this case is

$$(x,y) = (1,0).$$

In the second case, the equation (2.1.3.3) implies in

$$x^3 - 3x(-3x^2) = 1 \Rightarrow x^3 - 9x^3 = 1 \Rightarrow x = -\frac{1}{2},$$

So, in the second case the solutions of (2.1.3.3) and (2.1.3.4) are the pairs $(x,y) \in B_2(0)$ satisfying

$$y^2 = 3x^2 \text{ and } x = -\frac{1}{2}, \tag{2.1.3.5}$$

that is

$$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \text{ and } \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

The jacobian of the function $f_0 := H(\cdot,0)$ is

$$J_{f_0}(x,y) = \begin{vmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{vmatrix} \tag{2.1.3.6}$$

$$\begin{aligned} J_{f_0}(x,y) &= (3x^2 - 3y^2)(3x^2 - 3y^2) + (6xy)(6xy) \\ &= (3x^2 - 3y^2)(3x^2 - 3y^2) + 36(xy)^2 \\ &= [(3x^2)^2 - 18(xy)^2 + (3y^2)^2] + 36(xy)^2 \\ &= [(3x^2)^2 + 18(xy)^2 + (3y^2)^2] \\ &= [(3x^2)^2 + 2(3x^2)(3y^2) + (3y^2)^2] \\ &= (3x^2 + 3y^2)^2. \end{aligned}$$

Since $(0,0) \notin f_0^{-1}((1,0))$, whence it follows that $(1,0)$ is a regular value of f_0. Moreover $J_{f_0}(x,y) > 0$ for all solution (x,y), whence $\deg((H(\cdot,0), B_2(0), (1,0)) = 3$. Consequently, by defining $f_1 := H(\cdot,1)$ we have

$$\deg(f_1, B_2(0), (1,0)) = 3. \tag{2.1.3.7}$$

Now, being $\varphi(1) = 0$ it follows that

$$\begin{aligned} f_1(1,0) &= \left.(x^3 - 3xy^2 + 1\varphi(x),\ y^3 + 3x^2y)\right|_{(1,0)} \\ &= (1^3 - 3.1.0^2 + \varphi(1),\ 0^3 + 3.1^2.0) = (1,0). \end{aligned}$$

Moreover,

$$J_{f_1}(x,y) = \begin{vmatrix} 3x^2 - 3y^2 + \varphi'(x) & -6xy \\ 6xy & 3x^2 - 3y^2 \end{vmatrix},$$

and

$$\begin{aligned} J_{f_1}(1,0) &= (3.1^2 - 3.0^2 + \varphi'(1))(3.1^2 - 3.0^2) + (6.1.0)(6.1.0) \\ &= (3 + \varphi'(1))3 > 0. \end{aligned} \tag{2.1.3.8}$$

By the Inverse Function Theorem, it follows that $(1,0)$ is an isolated solution of $f_1(x,y) = (1,0)$. Then (2.1.3.8) gives

$$i(f_1,(1,0),(1,0)) = 1. \tag{2.1.3.9}$$

In light of (2.1.2.2), the relations (2.1.3.7) and (2.1.3.9) implies that there must be another solution $(\overline{x},\overline{y}) \neq (1,0)$ of $f_1(x,y) = (1,0)$ in $B_2(0)$. We assert that $\overline{y} \neq 0$. Indeed, suppose that $\overline{y} = 0$. Then $f_1(\overline{x},0) = (1,0)$ gives

$$\begin{aligned} (x^3 - 3xy^2 + 1\varphi(x))\Big|_{(\overline{x},0)} = 1 \;&\Rightarrow\; \overline{x}^3 - 3\overline{x}.0^2 + 1\varphi(\overline{x}) = 1 \\ &\Rightarrow\; \overline{x}^3 + \varphi(\overline{x}) = 1, \end{aligned}$$

but the function $x \mapsto x^3 + \varphi(x)$ is increasing and $x = 1$ is a solution of $x^3 + \varphi(x) = 1$. Consequently, $\overline{x} = 1$ and so $(\overline{x},\overline{y}) = (1,0)$, which is a contradiction. Then, we conclude that $\overline{y} \neq 0$. Observe that $(\overline{x}, -\overline{y})$ is also a solution of $f_1(x,y) = (1,0)$. Indeed, the fact $f_1(\overline{x},\overline{y}) = (1,0)$ implies in

$$\begin{aligned} (-y^3 + 3x^2y)\Big|_{(\overline{x},\overline{y})} = 0 \;&\Rightarrow\; -\overline{y}^3 + 3\overline{x}^2\overline{y} = 0 \\ &\overset{\overline{y}\neq 0}{\Rightarrow}\; -\overline{y}^2 + 3\overline{x}^2 = 0 \\ &\Rightarrow\; -(-\overline{y})^2 + 3\overline{x}^2 = 0 \\ &\Rightarrow\; -\overline{y}(-(-\overline{y})^2 + 3\overline{x}^2) = 0 \\ &\Rightarrow\; -(-\overline{y})^3 + 3\overline{x}^2(-\overline{y}) = 0 \end{aligned} \tag{2.1.3.10}$$

and

$$\begin{aligned} (x^3 - 3xy^2 + 1\varphi(x))\Big|_{(\overline{x},\overline{y})} = 0 \;&\Rightarrow\; \overline{x}^3 - 3\overline{x}\overline{y}^2 + 1\varphi(\overline{x}) = 0 \\ &\Rightarrow\; \overline{x}^3 - 3\overline{x}(-\overline{y})^2 + 1\varphi(\overline{x}) = 0. \end{aligned} \tag{2.1.3.11}$$

So, (2.1.3.10) and (2.1.3.11) means that $(\overline{x}, -\overline{y})$ is a solution of $f_1(x,y) = (1,0)$ as well. Therefore, there exist at least three solutions of (2.1.3.1).

Observe that the Brouwer degree is defined for functions $f : \overline{U} \to \mathbb{R}^N$ where U is a domain that lives in a finite dimensional euclidean space $\mathbb{R}^N$. Although, as we mentioned in the previous section, we want to work with degree to operators $F : F \to E$ that acts in a certain subspace F (a space of a infinite dimensional space). First, we state that if E is an infinite dimensional real Banach space and U a bounded open subset of E, then it is not possible to define a topological degree, that is, a function $(f,U,v) \mapsto d(f,U,v)$ P1) (Normalization), P3) (Invariance under homotopy) and P5) (Additivity) for the class $C(\overline{U};E)$ of all continuous maps $f : \overline{U} \to E$ where U is a bounded open subset of E. The argument is based on the following two results.

Theorem 2.1.1. *There is a unique function d that associates for each (f, U, v), where $f : \overline{U} \to \mathbb{R}^N$ is a continuous function and $v \in \mathbb{R}^N \setminus f(\partial U)$, an integer number $d(f, U, v)$, such that d satisfies P1) (Normalization), P3) (Invariance under homotopy) and P5) (Additivity).*

Proof. See §1 of Chapter 1 of Deimling [17]. □

And also

Theorem 2.1.2 (Brouwer's Fixed Point Theorem). *Let $D \subset \mathbb{R}^N$ be a nonempty compact convex set and $f : D \to D$ a continuous function. Then f has a fixed point.*

Proof. See Theorem 3.2 of Deimling [17]. □

Remark 2.1.2. *The proof of Theorem 2.1.2, as we can see in [17], follows just by using the properties P1) (Normalization), P3) (Invariance under homotopy) and P5) (Additivity).*

As a consequence of the above remark, it follows that if it was possible to define a topological degree for the class $C(\overline{U}; E)$, then Theorem 2.1.2 would holds for every $f \in C(\overline{U}; E)$, but in the introduction of Chapter 2 of [17], the following counter example is given. Let $E = c_0$ be the Banach space of real sequences $u = (u_n)$ that converges to zero, with norm $\|u\| = \sup_n |u_n|$ and $f : E \to E$ be a function defined by

$$\begin{cases} (f(u))_1 = (1 + \|u\|)/2 \\ (f(u))_{n+1} = u_n \quad n \geq 1. \end{cases}$$

Observe that

$$\begin{aligned} \|f(u) - f(v)\| &= \sup_n (|(1 + \|u\|)/2 - (1 + \|v\|)/2|, u_1 - v_1, u_2 - v_2, \ldots) \\ &\leq \sup_n (|\|u - v\|/2, |u_1 - v_1|, |u_2 - v_2|, \ldots) \\ &= \|u - v\| \end{aligned}$$

and so f is a continuous function. Moreover, f maps the compact convex set $\overline{B}_1(0)$ into itself. Although f has no fixed point since $u = f(u)$ implies

$$\begin{aligned} u_1 &= (f(u))_1 = (1 + \|u\|)/2 \Rightarrow \\ \Rightarrow u_2 &= (f(u))_2 = u_1 \Rightarrow \\ \Rightarrow & \quad \Rightarrow \\ \Rightarrow u_n &= u_1 = (1 + \|u\|)/2 \ \forall n \end{aligned}$$

and so $u \notin c_0$, which is a contradiction. We conclude then there it is not possible to define such a function for the class $C(\overline{U}; E)$. Although it is possible to extend the concept of Brouwer degree for a certain subclass of $C(\overline{U}; E)$, as we will see in the next chapter.

2.2 The Leray-Schauder degree

In 1934, Leray and Schauder [28] extended the concept of Brouwer degree to operators acting in bounded open subsets of Banach spaces with infinite dimension. The theory has been widely studied over the centuries by several authors, among which we quote

Deimling [17] (1985), Kesavan [23] (2004), et al.

The idea is to take advantage of Brouwer degree definition to construct one that covers a type of operator called compact perturbation of identity, that is, an operator $\Phi : \overline{U} \to E$ defined by $\Phi := I - K$, where I is the identity, K is a compact operator and U is an open bounded subset of a real Banach space E.

The first step is to consider operators of the type $I - T$, where $T : \overline{U} \to E$ is an operator such that $\mathrm{Im}(T) \subset F$, with F being a finite dimensional subspace, that is, T has finite rank, and define the Leray-Schauder degree of $I - T$ as the Brower degree of $I - T$ restricted to $\overline{U} \cap F$. After this, we show that any compact perturbation $\Phi = I - K$ can be approximated by an operator $I - T$, where T is an operator with finite rank, then we define the Leray-Schauder degree of Φ as the degree of $I - T$.

Moreover, we show that compact perturbations of the identity are proper functions and this feature allows us to establish the same properties of the Brouwer degree for the Leray-Schauder degree. We emphasize Theorem 2.2.1, that is an invariance under homotopy property that differs from the P3) when it consider a variation of the domain U, besides the variation of the homotopy parameter.

As well as the Brouwer degree, the Leray-Schauder degree, being an extension of it, also admits a limit version called "index of an isolated solution" which satisfies the Leray-Schauder Formula. This is a very useful tool in applying degree theory for obtain existence of bifurcation points.

Finally, we exemplify the use of the Leray-Schauder degree by showing existence of solutions for a sublinear problem of the type

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

2.2.1 The Leray-Schauder Degree

In order to make reading more fluid, let us define some objects that appears frequently in this section.

Definition 2.2.1. *Consider $T : \overline{U} \to E$ an operator in $C(\overline{U}; E)$ and denote by $\varphi : \overline{U} \to E$ the operator $\varphi := I - T$. We say that φ is a* **bounded T-perturbation of the identity** *when T has a finite rank (i.e., there exists a finite dimensional subspace F of E such that $T(\overline{U}) \subset F$).*

Remark 2.2.1. *For each finite dimensional subspace F of E considered, in this section, we are assuming that $U \cap F \neq \emptyset$.*

Definition 2.2.2. *Let $K \in Q(\overline{U}; E)$. Then the operator $\Phi : \overline{U} \to E$ defined as $\Phi(u) = u - K(u)$ is called a* **compact K-perturbation of the identity.**

Let $\varphi : \overline{U} \to E$ be a bounded T-perturbation of the identity and $\beta = \{v_1, \ldots, v_N\}$ be a basis of F. For each $u \in U \cap F$, we will denote by u_β, the coordinates of u relative to the basis β. Let us denote $U_\beta := \{u_\beta \,;\, u \in U \cap F\}$.

From Linear Algebra, we know that the application $R_\beta : F \to \mathbb{R}^N$ defined by $u \mapsto u_\beta$ is a linear transformation between finite dimensional spaces (and so continuous) as well as its inverse $R_\beta^{-1} : \mathbb{R}^N \to F$. Consequently, U_β is a bounded open subset of $\mathbb{R}^N$. Observe that U_β is a subset of $\mathbb{R}^N$ and for each $u_\beta \in U_\beta$ the vector

$$R_\beta^{-1}(u_\beta) = \sum_{i=1}^{N} \langle u_\beta, e_i \rangle v_i$$

belongs to $F \cap U$ and $(\varphi(R_\beta^{-1}(u_\beta)))_\beta$ lies in $\mathbb{R}^N$. Thus, it makes sense to consider the operator $\varphi_\beta : \overline{U_\beta} \to \mathbb{R}^N$ defined by $\varphi_\beta(u_\beta) = (\varphi(R_\beta^{-1}(u_\beta)))_\beta$.

Let us formalize the definition of the operator φ_β.

Definition 2.2.3. *Let F be a subspace of E with $\dim F = N$ and β be a basis of F. For each continuous function $\varphi : \overline{U} \to E$ such that $\varphi(\overline{U} \cap F) \subset F$, we define the β version of φ, denoted by φ_β, as the function*

$$
\begin{aligned}
\varphi_\beta : \overline{U_\beta} &\to \mathbb{R}^N \\
u_\beta &\mapsto (\varphi(R_\beta^{-1}(u_\beta)))_\beta .
\end{aligned}
$$

Since U_β is a bounded open subset of $\mathbb{R}^N$, then the Brouwer degree $deg(\varphi_\beta, U_\beta, v_\beta)$ is well defined for each $v_\beta \notin \varphi_\beta(\partial(U_\beta))$. Our aim is to use this number to define the degree of φ restricted to F. Therefore, we need to relate the points $v \in F$ satisfying $v \notin \varphi(\partial U)$ to those $v_\beta \in U_\beta$ satisfying $v_\beta \notin \varphi_\beta(\partial(U_\beta))$. For this, consider the following lemma:

Lemma 2.2.1. *Let F be a subspace of E with $\dim F = N$ and β a basis of F. Assume that $\varphi : \overline{U} \to E$ is a continuous function such that $\varphi(\overline{U} \cap F) \subset F$. Then, the β version $\varphi_\beta : \overline{U_\beta} \to \mathbb{R}^N$ of φ is a continuous function. Moreover, for each $v \in F$, it holds*

$$v \notin \varphi(\partial U) \text{ if, and only if, } v_\beta \notin \varphi_\beta(\partial(U_\beta)).$$

Proof. First, suppose by contradiction that $v \notin \varphi(\partial U)$, but $v_\beta \in \varphi_\beta(\partial(U_\beta))$. Thus, there exists $u_\beta \in \partial(U_\beta)$ such that $\varphi_\beta(u_\beta) = v_\beta$, that is,

$$(\varphi(u))_\beta = (\varphi(R_\beta^{-1}(u_\beta)))_\beta = \varphi_\beta(u_\beta) = v_\beta,$$

which implies in

$$\varphi(u) = v, \tag{2.2.1.1}$$

after applying R_β^{-1} in the equality. Moreover, since $u_\beta \in \partial(U_\beta)$, then there exist sequences $(u_\beta^n)_n$ in U_β and $(w_\beta^n)_n$ in $(U_\beta)^c$ converging to u_β. Observing that

$$
\begin{cases}
u_\beta^n \in U_\beta & \Rightarrow R_\beta^{-1}(u_\beta^n) \in U \text{ and} \\
w_\beta^n \notin (U_\beta)^c & \Rightarrow R_\beta^{-1}(w_\beta^n) \in U^c,
\end{cases}
$$

we have that $(R_\beta^{-1}(u_\beta^n))_n$ and $(R_\beta^{-1}(w_\beta^n))_n$ are sequences in U and U^c, respectively. Now, since R_β^{-1} is a continuous function, we get that both sequences converge to $R_\beta^{-1} u_\beta = u$, whence

$$u \in \partial U. \tag{2.2.1.2}$$

Combining (2.2.1.1) and (2.2.1.2) we conclude that $v \in \varphi(\partial U)$, which is a contradiction. Therefore, if $v \notin \varphi(\partial U)$, then $v_\beta \notin \varphi_\beta(\partial(U_\beta))$.

Conversely, suppose by contradiction that $v_\beta \notin \varphi_\beta(\partial(U_\beta))$, but $v \in \varphi(\partial U)$. So, there exists $u \in \partial U$ such that $\varphi(u) = v$. Since $u \in \partial U$ we can find sequences $(u^n)_n$ and $(w^n)_n$ in U and U^c, respectively, converging to u. Note that

$$
\begin{cases}
u^n \in U \Rightarrow R_\beta(u^n) \in U_\beta \text{ and} \\
w^n \in U^c \Rightarrow R_\beta(w^n) \in (U_\beta)^c.
\end{cases}
$$

By continuity of R, we deduce that

$$\begin{cases} R(u^n) \to R(u) = u_\beta & \text{and} \\ R(w^n) \to R(u) = u_\beta, \end{cases}$$

that is $u_\beta \in \partial U_\beta$. But,

$$\varphi(u) = v \Rightarrow \varphi_\beta(u_\beta) = R(\varphi(R^{-1}(u_\beta))) = R(v) = v_\beta$$

which contradicts $v_\beta \notin \varphi_\beta(\partial(U_\beta))$ and the lemma is proved. □

Now, we are able to propose the following definition:

Definition 2.2.4. *Let $T \in C(\overline{U}; E)$ be an operator with finite rank and let F be a finite dimensional subspace of E with dim$F = N$ such that $T(\overline{U}) \subset F$. Let $\varphi : \overline{U} \to E$ be the bounded T-perturbation of the identity, β a basis for F and the β version $\varphi_\beta : \overline{U_\beta} \to \mathbb{R}^N$ of φ. For each $v \in \varphi(\partial U)^c \cap F$, we define the Leray-Schauder degree of φ in $U \cap F$ at the point $v \in F$, relative to the basis β, as*

$$deg(\varphi|_{\overline{U \cap F}}, U \cap F, v) := deg(\varphi_\beta, U_\beta, v_\beta). \tag{2.2.1.3}$$

We state that the number $deg(\varphi_\beta, U_\beta, v_\beta)$ does not depends on the choice of the basis β.

Proposition 2.2.1. *Let $T \in C(\overline{U}; E)$ be an operator with finite rank, F a finite dimensional subspace of E with dim$F = N$ satisfying $T(\overline{U}) \subset F$ and $\varphi : \overline{U} \to E$ be the bounded T-perturbation of the identity. If $v \in (\varphi(\partial U))^c \cap F$ and β_1, β_2 are two different basis of F, then $v_{\beta_i} \notin \varphi_{\beta_i}(\partial(U_{\beta_i}))$ for each $i \in \{1, 2\}$ and*

$$deg\,(\varphi_{\beta_1}, U_{\beta_1}, v_{\beta_1}) = deg\,(\varphi_{\beta_2}, U_{\beta_2}, v_{\beta_2})\,, \tag{2.2.1.4}$$

where $v \notin f(\partial U)$ and for each $i \in \{1, 2\}$, $\varphi_{\beta_i} : \overline{U_{\beta_i}} \to \mathbb{R}^N$ is the β_i version of φ.

Proof. The statement about v_{β_i} follows directly from Lemma 2.2.1. Let M be the change of basis matrix from basis β_1 to β_2. Observe that

$$\varphi_{\beta_2}(u_{\beta_2}) = M(\varphi_{\beta_1}(M^{-1}u_{\beta_2})) \tag{2.2.1.5}$$

and so by Chain rule,

$$\varphi'_{\beta_2}(u_{\beta_2}) = M\varphi'_{\beta_1}(M^{-1}u_{\beta_2})M^{-1} = \varphi'_{\beta_1}(u_{\beta_1})$$

by whence

$$J_{\varphi_{\beta_2}}(u_{\beta_2}) = J_{\varphi_{\beta_1}}(u_{\beta_1})$$

and consequently

$$\deg\,(\varphi_{\beta_1}, U_{\beta_1}, v_{\beta_1}) = \deg\,(\varphi_{\beta_2}, U_{\beta_2}, v_{\beta_2})\,.$$

□

As a consequence of this result we can improve Definition 2.2.4 as follows:

Definition 2.2.5. *Let $T \in C(\overline{U}; E)$ be an operator such that there exists a finite dimensional subspace F of E, with dim$F = N$, satisfying $T(\overline{U}) \subset F$. Let $\varphi : \overline{U} \to E$ be the bounded T-perturbation of the identity. For each $v \in \varphi(\partial U)^c \cap F$, we define the Brouwer topological degree of φ in $U \cap F$, at the point $v \in F$, as*

$$deg(\varphi|_{\overline{U} \cap F}, U \cap F, v) := deg(\varphi_\beta, U_\beta, v_\beta),$$

where β is any basis of F and $\varphi_\beta : \overline{U_\beta} \to \mathbb{R}^N$ is the β version of φ.

Furthermore, the definition needs to be independent on the choice of the subspace F. This is what we will proof after the following lemma:

Lemma 2.2.2. *Let $T \in C(\overline{U}, E)$ be an operator such that there exists subspaces F_i of E with dimension dim$F_i = N_i$ such that $T(\overline{U}) \subset F_1 \subset F_2$, for each $i \in \{1, 2\}$. Let $\varphi : \overline{U} \to E$ be the bounded T-perturbation of the identity. If $N_1 < N_2$ and $v \in (\varphi(\partial U))^c \cap (F_1 \cap F_2)$, then*

$$deg(\varphi_{\overline{U} \cap F_1}, \overline{U} \cap F_1, v) = deg(\varphi_{\overline{U} \cap F_2}, \overline{U} \cap F_2, v). \tag{2.2.1.6}$$

Proof. Let

$$\begin{cases} \beta_1 = \{v_1, v_2, \dots, v_{N_1}\} \text{ be a basis for } F_1 \text{ and} \\ \beta_2 = \{v_1, v_2, \dots, v_{N_1}, v_{N_1+1}, \dots, v_{N_1+N_2}\} \text{ a basis for } F_2. \end{cases} \tag{2.2.1.7}$$

So we have

$$u_{\beta_2} = (u_{\beta_1}, w), \ w \in \mathbb{R}^{N_2 - N_1}, \tag{2.2.1.8}$$

for all $u \in F_2$.

For each $i \in \{1, 2\}$, consider the β_i version $\varphi_{\beta_i} : \overline{U_{\beta_i}} \to \mathbb{R}^{N_i}$ of φ and $u_{\beta_2} \in U_{\beta_2}$. Since, $T(\overline{U}) \subset F_1$, we have $T(u) \in F_1$ for each $u \in \overline{U}$ and hence the description (2.2.1.8) implies that

$$(T(u))_{\beta_2} = ((T(u))_{\beta_1}, \mathbf{0}) \tag{2.2.1.9}$$

where $\mathbf{0} := (0, 0, \dots, 0) \in \mathbb{R}^{N_2 - N_1}$.

Then,

$$\begin{aligned}
\varphi_{\beta_2}(u_{\beta_2}) &= (\varphi(R_{\beta_2}^{-1}(u_{\beta_2})))_{\beta_2} \\
&= (\varphi(u))_{\beta_2} \\
&= (u - T(u))_{\beta_2} \\
&= u_{\beta_2} - (T(u))_{\beta_2} \\
&\overset{(2.2.1.9)}{=} u_{\beta_2} - ((T(u))_{\beta_1}, \mathbf{0}) \\
&\overset{(2.2.1.8)}{=} (u_{\beta_1}, w) - ((T(u))_{\beta_1}, \mathbf{0}) \\
&= (u_{\beta_1} - (T(u))_{\beta_1}, w - \mathbf{0}) \\
&= ((u - T(u))_{\beta_1}, w) \\
&= (\varphi_{\beta_1}(u_{\beta_1}), w).
\end{aligned}$$

It follows from the first and the last expressions that

$$\varphi_{\beta_2}(u_{\beta_2}) = (\varphi_{\beta_1}(u_{\beta_1}), w) \tag{2.2.1.10}$$

and by derivating (2.2.1.10) with respect to u_2, we have

$$\varphi'_2(u_2) = \varphi'_2((u_1, w)) = \begin{bmatrix} \varphi'_1(u_1) & 0 \\ 0 & I_{N_2 - N_1}, \end{bmatrix}$$

so that

$$J_{\varphi_{\beta_2}}(u_2) = J_{\varphi_{\beta_2}}((u_1, w)) = J_{\varphi_{\beta_1}}(u_1) J_{I_{N_2-N_1}}(w) = J_{\varphi_{\beta_1}}(u_1),$$

that is,

$$J_{\varphi_{\beta_2}}(u_2) = J_{\varphi_{\beta_1}}(u_1). \tag{2.2.1.11}$$

Consequently, by Definition 2.2.5, we conclude that (2.2.1.6) holds. □

Naturally, Lemma (2.2.2) induces the following proposition.

Proposition 2.2.2. *Let F and G be two finite dimensional subspaces of a real Banach space E, $T \in C(\overline{U}; E)$ an operator such that $T(\overline{U})$ is contained in both F and G. Let $\varphi : \overline{U} \to E$ be the bounded T-perturbation of the identity. Then, for each $v \in \varphi(\partial U)^c \cap F \cap G$,*

$$deg(\varphi|_{\overline{U} \cap F}, U \cap F, v) = deg(\varphi|_{\overline{U} \cap G}, U \cap G, v). \tag{2.2.1.12}$$

Proof. We know that $F \cap G$ is a subspace of F. Now, using the last result for $F_2 := F$ and $F_1 := F \cap G$, we get for each $v \notin \varphi(\partial U)$ that

$$\deg(\varphi|_{U \cap F}, U \cap F, v) = \deg(\varphi|_{\overline{U} \cap F \cap G}, U \cap F \cap G, v). \tag{2.2.1.13}$$

Again, by using the last result with $F_2 := G$ and $F_1 := F \cap G$, we obtain

$$\deg(\varphi|_{\overline{U} \cap F \cap G}, U \cap F \cap G, v) = \deg(\varphi|_{\overline{U} \cap G}, U \cap G, v). \tag{2.2.1.14}$$

So (2.2.1.13) and (2.2.1.14) imply (2.2.1.12). □

In other words, Definition 2.2.5 does not depend on the choice of subspace F containing $T(\overline{U})$ and v. This leads us to the following definition of degree for bounded perturbations of identity.

Definition 2.2.6. *Let $T \in C(\overline{U}; E)$ be an operator and $v \in E$ such that the family*

$$\mathcal{F} := \{F \text{ is a finite dimensional subspace of } E \text{ containing } T(\overline{U}) \text{ and } v\}$$

is not empty and the bounded T-perturbation of the identity $\varphi : \overline{U} \to E$ is such that $v \notin \varphi(\partial U)$. We define the Leray-Schauder degree of φ, in U, at the point v by

$$deg(\varphi, U, v) = deg(\varphi_{\overline{U} \cap F}, U \cap F, v)$$

where F is any element of $\mathcal{F}$ containing v.

Now, our aim is to show that compact perturbations of identity can be approximated by this bounded perturbations of identity. In this sense, the following proposition will be useful.

Proposition 2.2.3 (Approximation by continuous functions). *Let Z be a compact subset of E. Given $\varepsilon > 0$, there exists a subspace $F_\varepsilon \subset E$, with $\dim F_\varepsilon < \infty$, and a map $g_\varepsilon \in C(Z, F_\varepsilon)$ such that*

$$\|u - g_\varepsilon(u)\| < \varepsilon, \ \forall\, u \in Z.$$

Proof. Given $\varepsilon > 0$, it follows from the compactness of Z that there exists $\{u_1, u_2, \ldots, u_n\} \subset K$ such that

$$Z \subset \bigcup_{i=1}^{n} B_\varepsilon(u_i).$$

Let $F_\varepsilon := \langle u_1, \ldots, u_n \rangle$ and functions $f_i : Z \to \mathbb{R}$, $i = 1, \ldots, n$, defined by

$$f_i(u) = \begin{cases} \varepsilon - \|u - u_i\|, & u \in B_\varepsilon(u_i) \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_i(u) \geq 0$, for all $u \in E$, and $\sum_{i=1}^n f_i(u) > 0$, for all $u \in Z$.

By defining $g_\varepsilon : Z \to F_\varepsilon$ as

$$g_\varepsilon(u) = \frac{\sum_{i=1}^n f_i(u) u_i}{\sum_{i=1}^n f_i(u)},$$

we have that g_ε is a continuous function such that

$$\begin{aligned} \|u - g_\varepsilon(u)\| &= \left\| \frac{\sum_{i=1}^n f_i(u) u - \sum_{i=1}^n f_i(u) u_i}{\sum_{i=1}^n f_i(u)} \right\| \\ &\leq \frac{1}{\sum_{i=1}^n f_i(u)} \sum_{i=1}^n f_i(u) \|u - u_i\| \\ &< \frac{1}{\sum_{i=1}^n f_i(u)} \sum_{i=1}^n f_i(u) \varepsilon = \varepsilon, \ \forall u \in Z. \end{aligned}$$

This ends the proof. □

Compact perturbations of the identity satisfy a certain property that classify them as proper functions. This property is useful in proving the same properties of the Brouwer degree, for the Leray-Schauder degree.

Definition 2.2.7 (Proper Functions). *Let U and V be normed vector spaces and let $f : U \to V$ be a function. We say that f is a proper function when $f^{-1}(Z)$ is a compact subset of U whenever Z is a compact subset of V.*

See Chapter 10 of Bourbaki [9] for a more general definition that holds for topological spaces. For our interests, Definition 2.2.7 is sufficient.

Proposition 2.2.4. *Assume $K : \overline{U} \subset E \to E$ is a compact operator. Then, the compact K-perturbation $\Phi : \overline{U} \to E$ is a closed (that is, maps each closed subset of $\overline{U}$ to a closed subset of E) and proper function.*

Proof. Let F be a closed subset of $\overline{U}$ and (u_n) a sequence in F such that $\Phi(u_n) \to v \in E$. It follows from the compactness of the operator K that there exists a subsequence (u_{n_k}) of (u_n) such that $K(u_{n_k}) \to w \in E$ and so $u_{n_k} \to v + w \in F$, because F is a closed set. By the continuity of Φ,

$$\Phi(u_{n_k}) \to \Phi(v + w) \text{ in } E$$

and so the uniqueness of the limit implies $\Phi(v + w) = v$. Thus, $v \in \Phi(F)$ as we wanted.

For the second part, let $(u_n) \subset \Phi^{-1}(Z)$ where Z is a compact subset of E. Khen, there exists a sequence (v_n) in Z satisfying $v_n = u_n - Ku_n$. By the compactness of Z, $v_n \to v \in Z$ up to a subsequence. Since U is a bounded set, it follows that $Ku_n \to w \in Z$, up to a subsequence. Hence, $u_n \to v + w \in Z$ (up to a subsequence), as we wanted. □

Lemma 2.2.3. *Let $K \in Q(\overline{U}, E)$ be an operator. Given $\varepsilon > 0$, there exists a finite rank continuous operator $K_\varepsilon : \overline{U} \to E$ such that $\|K(u) - K_\varepsilon(u)\| < \varepsilon$, for all $u \in \overline{U}$.*

Proof. By taking $Z = \overline{K(\overline{U})}$ in Proposition 2.2.3, we have that there exist a finite dimensional $F_\varepsilon \subset E$ and a continuous function $g_\varepsilon : Z \to F_\varepsilon$ satisfying
$\|u - g_\varepsilon(u)\| < \varepsilon$, for all $u \in Z$. Defining $K_\varepsilon(u) = g_\varepsilon(K(u))$ for all $u \in \overline{U}$, we observe that $K_\varepsilon(\overline{U}) = g_\varepsilon(K(\overline{U})) \subset g_\varepsilon(Z) \subset F_\varepsilon$ and

$$\|K(u) - K_\varepsilon(u)\| = \|K(u) - g_\varepsilon(K(u))\| < \varepsilon \ \forall\, u \in \overline{U},$$

as we wanted. □

In order to establish a definition of degree for compact perturbations of identity, we will use Lemma 2.2.3 for a convenient $\varepsilon > 0$.

Remark 2.2.2. *Let Φ be the compact K-perturbation of the identity and φ_ε the bounded K_ε-perturbation of the identity. By Proposition 2.2.4, $\Phi(\partial U)$ is a closed subset of E, therefore $r := \rho(v, \Phi(\partial U)) > 0$. Take $\varepsilon := r/2$ in Lemma 2.2.3. We assert that $v \notin \varphi_\varepsilon(\partial U)$. Indeed, if $u \in \partial U$, then*

$$\begin{aligned}
\|v - \varphi_\varepsilon(u)\| &= \|(v - \Phi(u)) - (\varphi_\varepsilon(u) - \Phi(u))\| \\
&\geq \|v - \Phi(u)\| - \|\Phi(u) - \varphi_\varepsilon(u)\| \\
&\geq r - \|(u - Ku) - (u - K_\varepsilon(u))\| \\
&= r - \|K_\varepsilon(u) - K(u)\| \\
&\geq r - \frac{r}{2} = \frac{r}{2}.
\end{aligned}$$

By taking the infimum in $u \in \partial U$, in both sides of the previous inequality, we get $\rho(v, \varphi_\varepsilon(\partial U)) > 0$. Thus, it is well defined the degree

$$\deg(\varphi_\varepsilon, U, v) = \deg\left(\varphi_\varepsilon|_{\overline{U} \cap F_\varepsilon}, U \cap F_\varepsilon, v\right),$$

for $v \notin \Phi(\partial U)$ and $\varepsilon := r/2$.

Before setting the definition of Leray-Schauder degree for compact perturbations of the identity, we must show the independence on the choice of the approximation φ_ε above.

Lemma 2.2.4. *Let K be as is Lemma 2.2.3 and $\Phi : \overline{U} \to E$ the compact K-perturbation of the identity. Consider $v \notin \Phi(\partial U)$ and $K_{\varepsilon_i} : \overline{U} \to E$ continuous operators such that $K_{\varepsilon_i}(\overline{U}) \subset F_i$, $\dim F_i < \infty$, $v \in F_1 \cap F_2$ and*

$$\|K - K_{\varepsilon_i}(u)\| \leq \frac{r}{2}, \text{ for each } i \in \{1, 2\} \tag{2.2.1.15}$$

where $r = \rho(v, \Phi(\partial U))$, whose the existence is guaranteed by Lemma 2.2.3. Then

$$\deg(\varphi_{\varepsilon_1}|_{\overline{U} \cap F_1}, U \cap F_1, v) = \deg(\psi_{\varepsilon_2}|_{\overline{U} \cap F_2}, U \cap F_2, v), \tag{2.2.1.16}$$

where $\varphi_{\varepsilon_1} : \overline{U} \to E$ and $\psi_{\varepsilon_2} : \overline{U} \to E$ are the bounded K_{ε_1} and K_{ε_2}-perturbation of the identity, respectively.

Proof. Let F be a finite dimensional subspace of E containing F_1 and F_2. Let us denote by $N :=\dim F$. By Proposition 2.2.2,

$$\begin{cases} \deg(\varphi_{\varepsilon_1}|_{\overline{U}\cap F_1}, U \cap F_2, v) = \deg(\varphi_{\varepsilon_1}|_{\overline{U}\cap F}, U \cap F, v) \text{ and} \\ \deg(\psi_{\varepsilon_2}|_{\overline{U}\cap F_2}, U \cap F_1, v) = \deg(\psi_{\varepsilon_2}|_{\overline{U}\cap F}, U \cap F, v). \end{cases} \tag{2.2.1.17}$$

Now, let β be a basis for F and consider $\varphi_\beta : \overline{U_\beta} \to \mathbb{R}^N$ and $\psi_\beta : \overline{U_\beta} \to \mathbb{R}^N$ the β versions of φ_{ε_1} and ψ_{ε_2}, respectively. Then, by Definition 2.2.5 we have

$$\begin{cases} \deg(\varphi_{\varepsilon_1}|_{\overline{U}\cap F}, U \cap F, v) = \deg(\varphi_\beta, U_\beta, v_\beta), \text{ and} \\ \deg(\psi_{\varepsilon_2}|_{\overline{U}\cap F}, U \cap F, v) = \deg(\psi_\beta, U_\beta, v_\beta). \end{cases} \tag{2.2.1.18}$$

Let us define the homotopies

$$\begin{aligned} H : \overline{U} \times [0,1] &\to E \\ (u,t) &\mapsto t\varphi_{\varepsilon_1}(u) + (1-t)\psi_{\varepsilon_2}(u) \end{aligned}$$

and

$$\begin{aligned} H_\beta : \overline{U_\beta} \times [0,1] &\to \mathbb{R}^N \\ (u,t) &\mapsto t\varphi_\beta(u) + (1-t)\psi_\beta(u), \end{aligned}$$

and note that the function $H_\beta(\cdot,t) : \overline{U_\beta} \to \mathbb{R}^N$ is the β version of the function $H(\cdot,t) : \overline{U} \to E$ for each fixed $t \in [0,1]$. Consequently, by Lemma 2.2.1 we deduce that $v_\beta \notin H(\partial U_\beta, t)$ for each $v \notin H(\partial U, t)$. Let us show that $v \notin H(\partial U, t)$, for each $t \in [0,1]$.

In fact, let $t \in [0,1]$. Since $r = \rho(v, \varphi(\partial U))$, we have

$$\|(I-K)u - v\| \geq r. \tag{2.2.1.19}$$

Moreover, (2.2.1.15) implies that

$$\|(I-K_{\varepsilon_i})u - (I-K)u\| \leq r/2, \; i = 1,2. \tag{2.2.1.20}$$

Then, by adding

$$-t(I-K)u - (1-t)(I-K)u + (I-K)u = 0$$

in $\|H(u,t) - v\|$, rearranging the terms and using the triangular inequality, we get

$$\begin{aligned} (LHS) &:= \|H(u,t) - v\| \\ &= \|t(I-K_{\varepsilon_1})u + (1-t)(I-K_{\varepsilon_2})u - v\| \\ &= \|t(I-K_{\varepsilon_1})u + [-t(I-K)u - (1-t)(I-K)u + (I-K)u] + \\ &+ (1-t)(I-K_{\varepsilon_2})u - v\| \\ &= \|[(I-K)u - v] + t[(I-K_{\varepsilon_1})u - (I-K)u] + (1-t)[(I-K_{\varepsilon_2})u - (I-K)u]\| \\ &\geq \|(I-K)u - v\| - t\|(I-K_{\varepsilon_1})u - (I-K)u\| - (1-t)\|(I-K_{\varepsilon_2})u - (I-K)u\| \\ &\geq r - t\frac{r}{2} - (1-t)\frac{r}{2} = r/2 > 0. \end{aligned}$$

Since u was taking arbitrarily, we conclude from the previous inequality that $v \notin H(\partial U, [0,1])$. Thus, $v \notin H(\partial U, t)$ for each $t \in [0,1]$ and hence $v_\beta \notin H(\partial U_\beta, t)$. This allows us to use the invariance under homotopy property of Brouwer degree for H, to obtain:

$$\begin{aligned}\deg(\varphi, U, v) &= \deg(H(\cdot,1), U, v)\\ &= \deg(H(\cdot,0), U, v)\\ &= \deg(\psi, U, v).\end{aligned}$$

Therefore, (2.2.1.16) follows by combining (2.2.1.18) and (2.2.1.17). □

Now we are able to establish the following definition.

Definition 2.2.8 (Leray-Schauder Degree). *Let $K \in Q(\overline{U};E)$ be an operator, $\Phi : I - K$ and $v \notin \Phi(\partial U)$. We define the Leray-Schauder degree of Φ in U at the point v, by*

$$\deg(\Phi, U, v) = \deg(\varphi_{r/2}|_{\overline{U}\cap F}, U, v), \tag{2.2.1.21}$$

where $\varphi_{r/2} := I - K_{r/2}$, $K_{r/2} : \overline{U} \to E$ is any finite rank operator satisfying $K_{r/2}(\overline{U}) \subset F$ and $\|K(u) - K_{r/2}(u)\| < \frac{r}{2}$, for all $u \in \overline{U}$.

Remark 2.2.3. *Observe that existence of $K_{r/2}$ is guaranteed by Lemma 2.2.3 and the object in the right hand side of (2.2.1.21) is well defined by Remark 2.2.2. Moreover, the independence on the choice of the approximation $K_{r/2}$ is guaranteed by above lemma.*

2.2.2 Properties of Leray-Schauder Degree

As an extension of the Brouwer degree, Leray-Schauder degree satisfies the same properties of the Brouwer degree. We will see in this section that this is due to the fact that compact perturbations of the identity are proper functions.

Proposition 2.2.5. *Let $K \in Q(\overline{U};E)$, $\varphi : \overline{U} \to E$ defined by $\Phi := I - K$, $v \notin \Phi(\partial U)$, $H \in C(\overline{U} \times [0,1];E)$ a function defined by $H(u,t) = u - S(u,t)$, where $S \in Q(\overline{U} \times [0,1];E)$, $v \notin H(\partial U \times [0,1])$ and $X \subset \overline{U}$ a closed subset such that $v \notin \Phi(X) \cup \Phi(\partial U)$. Then, the following properties for the Leray-Schauder degree holds.*

P1) (Normalization). $\deg(I,U,v) = \begin{cases} 1, & v \in U \\ 0, & v \notin U. \end{cases}$

P2) (Continuity relative to K). There exists a neighborhood $\mathcal{U}$ of K in the topology $(Q(\overline{U};E), \|\cdot\|_\infty)$ such that for all $G \in \mathcal{U}$ we have $v \notin (I - G)(\partial U)$ and

$$\deg(I - K, U, v) = \deg(I - G, U, v). \tag{2.2.2.1}$$

P3) (Invariance under homotopy). If $H \in C(\overline{U} \times [0,1];E)$ is the function defined by $H(u,t) = u - S(u,t)$, where $S \in Q(\overline{U} \times [0,1],E)$ and $v \notin H(\partial\overline{U} \times [0,1])$, then $\deg(H(\cdot,t),U,v)$ is constant in $[0,1]$.

P4) (Invariance under translations)

$$\deg(\Phi, U, v) = \deg(\Phi - v, U, 0). \tag{2.2.2.2}$$

P5) (Constant in connected components of $E \setminus \Phi(\partial U)$). The function $\deg(\Phi, U, \cdot)$ is constant in each connected component of $E \setminus \Phi(\partial U)$.

P6) *(Additivity). If $U = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$ where U_i is a bounded open subset of E, for $i = 1, 2$ and $v \notin \Phi(\partial U_1) \cup \Phi(\partial U_2)$, then*

$$\deg(\Phi, U, v) = \deg(\Phi, U_1, v) + \deg(\Phi, U_2, v).$$

P7) *(Existence of Solution). If $v \notin \Phi(\overline{U})$, then $\deg(\Phi, U, v) = 0$.*

P8) *(Excision). Let $X \subset \overline{U}$ be a closed subset and $v \notin \Phi(X) \cup \Phi(\partial U)$. Then,*

$$\deg(\Phi, U, v) = \deg(\Phi, U \setminus X, v).$$

P9) *(Boundary Dependence). If Ψ is a S-compact perturbation of the identity such that $\Phi|_{\partial U} = \Psi|_{\partial U}$, then*

$$\deg(\Phi, U, v) = \deg(\Psi, U, v). \tag{2.2.2.3}$$

Proof. P1): Take $F = \langle v \rangle$ and $K \equiv 0$. Then, $\varphi = I$ and

$$\deg(I, U, v) = \deg(I|_{\overline{U} \cap F}, U \cap F, v) = \begin{cases} 1, & \text{if } v \in U \cap F, \\ 0, & \text{if } v \notin U \cap F, \end{cases}$$

where the last equality follows from P1) property of Brouwer Degree

P2): Let us set $r = \rho(v, \Phi(\partial U)) > 0$ and $\mathcal{U} = \{G \in Q(\overline{U}, E); \|K - G\|_\infty < \frac{r}{2}\}$. Let $G \in \mathcal{U}$ and consider the operators $K_1, G_1 \in C(\overline{U}; E)$ with finite rank such that

$$\|K - K_1\|_\infty, \|G - G_1\|_\infty < \frac{r}{4}, \tag{2.2.2.4}$$

whose the existence are guaranteed by Lemma 2.2.3. Let $\varphi_{\frac{r}{4}} : \overline{U} \to E$ be the bounded K_1-perturbation of the identity and $\psi_{\frac{r}{4}} : \overline{U} \to E$ the bounded G_1-perturbation of the identity. Take a subspace F of E containing $K_1(\overline{U}), G_1(\overline{U})$ and v and let us denote by $N := \dim F$. Consider β a basis for F and $\varphi_\beta : \overline{U_\beta} \to \mathbb{R}^N$, $\psi_\beta : \overline{U_\beta} \to \mathbb{R}^N$ the β versions of $\varphi_{\frac{r}{4}}$ and $\psi_{\frac{r}{4}}$, respectively.

By Definition 2.2.8,

$$\left.\begin{aligned} \deg(I - K, U, v) &= \deg(\varphi_{\frac{r}{4}}|_{\overline{U} \cap F}, U \cap F, v) = \deg(\varphi_\beta, U_\beta, v_\beta) \\ \deg(I - G, U, v) &= \deg(\psi_{\frac{r}{4}}|_{\overline{U} \cap F}, U \cap F, v) = \deg(\psi_\beta, U_\beta, v_\beta) \end{aligned}\right\}. \tag{2.2.2.5}$$

Let us define the homotopies

$$\begin{aligned} H : \overline{U} \times [0,1] &\to E \\ (u,t) &\mapsto t\varphi_{\frac{r}{4}}(u) + (1-t)\psi_{\frac{r}{4}}(u) \end{aligned}$$

and

$$\begin{aligned} H_\beta : \overline{U_\beta} \times [0,1] &\to \mathbb{R}^N \\ (u,t) &\mapsto t\varphi_\beta(u) + (1-t)\psi_\beta(u). \end{aligned}$$

Observe that for each fixed $t \in [0,1]$, the function $H_\beta(\cdot, t) : \overline{U_\beta} \to \mathbb{R}^N$ is the β version of the function $H(\cdot, t) : \overline{U} \to E$, so by Lemma 2.2.1 we have $v_\beta \notin H(\partial(U_\beta), t)$, for each $v \notin H(\partial U, t)$. Let us show that $v \notin H(\partial U, [0,1])$. Take $u \in \partial U$ and $t \in [0,1]$. Since $r = \rho(v, \Phi(\partial U))$, we have

$$\|(I - K)u - v\| \geq r.$$

Moreover, (2.2.2.4) implies

$$\|(I-G_1)u-(I-G)u\| \le r/4 \text{ and}$$
$$\|(I-K_1)u-(I-K)u\| \le r/4.$$

By adding

$$-t(I-K)u-(1-t)(I-K)u+(I-K)u=0$$

in $\|H(u,t)-v\|$ and rearranging the terms, we get

$$\begin{aligned}
(LHS) &:= \|H(u,t)-v\| \\
&= \|t(I-K_1)u+(1-t)(I-G_1)u-v\| \\
&= \|t(I-K_1)u+[-t(I-K)u-(1-t)(I-K)u+(I-K)u]+ \\
&+ (1-t)(I-G_1)u-v\| \\
&= \|[(I-K)u-v]+t[(I-K_1)u-(I-K)u]+(1-t)[(I-G_1)u-(I-K)u]\|.
\end{aligned}$$

Similarly, by adding

$$(1-t)(I-G)u-(1-t)(I-G)u=0$$

in the last expression and rearranging the therms, we obtain

$$\begin{aligned}
(LHS) &= \|[(I-K)u-v]+t[(I-K_1)u-(I-K)u]+(1-t)[(I-G_1)u-(I-K)u]+ \\
&+ [(1-t)(I-G)u-(1-t)(I-G)u]\| \\
&= \|[(I-K)u-v]+t[(I-K_1)u-(I-K)u]+(1-t)[(I-G_1)u-(I-G)u]+ \\
&+ [(1-t)[(I-G)u-(I-K)u]\|
\end{aligned}$$

which combined with the triangular inequality gives us the following inequality

$$\begin{aligned}
(LHS) &\ge \|(I-K)u-v\|-t\|(I-K_1)u-(I-K)u\| \\
&-(1-t)\|(I-G_1)u-(I-G)u\|-(1-t)\|(I-G)u-(I-K)u\| \\
&\ge r-t\frac{r}{4}-(1-t)\frac{r}{4}-(1-t)\frac{r}{2}=3\frac{r}{4}-(1-t)\frac{r}{2}\ge 3\frac{r}{4}-\frac{r}{2}=\frac{r}{4}>0.
\end{aligned}$$

Since u was taken arbitrarily in ∂U, we can conclude that $v\notin H(\cdot,t)(\partial U)$. Consequently, $v\notin H(\partial(U), [0,1])$, whence we infer by invariance under homotopy property for Brouwer degree that

$$\begin{aligned}
\deg(\varphi, U, v) &= \deg(H(\cdot,1),U,v) \\
&= \deg(H(\cdot,0),U,v) \\
&= \deg(\psi, U, v).
\end{aligned}$$

which implies (2.2.2.1) after using (2.2.2.5).

P3): By P2), $f(t):=\deg(H(\cdot,t),U,v)$ is locally constant in $[0,1]$ and so continuous. Since $f([0,1])\subset\mathbb{Z}$ and $f([0,1])$ is connected, it follows that f is constant;

P4): Let $\varphi_{\frac{r}{2}} : \overline{U} \to E$ be the bounded $K_{\frac{r}{2}}$-perturbation of the identity and $\varphi_\beta : \overline{U_\beta} \to \mathbb{R}^N$ the β version of $\varphi_{\frac{r}{2}}$. Then,

$$\begin{aligned}
\deg(\Phi, U, v) &= \deg(\varphi_{\frac{r}{2}}|_{\overline{U} \cap F}, U \cap F, v) \\
&= \deg(\varphi_\beta, U_\beta, v_\beta) \\
&= \deg(\varphi_\beta - v_\beta, U_\beta, 0) \\
&= \deg((\Phi - v)_\beta, U_\beta, 0) \\
&= \deg((\varphi_{\frac{r}{2}} - v)|_{\overline{U} \cap F}, U \cap F, 0) \\
&= \deg(\Phi - v, U, 0).
\end{aligned}$$

which proves (2.2.2.2).

P5): Let C be a connected component of the set $\Phi(\partial U) \setminus E$ and $f : C \to \mathbb{Z}$ the function defined by $f(v) = \deg(\Phi, U, v)$. Given $p \in C$, consider $g : \overline{U} \to E$ defined by $g(u) = \Phi(u) - (v - p)$.

So it follows from P4) that

$$\begin{aligned}
\deg(\Phi, U, v) &= \deg(\Phi - v, U, 0) = \deg(\Phi - (v - p), U, p) \\
&= \deg(g, U, p) \qquad (2.2.2.6)
\end{aligned}$$

holds for any $v \in C$.

Now, if $v \in E$ is such that $\|p - v\| \leq r/2$, then $\|\Phi - g\|_\infty \leq r/2$ and consequently by P2)

$$\deg(g, U, p) = \deg(\Phi, U, p). \qquad (2.2.2.7)$$

Thus, by combining (2.2.2.6) and (2.2.2.7), we conclude that f is locally constant and so continuous. Since C is a connected set, we get that $f(C)$ is a connected set in $\mathbb{Z}$, so f is constant.

P6): Let $K_{r/2} : \overline{U} \to E$ be the continuous operator as in Definition 2.2.8, that is, $K_{r/2}(\overline{U}) \subset F$ with $\dim F < \infty$ such that $\|K - K_{r/2}\|_\infty < r/2$. So $U \cap F = (U_1 \cap F) \cup (U_2 \cap F)$ and by the definition of Leray-Schauder degree,

$$\begin{aligned}
\deg(\Phi, U, v) &= \deg(\varphi_{\frac{r}{2}}|_{\overline{U} \cap F}, U \cap F, v) \\
&= \deg(\varphi_{\frac{r}{2}}|_{\overline{U} \cap F}, U_1 \cap F, v) + \deg(\varphi_{\frac{r}{2}}|_{\overline{U} \cap F}, U_2 \cap F, v) \\
&= \deg(\Phi, U_1, v) + \deg(\Phi, U_2, v),
\end{aligned}$$

this ends the proof.

P7): The definition of Leray-Schauder degree says

$$\deg(\Phi, U, v) = \deg(\varphi_{r/2}|_{\overline{U} \cap F}, U \cap F, v), \qquad (2.2.2.8)$$

where $\varphi_{r/2}$ is defined as in Definition 2.2.8.

On the other hand, $v \notin \varphi_{r/2}(\overline{U})$ because

$$\|v - \varphi_{r/2}(u)\| \geq \|\Phi(u) - v\| - \|\varphi_{r/2}(u) - \Phi(u)\| \geq r - \frac{r}{2}$$

and so the existence of solution property of the Brouwer degree implies

$$\deg(\varphi_{\frac{r}{2}}|_{\overline{U} \cap F}, U \cap F, v) = 0.$$

Thus, (2.2.2.8) results in

$$\deg(\Phi, U, v) = 0,$$

as we wanted.

P8): Observe that $(U \setminus X) \cap F = (U \cap F) \setminus X$ and since F is a closed subset of E, $\overline{U \setminus X} \cap F = \overline{(U \cap F) \setminus X}$ and so

$$\begin{aligned}
\deg(\Phi, U, v) &= \deg(\varphi_{\frac{r}{2}} j_{\overline{U} \cap F}, (U \cap F), v) \\
&= \deg(\varphi_{\frac{r}{2}} j_{\overline{U} \cap F}, (U \cap F) \setminus X, v) \\
&= \deg(\varphi_{\frac{r}{2}} j_{\overline{U} \cap F}, (U \setminus X) \cap F, v) \\
&= \deg(\varphi_{\frac{r}{2}} j_{\overline{U \setminus X} \cap F}, (U \setminus X) \cap F, v) \\
&= \deg(\Phi, U \setminus X, v),
\end{aligned}$$

which ends the proof.

P9): The homotopy $H(u,t) : \overline{U} \times [0,1] \to E$ defined by $H(u,t) = t\Phi(u) + (1-t)\Psi(u)$ is such that $H(u,t) = t\Phi(u) + (1-t)\Phi(u) = \Phi(u) \neq v$ for all $u \in \partial U$. So by invariance under homotopy property of Leray-Schauder degree, (2.2.2.3) holds. $\square$

Let E be a Banach space, $[a,b] \subset \mathbb{R}$, an operator $K : [a,b] \times E \to E$ and $\mathcal{U}$ a bounded open subset of $[a,b] \times E$. Consider in $[a,b] \times E$ the norm $\|(\lambda,u)\| = (|\lambda|^2 + \|u\|^2)^{1/2}$ and denote by

$$\begin{aligned}
\mathcal{U}_\lambda &:= \{u \in E; (\lambda,u) \in \mathcal{U}\}, \\
\Phi_\lambda(u) &:= u - K(\lambda,u).
\end{aligned}$$

Under these assumptions, we have the following invariance under homotopy.

Theorem 2.2.1. *Assume K is a compact operator and that the equation*

$$\Phi(\lambda,u) := u - K(\lambda,u) = 0 \tag{2.2.2.9}$$

does not admit zeros in $\partial\mathcal{U}$. Then

$$deg(\Phi_\lambda, \mathcal{U}_\lambda, 0) \text{ is constant in } \lambda \in [a,b]. \tag{2.2.2.10}$$

By the compactness of $[a,b]$, in order to prove Theorem 2.2.1, it is sufficient to prove the following lemma.

Lemma 2.2.5. *Let $\lambda_0 \in [a,b]$. Then, there exists $\varepsilon > 0$ such that*

$$deg(\Phi_\lambda, \mathcal{U}_\lambda, 0) \quad \text{constant for all } \lambda \text{ such that } |\lambda - \lambda_0| < \varepsilon.$$

Indeed, by using this lemma for each $\lambda \in [a,b]$, we deduce that there exist $\varepsilon(\lambda) > 0$ such that

$$deg(\Phi_\lambda, \mathcal{U}_\lambda, 0) \quad \text{constant for all } \lambda \text{ such that } |\lambda - \lambda_0| < \varepsilon(\lambda).$$

In fact, the claim of Lemma 2.2.5 means that the function $\lambda \mapsto deg(\Phi_\lambda, \mathcal{U}_\lambda, 0)$ is locally constant, and since the degree is an integer, it follows that the claim of Theorem 2.2.1 holds.

Our proof of Lemma 2.2.5 is based on the proof of Theorem 4.1 of [2]. Before doing it, consider the following remark.

Remark 2.2.4 (Relating ∂U_λ and ∂U). *Let $u \in \partial U_\lambda$, then, there exist sequences (u_n) in U_λ and $(v_n) \subset (U_\lambda)^c$ converging to u. Thus $((\lambda, u_n))$ and $((\lambda, v_n))$ are sequences in U and $(U)^c$, respectively, converging to (λ, u), i.e., $(\lambda, u) \in \partial U$.*

Remark 2.2.5. *Observe that by assuming the hypotheses of Theorem 2.2.1, it follows by the above remark that the degree $deg(\Phi_\lambda, U_\lambda, 0)$ is well defined.*

Proof. Fix $\lambda_0 \in (a, b)$ and let S_{λ_0} denote the set $\{u \in U_{\lambda_0} ;\ \Phi(\lambda_0, u) = 0\}$. By the compactness of K, it follows that S_{λ_0} is compact and since ∂U does not contain any solution of $\Phi(\lambda, u) = 0$, we imply in light of Remark 2.2.4 that $S_{\lambda_0} \cap \partial(U_{\lambda_0}) = \emptyset$. Thus, there exists an open neighbourhood O_{λ_0} of S_{λ_0} and $\varepsilon > 0$ such that

$$[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \times O_{\lambda_0} \subset U.$$

Moreover, we claim that if ε is sufficiently small, one holds

$$\{(\lambda, u);\ \Phi(\lambda, u) = 0,\ \lambda_0 - \varepsilon \le \lambda \le \lambda_0 + \varepsilon\} \subset [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \times O_{\lambda_0}. \tag{2.2.2.11}$$

Indeed, suppose the contrary. Then there exist sequences $\varepsilon_n \to 0$ and (λ_n, u_n) such that

$$|\lambda_n - \lambda_0| \le \varepsilon_n,\quad \Phi(\lambda_n, u_n) = 0,\ (\lambda_n, u_n) \notin [\lambda_0 - \varepsilon_n, \lambda_0 + \varepsilon_n] \times O_{\lambda_0}.$$

By the compactness of K we can assume that, up to a subsequence, $(\lambda_n, u_n) \to (\lambda_0, u_0)$. By continuity, $\Phi(\lambda_0, u_0) = \lim \Phi(\lambda_n, u_n) = 0$ and so $u_0 \in S_{\lambda_0}$. On the other hand, since $(\lambda_n, u_n) \notin [\lambda_0 - \varepsilon_n, \lambda_0 + \varepsilon_n] \times O_{\lambda_0}$ and $\lambda_0 \in [\lambda_0 - \varepsilon_n, \lambda_0 + \varepsilon_n]$ for all n, we must have $u_0 \notin O_{\lambda_0}$. But $(\lambda_0, u_0) \in S_{\lambda_0} \subset O_{\lambda_0}$ and so we get a contradiction.

Observe that (2.2.2.11) shows that Φ is an admissible homotopy in $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \times O_{\lambda_0}$ and so, by the standard homotopy invariance P3), we deduce that

$$deg(\Phi_\lambda, O_{\lambda_0}, 0) \equiv \text{constant},\ \forall\, \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]. \tag{2.2.2.12}$$

Furthermore, (2.2.2.11) implies that $S_\lambda \subset O_{\lambda_0}$ for all $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$. Now, observe that

$$U_\lambda \setminus (\partial(S_\lambda) \cup \partial(O_{\lambda_0})) = \left[U_\lambda \setminus \left(\overline{O_{\lambda_0} \setminus S_\lambda}\right)\right] \cup [O_{\lambda_0} \setminus S_\lambda].$$

By applying the properties of additivity and excision, we deduce that

$$deg(\Phi_\lambda, U_\lambda, 0) = deg(\Phi_\lambda, O_{\lambda_0}, 0),\ \forall\, \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]. \tag{2.2.2.13}$$

Finally, by combining (2.2.2.12) and (2.2.2.13), we conclude that the claim of Lemma 2.2.5 holds. The case where $\lambda_0 \in \{a, b\}$ is analogous. □

2.2.3 The Index of Isolated Solutions

For isolated solutions, there is the concept of index of an isolated solution, which is a limit of the degree in certain sense that will be clear soon. The index of an isolated solution satisfies an identity that is sometimes called Leray-Schauder formula. As we will see in the next chapter, this formula is useful in proving the existence of bifurcation points.

Let $K \in Q(\overline{U}; E)$, Φ the K-perturbation of identity and $v \notin \Phi(\partial U)$. Suppose $u \in U$ is an isolated solution of $\Phi(u) = v$, that is, there exists $r > 0$ such that u is the only solution of $\Phi(u) = v$ in $B_r(u) \subset U$. Then, by applying the excision property with the closed set $\overline{(B_r(u) \setminus B_\varepsilon(u))}$, we have

$$\begin{aligned}
\deg(\Phi, B_r(u), v) &= \deg(\Phi, B_r(u) \setminus \overline{(B_r(u) \setminus B_\varepsilon(u))}, v) \\
&= \deg(\Phi, B_\varepsilon(u), v),\ \forall\, 0 < \varepsilon < r.
\end{aligned} \tag{2.2.3.1}$$

Remark 2.2.6. *The equality (2.2.3.1) implies that*

$$\lim_{\varepsilon \to 0} \deg(\varphi, B_\varepsilon(u), v) = \deg(\varphi, B_r(u), v).$$

Thus, we define the Leray Schauder Index of an isolated solution of φ in u relative to the point v, by

$$i(\varphi, u, v) = \lim_{\varepsilon \to 0} \deg(\varphi, B_\varepsilon(u), v). \tag{2.2.3.2}$$

Before enunciate the Leray-Schauder formula, we need the following lemma.

Proposition 2.2.6. *Let E be a Banach space and $K \in Q(\overline{U}; E)$, where $U \subset E$ is a neighbourhood of the origin. Assume that K is differentiable at 0. Then $K'(0) : E \to E$ is a compact linear operator.*

Proof. See Proposition 3.5.2 in Kesavan [23]. □

Lemma 2.2.6. *Let U be an open neighbourhood of the origin and $K \in Q(\overline{U}; E)$ differentiable at the origin such that $K(0) = 0$. If 1 is not a characteristic value of $K'(0)$, then 0 is an isolated solution of $(I - K)u = 0$.*

Proof. It follows from Proposition 2.2.6 that $K'(0)$ is a compact linear operator. Suppose by contradiction that 0 is not an isolated solution of $I - K$, that is, there exists a sequence (v_n) in $E \setminus \{0\}$ converging to 0 and satisfying $v_n = Kv_n$. Then,

$$\frac{v_n}{\|v_n\|} - K'(0)\frac{v_n}{\|v_n\|} = \frac{K(0 + v_n) - K(0) - K'(0)v_n}{\|v_n\|} \to 0$$

on the other hand, by the compactness of $K'(0)$ and the fact that $v_n\|v_n\|^{-1}$ is a bounded sequence, it follows that there exists $w \in E$ such that

$$K'(0)\frac{v_n}{\|v_n\|} \to w, \text{ up to a subsequence,}$$

and so $v_n\|v_n\|^{-1} \to w \neq 0$. Thus by the continuity of $K'(0)$, we obtain $w = K'(0)w$. This contradicts the hypothesis that 1 is not a characteristic value of $K'(0)$. □

Theorem 2.2.2 (Leray-Schauder Formula). *Let E be a Banach space, U a bounded open subset of E and $K \in Q(\overline{U}; E)$ differentiable at the origin such that $K(0) = 0$. If 1 is not a characteristic value of $K'(0)$, then*

$$i(\varphi, 0, 0) = (-1)^\beta,$$

where $\varphi := I - K$ and β is the sum of the algebraic multiplicities of the characteristic values of $K'(0)$ contained in the interval $(0, 1)$.

Proof. See Proposition 3.5.3 in Kesavan [23]. The original one appears in [28]. □

Remark 2.2.7. *The Leray-Schauder Formula is also known as the calculation of the index via linearization, because it depends only on the "linear part" $K'(0)$ of K.*

2.2.4 Application

As we mentioned before, one of the motivations for constructing the Leray-Schauder degree is to obtain a tool that allows us to get information about the existence of solutions for equations of the type $\Phi(u) = 0$, where $\Phi(u) = u - K(u)$, $K : \overline{U} \to E$ is a compact operator and U is an open bounded subset of an arbitrary Banach space E. So, the following example illustrate an application of the Leray-Schauder degree in studying existence of solutions for a Dirichlet problem, whose formulation involves an operator acting in the infinite dimensional Banach space $E = L^2(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^N$.

Theorem 2.2.3 (Theorem 3.23 of [2]). *Let Ω be a bounded open subset of $\mathbb{R}^N$ and consider the following boundary value problem*

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.4.1}$$

where $f : \overline{\Omega} \times \mathbb{R} \mapsto \mathbb{R}$ is a locally Hölder continuous functions satisfying

$$\lim_{|s| \to \infty} \frac{f(x,s)}{s} = 0, \text{ uniformly with respect to } x \in \overline{\Omega}. \tag{2.2.4.2}$$

Then, (2.2.4.1) has a classical solution.

Proof. Given $\varepsilon > 0$, it follows from (2.2.4.2) that exists $s_0 > 0$ such that

$$\left| \frac{f(x,s)}{s} \right| < \varepsilon, \ \forall\, s \geq s_0 \text{ and } \forall\, x \in \overline{\Omega},$$

so

$$|f(x,s)| \leq s\varepsilon \ \forall\, s \geq s_0, \text{ and } \forall\, x \in \overline{\Omega}.$$

Noting that

$$s \leq |s| + s_0 \ \forall\, s \in \mathbb{R} \Rightarrow s\varepsilon \leq |s|\varepsilon + s_0\varepsilon \ \forall s \in \mathbb{R},$$

we obtain

$$|f(x,s)| \leq C_\varepsilon + |s|\varepsilon, \ \forall\, s \in \mathbb{R} \text{ and } \forall\, x \in \overline{\Omega}, \tag{2.2.4.3}$$

where $C_\varepsilon = s_0\varepsilon$.
Then, for each fixed $\varepsilon \leq 1$, we have

$$|f(x,s)| \leq C_1 + |s|, \ \forall\, s \in \mathbb{R} \text{ and } \forall\, x \in \overline{\Omega},$$

that is, for each fixed $\varepsilon \leq 1$, the function f satisfies the hypothesis (5.3.1.4) of Theorem 5.3.9 for $g \equiv C_1$ and $p = 2$ and the hypothesis (5.4.0.6) of Theorem (5.4.5) for $a_1 \equiv C_1 > 0$, $a_2 \equiv 1$ and $p = 2$. So the Nemytskii operator F of the function f is a continuous and bounded function from $L^2(\Omega)$ to $L^2(\Omega)$.

Let $S : L^2(\Omega) \to L^2(\Omega)$ be the solution operator of the problem

$$\begin{cases} -\Delta u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.4.4}$$

whose the existence is guaranteed by Theorem 5.4.4, in Appendix. Moreover, Theorem 5.4.4 says that S is a compact linear operator. Since F is bounded and continuous, it follows that $S \circ F : L^2(\Omega) \to L^2(\Omega)$ is a compact operator. Note that given $v \in L^2(\Omega)$, we have that $S(F(v))$ is the solution operator of the problem

$$\begin{cases} -\Delta u = f(x,v) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us define the homotopy

$$\begin{aligned} H : [0,1] \times L^2(\Omega) &\to L^2(\Omega) \\ (t,u) &\mapsto u - tS(F(u)). \end{aligned}$$

Observe that u is a weak solution of (2.2.4.1) if, and only if, $H(1,u) = 0$, because S is the solution operator of (2.2.4.4) and

$$F(u)(x) = f(x,u(x)).$$

We claim that H is admissible, that is, there exists $R > 0$ such that $H(t,\partial B_R(0)) \not\ni 0$ for all $t \in [0,1]$. Indeed, suppose there is no such $R > 0$. Hence, there would exist a sequence $(u_n) \in L^2(\Omega)$ and $(t_n) \in [0,1]$ satisfying $\|u_n\|_2 > n$ and $u_n = t_n S(F(u))$. By the linearity of S, we can say that $u_n = S(t_n F(u))$, that is,

$$\int_\Omega \nabla u_n \nabla \varphi = \int_\Omega t_n f(x,u_n)\varphi, \quad \forall \varphi \in H_0^1(\Omega).$$

By taking $\varphi = u_n$ as a test function and using (2.2.4.3), we get

$$\int_\Omega |\nabla u_n|^2 = t_n \int_\Omega f(x,u_n)u_n \leq \int_\Omega |f(x,u_n)u_n| \leq C_\varepsilon \int_\Omega |u_n| + \varepsilon \int_\Omega |u_n|^2,$$

that is,

$$\int_\Omega |\nabla u_n|^2 \leq C_\varepsilon \|u_n\|_1 + \varepsilon \|u_n\|_2^2.$$

But, since $L^2(\Omega) \hookrightarrow L^1(\Omega)$, we have $\|u_n\|_1 \leq D\|u_n\|_2$, for some constant D. Then, by calling $\tilde{C}_\varepsilon := DC_\varepsilon$, we get

$$\int_\Omega |\nabla u_n|^2 \leq \tilde{C}_\varepsilon \|u_n\|_2 + \varepsilon \|u_n\|_2^2.$$

On the other hand, by using Theorem 5.5.1,

$$\lambda_1 \|u_n\|_2^2 \leq \int_\Omega |\nabla u_n|^2$$

and consequently

$$\lambda_1 \|u_n\|_2^2 \leq \tilde{C}_\varepsilon \|u_n\|_2 + \varepsilon \|u_n\|_2^2,$$

that is

$$\|u_n\|_2 \leq \frac{\tilde{C}_\varepsilon}{\lambda_1 - \varepsilon}$$

for $\varepsilon < \min\{1, \lambda_1\}$, which contradicts $\|u_n\|_2 \to \infty$.

Thus, for some $R > 0$ we have $H(t,\partial B_R(0)) \not\ni 0$, for all $t \in [0,1]$, whence degree

$\deg(I - t(S \circ F), B_R(0), 0)$ is well defined for all $t \in [0,1]$. By homotopy invariance property, we obtain

$$\begin{aligned}
\deg(I - S \circ F, B_R(0), 0) &= \deg(H(1, \cdot), B_R(0), 0) \\
&= \deg(H(0, \cdot), B_R(0), 0) \\
&= \deg(I, B_R(0), 0) \\
&= 1,
\end{aligned}$$

which implies, after using the solution existence property of the Leray-Schauder degree, that there exists $u \in L^2(\Omega)$ such that $u = S(F(u))$, that is, u is a weak solution of (2.2.4.1). By Theorem 5.4.5, in Appendix, u is a classical solution. □

2.3 Final comments

In this chapter, we raised the background in Degree Theory which is necessary to the study of the bifurcation results of the next chapter. The above application, already illustrate the power of the Leray-Schauder degree in solving problems that can be formulated by therms of an operator that is a compact perturbation of the identity. This is an existence result for a sublinear problem. In the next chapter, we will use the Leray-Schauder degree theory to prove Theorem A, which is our main result. By applying Theorem A, we will prove in Chapter 4 a global bifurcation existence result for a quasilinear problem.

Chapter 3

Global Bifurcation for General Compact Perturbations of the Identity

3.1 Introduction

In 1964, it was published the english version [24] of the russian book [25] of Mark Krasnosel'skii. This book contains a result that gives a sufficient condition for the existence of a bifurcation point from an eigenvalue problem. The literature attributes to this result the rank of the first bifurcation existence theorem. Seven years later, Paul H. Rabinowitz proved Theorem 1.3 of [33], sometimes called the "Global Bifurcation Alternative of Rabinowitz" that establishes a global feature of the bifurcation result introduced by Krasnosel'skii. Both results dealt with a problem of the type $(\lambda, u) = 0$. The fact that is a compact perturbation of the identity allows the use of the Leray-Schauder degree theory in the proofs, as done by the two authors. We emphasize that the authors assumed that K is not a general compact operator, but of the type (1.0.0.6) with $L(\lambda) = \lambda L$, where L is a linear compact operator. So they proved that if $\lambda_0 \in \mathbb{R}$ is a characteristic value of L with odd algebraic multiplicity, then $(\lambda_0, 0)$ is a bifurcation point. The argument is based on the fact that the oddness of the multiplicity implies, by using the Leray-Schauder Formula, implies that the index changes its sign when λ crosses λ_0.

The argument of the "index sign change" in the proof of Theorem 1.3 was explored by many authors later in order to establish more general existence bifurcation results. Julián López-Gómez (the most notable among them) besides having contributed in this sense by developing a generalization of the Leray-Schauder Formula based in a generalized algebraic multiplicity, also developed a great historical overview of the bifurcation theory in his book [30].

All bifurcation results developed by these authors (see Section 3.3) maintain the hypothesis on of having a linear (or homogeneous) part and are general bifurcation results, in the sense that its formulations involves more general operators than those that are associated to some specific differential problem. Although, Antonio Ambrosetti and Peter Hess [1] (1980) took advantage of this "index sign change" argument and applied it in order to obtain a global bifurcation result for an asymptotic linear elliptic eigenvalue problem. By constructing a solution operator, they managed to formulate the problem as $(\lambda, u) = 0$ where $(\lambda, u) = u$ $K(\lambda, u)$ and K is a compact operator. However, they not needed K to be as the operator G in [33], to guarantee the index sign change, but used the homotopy invariance of the degree to calculate the index and show that it changes its value (not necessarily the sign) when the parameter crosses a certain characteristic value λ_1 of the assymptotic associated problem, then deduced the existence of a global bifurcation from infinity in the problem. In the proof, the authors just mentioned that they were using an adaptation of the Theorem 1.3 of [33].

Later, David Arcoya, José Carmona and Benedetta Pellacci [4] (2001) established a global bifurcation result for a quasilinear elliptic problem by following the ideas of [1]. The authors used the existence result by Leray and Lions [27] (1965) and the uniqueness result by Artola [6] in order to obtain a solution operator that allows to formulate the problem as $(\lambda, u) = 0$, where $(\lambda, u) = u \quad K(\lambda, u)$, $K : [0, +\infty) \quad E \to E$ is a compact operator and E is some appropriated real Banach space (see the discussion about the choice of the space E in Chapter 1). As in [1], the proof did not required K to be as the operator G in [33] to ensure the index sign change, but the authors used the invariance under homotopy property and showed that the index changes its value when λ crosses a certain characteristic value λ_1 of the assymptotic associated problem, then they mentioned that the conclusion follows from the argument in [1].

So it aroused a demand for a new formulation of global bifurcation theorem involving a more general compact perturbation of the identity than that one in Rabinowitz [33] and that makes it possible of being applied to deduce global bifurcation existence for problems like those studied in [4] and [1]. Motivated by this lack, we propose the main theorem of this work: Theorem A. Our goal in this chapter is to demonstrate it and to deduce the Global Bifurcation Alternative of Rabinowitz as a corollary of it. Before it, let us present the bifurcation existence results by Krasnosel'skii and Rabinowitz and also Theorem B, which is inspired by Theorem 1 of [12].

3.2 The first bifurcation theorems

In the preface of [35], David H. Sattinger says "In analyzing the dynamics of a physical system governed by nonlinear equations the following questions present themselves: Are there equilibrium states of the system? How many are there? Are they stable or unstable? What happens as external parameters are varied? As parameters are varied, a given equilibrium may lose its stability (although it may continue to exist as a mathematical solution of the problem) and other equilibria or time periodic oscillations may branch off. Thus, bifurcation is a phenomenon closely related to the loss of stability in non linear physical systems". Also in the introduction of [24], Krasnosel'skii mentioned that the bifurcation theory concept arises from the stability theory, and the origin dates back to A.M. Lyapunov and Poincaré. Although, we begin here from the first well known definition of bifurcation point of Mark Krasnosel'skii [24] (1964). We point out that his definition differs from the modern one when it is formulated for eigenvalue problems in the form $u = \lambda Au$, while the modern definition is more general and it is formulated for abstract problems like $F(\lambda, u) = 0$.

For chronological reasons, we will present in the subsection 3.2.1 the Krasnosel'skii's concept of bifurcation and the modern bifurcation concept. In the subsection 3.2.2 we construct a real Banach space of the compact homogeneous operators which is the counter domain of the operator L, presented at Chapter 1. In the subsection 3.2.3 we enunciate Krasnosel'skii Theorem, which is the first well known bifurcation result in the literature, prove Lemma B, which is a generalization of Lema 2.1 of [24] and also Lemma C, which is a formalization of the statement made by Dai in the introduction of [12][1]. Finally,

[1]we refer to Dai's citation made in Chapter 1 of this present work.

in the subsection 3.2.4, we enunciate the Rabinowitz Global Bifurcation Alternative and then we prove Corollary B that states that Krasnosel'skii's Theorem (Theorem 3.2.1) is a consequence of Rabinowitz Theorem (Theorem 3.2.2).

3.2.1 The definition of bifurcation point of Krasnosel'skii and the modern one

Let $A : E \to E$ be an abstract operator such that $A(0) = 0$ and consider the problem of solving

$$\lambda A(u) = u. \tag{3.2.1.1}$$

Observe that $u = 0$ is a solution for all $\lambda \in \mathbb{R}$ which is called the trivial solution. One may ask if there exists any non trivial solution to (3.2.1.1). One way to answer this question is considering a neighbourhood U of a trivial solution $(\lambda, 0)$ and try to show that there exists a non trivial solution in U. In this sense, in [24] was proposed the following definition.

Definition 3.2.1 (Krasnosel'skii's bifurcation point definition). *The number $\lambda_0 \in \mathbb{R}$ is called a bifurcation point of the problem (3.2.1.1) if, for any $\varepsilon, \delta > 0$, there exists a characteristic value λ of the operator A such that $|\lambda - \lambda_0| < \varepsilon$ and such that this characteristic value has at least one eigenfunction u, that is, a nontrivial solution of (3.2.1.1) with norm less then δ.*

After Krasnosel'skii, the mathematicians began to study the bifurcation phenomenon in a context of more general problems than the eigenvalue problem. In this sense we present the following exposition.

A lot of PDE problems with Dirichlet boundary condition as

$$\begin{cases} u = u^p & , \\ u = 0 & \partial \ , \end{cases}$$

admits the trivial solution, that is, admits $u = 0$ as a solution. More generally, problems like

$$\begin{cases} Lu = f(u) & , \\ u = 0 & \partial \ , \end{cases}$$

where L is a differential operator satisfying $L0 = 0$ and f is a function satisfying $f(0) = 0$, admits the trivial solution. If we go beyond by adding a real parameter λ and considering problems like

$$\begin{cases} Lu = f(\lambda, u) & , \\ u = 0 & \partial \ , \end{cases}$$

such that $(\lambda, 0)$ is a solution for all $\lambda \in \mathbb{R}$, then we can ask if, by taking advantage of the easy (or trivial) existent curve of solutions $\{(\lambda, 0); \lambda \in \mathbb{R}\}$, it is possible to find nontrivial solutions (λ, u) with $u \neq 0$ as Krasnosel'skii done for eigenvalue problems. A natural way to do it, consists in investigating the behaviour of L and f near the curve of trivial solutions and this is what about the modern bifurcation theory is concerned.

A physical problem that motivates the study is, for example, the Euler Buckling, which deals with how an external force applied in a column induces different profiles of its deformation. A simple search with the words "Euler Buckling" in any science research platform gives several results about it.

Let us formalize the modern bifurcation concept.

Assume that X and Y are Banach spaces and consider the problem of finding (λ, u) in $\mathbb{R} \times X$ such that

$$F(\lambda, u) = 0, \tag{3.2.1.2}$$

where $F : \mathbb{R} \times X \to Y$ is an operator satisfying

$$F(\lambda, 0) = 0 \quad \forall \lambda \in \mathbb{R}.$$

We call $\{(\lambda, 0);\ \lambda \in \mathbb{R}\}$ the curve of the trivial solutions to (3.2.1.2) and naturally the set

$$\Sigma_F = \{(\lambda, u) \in \mathbb{R} \times X;\ u \neq 0,\ F(\lambda, u) = 0\}$$

is called the set of nontrivial solutions of (3.2.1.2).

So we have the following definition of bifurcation point for the problem (3.2.1.2).

Definition 3.2.2. *Let $\lambda_0 \in \mathbb{R}$. We say that $(\lambda_0, 0)$ is a bifurcation point from the curve of trivial solutions of the problem (3.2.1.2) when there exists a sequence $((\lambda_n, u_n))$ in Σ_F that converges to $(\lambda_0, 0)$.*

Observe that when $(\lambda_0, 0)$ is a bifurcation point, then $(\lambda_0, 0) \in \overline{\Sigma_F}$. So it is convenient to establish the following notation:

$$S_F := \overline{\Sigma_F}.$$

So $(\lambda_0, 0)$ is a bifurcation point of $F = 0$ if and only if $(\lambda_0, 0) \in S_F$.

Now, let E be a real Banach space, $F : \mathbb{R} \times E \to Y$ an operator as above an assume that $X = Y = E$ and define the operator $\tilde{F} : \mathbb{R} \times E \to E$ by

$$\tilde{F}(\lambda, u) = \begin{cases} \left(\lambda, \dfrac{u}{\|u\|^2}\right) & \text{if } u \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $(\lambda_0, 0)$ is a bifurcation point from the curve of trivial solutions of $\tilde{F}$, then there exists a sequence of solutions (λ_n, u_n) of $\tilde{F} = 0$ that converges to $(\lambda_0, 0)$. Observe that the sequence $z_n := u_n / \|u_n\|^2$ is such that $\|z_n\| \to +\infty$. So we get a sequence (λ_n, z_n) satisfying

$$F(\lambda_n, z_n) = 0,$$

such that $\lambda_n \to \lambda_0$ and $\|z_n\| \to +\infty$. This means that, by using results about the existence of bifurcation points from the curve of trivial solutions and the above change of variable, we can deduce the existence of points that satisfies a type of bifurcation that we call bifurcation from infinity, which leads us to the following definition.

Definition 3.2.3. *Let $(E, \|\ \|)$ be a real Banach space and $F : \mathbb{R} \times E \to E$ be an abstract operator. We say that $\lambda_\infty \in \mathbb{R}$ is a bifurcation point from infinity, of solutions of $F = 0$, when there exists a sequence (λ_n, z_n) of solutions of $F = 0$ such that $\lambda_n \to \lambda_\infty$ and $\|z_n\| \to \infty$.*

Remark 3.2.1. *Observe that the change of variable $z = u/\|u\|^2$ was introduced above as a motivation for the definition of bifurcation from infinity, although it is also play a role of work around with the restrictive feature of the Leray-Degree to impose boundedness on the domain Ω where the degree is calculated.*

In the next chapter we will prove the existence of a global bifurcation from infinity for a quasilinear problem by applying Theorem 3.4 and the above change of variable. Before presenting the results, the next subsection is dedicated to construct a Banach space of compact homogeneous operators which is the space where the family $L(\lambda)$ belongs.

3.2.2 The Banach Space of Homogeneous Compact Operators

Let $H(E)$ be the family of all homogeneous operators $H : E \to E$ of degree 1 equipped with the usual function operations of sum and product by scalar. Observe that $H(E)$ is a vector space.

Along this subsection, we will consider the following subspaces of $H(E)$.

$$H_f(E) := \left\{ H \in H(E);\ \sup_{E\setminus\{0\}} \left\| \frac{H(u)}{\|u\|} \right\| < \infty \right\}$$

and

$$H_c(E) := \{ H \in H(E);\ H \text{ is a compact operator} \}$$

Lemma 3.2.1. *The function*

$$\begin{aligned} \|\ \| : H &\to \mathbb{R} \\ H &\mapsto \|H\| = \sup_{E\setminus\{0\}} \left\| \tfrac{H(u)}{\|u\|} \right\| \end{aligned}$$

defines a norm in the subspace $H_f(E)$ of $H(E)$.

Proof. Observe that if $H = 0$, then $\|H\| = 0$. Conversely, if $\|H\| = 0$, then

$$\left\| \frac{H(u)}{\|u\|} \right\| \le \|H\| = 0 \Rightarrow H(u) = 0 \ \forall\, u \in E,$$

that is $H = 0$.

As a consequence of the properties of the supremmum, the norm $\|\ \|$ in $H_f(E)$ inherits the multiplicative property and triangular inequality from the norm $\|\ \|$ in E and so we conclude that $(H_f(E), \|\ \|)$ is a normed space. □

Remark 3.2.2. *As a consequence of the definition of the space $H_f(E)$, we have that*

$$\|H(u)\| \le \|H\|\|u\| \ \forall\, u \in E$$

for each $H \in H_f(E)$.

Lemma $\mathbf{A_0}$. *$H_f(E)$ is a Banach subspace of $H(E)$.*

Proof. The fact that $H_f(E)$ is a subspace of $H(E)$ is clear. In order to prove that $H_f(E)$ is a Banach space, consider a Cauchy sequence $(H_n)_n$ in $H_f(E)$. Observe that

$$\|H_n(u) - H_m(u)\| \le \|(H_n - H_m)(u)\| \le \|H_n - H_m\|\|u\|, \text{ for each } u \in E \tag{3.2.2.1}$$

and so $(H_n(u))$ is a Cauchy sequence in the Banach space E, thus there exists some $H(u) \in E$ such that

$$H_n(u) \to H(u) \ \text{ in } E.$$

Observe that the operator H defined by $u \mapsto H(u)$ is homogeneous as a consequence of the limit properties.

Let $\varepsilon > 0$. It follows from the fact that (H_n) is a Cauchy sequence that there exists $n_0 \in \mathbb{N}$ such that

$$\|H_n - H_m\|\|u\| < \varepsilon\|u\| \ \ n, m \ge n_0 \tag{3.2.2.2}$$

whence follows that

$$\|H_n(u) - H(u)\| = \|(H_n - H)(u)\| \leq \varepsilon\|u\| \quad n \geq n_0, \forall u \in E, \tag{3.2.2.3}$$

by taking the limit in (3.2.2.1) as $m \to +\infty$. This inequality implies that $H - H_{n_0} \in H_f(E)$ and so $H = (H - H_{n_0}) + H_{n_0} \in H_f(E)$. Also the inequality implies implies that $H_n \to H$ in $H_f(E)$, as we wanted. □

Lemma A. *$H_c(E)$ is a closed subspace of $H_f(E)$.*

Proof. First, let us show that $H_c(E)$ is a subspace of $H_f(E)$. Since the family of compact operators is a vector space, so is the family $H_c(E)$. Moreover, $H_c(E) \subset H_f(E)$. Indeed, suppose the contrary, that is, there exists $H \in H_c(E) \setminus H(E)$. Then there exists a sequence $(u_n)_n$ in $E \setminus \{0\}$ such that

$$\left\| \frac{H(u_n)}{\|u_n\|} \right\| > n.$$

Since H is homogeneous, it follows that

$$\left\| H\left(\frac{u_n}{\|u_n\|}\right) \right\| > n,$$

on the other hand, the sequence $(u_n/\|u_n\|)$ is bounded and by the compactness of H the sequence

$$\left(H\left(\frac{u_n}{\|u_n\|}\right) \right)_n$$

converges in E up to a subsequence, which is a contradiction and that leads us to conclude that $H_c(E)$ is a subspace of $H_f(E)$.

Consider a sequence $(H_n)_n$ in $H_c(E)$ such that $H_n \to H$ in $H_f(E)$ for some $H \in H_f(E)$. Let $(u_n)_n$ be a bounded sequence in E and $C > 0$ be a number such that $\|u_n\| \leq C$ for all n. Since $H_1 \in H_c(E)$, then there exists a subsequence $(u_{1;n})_n$ of (u_n) such that $(H_1(u_{1;n}))_n$ converges in E up to a subsequence and so $(H_1(u_{1;n}))_n$ is a Cauchy sequence. Since $(u_{1;n})_n$ is a bounded subsequence and $H_2 \in H_c(E)$, it follows that $(u_{1;n})_n$ admits a subsequence $(u_{2;n})_n$ such that $(H_2(u_{2;n}))_n$ converges in E up to a subsequence and hence $(H_2(u_{2;n}))_n$ is a Cauchy sequence. By proceeding recursively and choosing the diagonal sequence (v_n), defined by $v_n := (u_{n;n})_n$ we obtain that

$$(H_m(v_n))_n$$

is a Cauchy sequence for each $m \in \mathbb{N}$ and $\|v_n\| \leq C$. Let $\varepsilon > 0$ and take n_0, n_1 such that

$$\|H - H_{n_0}\| \leq \frac{\varepsilon}{3C}$$

and

$$\|H_{n_0}(v_i) - H_{n_0}(v_j)\| \leq \frac{\varepsilon}{3} \ \forall\, i, j \geq n_1.$$

Observe that

$$\begin{aligned} \|H(v_i) - H(v_j)\| &\leq \|H(v_i) - H_{n_0}(v_i)\| + \|H_{n_0}(v_i) - H_{n_0}(v_j)\| + \|H_{n_0}(v_j) - H(v_j)\| \\ &\leq \|H - H_{n_0}\|\|v_i\| + \frac{\varepsilon}{3} + \|H_{n_0} - H\|\|v_j\| < \frac{\varepsilon}{3C}C + \frac{\varepsilon}{3} + \frac{\varepsilon}{3C}C = \varepsilon, \end{aligned}$$

that is, $(H(v_n))$ is a Cauchy sequence in the Banach space E and so it converges up to a subsequence. Since $(v_n)_n$ is a subsequence of $(u_n)_n$, we deduce that $H \in H_c(E)$ as we wanted. □

Remark 3.2.3. *In particular, by combining the Lemma A_0 and Lemma A we deduce that $(H_c(E), \|\cdot\|)$ is a Banach space.*

3.2.3 Theorem of Krasnosel'skii and necessary conditions

In [24], Krasnosel'skii proved a necessary condition for a number λ_0 being a bifurcation point of the problem (3.2.1.1) (see Lemma 2.1 of [31]). This result was used by Rabinowitz in [33]. Although the problem treated by Rabinowitz is slightly more general than the problem studied by Krasnosel'skii (the difference is in the structure of the operator H, compare the operator H in Theorem 3.2.1 with the operator H in Theorem 3.2.2) so that the necessary condition, as formulated in [24], does not apply to it. In [33], the author just referenced the Krasnosel'skii's necessary condition, without justifying that it is possible to obtain the same result for his problem. Motivated by this lack, we're going to open a parenthesis in the exposition of the Krasnosel'skii's bifurcation approach by proving the following generalized version of the Krasnosel'skii's necessary condition that applies not only to (3.2.1.1), but also to the problem in [33].

Lemma B (Generalization of Krasnosel'skii's necessary condition). *Suppose that the operator $I - L(\lambda_0) : E \to E$ admits an inverse operator $(I - L(\lambda_0))^{-1} \in H_f(E)$. Then there exists a ball $B \subset E$ centered at 0, such that*

$$\Phi(\lambda, u) = 0$$

does not admit any nontrivial solution in B for λ lying in a interval $(\lambda_0 - \xi, \lambda_0 + \xi)$ where ξ is a positive number depending only on L and λ_0. In particular, $(\lambda_0, 0)$ is not a bifurcation point from the curve of trivial solutions of $\Phi(\lambda, u) = 0$.

Proof. Since there exists $(I - L(\lambda_0))^{-1} \in H_f(E)$, it follows by Remark 3.2.2 that

$$\|(I - L(\lambda_0))^{-1}u\| \le \|(I - L(\lambda_0))^{-1}\|\|u\| \text{ for all } u \in E. \tag{3.2.3.1}$$

Moreover, it follows from the continuity of $\lambda \mapsto L(\lambda)$ that it is possible to take a $\xi > 0$ sufficiently small such that

$$\|L(\lambda) - L(\lambda_0)\| < \frac{1}{3\|(I - L(\lambda_0))^{-1}\|} \text{ for all } \lambda \in (\lambda_0 - \xi, \lambda_0 + \xi) \tag{3.2.3.2}$$

The fact that $H(\lambda, u) = o(\|u\|)$ uniformly in each bounded interval of parameters λ implies that there exists a ball $B \subset E$ centered at zero such that

$$\|H(\lambda, u)\| \le \frac{\|u\|}{3\|(I - L(\lambda_0))^{-1}\|}, \ \forall \lambda \in [\lambda_0 - \xi, \lambda_0 + \xi], \ \forall u \in B.$$

Now, suppose that there exists $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ and $u \in E$ such that

$$\Phi(\lambda, u) = 0.$$

Then,

$$\begin{aligned}
\|u\| &= \|(I - L(\lambda_0))^{-1}(I - L(\lambda_0))u\| \\
&\le \|(I - L(\lambda_0))^{-1}\|\|(I - L(\lambda_0))u\| \\
&\le \|(I - L(\lambda_0))^{-1}\|\|[I - (L(\lambda_0) - L(\lambda))u] - L(\lambda)u\| \\
&\le \|(I - L(\lambda_0))^{-1}\|\|[I - (L(\lambda_0) - L(\lambda))u] - (\Phi(\lambda, u) - H(\lambda, u))\| \\
&\le \|(I - L(\lambda_0))^{-1}\|\|(u - \Phi(\lambda, u)) + (L(\lambda) - L(\lambda_0))u + H(\lambda, u)\| \\
&\le \|(I - L(\lambda_0))^{-1}\|\|(L(\lambda) - L(\lambda_0))u + H(\lambda, u)\| \\
&\le \|(I - L(\lambda_0))^{-1}\|\|L(\lambda) - L(\lambda_0)\|\|u\| + \|(I - L(\lambda_0))^{-1}\|\|H(\lambda, u)\| \\
&\le \frac{2}{3}\|u\|
\end{aligned}$$

which implies that $u = 0$ and the lemma is proved. □

Corollary A. *If $L(\lambda)$ is a family of linear operators and $\lambda_0 \notin r(L_0)$, then the same conclusion of Lemma B holds.*

Proof. Indeed, if $\lambda_0 \notin r(L_0)$, then $I - L(\lambda_0)$ is injective and since $L(\lambda) \in H_c(E)$, it follows (in particular) that $L(\lambda)$ is a compact linear operator and so $I - L(\lambda_0)$ admits a continuous inverse due to Theorem 5.3.3. Consequently $(I - L(\lambda_0))^{-1} \in H_f(E)$. □

Lemma C. *If $\lambda_0 \in \mathbb{R}$ is such that $L(\lambda_0)$ is a contraction, then the same conclusion of Lemma B holds.*

Proof. Lemma C is a corollary of the proof of Lemma B. Indeed, by Theorem 5.3.8, if $L(\lambda_0)$ is a contraction, then it holds the inequality (3.2.3.1). □

Remark 3.2.4. *By Theorem 5.3.3, $r(L_0)$ is a subset of the spectre, and by Theorem 5.3.1, the spectre minus $\{0\}$ is a discrete set. Hence, the set of all bifurcation points of $\Phi(\lambda, u) = 0$ is a discrete set as a consequence of Corollary A.*

Observe that Corollary A gives us the necessary condition: if $(\lambda_0, 0)$ is bifurcation point from the curve of trivial solutions of $\Phi(\lambda, u) = 0$, then $\lambda_0 \in r(L_0)$. This result assumes, additionally to the general hypotheses under the operator L (presented in Chapter 1), the hypothesis that $L(\lambda)$ is a family of linear operators. Although, the following lemma proves that this necessary condition also holds when the operator L satisfies only the general hypothesis.

Lemma D. *If $(\lambda_0, 0)$ is a bifurcation point from the curve of trivial solutions of $\Phi(\lambda, u) = 0$, then $\lambda_0 \in r(L_0)$.*

Proof. Let $(\lambda_0, 0)$ be a bifurcation point from the curve of trivial solutions of $\Phi(\lambda, u) = 0$. Then there exists a sequence of solutions (λ_n, u_n) of $\Phi(\lambda, u)$ in $\mathbb{R} \times (E \setminus \{0\})$ such that

$$(\lambda_n, u_n) \to (\lambda_0, 0).$$

Thus,

$$u_n = L(\lambda_n)u_n + H(\lambda_n, u_n) \ \forall n$$

and by diviving by $\|u_n\|$ we obtain

$$\frac{u_n}{\|u_n\|} = L(\lambda_n)\frac{u_n}{\|u_n\|} + \frac{H(\lambda_n, u_n)}{\|u_n\|}. \tag{3.2.3.3}$$

Observe that

$$\left\| L(\lambda_n)\frac{u_n}{\|u_n\|} - L(\lambda_0)\frac{u_n}{\|u_n\|} \right\| \leq \|L(\lambda_n) - L(\lambda_0)\| \to 0, \tag{3.2.3.4}$$

on the other hand, it follows from the compactness of the operator $L(\lambda_0)$ that there exists some $v \in E$ such that

$$L(\lambda_0)\frac{u_n}{\|u_n\|} \to v \text{ up to a subsequence.} \tag{3.2.3.5}$$

By combining (3.2.3.4) and (3.2.3.5), we deduce that

$$L(\lambda_n)\frac{u_n}{\|u_n\|} \to v \text{ up to a subsequence.}$$

Now, since $\lambda_n \to \lambda_0$, it follows that (λ_n) is a bounded sequence and so $H(\lambda_n, u_n)/\|u_n\| \to 0$, which implies that

$$\frac{u_n}{\|u_n\|} \to v \neq 0$$

up to a subsequence, by (3.2.3.3). Consequently we obtain that v satisfies

$$v = L(\lambda_0)v,$$

by passing to the limit when $n \to \infty$ in (3.2.3.3), as we wanted. □

Definition 3.2.4 (Continuous branch of eigenvectors). *Let $A : E \to E$ be a compact operator such that $A(0) = 0$ with Fréchet derivative at zero $A'(0)$. Assume that $\lambda_0 \in \mathbb{R}$ be a bifurcation point of $\Phi(\lambda, u) = 0$, where we are identifying the operators L and H from hypothesis H3) as $L(\lambda) = \lambda A'(0)$, $H(\lambda, u) = \lambda H(u)$ and $H(u) = A(u) - A'(0)u$ with $H(u) = o(\|u\|)$. Consider $\varepsilon > 0$ small enough such that*

$$(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \cap (r(L_0) \setminus \{\lambda_0\}) = \emptyset$$

and V be the subset of E constituted by the eigenvectors of the operator A that corresponds to a characteristic value in $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. Assume that there exists a ball $0 \in B \subset E$ such that each open neighbourhood U of $u = 0$ contained in B satisfies $\partial U \cap V \neq \emptyset$. In this situation we say that V form a continuous branch of eigenvectors corresponding to the bifurcation point λ_0.

Finally, we enunciate the bifurcation theorem of Krasnosel'skii.

Theorem 3.2.1 (Krasnosel'skii's Theorem). *If $A : E \to E$ is a compact operator with Fréchet derivative $A'(0)$ such that $A(0) = 0$, if $\lambda_0 \in r(L_0)$ is of odd algebraic multiplicity, then $(\lambda_0, 0)$ is a bifurcation point of $\Phi(\lambda, u) = 0$, where we are identifying the operators L and H from hypothesis H3) as $L(\lambda) = \lambda A'(0)$, $H(\lambda, u) = \lambda H(u)$ and $H(u) = A(u) - A'(0)u$ with $H(u) = o(\|u\|)$.*

Moreover, associated with λ_0 there exists a continuous branch of eigenvectors of the operator A.

Remark 3.2.5. *Observe that*

$$\begin{aligned}
\Phi(\lambda, u) = 0 \quad &\Leftrightarrow \quad u - [L(\lambda)u + H(\lambda, u)] = 0 \\
&\Leftrightarrow \quad u - [\lambda A'(0)u + \lambda H(u)] = 0 \\
&\Leftrightarrow \quad u - \lambda[A'(0)u + H(u)] = 0 \\
&\Leftrightarrow \quad u - \lambda A(u) = 0 \\
&\Leftrightarrow \quad u = \lambda A(u).
\end{aligned}$$

We will prove later that this theorem follows as a corollary of Theorem 3.2.2 (Global Bifurcation Alternative of Rabinowitz).

3.2.4 Global bifurcation alternative of Rabinowitz

Since we established the modern concept of bifurcation, we are able to state the Global bifurcation alternative of Rabinowitz. This name is attributed to Theorem 1.3 in [33] (1971), that we will enunciated and proved and after this the proof of Krasnosel'skii's bifurcation theorem (Theorem 3.2.1) will be obtained as a corollary of Rabinowitz result. In the next section we prove that the Global Bifurcation Alternative of Rabinowitz follows as a corollary of Theorem 3.4 of the Theorem 1.3 of [33].

In [33], Rabinowitz established the following remarkable result.

Theorem 3.2.2 (Global bifurcation alternative of Rabinowitz). *Suppose that the operators $L : \mathbb{R} \to H_c(E)$ and $H : \mathbb{R} \times E \to E$ of (1.0.0.6) are given by $L(\lambda) = \lambda L$, where L is a compact linear operator and $H : \mathbb{R} \times E \to E$ is a compact operator. If $\lambda_0 \in r(L_0)$ is of odd algebraic multiplicity, then there exists a maximal continuum $\mathcal{C}_0$ of $\mathcal{S}$ (connected and closed subset of $\mathcal{S}$) containing $(\lambda_0, 0)$ such that $\mathcal{C}_0$ satisfies, at least, one of the following (non-excluding) alternatives:*

i) $\mathcal{C}_0$ *is unbounded,*

ii) $\mathcal{C}_0 \cap \{(\lambda_1, 0)\} \neq \emptyset$ *for some* $\lambda_1 \neq \lambda_0$,

where

$$\mathcal{S} := \overline{\{(\lambda, u);\quad \Phi(\lambda, u) = 0,\ u \neq 0\}}$$

Remark 3.2.6. *By "maximal" we mean that* $\mathcal{C}_0$ *is not a proper continuum of any continuum of* $\mathcal{S}$.

Remark 3.2.7. *By "non-excluding" we mean that it is possible to occur simultaneously the alternatives i) and ii).*

Remark 3.2.8. *In particular, Theorem 3.2.2 implies that λ_0 is a bifurcation point because* $\mathcal{C}_0$ *is a continuum of* $\mathcal{S}$ *containing* $(\lambda_0, 0)$.

Consider the following claim.

Corollary B. *Theorem 3.2.1 of Krasnosel'skii is a corollary of Theorem 3.2.2 of Rabinowitz.*

Proof. Let $L : E \to E$, $H : \mathbb{R} \times E \to E$ and λ_0 as in Theorem 3.2.1 and let us denote by B_ε the open ball in E of radius ε and centered at $(\lambda_0, 0)$. Since $H(\lambda, u) = \lambda H(u)$ and $H(u) = 0(\|u\|)$ near $u = 0$, then Theorem 3.2.2 applies to the operator $\Phi(\lambda, u) = u - [L(\lambda)u + H(\lambda, u)]$ and so there exists a maximal subcontinumm $\mathcal{C}_0$ of $\mathcal{S}$ containing $(\lambda_0, 0)$ satisfying, at least, one of the alternatives i) and ii) of Theorem 3.2.2. Suppose that the conclusion of Theorem 3.2.1 does not hold. Since λ_0 is a bifurcation point, the only possibility is that there is no continuous branch of eigenvectors of A associated to the characteristic value λ_0. The idea is to obtain a contradiction with the connectedness of a certain subcontinumm of $\mathcal{C}_0$. Let $\delta > 0$ such that

$$(\lambda_0 - \delta, \lambda_0 + \delta) \cap (r(L_0) \setminus \{\lambda_0\}) = \emptyset.$$

Let $\mathcal{D}_0 \neq \{(\lambda_0, 0)\}$ be a proper continuum of $\mathcal{C}_0$, containing $(\lambda_0, 0)$, such that

$$\operatorname{proj} \mathcal{D}_0 \subset (\lambda_0 - \delta, \lambda_0 + \delta). \tag{3.2.4.1}$$

By the fact that there is no continuous branch of eigenvectors corresponding to λ_0, it follows that, independently on which of the two alternatives i) and ii) occurs, there exists an open neighbourhood U of $u = 0$ such that

$$[(\lambda_0 - \delta, \lambda_0 + \delta) \times U^c] \cap \mathcal{D}_0 \neq \emptyset \tag{3.2.4.2}$$

and

$$\partial U \cap V = \emptyset, \tag{3.2.4.3}$$

where V is the set of the eigenvalues of A that corresponds to some characteristic value lying in the interval $(\lambda_0-\delta,\lambda_0+\delta)$.

Observe that

$$[(\lambda_0-\delta,\lambda_0+\delta)\times\partial U]\cap\mathcal{D}_0=\emptyset, \tag{3.2.4.4}$$

on the contrary, if there would exist $(\lambda,u)\in[(\lambda_0-\delta,\lambda_0+\delta)\times\partial U]$, then $u\neq 0$ because U is a neighbourhood of $u=0$ and

$$u=\lambda A(u),$$

for some $\lambda\in(\lambda_0-\delta,\lambda_0+\delta)$, but since $\eta<\delta$, it follows that $u\in\partial U$ is an eigenvalue of A corresponding to the characteristic value $\lambda\in(\lambda_0-\delta,\lambda_0+\delta)$. By (3.2.4.3), this can not happen and so we conclude that (3.2.4.4) is true. Thus,

$$(\lambda_0,0)\in\mathcal{D}_0\cap[(\lambda_0-\delta,\lambda_0+\delta)\times U]\neq\emptyset$$

and

$$\mathcal{D}_0\cap[(\lambda_0-\delta,\lambda_0+\delta)\times U]^c\neq\emptyset,$$

because

$$\emptyset\overset{(3.2.4.2)}{\neq}([(\lambda_0-\delta,\lambda_0+\delta)\times U^c]\cap\mathcal{D}_0)\subset([(\lambda_0-\delta,\lambda_0+\delta)\times U]^c\cap\mathcal{D}_0)$$

and so by the connectedness of $\mathcal{D}_0$, it follows that

$$\begin{aligned}\emptyset &\neq \mathcal{D}_0\cap\partial([(\lambda_0-\delta,\lambda_0+\delta)\times U])\\ &= \mathcal{D}_0\cap\{[\partial(\lambda_0-\delta,\lambda_0+\delta)\times U]\cup[(\lambda_0-\delta,\lambda_0+\delta)\times\partial U]\}\\ &= (\mathcal{D}_0\cap[\partial(\lambda_0-\delta,\lambda_0+\delta)\times U])\cup(\mathcal{D}_0\cap[(\lambda_0-\delta,\lambda_0+\delta)\times\partial U])\end{aligned}$$

but

$$\mathcal{D}_0\cap[\partial(\lambda_0-\delta,\lambda_0+\delta)\times U]=\emptyset,$$

by (3.2.4.1) and

$$\mathcal{D}_0\cap[(\lambda_0-\delta,\lambda_0+\delta)\times\partial U]=\emptyset,$$

by (3.2.4.4). Thus we get the contradiction $\emptyset\neq\emptyset$ and the Corollary A is proved. □

Now, we will present a generalization of Theorem 3.2.2 whose the hypotheses are inspired by the unilateral bifurcation result given by Theorem 1 of Dai and Feng [12] (2019).

Theorem B (A type of Dai's Theorem on a strip). *Assume that $\lambda_0\in r(L_0)$ is isolated such that*

$$i(\ (\lambda_0-\eta,\ ),0)\neq i(\ (\lambda_0+\xi,\ ),0) \tag{3.2.4.5}$$

for sufficiently small positive numbers ξ and η. Then there exists a continuum $\mathcal{C}_0$ of

$$\mathcal{S}:=\overline{\{(\lambda,u)\in I\times(E\setminus\{0\});\ \ (\lambda,u)=0\}}$$

containing $(\lambda_0,0)$ such that $\mathcal{C}_0$ satisfies, at least, one of the following (non-excluding) alternatives:

i) $\mathcal{C}_{\lambda_0}$ *is unbounded;*

ii) $\mathcal{C}_{\lambda_0}$ *intercepts some* $(d,u)\in I\times E$ *where* d *is an extremity of the interval* I *(if* I *possesses some extremity) for some* $u\in E$ *or (not exclusive) intercepts some* $(\hat{\lambda},0)$ *with* $\hat{\lambda}\in I$*.*

In order to prove that Theorem 3.2.2 is a corollary of Theorem B, consider the following lemma.

Lemma 3.2.2. *Let* $L:\mathbb{R}\times E\to E$ *and* $H:\mathbb{R}\times E\to E$ *as in Theorem 3.2.2. If* $\lambda_0\in r(L_0)$ *is of odd algebraic multiplicity, then there exists a neighbourhood* $(\lambda_0-\delta,\lambda_0+\delta)$ *such that*

$$i(\Phi_{\underline{\lambda}},0)=-i(\Phi_{\overline{\lambda}},0)$$

for all $\lambda_0-\delta<\underline{\lambda}<\lambda_0<\overline{\lambda}<\lambda_0+\delta$*.*

Proof. The idea is to use the Leray-Schauder Formula. Let $\delta<\operatorname{dist}(r(L_0)\setminus\{\lambda_0\},\{\lambda_0\})$ and $\lambda_0-\delta<\underline{\lambda}<\lambda_0$. Then $\underline{\lambda}\notin r(L_0)$, in other words, the operator $T:E\to E$ defined by

$$T(u)=\Phi(\underline{\lambda},u)=\Phi_{\underline{\lambda}}(u)$$

is such that the operator

$$T'(0)=\overline{\lambda}L$$

does not admits 1 as a characteristic value. By Theorem 2.2.2, it follows that

$$i(\Phi_{\underline{\lambda}},0)=(-1)^{\beta}$$

where β is the sum the algebraic multiplicities of the characteristics values of $\overline{\lambda}L$ lying in the interval $(0,1)$, clearly, this corresponds to the sum of the characteristic values of the operator L lying in the interval $(0,\overline{\lambda})$.

On the other hand, since the only characteristic value of L lying in $(\underline{\lambda},\overline{\lambda})$ is λ_0, we deduce that

$$i(\Phi_{\overline{\lambda}},0)=(-1)^{\beta+m(\lambda_0)}\quad\text{where } m(\lambda_0):=\text{algebraic multiplicity of }\lambda_0,$$

by applying Theorem 2.2.2 to the operator $S:E\to E$ defined by

$$S(u)=\Phi(\overline{\lambda},u)=\Phi_{\overline{\lambda}}(u).$$

Since $m(\lambda_0)$ is odd, it follows that

$$i(\Phi_{\underline{\lambda}},0)=-i(\Phi_{\overline{\lambda}},0)\neq 0$$

and the lemma is proved. □

Let $L:\mathbb{R}\times E\to E$ and $H:\mathbb{R}\times E\to E$ as in Theorem 3.2.2. By taking $I=\mathbb{R}$ in Theorem B, we see that $\lambda_0\in r(L_0)$ is isolated and, by Lemma 3.2.2 the hypothesis (3.2.4.5) is also satisfied and so we just proved the following corollary.

Corollary C. *The Global Bifurcation Alternative of Rabinowitz (Theorem 3.2.2) is a corollary of Theorem B.*

In the next section we will prove Theorem A, which is our main result and it is a generalization of Theorem (3.2.4) which requires just compactness of the operator $K:I\times E\to E$, instead of requiring K to have some homogeneous part.

3.3 Further results and approaches that "bifurcate" from Rabinowitz theorem

There are a lot of results that "bifurcates" from Theorem 1.3 of [33] in the sense that their approaches are based on the main argument of the proof of Theorem 1.3, that is, by using the "index sign change" and taking a convenient open neighbourhood of the bifurcation point candidate, it is deduced the global alternatives of bifurcation. All of these results deals with an operator $G : \mathbb{R} \times E \to E$ in the form

$$G(\lambda, u) = L(\lambda)u + H(\lambda, u) \tag{3.3.0.1}$$

where

$$H(\lambda, u) = o(\|u\|) \text{ near } u = 0, \text{ uniformly on each compact interval of } \lambda. \tag{3.3.0.2}$$

and the operator L is well behaved in some sense. For example, Lopez Gomez in [30] (2001), studied the existence of bifurcation at a point $(\lambda_0, 0)$ for the problem

$$G(\lambda, u) = 0, \tag{3.3.0.3}$$

by assuming the hypotheses

HL1) $\lambda \mapsto L(\lambda)$ is a Fredholm operator with index 0 for each λ near λ_0,

HL2) $\lambda \mapsto L(\lambda)$ is a C^r function,

HL3) $(\lambda, u) \mapsto H(\lambda, u)$ is a C^k function,

where $r \geq n$ and $k \geq m$ and the numbers m and n can assume any value in the set $\{0, 1, 2\}$, depending on the result. Dai and Zhaosheng Feng [12] (2019), established some bifurcation results for the problem of solving

$$G(\lambda, u) = u,$$

by assuming

HD1) $L(\lambda) \in H_c(E)$ for all λ.

HD2) $\lambda \mapsto L(\lambda)$ is a continuous function.

So these results do not play a crucial role in the development of the main idea of this text, which is to obtain (as we will do in the next section by presenting Theorem A) a generalization of the Global Bifurcation Alternative of Rabinowitz that applies to general compact perturbations of the identity $\Phi : I \times E \to E$ defined by $\Phi(\lambda, u) = u - K(\lambda, u)$, where $K : I \times E \to E$ is a compact operator and I is a closed interval with non-empty interior, in order to deduce the existence of global bifurcation for a class of quasilinear problems. However, we will comment these results to show how broad can be the bifurcation theory.

Dancer proved that the statement of Theorem 3.2.2 of Rabinowitz is actually stronger, in Theorem 1 of [15] (1974). Later in [30] (2001), Lopez Gomez formulated a more general version of Dancer's theorem in therms of the index sign change. So we will enunciate the refinement of Lopez Gomez. In this case, the hypotheses $HL1$) and $HL2$) are substituted by the following.

Let J be an interval (possibly unbounded) and assume that $L(\lambda) : E \to E$ is an operator in the form

$$L(\lambda) := I - K(\lambda),$$

where $K(\lambda)$ is a continuous family of compact operators for each λ lying in J. Also we assume that $H : J \times E \to E$ is a compact operator on bounded sets satisfying the hypotheses $HL3$) with $k = 0$, and (3.3.0.2) uniformly in any compact interval of J. Let us denote

$$\Sigma(L) = \{\lambda \in (a, b);\ \dim N[L(\lambda)] \geq 1\}.$$

Under these conditions, Theorem 6.3.1 of [30] states that the following result holds for the problem of solving (3.3.0.3).

Theorem 3.3.1. *Assume that $\Sigma(L)$ is discrete in J and let $\mathfrak{C}$ be a bounded component of $\mathfrak{S} := \{(\lambda, u) \in J \times (E \setminus \{0\}); G(\lambda, u) = 0\}$. Then, the set*

$$D := \mathfrak{C} \cap \{(\lambda, 0) :\ \lambda \in \Sigma(L)\}$$

is finite, possibly empty. Moreover, if for some natural number $N \geq 1$, there exist $\lambda_1, \ldots, \lambda_N \in \Sigma(L)$ for which

$$D = \{(\lambda_1, 0), \ldots, (\lambda_N, 0)\}$$

then

$$\sum_{j=1}^{N} P(\lambda_j) = 0. \tag{3.3.0.4}$$

The object P in equation (3.3.0.4) is a function called parity map, assumes the values 1 and -1 and another properties that leads Lopez Gomez to prove the following corollary of Theorem 3.3.1.

Corollary 3.3.1. *Assume $J = \mathbb{R}$ and $\Sigma(L)$ discrete. Let $\lambda_0 \in \Sigma(L)$ such that $i(K(\lambda), 0)$ changes sign as λ crosses λ_0. Let $\mathfrak{C}$ denote the component of $\mathfrak{S}$ emanating from $(\lambda, 0)$ at $(\lambda_0, 0)$, whose existence is guaranteed by Theorem 6.2.1 of [30]. Then, one of the following non-excluding options occurs. Either*

i) $\mathfrak{C}$ is unbounded in $\mathbb{R} \times E$.

ii) There exists $\lambda_1 \in \Sigma(L) \setminus \{\lambda_0\}$ such that $(\lambda_1, 0) \in \mathfrak{C}$.

We quote another Rabinowitz's result that is widely studied in the literature, which is called the unilateral global bifurcation. So consider (3.3.0.1) in the context of [33], that is, $L(\lambda) := \lambda L$, where L is a linear operator, H is an operator satisfying (3.3.0.2) and consider the problem of solving $G(\lambda, u) = u$. Additionally to the hypothesis of Theorem 3.2.2, by assuming that λ_0 is a simple eigenvalue, that is, has geometric multiplicity equals 1, Theorem 1.25 of [33] states that

Theorem 3.3.2 (Theorem 1.25 of [33]). *$\mathfrak{C}_0$ posseses a subcontinumm in $K^{+}_{\zeta} \cup \{(\lambda_0, 0)\}$ and in $K^{-}_{\zeta} \cup \{(\lambda_0, 0)\}$ each of which intercept $(\lambda_0, 0)$ and ∂B_{ζ} for all $\zeta > 0$ sufficiently small.*

where $\mathcal{C}_0$ is the maximal continuum which the existence is guaranteed by Theorem 1.3 of [33] and

$$K^+_{\xi,\eta} = \{(\lambda,u) \in E;\ |\lambda - \lambda_0| < \xi,\ |\langle \ell, u\rangle| > \eta\|u\|\},$$
$$K^-_{\xi,\eta} = \{(\lambda,u) \in E;\ |\lambda - \lambda_0| < \xi,\ |\langle \ell, u\rangle| < -\eta\|u\|\}$$

where $\ell \in E'$ is the corresponding eigenvector of the adjoint operator L^T of L.

Moreover, it derives from these two subcontinua, two correspondent subcontinua $\mathcal{C}_0^+$ and $\mathcal{C}_0^-$, which satisfies the following result.

Theorem 3.3.3 (theorem 1.27 of [33]). *Each of $\mathcal{C}_0^+$ and $\mathcal{C}_0^-$ either satisfies the alternatives of Theorem 3.2.2 or (iii) contains a pair of points $(\lambda,u),(\lambda,-u)$, $u \neq 0$.*

Also

Theorem 3.3.4. *Each of $\mathcal{C}_0^+$ and $\mathcal{C}_0^-$ intercepts $(\mu,0)$ and either*

i) is unbounded, or

ii) intercepts $(\lambda_1,0)$, where $\lambda_0 \neq \lambda_1 \in r(L)$.

Although, [14] (2002) as well as López-Gómez in [30] (2001), pointed out that the theorems 3.3.3 and 3.3.4 contain gaps. López-Gómez solved the gap by presenting a more precise (and weaker) conclusion of Theorem 1.27 that actually holds. So let us enunciate it.

Let Y be a closed subspace of U such that

$$U = N[L(\lambda_0)] \oplus Y.$$

Under the same assumptions of Theorem 3.3.1, Theorem 6.4.3 of [30] states that

Theorem 3.3.5. *Assume that $\mathrm{r}(L)$ is discrete, $\lambda_0 \in \mathrm{r}(L)$ satisfies*

$$N[L(\lambda_0)] = span[\varphi_0]$$
$$\|\varphi_0\| = 1$$

and the index $i(K(\lambda),0,0)$ changes sign as λ crosses λ_0. Then, each of the components, $\mathcal{C}^+$ and $\mathcal{C}^-$, either satisfies the alternatives of Corollary 6.3.2 or contains a point

$$(\lambda,y) \in \mathbb{R} \times (Y \setminus \{0\})$$

where Y is the complement of $N[L(\lambda_0)]$ in U.

Moreover, Dancer exhibited a counterexample of Theorem 3.3.4, in [15].

The idea of working only with the hypothesis of the index sign change, in order to work around with the requirement of Theorem 1.3 of [33] for G to have a linear part, carries its power in allowing the deduction of the existence of global bifurcation for general compact perturbations of the identity, although, even by keeping the assumption about the operator G to have a linear part, López-Gómez [30] contributed with some results by adopting an approach based on a "generalized algebraic multiplicity". Among them we will enunciate Theorem 5.6.2 of [30]. Before it, let us introduce the following concept which was adopted by Lopez Gomez, in [30], besides the usual concept of bifurcation point.

Definition 3.3.1 (Nonlinear eigenvalue). *We say that λ_0 is a nonlinear eigenvalue of $L(\lambda)$ if $(\lambda_0, 0)$ is a bifurcation point of (3.3.0.3) from the curve of trivial solutions for any C^k function (for some $k \geq 2$) $(\lambda, u) \mapsto H(\lambda, u)$.*

Theorem 3.3.6. *Assume that the function $\lambda \mapsto L(\lambda)$ satisfies HL1), HL2) with $r \geq 1$, the operator $L(\lambda)$ has the form*

$$L(\lambda) := I - K(\lambda)$$

where $K(\lambda)$ is a family of compact linear operators and that $1 \leq \nu < r$, where ν is the order of λ_0 as an element of $\Sigma(L)$. Under these conditions, there exists

$$\eta \in \{-1, 1\}$$

such that

$$i(K(\lambda), 0) = \eta sgn(\lambda - \lambda_0)^{\chi[L(\lambda);\lambda_0]} \tag{3.3.0.5}$$

for each $0 < |\lambda - \lambda_0| < \delta$. In particular,

$$P[L(\lambda); \lambda_0] = 1$$

if and only if

$$\chi[L(\lambda); \lambda_0] \in 2\mathbb{N} + 1.$$

Therefore, λ_0 is a nonlinear eigenvalue of $L(\lambda)$ if, and only if, the parity of the crossing number of $L(\lambda)$ at λ_0 is 1. In other words, λ_0 is a nonlinear eigenvalue of $L(\lambda)$ only if an odd number of eigenvalues of $L(\lambda)$ cross the imaginary axis in $\mathbb{C}$ as λ passes through λ_0.

The number $\chi[L(\lambda); \lambda_0]$ is a generalization of the concept of algebraic multiplicity and $P[L(\lambda); \lambda_0]$ is a function called "parity of the crossing number", these are two concepts of the main objects studied by Lopez Gomez in [30].

By weakening the linearity hypothesis under the operator $L(\lambda)$ by assuming the general assumptions of Chapter 3, Guowei Dai in [11] (2015) and in [12] (2019), together with Zhaosheng Feng, established a unilateral bifurcation existence result analogous to Theorem 3.3.2.

We point out that the Theorem A is not the only one in the literature that proves the existence of global bifurcation by requiring just compactness of the operator K and some information about the index (or degree), in fact, Fleckinger [19] (2005) proved the following theorem.

Theorem 3.3.7. *Let E be a real Banach space, $K : \mathbb{R} \times E \to E$ a compact operator and (λ_0, u_0) a solution of (3.3.0.3). Suppose $\mathcal{U} \subset E$ is an open bounded set such that $u_0 \in \mathcal{U}$ and*

i) for fixed λ_0 there is no other solution in $\overline{\mathcal{U}}$,

ii) $deg(I - K(\lambda_0, \cdot), \mathcal{U}, 0) \neq 0$.

Then there exists a continuum $\mathcal{C}^+ \subset [\lambda_0, \infty) \times E$ of solutions of $K(\lambda, u) = 0$ with

$$(\lambda_0, u_0) \in \mathcal{C}^+$$

satisfying one of the following two alternatives holds:

a) $\mathcal{C}^+$ *is unbounded;*
or

b) $\mathcal{C}^+ \cap (\{\lambda_0\} \times E \setminus \overline{\mathcal{U}}) \neq \emptyset$.

Out of the general context, that is, when K is the solution operator of some problem involving a specific differential operator as the p-laplacian for example, there are lot of authors among which we cite Lee and Sim [26] (2006), Girg and Takáč, Dai and Ma [13] and Yang et al. [37].

3.4 Global bifurcation for general compact perturbations of the identity

Let us recall that the purpose of this chapter is to obtain a more general version of Theorem 3.2.2 that applies for general compact perturbations of the identity $\Phi(\lambda, u) = u - K(\lambda, u)$, without requiring K to have some linear part (as Theorem 3.2.2 does) or some homogeneous part (as Theorem 3.2.4 does). For We will denote by $\mathcal{B}_\varepsilon$ and B_ε the open balls in $\mathbb{E} := I \times E$ and E of radius ε and centered at $(\lambda_0, 0)$ and 0, respectively, where $I \subset \mathbb{R}$ is a closed (not necessarily bounded) interval.

So we will prove a type of Rabinowitz Theorem (Theorem 3.2.2) for problems like

$$\Phi(\lambda, u) = 0,$$

where $\Phi : I \times E \to E$ is an operator defined by

$$\Phi(\lambda, u) := u - K(\lambda, u),$$

$K : I \times E \to E$ is an abstract compact operator and I is a closed interval (not necessarily bounded) of $\mathbb{R}$.

Let us define

$$\Phi_\lambda(u) = \Phi(\lambda, u)$$

for each $(\lambda, u) \in I \times E$. We will denote by $\mathcal{S}$ the closure of the set

$$\{(\lambda, u); \ \Phi(\lambda, u) = 0, \ u \neq 0\}.$$

By int(I) (int$(I \times E)$) we denote the interior of I ($I \times E$) in the topology of $\mathbb{R}$ ($\mathbb{R} \times E$).

Consider the following result.

Theorem A (A type of Rabinowitz Theorem in a strip). *Let E be a Banach space, I a closed interval (not necessarily bounded) with non empty interior, $K : I \times E \to E$ an abstract operator and $\lambda_0 \in$ int(I) satisfying the following hypothesis:*

1) K *is a compact operator;*

2) there exists an interval $(a, b) \subset I$ *such that*

$$[((a, b) \times \{0\}) \setminus \{(\lambda_0, 0)\}] \cap \mathcal{S} = \emptyset; \tag{3.4.0.1}$$

3)

$$i(I - K(\lambda_0 - \eta, \cdot), 0) \neq i(I - K(\lambda_0 + \xi, \cdot), 0) \tag{3.4.0.2}$$

for η and ξ positive numbers small enough.

Then there exists a continuum $\mathcal{C}_0$ *of* $\mathcal{S}$ *containing* $(\lambda_0, 0)$ *such that* $\mathcal{C}_0$ *satisfies, at least, one of the following (non-excluding) alternatives:*

i) $\mathcal{C}_0$ *is unbounded,*

ii) $\mathcal{C}_0$ *intercepts some* $(d, u) \in I \times E$ *where* d *is an extremity of the interval* I *(if* I *possesses some extremity) for some* $u \in E$ *or (not exclusive) intercepts some* $(\lambda_1, 0)$ *with* $\lambda_1 \in I$.

Remark 3.4.1. *By using the argument that was used to prove Corollary B, it follows that Theorem A generalizes Theorem 3.2.2 when we take* $I = \mathbb{R}$ *in the hypothesis Theorem A.*

Idea of the proof: The argument is proving by absurd. First, the hypothesis (3.4.0.1) combined with the compactness of K guarantees that if $(\lambda_0, 0)$ is not a bifurcation point, then $\Phi(\lambda, u) : [\lambda_0 - \delta, \lambda_0 + \delta] \times \overline{B_{\rho_0}} \to E$ is a admissible homotopy for t near λ_0 and that for each $\lambda \in [\lambda_0, -\delta, \lambda_0 + \delta]$, $u = 0$ is an isolated solution of $\Phi_\lambda(u) = 0$. By the invariance under homotopy,

$$\begin{aligned} i(\Phi(\lambda - \eta, \cdot), 0) &= \deg(\Phi(\lambda - \eta, \cdot), B_{\rho_0}, 0) \\ &= \deg(\Phi(\lambda + \xi, \cdot), B_{\rho_0}, 0) \\ &= i(\Phi(\lambda + \xi, \cdot), B_{\rho_0}, 0) \end{aligned}$$

but this contradicts (3.4.0.2). So $(\lambda_0, 0)$ is a bifurcation point.

This fact ensures the existence of a maximal continuum $\mathcal{C}_0$ of $\mathcal{S}$ containing $(\lambda_0, 0)$. Then we suppose by absurd that this continuum does not satisfies none of the alternatives i) or ii). Since i) is not satisfied, we can take a bounded neighbourhood $\mathcal{O}$ of $\mathcal{C}_0$. On the other hand, it follows from the assumption that $\mathcal{C}_0$ does not satisfies $ii)$ that $\mathcal{O}$ can be taken such that $\mathcal{O} \subset \operatorname{int}(I \times E)$. Moreover, by using a result about "separation of compact sets" $\mathcal{O}$ can be taken such that $\partial\mathcal{O} \cap \mathcal{S} = \emptyset$.

Finally, by the fact that $\mathcal{C}_0$ does not intercepts the curve $\{(\lambda, 0); \lambda \in I \setminus \{\lambda_0\}\}$, we can deduce that

$$\deg(\Phi_\theta, \mathcal{O}_\theta, 0)$$

is constant for all $\theta \in [\lambda_0 - \delta, \lambda_0 + \delta]$, due to Theorem 2.2.1 for a certain small $\delta > 0$, but by using some properties of the degree and the hypothesis (3.4.0.2), one can deduce that

$$\deg(\Phi_{\underline{\lambda}}, \mathcal{O}_{\underline{\lambda}}, 0) \neq \deg(\Phi_{\overline{\lambda}}, \mathcal{O}_{\overline{\lambda}}, 0)$$

for certain $\underline{\lambda}, \overline{\lambda} \in [\lambda_0 - \delta, \lambda_0 + \delta]$, which is a contradiction.

Before actually proving Theorem A, we need the following lemmas.

Lemma 3.4.1. *Let* Z *be a compact metric space and* A *and* B *be disjoint closed subsets of* Z. *Then either there exists a continuum of* Z *intersecting both* A *and* B *or*

$$Z = Z_A \cup Z_B,$$

where Z_A *and* Z_B *are two disjoint compact subsets of* Z *containing* A *and* B*, respectively.*

Proof. See [36]. □

Lemma 3.4.2. *Under the conditions of Theorem 3.4,* $(\lambda_0, 0) \in \mathcal{S}$.

Proof. Suppose that $(\lambda_0, 0) \notin \mathcal{S}$.

Claim. *Let $0 < \delta < \min\{\lambda_0 - a, b - \lambda_0\}$ such that $(\lambda_0 - \delta, \lambda_0 + \delta) \in int(I)$. Then there exists $\rho_0 > 0$ such that*

$$u \in \overline{B_{\rho_0}}, \quad \Phi(u) = 0 \Rightarrow u = 0 \ \forall \nu \in [\lambda_0 - \delta/2, \lambda_0 + \delta/2].$$

In particular, for each $\nu \in [\lambda_0 - \delta/2, \lambda_0 + \delta/2]$, $u = 0$ is an isolated solution of $\Phi = 0$, consequently

$$i(\Phi(\nu, \cdot), 0) = deg(\Phi(\nu, \cdot), B_{\rho_0}, 0) \text{ for all } \nu \in [\lambda_0 - \delta/2, \lambda_0 + \delta/2].$$

Proof. In fact, suppose by absurd that the claim is false. Then there exist a sequence (ν_n) in $[\lambda_0 - \varepsilon_0/2, \lambda_0 + \varepsilon_0/2]$, a decreasing sequence (ρ_n) of positive numbers that converges to zero and a sequence (u_n) in $\overline{B_{\rho_n}} \setminus \{0\}$ satisfying

$$u_n - K(\nu_n, u_n) = \Phi_{\nu_n}(u_n) = 0 \ \forall n.$$

Since (ν_n) is bounded, there exists

$$\nu \in [\lambda_0 - \delta/2, \lambda_0 + \delta/2]$$

such that

$$\nu_n \to \nu \text{ up to a subsequence}$$

and so $(u_n, \nu_n) \to (0, \nu)$ up to a subsequence. Moreover, $u_n \neq 0$, hence $(\nu, 0) \in$ S, but this is impossible because

$$\nu \in [\lambda_0 - \delta/2, \lambda_0 + \delta/2] \subset (a, b),$$

so the claim is proved. □

By the claim, it follows that the homotopy

$$\Phi|_J : [\lambda_0 - \delta/2, \lambda_0 + \delta/2] \times \overline{B_{\rho_0}} \to E$$

($J := [\lambda_0 - \delta/2, \lambda_0 + \delta/2] \times \overline{B_{\rho_0}}$) is admissible and by the invariance under homotopy property of Leray Schauder degree (P3), we deduce that

$$deg(\Phi(\lambda, \cdot), B_{\rho_0}, 0) \quad \text{constant for } \lambda \in [\lambda_0 - \delta/2, \lambda_0 + \delta/2]. \tag{3.4.0.3}$$

Let ξ and η be small positive numbers such that $\lambda_0 + \xi, \lambda_0 - \eta \in [\lambda_0 - \delta/2, \lambda_0 + \delta/2]$ and satisfies (3.4.0.2). Then

$$i(\Phi(\lambda_0 - \eta, \cdot), 0) = deg(\Phi(\lambda_0 - \eta, \cdot), B_{\rho_0}, 0) = deg(\Phi(\lambda_0 + \xi, \cdot), B_{\rho_0}, 0) = i(\Phi(\lambda_0 + \xi, \cdot), 0),$$

which contradicts (3.4.0.2) and so the lemma is proved. □

Lemma 3.4.3. *Assume that the hypotheses of Theorem 3.4 hold. If does not exist a continuum C_{λ_0} of S containing $(\lambda_0, 0)$ and satisfying, at least, one of the following (non-excluding) alternatives*

i) C_{λ_0} *is unbounded,*

ii) C_{λ_0} *intercepts some $(d, u) \in I \times E$ where d is an extremity of the interval I (if I possesses some extremity) for some $u \in E$ or (not exclusive) intercepts some $(\lambda_1, 0)$ with $\lambda_1 \in I$,* *then, for each* $0 < \delta < \min\{\lambda_0 - a, b - \lambda_0\}$ *such that* $(\lambda_0 - \delta, \lambda_0 + \delta) \in$ *int*(I)*, there exists a bounded open neighbourhood* $\mathcal{O} \subset$ *int*$(I \times E)$ *of* $(\lambda_0, 0)$ *such that*

1) $\partial\mathcal{O} \cap \mathcal{S} = \emptyset$*; and*

2) $|\lambda - \lambda_0| < \delta$ *for any* $(\lambda, 0) \in \overline{\mathcal{O}}$*.*

Proof. Suppose that none of the alternatives i) and ii) occurs. Let $\mathcal{C}_0$ be the maximal connected component subset of $\mathcal{S}$ that contains $(\lambda_0, 0)$ and note that $\mathcal{C}_0$ is closed. Indeed, $\overline{\mathcal{C}_0}$ is connected and since $\mathcal{S}$ is a closed set, it follows that $\overline{\mathcal{C}_0} \subset \mathcal{S}$ and so the maximality of $\mathcal{C}_0$ implies that

$$\overline{\mathcal{C}_0} \subset \mathcal{C}_0.$$

Thus, $\mathcal{C}_0$ is a continuum of $\mathcal{S}$ which meets $(\lambda_0, 0)$. By hypothesis, this set does not verify i), that is, $\mathcal{C}_0$ is a bounded subset of E. We state that $\mathcal{C}_0$ is a compact set. Indeed, since K is continuous and $\mathcal{C}_0 \subset \mathcal{S}$, then

$$u = K(\lambda, u) \quad \forall\, (\lambda, u) \in \mathcal{C}_0. \tag{3.4.0.4}$$

Let (λ_n, u_n) be a sequence in $\mathcal{C}_0$. Then

$$u_n = K(\lambda_n, u_n)$$

and by the compactness of K, the sequence $u_n = K(\lambda_n, u_n)$ converges to some $u \in E$, up to a subsequence, and by the boundedness of (λ_n), we have that λ_n converges for some $\lambda \in I$, thus

$$(\lambda_n, u_n) \to (\lambda, u) \;\text{ up to a subsequence}$$

and so $(\lambda, u) \in \overline{\mathcal{C}_0} \subset \mathcal{C}_0$, which proves the compactness of $\mathcal{C}_0$.

Let $\delta < \min\{\lambda_0 - a, b - \lambda_0\}$ such that $(\lambda_0 - \delta, \lambda_0 + \delta) \subset \text{int}(I)$. Since $\mathcal{C}_0$ does not satisfies ii) as well and (3.4.0.1) holds, it follows that there exists a bounded open neighbourhood $\mathcal{U} \subset \text{int}(I \times E)$ of $\mathcal{C}_0$ (that is, a bounded open subset $\mathcal{U}$ of E such that $\mathcal{C}_0 \subset \mathcal{U} \subset \text{int}(I \times E)$) such that

$$\mathcal{U} \text{ contains no solutions } (\lambda, 0) \text{ of } \Phi(\lambda, u) = 0 \text{ for } |\lambda - \lambda_0| \geq \delta. \tag{3.4.0.5}$$

Define

$$Z := \overline{\mathcal{U}} \cap \mathcal{S}$$

and observe that this set is not empty because it contains the point $(\lambda_0, 0)$. Also, Z is a bounded closed subset of the set $\mathcal{S}$, which implies by the compactness of Φ that Z is a compact set. Moreover, the sets

$$\begin{cases} A := \mathcal{C}_0, \\ B := \partial\mathcal{U} \cap \mathcal{S} \end{cases}$$

are compact. Indeed, we just showed the compactness of A and since B is a bounded closed subset of the set $\mathcal{S}$, it follows by the compactness of Φ that B is compact. Indeed, let (λ_n, u_n) be a bounded subsequence in $B \subset \mathcal{S}$. Since Φ is continuous, it follows that

$$u_n = K(\lambda_n, u_n) \;\text{ for all } n.$$

By the compactness of K, the sequence $u_n = K(\lambda_n, u_n)$ converges up to a subsequence and so (λ_n, u_n) converges up to a subsequence, in other words B is sequentially compact, but since $\mathcal{E}$ is a metric space, it follows that B is a compact set.

Observe also that there is no continuum of $\mathcal{S}$ linking A and B. In fact, suppose that there exists a continuum $\mathcal{D}$ of $\mathcal{S}$ linking A and B. Then there exists $p \in \mathcal{D} \cap B$ and so $p \in \partial\mathcal{U}$. Since $\mathcal{U}$ is a neighborhood of the compact set A, it follows that $p \notin A = \mathcal{C}_0$, and so $\mathcal{C}_0$ is a proper subset of the continuum $\mathcal{D} \cup \mathcal{C}_0$ of $\mathcal{S}$, but this contradicts the maximality of $\mathcal{C}_0$. So, Lemma 3.4.1 implies that there exists two compact sets Z_A and Z_B containing A and B, respectively, satisfying

$$Z = Z_A \cup Z_B$$

and

$$Z_A \cap Z_B = \emptyset. \tag{3.4.0.6}$$

Observe that $Z_A \subset Z = \overline{\mathcal{U}} \cap \mathcal{S}$ and so

$$Z_A \subset \overline{\mathcal{U}} \tag{3.4.0.7}$$

and

$$Z_A \subset \mathcal{S}. \tag{3.4.0.8}$$

By (3.4.0.8) and (3.4.0.6), it follows that

$$Z_A \cap \partial\mathcal{U} = \emptyset \tag{3.4.0.9}$$

and so (3.4.0.7) and (3.4.0.9) implies that

$$Z_A \subset \mathcal{U}. \tag{3.4.0.10}$$

Now, by combining (3.4.0.10) with (3.4.0.6) and the compactness of Z_A, it is possible to take a neighbourhood $\mathcal{O}$ of Z_A such that

a) $\overline{\mathcal{O}} \cap Z_B = \emptyset$,

b) $\overline{\mathcal{O}} \subset \mathcal{U}$.

Observe that b) implies that

$$|\lambda - \lambda_0| < \delta \quad \forall\, (\lambda, 0) \in \overline{\mathcal{O}}. \tag{3.4.0.11}$$

Moreover, it also satisfies

$$\partial\mathcal{O} \cap \mathcal{S} = \emptyset. \tag{3.4.0.12}$$

Indeed, suppose that there exists

$$q \in \partial\mathcal{O} \cap \mathcal{S}, \tag{3.4.0.13}$$

then

$$q \in \partial\mathcal{O} \subset \overline{\mathcal{O}} \subset \mathcal{U} \subset \overline{\mathcal{U}} \stackrel{(3.4.0.13)}{\Rightarrow} q \in \overline{\mathcal{U}} \cap \mathcal{S} = Z.$$

By a) and $q \in \partial\mathcal{O}$, it follows that $q \notin Z_B$. On the other hand $q \notin Z_A$ because $Z_A \subset \mathcal{O}$ and $q \in \partial\mathcal{O}$, so this lead a contradiction because $q \in Z = Z_A \cup Z_B$.

Finally, observe that

$$(\lambda_0, 0) \in \mathcal{C}_0 = A \subset Z_A \subset \mathcal{O} \subset \mathcal{U}$$

and so

$$\mathcal{O} \cap \operatorname{int}(I \times E) \text{ is an open neighbourhood of } (\lambda_0, 0). \tag{3.4.0.14}$$

So (3.4.0.11), (3.4.0.12) and (3.4.0.14) proves the statement of the theorem. □

Now we are able to prove Theorem 3.4.

Proof of Theorem 3.4. Suppose that does not exist C_{0} as in Theorem 3.4 and let

$$0 < \delta < \min\{\lambda_0 - a, b - \lambda_0\}$$

such that $(\lambda_0 - \delta, \lambda_0 + \delta) \in \mathrm{int}(I)$ and O as given in Lemma 3.4.3.

Let us define

$$O := \{u \in E; (\lambda, u) \in O\}.$$

In order to ensure the well definition of the Leray-Schauder degree of for certain parameters ν we will need the following claims.

Claim 3.4.1. *For each $\lambda_0 < \lambda < \lambda_0 + \delta$, there exists $\rho_1 > 0$ such that it holds*

$$u \in \overline{B}_{1}, \quad (u) = 0 \Rightarrow u = 0 \ \forall \nu \in [\lambda, \lambda_0 + \delta].$$

In particular, $(u) \neq 0$ for all $u \in \partial(O \setminus \overline{B}_{0})$.

Proof. Suppose by absurd that the statement is false. Then there exists a sequence (ν_n) in $[\lambda, \lambda + \delta]$, a decreasing sequence (ρ_n) of positive numbers that converges to zero and a sequence (u_n) in $\overline{B}_{n} \setminus \{0\}$ such that

$$u_n - K(\nu_n, u_n) = {}_{n}(u_n) = 0 \ \forall n.$$

Since $u_n \to 0$ and

$$\nu_n \to \nu \in [\lambda, \lambda_0 + \delta] \text{ up to a subsequence,}$$

we have that $(\nu, 0)$ is a bifurcation point, which contradicts hypothesis (3.4.0.1) because $\nu \in [\lambda, \lambda_0 + \delta]$ and $\delta < \min\{b - \lambda_0, \lambda_0 - a\}$, so the first statement of Claim 3.4.1 is proved.

Now, suppose that there exists $\nu \in [\lambda, \lambda_0 + \delta]$ and $u \in \partial(O \setminus \overline{B}_{0})$ such that $(u) = 0$. Observe that

$$u \in \partial(O \setminus \overline{B}_{0}) \Rightarrow u \neq 0 \tag{3.4.0.15}$$

and consequently

$$(\nu, u) \in S.$$

The idea is to prove that

$$(\nu, u) \in \partial O,$$

what would imply in a contradiction with the fact that $\partial O \cap S = \emptyset$.

Since $u \in \partial(O \setminus \overline{B}_{0})$, there exist sequences

$$\begin{cases} u_n \in \left[O \setminus \left(\overline{B}_{0}\right)\right]^c = \left[O \cap \left(\overline{B}_{0}\right)^c\right]^c = O^c \cup \left(\overline{B}_{0}\right), \\ v_n \in O \setminus \left(\overline{B}_{0}\right) = O \cap \left(\overline{B}_{0}\right)^c \end{cases} \tag{3.4.0.16}$$

that converges to u. We state that there exists, at most, a finite amount of indices n such that $u_n \in \overline{B}_{0}$. Indeed, suppose that there exists a subsequence of (u_n) (which we will denote in the same way) lying in $\overline{B}_{0}$ and such that $u_n \to u$. Then $u \in \overline{B}_{0}$, because this is a closed set, so $(\nu, u) \in \{\nu\} \times \overline{B}_{0}$ is a solution of $(u) = 0$ and hence

$$u = 0$$

by the first statement of Claim 3.4.1, which contradicts (3.4.0.15) and we conclude that, at most, a finite amount of indices n satisfies $u_n \in \overline{B}_{0}$. This fact together with (3.4.0.16) implies that there exists an infinite amount of indices n such that $u_n \in O^c$, then $u \in \overline{O^c}$. Now the convergence $v_n \to u$ implies that $u \in \overline{O}$, so we conclude that $u \in \partial(O)$ and consequently

$$(\nu, u) \in \partial O$$

(see Remark 2.2.4), as we wanted. So the claim is proved. □

Remark 3.4.2. *Claim 3.4.1 also holds for each* $0 < \rho_1' < \rho_1$.

Since $O \subset \mathrm{int}(I \times E)$ is a bounded set, then there exists $\lambda < \lambda_0 - \delta$ and $\lambda > \lambda_0 + \delta$ such that $[\lambda, \lambda] \subset I$ and

$$(\xi, w) \in \overline{O} \Rightarrow \lambda < \xi < \lambda . \tag{3.4.0.17}$$

Claim 3.4.2. *There exists* $\rho_0 > 0$ *such that it holds*

$$\overline{O} \cap \overline{B}_{0} = \emptyset,$$

for any $\nu \in [\lambda_0 + \delta, \lambda]$ *given. In particular,* $(u) \neq 0$ *for all* $u \in \partial(O \setminus \overline{B}_{0})$.

Proof. Suppose that the statement is not true. Then there exists a sequence

$$(\nu_n) \text{ in } [\lambda_0 + \delta, \lambda],$$

a decreasing sequence (ρ_n) of positive numbers that converges to zero and a sequence (u_n) in $\overline{O}_{n} \cap \overline{B}_{n}$. Since $u_n \to 0$,

$$\nu_n \to \nu \in [\lambda_0 + \delta, \lambda] \text{ up to a subsequence,}$$

and $u_n \in \overline{O}_{n}$, it follows that there exists a sequence $v_n \in O_{n}$ such that $v_n \to 0$. Hence (ν_n, v_n) is a sequence in O such that $(\nu_n, v_n) \to (\nu, 0)$, which means that $(\nu, 0) \in \overline{O}$, but this is impossible since the set $\overline{O}$ does not contain any pair $(\theta, 0)$ with $|\theta - \lambda_0| \geq \delta$, by item 2) of the statement of Lemma 3.4.3. So the first statement of Claim 3.4.2 is proved.

Now, suppose that there exists $\nu \in [\lambda_0+\delta, \lambda]$ and $u \in \partial(O \setminus \overline{B}_{0})$ such that $(u) = 0$. By i), $O \cap \overline{B}_{0} = \emptyset$ and so $O \setminus \overline{B}_{0} = O$, thus

$$u \in \partial(O \setminus \overline{B}_{0}) = \partial O \subset \overline{O},$$

implies

$$u \notin \overline{B}_{0}. \tag{3.4.0.18}$$

Now, the fact that $u \in \partial O$ implies that $(\nu, u) \in \partial O$, together with (3.4.0.18) imply $(\nu, u) \in S$. So $(\nu, u) \in \partial O \cap S = \emptyset$, which is a contradiction and so the claim is proved. □

Remark 3.4.3. *Claim 3.4.2 also holds for each* $0 < \rho_0' < \rho_0$.

Let $\lambda_0 < \lambda < \lambda_0 + \delta$ and $\rho_1(\lambda), \rho_0(\lambda)$ be the positive numbers given by Claim 3.4.1 and Claim 3.4.2. For each

$$0 < \rho_1'(\lambda) < \rho_1(\lambda) \text{ and } 0 < \rho_0'(\lambda) < \rho_0(\lambda),$$

define

$$\rho_e(\lambda) := \min\{\rho_0'(\lambda), \rho_1'(\lambda)\}.$$

Claim 3.4.3. *Let* $\lambda_0 < \lambda < \lambda_0 + \delta$, $\hat{E} := [\lambda, \lambda\ ]\quad E$ *and* $\rho_e := \rho_e(\lambda)$. *The boundary*

$$\partial \mathcal{U} \text{ (in } \hat{E})$$

of the open set $\mathcal{U} := \mathcal{O} \setminus ([\lambda, \lambda\ ]\quad \overline{B\ _e})$ *does not have zeros of* .

Proof. Suppose, by contradiction that there exists a solution of $\ = 0$ on $\partial\mathcal{U}$. Then there exists a sequence $(\alpha_n, v_n) \in \mathcal{U} \cap \hat{E}$ that converges to (ν, u). Since

$$v_n \notin \overline{B\ _e}, \tag{3.4.0.19}$$

we deduce that

$$(\nu, u) \in \mathcal{S}.$$

The idea is to prove that

$$(\nu, u) \in \partial\mathcal{O}$$

that would imply the contradiction

$$(\nu, v) \in \mathcal{S} \cap \partial\mathcal{O} = \emptyset.$$

Since $(\alpha_n, v_n) \in \mathcal{O}$ for all n, then

$$(\nu, u) \in \overline{\mathcal{O}}$$

and $\nu \in [\lambda, \lambda\ ]$ because (α_n) is a sequence in the closed set $[\lambda, \lambda\ ]$.

We claim that $(\nu, u) \notin \mathcal{O}$. Indeed, suppose that $(\nu, u) \in \mathcal{O}$. Hence

$$u \in \mathcal{O} \tag{3.4.0.20}$$

and by the fact that $(\nu, u) \in \partial\mathcal{U}$, we deduce that there exists a sequence (η_n, w_n) in $\hat{E} \cap \mathcal{U}^c$ that converges to (ν, u). Observe that

$$\begin{aligned}
\hat{E} \cap \mathcal{U}^c &= \hat{E} \cap \left[\mathcal{O} \cap ([\lambda, \lambda\ ]\quad \overline{B\ _e})^c\right]^c \\
&= \hat{E} \cap \left[\mathcal{O}^c \cup ([\lambda, \lambda\ ]\quad \overline{B\ _e})\right] \\
&= [\hat{E} \cap \mathcal{O}^c] \cup [\hat{E} \cap ([\lambda, \lambda\ ]\quad \overline{B\ _e})]
\end{aligned} \tag{3.4.0.21}$$

whence together with (3.4.0.21), $(\eta_n, w_n) \to (\nu, u) \in \mathcal{O}$ and that $\mathcal{O}$ is an open subset, we obtain that there exists an n_0 such that (η_n, w_n) lies in $\hat{E} \cap ([\lambda, \lambda\ ]\quad \overline{B\ _e})$ for all $n \quad n_0$, consequently

$$u \in \overline{B\ _e} \tag{3.4.0.22}$$

and combining it with (3.4.0.20) we deduce that

$$u \in \overline{\mathcal{O}\ } \cap \overline{B\ _e},$$

which by Claim (3.4.2) implies that

$$\nu < \lambda_0 + \delta. \tag{3.4.0.23}$$

So by (3.4.0.23) and (3.4.0.22), we deduce that $u = 0$, which is impossible due to $v_n \to u$, and according to (3.4.0.19), $v_n \notin \overline{B\ _e}$. That is $(\nu, u) \notin \mathcal{O}$ and then $(\nu, u) \in \partial\mathcal{O}$ as we wanted and the claim is proved. □

Remark 3.4.4. *For each $\lambda_0 \quad \delta < \lambda < \lambda_0$, there holds three analogous claims to those ones in 3.4.1, 3.4.2 and 3.4.3 above.*

Since O is a neighbourhood of $(\lambda_0, 0)$, we can take a ball B O. Let $\xi < \delta$ and $\eta < \delta$ be the positive numbers such that

$$i(I \quad K(\lambda_0 \quad \eta, \quad), 0) \neq i(I \quad K(\lambda_0 + \xi, \quad), 0), \tag{3.4.0.24}$$

whose the existence is guaranteed by the hypothesis. The hypothesis states that ξ and η can be taken small enough, so we can assume that

$$(\overline{\lambda}, 0), (\underline{\lambda}, 0) \; 2 \; \mathrm{B},$$

where $\overline{\lambda} := \lambda_0 + \xi$ and $\underline{\lambda} := \lambda_0 \quad \eta$, then we can write

$$i(\quad _{\_}, 0) \neq i(\quad _{-}, 0). \tag{3.4.0.25}$$

Moreover, since the numbers $\rho_e(\overline{\lambda})$ and $\rho_e(\underline{\lambda})$ can be taken small enough (by remarks 3.4.2 and 3.4.3), we can assume that

$$\begin{array}{lll} f\underline{\lambda}g & B_{e(\_)} & \mathrm{B} \text{ and} \\ f\overline{\lambda}g & B_{e(\bar{})} & \mathrm{B} \end{array}$$

and consequently

$$\begin{cases} B_{e(\_)} & O_{\_}, \\ B_{e(\bar{})} & O_{-}. \end{cases} \tag{3.4.0.26}$$

Furthermore, by the fact that $\rho_e(\underline{\lambda}) \quad \rho_1(\underline{\lambda})$ and $\rho_e(\overline{\lambda}) \quad \rho_1(\overline{\lambda})$, it follows that

$$\begin{cases} deg(\quad _{-}, B_{e(\bar{})}, 0) & = i(\quad _{-}, 0), \\ deg(\quad _{\_}, B_{e(\_)}, 0) & = i(\quad _{\_}, 0). \end{cases} \tag{3.4.0.27}$$

Since (3.4.0.26), then

$$\begin{array}{rcl} O_{-} \, n \, \partial B_{e(\bar{})} & = & B_{e(\bar{})} \; [\; (O_{-} \, n \, \overline{B_{e(\bar{})}}), \\ O_{\_} \, n \, \partial B_{e(\_)} & = & B_{e(\_)} \; [\; (O_{\_} \, n \, \overline{B_{e(\_)}}) \end{array}$$

and by the excision property

$$\begin{array}{r} deg(\quad _{-}, O_{-}, 0) = deg(\quad _{-}, O_{-} \, n \, \partial B_{e(\bar{})}, 0), \\ deg(\quad _{\_}, O_{\_}, 0) = deg(\quad _{\_}, O_{\_} \, n \, \partial B_{e(\_)}, 0) \end{array}$$

and so by the additivity of deg

$$\begin{array}{rcl} deg(\quad _{-}, O_{-}, 0) & = & deg(\quad _{-}, B_{e(\bar{})}, 0) + deg(\quad _{-}, O_{-} \, n \, \overline{B_{e(\bar{})}}, 0), \\ deg(\quad _{\_}, O_{\_}, 0) & = & deg(\quad _{\_}, B_{e(\_)}, 0) + deg(\quad _{\_}, O_{\_} \, n \, \overline{B_{e(\_)}}, 0), \end{array}$$

which by (3.4.0.27) can be rewritten as

$$\begin{cases} deg(\quad _{-}, O_{-}, 0) & = i(\quad _{-}, 0) + deg(\quad _{-}, O_{-} \, n \, \overline{B_{e(\bar{})}}, 0), \\ deg(\quad _{\_}, O_{\_}, 0) & = i(\quad _{\_}, 0) + deg(\quad _{\_}, O_{\_} \, n \, \overline{B_{e(\_)}}, 0). \end{cases} \tag{3.4.0.28}$$

Applying Claim 3.4.3 to

$$\mathcal{U} = \mathcal{O} \setminus ([\lambda \;, \underline{\lambda}] \quad \overline{B_{e(\underline{\ })}}) \quad \left(\mathcal{U} = \mathcal{O} \setminus ([\overline{\lambda}, \lambda\;] \quad \overline{B_{e(\overline{\ })}})\right)$$

and Theorem 2.2.1 with the open set $\ = \mathcal{U}$ and the interval $\left[\,\overline{\lambda}, \lambda\;\right]$ $([\,\lambda\;, \underline{\lambda}\,])$, we obtain

$$deg(\ \;, \mathcal{U}\;, 0) \quad \text{constant in } \theta \in \left[\,\overline{\lambda}, \lambda\;\right] \; ([\,\lambda\;, \underline{\lambda}\,]).$$

Note that

$$\mathcal{U}\;\; = \mathcal{O}\;\; \setminus \overline{B_{e(\overline{\ })}} \quad (\mathcal{O}\;\; \setminus \overline{B_{e(\underline{\ })}})$$

and so we can write

$$\begin{cases} deg(\ \;, \mathcal{O}\;\; \setminus \overline{B_{e(\overline{\ })}}, 0) & \text{constant in } \theta \in \left[\,\overline{\lambda}, \lambda\;\right], \\ deg(\ \;, \mathcal{O}\;\; \setminus \overline{B_{e(\underline{\ })}}, 0) & \text{constant in } \theta \in [\,\lambda\;, \underline{\lambda}\,]. \end{cases} \tag{3.4.0.29}$$

By the definition of $\lambda\;\; (\lambda\;)$ (see (3.4.0.17)), we have

$$\begin{cases} \mathcal{O}\;\; = \emptyset, \\ \mathcal{O}\;\; = \emptyset, \end{cases}$$

and so

$$\begin{aligned} deg(\ \;, \mathcal{O}\;\; \setminus \overline{B_{e(\overline{\ })}}) &= 0, \\ deg(\ \;, \mathcal{O}\;\; \setminus \overline{B_{e(\underline{\ })}}) &= 0 \end{aligned}$$

and by (3.4.0.29), we deduce that

$$\begin{aligned} deg(\ _{-}, \mathcal{O}_{-} \setminus \overline{B_{e(\overline{\ })}}) &= 0, \\ deg(\ _{\_}, \mathcal{O}_{\_} \setminus \overline{B_{e(\underline{\ })}}) &= 0. \end{aligned}$$

So by using these two equalities in (3.4.0.28), we obtain

$$\begin{aligned} deg(\ _{-}, \mathcal{O}_{-}, 0) &= i(\ _{-}, 0), \\ deg(\ _{\_}, \mathcal{O}_{\_}, 0) &= i(\ _{\_}, 0). \end{aligned}$$

Since (3.4.0.24) holds, then

$$deg(\ _{-}, \mathcal{O}_{-}, 0) \neq deg(\ _{\_}, \mathcal{O}_{\_}, 0),$$

moreover, the fact $\partial\mathcal{O} \cap \mathcal{S} = \emptyset$ allows us to use Theorem 2.2.1 with the open set $\ = \mathcal{O}$, in order to deduce that

$$deg(\ \;, \mathcal{O}\;, 0) \quad \text{constant in } [\lambda_0 \quad \delta, \lambda_0 + \delta]. \tag{3.4.0.30}$$

In particular,

$$deg(\ _{-}, \mathcal{O}_{-}, 0) = deg(\ _{\_}, \mathcal{O}_{\_}, 0)$$

which is a contradiction and the proof is complete. □

Remark 3.4.5. *Observe that the condition that there exists some $\zeta > 0$ such that*

$$i(\ _{\_}, 0) \neq i(\ _{-}, 0)$$

for every $\underline{\lambda}$ 2 $(\lambda_0 \quad \zeta, \lambda_0)$ and every $\overline{\lambda}$ 2 $(\lambda_0, \lambda_0 + \zeta)$, is equivalent to the condition (3.4.0.2). Indeed, it is su cient to prove that, under the hypotheses of Theorem A, there exists some $\zeta > 0$ such that

$$i(\ _{-}, 0) \quad = \quad \textit{constant } 8\ \overline{\lambda}\ 2\ (\lambda_0, \lambda_0 + \zeta), \qquad (3.4.0.31)$$

$$i(\ _{\_}, 0) \quad = \quad \textit{constant } 8\ \overline{\lambda}\ 2\ (\lambda_0 \quad \zeta, \lambda_0), \qquad (3.4.0.32)$$

so this is what we will prove now.

Since O is a neighbourhood of $(\lambda_0, 0)$, it follows that there exists $0 < \varepsilon < \delta$ such that the ball B" is contained in O. Take $\zeta < \varepsilon/2$, consider θ and ν in $(\lambda_0, \lambda_0 + \zeta)$ with $\theta < \nu$ and let $\rho_e(\theta)$ be small enough such that $\rho_e(\theta) < \varepsilon/2$. Thus,

$$deg(\quad , B_{\ e(\ )}, 0) = i(\quad , 0)$$

and since ν 2 $[\theta, \lambda\ ]$, we deduce that

$$deg(\quad , B_{\ e(\ )}, 0) = i(\quad , 0).$$

Note that since $\rho_e(\theta) < \varepsilon/2$ and ν, θ 2 $(\lambda_0, \lambda_0 + \zeta)$, then

$$k(\nu, u) \quad (\lambda_0, 0)k = (j\nu \quad \lambda_0 j^2 + kuk^2)^{1=2} = (\varepsilon^2/2)^{1=2} \quad \varepsilon \quad 8\ u\ 2\ B_{\ e(\ )},$$

$$k(\theta, u) \quad (\lambda_0, 0)k = (j\theta \quad \lambda_0 j^2 + kuk^2)^{1=2} = (\varepsilon^2/2)^{1=2} \quad \varepsilon \quad \varepsilon \quad 8\ u\ 2\ B_{\ e(\ )},$$

which implies that

$$B_{\ e(\ )} \quad O \quad \backslash\ O$$

and so, similarly as we done in the proof of Theorem A, we can deduce that

$$deg(\quad , O \ , 0) = deg(\quad , B_{\ e(\ )}, 0)$$

and

$$deg(\quad , O \ , 0) = deg(\quad , B_{\ e(\ )}, 0),$$

hence

$$i(\quad , 0) = i(\quad , 0)$$

and we just proved (3.4.0.31). For analogy, it also holds (3.4.0.32).

3.5 Final comments of Chapter 3

In this final section, we will comment on the relevance of the most import results with alphabetical naming.

The propose of the combination of Lemma A_0 and Lemma A, helps understand the meaning of the hypothesis about the continuity of the function λ 7! $L(\lambda)$, given in [12], since in the paper it is not clear which topology is considered in the space where the family of homogeneous operators (with homogeneity degree 1) $fL(\lambda)g$ lies.

Lemma B generalizes the necessary condition given in Lemma 2.1 of [24] and makes room for the study of the candidates of bifurcation point for problems $\quad (\lambda, u) = 0$ where is an operator satisfying H1), H2) and H3). Lemma C provides an alternative necessary condition for λ_0 being a bifurcation point that is formulated for the case when $L(\lambda_0)$ is a contraction and this is promising when we take into account that there exists a wide study about contractions in the nonlinear functional analysis literature. The result given by Lemma D, is a formalization of the statement "[...] the Leray-Schauder degree, $\deg(I - L(\lambda), B_r, 0)$, is well defined for arbitrary $r-$ ball B_r and $\lambda \notin r(L_0)$[...]" done in the introduction of [12] when it explains why the necessary condition "λ_0 bifurcation point implies $\lambda_0 \in r(L_0)$" also holds in the case where $L(\lambda)$ is just a compact homogeneous operator with homogeneity degree 1, instead of the strong condition of being $L(\lambda) = \lambda L$, where L is a linear compact operator, as the Rabinowitz and Kranosel'skii's classic results require.

Corollary B is a pure theoretical result, but it gives an interesting point of view of the implication "Rabinowitz $\Rightarrow$ Krasnosel'skii" when we pay attention to the fact that its proof is based just on topological arguments without the need of degree theory tools.

Finally, we must point out that our contribution in Theorem A lies in the fact that its formulation includes closed intervals $I \subset \mathbb{R}$ different from $\mathbb{R}$, since Theorem 11 of [3], given by Arcoya, already states our claim for the case where $I = \mathbb{R}$. This generalization is interesting because it can happen in some problems that just some strip of values of the bifurcation parameter λ is interesting. For example in the problem (P), where the strip of interest is $I = [0, +\infty)$.

Chapter 4

Global bifurcation from infinity for a quasilinear Leray-Lions problem

Consider the following quasilinear elliptic problem

$$\begin{cases} -\mathrm{div}(A(x,u)\nabla u) = f(\lambda,x,u) & x\in\Omega, \\ u = 0 & x\in\partial\Omega, \end{cases} \tag{P}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded open set with smooth boundary $\partial\Omega$ and

$$\begin{array}{rccc} f: & \mathbb{R}\times\Omega\times\mathbb{R}^+ & \to & \mathbb{R} \\ & (\lambda,x,s) & \mapsto & f(\lambda,x,s) \end{array}$$

is a Carathéodory function, i.e.:

i) the function $x\mapsto f(\lambda,x,s)$ is measurable for all fixed $(\lambda,s)\in\mathbb{R}\times\mathbb{R}^+$;

ii) the function $(\lambda,s)\mapsto f(\lambda,x,s)$ is continuous for almost every $x\in\Omega$.

Assume the following hypothesis about the signal of f.

Since we are interested in positive solutions, we will assume that

$$f(\lambda,x,0)\geq 0,\ \forall\lambda\in[0,+\infty). \tag{f_0}$$

We will also need the following additional hypothesis. *There exists* $s_0>0$ *and a function* $C_0\in L^r(\Omega)$ *such that*

$$f(0,x,s)\geq C_0(x) \text{ a.e. in } \Omega \text{ for every } s\geq s_0. \tag{f_0'}$$

Remark 4.0.1. *The hypothesis* (f_0') *is not required in [4], but we will need it in order to prove Claim 4.2.4.*

Moreover, for some $r>N/2$, the function f satisfies the following condition: *for every bounded subset* Λ *in* $\mathbb{R}$ *and every* $s_0>0$, *there exists a positive function* $C(x)\in L^r(\Omega)$ *such that*

$$|f(\lambda,x,s)|\leq C(x) \text{ for a.e. } x\in\Omega,\ \forall\lambda\in\Lambda \text{ and } \forall s\in[0,s_0]. \tag{f_1}$$

For each $(x,s)\in\Omega\times\mathbb{R}^+$, $A(x,s)$ is a symmetric matrix of order N whose coefficients

$$\begin{array}{rccc} a_{ij}: & \Omega\times\mathbb{R}^+ & \to & \mathbb{R} \\ & (x,s) & \mapsto & a_{ij}(x,s) \end{array}$$

are Carathéodory functions. Moreover, we assume that there exist positive constants γ and β satisfying

$$|A(x,s)| \leq \beta, \tag{A_1}$$
$$A(x,s)\xi \cdot \xi \geq \gamma|\xi|^2 \tag{A_2}$$

for each $(s,\xi) \in \mathbb{R}^+ \times \mathbb{R}^N$ and almost every $x \in \Omega$.

In order to construct a solution operator to the problem (P) we will also need the following hypothesis about the matrix A.

$$|A(x,s) - A(x,t)| \leq \omega(|s-t|) \quad \forall\, s,t \in \mathbb{R} \tag{A_3}$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}$ *is an Osgood function.*

As a space of solutions, we will adopt the space $H_0^1(\Omega)$. According to Section 5.4 of Appendix, a weak solution for (P) is a function $u \in H_0^1(\Omega)$ that satisfies

$$\int_\Omega A(x,u)\nabla u \nabla v dx = \int_\Omega f(\lambda,x,u)v dx, \quad \forall v \in H_0^1(\Omega),$$

where ∇ stands for the weak gradient. For a positive weak solution of (P), we mean a non-negative and non-zero function $u \in H_0^1(\Omega)$ that satisfies the above equation.

Now, one can ask what are the bifurcation point candidates of the problem (P). To answer it, consider the following explanation. Our approach will be to construct a solution operator S of a certain auxiliar problem (see Section 4.1) that will allow us to formulate a problem in the form

$$F(\lambda,u) = 0,$$

where $F : [0,+\infty) \times E \to E$ (see Remark 4.2.8) is a perturbation of the identity $F(\lambda,u) = u - S(\lambda,u)$, and E is one of the Banach spaces $H_0^1(\Omega)$ or $C_0(\overline{\Omega})$. As in [1], we do not have a priori that the operator S has a structure of the type $S(\lambda,z) = L(\lambda)u + H(\lambda,u)$ with L and H satisfying the general hypotheses H1), H2), H3) and H∞) of Chapter 3, and so we can not apply Lemma B to find the candidates. However, we can proceed analogously with the formal argument given in the proof of Lemma D and this is what we will do now. Suppose that $(\lambda_\infty, 0)$ is a bifurcation point from infinity of $F(\lambda,u) = 0$. Then there exists a sequence $(\lambda_n,u_n) \in [0,+\infty) \times E$ such that u_n is a solution of (P) for $\lambda = \lambda_n$, with $\|u_n\|_E \to +\infty$ and $\lambda_n \to \lambda_\infty$. Hence

$$\int_\Omega A(x,u_n)\nabla u_n \nabla v dx = \int_\Omega f(\lambda_n,x,u_n)v dx \quad \forall v \in H_0^1(\Omega).$$

Analogously as done in the proof of Lemma D, we now divide the above equation by $\|u_n\|_E$

$$\int_\Omega A(x,u_n)\nabla\left(\frac{u_n}{\|u_n\|_E}\right)\nabla v dx = \int_\Omega \frac{f(\lambda_n,x,u_n)}{\|u_n\|_E} v dx \quad \forall v \in H_0^1(\Omega). \tag{4.0.0.1}$$

The idea is passing to the limit the above equation so that we can relate λ and λ_∞. We warn that what we will do now is just a sketch and for details see Lemma 4.3.5 (Lemma 4.4.5) and Remark 4.4.2 (Remark 4.3.2) for the case where $E = H_0^1(\Omega)$ ($E = C_0(\overline{\Omega})$). Since $\|u_n\|_E \to +\infty$, one can infer that if u_n is a positive solution, then the sequence (u_n) converges to $+\infty$ a.e. in some subset Ω^+ of Ω, with $m(\Omega^+) > 0$. These facts lead us naturally to the following additional hypotheses about the asymptotic behaviour of f and A at $s = +\infty$.

$$\exists \lim_{s \to +\infty} A(x,s) = A(x,+\infty) \ \text{ a.e. in } \Omega. \tag{A_4}$$

There exists a positive function $f_\infty^0 \in L^r(\Omega)$ *bounded away from zero a.e. such that*

$$\lim_{s \to +\infty} \frac{f(\lambda,x,s)}{s} = \lambda f_\infty^0(x) \tag{f_2}$$

uniformly with respect to $x \in \Omega$ *and for each fixed* $\lambda \in [0,+\infty)$.

Remark 4.0.2. *The hypotheses about* f_∞^0 *being bounded away from zero is not required in [4], but we will need it in order to prove Claim 4.2.4.*

As a consequence of (f_2), we have that the function $s \mapsto f(\lambda,x,s)$ is asymptotically linear at infinity and uniformly in $x \in \Omega$ for each fixed $\lambda \in \mathbb{R}^+$.

In order to guarantee the existence of

$$\lim_{n \to +\infty} \int_\Omega \frac{f(\lambda,x,u_n)}{\|u_n\|_E} dx$$

when $0 \leq \|u_n\|_E \to +\infty$ we will assume the following hypothesis.
There exist positive functions $K_1, K_2 \in L^r(\Omega)$ *such that*

$$|f(\lambda,x,s) - f(\overline{\lambda},x,s)| \leq |\lambda - \overline{\lambda}|[K_1(x)s + K_2(x)] \tag{f_3}$$

for each $\lambda, \overline{\lambda}, s \in \mathbb{R}^+$ *and a.e.* $x \in \Omega$.

Note that the conditions (f_1)-(f_3) implies that for each bounded subset $\Lambda \subset \mathbb{R}$ given, there exists positive functions C_1, C_2 in $L^r(\Omega)$ such that

$$|f(\lambda,x,s)| \leq C_1(x)s + C_2(x) \tag{4.0.0.2}$$

for all $s \in \mathbb{R}^+$, $\lambda \in \Lambda$ and a.e. $x \in \Omega$. Indeed, by (f_3) and the triangular inequality

$$|f(\lambda,x,s)| \leq |f(0,x,s)| + |\lambda|[K_1(x)s + K_2(x)] \tag{4.0.0.3}$$

for all $\lambda, s \in \mathbb{R}^+$ and a.e. $x \in \Omega$. Since Λ is bounded set, we deduce that there exists a constant $D > 0$ such that

$$|f(\lambda,x,s)| \leq |f(0,x,s)| + D(K_1(x)s + K_2(x)) \tag{4.0.0.4}$$

for all $\lambda \in \Lambda$, $s \in \mathbb{R}^+$ and a.e. $x \in \Omega$.

On the other hand, it follows from (f_2) that there exists $s_0 > 0$ such that

$$\left|\frac{f(0,x,s)}{s}\right| \leq \left|\frac{f(0,x,s)}{s} - 0 f_\infty^0(x)\right| < 1, \quad \forall s > s_0 \text{ and } \forall\, x \in \Omega. \tag{4.0.0.5}$$

Hence

$$|f(0,x,s)| \leq s, \ \forall\, s > s_0 \text{ and } x \in \Omega \tag{4.0.0.6}$$

which combined with (4.0.0.4) implies in (4.0.0.2).

Since $f_\infty^0(x) \not\equiv 0$, Theorem 5.5.2 states that the eigenvalue problem with weight

$$\begin{cases} -\operatorname{div}(A(x,+\infty)\nabla u) = \lambda f_\infty^0(x) u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega \end{cases} \tag{4.0.0.7}$$

admits a first positive eigenvalue $\lambda_1(f^0_\infty)$, which we will denote by

$$\lambda_\infty = \lambda_1(f^0_\infty). \tag{4.0.0.8}$$

The first positive eigenfunction associated with λ_∞ will be denoted by ψ and considered normalized from now on, that is, $\|\psi\| = 1$, where $\|\psi\| := \|\nabla\psi\|_2$.

Now, going back to the problem of passing to the limit the equation (4.0.0.1), we state that the normalized sequence $(u_n/\|u_n\|_E)_n$ converges weakly to some positive $z \in E$ and so, by passing to the limit in equation (4.0.0.1), we deduce that z is a non trivial and non negative solution of

$$\mathrm{div}(A(x,\infty)\nabla z) = \lambda f^0_\infty(x) z. \tag{4.0.0.9}$$

By testing against ψ and using that λ_∞ is an eigenvalue of (4.0.0.7), we obtain

$$\begin{aligned} \int \lambda_\infty f^0_\infty z\psi &= \int A(x,\infty)\nabla z\nabla\psi \\ &= \int \lambda f^0_\infty z\psi, \end{aligned}$$

hence $\lambda = \lambda_\infty$. So we conclude that the only $\lambda \geq 0$ that is possibly a bifurcation point from infinity of problem (P) is $\lambda = \lambda_\infty$.

Finally, in order to construct the solution operator of the problem (P), we will need to extend the function f and the matrix A for negative values of s by

$$a_{ij}(x,s) = a_{ij}(x,0) \textit{ for every } x \in \Omega \textit{ and every } s < 0 \tag{$A\;$}$$

(note that the hypotheses (A_1) and (A_2) remains valid) and

$$f(\lambda,x,s) = f(\lambda,x,0) \textit{ for all } x \in \Omega,\ \lambda \in \mathbb{R},\ s < 0 \tag{$f\;$}$$

(note that f is still a Carathéodory function satisfying the hypothesis (f_1)).

Therefore, it holds the two main theorems of this chapter which are the following.

Theorem C. *Let A be a matrix satisfying $(A\;)$, (A_1)-(A_4) and f be a function satisfying $(f\;)$, (f_0), (f_0)-(f_3). Then, it emanates from $(\lambda_\infty, 0)$ a continuum $\mathcal{C}_\infty$ of*

$$cl\left(\left\{(\lambda,z) \in [0,+\infty) \times H^1_0(\Omega);\ \frac{z}{\|z\|^2} \text{ is a positive solution of } (P)\right\}\right).$$

Moreover, if

$$f(0,x,s) = 0\ \forall x \in \Omega,\ \forall s \geq 0, \tag{4.0.0.10}$$

then $\mathcal{C}_\infty$ is unbounded.

Theorem D. *Let A be a matrix satisfying $(A\;)$, (A_1)-(A_4) and f be a function satisfying $(f\;)$, (f_0), (f_0)-(f_3). Then, it emanates from $(\lambda_\infty, 0)$ a continuum $\mathcal{C}_\infty$ of*

$$cl\left(\left\{(\lambda,z) \in [0,+\infty) \times C_0(\overline{\Omega});\ \frac{z}{\|z\|_0^2} \text{ is a positive solution of } (P)\right\}\right).$$

Moreover, if

$$f(0,x,s) = 0\ \forall x \in \Omega,\ \forall s \geq 0, \tag{4.0.0.11}$$

then $\mathcal{C}_\infty$ is unbounded.

Remark 4.0.3. *These two theorems are based on the first two statements of Theorem 3.4 of [4], but it differs from the original in the following points.*

i) Theorem C (repectively, Theorem D) provides the information that the continuum $\mathcal{C}_\infty$ lies in $[0,+\infty)\times H_0^1(\Omega)$ (respectively, $[0,+\infty)\times C_0(\overline{\Omega})$). This additional information is motivated by question Q_1) proposed in Chapter 1.

ii) The solutions of the continuum $\mathcal{C}_\infty$ of Theorem C are not necessarily continuous.

The next two subsections will be dedicated to enunciate and apply, to our problem, Leray and Lion's existence result given by Théorème 2 of [27] and Artola and Boccardo's comparison result given by Theorem 1 of [7].

4.1 The solution operator of the auxiliar problem $Q(u) = h$

This section is dedicated to construct a solution operator S, which associates for each $(\lambda,u)\in[0,+\infty)\times H_0^1(\Omega)$, solution $v\in H_0^1(\Omega)$ of the problem

$$\begin{cases} -\operatorname{div}(A(x,v)\nabla v) = f(\lambda,x,u) & x\in\Omega, \\ u = 0 & x\in\partial\Omega. \end{cases}$$

4.1.1 Leray and Lions existence Result and Artola and Boccardo's comparisson result

Below, to convenience of the readers, we will enunciate the existence result given by Théoreme 2 of [27]. Let Ω be a bounded open set in $\mathbb{R}^N$ and V a closed subset of $W^{m,p}(\Omega)$ satisfying the following conditions:

$$W_0^{m,p}(\Omega)\subset V\subset W^{m,p}(\Omega), \tag{HV1}$$
$$V\hookrightarrow W^{m-1,p}(\Omega) \text{ is a compact embedding,} \tag{HV2}$$

for $1<p<\infty$.

We will say that V satisfies the hypothesis (HV) when (HV1) and (HV2) hold. Consider a family of functions indexed by multi indexes α (of order N)

$$\begin{aligned} A_\alpha : \quad & \Omega\times\mathbb{R}^{N_1}\times\mathbb{R}^{N_2} \to \mathbb{R} \\ & (x,\eta,\xi) \mapsto A_\alpha(x,\eta,\xi), \end{aligned}$$

where N_1 (respectively, N_2) is the number of derivations D^α in $\mathbb{R}^N$ of order less than or equal $m-1$ (respectively of order equal m), and fix an order of multi indices

$$\beta_1,\beta_2,\ldots,\beta_{N_2},\quad |\beta_i| = m.$$

Thus, each multi index α with $|\alpha| = m$ corresponds to a $j\in\{1,2,\ldots,N_2\}$ such that $\alpha=\beta_j$. So for each $\xi\in\mathbb{R}^{N_2}$, we denote $\xi_\alpha = \xi_j := \langle\xi,e_j\rangle$.

Suppose that each A_α is Carathéodory function, i.e.:

i) the function $(\eta,\xi)\mapsto A_\alpha(x,\eta,\xi)$ is continuous over $\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}$ for a.e. $x\in\Omega$;

ii) the function $x \mapsto A_\beta(x,\eta,\xi)$ is measurable for all fixed $(\eta,\xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$.

We will use the following notation for the derivatives:

$$D^k u := (D^{\beta_1}u, D^{\beta_2}u, \dots, D^{\beta_{J_k}}u),$$

where $\{D^{\beta_j}u;\ j = 1,2,\dots,J_k\}$ denotes the set of all u derivatives of order $|\beta_j| = k$,

$$\delta u := (u, Du, \dots, D^{m-1}u),$$

$$A_\beta(x,\delta u, D^m v) : x \mapsto A_\beta(x, \delta u(x), D^m v(x))$$

and

$$v_\alpha := D^\alpha v$$

for each $v \in V$ and each multi index α with $|\alpha| = m$.

Moreover, we will need the following integrability hypothesis over A_β:

$$A_\beta(x,\delta u, D^m v) \in L^q(\Omega) \text{ for all } u,v \in W^{m,p}(\Omega), \text{ where } \frac{1}{p} + \frac{1}{q} = 1. \qquad \text{(HIA)}$$

In this case, it makes sense to define the operator

$$a(u,w) = \sum_{|\beta| \le m} \int_\Omega A_\beta(x,\delta u, D^m u) D^\beta w\, dx \qquad (4.1.1.1)$$

for $u,w \in V$. Observe that by the linearity of the integral and derivation operator and Hölder's inequality, we obtain that the function $w \mapsto a(u,w)$ is linear and continuous for each fixed $u \in V$. In other words,

$$a(u,w) = \langle A(u), w\rangle,\ A(u) \in V', \qquad (4.1.1.2)$$

where V' represents the topological dual space of V.

Given $f \in V'$, we want to prove the existence of $u \in V$ satisfying

$$A(u) = f \qquad (4.1.1.3)$$

or equivalently

$$a(u,w) = (f,w), \quad \forall\, w \in V. \qquad (4.1.1.4)$$

Theorem 4.1.1 (Théorème 2 [27]). *Let V be a closed subset of $W^{m,p}(\Omega)$ $(1 < p < +\infty)$ and $A_\beta : \Omega \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ a family of functions such that (HV) and (HIA) are satisfied. Assume also that the following hypotheses hold:*

$$\frac{|a(v,v)|}{\|v\|} \to \infty, \quad \text{as } \|v\| \to \infty,\ v \in V; \qquad (4.1.1.5)$$

for $\eta \in \mathbb{R}^{N_1}$ and $\xi \in \mathbb{R}^{N_2}$,

$$\sum_{|\alpha|=m} \frac{A_\alpha(x,\eta,\xi)\xi_\alpha}{|\xi| + |\xi|^{p-1}} \to \infty \quad \text{if } |\xi| \to \infty, \qquad (4.1.1.6)$$

for a.e. fixed $x \in \Omega$ and uniformly in each bounded subset of η in $\mathbb{R}^{N_1}$, and

$$\sum_{|\alpha|=m} [A_\alpha(x,\eta,\xi^*) - A_\alpha(x,\eta,\xi)][\xi^*_\alpha - \xi_\alpha] > 0 \qquad (4.1.1.7)$$

for a.e. x in Ω and $\xi \neq \xi^*$. Then, there exists a solution $u \in V$ of (4.1.1.3).

In order to construct the solution operator for the problem (P), we will also need the following comparison result by Artola and Boccardo [7] (1996).

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 1$ and consider a $N \times N$ matrix $B(x,s) = (b_{ij}(x,s))$, $1 \leq i,j \leq N$, where the coefficients $b_{ij}(x,s)$ are Carathéodory functions such that

$$b_{ij}(x,s) \in L^{\infty}(\Omega \times \mathbb{R}) \tag{4.1.1.8}$$

$$\exists\, \gamma, \beta > 0 : \ \gamma|\xi|^2 \leq B(x,s)\xi\xi \leq \beta|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^N \tag{4.1.1.9}$$

and

$$|B(x,s) - B(x,t)| \leq \omega(|s-t|) \quad \forall\, s,t \in \mathbb{R} \tag{4.1.1.10}$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}$ is an Osgood function, that is,

$$\omega \text{ is non decreasing, } \omega(0) = 0 \text{ and } \int_{0^+} \frac{ds}{\omega(s)} = +\infty.$$

Theorem 4.1.2. *Let B be a matrix satisfying the above conditions and $h_i \in H^{-1}(\Omega)$, $i = 1,2$ functionals satisfying $h_1(x) \leq h_2(x)$ for a.e. $x \in \Omega$. If $u_i \in H_0^1(\Omega)$ is a solution of*

$$Q(u_i) = h_i \ \text{ for } i = 1,2, \tag{4.1.1.11}$$

then $u_1(x) \leq u_2(x)$ for a.e. $x \in \Omega$, where

$$\begin{aligned} Q : \ & H_0^1(\Omega) \to H^{-1}(\Omega) \\ & u \mapsto Q(u) : \ H_0^1(\Omega) \to \mathbb{R} \\ & \qquad\qquad v \mapsto \int_\Omega B(x,u)\nabla u \nabla v. \end{aligned}$$

4.1.2 Applying Leray and Lions and Artola and Boccardo's results for (P_λ)

The hypotheses (A_1) and (A_2) make the problem of finding a $u \in H_0^1(\Omega)$ satisfying

$$Q(u) = h, \ \text{ for each } h \in H^{-1}(\Omega), \tag{4.1.2.1}$$

a Leray-Lions type problem, where

$$\begin{aligned} Q : \ & H_0^1(\Omega) \to H^{-1}(\Omega) \\ & u \mapsto Q(u) : \ H_0^1(\Omega) \to \mathbb{R} \\ & \qquad\qquad v \mapsto \int_\Omega A(x,u)\nabla u \nabla v. \end{aligned} \tag{4.1.2.2}$$

In other words, (4.1.2.1) satisfies the hypotheses of Théorème 2 of [27], so for each $h \in H^{-1}(\Omega)$ given, the problem (4.1.2.1) admits a solution.

Moreover, by assuming the hypothesis

$$|A(x,s) - A(x,t)| \leq \omega(|s-t|) \quad \forall\, s,t \in \mathbb{R} \tag{A_3}$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}$ is an Osgood function, we can deduce that such u is unique, after applying Artola and Boccardo's comparison result given in Theorem 4.1.2.

By combining existence and the uniqueness of solution for (4.1.2.1), we conclude that there exists a well defined operator $S := Q^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$, which associates for each $h \in H^{-1}(\Omega)$ the only solution $u \in H_0^1(\Omega)$ of (4.1.2.1).

In what follows, we will prove our claim that the problem (4.1.2.1) satisfies the hypotheses of the existence result by Leray and Lions given by Theorem 4.1.1 and the hypotheses of the comparison result by Artola and Boccardo given by Theorem 4.1.2 so that we can prove the following lemma.

Lemma 4.1.1 (Existence and uniqueness of weak solution). *Let A be a matrix as defined above satisfying (A_1) and (A_2). For each $h \in H^{-1}(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ satisfying*

$$Q(u) = h. \tag{4.1.2.3}$$

Proof. First, we have to show that (4.1.2.2) is well defined, that is, $Q(u) \in H^{-1}(\Omega)$ for each $u \in H_0^1(\Omega)$. Let $u \in H_0^1(\Omega)$. The linearity of the operator $v \mapsto Q(u)v$ follows directly from the linearity of the integral and of the weak gradient operator. As a consequence of the linearity, it is sufficient to prove the continuity of $v \mapsto Q(u)v$ at $v = 0 \in H_0^1(\Omega)$, but this is a consequence of Hölder inequality, as we show below.

$$\begin{aligned}
|Q(u)v| = \left|\int_\Omega A(x,u)\nabla u \nabla v\,dx\right| &\le \int_\Omega |A(x,u)\nabla u \nabla v|\,dx \\
&\overset{(A_1)}{\le} \int_\Omega \beta|\nabla u||\nabla v|\,dx \\
&\le \beta\left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}\left(\int_\Omega |\nabla v|^2 dx\right)^{1/2} \\
&= \beta\left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}\|v\|.
\end{aligned}$$

Therefore, $Q(u) \in H^{-1}(\Omega)$, thus (4.1.2.1) is well defined.

Now, let us set an appropriated V and $\mathcal{A}_\alpha$ such that the problem (4.1.1.3) can be formulated as (4.1.2.3). By taking $m = 1$, $p = 2$ and $V = H_0^1(\Omega)$, we obtain that (HV1) is clearly satisfied and by Rellich Kondrachov's Theorem, (HV2) also holds.

Observe that the choice $m = 1$ obliges N_1 to be equal 1 and N_2 to be equal N. So define

$$\begin{aligned}
\mathcal{A}_\alpha : \Omega \times \mathbb{R} \times \mathbb{R}^N &\to \mathbb{R} \\
(x,\eta,\xi) &\mapsto \begin{cases} \displaystyle\sum_{i=1}^N a_{ij}(x,\eta)\langle \xi, e_i\rangle & \text{if } \alpha = e_j,\ 1 \le j \le N, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Since a_{ij} is Carathéodory, it follows that $\mathcal{A}_\alpha$ is Carathéodory. Moreover, since $N_1 = 1$ we have $\delta u = \{u\}$ and consequently

$$\mathcal{A}_\alpha(x,\delta u, D^m v) = \mathcal{A}_\alpha(x,u,Dv) = \begin{cases} \displaystyle\sum_{i=1}^N a_{ij}(x,u)\langle Dv, e_i\rangle & \text{if } \alpha = e_j,\ 1 \le j \le N, \\ 0 & \text{otherwise,} \end{cases}$$

for each $u, v \in V = H_0^1(\Omega)$.

Thus the hypothesis (HIA) (for $q = 2$) is satisfied as a direct consequence of (A_1) and the fact that $v \in H_0^1(\Omega)$.

The definition (4.1.1.1) says that

$$\begin{aligned} a(u,v) &= \sum_{|\alpha|\le m}\int_\Omega A_\alpha(x,\delta u, D^m u)D^\alpha v\,dx \\ &= \sum_{|\alpha|\le 1}\int_\Omega A_\alpha(x,\delta u, Du)D^\alpha v\,dx \\ &= \int_\Omega A(x,u)\nabla u\nabla v, \qquad (4.1.2.1) \end{aligned}$$

which means that our definition of A_α is exactly what it needed to be.

It remains to verify the hypotheses (4.1.1.5) and (4.1.1.6). The hypothesis (4.1.1.5) is a consequence of (A_2) as follows:

$$\begin{aligned} \frac{|a(u,u)|}{\|u\|} = \frac{1}{\|u\|}\int_\Omega A(x,u)\nabla u\cdot\nabla u &\ge \frac{1}{\|u\|}\int_\Omega \gamma|\nabla u|^2 \\ &\ge \frac{\gamma}{\|u\|}\int_\Omega |\nabla u|^2 \\ &\ge \gamma\frac{\|u\|^2}{\|u\|} \to +\infty \text{ as } \|u\|\to\infty. \end{aligned}$$

To verify the hypothesis (4.1.1.6) observe that by (A_2),

$$\begin{aligned} \sum_{|\alpha|=m}\frac{A_\alpha(x,\eta,\xi)\xi_\alpha}{|\xi|+|\xi|^{p-1}} &= \frac{A(x,\eta)\xi\cdot\xi}{2|\xi|} \\ &\ge \frac{\gamma|\xi|^2}{2|\xi|} \to \infty \text{ as } |\xi|\to\infty. \end{aligned}$$

Finally, the hypothesis (4.1.1.7) is verified as follows:

$$\begin{aligned} \sum_{|\alpha|=m}[A_\alpha(x,\eta,\xi') - A_\alpha(x,\eta,\xi)][\xi'_\alpha - \xi_\alpha] &= \sum_{j=1}^N\sum_{i=1}^N[a_{ij}(x,\eta)\xi'_i - a_{ij}(x,\eta)\xi_i][\xi'_j - \xi_j] \\ &= \sum_{j=1}^N\sum_{i=1}^N[a_{ij}(x,\eta)(\xi'_i - \xi_i)][\xi'_j - \xi_j] \\ &= A(x,\eta)(\xi'-\xi)\cdot(\xi'-\xi) \\ &\ge \gamma|\xi'-\xi|^2 > 0 \text{ if } \xi' \ne \xi. \end{aligned}$$

So we conclude that the hypotheses of Theorem 4.1.1 are satisfied and consequently for each $h \in H^{-1}(\Omega)$, there exists $u \in H_0^1(\Omega)$ such that $Q(u) = h$. So Lemma 4.1.1 is proved.

In order to complete the construction of the solution operator of the problem (P), it remains to prove the uniqueness of the solution, which is obtained by applying Theorem 4.1.2 for the matrix $B = A$.

By (A_1), it follows that (4.1.1.8) is satisfied, on the other hand the hypothesis (4.1.1.9) is a consequence of (A_1) and (A_2). So, by assuming the hypothesis (4.1.1.10) over A, we conclude that if $u_1, u_2 \in H_0^1(\Omega)$ are solutions of

$$Q(u) = h,$$

then $u_1 \quad u_2$ and $u_1 \quad u_2$, due to Theorem 4.1.2 and the uniqueness of the solution of the problem $Q(u) = h$ is proved. □

Once proved the existence and the uniqueness of solution for $Q(u) = h$, we can define the solution operator $S := Q^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$, which associates for each $h \in H^{-1}(\Omega)$ the only solution $u \in H_0^1(\Omega)$ of $Q(u) = h$.

4.2 Proof of Theorems C and D

4.2.1 The Homotopies

Let E be the one of the spaces $H_0^1(\Omega)$ or $C_0(\overline{\Omega})$. To handle the problem of calculating the index $i(\quad, 0, 0)$ for $\lambda < \lambda_1$, we will prove that the function

$$H_1(\tau, z) = \begin{cases} z - \|z\|_E^2 S\left(\tau f\left(\lambda, x, \dfrac{z}{\|z\|_E^2}\right)\right) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0 \end{cases}$$

maps $[0,1] \times E$ into E, the operator

$$(\tau, z) \mapsto \|z\|_E^2 S\left(\tau f\left(\lambda, x, \frac{z}{\|z\|_E^2}\right)\right) \tag{4.2.1.1}$$

is compact in $[0,1] \times E$ and

$$H_1(\tau, z) \neq 0 \quad \text{in } [0,1] \times \partial B_R(0),$$

for some $R > 0$ such that $z = 0$ is the only solution of $\quad = 0$ in $B_R(0)$. So

$$H_1 : [0,1] \times \overline{B_R}(0) \to E \tag{4.2.1.2}$$

is an admissible homotopy and we can deduce that

$$\begin{aligned} i(\quad, 0) &= deg(H_1(1, \cdot), B_R(0), 0) \\ &= deg(H_1(0, \cdot), B_R(0), 0) \\ &= deg(I, B_R(0), 0) \\ &= 1. \end{aligned}$$

On the other hand, for $\lambda > \lambda_1$ we will prove that the function

$$H_2(t, z) = \begin{cases} z - \|z\|_E^2 S\left(f\left(\lambda, x, \dfrac{z}{\|z\|_E^2}\right) + \dfrac{t\phi}{\|z\|_E^2}\right) & \text{if } z \neq 0, \\ \quad_t & \text{if } z = 0 \end{cases}$$

maps $[0,1] \times E$ into E, where $\phi \in C_0^1(\overline{\Omega})$ is a positive function bounded away from zero and $\quad_t : \overline{\Omega} \to \mathbb{R}$ is the unique weak solution of

$$\begin{cases} -\operatorname{div}(A(x, 1)\nabla u) = t\phi \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.2.1.3}$$

the operator

$$(t,z) \mapsto \|z\|_E^2 S\left(f\left(\lambda, x, \frac{z}{\|z\|_E^2}\right) + \frac{t\phi}{\|z\|_E^2}\right) \tag{4.2.1.4}$$

is compact in $[0,1] \times E$ and

$$H_2(\tau, z) \neq 0 \;\text{ in } [0,1] \times \partial B_R(0)$$

for some $R > 0$ such that $z = 0$ is the unique solution of $\;= 0$ in $B_R(0)$. So

$$H_2 : [0,1] \times \overline{B_R}(0) \to E$$

is an admissible homotopy and we can deduce that

$$\begin{aligned} i(\;, 0) &= deg(H_2(0, \;), B_R(0), 0) \\ &= deg(H_2(1, \;), B_R(0), 0) \\ &= 0, \end{aligned}$$

where the last equality follows from the fact that $\| \;_1\|_E > 0$. In fact, in this case it is possible to assume R small enough such that

$$\|H_2(1,z) - H_2(1,0)\|_E > 0,$$

for all $z \in \overline{B_R}(0)$, due to the continuity of H_2, and so

$$H_2(1,z) \neq 0, \;\forall\, z \in \overline{B_R(0)}. \tag{4.2.1.5}$$

The compactness of the operators in (4.2.1.1) and (4.2.1.4) will be verified simultaneously by proving that, for each fixed $\lambda \in [0, +\infty)$, the operator

$$(\tau, t, z) \mapsto T(\tau, t, z) = \begin{cases} \|z\|_E^2 S\left(\tau f\left(\lambda, x, \frac{z}{\|z\|_E^2}\right) + \frac{t\phi}{\|z\|_E^2}\right) & \text{if } z \neq 0, \\ \qquad\qquad t & \text{if } z = 0 \end{cases}$$

is compact in $[0,1] \times [0,1] \times E$ and maps $[0,1] \times [0,1] \times E$ into E.

In order to prove the compactness of the operators T and K some estimates will be useful and so we will compile them in the following subsection.

4.2.2 Estimates

From now on, we will adopt the following notations and conventions.

i) $1' = \infty$ and $\infty' = 1$.

ii) $2^* = 2N/(N-2)$, if $N > 2$ and $2^* = \infty$, if $N \leq 2$.

iii) If q is a positive number, then $q\infty = \infty$.

Claim 4.2.1. *$H^1_0(\Omega) \hookrightarrow L^{2r'}(\Omega)$ is a compact embedding.*

Proof. Case $N=1$. In this case $N = 1 < 2$ and so item, by b) of Theorem 5.6.2 (Rellich Kondrachov), we deduce that

$$H_0^1(\Omega) \hookrightarrow C^0(\overline{\Omega})$$

is a compact embedding. If $r = 1$, then $2r' = 2\cdot\infty = \infty$, $L^{2r'}(\Omega) = L^\infty(\Omega)$ and

$$C^0(\overline{\Omega}) \hookrightarrow L^\infty(\Omega)$$

is a compact embedding by Arzelà-Ascoli Theorem. On the other hand, if $r > 1$, then $2r' < \infty$ and, by Arzelà-Ascoli Theorem and the Lebesgue's Dominated Convergence Theorem, we deduce that

$$C^0(\overline{\Omega}) \hookrightarrow L^{2r'}(\Omega)$$

is a compact embedding. Thus, the case $N = 1$ is concluded.
Case $N>1$. Since $r > N/2$, it follows that $r > 1$ and so $r' > 1$ is a positive number. If $N = 2$, then $2^* = \infty$ and so $2r' < 2^*$, whence by item a) of Theorem 5.6.2 (Rellich Kondrachov), we deduce that $H_0^1(\Omega) \hookrightarrow L^{2r'}(\Omega)$ is a compact embedding. On the other hand, if $N > 2$ then,

$$\begin{aligned}
2r' &= 2\left(\frac{r}{r-1}\right)\\
&= 2\frac{(r-1)+1}{r-1}\\
&= 2\left(1+\frac{1}{r-1}\right)\\
&< 2\left(1+\frac{1}{\frac{N}{2}-1}\right)\\
&= 2\left(1+\frac{2}{N-2}\right)\\
&= \frac{2N}{N-2} = 2^*,
\end{aligned}$$

whence by item a) of Theorem 5.6.2, $H_0^1(\Omega) \hookrightarrow L^{2r'}(\Omega)$ is a compact embedding. □

Claim 4.2.2. *If $r = 1$, then $(2r')' = r = 1$. On the other hand, if $r > 1$ then $(2r')' < r$.*

Proof. If $r = 1$ then $2r' = \infty$ and so $(2r')' = 1 = r$. Now, assume $r > 1$. Then $r' = r/(r-1)$ and

$$(2r')' = \frac{2r'}{2r'-1} = \frac{\frac{2r}{r-1}}{\frac{2r}{r-1}-1} = \frac{2r}{2r-(r-1)} = \frac{2r}{r+1},$$

that is,

$$(2r')' = \frac{2r}{r+1}. \tag{4.2.2.1}$$

So

$$r > 1 \Rightarrow r+1 > 2 \Rightarrow \frac{2}{r+1} < 1 \Rightarrow (2r')' = \frac{2r}{r+1} < r.$$

□

Remark 4.2.1. *Assume that $N \geq 2$ and define*

$$p := \frac{r+1}{2} > 1. \tag{4.2.2.2}$$

Observe that

$$(2r')'p \overset{(4.2.2.2)}{=} (2r')'\frac{r+1}{2} \overset{(4.2.2.1)}{=} \frac{2r}{r+1}\frac{r+1}{2} = r,$$

that is,

$$(2r')'p = r. \tag{4.2.2.3}$$

Moreover,

$$p' = \frac{p}{p-1} = \frac{\frac{r+1}{2}}{\frac{r+1}{2}-1} = \frac{r+1}{r-1}$$

and so

$$(2r')'p' \overset{(4.2.2.1)}{=} \left(\frac{2r}{r+1}\right)p' = \frac{2r}{r+1}\frac{r+1}{r-1} = 2r',$$

that is

$$(2r')'p' = 2r'. \tag{4.2.2.4}$$

Lemma 4.2.1 (Estimate I). *Let $D \in L^r(\Omega)$ be a positive function. If (u_n) is a bounded sequence in $H_0^1(\Omega)$, then there exists $h \in L^{2r'}(\Omega)$ such that*

$$|D(x)u_n(x)| \leq D(x)h(x) \text{ a.e. in } \Omega. \tag{4.2.2.5}$$

Moreover, $Dh \in L^{(2r')'}(\Omega)$.

Proof. Case $N=1$: Observe that $N = 1 < 2$ and so by item $b)$ of Theorem 5.6.2 and the boundedness of (u_n) in $H_0^1(\Omega)$, it follows that

$$\|u_n\|_{C^0(\overline{\Omega})} \leq M,$$

for some constant $M < 0$. Take then $h \equiv M$. So $h \in L^{(2r')'}(\Omega)$ and $Dh = DM \in L^r(\Omega)$. By Claim 4.2.2, it follows that $(2r')' \leq r$ and since $D \in L^r(\Omega)$, we conclude that $Dh \in L^{(2r')'}(\Omega)$.

Case $N > 1$. Since $r > N/2$, then $r > 1$. As we saw in the proof of Claim 4.2.1, the fact $r > 1$ implies that $2r' < 2^*$. By item $a)$ of Theorem 5.6.2, it follows that $u_n \to \overline{u}$ for some $\overline{u} \in L^{2r'}(\Omega)$. As a consequence of Proposition 5.3.2, we deduce that

$$|u_n(x)| \leq h(x) \text{ a.e. in } \Omega$$

for some $h \in L^{2r'}(\Omega)$. By multiplying by $D(x)$ we obtain (4.2.2.5).

Now let us prove that $Dh \in L^{(2r')'}(\Omega)$. Observe that by (4.2.2.3) and (4.2.2.4), we have

$$\begin{cases} \left(D^{(2r')'}\right)^p & = C_1^r \\ \left(h^{(2r')'}\right)^{p'} & = h^{2r'} \end{cases}$$

and so, by using Hölder's inequality for

$$D^{(2r')'} \quad \text{and} \quad h^{(2r')'}$$

with p and p', we obtain

$$\begin{aligned}
\int |Dh|^{(2r')'} &\le \left(\int |D|^r\right)^{1/p}\left(\int |h|^{2r'}\right)^{1/p'}\\
&\le \left[\left(\int |D|^r\right)^{\frac{1}{r}}\right]^{\frac{r}{p}}\left[\left(\int |h|^{2r'}\right)^{1/(2r')}\right]^{\frac{2r'}{p'}}\\
&\le \|D\|_r^{r/p}\|h\|_{2r'}^{2r'/p'} < \infty,
\end{aligned}$$

thus, $Dh \in L^{(2r')'}(\Omega)$ as we wanted. $\square$

Lemma 4.2.2 (Estimate II). *Let $D \in L^r(\Omega)$ and $z \in H_0^1(\Omega)$. Then the following estimates hold.*

$$\int |Dz| \le M\|D\|_{(2r')'}\|z\|_{2r'}, \ \forall z \in H_0^1(\Omega). \tag{4.2.2.6}$$

In particular, as a consequence of Claim 4.2.1, we have

$$\int |Dz| \le M\|D\|_{(2r')'}\|z\|, \ \forall z \in H_0^1(\Omega). \tag{4.2.2.7}$$

Proof. Case $r=1$: In this case $2r' = \infty$, which implies by Claim 4.2.1, that $z \in L^\infty(\Omega)$ and so

$$\begin{aligned}
\int |Dz| &\le \|z\|_\infty \int |D|\\
&= \|z\|_{2r'}\int |D|.
\end{aligned}$$

Moreover, by Claim 4.2.2, $(2r')' \le r$, whence

$$\int |D| \le M\|D\|_{(2r')'},$$

for some constant M and the estimate (4.2.2.6) holds.
Case $r>1$: By Hölder's inequality, we have

$$\int |Dz| \le \|D\|_{(2r')'}\|z\|_{2r'}.$$

$\square$

Lemma 4.2.3 (Estimate III). *Let $D \in L^r(\Omega)$. Then the following estimates holds*

$$\int |Dzy| \le \|D\|_r^{r/(p(2r')')}\|z\|_{2r'}^{2r'/(p'(2r')')}\|y\|_{2r'}, \ \forall z, y \in H_0^1(\Omega). \tag{4.2.2.8}$$

In particular, by Claim 4.2.1,

$$\int |Dzy| \le M\|D\|_r^{r/(p(2r')')}\|z\|_{2r'}^{2r'/(p'(2r')')}\|y\|, \ \forall z, y \in H_0^1(\Omega).$$

Proof. Case **$N > 1$:** Since $r > N/2$, it follows that $r > 1$ and so $r' > 1$ is a positive number and as we saw in the proof of Claim 4.2.1, the fact $r > 1$ this implies $2r' < 2^*$. Since $z \in H_0^1(\Omega)$, it follows by Rellich Kondrachov's Theorem that $z \in L^{2r'}(\Omega)$. Now, observe that

$$\begin{cases} \left|D^{(2r')'}\right|^p = D^r, \\ \left(|z|^{(2r')'}\right)^{p'} = |z|^{2r'}, \end{cases}$$

due to (4.2.2.3) and (4.2.2.4), where p is the number defined in Remark 4.2.1. Thus, by applying Hölder inequality to

$$D^{(2r')'} \quad \text{and} \quad (z)^{(2r')'}$$

with p and p', we deduce that

$$\begin{aligned} \int |Dz|^{(2r')'} &\le \left(\int |D|^r\right)^{1/p} \left(\int |z|^{2r'}\right)^{1/p'} \\ &\le \left[\left(\int |D|^r\right)^{\frac{1}{r}}\right]^{\frac{r}{p}} \left[\left(\int |z|^{2r'}\right)^{1/(2r')}\right]^{\frac{2r'}{p'}} \\ &\le \|D\|_r^{r/p} \|z\|_{2r'}^{2r'/p'}. \end{aligned}$$

Thus,

$$\left(\int |Dz|^{(2r')'}\right)^{\frac{1}{(2r')'}} \le \|D\|_r^{r/(p(2r')')} \|z\|_{2r'}^{2r'/(p'(2r')')}. \tag{4.2.2.9}$$

On the other hand, by applying Hölder's inequality for

$$|Dz| \quad \text{and} \quad |y|$$

with $(2r')'$ and $2r'$, we deduce that

$$\begin{aligned} \int |Dzy| &\le \left(\int |Dz|^{(2r')'}\right)^{\frac{1}{(2r')'}} \|y\|_{2r'} \\ &\overset{(4.2.2.9)}{\le} \|D\|_r^{r/(p(2r')')} \|z\|_{2r'}^{2r'/(p'(2r')')} \|y\|_{2r'}, \end{aligned}$$

that is, the estimate (4.2.2.8) holds.
Case **$N = 1$:** Observe that $N = 1 < 2$, so by item b) of Theorem 5.6.2, it follows that $z \in L^\infty(\Omega)$, in particular $z \in L^{2r'}(\Omega)$ and so we can proceed with the same arguments used in the previous case in order to conclude that the estimate (4.2.2.8) holds. □

4.2.3 $L^r(\Omega)$ and $L^{(2r')'}(\Omega)$-boundedness of the function f

In order to prove the compactness of the operators T and K we will also need the following claims.

Claim 4.2.3. *Let Ω be a bounded subset of* $\mathbb{R}^N$ *and (u_n) a sequence in $H_0^1(\Omega)$. Then, there exist positive functions $C, C_1 \in L^r(\Omega)$ such that*

$$|f(\lambda, x, u_n(x))| \le C(x) + C_1(x)u_n^+ \quad \forall\, x \in \Omega, \ n \in \mathbb{N}. \tag{4.2.3.1}$$

Proof. we deduce by (f_1) that there exists some positive function $C \in L^r(\Omega)$ such that for each $x \in \Omega$ such that $u_n(x) \le 0$,

$$|f(\lambda, x, u_n(x))| = f(\lambda, x, 0) \le C(x) \tag{4.2.3.2}$$

for all $\lambda \in \mathbb{R}$. On the other hand, if $u_n(x) > 0$ then, by (f_1), there exists positive functions $C_1, C_2 \in L^r(\Omega)$ such that

$$|f(\lambda, x, u_n(x))| = |f(\lambda, x, u_n^+(x))| \le C_1(x)u_n^+ + C_2(x).$$

So we conclude that (4.2.3.1) holds. □

Claim 4.2.4. *Let $\lambda \ge 0$. Then there exists a function $D \in L^r(\Omega)$ such that*

$$f(\lambda, x, s) \ge -D(x) \text{ a.e. in } \Omega, \forall s \ge 0.$$

Proof. Case $\lambda = \mathbf{0}$**:** By (f_0), we have that there exists some $s_0 > 0$ such that

$$f(0, x, s) \ge C_0(x) \text{ a.e. in } \Omega \ \forall s \ge s_0.$$

On the other hand by (f_1), we deduce that there exist a positive function $C \in L^r(\Omega)$ such that

$$f(0, x, s) \ge -C(x) \text{ a.e. in } \Omega \ \forall s \in [0, s_0].$$

So the case $\lambda = 0$ is concluded.
Case $\lambda > 0$: Since f_1' is bounded away from zero a.e., it follows that there exists $v > 0$ such that

$$f_1'(x) > v \text{ a.e. in } \Omega.$$

Moreover, by (f_2), there exists $s_1 > 0$ such that

$$\frac{f(\lambda, x, s)}{s} > \lambda f_1'(x) \ge \lambda v > 0, \ \forall s \ge s_1, \text{ a.e. in } \Omega.$$

In particular,

$$f(\lambda, x, s) > 0, \text{ a.e. in } \Omega, \ \forall s \ge s_1.$$

On the other hand, by (f_1), there exist a positive function $C \in L^r(\Omega)$ such that

$$f(\lambda, x, s) > -C(x), \text{ a.e. in } \Omega, \ \forall s \in [0, s_1].$$

Thus, Case 2 is concluded. □

4.2.4 Constructing the operator

As we will see in this section, the choice of the space E as being $H_0^1(\Omega)$ or $C_0(\overline{\Omega})$ depend on the integrability of the function

$$f\left(\lambda, x, \frac{z}{\|z\|_E}\right),$$

for a given $z \in E$.

Let $q \in \{r, (2r')'\}$ and define the space E by

$$\begin{cases} E(q) = H_0^1(\Omega) & \text{if } q = (2r')', \\ E(q) = C_0(\Omega) & \text{if } q = r. \end{cases} \tag{4.2.4.1}$$

Remark 4.2.2. *Observe that if $g \in L^q(\Omega)$, then the function $\varphi : H_0^1(\Omega) \to \mathbb{R}$ defined by*

$$\begin{array}{rccc} \varphi(g): & H_0^1(\Omega) & \to & \mathbb{R} \\ & v & \mapsto & \int_\Omega gv\,dx, \end{array}$$

is an operator in $H^{-1}(\Omega)$, due to Claim 4.2.1. Moreover, $q = r$, one follows by Theorem 5.4.3 that

$$S(\varphi(g)) \in C_0(\overline{\Omega}) = E(r).$$

Remark 4.2.3. *In order to make the reading more comfortable, we will adopt the following notation abuse*

$$S(g(x)) := S(\varphi(g)),$$

for each $g \in L^q(\Omega)$.

Remark 4.2.4. *By Hölder inequality, the linear application $g \mapsto \varphi(g)$ is a compact operator.*

Remark 4.2.5. *Let $u \in H_0^1(\Omega)$, then $u \in L^{2r'}(\Omega)$, by Claim 4.2.1. Combining Claim 4.2.3 and the second statement of Estimate I, we deduce that $f(\lambda, x, u(x)) \in L^{(2r')'}(\Omega)$. Thus, by Remark 4.2.2, we have that $\varphi(g) \in H^{-1}(\Omega)$.*

Remark 4.2.6. *Note that for each fixed $(\lambda, u) \in \mathbb{R} \times C_0(\overline{\Omega})$ the function*

$$x \mapsto f(\lambda, x, u(x))$$

belongs to $L^r(\Omega)$, due to (f_1) and so by Remark 4.2.2,

$$S(f(\lambda, x, u)) \in E(r) = C_0(\overline{\Omega}).$$

By Remarks 4.2.6 and 4.2.5, we deduce that it makes sense to consider the perturbation of the identity

$$\begin{array}{rccc} F: & [0, +\infty) \times E(q) & \to & E(q) \\ & (\lambda, u) & \mapsto & u - S(f(\lambda, x, u)). \end{array}$$

Observe that $u \in E(q)$ is a weak solution of (P) if, and only if,

$$F(\lambda, u) = 0. \tag{4.2.4.2}$$

By applying the change of variable introduced in Chapter 3 (see Remark 3.2.1) to the operator F, we obtain the operator $\overline{F} : [0, +\infty) \times E(q) \to E(q)$ defined by

$$\overline{F}(\lambda, z) = \begin{cases} \dfrac{z}{\|z\|^2_{E(q)}} - S\left(f\left(\lambda, x, \dfrac{z}{\|z\|^2_{E(q)}}\right)\right), & \text{if } z \neq 0, \\ 0 & , \text{ if } z = 0. \end{cases} \tag{4.2.4.3}$$

Remark 4.2.7. *Fix $\lambda \in [0, +\infty)$, $(\tau, t, z) \in [0,1] \times [0,1] \times C_0(\overline{\Omega})$. Let $\phi \in C_0^\infty(\overline{\Omega})$ be a positive function bounded away from zero and Φ_t as defined in (4.2.1.3). For $z = 0$, we already know that $T(\tau, t, 0) = \Phi_t \in C_0(\overline{\Omega})$. If $z \neq 0$, it follows from (4.0.0.2) that*

$$\begin{aligned} \left|\tau f\left(\lambda, x, \frac{z}{\|z\|_0^2}\right) + \frac{t\phi}{\|z\|_0^2}\right| &\le \tau\left(C_1(x)\frac{z}{\|z\|_0^2} + C(x)\right) + \frac{t\|\phi\|_\infty}{\|z\|_0^2} \\ &\le \tau\left(C_1(x)\frac{\|z\|_0}{\|z\|_0^2} + C(x)\right) + \frac{t\|\phi\|_\infty}{\|z\|_0^2} \in L^r(\Omega), \end{aligned}$$

and so by Remark 4.2.2 we obtain that $T(\tau, t, z) \in C_0(\overline{\Omega})$, whence T is an operator that maps $[0,1] \times [0,1] \times C_0(\overline{\Omega}) = E(r)$ into $C_0(\overline{\Omega})$.

As discussed in the motivation of Definition 3.2.3, for each bifurcation point from the curve of trivial solutions of $\tilde{F} = 0$, one corresponds a bifurcation point from the infinity of the problem $F = 0$.

Since our approach is based on Leray-Schauder degree theory and Theorem 3.4, it is appropriated to rewrite the equation $\tilde{F}(\lambda, z) = 0$ in the form

$$z - K(\lambda, z) = 0$$

where $K : [0, +\infty) \times E(q) \to E(q)$ is a compact operator.

Remark 4.2.8. *Observe that $(\lambda, z) \in [0, +\infty) \times E(q)$ is a solution of $\tilde{F}(\lambda, z) = 0$ if, and only if, (λ, z) is a solution of*

$$\Phi(\lambda, z) := z - K(\lambda, z) = 0,$$

where $K : [0, +\infty) \times E(q) \to E(q)$ defined by

$$K(\lambda, z) = \begin{cases} \|z\|^2_{E(q)} S\left(f\left(\lambda, x, \dfrac{z}{\|z\|^2_{E(q)}}\right)\right) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

is a compact operator (as we will see in the following two sections).

Let us define $\Phi_\lambda(z) := \Phi(\lambda, z)$, for each $(\lambda, z) \in [0, +\infty) \times E(q)$. In the next two sections, we will prove the following statements

$S_1)$ $i(\Phi_\lambda, 0) = 1$, for each $0 < \lambda < \lambda_\infty$,

$S_2)$ $i(\Phi_\lambda, 0) = 0$, for each $\lambda > \lambda_\infty$,

for each case $E = E((2r')') = H_0^1(\Omega)$ and $E = E(r) = C_0(\bar{\Omega})$.

4.3 Compactness of the operators K and T for the case $E = H_0^1(\Omega)$

Consider the Banach space $E := H_0^1(\Omega)$. In this section, we will prove the compactness of the operators K and T, in order to deduce Theorem C. So let us start from the following lemma.

Lemma 4.3.1. *If (τ_n, z_n) is a bounded sequence in $[0, 1] \times (E \setminus \{0\})$ such that $z_n \to 0$ in E, then*

$$\|z_n\|^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) \to 0 \text{ in } L^{(2r')'}(\Omega), \text{ up to a subsequence.} \tag{4.3.0.1}$$

Proof. The idea is to use the Dominated Lebesgue's Convergence Theorem. First, by using (4.2.3.1) for $u_n = z_n/\|z_n\|^2$, we get

$$\left|\|z_n\|^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right)\right| \le \tau_n (C_1(x) z_n^+ + \|z_n\|^2 C(x)). \tag{4.3.0.2}$$

So for n large enough we have

$$\left|\|z_n\|^2\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right)\right| \le \tau_n(C_1(x)z_n^+ + C(x)) \le C_1(x)z_n^+ + C(x)$$

because $\|z_n\| \to 0$. Since (z_n) is bounded in E, it follows by Remark (5.4.1) that the sequence (z_n^+) is bounded in E. By Estimate I (Lemma 4.2.1) it follows that

$$C_1(x)z_n^+(x) \le C_1(x)h(x) \text{ a.e. in } \Omega,$$

for some $h \in L^{2r'}(\Omega)$ and $C_1 h \in L^{(2r')'}(\Omega)$. Thus,

$$\left|\|z_n\|^2\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right)\right| \le C_1(x)h + C(x). \tag{4.3.0.3}$$

By Claim 4.2.2, it follows that $(2r')' \le r$ and so $C_1 h + C \in L^{(2r')'}(\Omega)$ and the domination hypothesis is verified.

Now, let us prove the hypothesis about the convergence a.e. in Ω. Let us denote

$$u_n := \frac{z_n}{\|z_n\|^2}.$$

Since $z_n \to 0$ in $H_0^1(\Omega)$, it follows that $z_n \rightharpoonup 0$ in $H_0^1(\Omega)$ and so

$$z_n \to 0 \text{ in } L^q(\Omega), \text{ for some } q \ge 1,$$

due to Rellich Kondrachov Theorem and hence

$$z_n \to 0 \text{ a.e. in } \Omega, \text{ up to a subsequence,} \tag{4.3.0.4}$$

due to Proposition (5.3.2).

Let $x \in \Omega$. There are two possibilities for the sequence $(u_n(x))_n$: it is bounded in R or not. In the first case, there exists some $v(x)$ such that $u_n(x) \to v(x)$, up to a subsequence, and so

$$\|z_n\|^2\tau_n f\left(\lambda, x, \frac{z_n(x)}{\|z_n\|^2}\right) \to 0.\tau f(\lambda, x, v(x)) = 0,$$

where τ is the limit (up to a subsequence) of τ_n. On the other hand, if the sequence $(u_n(x))_n$ is unbounded, then at least one of the following alternatives must occur:

i) $u_n(x) \to +\infty$ up to a subsequence;

ii) $u_n(x) \to -\infty$ up to a subsequence.

In the case i), we deduce that

$$\|z_n\|^2\tau_n f\left(\lambda, x, \frac{z_n(x)}{\|z_n\|^2}\right) = \frac{\tau_n f(\lambda, x, u_n(x))}{\frac{z_n}{\|z_n\|^2}} z_n(x) = \frac{\tau_n f(\lambda, x, u_n(x))}{u_n(x)} z_n(x)$$

and so by (f_2) we have

$$\|z_n\|^2\tau_n f\left(\lambda, x, \frac{z_n(x)}{\|z_n\|^2}\right) = \frac{\tau_n f(\lambda, x, u_n(x))}{u_n(x)} z_n(x) \to \lambda\tau f'_\infty(x).0.$$

In the case ii), we deduce that

$$\frac{\tau_n f(\lambda, x, u_n(x))}{u_n(x)} z_n(x) \to \tau\lambda f(\lambda, x, 0).0$$

by using the hypothesis $(f\;)$ and the fact (4.3.0.4).

By Lebesgue Dominated Convergence Theorem, we conclude that holds (4.4.0.1) and the lemma is proved. □

Lemma 4.3.2 (Compactness of the operator T). *Let $\phi \in C_0^1(\overline{\Omega})$ be a function such that $\phi > 0$ in Ω. Then, the operator*

$$T(\tau, t, z) = \begin{cases} \|z\|^2 S\left(\tau f\left(\lambda, x, \frac{z}{\|z\|^2}\right) + t\frac{\phi}{\|z\|^2}\right) & \text{if } z \neq 0, \\ {}_t & \text{if } z = 0, \end{cases} \tag{4.3.0.5}$$

with ${}_t$ as defined in (4.2.1.3), is compact in $[0,1] \times [0,1] \times E$.

Proof. Let $((\tau_n, t_n, z_n))_n$ be a bounded sequence in $[0,1] \times [0,1] \times E$. Without loss of generality, we can assume that $z_n \neq 0$ for all n and $(\tau_n, t_n) \to (\tau, t) \in [0,1]^2$. Define

$$w_n := S\left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) + \frac{t_n\phi}{\|z_n\|^2}\right) \tag{4.3.0.6}$$

that is,

$$\int A(x, w_n)\nabla w_n \nabla v = \int \left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) + \frac{t_n\phi}{\|z_n\|^2}\right) v, \quad \forall\, v \in H_0^1(\Omega). \tag{4.3.0.7}$$

By multiplying (4.4.0.7) by $\|z_n\|^2$ we get

$$\begin{aligned} \int A(x, w_n)\nabla(\|z_n\|^2 w_n)\nabla v\,dx &= \int \left(\|z_n\|^2\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) + t_n\phi\right) v \\ &= \int \left[\|z_n\|^2\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) v + t_n\phi v\right] \end{aligned} \tag{4.3.0.8}$$

for all $v \in H_0^1(\Omega)$. Let us define $y_n := z_n\|w_n\|$.

Take $v = y_n$ as a test function in (4.4.0.8). So we obtain from (A_2) that

$$\begin{aligned} \gamma\|y_n\|^2 &\leq \int A(x, w_n)\nabla y_n \nabla y_n \\ &= \int \|z_n\|^2\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) y_n + \int t_n\phi y_n \\ &\overset{(4{:}2{:}3{:}1)}{\leq} \int \tau_n(\|z_n\|^2 C(x) + C_1(x) z_n^+) y_n + \int t_n\phi y_n \\ &\leq \|z_n\|^2\int |C(x) y_n| + \int |C_1(x) z_n^+ y_n| \\ &+ \int |t_n\phi y_n|. \end{aligned} \tag{4.3.0.9}$$

So

$$\gamma\|y_n\|^2 \leq \|z_n\|^2\int |C(x) y_n| + \int |C_1(x) z_n^+ y_n| + \int |t_n\phi y_n|. \tag{4.3.0.10}$$

Now, it remains to estimate

$$\int |C_1(x) z_n^+ y_n|.$$

By Estimates II and III, we obtain

$$\begin{cases} \int |C(x) y_n| \leq M\|C\|_{(2r')'}\|y_n\|, \quad \text{(Estimate II)} & (4.3.0.11) \\ \int |C_1 z_n^+ y_n| \leq M\|C_1\|_r^{r/[p(2r')']}\|z_n^+\|^{2r'/[p'(2r')']}\|y_n\| \quad \text{(Estimate III).} & (4.3.0.12) \end{cases}$$

Moreover, since $\phi \in C_0^1(\overline{\Omega})$, we deduce that

$$\int |t_n \phi y_n| \leq t_n\|\phi\|_2\|y_n\|_2 \leq t_n M\|y_n\|, \tag{4.3.0.13}$$

for some constant $M > 0$.

By applying the estimates (4.3.0.11), (4.3.0.12) and (4.3.0.13) in (4.3.0.9), we obtain

$$\gamma\|y_n\|^2 \leq M(\|z_n\|^2\|y_n\| + \|z_n^+\|^{2r'/[p'(2r')']}\|y_n\| + t_n\|y_n\|),$$

so by diving both sides of the inequality by $\|y_n\| \neq 0$ and using Remark 5.4.1, we have

$$\gamma\|y_n\| \leq M(\|z_n\|^2 + \|z_n\|^{2r'/[p'(2r')']} + t_n). \tag{4.3.0.14}$$

Since (z_n) is a bounded sequence in E and $t_n \to 1$, it follows that

$$(y_n) \text{ is a bounded sequence in } E, \tag{4.3.0.15}$$

due to the estimate (4.4.0.10). Since $H_0^1(\Omega)$ is a reflexive space it follows that there exists some $y \in H_0^1(\Omega)$ such that

$$y_n \rightharpoonup y \text{ in } E.$$

By Claim 4.2.1, it follows that

$$y_n \to y \text{ in } L^{2r'}(\Omega). \tag{4.3.0.16}$$

By taking $v = y_n - y$ as a test function in (4.4.0.8), we obtain

$$\begin{aligned} \int A(x, w_n)\nabla y_n \nabla(y_n - y) &= \int \|z_n\|^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right)(y_n - y) + \\ &+ \int t_n \phi(y_n - y). \end{aligned}$$

Subtracting

$$\int A(x, w_n)\nabla y \nabla(y_n - y),$$

in both sides, using (A_2) and Estimates II and III, we obtain that

$$
\begin{aligned}
\gamma k y_n \quad y k^2 \qquad & \int A(x, w_n)\, r\, (y_n \quad y)\, r\, (y_n \quad y) \\
= \quad & \int k z_n k^2 \tau_n f\left(\lambda, x, \frac{z_n}{k z_n k^2}\right)(y_n \quad y) + \\
+ \quad & \int t_n \phi (y_n \quad y) \qquad \int A(x, w_n)\, r\, y\, r\, (y_n \quad y) \\
& \int j k z_n k^2 C(x) + C_1(x) z_n^+ j j y_n \quad y j + \\
+ \quad & \int t_n \phi (y_n \quad y) \qquad \int A(x, w_n)\, r\, y\, r\, (y_n \quad y), \\
& M (k z_n k^2 k C k_{(2r^0)^0} k y_n \quad y k_{2r^0} + \\
+ \quad & k C_1 k_r^{r=(p(2r^0)^0)} k z_n k_{2r^0}^{2r^0=(p^0(2r^0)^0)} k y \quad y_n k_{2r^0}) + t_n k \phi k_{\,1}\, k y_n \quad y k_1 + \\
+ \quad & \left| \int A(x, w_n)\, r\, y\, r\, (y_n \quad y) \right|.
\end{aligned}
$$

So

$$
\begin{aligned}
\gamma k y_n \quad y k^2 \qquad & M (k z_n k^2 k C k_{(2r^0)^0} k y_n \quad y k_{2r^0} + \\
+ \quad & k C_1 k_r^{r=(p(2r^0)^0)} k z_n k_{2r^0}^{2r^0=(p^0(2r^0)^0)} k y \quad y_n k_{2r^0}) + t_n k \phi k_{\,1}\, k y_n \quad y k_1 + \\
+ \quad & \left| \int A(x, w_n)\, r\, y\, r\, (y_n \quad y) \right|. \qquad (4.3.0.17)
\end{aligned}
$$

By the convergence $y_n \ ! \ y$ in $L^{2r^0}(\ \ )$ (see (4.3.0.16) we imply that the therms

$$
\begin{cases}
k z_n k^2 k C k_{(2r^0)^0} k y_n \quad y k_{2r^0}, \\
k C_1 k_r^{r=(p(2r^0)^0)} k z_n k_{2r^0}^{2r^0=(p^0(2r^0)^0)} k y \quad y_n k_{2r^0}, \\
t_n k \phi k_{\,1}\, k y_n \quad y k_1
\end{cases}
$$

converges to 0 up to a subsequence. Now, in order to prove the convergence of the last therm to zero, that is,

$$\left| \int A(x, w_n)\, r\, y\, r\, (y_n \quad y) \right| \ ! \ 0,$$

we will study the following two cases separately:

1) $k z_n k$ is not bounded away from 0;

2) $k z_n k$ is bounded away from 0.

Case 1): In this case, we can assume that $z_n \ ! \ 0$ in $H^1_0(\ \ )$.
Case 1a) $t = 0$**:** Since $t_n \ ! \ t = 0$, the estimate (4.4.0.10) implies that $y_n \ ! \ 0$ in E. In particular, if $(\tau_n, t_n, z_n) \ ! \ (\tau, 0, 0)$, then

$$T(\tau_n, t_n, z_n) = y_n \ ! \ 0 = \quad _0 = T(\tau, 0, 0),$$

that is, for every τ 2 $[0, 1]$ the operator T is continuous in $(\tau, 0, 0)$. So we just proved that if (τ_n, t_n, z_n) is a bounded sequence converging to $(\tau, 0, 0)$, then the sequence $T(\tau_n, t_n, z_n)$ converges to $T(\tau, 0, 0)$ in E, up to a subsequence. Therefore, Case 1a) is concluded.
Case 1b) $t > 0$**:** To prove the case t 2 $(0, 1]$, consider the following claim.

Claim 4.3.1. *If* t *2* $(0, 1]$*, then the following statements are true:*

i) there exists an infinite subset J *of* N *such that* $kw_n k$! + *1 for* n *2* J*,*

ii) there exists an n_0 *such that* w_n *is non negative for all* n n_0 *such that* n *2* J*.*

Proof. Let us prove i). We state that $kw_n k$! + *1* , up to a subsequence. In fact, suppose, by contradiction that $kw_n k$ is bounded. By testing (4.4.0.7) against a positive function v *2* $C_0^1(\bar{\ })$ such that $kvk>0$, we obtain

$$\int \left(\tau_n f\left(\lambda, x, \frac{z_n}{kz_n k^2}\right) + \frac{t_n \phi}{kz_n k^2}\right) v \qquad \int jA(x, w_n)\, r\, w_n\, r\, vj$$
$$\beta k w_n k k v k \quad M,$$

for some constant $M > 0$, so

$$\int \left(\tau_n f\left(\lambda, x, \frac{z_n}{kz_n k^2}\right) + \frac{t_n \phi}{kz_n k^2}\right) v \quad M, \ 8\ n. \tag{4.3.0.18}$$

On the other hand, since

$$\left(\tau_n f\left(\lambda, x, \frac{z_n}{kz_n k^2}\right) + \frac{t_n \phi}{kz_n k^2}\right) = \frac{1}{kz_n k^2}\left(kz_n k^2 \tau_n f\left(\lambda, x, \frac{z_n}{kz_n k^2}\right) + t_n \phi\right),$$

we can rewrite the (4.4.0.15) as

$$\frac{1}{kz_n k^2}\left[\int kz_n k^2 \tau_n f\left(\lambda, x, \frac{z_n}{kz_n k^2}\right) v + \int t_n \phi v\right] \quad M, \ 8n. \tag{4.3.0.19}$$

Now, observe that

$$\left|\int kz_n k^2 \tau_n f\left(\lambda, x, \frac{z_n}{kz_n k^2}\right) v\right| \qquad kvk_1 \int \left|kz_n k^2 \tau_n f\left(\lambda, x, \frac{z_n}{kz_n k^2}\right)\right|$$
$$M kvk_1 \left\|kz_n k^2 \tau_n f\left(\lambda, x, \frac{z_n}{kz_n k^2}\right)\right\|_{(2r^0)^0} \ ! \ 0,$$

due to Lemma 4.4.1. But

$$\int t_n \phi v \ ! \ \int t \phi v > 0$$

and so

$$\frac{1}{kz_n k^2}\int \left(kz_n k^2 \tau_n f\left(\lambda, x, \frac{z_n}{kz_n k^2}\right) v + t_n \phi v\right) \stackrel{n\,!\,1}{!} +1,$$

which contradicts (4.4.0.16) and so we conclude that $kw_n k$ is unbounded and hence there exists a set of indexes J such that $kw_n k$! + *1* (n *2* J), which proves i).

Let us prove ii) by using the weak formulation of Maximum Principle in (4.4.0.6). By Claim 4.2.4, it follows that

$$\tau_n f\left(\lambda, x, \frac{z_n}{kz_n k^2}\right) + t_n \frac{\phi}{kz_n k^2} \quad 0, \text{ up to a subsequence.} \tag{4.3.0.20}$$

So we deduce, by Theorem 5.4.1, that the function w_n is non negative for all n n_0 in J. $\square$

Moreover, there exists a number $\xi \geq 0$ such that

$$\|y_n\| \to \xi \quad n \in J \text{ up to a subsequence.} \tag{4.3.0.21}$$

We claim that $\xi > 0$. Indeed, let $v \in H_0^1(\Omega)$ be a positive function. By (A_1) and the Hölder inequality,

$$\left|\int A(w_n)\nabla(\|z_n\|^2 w_n)\nabla v\right| \leq \beta\|y_n\|\|v\|$$

and combining it with (4.4.0.8) we obtain

$$\left|\int \left(\|z_n\|^2\tau f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right) + t\phi\right) v\right| \leq \beta\|y_n\|\|v\|,$$

so we conclude by (4.4.0.17) that $\|y_n\| \nrightarrow 0$ and consequently $\xi > 0$.
Now, observe that since $y \geq 0$, it follows that

$$\left|\int A(x,w_n)\nabla y\nabla(y_n - y)\right| = \left|\int_{\Omega^+} A(x,w_n)\nabla y\nabla(y_n - y)\right|,$$

where

$$\Omega^+ := \{x \in \Omega\ ;\ y(x) > 0\}.$$

We claim that $m(\Omega^+) > 0$, on the contrary

$$\left|\int_{\Omega^+} A(x,w_n)\nabla y\nabla(y_n - y)\right| = 0$$

and by the convergence $y_n \to y$, in $L^{2r'}(\Omega)$ (see (4.3.0.16)), we imply that (RHS) of (4.3.0.17) converges to 0 and so we would obtain $\|y_n - y\| \to 0$, by whence $\|y_n\| \to \|y\|$, but since $\xi > 0$, it would imply that $\|y\| > 0$, but y is non negative and so it would imply $m(\Omega^+) > 0$, which is a contradiction. Thus, $m(\Omega^+) > 0$.

In order to prove the convergence

$$\left|\int_{\Omega^+} A(x,w_n)\nabla y\nabla(y_n - y)\right| \to 0,$$

consider the following claim.

Claim 4.3.2. *Let $v \in H_0^1(\Omega)$. Then the following convergence holds (up to a subsequence and for $n \in J$):*

$$\int_{\Omega^+} A(x,w_n)\nabla y_n\nabla v\,dx \to \int_{\Omega^+} A(x,+\infty)\nabla y\nabla v\,dx. \tag{4.3.0.22}$$

Proof. Consider the sequence of functionals $\varphi_n : H_0^1(\Omega^+) \to \mathbb{R}$ defined by

$$\varphi_n(v) = \int_{\Omega^+} A(x,w_n)\nabla y\nabla v.$$

By (A_1) and the fact $w_n = \|w_n\|\bar{w}_n \to +\infty$ a.e. in Ω^+, we deduce that

$$A(x,w_n)\nabla y \to A(x,+\infty)\nabla y \quad \text{in } (L^2(\Omega^+))^N,$$

by Lesbesgue Dominated Convergence Theorem, which implies that

$$\varphi_n \to \varphi \text{ in } H^{-1}(\Omega^+)$$

where $\varphi : H_0^1(\Omega^+) \to \mathbb{R}$ is the functional defined by

$$\varphi(v) = \int_{\Omega^+} A(x, +\infty)\nabla y \nabla v$$

and since $y_n \rightharpoonup y$, the Proposition (5.3.1) implies the convergence of the claim. □

By Claim 4.4.2, we conclude that $y_n \to y$ in $H_0^1(\Omega)$. Then, we just proved that the sequence $T(\tau_n, t_n, z_n)$ converges to some $y \in E$, in E, up to a subsequence, whenever (τ_n, t_n, z_n) is a bounded subsequence, $t_n \to t > 0$ and $\|z_n\|$ is not bounded away from zero. Now, let us prove the continuity of T. Since $y \geq 0$, it follows that

$$\begin{aligned}
\lim_{n\to+\infty}\int_\Omega A(x,w_n)\nabla y_n \nabla v dx &= \lim_{n\to+\infty}\int_{\Omega^+} A(x,w_n)\nabla y_n \nabla v dx \\
&= \int_{\Omega^+} A(x,+\infty)\nabla y \nabla v dx \\
&= \int_\Omega A(x,+\infty)\nabla y \nabla v dx, \ \forall v \in E. \quad (4.3.0.23)
\end{aligned}$$

On the other hand,

$$\lim_{n\to+\infty}\int_\Omega \left(\tau_n\|z_n\|^2 f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) + t_n\phi\right)v = \int_\Omega t\phi, \ \forall v \in E. \quad (4.3.0.24)$$

Indeed, Lemmas 4.2.1, 4.4.1 and Estimate II, we deduce that

$$\int_\Omega \left|\tau_n\|z_n\|^2 f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right)v\right| \leq \left\|\tau_n\|z_n\|^2 f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right)\right\|_{(2r')'}\|v\|_{2r'} \to 0,$$

so (4.4.0.21) holds. By combing (4.4.0.20) and (4.4.0.21), we deduce by passing to the limit the equation (4.4.0.8) that

$$\int_\Omega A(x,+\infty)\nabla y \nabla v dx = \int_\Omega t\phi v \ \forall v \in E,$$

which means that $y = T(\tau, t, 0)$. We just proved that $T(\tau_n, t_n, z_n) \to T(\tau, t, 0)$ whenever (τ_n, t_n, z_n) is a bounded sequence such that $(\tau_n, t_n, z_n) \to (\tau, t, 0)$, up to a subsequence, with $t > 0$. This proves the Case 1b) and so Case 1 is concluded.

Case 2: In this case, since (z_n) is a bounded sequence, then the sequence $z_n/\|z_n\|^2$ is bounded in $H_0^1(\Omega)$. So by using Lemma 4.2.3 and Estimate I, we deduce that

$$\begin{aligned}
\left|\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) + \frac{t_n\phi}{\|z_n\|^2}\right| &\leq C(x) + C_1(x)\frac{z_n^+}{\|z_n\|^2} + \frac{t_n\phi}{\|z_n\|^2} \\
&\leq C(x) + C_1(x)h(x) + \frac{t_n\phi}{\|z_n\|^2}, \quad (4.3.0.25)
\end{aligned}$$

for some $h \in L^{2r'}(\Omega)$ and $C_1 h \in L^{(2r')'}(\Omega)$. Consequently,

$$\left(\left\|\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) + \frac{t_n\phi}{\|z_n\|^2}\right\|_{(2r')'}\right)_n$$

is a bounded sequence. By taking $v = w_n$ as a test function in (4.4.0.7), using (A_2), Estimate II and Claim 4.2.1, we obtain

$$\begin{aligned} \gamma\|w_n\|^2 &\leq \int A(x, w_n)\,\nabla w_n \nabla w_n \\ &= \int \left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) + \frac{t_n\phi}{\|z_n\|^2}\right) w_n \\ &\leq \left\|\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) + \frac{t_n\phi}{\|z_n\|^2}\right\|_{(2r')'} M\|w_n\|, \end{aligned} \tag{4.3.0.26}$$

which means that (w_n) is a bounded sequence in $H_0^1(\Omega)$ and so

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega),$$

for some $w \in H_0^1(\Omega)$. Moreover, by Claim 4.2.1, it follows that

$$w_n \to w \text{ in } L^{2r'}(\Omega) \text{ up to a subsequence,} \tag{4.3.0.27}$$

in particular $w_n \to w$ a.e. in Ω up to a subsequence and so, by Lebesgue Dominated Convergence, we deduce that

$$A(x, w_n)\,\nabla y \to A(x, w)\,\nabla y \text{ in } L^2(\Omega)$$

and similarly as we argued in Claim 4.4.2, we deduce that

$$\int A(x, w_n)\,\nabla y \nabla y_n \to \int A(x, w)\,\nabla y \nabla y$$

and so

$$\left|\int A(x, w_n)\,\nabla y\,\nabla (y_n - y)\right| \to 0, \tag{4.3.0.28}$$

whence we conclude that

$$T(\tau_n, t_n, z_n) = y_n \to y \text{ in } H_0^1(\Omega). \tag{4.3.0.29}$$

Thus, we just proved that $(T(\tau_n, t_n, z_n))$ converges to some $y \in E$, up to a subsequence, whenever (τ_n, t_n, z_n) is a bounded sequence and $\|z_n\|$ is bounded away from zero. Now, let us prove the continuity of T. By taking $v = w_n - w$ as a test function in (4.4.0.7) we have

$$\int A(x, w_n)\,\nabla w_n \nabla (w_n - w) = \int \left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) + \frac{t_n\phi}{\|z_n\|^2}\right)(w_n - w)$$

and by subtracting

$$\int A(x, w_n)\,\nabla w\,\nabla (w_n - w),$$

using (A_2) and Estimates II and III, we obtain that

$$
\begin{aligned}
\gamma\|w_n - w\|^2 &\leq \int A(x,w_n)\nabla(w_n - w)\nabla(w_n - w)\\
&= \int \|z_n\|^2\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right)(w_n - w) +\\
&+ \int t_n\frac{\phi}{\|z_n\|^2}(w_n - w) - \int A(x,w_n)\nabla w\nabla(w_n - w)\\
&\leq \int \left|C(x) + C_1(x)\frac{z_n^+}{\|z_n\|^2}\right| |w_n - w| +\\
&+ \int t_n\frac{\phi}{\|z_n\|^2}(w_n - w) - \int A(x,w_n)\nabla w\nabla(w_n - w),\\
&\leq M(\|C\|_{(2r')'}\|w_n - w\|_{2r'} +\\
&+ \|C_1\|_r^{r/(p(2r')')}\|z_n\|_{2r'}^{2r'/(p'(2r')')}\|w - w_n\|_{2r'}) + t_n\|\phi\|\|w_n - w\|_1 +\\
&+ \left|\int A(x,w_n)\nabla w\nabla(w_n - w)\right|.
\end{aligned}
$$

By the convergences (4.4.0.27) and (4.4.0.28), it follows that the (RHS) converges to zero up to a subsequence, thus,

$$w_n \to w \text{ in } E. \tag{4.3.0.30}$$

By the same argument used to prove (4.4.0.28), we can deduce that

$$\lim_{n\to+\infty}\int A(x,w_n)\nabla w_n\nabla v = \int A(x,w)\nabla w\nabla v \quad \forall v \in E. \tag{4.3.0.31}$$

Moreover, as we argueed in (4.4.0.22), we deduce that there exists some $D \in L^{(2r')'}(\Omega)$ such that

$$\left|\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right)\right| \leq D(x) \text{ a.e. in } \Omega.$$

So by Lebesgue's Dominated Convergence Theorem, we deduce that

$$\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) \to \tau f\left(\lambda, x, \frac{z}{\|z\|^2}\right) \text{ in } L^{(2r')'}(\Omega).$$

But, by Estimate II, the expression

$$\left|\int\left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) - \tau f\left(\lambda, x, \frac{z}{\|z\|^2}\right)\right)v\right|$$

is estimated from above by

$$\left\|\left(\tau_n\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) - \tau f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right)\right)\right\|_{(2r')'}\|v\|_{2r'},$$

consequently,

$$\lim_{n\to+\infty}\int\left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|^2}\right) + t_n\frac{\phi}{\|z_n\|}\right)v = \int\left(\tau f\left(\lambda, x, \frac{z}{\|z\|^2}\right) + t\frac{\phi}{\|z\|}\right)v. \tag{4.3.0.32}$$

Now, by combining (4.4.0.30) and (4.4.0.31), we deduce that by passing to the limit the equation (4.4.0.7), we obtain

$$\int_\Omega A(x,w)\nabla w \nabla v = \int_\Omega \left(\tau f\left(\lambda, x, \frac{z}{\|z\|^2}\right) + t\frac{\phi}{\|z\|}\right) v, \ \forall v \in E,$$

in other words,

$$w = S(\tau, t, z). \tag{4.3.0.33}$$

But $w_n = S(\tau_n, t_n, z_n)$ and so, the convergence (4.3.0.30) proves the continuity of S in (τ, t, z) for $z \neq 0$. Since $z \mapsto \|z\|^2$ is a continuous function, it follows that T is continuous in (τ, t, z). □

In order to prove the compactness of the operator

$$K : [0, +\infty) \times E \to E$$

consider the following lemma.

Lemma 4.3.3. *If (λ_n, z_n) is a bounded sequence in $[0, +\infty) \times E$ such that $z_n \to 0$, then*

$$\|z_n\|^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right) \to 0 \text{ in } L^{(2r')'}(\Omega), \text{ up to a subsequence.} \tag{4.3.0.34}$$

Proof. Let (λ_n, z_n) be a bounded sequence in $[0, +\infty) \times E$. Then there exists $\lambda \in [0, +\infty)$ such that $\lambda_n \to \lambda$, up to a subsequence. The idea to use the Dominated Lebesgue's Convergence Theorem. So let us show that the domination hypothesis is verified. By using Claim 4.2.3, we deduce that

$$\left|\|z_n\|^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right)\right| \leq \|z_n\|^2 C(x) + C_1(x) z_n^+, \tag{4.3.0.35}$$

since $\|z_n\| \to 0$, we assume n large enough so that

$$\left|\|z_n\|^2 \ f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right)\right| \leq C(x) + C_1(x) z_n^+ \tag{4.3.0.36}$$

By Estimate I, we can deduce that there exists some $D \in L^{(2r')'}(\Omega)$ such that

$$\left|\|z_n\|^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right)\right| \leq D(x) \text{ up to a subsequence}$$

and so the domination hypothesis is verified.
Let us denote

$$u_n := \frac{z_n}{\|z_n\|^2}.$$

Since $z_n \to 0$ in $H_0^1(\Omega)$, it follows that

$$z_n \to 0 \text{ a.e. in } \Omega \text{ up to a subsequence,} \tag{4.3.0.37}$$

due to Claim 4.2.1 and Proposition 5.3.2.

Let $x \in \Omega$. There are two possibilities for the sequence $(u_n(x))_n$, it is bounded or not. In the first case, there exists some $v(x)$ such that $u_n(x) \to v(x)$ up to a subsequence and so

$$\|z_n\|^2 f\left(\lambda_n, x, \frac{z_n(x)}{\|z_n\|^2}\right) \to 0.f(\lambda, x, v(x)) = 0 \ \text{ up to a subsequence,}$$

In the second case, at least one of the following alternatives must occur:

i) $u_n(x) \to +\infty$ up to a subsequence;

ii) $u_n(x) \to -\infty$ up to a subsequence.

In the case i), we deduce that

$$\|z_n\|^2 f\left(\lambda_n, x, \frac{z_n(x)}{\|z_n\|^2}\right) = \frac{f(\lambda_n, x, u_n(x))}{\frac{z_n}{\|z_n\|^2}} z_n(x) = \frac{f(\lambda_n, x, u_n(x))}{u_n(x)} z_n(x)$$

and so

$$\|z_n\|^2 f\left(\lambda_n, x, \frac{z_n(x)}{\|z_n\|^2}\right) = \frac{f(\lambda_n, x, u_n(x))}{u_n(x)} z_n(x) \to \lambda f_+(x).0$$

by (f_3).

In the case ii), we deduce that

$$\frac{f(\lambda_n, x, u_n(x))}{u_n(x)} z_n(x) \to \lambda f(\lambda, x, 0).0$$

by using the hypothesis (f) and the fact (4.4.0.36).

By Lebesgue Dominated Convergence Theorem, we conclude that holds (4.4.0.33) and the lemma is proved. □

Lemma 4.3.4 (Compactness of the operator K). *The operator*

$$K : [0, +\infty) \times E \to E$$

defined in Remark 4.2.8 is compact.

Proof. Let (λ_n, z_n) be a bounded sequence in $[0, +\infty) \times E$. Without loss, we can assume that $z_n \neq 0$ for all n and $\lambda_n \to \lambda \in [0, +\infty)$. Define

$$w_n := S\left(f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right)\right)$$

that is,

$$\int A(x, w_n) \nabla w_n \nabla v\,dx = \int f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right) v\,dx, \quad \forall\, v \in H_0^1(\Omega). \tag{4.3.0.38}$$

and $y_n := \|z_n\|^2 w_n$, thus

$$K(\lambda_n, z_n) = \|z_n\|^2 w_n = y_n.$$

We will study the following two cases separately:

1) $\|z_n\|$ is not bounded away from 0;

2) $\|z_n\|$ is bounded away from 0.

Case 1): In this case we can assume that $z_n \to 0$.

By multiplying (4.4.0.37) by $\|z_n\|^2$ we get

$$\int A(x, w_n) \nabla (\|z_n\|^2 w_n) \nabla v\,dx = \int \|z_n\|^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right) v\,dx \text{ for all } v \in H_0^1(\Omega). \tag{4.3.0.39}$$

and by taking $v = y_n$ as a test function in (4.4.0.38), using Claim (4.2.1) and Estimate II, we obtain

$$
\begin{aligned}
\gamma \|y_n\|^2 &\leq \int A(x, w_n) \nabla y_n \nabla y_n \\
&= \int \|z_n\|^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right) y_n \\
&\leq \left\| \|z_n\|^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right) \right\|_{(2r')'} \|y_n\|_{2r'} \\
&\leq M \left\| \|z_n\|^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right) \right\|_{(2r')'} \|y_n\|
\end{aligned}
$$

by whence

$$
\gamma \|y_n\| \leq M \left\| \|z_n\|^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right) \right\|_{(2r')'},
$$

but, by Lemma 4.4.0.33 the (RHS) converges to zero and so we conclude that $K(\lambda_n, z_n) = y_n \to 0 = K(\lambda, 0)$ in E. Thus, Case 1 is concluded.
Case 2): By taking $v = y_n$ as a test function in (4.4.0.38) and using Claim 4.2.3, we obtain

$$
\begin{aligned}
\gamma \|y_n\|^2 &\leq \int A(x, w_n) \nabla y_n \nabla y_n \\
&= \int \|z_n\|^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right) y_n \\
&\leq \int \|z_n\|^2 C(x) y_n + C_1(x) z_n^+ y_n.
\end{aligned}
\tag{4.3.0.40}
$$

By using Estimates II and III, we obtain

$$
\gamma \|y_n\|^2 \leq M(\|z_n\|^2 \|y_n\| + \|z_n^+\|^{2r'/[p'(2r')']} \|y_n\|)
$$

and since (z_n) is bounded in E, it follows that

$$
\gamma \|y_n\|^2 \leq M \|y_n\|, \tag{4.3.0.41}
$$

which means that the sequence (y_n) is bounded in E and consequently there exist some $y \in E$ such that

$$
y_n \rightharpoonup y \text{ in } E.
$$

By Claim 4.2.1, it follows that

$$
y_n \to y \text{ in } L^{2r'}(\Omega), \text{ up to a subsequence.} \tag{4.3.0.42}
$$

Moreover, since $\|z_n\|$ is bounded away from zero and (y_n) is a bounded sequence, it follows that the sequence $w_n = \|z_n\|^{-2} y_n$ is bounded E. As we argued in Case 2 of the proof of Lemma 4.4.2, we have

$$
\int A(x, w_n) \nabla y \nabla (y_n - y) \to 0, \tag{4.3.0.43}
$$

due to item b) of Proposition 5.3.1.

By taking $v = y_n - y$ as a test function in (4.4.0.8), we obtain

$$\int A(x, w_n)\nabla y_n \nabla(y_n - y) = \int \|z_n\|^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right)(y_n - y),$$

by subtracting

$$\int A(x, w_n)\nabla y \nabla(y_n - y)$$

and using (A_2), we get

$$\begin{aligned}
\gamma\|y_n - y\|^2 &\leq \int A(x, w_n)\nabla(y_n - y_n)\nabla(y_n - y) \\
&= \int \|z_n\|^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|^2}\right)(y_n - y) - \int A(x, w_n)\nabla y \nabla(y_n - y) \\
&\leq \int |\|z_n\|^2 C(x) + C_1(x) z_n^+||y_n - y| \\
&\quad - \int A(x, w_n)\nabla y \nabla(y_n - y) \\
&\leq M(\|z_n\|^2 \|C\|_{(2r')'}\|y_n - y\|_{2r'} + \\
&\quad + \|C_1\|_r^{r/(p(2r')')}\|z_n\|_{2r'}^{2r'/(p'(2r')')}\|y_n - y\|_{2r'}) + \\
&\quad + \left|\int A(x, w_n)\nabla y \nabla(y_n - y)\right|.
\end{aligned}$$

By the convergences $y_n \to y$ in $L^{2r'}(\Omega)$ (see (4.3.0.42)) and (4.3.0.43), it follows that the (RHS) converges to zero, up to a subsequence and consequently $y_n \to y$ in E, that is, $K(\lambda_n, z_n) = y_n \to y$ in E. The proof of the continuity follows similarly as we done in the proof of Case 2 of Lemma 4.4.2. □

The following lemma proves that if $0 \leq \lambda < \lambda_1$, then $(\lambda, 0)$ is not a bifurcation point from the curve of trivial solutions of $\Phi(\lambda, z) = 0$.

Lemma 4.3.5. *Assume the hypotheses (f), (A_1)-(A_4) and (f_0)-(f_3) and let $\Lambda \subset [0, \lambda_1)$ be a compact interval. Then there exists a number $R > 0$ such that*

$$u \neq S(tf(\lambda, x, u)) \tag{4.3.0.44}$$

for all $u \in E$ with $\|u\| \geq R$, all $\lambda \in \Lambda$ and all $t \in [0, 1]$.

Proof. Suppose that there exists sequences (λ_n) in Λ, (t_n) in $[0, 1]$ and (u_n) in E with $\|u\| \to \infty$ such that

$$u_n = S(t_n f(\lambda_n, x, u_n)). \tag{4.3.0.45}$$

Since Λ and $[0, 1]$ are compact sets, we deduce the existence of $\lambda \in \Lambda$ and $t \in [0, 1]$ such that $\lambda_n \to \lambda$ and $t_n \to t$, up to a subsequence. Let us define the normalized sequence $z_n := u_n\|u_n\|^{-1}$ and note that by dividing (4.4.0.43) by $\|u_n\|$, we obtain that z_n satisfies the equation

$$\int A(x, u_n)\nabla z_n \nabla v = t_n \int \frac{f(\lambda_n, x, u_n)}{\|u_n\|} v, \quad \forall\, v \in H_0^1(\Omega). \tag{4.3.0.46}$$

By taking $v = u_n$ as a test function and using (A_2), (f_∞) and (f_0), we deduce that

$$\gamma\frac{\|u_n\|^2}{\|u_n\|} - \int t_n \frac{f(\lambda_n, x, u_n)}{\|u_n\|} u_n \leq 0 \tag{4.3.0.47}$$

and so $u_n \leq 0$ for all n.

Since the sequence (z_n) is bounded in the reflexive space E, it follows that there exists some $z \in E$ such that

$$z_n \rightharpoonup z \quad \text{in } E \tag{4.3.0.48}$$

and so by Claim 4.2.1,

$$z_n \to z \text{ in } L^{2r'}(\Omega), \text{ up to a subsequence.} \tag{4.3.0.49}$$

By taking $v = z_n - z$ as a test function in (4.3.0.46), (A_2) and subtracting

$$\int A(x, u_n) \nabla z \nabla(z_n - z),$$

in both sides of (4.3.0.46) we obtain

$$\begin{aligned}
\gamma\|z_n - z\|^2 &\leq \int A(x, u_n) \nabla(z_n - z) \nabla(z_n - z) \\
&\leq t_n \int \frac{f(\lambda_n, x, u_n)}{\|u_n\|}(z_n - z) - \int A(x, u_n) \nabla z \nabla(z_n - z) \\
&= t_n \int \frac{f(\lambda_n, x, u_n)}{\|u_n\|}(z_n - z) - \int_{\Omega^+} A(x, u_n) \nabla z \nabla(z_n - z),
\end{aligned}$$

where

$$\Omega^+ = \{x \in \Omega;\ z(x) > 0\}.$$

By using Claim 4.2.3 and the facts that $\|u_n\| \to \infty$ and $t_n \in [0,1]$, we deduce that

$$\begin{aligned}
\gamma\|z_n - z\|^2 &\leq \int \left| C(x) + C_1(x)\frac{u_n^+}{\|u_n\|} \right| |z_n - z| + \\
&+ \left| \int_{\Omega^+} A(x, u_n) \nabla z \nabla(z_n - z) \right|.
\end{aligned}$$

By using the Estimates II and III, we deduce that

$$\gamma\|z_n - z\|^2 \leq M\|z_n - z\|_{2r'} + \left| \int_{\Omega^+} A(x, u_n) \nabla z \nabla(z_n - z) \right|$$

Similarly as we done in Lemma 4.4.6, we deduce that $m(\Omega^+) > 0$ (observe that $\|z_n\| = 1$). Note that
$u_n = \|u_n\| z_n \to +\infty$ a.e. in Ω^+, then $A(x, u_n)\nabla z \to A(x, \infty)\nabla z$ in $L^2(\Omega^+)^N$, due to (A_1) and the Dominated Convergence Lebesgue Theorem. Since $z_n \rightharpoonup z$ (see (4.4.0.64)), it follows by Proposition 5.3.1 that

$$\lim_{n\to\infty} \int_{\Omega^+} A(x, u_n) \nabla z_n \nabla v = \lim_{n\to\infty} \int_{\Omega^+} A(x, +\infty) \nabla z \nabla v,\ \forall\, v \in E. \tag{4.3.0.50}$$

Moreover, by using Claim 4.2.3, Estimates I, II and III, we can deduce that there exist some $D \in L^{(2r')'}(\Omega)$ such that

$$\left|\frac{t_n f(\lambda_n, x, u_n)}{\|u_n\|}\right| \leq D(x), \text{ a.e. in } \Omega.$$

Moreover, for every $x \in \Omega$ such that $u_n(x) > 0$, we have

$$\frac{t_n f(\lambda_n, x, u_n)}{\|u_n\|} = \frac{t_n f(\lambda_n, x, u_n)}{u_n} z_n(x)$$

thus,

$$\frac{t_n f(\lambda_n, x, u_n)}{\|u_n\|} \to t\lambda f'_\infty z, \text{ a.e. in } \Omega.$$

Consequently, by Lebesgue's Dominated Convergence Theorem and the hypothesis (f_2), we deduce that

$$\frac{t_n f(\lambda_n, x, u_n)}{\|u_n\|} \to t\lambda f'_\infty z \text{ in } L^{(2r')'}(\Omega).$$

By using the Estimate II, we deduce that

$$\left|\int_\Omega \left(\frac{t_n f(\lambda_n, x, u_n)}{\|u_n\|} - t\lambda f_\infty z\right) v\right| \leq \left\|\frac{t_n f(\lambda_n, x, u_n)}{\|u_n\|} - t\lambda f_\infty z\right\|_{(2r')'} \|v\|_{2r'}, \ \forall\, v \in E.$$

So

$$\lim_{n \to +\infty} \frac{t_n f(\lambda_n, x, u_n)}{\|u_n\|} v = t\lambda f'_\infty z v, \ \forall\, v \in E. \tag{4.3.0.51}$$

Combining (4.4.0.49) and (4.4.0.50) we deduce, by passing to the limit in equation (4.3.0.46), that z is a non trivial and non negative solution of

$$-\operatorname{div}(A(x, \infty) \nabla z) = t\lambda f'_\infty(x) z. \tag{4.3.0.52}$$

By testing against ψ and using that λ_∞ is an eigenvalue of (4.0.0.7), we obtain

$$\begin{aligned} \int_\Omega \lambda_\infty f'_\infty z\psi &= \int_\Omega A(x, \infty) \nabla z \nabla \psi \\ &= \int_\Omega t\lambda f'_\infty z\psi. \end{aligned}$$

Hence $t\lambda = \lambda_\infty$ and $z = \psi$, where ψ is the eigenfunction (associated to λ_∞) of the Dirichlet eigenvalue problem with weight (4.0.0.7), as we defined before, but this contradicts the hypothesis $\lambda < \lambda_\infty$. □

Remark 4.3.1. *If we consider the sequence $t_n = 1$ for all n, then the argument in (4.3.0.47) shows that every solution u of $\Phi(u) = 0$ is non negative.*

Remark 4.3.2 (Necessary condition for some λ^* to be a bifurcation point from infinity of (P)). *Let $\lambda^* \geq 0$ be such that λ^* is a bifurcation point from infinity of the problem (P). If we take $\Lambda = \{\lambda_n\}$, where λ_n is a sequence in $[0, +\infty)$ such that $\lambda_n \to \lambda^*$ and (t_n) be the sequence defined by $t_n = 1$ for all n, then the same arguments leads us to (4.3.0.52) with $t = 1$ and so we conclude that $\lambda^* = \lambda_\infty$.*

Corollary 4.3.1. *As a consequence of Remark (5.4.1), we can replace $\|u\| \geq R$ by $\|u\|_0 \geq R$ in the conclusion of Lemma 4.3.5.*

Remark 4.3.3. *Let $0 \leq \lambda < \lambda_1$ and $R > 0$ as in Lemma 4.3.5. Then,*

$$z - \|z\|^2 S\left(tf\left(\lambda, x, \frac{z}{\|z\|^2}\right)\right) \neq 0 \quad \text{in } \overline{B_{R^{-1}}}(0) \setminus \{0\}\ t \in [0,1].$$

Thus the homotopy (4.2.1) is admissible in $[0,1] \times \overline{B}_{R^{-1}}(0)$ and

$$i(H_1(t, \cdot), 0) = deg(H_1(t, \cdot), B_{R^{-1}}(0), 0)$$

for all $t \in [0,1]$, hence by the invariance under homotopy

$$\begin{aligned} i(\Phi, 0) &= i(H_1(1, \cdot), 0) \\ &= deg(H_1(0, \cdot), 0, 0) \\ &= deg(I, 0, 0) \\ &= 1 \end{aligned}$$

for each $0 \leq \lambda < \lambda_1$.

The following lemma proves that H_2 is admissible in some $H_0^1(\Omega)$-ball. Moreover, if $\lambda > \lambda_1$, then $(\lambda, 0)$ is not a bifurcation point from the curve of trivial solutions of $z - \Phi(\lambda, z) = 0$.

Lemma 4.3.6. *Suppose that conditions (f_∞), (A_1)-(A_4) and (f_0)-(f_3) are satisfied and let ϕ be a positive function in $C_0^1(\overline{\Omega})$ bounded away from zero. If $\lambda > \lambda_1$, then there exists $R > 0$ such that*

$$u - S(f(\lambda, x, u) + \tau\|u\|^2\phi) \neq 0, \tag{4.3.0.53}$$

for all $u \in E$ with $\|u\| \leq R$, for all $\tau \geq 0$.

Proof. Suppose that there exists a sequence (u_n) in E such that $\|u_n\| \to 1$ and for a non negative sequence of numbers (τ_n), we have

$$-\text{div}(A(x, u_n)\nabla u_n) = f(\lambda, x, u_n) + \tau_n\|u_n\|^2\phi. \tag{4.3.0.54}$$

By using u_n as test function, we obtain

$$\int A(x, u_n)\nabla u_n \cdot \nabla u_n = \int f(\lambda, x, u_n)u_n + \tau_n \int u_n\|u_n\|^2\phi \tag{4.3.0.55}$$

and since $\tau_n \geq 0$ and ϕ is positive,

$$\gamma\|u_n\|^2 \leq \int (f(\lambda, x, 0) + \tau_n\|u_n\|^2\phi)u_n \leq 0,$$

where in the last inequality we use (f_∞) and (A_2), thus $u_n \leq 0$.

Defining the normalized function $z_n := u_n\|u_n\|^{-1}$, we can assume that

$$z_n \rightharpoonup z \text{ in } E, \tag{4.3.0.56}$$

for some $z \in E$. Then,

$$z_n \to z \quad \text{in } L^{(2r')}(\Omega) \tag{4.3.0.57}$$

due to Claim 4.2.1 and so $z_n \to z$ a.e. in Ω, up to a subsequence, as a consequence of Proposition 5.3.2. Consequently $z \geq 0$.

By taking $\phi/\|u_n\|$ as a test function in (4.4.0.53), we get

$$\int A(x,u_n)\nabla\left(\frac{u_n}{\|u_n\|}\right)\nabla\phi = \int \frac{f(\lambda,x,u_n)}{\|u_n\|}\phi + \tau_n\|u_n\|\int \phi^2$$

and by using the conditions (A_1), (4.2.3), Estimates II and Claim 4.2.1 we obtain

$$\begin{aligned}
\tau_n\|u_n\|\int \phi^2 &= \int A(x,u_n)\nabla\left(\frac{u_n}{\|u_n\|}\right)\nabla\phi \\
&\quad - \int \frac{f(\lambda,x,u_n)}{\|u_n\|}\phi \\
&\leq \beta\|\phi\|^2 + \int\left(C_1(x)z_n + \frac{C_2(x)}{\|u_n\|}\right)\phi \\
&\leq \beta\|\phi\|^2 + M\|\phi\|_\infty\left(\|C_1\|_{(2r')'}\|z_n\| + \frac{\|C_2\|_{(2r')'}}{\|u_n\|}\right) \\
&\leq \beta\|\phi\|^2 + M\|\phi\|_\infty\left(\|C_1\|_{(2r')'} + \|C_2\|_{(2r')'}\right)
\end{aligned}$$

hence $\tau_n\|u_n\|$ is a bounded sequence and so there exists some $\tau \geq 0$ such that $\tau_n\|u_n\|$ converges to τ, up to a subsequence.

Taking $(z_n - z)/\|u_n\|$ as a test function in (4.4.0.53), we obtain

$$\int A(x,u_n)\nabla z_n \nabla(z_n - z) = \int \frac{f(\lambda,x,u_n)}{\|u_n\|}(z_n - z) + \tau_n\|u_n\|\int \phi(z_n - z).$$

Subtracting

$$\int A(x,u_n)\nabla z \nabla(z_n - z),$$

in both sides of the above equation, we get from (A_2) and (4.0.0.2) that

$$\begin{aligned}
\gamma\|z_n - z\|^2 &\leq \int A(x,u_n)\nabla(z_n - z)\nabla(z_n - z) \\
&= \int \frac{f(\lambda,x,u_n)}{\|u_n\|}(z_n - z) + \\
&\quad + \tau_n\|u_n\|\int \phi(z_n - z) - \int A(x,u_n)\nabla z \nabla(z_n - z) \\
&\leq \int\left(C_1(x)z_n + \frac{C_2(x)}{\|u_n\|}\right)|z_n - z| \\
&\quad + \tau_n\|u_n\|\int \phi(z_n - z) - \int A(x,u_n)\nabla z \nabla(z_n - z) \\
&= \int\left(C_1(x)z_n + \frac{C_2(x)}{\|u_n\|}\right)|z_n - z| \\
&\quad + \tau_n\|u_n\|\int \phi(z_n - z) - \int_{\Omega^+} A(x,u_n)\nabla z \nabla(z_n - z), \quad (4.3.0.58)
\end{aligned}$$

where

$$\Omega^+ = \{x \in \Omega : z(x) > 0\}.$$

By applying the Estimates II, III and using the convergence $z_n \to z$ in $L^{2r'}(\Omega)$ (see (4.3.0.57)), we deduce that

$$\begin{cases} \displaystyle\int_\Omega \left(C_1(x) z_n + \frac{C_2(x)}{\|u_n\|}\right) |z_n - z| \to 0, & (4.3.0.59) \\ \displaystyle\tau_n \|u_n\| \int_\Omega \phi(z_n - z) \to 0. & (4.3.0.60) \end{cases}$$

We claim that $m(\Omega^+) > 0$. Indeed, on the contrary we would have

$$\int_{\Omega^+} A(x, u_n) \nabla z \nabla (z_n - z) = 0$$

and so the inequality (4.4.0.59) and the convergences (4.3.0.59) and (4.3.0.60) imply that $z_n \to z$ in E, consequently $1 = \|z_n\| \to \|z\| = 1$, hence $0 \le z \not\equiv 0$, but since $z \in E$, it follows that $m(\Omega^+) > 0$, which is a contradiction.

Note that the $u_n = z_n \|u_n\| \to +\infty$ a.e. in Ω^+ and so the conditions (A_1) and (A_4) implies, by Dominated Convergence Lebesgue Theorem, that $A(x, u_n) \nabla z$ converges strongly to $A(x, \infty) \nabla z$ in $L^2(\Omega^+)$. So dividing (4.4.0.53) by $\|u_n\|$ and taking the limit when $n \to +\infty$ (similarly as we done in the previous lemma), we deduce that z satisfies the equation

$$-\operatorname{div}(A(x, \infty) \nabla z) = \lambda f_\infty'(x) z + \tau\phi.$$

So, testing the above equations against ψ, we have

$$\begin{aligned} \int_\Omega A(x, \infty) \nabla z \nabla \psi &= \lambda \int_\Omega f_\infty'(x) z\psi + \int_\Omega \tau\phi\psi \\ &\ge \lambda \int_\Omega f_\infty'(x) z\psi. \end{aligned}$$

On the other hand,

$$\lambda_\infty \int_\Omega f_\infty'(x) z\psi = \int_\Omega A(x, \infty) \nabla z \nabla \psi,$$

then

$$\lambda_\infty \int_\Omega f_\infty'(x) z\psi \ge \lambda \int_\Omega f_\infty'(x) z\psi, \tag{4.3.0.61}$$

but since $z \neq 0$, this inequality implies that $\lambda_\infty \ge \lambda$, which is a contradiction. $\square$

Corollary 4.3.2. *As a consequence of Remark (5.4.1), we can replace $\|u\| \le R$ by $\|u\|_0 \le R$ in the conclusion of Lemma 4.4.6.*

Remark 4.3.4. *Let $\lambda > \lambda_\infty$ and ϕ be a positive function and $R > 0$ as in the Lemma 4.4.6. Then*

$$z - \|z\|^2 S\left(f\left(\lambda, x, \frac{z}{\|z\|^2}\right) + \frac{\tau\phi}{\|z\|^2}\right) \not\equiv 0 \quad \text{in } \overline{B_{R^{-1}}}(0) \setminus \{0\}, \ \tau \in [0,1],$$

thus the homotopy (4.2.1) is admissible in $[0,1] \times \overline{B}_{R^{-1}}(0)$ and

$$i(H_2(\tau, \cdot), 0) = deg(H_2(\tau, \cdot), B_{R^{-1}}(0), 0)$$

for all $\tau \in [0,1]$.

Observe that by Lemma 4.4.6,

$$\frac{z}{\|z\|^2} \quad S\left(f\left(\lambda, x, \frac{z}{\|z\|^2}\right) + \tau \left\|\frac{z}{\|z\|^2}\right\|^2 \phi\right) \neq 0$$

for each $z \neq 0$ with

$$\left\|\frac{z}{\|z\|^2}\right\| \quad R \ , \ \|z\| \quad R^{-1},$$

moreover $H_2(1,0) = \ _1 \neq 0$, hence $H_2(1, \ ) \neq 0$ in $\overline{B_{R^{-1}}}(0)$ and so by the property of existence of solution of the Leray-Schauder degree, we deduce that

$$deg(H_2(1, \ ), 0, 0) = 0.$$

By the invariance under homotopy,

$$\begin{aligned} i(\ , 0) &= i(H_2(0, \ ), 0, 0) \\ &= \deg(H_2(0, \ ), 0, 0) \\ &= \deg(H_2(1, \ ), 0, 0) \\ &= 0 \end{aligned}$$

for each $\lambda > \lambda_1$, where the last equation folllows from the argument in (4.2.1.5).

Finally, by applying Theorem A we deduce that it holds Theorem C as follows.

Proof. The existence of the continuum follows directly by applying Theorem A for $I = [0, +\infty)$, $\mu = \lambda_1$ and (a, b) being any interval containing λ_1 with $a > 0$. Now, by combining the additional hypothesis (4.0.0.10) with Remark (4.4.1), we obtain that $u = 0$ is the only solution of $(P \ )$ for $\lambda = 0$. But by Lemma 4.4.6, $(0,0)$ is not a bifurcation point and so C_1 does not satisfies the alternative ii) of Theorem A. So we conclude that it must satisfy i), that is, C_1 is unbounded. □

Remark 4.3.5. *Observe that*

$$= \ _F,$$

by Remark 4.2.8.

4.4 Compactness of the operators K and T for the case $E = C_0(\overline{\ })$

Consider the Banach space $E := C_0(\overline{\ })$ of the continuous functions $u : \overline{\ } \to \mathbb{R}$ that vanishes at ∂ . In this section, we will the prove the compactness of the operators K (see Chapter 1) and T in order to deduce Theorem D.

Lemma 4.4.1. *If (τ_n, z_n) is a bounded sequence in $[0,1] \quad (E \setminus \{0\})$ such that $z_n \to 0$, then*

$$\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) \to 0 \text{ in } L^r(\ ) \text{ up to a subsequence.} \tag{4.4.0.1}$$

Proof. The idea is to use the Dominated Lebesgue's Convergence Theorem. So let us show that the domination hypothesis is verified. By using (4.2.3.1) for $u_n = z_n/\|z_n\|_0^2$ we get

$$\left|\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)\right| \le \tau_n(\|z_n\|_0^2 C(x) + C_1(x) z_n^+), \tag{4.4.0.2}$$

which implies for n large enough

$$\begin{aligned}\left|\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)\right| &\le \tau_n(C(x) + C_1(x) z_n^+) \\ &\le C(x) + C_1(x) z_n^+,\end{aligned}$$

because $\|z_n\|_0 \to 0$. Since (z_n) is bounded in E, it follows that

$$\left|\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)\right| \le C(x) + C_1(x) M \in L^r(\Omega), \tag{4.4.0.3}$$

for some constant $M > 0$ and the domination hypothesis is verified.

Now, let us prove the hypothesis about the convergence a.e. in Ω. Let us denote

$$u_n := \frac{z_n}{\|z_n\|_0^2}.$$

Since $z_n \to 0$ in E, it follows that

$$z_n \to 0 \text{ a.e. in } \Omega \text{ up to a subsequence,} \tag{4.4.0.4}$$

due to Proposition (5.3.2).

Let $x \in \Omega$. There are two possibilities for the sequence $(u_n(x))_n$, it is bounded or not. In the first case, there exists some $v(x)$ such that $u_n(x) \to v(x)$ up to a subsequence and so

$$\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n(x)}{\|z_n\|_0^2}\right) \to 0.\tau f(\lambda, x, v(x)) = 0, \quad \text{up to a subsequence,}$$

where τ is the limit (up to a subsequence) of τ_n. On the other hand if the sequence $(u_n(x))_n$ is unbounded, then at least one of the following alternatives must occur:

i) $u_n(x) \to +\infty$ up to a subsequence;

ii) $u_n(x) \to -\infty$ up to a subsequence.

In the case i), we deduce that

$$\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n(x)}{\|z_n\|_0^2}\right) = \frac{\tau_n f(\lambda, x, u_n(x))}{\frac{z_n}{\|z_n\|_0^2}} z_n(x) = \frac{\tau_n f(\lambda, x, u_n(x))}{u_n(x)} z_n(x)$$

and so

$$\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n(x)}{\|z_n\|_0^2}\right) = \frac{\tau_n f(\lambda, x, u_n(x))}{u_n(x)} z_n(x) \to \lambda \tau f_+(x).0$$

by (f_3).

In the case ii), we deduce that

$$\frac{\tau_n f(\lambda, x, u_n(x))}{u_n(x)} z_n(x) \to \tau \lambda f(\lambda, x, 0).0$$

by using the hypothesis $(f\ )$ and the fact (4.4.0.36).

By Lebesgue Dominated Convergence Theorem, we conclude that holds (4.4.0.1) and the lemma is proved. □

Lemma 4.4.2 (Compactness of the operator T). *Let $\phi \in C_0^1(\overline{\Omega})$ be a function such that $\phi(x) > 0$ for all $x \in \Omega$. Then the operator*

$$T(\tau, t, z) = \begin{cases} \|z\|_0^2 S\left(\tau f\left(\lambda, x, \frac{z}{\|z\|_0^2}\right) + t\frac{\phi}{\|z\|_0^2}\right) & \text{if } z \neq 0, \\ \Theta_t & \text{if } z = 0, \end{cases} \tag{4.4.0.5}$$

with Θ_t as defined in (4.2.1.3), is compact in $[0,1] \times [0,1] \times E$.

Proof. Let $((\tau_n, t_n, z_n))_n$ be a bounded sequence in $[0,1] \times [0,1] \times E$. Without loss of generality, we can assume that $z_n \neq 0$ for all n and $(\tau_n, t_n) \to (\tau, t) \in [0,1]^2$. Define

$$w_n := S\left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) + \frac{t_n \phi}{\|z_n\|_0^2}\right) \tag{4.4.0.6}$$

that is,

$$\int_\Omega A(x, w_n)\nabla w_n \nabla v = \int_\Omega \left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) + \frac{t_n\phi}{\|z_n\|_0^2}\right) v, \quad \forall\, v \in H_0^1(\Omega). \tag{4.4.0.7}$$

By multiplying (4.4.0.7) by $\|z_n\|_0^2$ we get

$$\begin{aligned} \int_\Omega A(x, w_n)\nabla(\|z_n\|_0^2 w_n)\nabla v\,dx &= \int_\Omega \left(\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) + t_n\phi\right) v \\ &= \int_\Omega \left[\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) v + t_n\phi v\right] \end{aligned} \tag{4.4.0.8}$$

for all $v \in H_0^1(\Omega)$. Let us define $y_n := z_n\|w_n\|_0$.

Take $v = y_n$ as a test function in (4.4.0.8). So we obtain from (A_2) that

$$\begin{aligned} \gamma\|y_n\|^2 &\le \int_\Omega A(x, w_n)\nabla y_n \nabla y_n \\ &= \int_\Omega \|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) y_n + \int_\Omega t_n\phi y_n \\ &\le M\left\|\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)\right\|_r \|y_n\|_{r'} + t_n\|\phi\|_r\|y_n\|_{r'}, \end{aligned} \tag{4.4.0.9}$$

where the last inequality follows from Hölder's inequality. But $r' < 2r'$ so by using Claim 4.2.1 we obtain $\|y_n\|_{r'} \le M\|y_n\|$ for some constant $M > 0$. Thus,

$$\gamma\|y_n\|^2 \le M\left\|\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)\right\|_r \|y_n\| + t_n\|\phi\|_\infty\|y_n\|$$

so by diving both sides of the inequality by $\|y_n\| \neq 0$ and using Remark 5.4.1, we have

$$\gamma\|y_n\| \le M\left\|\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)\right\|_r + t_n\|\phi\|_\infty. \tag{4.4.0.10}$$

But by using (4.2.3.1) for $u_n = z_n/\|z_n\|_0^2$ we get

$$\begin{aligned} \left|\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)\right| &\le \tau_n(\|z_n\|_0^2 C(x) + C_1(x)z_n^+) \\ &\Rightarrow \left|\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)\right| \\ &\le \tau_n(C(x) + C_1(x)z_n^+) \\ &\le C(x) + C_1(x)z_n^+ \\ &\le C(x) + MC_1(x), \end{aligned}$$

for some constant $M > 0$, because $(\|z_n\|_0)$ is a bounded sequence. Whence

$$(y_n) \text{ is a bounded sequence in } H_0^1(\Omega)$$

and so by the reflexivity of the space $H_0^1(\Omega)$, it follows that there exists some $y \in H_0^1(\Omega)$ such that

$$y_n \rightharpoonup y \ \text{ in } H_0^1(\Omega). \tag{4.4.0.11}$$

Moreover, by Claim 4.2.1, we have that

$$y_n \to y \ \text{ in } L^{r'}(\Omega) \tag{4.4.0.12}$$

and by Proposition 5.3.2, we deduce that

$$y_n \to y \ \text{ a.e. in } \Omega.$$

By Theorem 5.4.2, it follows that

$$(y_n) \text{ is a bounded sequence in } E, \tag{4.4.0.13}$$

which implies by Arzelà-Ascoli, that there exist some $\overline{y} \in E$ such that

$$y_n \to \overline{y} \text{ in } E.$$

But by (4.4), we deduce that $\overline{y} = y$.

By taking $v = y_n - y$ as a test function in (4.4.0.8), we obtain

$$\begin{aligned} \int_\Omega A(x, w_n) \nabla y_n \nabla (y_n - y) &= \int_\Omega \|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)(y_n - y) + \\ &+ \int_\Omega t_n \phi(y_n - y), \end{aligned}$$

whence, by subtracting

$$\int_\Omega A(x, w_n) \nabla y \nabla (y_n - y),$$

using (A_2) and Hölder's inequality we obtain that

$$\begin{aligned} \gamma \|y_n - y\|^2 &\le \int_\Omega A(x, w_n) \nabla (y_n - y_n) \nabla (y_n - y) \\ &= \int_\Omega \|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)(y_n - y) + \\ &+ \int_\Omega t_n \phi(y_n - y) - \int_\Omega A(x, w_n) \nabla y \nabla (y_n - y) \\ &\le M \left(\left\| \|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) \right\|_r \|y_n - y\|_{r'} + t_n \|\phi\|_1 \|y_n - y\|_{r'} + \right. \\ &+ \left. \left| \int_\Omega A(x, w_n) \nabla y \nabla .(y_n - y) \right| \right). \end{aligned} \tag{4.4.0.14}$$

By the convergence $y_n \to y$ in $L^{r'}(\Omega)$ (see (4.4.0.12)) we imply that the first two therms on the (RHS) converges to zero up to a subsequence. Now, in order to prove the convergence of the last therm to zero, that is,

$$\left| \int_\Omega A(x, w_n) \nabla y \nabla (y_n - y) \right| \to 0,$$

we will study the following two cases separately:

1) $\|z_n\|_0$ is not bounded away from 0;

2) $\|z_n\|_0$ is bounded away from 0.

Case 1): In this case we can assume that $z_n \to 0$.
Case 1a) $t=0$: Since $t_n \to t = 0$, the estimate (4.4.0.10) and Lemma 4.4.1 implies that $y_n \to 0$ in E. In particular, if $(\tau_n, t_n, z_n) \to (\tau, 0, 0)$, then

$$T(\tau_n, t_n, z_n) = y_n \to 0 = {}_0 = T(\tau, 0, 0),$$

that is, for every $\tau \in [0,1]$ the operator T is continuous in $(\tau, 0, 0)$. So we just proved that if (τ_n, t_n, z_n) is a bounded sequence converging to $(\tau, 0, 0)$, then the sequence $T(\tau_n, t_n, z_n)$ converges to $T(\tau, 0, 0)$ in E, up to a subsequence. Therefore, Case 1a) is concluded.
Case 1b) $t>0$: To prove the case $t \in (0,1]$, consider the following claim.

Claim 4.4.1. *If $t \in (0,1]$, then the following statements are true:*

i) there exists an infinite subset J of N *such that $\|w_n\| \to +\infty$ for $n \in J$,*

ii) there exists an n_0 such that w_n is non negative for all $n \geq n_0$ such that $n \in J$.

Proof. Let us prove i). We state that $\|w_n\| \to +\infty$, up to a subsequence. In fact, suppose, by contradiction that $\|w_n\|$ is bounded. By testing (4.4.0.7) against a positive function $v \in C_0^{\infty}(\overline{\Omega})$ such that $\|v\| > 0$, we obtain

$$\begin{aligned}\int \left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) + \frac{t_n \phi}{\|z_n\|_0^2}\right) v &\leq \int |A(x, w_n) \nabla w_n \nabla v| \\ &\leq \beta \|w_n\| \|v\| \leq M,\end{aligned}$$

for some constant $M > 0$, so

$$\int \left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) + \frac{t_n \phi}{\|z_n\|_0^2}\right) v \leq M, \ \forall n. \tag{4.4.0.15}$$

On the other hand, since

$$\left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) + \frac{t_n \phi}{\|z_n\|_0^2}\right) = \frac{1}{\|z_n\|_0^2}\left(\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) + t_n \phi\right),$$

we can rewrite the (4.4.0.15) as

$$\frac{1}{\|z_n\|_0^2}\left[\int \|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) v + \int t_n \phi v\right] \leq M, \ \forall n. \tag{4.4.0.16}$$

Now, observe that

$$\begin{aligned}\left|\int \|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) v\right| &\leq \|v\|_\infty \int \left|\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)\right| \\ &\leq M \|v\|_\infty \left\|\|z_n\|_0^2 \tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)\right\|_r \to 0,\end{aligned}$$

due to Lemma 4.4.1. But

$$\int t_n \phi v \to \int t \phi v > 0$$

and so

$$\frac{1}{\|z_n\|_0^2}\int_\Omega \left(\|z_n\|_0^2\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) v + t_n\phi v\right) \xrightarrow{n\to\infty} +\infty$$

which contradicts (4.4.0.16) and so we conclude that $\|w_n\|$ is unbounded and hence there exists a set of indexes J such that $\|w_n\| \to +\infty$ $(n \in J)$, which proves i).

Let us prove ii) by using the weak formulation of Maximum Principle in (4.4.0.6). By Claim 4.2.4, it follows that

$$\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) + t_n\frac{\phi}{\|z_n\|_0^2} \geq 0, \text{ up to a subsequence.} \tag{4.4.0.17}$$

So we deduce, by Theorem 5.4.1, that the function w_n is non negative for all $n \geq n_0$ in J. □

Moreover, there exists a number $\xi \geq 0$ such that

$$\|y_n\| \to \xi \quad n \in J \text{ up to a subsequence.} \tag{4.4.0.18}$$

We claim that $\xi > 0$. Indeed, let $v \in H_0^1(\Omega)$ be a positive function. By (A_1) and the Hölder Inequality,

$$\left|\int_\Omega A(w_n)\nabla(\|z_n\|_0^2 w_n)\nabla v\right| \leq \beta\|y_n\|\|v\|$$

and combining it with (4.4.0.8) we obtain

$$\left|\int_\Omega \left(\|z_n\|_0^2\tau f\left(\lambda_n, x, \frac{z_n}{\|z_n\|_0^2}\right) + t\phi\right) v\right| \leq \beta\|y_n\|\|v\|,$$

so we conclude by (4.4.0.17) that $\|y_n\| \nrightarrow 0$ and consequently $\xi > 0$.
Now, observe that since $y \geq 0$, it follows that

$$\left|\int_\Omega A(x, w_n)\nabla y\nabla(y_n - y)\right| = \left|\int_{\Omega^+} A(x, w_n)\nabla y\nabla(y_n - y)\right|,$$

where

$$\Omega^+ := \{x \in \Omega\,;\, y(x) > 0\}.$$

We claim that $m(\Omega^+) > 0$, on the contrary

$$\left|\int_{\Omega^+} A(x, w_n)\nabla y\nabla(y_n - y)\right| = 0$$

and by the convergence $y_n \to y$, in $L^{r'}(\Omega)$ (see (4.4.0.12)), we imply that (RHS) of (4.4.0.14) converges to 0 and so we would obtain $\|y_n - y\| \to 0$, by whence $\|y_n\| \to \|y\|$, but since $\xi > 0$, it would imply that $\|y\| > 0$, but y is non negative and so it would imply $m(\Omega^+) > 0$, which is a contradiction. Thus, $m(\Omega^+) > 0$.

In order to prove the convergence

$$\left|\int_{\Omega^+} A(x, w_n)\nabla y\nabla(y_n - y)\right| \to 0,$$

consider the following claim.

Claim 4.4.2. *Let $v \in H_0^1(\Omega)$ and $t \in (0,1]$. Then the following convergence holds (up to a subsequence and for $n \in J$):*

$$\int_{\Omega^+} A(x,w_n)\nabla y_n \nabla v\,dx \;\to\; \int_{\Omega^+} A(x,+\infty)\nabla y \nabla v\,dx. \qquad (4.4.0.19)$$

Proof. Consider the sequence of functionals $\varphi_n : H_0^1(\Omega^+) \to \mathbb{R}$ defined by

$$\varphi_n(v) = \int_{\Omega^+} A(x,w_n)\nabla y \nabla v.$$

By (A_1) and the fact $w_n = \|w_n\|\bar{w}_n \to +\infty$ a.e. in Ω^+, we deduce that

$$A(x,w_n)\nabla y \;\to\; A(x,+\infty)\nabla y \quad \text{in } (L^2(\Omega^+))^N,$$

by Lesbesgue Dominated Convergence Theorem, which implies that

$$\varphi_n \to \varphi \quad \text{strongly in } H^{-1}(\Omega^+)$$

where $\varphi : H_0^1(\Omega^+) \to \mathbb{R}$ is the functional defined by

$$\varphi(v) = \int_{\Omega^+} A(x,+\infty)\nabla y \nabla v$$

and since $y_n \rightharpoonup y$ (see (4.4.0.11)), the Proposition (5.3.1) implies the convergence of the claim. $\square$

Thus, Claim 4.4.2 proves that the last therm of (RHS) (4.4.0.14) converges to zero up to a subsequence. But, we already know that the others also converges to zero up to a subsequence, due to the convergence $y_n \to y$ (see (4.4.0.12)) and so we conclude that $y_n \to y$ in $H_0^1(\Omega)$. Then, we just proved that the sequence $T(\tau_n, t_n, z_n)$ converges to some $y \in E$, in E, up to a subsequence, whenever (τ_n, t_n, z_n) is a bounded subsequence, $t_n \to t > 0$ and $\|z_n\|_0$ is not bounded away from zero. Now, let us prove the continuity of T. Since $y \ge 0$, it follows that

$$\begin{aligned}
\lim_{n\to+\infty} \int_{\Omega} A(x,w_n)\nabla y_n \nabla v\,dx &= \lim_{n\to+\infty} \int_{\Omega^+} A(x,w_n)\nabla y_n \nabla v\,dx \\
&= \int_{\Omega^+} A(x,+\infty)\nabla y \nabla v\,dx \\
&= \int_{\Omega} A(x,+\infty)\nabla y \nabla v\,dx, \;\; \forall\, v \in H_0^1(\Omega). \qquad (4.4.0.20)
\end{aligned}$$

On the other hand,

$$\lim_{n\to+\infty} \int_{\Omega} \left(\tau_n \|z_n\|_0^2 f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) + t_n\phi\right) v = \int_{\Omega} t\phi, \;\; \forall\, v \in E. \qquad (4.4.0.21)$$

Indeed, by Hölder's inequality, we have that

$$\int_{\Omega} \left|\tau_n \|z_n\|_0^2 f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) v\right| \le \left\|\tau_n \|z_n\|_0^2 f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)\right\|_r \|v\|_{r'},$$

bu the (RHS) converges to zero up to a subsequence, due to Lemma 4.4.1, so (4.4.0.21) holds. By combing (4.4.0.20) and (4.4.0.21), we deduce by passing to the limit the equation (4.4.0.8) that

$$\int A(x,+\infty)\nabla y\nabla v\,dx = \int t\phi v \quad \forall v \in E,$$

which means that $y = T(\tau,t,0)$. We just proved that $T(\tau_n,t_n,z_n) \to T(\tau,t,0)$ whenever (τ_n,t_n,z_n) is a bounded sequence such that $(\tau_n,t_n,z_n) \to (\tau,t,0)$, up to a subsequence, with $t>0$. This proves the Case 1b) and so Case 1 is concluded.

Case 2: In this case, since the sequence (z_n) is bounded in E we deduce that

$$\begin{aligned}\left|\tau_n f\left(\lambda,x,\frac{z_n}{\|z_n\|_0^2}\right)\right| &\le C(x)+C_1(x)\frac{z_n^+}{\|z_n\|_0^2}\\ &\le M(C(x)+C_1(x)),\end{aligned}\tag{4.4.0.22}$$

for some constant $M>0$. Consequently,

$$\left(\left\|\tau_n f\left(\lambda,x,\frac{z_n}{\|z_n\|_0^2}\right)\right\|_r\right)_n$$

is a bounded sequence. By taking $v = w_n$ as a test function in (4.4.0.7), using (A_2), Hölder's inequality and Claim 4.2.1, we obtain

$$\begin{aligned}\gamma\|w_n\|^2 &\le \int A(x,w_n)\nabla w_n\nabla w_n\\ &= \int\left(\tau_n f\left(\lambda,x,\frac{z_n}{\|z_n\|_0^2}\right)+\frac{t_n\phi}{\|z_n\|_0^2}\right)w_n\\ &\le \left\|\tau_n f\left(\lambda,x,\frac{z_n}{\|z_n\|_0^2}\right)\right\|_r M\|w_n\|_{r'}+\|\phi\|_r\|w_n\|_{r'} \qquad (4.4.0.23)\\ &\le \left\|\tau_n f\left(\lambda,x,\frac{z_n}{\|z_n\|_0^2}\right)\right\|_r M\|w_n\|+\|\phi\|_r\|w_n\|, \qquad (4.4.0.24)\end{aligned}$$

which means that (w_n) is a bounded sequence in $H_0^1(\Omega)$. The regularity given by Theorem 5.4.2 implies that the sequence (w_n) is bounded in E, whence, by Arzelà-Ascoli Theorem, we deduce that there exist some $\overline{w} \in E$ such that

$$w_n \to \overline{w} \text{ in } E, \text{ up to a subsequence.}\tag{4.4.0.25}$$

Also by the boundedness of (w_n) in the reflexive space $H_0^1(\Omega)$, we imply that there exist $w \in H_0^1(\Omega)$ such that

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega),\tag{4.4.0.26}$$

for some $w \in H_0^1(\Omega)$, up to a subsequence. Moreover, by Claim 4.2.1, it follows that

$$w_n \to w \text{ in } L^{r'}(\Omega) \text{ up to a subsequence,}\tag{4.4.0.27}$$

in particular $w_n \to w$ a.e. in Ω up to a subsequence and so $\overline{w} = w$ and, by Lebesgue Dominated Convergence, we deduce that

$$A(x,w_n)\nabla y \to A(x,w)\nabla y \text{ in } L^2(\Omega)$$

and similarly as we argued in Claim 4.4.2, we deduce by (4.4.0.26), that

$$\int_\Omega A(x,w_n)\nabla y \nabla y_n \to \int_\Omega A(x,w)\nabla y \nabla y$$

and so

$$\left|\int_\Omega A(x,w_n)\nabla y \nabla (y_n - y)\right| \to 0, \tag{4.4.0.28}$$

whence we conclude that

$$T(\tau_n, t_n, z_n) = y_n \to y \text{ in } H_0^1(\Omega). \tag{4.4.0.29}$$

Thus, we just proved that $(T(\tau_n, t_n, z_n))$ converges to some $y \in E$, up to a subsequence, whenever (τ_n, t_n, z_n) is a bounded sequence and $\|z_n\|_0$ is bounded away from zero. Now, let us prove the continuity of T. Let (τ_n, t_n, z_n) be a sequence in $[0,1]\times[0,1]\times E$ converging to some (τ, t, z) in $[0,1]\times[0,1]\times(E\setminus\{0\})$.

By using the same argument used to prove (4.4.0.28), we can deduce that

$$\lim_{n\to+\infty}\int_\Omega A(x,w_n)\nabla w_n \nabla v = \int_\Omega A(x,w)\nabla w \nabla v \quad \forall v \in H_0^1(\Omega). \tag{4.4.0.30}$$

Moreover, by Lebesgue's Dominated Convergence Theorem, we deduce that

$$\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) \to \tau f\left(\lambda, x, \frac{z}{\|z\|^2}\right) \text{ in } L^r(\Omega).$$

But, the expression

$$\left|\int_\Omega \left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) - \tau f\left(\lambda, x, \frac{z}{\|z\|^2}\right)\right) v\right|$$

is estimated from above by

$$\left\|\left(\tau_n \left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) - \tau f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right)\right)\right\|_r \|v\|_{r'},$$

consequently,

$$\lim_{n\to+\infty}\int_\Omega \left(\tau_n f\left(\lambda, x, \frac{z_n}{\|z_n\|_0^2}\right) + t_n\frac{\phi}{\|z_n\|_0}\right) v = \int_\Omega \left(\tau f\left(\lambda, x, \frac{z}{\|z\|^2}\right) + t\frac{\phi}{\|z\|}\right) v. \tag{4.4.0.31}$$

Now, by combining (4.4.0.30) and (4.4.0.31), we deduce that by passing to the limit the equation (4.4.0.7), we obtain

$$\int_\Omega A(x,w)\nabla w \nabla v = \int_\Omega \left(\tau f\left(\lambda, x, \frac{z}{\|z\|^2}\right) + t\frac{\phi}{\|z\|}\right) v, \ \forall\, v \in H_0^1(\Omega),$$

in other words,

$$w = S(\tau, t, z). \tag{4.4.0.32}$$

But $w_n = S(\tau_n, t_n, z_n)$ and so the convergence (4.4.0.25), combined with the fact that $w = \overline{w}$, proves the continuity of S in (τ, t, z) for $z \neq 0$. Since $z \mapsto \|z\|_0^2$ is a continuous function, it follows that T is continuous in (τ, t, z). □

In order to prove the compactness of the operator

$$K : [0, +\infty) \times E \to E$$

consider the following lemma.

Lemma 4.4.3. *If (λ_n, z_n) is a bounded sequence in $[0, +\infty) \times E$ such that $z_n \to 0$, then*

$$\|z_n\|_0^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|_0^2}\right) \to 0 \text{ in } L^r(\Omega), \text{ up to a subsequence.} \tag{4.4.0.33}$$

Proof. Let (λ_n, z_n) be a bounded sequence in $[0, +\infty) \times E$. Then there exists $\lambda \in [0, +\infty)$ such that $\lambda_n \to \lambda$, up to a subsequence. The idea to use the Dominated Lebesgue's Convergence Theorem. So let us show that the domination hypothesis is verified. By using Claim 4.2.3, we deduce that

$$\left|\|z_n\|_0^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|_0^2}\right)\right| \leq \|z_n\|_0^2 C(x) + C_1(x) z_n^+, \tag{4.4.0.34}$$

since $\|z_n\|_0 \to 0$, we assume n large enough so that

$$\left|\|z_n\|_0^2\, f\left(\lambda_n, x, \frac{z_n}{\|z_n\|_0^2}\right)\right| \leq C(x) + C_1(x) M, \tag{4.4.0.35}$$

for some constant $M > 0$ and so the domination hypothesis is verified.

Let us denote

$$u_n := \frac{z_n}{\|z_n\|_0^2}.$$

Since $z_n \to 0$ in E, in particular, we have that

$$z_n \to 0 \text{ a.e. in } \Omega \text{ up to a subsequence.} \tag{4.4.0.36}$$

Let $x \in \Omega$. There are two possibilities for the sequence $(u_n(x))_n$, it is bounded or not. In the first case, there exists some $v(x)$ such that $u_n(x) \to v(x)$ up to a subsequence and so by the continuity of $(\lambda, s) \mapsto f(\lambda, x, s)$ (for a.e. fixed x in Ω), we deduce that

$$\|z_n\|_0^2 f\left(\lambda_n, x, \frac{z_n(x)}{\|z_n\|_0^2}\right) \to 0.f(\lambda, x, v(x)) = 0 \text{ up to a subsequence,}$$

In the second case, at least one of the following alternatives must occur:

i) $u_n(x) \to +\infty$ up to a subsequence;

ii) $u_n(x) \to -\infty$ up to a subsequence.

In the case $i)$, we deduce that

$$\|z_n\|_0^2 f\left(\lambda_n, x, \frac{z_n(x)}{\|z_n\|_0^2}\right) = \frac{f(\lambda_n, x, u_n(x))}{\frac{z_n}{\|z_n\|_0^2}} z_n(x) = \frac{f(\lambda_n, x, u_n(x))}{u_n(x)} z_n(x)$$

and so

$$\|z_n\|_0^2 f\left(\lambda_n, x, \frac{z_n(x)}{\|z_n\|_0^2}\right) = \frac{f(\lambda_n, x, u_n(x))}{u_n(x)} z_n(x) \to \lambda f_+(x).0$$

by (f_3).

In the case ii), we deduce that

$$\frac{f(\lambda_n, x, u_n(x))}{u_n(x)} z_n(x) \to \lambda f(\lambda, x, 0).0$$

by using the hypothesis $(f\ )$ and the fact (4.4.0.36).

By Lebesgue Dominated Convergence Theorem, we conclude that holds (4.4.0.33) and the lemma is proved. $\square$

Lemma 4.4.4 (Compactness of the operator K). *The operator*

$$K : [0, +\infty) \times E \to E$$

defined in Remark 4.2.8 is compact.

Proof. Let (λ_n, z_n) be a bounded sequence in $[0, +\infty) \times E$. Without loss, we can assume that $z_n \neq 0$ for all n and $\lambda_n \to \lambda \in [0, +\infty)$. Define

$$w_n := S\left(f\left(\lambda_n, x, \frac{z_n}{\|z_n\|_0^2}\right)\right)$$

that is,

$$\int_\Omega A(x, w_n)\nabla w_n \nabla v\,dx = \int_\Omega f\left(\lambda_n, x, \frac{z_n}{\|z_n\|_0^2}\right) v\,dx, \quad \forall v \in H_0^1(\Omega). \tag{4.4.0.37}$$

and $y_n := \|z_n\|_0^2 w_n$, thus

$$K(\lambda_n, z_n) = \|z_n\|_0^2 w_n = y_n.$$

By multiplying (4.4.0.37) by $\|z_n\|_0^2$ we get

$$\int_\Omega A(x, w_n)\nabla(\|z_n\|_0^2 w_n)\nabla v\,dx = \int_\Omega \|z_n\|_0^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|_0^2}\right) v\,dx \text{ for all } v \in H_0^1(\Omega). \tag{4.4.0.38}$$

Taking $v = y_n$ as a test function in (4.4.0.38), using (A_2), Claims 4.2.3 and 4.2.1, we obtain

$$\begin{aligned}
\gamma\|y_n\|^2 &\leq \int_\Omega A(x, w_n)\nabla y_n \nabla y_n \\
&= \int_\Omega \|z_n\|_0^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|_0^2}\right) y_n \\
&\leq \left\|\|z_n\|_0^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|_0^2}\right)\right\|_r \|y_n\|_{r'} \\
&\leq M \left\|\|z_n\|_0^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|_0^2}\right)\right\|_r \|y_n\|
\end{aligned}$$

by whence

$$\gamma\|y_n\| \leq M \left\|\|z_n\|_0^2 f\left(\lambda_n, x, \frac{z_n}{\|z_n\|_0^2}\right)\right\|_r. \tag{4.4.0.39}$$

Thus, (y_n) is bounded in $H_0^1(\Omega)$. The regularity given by Theorem 5.4.2, implies that

$$(y_n) \text{ is bounded in } E. \tag{4.4.0.40}$$

By Arzelà-Ascoli Theorem, we imply that there exists some $\overline{y} \in E$ such that

$$y_n \to \overline{y} \text{ in } E, \text{ up to a subsequence.} \tag{4.4.0.41}$$

Also by the boundedness of (y_n) in $H_0^1(\Omega)$ we imply that there exists some $y \in H_0^1(\Omega)$ such that

$$y_n \rightharpoonup y, \text{ up to a subsequence,}$$

whence, by Claim 4.2.1, we imply that

$$y_n \to y \text{ in } L^{r'}(\Omega)$$

and so by Proposition 5.3.2,

$$y_n \to y \text{ a.e. in } \Omega, \text{ up to a subsequence.}$$

Consequently $y = \overline{y}$.

Now, we will study the continuity for the following two cases separately:

1) $\|z_n\|_0$ is not bounded away from 0;

2) $\|z_n\|_0$ is bounded away from 0.

Case 1): In this case we can assume that $z_n \to 0$. By Lemma 4.4.3 the (RHS) of (4.4.0.39) converges to zero and, consequently, $y = 0$ and we conclude that

$$K(\lambda_n, z_n) = y_n \to 0 = K(\lambda, 0) \text{ in } E.$$

Thus, Case 1 is concluded.
Case 2): Moreover, since $\|z_n\|_0$ is bounded away from zero and (y_n) is a bounded sequence (see (4.4.0.40)), it follows that the sequence $w_n = \|z_n\|_0^{-2} y_n$ is bounded E. So the proof of the continuity follows similarly as we done in the proof of Case 2 of Lemma 4.4.2. □

The following lemma proves that if $0 \leq \lambda < \lambda_1$, then $(\lambda, 0)$ is not a bifurcation point from the curve of trivial solutions of $\Phi(\lambda, z) = 0$.

Lemma 4.4.5. *Assume the hypotheses (f_∞), (A_1)-(A_4) and (f_0)-(f_3) and let $\Lambda \subset [0, \lambda_1)$ be a compact interval. Then there exists a number $R > 0$ such that*

$$u \neq S(tf(\lambda, x, u)) \tag{4.4.0.42}$$

for all $u \in E$ with $\|u\|_0 \geq R$, all $\lambda \in \Lambda$ and all $t \in [0,1]$.

Proof. Suppose that there exists sequences (λ_n) in Λ, (t_n) in $[0,1]$ and (u_n) in E with $\|u\| \to \infty$ such that

$$u_n = S(t_n f(\lambda_n, x, u_n)). \tag{4.4.0.43}$$

Since Λ and $[0,1]$ are compact sets, we deduce the existence of $\lambda \in \Lambda$ and $t \in [0,1]$ such that $\lambda_n \to \lambda$ and $t_n \to t$ up to a subsequence. Let us define the normalized sequence $z_n := u_n \|u_n\|_0^{-1}$ and note that by dividing (4.4.0.43) by $\|u_n\|_0$, we obtain that z_n satisfies the equation

$$\int_\Omega A(x, u_n) \nabla z_n \nabla v = t_n \int_\Omega \frac{f(\lambda_n, x, u_n)}{\|u_n\|_0} v \quad \forall v \in H_0^1(\Omega). \tag{4.4.0.44}$$

By taking $v = u_n$ as a test function and using (A_2), (f_∞) and (f_0), we deduce that

$$\gamma\frac{\|u_n\|^2}{\|u_n\|_0} - \int_\Omega t_n\frac{f(\lambda_n,x,u_n)}{\|u_n\|_0}u_n \leq 0 \tag{4.4.0.45}$$

and so $u_n \leq 0$ for all n.

By taking z_n as a test function in (??), using (A_2) and Claim 4.2.3, we deduce that

$$\begin{aligned}
\gamma\|z_n\|^2 &= \int_\Omega A(x,u_n)\nabla z_n \nabla z_n \\
&= t_n\int_\Omega \frac{f(\lambda_n,x,u_n)}{\|u_n\|_0}z_n \\
&\leq \int_\Omega \frac{Cz_n}{\|u_n\|_0} + C_1 z_n^2 \\
&\leq \int_\Omega M(C + C_1) \\
&\leq M\|C + C_1\|_r.
\end{aligned}$$

Whence the sequence (z_n) is bounded in $H_0^1(\Omega)$ and so it follows that there exists some $z \in E$ such that

$$z_n \rightharpoonup z \quad \text{in } H_0^1(\Omega) \tag{4.4.0.46}$$

and so by Claim 4.2.1, we have that

$$z_n \to z \text{ in } L^{r'}(\Omega), \text{ up to a subsequence.} \tag{4.4.0.47}$$

So Proposition 5.3.2 implies that

$$z_n \to z \text{ a.e. in } \Omega, \text{ up to a subsequence.} \tag{4.4.0.48}$$

Since $\|z_n\|_0 = 1$, it follows by Arzelà-Ascoli Theorem, that there exists some $\overline{z} \in E$ such that

$$z_n \to \overline{z} \text{ in } E, \text{ up to a subsequence.}$$

Whence by (4.4.0.66), we deduce that $\overline{z} = z$.

Note that $u_n = \|u_n\|_0 z_n \to +\infty$ a.e. in Ω^+, where

$$\Omega^+ = \{x \in \Omega\ ;\ z(x) > 0\}$$

and so $A(x,u_n)\nabla z \to A(x,+\infty)\nabla z$ in $L^2(\Omega^+)^N$, due to (A_1) and the Dominated Convergence Lebesgue Theorem. Since $z_n \rightharpoonup z$ (see (4.4.0.64)), it follows by Proposition 5.3.1 that

$$\lim_{n\to\infty}\int_{\Omega^+} A(x,u_n)\nabla z_n \nabla v = \lim_{n\to\infty}\int_{\Omega^+} A(x,+\infty)\nabla z \nabla v,\ \forall v \in H_0^1(\Omega). \tag{4.4.0.49}$$

Moreover, by using Claim 4.2.3 we have that

$$\begin{aligned}
\left|\frac{f(\lambda_n,x,u_n)}{\|u_n\|_0}\right| &\leq \frac{C}{\|u_n\|_0} + C_1 z_n \\
&\leq M(C + C_1).
\end{aligned}$$

For every $x \in \Omega$ such that $u_n(x) > 0$, we have

$$\frac{f(\lambda_n, x, u_n)}{\|u_n\|_0} = \frac{f(\lambda_n, x, u_n)}{u_n} z_n(x)$$

thus,

$$\frac{f(\lambda_n, x, u_n)}{\|u_n\|_0} \to \lambda f_\infty^0 z, \text{ a.e. in } \Omega.$$

Consequently, by Lebesgue's Dominated Convergence Theorem and the hypothesis (f_2), we deduce that

$$\frac{f(\lambda_n, x, u_n)}{\|u_n\|_0} \to \lambda f_\infty^0 z \text{ in } L^r(\Omega).$$

By using Hölder's inequality and Claim, we deduce that

$$\left| \int \left(\frac{f(\lambda_n, x, u_n)}{\|u_n\|_0} - \lambda f_\infty z \right) v \right| \leq \left\| \frac{f(\lambda_n, x, u_n)}{\|u_n\|_0} - \lambda f_\infty z \right\|_r \|v\|_{r'}, \ \forall\, v \in H_0^1(\Omega).$$

So

$$\lim_{n \to +\infty} \frac{f(\lambda_n, x, u_n)}{\|u_n\|_0} v = \lambda f_\infty^0 z v, \ \forall\, v \in H_0^1(\Omega). \tag{4.4.0.50}$$

Combining (4.4.0.49) and (4.4.0.50) we deduce, by passing to the limit in equation (4.4.0.44), that z is a non trivial and non negative solution of

$$-\mathrm{div}(A(x, \infty) \nabla z) = t\lambda f_\infty^0(x) z, \tag{4.4.0.51}$$

hence $t\lambda = \lambda_\infty$ and $z = \psi$, where ψ is the eigenfunction (associated to λ_∞) of the Dirichlet eigenvalue problem with weight (4.0.0.7), as we defined before, but this contradicts the hypothesis $\lambda < \lambda_\infty$. □

Remark 4.4.1. *If we consider the sequence $t_n = 1$ for all n, then the argument in (4.4.0.45) shows that every solution u of $\Phi(u) = 0$ is non negative.*

Remark 4.4.2 (Necessary condition for some λ to be a bifurcation point from infinity of (P)). *Let $\lambda \geq 0$ be such that λ is a bifurcation point from infinity of the problem (P). If we take $\Lambda = \{\lambda_n\}$, where λ_n is a sequence in $[0, +\infty)$ such that $\lambda_n \to \lambda$ and (t_n) be the sequence defined by $t_n = 1$ for all n, then the same arguments leads us to (4.4.0.51) with $t = 1$ and so we conclude that $\lambda = \lambda_\infty$.*

Remark 4.4.3. *Let $0 \leq \lambda < \lambda_\infty$ and $R > 0$ as in Lemma 4.4.5. Then,*

$$z - \|z\|_0^2 S\left(tf\left(\lambda, x, \frac{z}{\|z\|_0^2}\right)\right) \neq 0 \ \text{ in } \overline{B_{R^{-1}}}(0) \setminus \{0\}\ t \in [0,1].$$

Thus, the homotopy (4.2.1) is admissible in $[0,1] \times \overline{B}_{R^{-1}}(0)$ and

$$i(H_1(t, \cdot), 0) = deg(H_1(t, \cdot), B_{R^{-1}}(0), 0)$$

for all $t \in [0,1]$, hence by the invariance under homotopy

$$\begin{aligned} i(\Psi, 0) &= i(H_1(1, \cdot), 0) \\ &= deg(H_1(0, \cdot), 0, 0) \\ &= deg(I, 0, 0) \\ &= 1 \end{aligned}$$

for each $0 \leq \lambda < \lambda_\infty$.

The following lemma proves that H_2 is admissible in some E-ball. Moreover, if $\lambda > \lambda_1$, then $(\lambda, 0)$ is not a bifurcation point from the curve of trivial solutions of $z - (\lambda, z) = 0$.

Lemma 4.4.6. *Suppose that conditions (f_∞), (A_1)-(A_4) and (f_0)-(f_3) are satisfied and let ϕ be a function in $C_0^1(\overline{\Omega})$ such that $\phi(x) > 0$ for all $x \in \Omega$. If $\lambda > \lambda_1$, then there exists $R > 0$ such that*

$$u - S(f(\lambda, x, u) + \tau \|u\|_0^2 \phi) \neq 0, \tag{4.4.0.52}$$

for all $u \in E$ with $\|u\|_0 \geq R$, for all $\tau \geq 0$.

Proof. Suppose that there exists a sequence (u_n) in E such that $\|u_n\|_0 \to \infty$ and for a non negative sequence of numbers (τ_n), we have

$$-\operatorname{div}(A(x, u_n)\nabla u_n) = f(\lambda, x, u_n) + \tau_n \|u_n\|_0^2 \phi. \tag{4.4.0.53}$$

By using u_n as test function, we obtain

$$\int A(x, u_n)\nabla u_n \cdot \nabla u_n = \int f(\lambda, x, u_n) u_n + \tau_n \int u_n \|u_n\|_0^2 \phi \tag{4.4.0.54}$$

and since $\tau_n \geq 0$ and ϕ is positive,

$$\gamma \|u_n\|^2 \leq \int (f(\lambda, x, 0) + \tau_n \|u_n\|_0^2 \phi) u_n \leq 0,$$

where in the last inequality we use (f_∞) and (A_2), thus $u_n \leq 0$.

Case 1 $(\|u_n\|)$ unbounded:

Define the normalized sequence $z_n := u_n \|u_n\|^{-1}$. By taking $\phi / \|u_n\|$ as a test function in (4.4.0.53), we get

$$\int A(x, u_n)\nabla\left(\frac{u_n}{\|u_n\|}\right) \cdot \nabla \phi = \int \frac{f(\lambda, x, u_n)}{\|u_n\|}\phi + \tau_n \|u_n\|_0^2 \int \phi \frac{\phi}{\|u_n\|}$$

and by using the conditions (A_1), (4.2.3), Hölder's inequality and Claim 4.2.1 we obtain

$$\begin{aligned}
\tau_n \frac{\|u_n\|_0^2}{\|u_n\|} \int \phi^2 &= \int A(x, u_n)\nabla\left(\frac{u_n}{\|u_n\|}\right) \cdot \nabla \phi \\
&\quad - \int \frac{f(\lambda, x, u_n)}{\|u_n\|}\phi \\
&\leq \beta \|\phi\|^2 + \int \left(\frac{C(x)}{\|u_n\|} + C_1(x) z_n\right)\phi \\
&\leq \beta \|\phi\|^2 + M \|\phi\|_\infty (\|C_1\|_r + \|C_2\|_r \|z_n\|_{r'}) \\
&\leq \beta \|\phi\|^2 + M \|\phi\|_\infty (\|C_1\|_r + \|C_2\|_r \|z_n\|)
\end{aligned}$$

and so $(\tau_n \|u_n\|_0^2 / \|u_n\|)$ is a bounded sequence. Thus, there exists some $\tau \geq 0$ such that $\tau_n \|u_n\|_0^2 / \|u_n\|$ converges to τ, up to a subsequence.

Now, observe that by dividing (4.4.0.53) by $\|u_n\|$, we deduce that z_n satisfies the equation

$$\int A(x, u_n)\nabla z_n \cdot \nabla v = \int \frac{f(\lambda, x, u_n)}{\|u_n\|} v + \tau_n \frac{\|u_n\|_0^2}{\|u_n\|} \int \phi v. \tag{4.4.0.55}$$

Since the sequence (z_n) is bounded in $H_0^1(\Omega)$, it follows that there exists some $z \in H_0^1(\Omega)$ such that

$$z_n \rightharpoonup z \quad \text{in } H_0^1(\Omega) \tag{4.4.0.56}$$

and so by Claim 4.2.1,

$$z_n \to z \text{ in } L^{2r'}(\Omega), \text{ up to a subsequence.} \tag{4.4.0.57}$$

So Proposition 5.3.2 implies that

$$z_n \to z \text{ a.e. in } \Omega, \text{ up to a subsequence.} \tag{4.4.0.58}$$

Since $\|z_n\|_0 = 1$, it follows, by Arzelà-Ascoli Theorem, that there exists some $\overline{z} \in E$ such that

$$z_n \to \overline{z} \text{ in } E, \text{ up to a subsequence,}$$

whence by (4.4.0.66), we deduce that $\overline{z} = z$.

Taking $v = z_n - z$ as a test function in (4.4.0.55), we obtain

$$\int A(x,u_n)\nabla z_n \nabla(z_n - z) = \int \frac{f(\lambda,x,u_n)}{\|u_n\|}(z_n - z) + \tau_n \frac{\|u_n\|_0^2}{\|u_n\|}\int \phi(z_n - z).$$

Subtracting

$$\int A(x,u_n)\nabla z \nabla(z_n - z),$$

in both sides of the above equation, we get from (A_2) and (4.0.0.2) that

$$\begin{aligned}
\gamma\|z_n - z\|^2 &\le \int A(x,u_n)\nabla(z_n - z)\nabla(z_n - z) \\
&= \int \frac{f(\lambda,x,u_n)}{\|u_n\|}(z_n - z) + \\
&+ \tau_n \frac{\|u_n\|_0^2}{\|u_n\|}\int \phi(z_n - z) - \int A(x,u_n)\nabla z \nabla(z_n - z) \\
&\le \int \left(\frac{C(x)}{\|u_n\|} + C_1(x) z_n\right)|z_n - z| \\
&+ \tau_n\|u_n\|\int \phi(z_n - z) - \int A(x,u_n)\nabla z \nabla(z_n - z) \\
&= \int \left(\frac{C(x)}{\|u_n\|} + C_1(x) z_n\right)|z_n - z| \\
&+ \tau_n \frac{\|u_n\|_0^2}{\|u_n\|}\int \phi(z_n - z) - \int_{\Omega^+} A(x,u_n)\nabla z \nabla(z_n - z) \\
&= M\left(\int (C(x) + C_1(x) z_n)|z_n - z|\right. \\
&+ \left.\int \phi(z_n - z) - \int_{\Omega^+} A(x,u_n)\nabla z \nabla(z_n - z)\right)
\end{aligned} \tag{4.4.0.59}$$

where

$$\Omega^+ = \{x \in \Omega : z(x) > 0\}.$$

By applying the Estimates II, III and using the convergence $z_n \to z$ in $L^{2r'}(\Omega)$ (see (4.4.0.65)), we deduce that

$$\begin{cases} \int C|z_n - z| \leq \|C\|_{(2r')'}\|z_n - z\|_{2r'} \text{ (Estimate II)} & (4.4.0.60)\\ \int C_1|z_n|z_n - z| \leq M\|C_1\|_r^{r/(p(2r')')}\|z_n\|_{2r'}^{2r'/(p'(2r')')}\|z_n - z\|_{2r'} \text{ (Estimate III)}. & (4.4.0.61) \end{cases}$$

But, by Claim 4.2.1, we have deduce that

$$\|z_n\|_{2r'}^{2r'/(p'(2r')')} \leq M\|z_n\|^{2r'/(p'(2r')')} = M$$

and by the convergence (4.4.0.65), conclude that the first two therms in (RHS) of (4.4.0.59) converges to zero, up to a subsequence.

We claim that $m(\Omega^+) > 0$. Indeed, on the contrary we would have

$$\int_{\Omega^+} A(x, u_n)\nabla z \nabla(z_n - z) = 0$$

and so the inequality (4.4.0.59) would imply $z_n \to z$ in $H_0^1(\Omega)$, consequently $1 = \|z_n\|_0 \to \|z\| = 1$, hence $0 \leq z \not\equiv 0$, but since $z \geq 0$, it would imply that $m(\Omega^+) > 0$, which is a contradiction.

Note that the $u_n = z_n\|u_n\| \to +\infty$ a.e. in $\Omega^+ := \{x \in \Omega; z(x) > 0\}$ and so the conditions (A_1) and (A_4) implies that $A(x, u_n)\nabla z$ converges strongly to $A(x, \infty)\nabla z$ in $L^2(\Omega^+)$, due to Dominated Convergence Lebesgue's Theorem. By the convergence $z_n \rightharpoonup z$ in $H_0^1(\Omega)$ (see (4.4.0.56)), we deduce that

$$\int_{\Omega^+} A(x, u_n)\nabla z \nabla(z_n - z) \to 0,$$

due to Proposition 5.3.2.

Finally, dividing (4.4.0.53) by $\|u_n\|$ and taking the limit when $n \to +\infty$, we deduce that z satisfies the equation

$$-\mathrm{div}(A(x, \infty)\nabla z) = \lambda f'_\infty(x)z + \tau\phi.$$

So, testing the above equations against ψ, we have

$$\begin{aligned} \int A(x, \infty)\nabla z \nabla\psi &= \lambda\int f'_\infty(x)z\psi + \int \tau\phi\psi \\ &\geq \lambda\int f'_\infty(x)z\psi. \end{aligned}$$

On the other hand,

$$\lambda_\infty\int f'_\infty(x)z\psi = \int A(x, \infty)\nabla z \nabla\psi,$$

then

$$\lambda_\infty\int f'_\infty(x)z\psi \geq \lambda\int f'_\infty(x)z\psi, \tag{4.4.0.62}$$

but since $z \neq 0$, this inequality implies that $\lambda_1 \quad \lambda$, which is a contradiction.
Case 2 ($\|u_n\|$) is a bounded sequence. Define the normalized sequence $z_n := u_n\|u_n\|_0^{-1}$. By taking $\phi/\|u_n\|_0$ as a test function in (4.4.0.53), we get

$$\int A(x, u_n)\, \nabla \left(\frac{u_n}{\|u_n\|_0}\right) \quad \nabla \phi = \int \frac{f(\lambda, x, u_n)}{\|u_n\|_0}\phi + \tau_n\|u_n\|_0\int \phi^2$$

and so, by using the hypotheses (A_1), (4.2.3), Hölder's inequality and Claim 4.2.1 we obtain

$$\begin{aligned}
\tau_n\|u_n\|_0\int \phi^2 \quad = & \quad \int A(x, u_n)\, \nabla \left(\frac{u_n}{\|u_n\|_0}\right) \quad \nabla \phi \\
& \quad \int \frac{f(\lambda, x, u_n)}{\|u_n\|}\phi \\
& \quad \beta\|\phi\|^2\frac{\|u_n\|^2}{\|u_n\|_0} + \int \left(\frac{C(x)}{\|u_n\|} + C_1(x)z_n\right)\phi \\
& \quad M\left(\beta\|\phi\|^2 + M\|\phi\|_1\left(\|C_1\|_r + \|C_2\|_r\right)\right).
\end{aligned}$$

So $\tau_n\|u_n\|_0$ is a bounded sequence. Thus, there exists some $\tau \quad 0$ such that $\tau_n\|u_n\|_0$ converges to τ up to a subsequence.

Now, observe that by dividing (4.4.0.53) by $\|u_n\|_0$, we deduce that z_n satisfies the equation

$$\int A(x, u_n)\, \nabla z_n\, \nabla v = \int \frac{f(\lambda, x, u_n)}{\|u_n\|_0}v + \tau_n\|u_n\|_0\int \phi v. \tag{4.4.0.63}$$

By taking $v = z_n$ as a test function and using (A_2), we obtain

$$\begin{aligned}
\gamma\|z_n\|^2 \quad & \quad \int A(x, u_n)\, \nabla z_n\, \nabla z_n \\
& \quad \int \frac{f(\lambda, x, u_n)}{\|u_n\|_0}z_n + \tau_n\|u_n\|_0\int \phi z_n \\
& \quad M\left(\int \frac{f(\lambda, x, u_n)}{\|u_n\|_0} + \int \phi\right) \\
& \quad M\left(\int \left(\frac{C(x)}{\|u_n\|_0} + C_1(x)z_n\right) + \bar{M}\right) \\
& \quad \hat{M}.
\end{aligned}$$

That is, the sequence (z_n) is bounded in the reflexive space $H_0^1(\ )$, then, there exists some $z \in H_0^1(\ )$ such that

$$z_n \rightharpoonup z \quad \text{in } H_0^1(\ ) \tag{4.4.0.64}$$

and so by Claim 4.2.1,

$$z_n \to z \text{ in } L^{2r'}(\ ), \text{ up to a subsequence.} \tag{4.4.0.65}$$

So Proposition 5.3.2 implies that

$$z_n \to z \text{ a.e. in } \ , \text{ up to a subsequence.} \tag{4.4.0.66}$$

Note that the $u_n = z_n\|u_n\| \to +\infty$ a.e. in $\Omega^+ := \{x \in \Omega; z(x) > 0\}$ and so the conditions (A_1) and (A_4) implies that $A(x,u_n)\nabla z$ converges strongly to $A(x,\infty)\nabla z$ in $L^2(\Omega^+)$, due to Dominated Convergence Lebesgue's Theorem. By the convergence $z_n \rightharpoonup z$ in $H_0^1(\Omega)$ (see 4.4.0.64), we deduce that

$$\int_{\Omega^+} A(x,u_n)\nabla z \nabla(z_n - z) \to 0,$$

due to Proposition 5.3.2.

Finally, by dividing (4.4.0.53) by $\|u_n\|_0$ and taking the limit when $n \to +\infty$, we deduce that z satisfies the equation

$$-\mathrm{div}(A(x,\infty)\nabla z) = \lambda f'_\infty(x)z + \tau\phi.$$

So, testing the above equation against ψ, we have

$$\int A(x,\infty)\nabla z\nabla\psi = \lambda\int f'_\infty(x)z\psi + \int \tau\phi\psi \tag{4.4.0.67}$$

$$\geq \lambda\int f'_\infty(x)z\psi. \tag{4.4.0.68}$$

On the other hand,

$$\lambda_\infty\int f'_\infty(x)z\psi = \int A(x,\infty)\nabla z\nabla\psi,$$

then

$$\lambda_\infty\int f'_\infty(x)z\psi \geq \lambda\int f'_\infty(x)z\psi, \tag{4.4.0.69}$$

but since $z \not\equiv 0$, this inequality implies that $\lambda_\infty \geq \lambda$, which is a contradiction. □

Remark 4.4.4. *Let $\lambda > \lambda_\infty$ and ϕ be a positive function and $R > 0$ as in the Lemma 4.4.6. Then*

$$z - \|z\|_0^2 S\left(f\left(\lambda, x, \frac{z}{\|z\|_0^2}\right) + \frac{\tau\phi}{\|z\|_0^2}\right) \neq 0 \quad \text{in } \overline{B_{R^{-1}}}(0)\setminus\{0\},\ \tau \in [0,1],$$

thus the homotopy (4.2.1) is admissible in $[0,1]\times\overline{B}_{R^{-1}}(0)$ and

$$i(H_2(\tau,\cdot),0) = deg(H_2(\tau,\cdot),B_{R^{-1}}(0),0)$$

for all $\tau \in [0,1]$.

Observe that by Lemma 4.4.6,

$$\frac{z}{\|z\|_0^2} - S\left(f\left(\lambda, x, \frac{z}{\|z\|_0^2}\right) + \tau\left\|\frac{z}{\|z\|_0^2}\right\|_0^2\phi\right) \neq 0$$

for each $z \neq 0$ with

$$\left\|\frac{z}{\|z\|_0^2}\right\|_0 \geq R\ ,\ \|z\|_0 \leq R^{-1},$$

moreover $H_2(1,0) = \ _1 \neq 0$, hence $H_2(1, \ ) \neq 0$ in $\overline{B_{R\ 1}}(0)$ and so by the property of existence of solution of the Leray-Schauder degree, we deduce that

$$deg(H_2(1, \ ), 0, 0) = 0.$$

By the invariance under homotopy,

$$\begin{aligned} i(\ , 0) &= i(H_2(0, \ ), 0, 0) \\ &= \deg(H_2(0, \ ), 0, 0) \\ &= \deg(H_2(1, \ ), 0, 0) \\ &= 0 \end{aligned}$$

for each $\lambda > \lambda_1$, where the last equation folllows from the argument in (4.2.1.5).

Finally, by applying Theorem A we deduce that it holds Theorem D as follows.

Proof. The existence of the continuum follows directly by applying Theorem A for $I = [0, +\infty)$, $\mu = \lambda_1$ and (a, b) being any interval containing λ_1 with $a > 0$. Now, by combining the additional hypothesis (4.0.0.11) with Remark (4.4.1), we obtain that $u = 0$ is the only solution of $(P\ )$ for $\lambda = 0$. But by Lemma 4.4.6, $(0, 0)$ is not a bifurcation point and so C $_1$ does not satisfies the alternative ii) of Theorem A. So we conclude that it must satisfy i), that is, C $_1$ is unbounded. □

Remark 4.4.5. *Observe that*

$$= \ _F,$$

by Remark 4.2.8.

4.5 Final comments

About the paper [4], we must comment that many other results are presented. Among them we quote the following. Based on the ideas from [5], the authors study the side of bifurcation. The bifurcation from the curve of trivial solutions was studied and it was proved the existence of a continuum C $_0$ emanating from

$$\lambda_0 = \begin{cases} \lambda_1(f_+^0(x, 0)), & \text{if } f_+^0(x, 0) \in L^r(\ ), \\ 0, & \text{otherwise,} \end{cases}$$

for a certain $f_+^0(x, 0) \in L^r(\ )$. The assumptions required in order to deduce this result were that [1] *for every bounded set of* R $\setminus \{0\}$ *and* $\lambda \in$,

$$\lim_{s \to 0^+} \frac{f(\lambda, x, s)}{s} = \lambda f_+^0(x, 0), \text{ uniformly in } (\lambda, x) \in \quad ,$$

with either $0 \quad f_+^0(x, 0) \in L^r(\ )$, $r > N/2$, *not identically zero or* $f_+^0(x, 0) = +\infty$ *a.e.* $x \in$ and

$$\exists \lim_{s \to 0^+} A(x, s) = A(x, 0) \text{ a.e. } x \in \ .$$

Moreover, by combining the two phenomena (bifurcation at infinity and from the curve of trivial solutions) many applications were presented. One of them is an existence result

[1] besides other technical hypotheses.

which states that, under certain conditions (see Theorem 5.12 of [4]), there exists $, \overline{\lambda}$ with $\overline{\lambda}$ λ_0 such that $(P$) has at least one positive solution for every λ $\overline{\lambda}$ and no positive solution if $\lambda >$. Also in Theorem 5.12 of [4], one have the multiplicity result that states that, under certain conditions, there exist at least two positive solutions for each λ 2 $(\lambda_0, \overline{\lambda})$. Moreover, the following open problem was proposed: does the continuum C $_0$ gives the maximal interval for which there exists a positive solution of $(P$)?

Chapter 5

Appendix

5.1 Topology

Proposition 5.1.1. *Let M and N be metric spaces and $f : M \to N$ a function. Under these conditions, the function f is continuous in $a \in M$ if for any given sequence $x_n \to a$, it follows that $f(x_n)$ converges to $f(a)$ up to a subsequence.*

Proof. See [29]. □

Definition 5.1.1. *A space Y is locally connected if it has a basis consisting of connected (open) sets.*

See Section 4 in Dugundji [18].

Example 5.1.1. *The euclidean space $\mathbb{R}^N$ is a locally connected space. Indeed, the balls are connected sets which constitute a basis for $\mathbb{R}^N$.*

See Section 4 in Dugungji [18].

Definition 5.1.2. *A space Y is path-connected (or: pathwise connected) if each pair of its points can be joined by a path.*

See Definition 5.1 in Dugundji [18].

Theorem 5.1.1. *An open set in $\mathbb{R}^N$ is connected if and only if it is path-connected.*

Proof. See [29]. □

Theorem 5.1.2. *Y is locally connected if and only if the components of each open set are open sets.*

Proof. See Section 4 in Dugundji [18]. □

5.2 Measure Theory

Theorem 5.2.1 (Lebesgue's Dominated Convergence Theorem). *Assume $f_1, f_2, \ldots, f_n, \ldots$ are μ-measurable functions on $\mathbb{R}^N$. Assume $g \geq 0$, $g \in L^1$. Assume*

$$\lim_{k \to \infty} f_k(x) \quad \text{exists for all } x \in \mathbb{R}^N$$

and

$$|f_k(x)| \le g(x) \quad \forall\, x \in \mathbb{R}^N.$$

Then $\lim_{k\to\infty} f_k \in L^1$ *and*

$$\int \left(\lim_{k\to\infty} f_k\right) d\mu = \lim_{k\to\infty} \int f_k d\mu.$$

Proof. See Section B of Jones [22]. □

Theorem 5.2.2. *Let* $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$, *$\mu$-measurable functions defined a.e. on* $\mathbb{R}^N$*, and such that*

$$\lim_{k\to\infty} \varphi_k(x) = \varphi(x) \text{ exists for a.e. } x \in \mathbb{R}^N.$$

Also assume $g_1, g_2, \ldots, g_n, \ldots$ *are nonnegative functions in* $L^1(\mathbb{R}^N)$*, that*

$$|\varphi_k(x)| \le g_k(x) \text{ for a.e. } x \in \mathbb{R}^N,$$

that

$$\lim_{k\to\infty} g_k(x) = g(x) \text{ exists for a.e. } x \in \mathbb{R}^N,$$

that $g \in L^1(\mathbb{R}^N)$*, and that*

$$\int_{\mathbb{R}^N} g d\mu = \lim_{k\to\infty} \int_{\mathbb{R}^N} g_k d\mu.$$

Then,

$$\int_{\mathbb{R}^N} \varphi d\lambda = \lim_{k\to\infty} \int_{\mathbb{R}^N} \varphi_k d\lambda.$$

Proof. See Problem 19 of [22]. □

5.3 Functional Analysis

Definition 5.3.1 (General compact operators)**.** *Let E and F be Banach spaces. A continuous operator $T : E \to F$ is said to be compact if $\overline{T(A)} \subset F$ is a compact subset whenever $A \subset E$ is bounded. Or equivalently, if the sequence $(T(u_n))_n$ converges up to a subsequence whenever $(u_n)_n$ is a bounded sequence in E.*

See Definition 9.5.1 in [8]. When $E = F$ and X is a subset of E, we denote by $Q(X; E)$ the family of all compact operators $T : X \to E$.

5.3.1 Spectral Theory

Here we present some concepts that can be found in Section 7.1 of [8].

Let E a $\mathbb{K}$ normed space and $T \in \mathcal{L}(E, E)$.

Definition 5.3.2 (Eigenvalue and Characteristic Value)**.** *Let $\lambda \in \mathbb{K}$. We say that λ is an eigenvalue of T when $\mathrm{Ker}(T - \lambda I) \neq \{0\}$. If $\lambda \neq 0$ is an eigenvalue of T, then its reciprocal λ^{-1} is called characteristic value of T.*

Suppose that $\lambda \in \mathbb{K}$ is not an eigenvalue of T. Thus, $\mathrm{Ker}(T - \lambda I) = \{0\}$ and it makes sense to consider the operator

$$(T - \lambda I)^{-1} : (T - \lambda I)(E) \subset E \to E,$$

which is an injective and linear operator.

Definition 5.3.3. *An scalar λ is said to be a regular value of the operator T is $(T - \lambda I)$ is bijective and*

$$(T - \lambda I)^{-1} : E \to E$$

is continuous. The set of all regular values of T is called the resolvent of T and denoted by $\rho(T)$. It complementar set $(\mathbb{K} \setminus \rho(T))$ is called the spectrum of T and it is denoted by $\sigma(T)$.

Remark 5.3.1. *If E is a Banach space, the Open Function Theorem implies that*

$$\rho(T) = \{\lambda \in \mathbb{K};\ (T - \lambda I) \text{ is bijective}\}.$$

Remark 5.3.2. *From the definition, it follows that every eigenvalue of T belongs to the spectrum of T.*

Theorem 5.3.1. *Let E be a Banach space. The spectre of a compact linear operator $T : E \to E$ is enumerable and its only accumulation point is zero.*

Theorem 5.3.2. *Let E be a Banach space, $T : E \to E$ a compact operator and $\lambda \in \mathbb{K}$, $\lambda \neq 0$. Then, the operator $(T - \lambda I)$ is injective if and only if it is surjective.*

Proof. See Lema 7.3.2 in [8]. □

Theorem 5.3.3. *Let E be an infinite dimensional Banach space and $T : E \to E$ a compact operator linear operator. Then,*

$$\sigma(T) = \{0\} \cup \{\lambda;\ \lambda \text{ is an eigenvalue of } T\}.$$

Proof. See Teorema 7.3.6 in [8]. □

Fredholm Operators

Definition 5.3.4. *Let U and V be Banach spaces and $T \in \mathbf{L}(U, V)$. We say that T is a Fredholm Operator if*

$$\dim N[T] < \infty \quad \text{and} \quad \operatorname{codim} R[T] < \infty .$$

Under these conditions it is possible to show that $R[T]$ is a closed subspace of V. Moreover we denote the index of T as

$$\operatorname{ind}[T] := \dim N[T] - \operatorname{codim} R[T].$$

By $\mathrm{Fred}_0(U, V)$ we denote the subfamily of $L(U, V)$ constituted by the Fredholm operadors of index 0.

Proposition 5.3.1. *Let E be a normed space.*

(a) *If $x_n \rightharpoonup x$ in E, then the sequence $(\|x_n\|)_{n=1}^{\infty}$ is bounded and $\|x\| \leq \liminf_n \|x_n\|$.*

(b) *If $x_n \rightharpoonup x$ in E and $\varphi_n \to \varphi$ in E', then $\varphi_n(x_n) \to \varphi(x)$ in $\mathbb{K}$.*

Proof. See Proposition 6.2.5 of [8]. □

Proposition 5.3.2. *If $f_n \to f$ in $L^p(X)$, $1 \leq p < \infty$, then there exists a subsequence $(f_{n_j})_{j=1}^{\infty}$ and a function $g \in L_p(X)$ such that:*

i) $f_{n_j}(x) \to f(x)$ μ *a.e. and*

ii) $|f_{n_j}(x)| \leq g(x)$ μ *a.e.*

Proof. See Exercise 9.9.1 of [8]. □

Theorem 5.3.4 (Teorema 13.44 of [21]). *Suppose that* $1 < p < \infty$ *and that* $(\|f_n\|_p)_{n=1}^{\infty}$ *is a bounded sequence. If* $f_n \to f$ μ *a.e., then* $f_n \to f$ *weakly in* $L^p(\Omega)$*.*

Proof. See Theorem 13.44 of [21]. □

Theorem 5.3.5 (Hölder inequality). *Let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and let* $(X, \ , \mu)$ *be a measure space. If* $f \in L_p(X, \ , \mu)$ *and* $g \in L_q(X, \ , \mu)$*, then* $fg \in L_1(X, \ , \mu)$ *and*

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q.$$

Proof. See Theorem 1.2.1 in [8]. □

Theorem 5.3.6 (Implicit Function Theorem). *Let* E_1, E_2 *and* F *be normed linear spaces and assume that* E_2 *is complete. Let* $\Omega \subseteq E_1 \times E_2$ *be an open set and let* $f : \Omega \to F$ *be a function such that*

i) f *is continuous;*

ii) for every $(u, v) \in \Omega$, $\frac{\partial f}{\partial v}(u, v)$ *exists and is continuous on* Ω*;*

iii) $f(a, b) = 0$ *and* $\frac{\partial f}{\partial v}(a, b)$ *is invertible with continuous inverse.*

Then, there exist neighbourhoods U of a *and V of* b *and a continuous function* $\varphi : U \to V$ *such that* $\varphi(a) = b$*,*

$$f(u, \varphi(u)) = 0$$

and these are the only solutions of $f(u, v) = 0$ *in* $U \times V$*.*

Proof. See Theorem 1.3.1 in Kesavan [23]. □

Theorem 5.3.7 (Inverse Function Theorem). *Let* E *and* F *be Banach spaces and* $f : \Omega \subseteq E \to F$ *be a* C^p*-map, for some* $p \geq 1$*. Let* $a \in \Omega$ *with* $f(a) = b$ *and let* $f'(a) : E \to F$ *be an isomorphism. Then there exists a neighbourhood V of* b *in* F *and a unique* C^p *function* $g : V \to E$ *such that*

$$\begin{cases} a = g(b) \\ f(g(y)) = y \end{cases}$$

for every $y \in V$*.*

Proof. See Theorem 1.3.2 in Kesavan [23]. □

Theorem 5.3.8. *Let* E *be a Banach space. If* $T : E \to E$ *is a contraction, then the operator* $T - I$ *is a Lipschitz bijection with Lipschitz inverse.*

Nemmytskii Operator.

Definition 5.3.5 (Caratheódory Functions[1]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset. We say that the function*

$$\Omega \times \mathbb{R}^M \ni (x,u) \mapsto f(x,u) \in \mathbb{R} \tag{5.3.1.1}$$

satisfies the Carathéodory Conditions if

$$u \mapsto f(x,u) \text{ is continuous for almost every } x \in \Omega. \tag{5.3.1.2}$$

and

$$x \mapsto f(x,u) \text{ is mensurable for all } u \in \mathbb{R}^M. \tag{5.3.1.3}$$

Definition 5.3.6 (Nemytskii Operator[2]). *Let Ω be a bounded open subset of $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R}^M$ satisfying the Carathéodory Conditions. Let E a certain family of functions $u : \Omega \to \mathbb{R}^M$, then, the function*

$$\begin{aligned} F : E &\to F(E) \\ u &\mapsto F(u) : \Omega \to \mathbb{R} \\ &\qquad\quad x \mapsto F(u)(x) := f(x,u(x)) \end{aligned}$$

is called Nemytskii operator of function f.

Theorem 5.3.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset and let*

$$\begin{aligned} f : \Omega \times \mathbb{R}^M &\to \mathbb{R} \\ (x,\xi) &\mapsto f(x,\xi) \end{aligned}$$

a function satisfying the Carathéodory Conditions. Additionally, let $p \in (1,\infty)$ and $g \in L^q(\Omega)$, where q is the conjugated of p, and suppose that f satisfies

$$|f(x,\xi)| \leq C|\xi|^{p-1} + g(x). \tag{5.3.1.4}$$

Then, the Nemytskii operator F of the function f is a function over $L^P(\Omega)$ to $L^q(\Omega)$ which is continuous and bounded.

5.4 Elliptic problems

In this section we present some concepts and results in elliptic problems as existence of solutions and regularity. For more details see [20].

Let Ω a bounded domain in $\mathbb{R}^N$, $a, b_i, c_i, d : \Omega \to \mathbb{R}$, be $i = 1, 2, \ldots, N$ measurable functions and consider a differential operator L in the form

$$Lu = D_i(a_{ij}(x)D_j u + b^i(x)u) + c^i(x)D_i u + d(x)u.$$

Assume that the function u is weakly differentiable and that the functions $a^{ij}D_j u + b^i u,\ c^i D_i u + du,\ i = 1, \ldots, n$ are locally integrable, then, we say that u satisfies $Lu = 0$, ($u \geq 0$, $u \leq 0$) in weak (or generalized), respectively in Ω when

$$\mathrm{L}(u,v) = \int_\Omega [(a^{ij}D_j u + b^i u)D_i v - (c^i D_i u + du)v]dx = 0 \ (\leq 0, \ \geq 0)$$

[1] see [34].

[2] see [34].

for all non negative function $v \in C_0^1(\Omega)$.

Let $f^i,\ g,\ i = 1,\dots,\ n$ be locally integrable (in Ω) functions. Then, a weakly differentiable function u is called weak (or generalized) solution of the non homogeneous equation

$$Lu = f \tag{5.4.0.1}$$

in Ω, if

$$\mathsf{L}(u,v) = F(v) = \int fv\,dx \ \forall\, v \in C_0^1(\Omega).$$

where $C_0^1(\Omega)$ denotes the space of test functions.

Consider the generalized Dirichlet problem for the equation (5.4.0.1). Let us assume that L is strictly elliptic in em Ω, that is, there exist a positive number λ such that

$$a^{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2,\ \forall\, x \in \Omega,\ \xi \in \mathbb{R}^n. \tag{5.4.0.2}$$

Suppose, also, that L satisfies the following limitation condition over its coefficient: there exist constants Λ and $\nu \geq 0$ such that for all $x \in \Omega$,

$$\sum |a^{ij}(x)|^2 \leq \Lambda^2,\quad \lambda^{-2}\sum(|b^i(x)|^2+|c^i(x)|^2) + \gamma^{-1}|d(x)| \leq \nu^2. \tag{5.4.0.3}$$

In order to enunciate the Weak Maximum Principle, we require a notion of inequality at the boundary for function in the Sobolev space $W^{1,2}(\Omega)$. We say that $u \in W^{1,2}(\Omega)$ satisfies $u \leq 0$ on $\partial\Omega$ if its positive part $u^+ = \max\{u, 0\} \in W_0^{1,2}(\Omega)$. If u is continuous in a neighborhood of $\partial\Omega$, then, u satisfies $u \leq 0$ on $\partial\Omega$ if the inequality holds in the classical pointwise sense. We say that $u \geq 0$ on $\partial\Omega$ if $-u \leq 0$ on $\partial\Omega$ and for $v \in W^{1,2}(\Omega)$ we say that $u \leq v$ on $\partial\Omega$ if $u - v \leq 0$ on ∂. Moreover,

$$\sup_{\partial\Omega} u = \inf\{k;\ u \leq k \text{ on } \partial\Omega, k \in \mathbb{R}\};\quad \inf_{\partial\Omega} u = -\sup_{\partial\Omega}(-u).$$

Theorem 5.4.1. *Let $u \in W^{1,2}(\Omega)$ satisfy $Lu \geq 0$ (≤ 0) in Ω. Then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+;\quad \inf_{\Omega} u \geq \inf_{\partial\Omega} u^-.$$

Proof. See Theorem 8.1 in [20]. □

Corollary 5.4.1. *Let L be an elliptic operator defined as in the above definition of weak solutions. Let $u \in W_0^{1,2}(\Omega)$ satisfy $Lu = 0$ in the weak sense. Then $Lu = 0$ in Ω in the weak sense.*

Theorem 5.4.2. *Suppose that the operator L satisfies conditions (5.4.0.2) and (5.4.0.3) and that $f^i \in L^q(\Omega),\ i = 1,\dots,n,\ g \in L^{q/2}(\Omega)$ for some $q > n$. Then, if u is a subsolution (supersolution) of equation (5.4.0.1) in $W^{1,2}(\Omega)$, satisfying $u \leq (\geq 0)$ in $\partial\Omega$, then, it holds*

$$\sup_{\Omega} u(-u) \leq C(\|u^+(u^-)\|_2 + k)$$

where $k = \gamma^{-1}(\|f\|_q + \|g\|_{q/2})$ and $C = C(n, \nu, q, |\Omega|)$.

Proof. See Theorem 8.15 in Gilbarg and Trudinger [20]. □

Remark 5.4.1. *Observe that if $p \geq 1$ then*

$$
\begin{aligned}
\|u^+\|_p^p &= \int |u^+|^p \\
&= \int_{\Omega_+} |u|^p dx \\
&\leq \int_\Omega |u|^p dx \\
&= \|u\|_p^p .
\end{aligned}
$$

Theorem 5.4.3. *Let L be an operator satisfying the conditions (5.4.0.2) and (5.4.0.3), $f \in L^r(\Omega)$ with $r > N/2$ and suppose that Ω satisfies a uniform exterior cone condition on a boundary portion T. Then if $u \in H(\Omega)$ satisfies the equation (5.4.0.1), then $u \in C^\alpha(\Omega \cup T)$ for some $\alpha > 0$.*

See Theorem 8.29 of [20].

Theorem 5.4.4 (Solution Operator Δ^{-1}). *Let Ω be a bounded open subset of $\mathbb{R}^N$. Then the Dirichlet boundary value problem*

$$
\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4.0.4}
$$

where $h \in L^2(\Omega)$, admits a solution operator $S : L^2(\Omega) \to H_0^1(\Omega)$, defined by $K(f) = u$, where u is the unique solution of (5.4.0.4). Moreover, S is a compact linear operator.

Proof. See Theorem 1.10 of [2]. □

Consider the following Dirichlet boundary Value Problem

$$
\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.4.0.5}
$$

Theorem 5.4.5. *Let $\partial\Omega$ be smooth and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a locally Hölder continuous, satisfying*

$$
|f(x,s)| \leq a_1(x) + a_2|s|^p, \quad \forall (x,u) \in \Omega \times \mathbb{R} \tag{5.4.0.6}
$$

for some $a_1 \in L^{2n/(n+2)}(\Omega)$ and $a_2 > 0$, with $1 < p \leq (n+2)/(n-2)$. Then every $u \in H_0^1(\Omega)$ which is a weak solution of (5.4.0.5) is a C^2-solution of (5.4.0.5).

Proof. See Theorem 1.16 and Remark 1.17 in [2]. □

Remark 5.4.2. *Observe that the hypothesis $1 < p \leq (n+2)/(n-2)$ just makes sense when $n > 2$. But for $n \in \{1,2\}$ the result remains valid. In fact, the idea of the proof in [2] is to use bootstrap method to deduce that if $u \in H_0^1(\Omega)$ is a weak solution of (5.4.0.5), then $u \in W^{2,\gamma}(\Omega)$ with $\gamma \geq n/2$ and this fact is sufficient for the conclusion of the proof. But, $u \in H_0^1(\Omega)$ implies that $u \in W^{2,1}(\Omega)$ and for $n \in \{1,2\}$ one have $n/2 \leq 1$ and so $\gamma = 1 \geq n/2$ satisfies that $u \in W^{1,\gamma}(\Omega)$, then it follows the proof.*

5.5 The eigenvalue problem and the eigenvalue problem with weight

Consider the linear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.5.0.1}$$

By Theorem 5.4.4 we know that (5.5.0.1) is equivalent to $u = Ku$, $u \in L^2(\Omega)$.

Theorem 5.5.1. *The problem (5.5.0.1) has a sequence of eigenvalues λ_k such that*

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots, \; \lambda_k \to +\infty$$

and

$$\lambda_1 \int u^2 dx \leq \int |\nabla u|^2 dx.$$

Proof. See Theorem 1.13 of [2]. □

Let us consider the eigenvalue problem

$$\begin{cases} Lu = \mu m u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.5.0.2}$$

where

$$Lu = -\sum_{i,j=1}^{N} D_j(a_{ij}(x)D_i) + a_0(x)u$$

is a strictly elliptic operator defined in a bounded open subset Ω of $\mathbb{R}^N$. We assume $a_{ij} = a_{ji}$ and $a_0 \geq 0$. The weight function $m : \overline{\Omega} \to \mathbb{R}$ lies in $L^r(\Omega)$ with $r \geq N/2$. The coefficients $a_{ij} \in L^\infty(\Omega)$ and $a_0 \in L^{N/2}(\Omega)$.

Theorem 5.5.2. *The eigenvalue problem (5.5.0.2) has double sequence of eigenvalues*

$$\ldots \leq \mu_{-2} \leq \mu_{-1} < 0 < \mu_1 \leq \mu_2 \leq \ldots$$

whose variational characterizations are

$$\frac{1}{\mu_n} = \sup_{F_n} \inf \left\{ \int m u^2; \; \|u\| = 1, \; u \in F_n \right\}, \quad \frac{1}{\mu_{-n}} = \inf_{F_n} \sup \left\{ \int m u^2; \; \|u\| = 1, \; u \in F_n \right\}$$

where F_n varies over all n-dimensional subspaces of H_0^1. The corresponding eigenfunctions ϕ_n are such that

$$a(\phi_n, v) = \mu_n \int m \phi_n v \quad \forall v \in H_0^1$$

and

$$a(\phi_n, \phi_n) = 1 \Rightarrow \frac{1}{\mu_n} = \int m \phi_n^2.$$

Proof. See Proposition 1.10 of [16]. □

Theorem 5.5.3. *Let $m : \Omega \to \mathbb{R}$ be an L^r-function, with $r > N/2$, (not necessarily positive). Suppose that $m > 0$ on a subset of Ω with positive measure. Then the first positive eigenvalue μ_1 of (5.5.0.2) is simple and ϕ_1 can be taken positive in Ω. A similar statement holds if $m < 0$ on a subset of Ω with positive measure.*

Proof. See Theorem 1.13 of [16]. □

5.6 Sobolev Spaces

Theorem 5.6.1 (Poincaré Inequality). *Let Ω be an open subset of $\mathbb{R}^n$ bounded in some direction. Under these conditions the expression*

$$\|u\| = \left(\sum_{i=1}^{n} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 dx \right)^{1/2} \tag{5.6.0.1}$$

defines a norm in $H_0^1(\Omega)$ which is equivalent to the norm $\|u\|_1 = \|u\|_{H^1(\Omega)}$.

Proof. See Theorem 2.2.4 in Section 2.2 of [32]. □

Theorem 5.6.2 (Rellich-Kondrachov Theorem). *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1 and $1 \leq p \leq \infty$. Then the following embeddings are compact:*

a) $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $1 \leq q < p^*$, *if* $p \leq n$;

b) $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ *if* $p > n$,

where

$$\begin{cases} p^* = \dfrac{np}{n-p} & \text{if } p < n, \\ p^* = \infty & \text{if } p = n. \end{cases}$$

Proof. See Theorem 2.5.4 of [32]. □

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