Universidade de Brasília

Instituto de Ciências Exatas

Departamento de Matemática

Abelian transitive state-closed subgroups of automorphisms of the m-ary tree

Autor: Bruno Zaban Silva

Orientador: Prof. Dr. Alex Carrazedo Dantas

Brasília

2021

Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Abelian transitive state-closed subgroups of automorphisms of the m-ary tree

por

Bruno Zaban Silva

Dissertação apresentada ao Departamento de Matemática da Universidade de Brasília, como parte dos requisitos para obtenção do grau de

MESTRE EM MATEMÁTICA

Brasília, 13 de outubro de 2021.

Comissão Examinadora:

Prof. Dr. Alex Carrazedo Dantas- MAT/UnB (Orientador)

Emerson Ferreira de Melo.

Prof. Dr. Emerson Ferreira de Melo– MAT/UnB (Membro)

Francesco Matucci

Prof. Dr. Francesco Matucci– UNIMIB (Membro)

O homem é uma corda, estendida entre o animal e o super-homem - uma corda por sobre um abismo.
Um perigoso atravessar, um perigoso olhar para trás, um perigoso arrepiar-se e estacar.
O que é grandioso no homem é que ele seja uma ponte, e não um fim: o que pode ser amado no homem é que ele seja uma passagem e um ocaso.
Eu amo aqueles que não sabem viver a não ser como poentes, pois eles são os que atravessam.

F.Nietzsche; Assim falou Zaratustra

Abstract

In this work we examine the abelian transitive state-closed subgroups of automorphisms groups of one-rooted regular trees. Its presentation, torsion subgroup, and the case of cyclic groups are shown in detail. Also, when the group of induced permutations is cyclic of prime order we obtain additional structural and topological properties. In the last section we study representations of the free abelian group of countable infinite rank.

Resumo

Neste trabalho examinamos os subgrupos abelianos transitivos e fechados por estado do grupo de automorfismos de árvores regulares unirraiz. Sua apresentação, subgrupo de torção e o caso de grupos cíclicos são vistos em detalhe. Ainda, quando o grupo de permutações induzidas é cíclico de ordem prima obtemos propriedades estruturais e topológicas adicionais. Na última seção, estudamos as representações do grupo abeliano livre de posto infinito enumerável.

Agradecimentos

Aos meus pais, Silas e Maria Luiza, e à minha irmã Maria Eugênia pelo apoio e incentivo constante; à minha filha Alyssa por adoçar a minha vida. À minha namorada Julia, pelo amor nos tempos do cólera.

Um agradecimento especial ao meu orientador Alex Carrazedo Dantas pela amizade, paciência e confiança dispensadas à mim. Esse trabalho foi muito mais recompensante com a sua orientação.

Aos professores Noraí Rocco, Aline Pinto, Igor Lima, Raimundo Bastos e Daniele Nantes pelo apoio e encorajamento nessa caminhada. Ao professor Lineu Neto, que há muitos anos atrás me apresentou apropriadamente à Rainha das Ciências.

Sob pena de deixar muitas pessoas queridas de fora, me arrisco a citar alguns amigos que a Matemática me deu e que foram apoio nos momentos difíceis e fonte de risadas nos momentos felizes: Vinícius, Katianny, Rômulo, Marcos, José Carlos e Renata.

Aos meus amigos de longa data Paulo Thirso, Laís, Natalia, Pedro e Ana Paula. Obrigado por me presentearem com a amizade de vocês. Ao meu caro amigo Éverton, por tudo.

Aos servidores do Departamento de Matemática e da Universidade de Brasília por nos propiciarem um ambiente em que o estudo pode florescer. Ao Chiquinho, o insubstituível livreiro da UnB.

Contents

Introduction

The definition of state-closed groups of automorphisms of a rooted tree was inspired by the interplay between the recursiveness of these automorphisms and Automata Theory, which was anticipated by the fact that the Burnside p-groups of Aleshin, Grigorchuk and Gupta-Sidki, among others, admit a faithful representation into the group of finite state automorphisms of a regular tree. In this work we explore the case when the state-closed groups are transitive and abelian, exploring its main features as we shall now describe in detail.

An automorphism of a one-rooted regular tree of degree m is a graph bijection which preserves vertex incidence. We can index the vertices of such tree by words on the alphabet $Y = \{1, 2, ..., m\}$, where the words appearing in the first level are $1, 2..., m$, the alphabet itself. In this case our tree will be denoted by $\mathcal{T}_m$. We notice that, an automorphism of a subtree can also be seen as an automorphism of the entire tree, since the subtree is isomorphic to $\mathcal{T}_m$. This observation leads us to the fact that the group of automorphisms of the tree, denoted by $\mathcal{A}_m$, can be written as $\mathcal{A}_m = (\mathcal{A}_m \times \mathcal{A}_m \times \ldots \times \mathcal{A}_m) \rtimes S_m$, where S_m is the symmetric group of degree m permuting the indexes of the copies of $\mathcal{A}_m$. Thus, the elements $\alpha \in \mathcal{A}_m$ can be expressed as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)\sigma$. The automorphisms α_i are called the *states* of α.

We define three important subgroups of a group $G \leq \mathcal{A}_m$: $Stab_G(n)$, the *stabilizer* of the n-th level, $Fix_G(u)$ the *fixator* of the word u and $P(G)$, the group of permutations on Y induced by the automorphisms in G. The group G will be called *transitive* if $P(G)$ is transitive.

In our investigations some closure operations applied to a group $G \leq \mathcal{A}_m$ will be of particular interest. First, we notice that the stabilizers $Stab_{\mathcal{A}_m}(n)$ are normal subgroups and provide $\mathcal{A}_m$ an inverse limit structure as $\mathcal{A}_m \simeq \varprojlim \frac{\mathcal{A}_m}{Stab_{\mathcal{A}_m}(n)}$; hence, the elements of $\mathcal{A}_m$ can be written as coherent infinite products. For a group $G \leq \mathcal{A}_m$, we define its *topological closure*, denoted by $\overline{G}$, as the set of well-defined infinite products of its elements. Now, consider a monomorphism defined recursively as $\alpha^{(0)} = \alpha$, $\alpha^{(1)} = (\alpha, \alpha, \ldots, \alpha)$ and $\alpha^{(n+1)} = (\alpha^{(n)})^{(1)}$. This diagonal map in some sense exhausts all the possibilities of actions of an automorphism α on every subtree. Then, the *diagonal closure* of G is the group $\widetilde{G} = \langle G^{(i)} \mid i \geq 0\rangle$. At last, the *state closure* $\widehat{G}$ will be the group generated by all states of the elements of G. In the abelian and transitive case, when considering the diagonal-topological closure A^*, writing $\alpha^{(i)}$ as α^{x^i} allows us to express its elements as products of expressions of the form

$$\alpha^{a_0}(\alpha^{a_1})^{(1)}(\alpha^{a_2})^{(2)} \ldots (\alpha^{a_n})^{(n)} \ldots = \alpha^{a_0+a_1x^1+a_2x^2+\ldots+a_nx^n+\ldots},$$

where the polynomial is in $\mathbb{Z}_m[[x]]$, the ring of formal sums over a m-adic ring.

A central question in the theory of state-closed groups (also called *self-similar*, depending on the approach) is whether a group G has a faithful representation in $\mathcal{A}_m$; i.e., if a group G acts on a one-rooted regular tree of some degree. One of the methods to tackle this question is using *virtual endomorphisms*, that is, a homomorphism $f : H \to G$ from a subgroup $H \leq G$ of finite index to G. In [NS04], the authors provide a method to find a representation $\varphi : G \to \mathcal{A}_m$:

Given a transversal $T = \{t_1, \ldots, t_m\}$ of H in G, every element $g \in G$ induces a permutation $\sigma(g) : Y \to Y$ with respect to T, given by

$$i^{\sigma(g)} = j \Leftrightarrow Ht_ig = Ht_j, \quad i, j = 1, \ldots, m.$$

Notice that $t_igt_j^{-1} \in H$, that is, $t_igt_{i^{\sigma(g)}}^{-1} \in H$. Now define $\varphi : G \to \mathcal{A}_m$ as:

$$g \mapsto ((t_1gt_{1^{\sigma(g)}}^{-1})^{f\varphi}, (t_2gt_{2^{\sigma(g)}}^{-1})^{f\varphi}, \ldots, (t_mgt_{m^{\sigma(g)}}^{-1})^{f\varphi})\sigma(g).$$

The function defined above gives us the remarkable property that G^{φ} is state-closed, transitive and

$$ker(\varphi) = \langle K \leq H \mid K \lhd G, K^f \leq K \rangle,$$

called the f-$core(H)$. If f-$core(H) = 1$ we say that f is *simple*, $G \simeq G^{\varphi}$ and the representation is faithful.

In possession of these tools, we proceed to investigate transitive abelian state-closed groups considering their diagonal-topological closure A^*. Notice that in the theorem below we give an explicit finite presentation for A^* considered as a module, even when A is not finitely presented:

Theorem A. *([BS10], p.460) Let A be a transitive abelian state-closed subgroup of degree m. Then A^* is additively a $\mathbb{Z}_m[[x]]$-module generated by $\{\beta_i \mid 1 \leq i \leq k\}$, subject to the set of relations*

$$\Big\{ r_i = \sum_{1 \leq j \leq k} m_i \beta_i - p_{ij} \beta_j x = 0 \mid 1 \leq i \leq k \Big\},$$

for some $p_{ij} \in \mathbb{Z}_m[[x]]$. Moreover, there exist $r, q \in \mathbb{Z}_m[[x]]$ such that $r = m - xq$ and $rA^* = (0)$. The elements of A^* can be represented additively as $\sum_{1 \leq i \leq k} p_i \beta_i$, where $p_i = \sum_{j \geq 0} p_{ij} x^j$, with $p_{ij} \in \mathbb{Z}$ and $0 \leq p_{ij} < m$.

Following up our discussion, we study the classical theme of torsion in abelian groups. The first step in this direction is proving that, if A is a transitive abelian state-closed group, then $tor(A)$ has finite exponent and is therefore a direct summand of A. Then, we prove the following result that adds an important topological consideration to torsion groups:

Theorem B. *([BS10], p.467) Suppose that A is a transitive state-closed abelian torsion group of degree m. Then A is conjugate to a subgroup of the topological closure of*

$$\widetilde{P(A)} = \langle \sigma^{(i)} \mid \sigma \in P(A),\ i \geq 0 \rangle.$$

Next, the special case of cyclic $\mathbb{Z}_m[[x]]$-modules is studied. We show an explicit form for A^* as a quotient module. The theorem confirms the example we provided of the diagonal-topological closure of the binary adding machine.

Theorem C. *([BS10],p.468) i) The expression $\alpha = (\alpha^{q_1}, \alpha^{q_2}, ..., \alpha^{q_m})\sigma$ is a well-defined automorphism of the m-ary tree;*
ii) Let A be the state closure of $\langle\alpha\rangle$. Then A^* is abelian, isomorphic to the quotient ring $\mathbb{Z}_m[[x]]/(r)$, where $r = m - qx$ and $q = q_1 + ... + q_m$.

In the case where the group $P(A)$ of induced permutations by elements of G has prime order, we can choose a single automorphism β such that $A^* = \langle\beta\rangle^*$ and A^* is topologically finitely generated.

Theorem D. *([BS10], p.470) Let m be a prime number. Let A be a torsion-free abelian transitive state-closed subgroup of $\mathcal{A}_m$ and let $\beta \in A \backslash Stab_A(1)$ such that $\zeta(\beta)$ is minimum. Then $A^* = \langle\beta\rangle^*$ and is topologically finitely generated.*

Now, let $\alpha = (e, ...e, \alpha^{x^{j-1}})\sigma \in \mathcal{A}_m$. Then $\alpha^m = \alpha^{x^j}$; that is, $\alpha^r = e$ where $r = m - x^j$. The states of α are $\alpha, \alpha^x, ..., \alpha^{x^{j-1}}$ and the group

$$D_m(j) = \langle \alpha, \alpha^x, ..., \alpha^{x^{j-1}} \rangle$$

is diagonally closed. Furthermore, the topological closure $\overline{D_m(j)}$ is isomorphic to the quotient ring $S = \frac{\mathbb{Z}_m[[x]]}{(r)}$, which is a free $\mathbb{Z}_m$-module of rank j.

With this special group in mind, we can state an extension of the previous theorem, providing even more structure to A^*:

Theorem E. *([BS10], p.470) In the same conditions of the previous theorem, we have that $A^* = \langle\beta\rangle^*$ is conjugate to $\overline{D_m(j)}$ for some $j \geq 1$.*

Then, we examine an open question that the authors state in [BS10], namely, whether a free abelian group of infinite rank admits a faithful transitive state-closed representation, even of prime degree. An answer was given in [BS20]:

Theorem F. *([BS20], p.108) Let $\mathbb{Z}^{(\omega)}$ the restricted product of countably many copies of the integers. Then there exists a faithful transitive state-closed action of $\mathbb{Z}^{(\omega)}$ into the binary tree.*

To bring our work to an end, we prove the case where is taken a direct sum of countably many copies of an abelian transitive state-closed group L and concluding that $L^{(\omega)} \wr C_2$ is also transitive and state-closed:

Theorem G. *([DS18], p.1062) Let L be an abelian transitive state-closed group and L^{ω} an infinite countable direct sum of copies of L. Then $L^{(\omega)} \wr C_2$ is also transitive and state-closed.*

Chapter 1

Preliminaries

In this chapter we present some fundamental algebraic preliminaries for our work. In the first section, definitions and examples about modules are established. They will play a major role on understanding presentations of abelian self-similar groups, since these groups will be seen as finitely presented modules over commutative rings, in contrast with the group presentation that is not, in general, finite. Next, we proceed to define the p-adic integers and profinite groups, as automorphism groups of trees inherit the topology of the tree, being isomorphic to an explicit inverse limit. Ending the chapter, we state some properties of abelian groups that will be used to investigate torsion elements in abelian self-similar groups.

1.1 Modules

The intuitive way of seeing modules is taking a vector space and weakening the condition on the field of coefficients; we only require the coefficients to be in a ring. We proceed with some standard definitions and useful examples.

Definition 1.1.1. Let R be a ring (maybe not commutative nor having identity). A *left R-module* or a *left module over R* is a set M together with:

i) a binary operation $+ : M \times M \to M$ under which it is an abelian group;

ii) An action of R on M (that is, a map $\cdot : R \times M \to M$), denoted by $r \cdot m$ (or shortly, rm), for all $r \in R$ and for all $m \in M$ which satisfies

a) $(r+s)m = rm + sm, \quad \text{for all } r, s \in R,\ m \in M;$

b) $(rs)m = r(sm), \quad \text{for all } r, s \in R,\ m \in M;$

c) $r(m+n) = rm + rn, \quad \text{for all } r \in R,\ m, n \in M;$

If the ring has a 1 we impose the additional axiom:

d) $1m = m, \quad \text{for all } m \in M.$

The definition of *right R-modules* is analogous.

Remark. If the ring R is commutative and M is a left R-module we can make M into a right R-module by defining $mr = rm$. In general, if R is not commutative, all the axioms for a right R-module are satisfied, with exception of the equivalent of axiom (ii)(b), that reads $m(rs) = (mr)s,\ m \in M,\ r, s \in R$.

More clearly: in a left R-module M, denote rm by mr, for $m \in M,\ r \in R$. Thus, we can also write $(sr)m = m(sr)$ and $s(rm) = (mr)s$. By the axiom *(ii)(b)* we have then

$$m(sr) = (mr)s.$$

Now, if R is commutative, it follows that $m(rs) = (mr)s$, satisfying the required condition. If R is a commutative ring, we shall omit the adjectives *left* and *right*, using only the term *R-modules*.

Definition 1.1.2. Let R be a ring and M be an R-module. An *R-submodule* of M is a subgroup N of M which is closed under the action of ring elements, i.e., $rn \in N$, for all $r \in R,\ n \in N$.

Example 1.1.3. a) Let R be a ring. Then $M = R$ is a left R-module over itself with the action being the multiplication in R. In this way, the left R-submodules are precisely the left ideals of R. We observe however that, if R is not commutative, M has both left and right R-module structures, but these may be different. For instance, consider the ring R of $n \times n$ matrices with entries from a field F. Now, taking M to be the set of $n \times n$ matrices with arbitrary elements of F in the first column and zeros elsewhere, we see that M is a submodule of R when R is considered as a left module over itself, but M is not a submodule of R when R is considered as a right R-module.

b) Let $R = \mathbb{Z}$ and A be any abelian group with the operation denoted by $+$. Then we make A into a $\mathbb{Z}$-module by defining

$$na = \begin{cases} a + a + a + \ \dots + a \ \ (n \text{ times}), & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -a - a - a - \ \dots - a \ \ (n \text{ times}), & \text{if } n < 0 \end{cases}$$

(0 is the identity of A). This definition of action of $\mathbb{Z}$ into A makes A an $\mathbb{Z}$-module; in fact this is the only possible action of $\mathbb{Z}$ on A, what allows us to conclude that *$\mathbb{Z}$-modules are the same as abelian groups.*

c) If A is an abelian group containing an element x of finite order n then $nx = 0$. Thus, a $\mathbb{Z}$-module may have nonzero elements x such that $nx = 0$ for some nonzero ring element n. In particular, if A has order m, then by Lagrange's Theorem $mx = 0$, for all $x \in A$. Note that in this case A is a module over $\mathbb{Z}/m\mathbb{Z}$.

d) More generally, if M is a R-module and for some two-sided ideal I of R we have $am = 0$, for all $a \in I$ and all $m \in M$, we say that M is *annihilated* by I. In this case we can make M into a $\frac{R}{I}$-module by defining the action of $\frac{R}{I}$ on M as $(r+I)m = rm$, for each $m \in M$ and $r + I$ in $\frac{R}{I}$.

e) Let R be a ring with 1 and $n \in \mathbb{Z}^+$. Defining $R^n = \{(a_1, a_2, ..., a_n) \ | \ a_i \in R\}$ with addition defined componentwise and multiplication as in the case of vector spaces, R^n is called the *free module of rank n over R.*

Definition 1.1.4. Let R be a ring and M and N be R-modules. A map $\varphi : M \to N$ is an *R-module homomorphism* if it respects the R-module structures of M and N, i.e.,

i) $\varphi(x + y) = \varphi(x) + \varphi(y)$, for all $x, y \in M$;

ii) $\varphi(rx) = r\varphi(x)$, for all $r \in R$ and $x \in M$.

Additionally, we define $ker(\varphi) = \{m \in M \mid \varphi(m) = 0\}$

Example 1.1.5. a) Let R be a ring, $n \in \mathbb{Z}^+$ and let $M = R^n$. For each $i \in \{1, ..., n\}$ the projection map

$$\pi_i : R^n \to R, \pi_i(x_1, ..., x_n) = x_i$$

is a surjective R-module homomorphism with kernel equal to the submodule of n-tuples which have a zero in position i.

b) If R is a ring and $M = R$ is a module over itself, then R-module homomorphisms (even from R to itself) need not be ring homomorphisms and ring homomorphisms need not be R-module homomorphisms. For example, when $R = \mathbb{Z}$, the $\mathbb{Z}$-module homomorphism $x \mapsto 2x$ is not a ring homomorphism (1 does not map to 1). In the other direction, when $R = F[x]$, F a field, the ring homomorphism

$$\varphi : f(x) \mapsto f(x^2)$$

is not an $F[x]$-module homomorphism as,if it were, we would have

$$x^2 = \varphi(x) = \varphi(x \cdot 1) = x\varphi(1) = x.$$

As one should expect, the definition of quotient modules, the Isomorphism Theorems and analogous facts for other structures also exist for modules. For further information the reader can consult [DF03].

Definition 1.1.6. Let M be an R-module and let $N_1, ..., N_n$ be submodules of M.

i) For any subset A of M let

$$RA = \{r_1a_1 + r_2a_2 + \ldots + r_ma_m \mid r_1, \ldots, r_n \in R,\ a_1, \ldots, a_m \in A,\ m \in \mathbb{Z}^+\}$$

(we define $RA = \{0\}$ if $A = \emptyset$); this is called the *submodule of M generated by A*. If $A = \{a_1, a_2, \ldots, a_n\}$ is a finite set we shall write $Ra_1 + Ra_2 + \ldots + Ra_n$ for RA. If N is a submodule of M (possibly $N = M$) with $N = RA$, A a subset of M, we say that N is *generated* by A and A is a *generating set* for N.

ii) A submodule N of M (possibly $N = M$) is *cyclic* if there exists an element $a \in M$ such that $N = Ra$, that is, N is generated by one element:

$$N = Ra = \{ra \mid r \in R\}.$$

It is assumed from now on that our modules are over commutative rings with identity 1. We remark that, for a finitely generated module, its submodules are not necessarily finitely generated. Consider for example R to be $F[x_1, x_2, \ldots]$ the polynomial ring in infinitely many variables over some field F and the R-module M to be R itself (a cyclic module, as $M = R = R1$). The submodule generated by $\{x_1, x_2, \ldots\}$ cannot be generated by any finite set.

Now we present a broader definition of free modules than the one on the previous Example 1.1.3. (e).

Definition 1.1.7. An R-module F is said to be *free* on the subset A of F if for every nonzero element $x \in F$, there exist unique nonzero elements $r_1, r_2, \ldots, r_n$ of R and unique $a_1, a_2, \ldots, a_n$ in A such that $x = r_1a_1 + r_2a_2 + \ldots + r_na_n$, for some positive integer n. We say A is a *basis* or a *set of free generators* for F. If R is a commutative ring the cardinality of A is called the *rank* of F.

As in other categories, the free R-module on a subset A satisfies a *universal property*, as follows: if M is any R-module and $\varphi : A \to M$ is any map of sets, then there is a unique R-module homomorphism $\Phi : F \to M$ such that $\Phi(a) = \phi(a)$, for all $a \in A$; i.e., the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, "\varphi"'] & F \arrow[d, "\Phi"] \\ & M \end{tikzcd}$$

where i is the inclusion map.

Example 1.1.8. a) Any ring R is a free module over itself, having any unit element as possible basis.

b) When $R = \mathbb{Z}$, the free module on a set A is called the *free abelian group on* A. If $|A| = n$, then it is called the free abelian group *of rank* n and is isomorphic as a group to $\mathbb{Z} \oplus \mathbb{Z} \oplus ... \oplus \mathbb{Z}$ (n times).

1.2 The p-adic Integers

The p-adic integers play a major role on Number Theory, mainly motivated by the so-called Local-Global Principle, attempting to find a solution to an equation "gluing" the solutions modulo powers of primes. We give a naive definition and then proceed to illustrate them in more concrete ways.

Definition 1.2.1. The *ring of p-adic integers* $\mathbb{Z}_p$ is defined as the set of formal power series $\sum_{i \geq 0} a_i p^i$ with integral coefficients a_i satisfying $0 \leq a_i \leq p - 1$.

With this definition, a p-adic integer $a = \sum_{i \geq 0} a_i p^i$ can be identified with the sequence $(a_i)_{i \geq 0}$ of its coefficients; therefore $\mathbb{Z}_p$ can be identified with the cartesian product

$$\mathbb{Z}_p = \prod_{i \geq 0} \{0, 1, \dots, p-1\} = \{0, 1, \dots, p-1\}^{\omega},$$

where ω stands for the cardinality of the integers.

To operate with these numbers, we consider the above formal sums paying attention on how to carry coefficients during calculations to keep the coefficients within the range $0 \leq a_i \leq p-1$. For example, in $\mathbb{Z}_2$ let $a = 1 = 1 \cdot 2^0 + 0 \cdot 2^1 + 0 \cdot 2^2 + \ldots$ and $b = 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + \ldots$. Then the addition $a+b$ is

$$\begin{aligned} 1 \ + \ (1+2+2^2+2^3+\ldots) &= 2+2+2^2+2^3+\ldots \\ &= 2^2+2^2+2^3+\ldots \\ &= 2^3+2^3+\ldots \\ &= \ \ldots \ = 0. \end{aligned}$$

Now, more generally, let

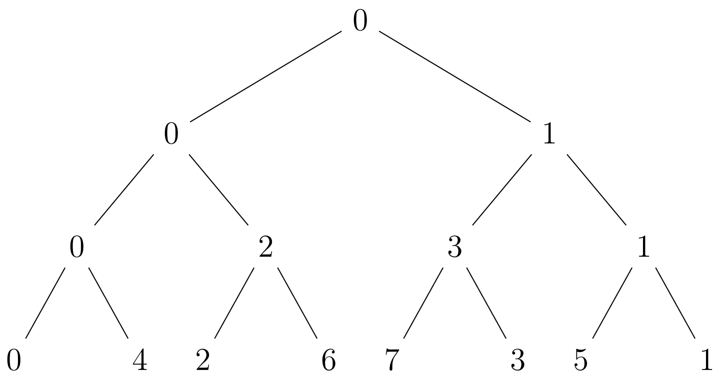
$$\begin{aligned} a = 1 = 1 \cdot p^0 + 0 \cdot p + 0 \cdot p^2 + \ldots \quad &\text{and} \\ b = (p-1) + (p-1)p + (p-1)p^2 + \ldots \quad &; \end{aligned}$$

The sum $a+b$ has first component 0, since $1 + (p-1) = p$. As we have to carry the coefficient 1 to the next component, we have that it is also 0. Continuing the process we find that all components vanish, and the result is $1 + b = 0$; therefore b is the additive inverse of 1, namely, $b = -1$. Therefore, for some p-adic $a = \sum_{i \geq 0} a_i p^i$, we define

$$\mu(a) = \sum_{i \geq 0} (p-1-a_i)p^i$$

so that $a + \mu(a) + 1 = 0$, or even $\mu(a) + 1 = -a$, the additive inverse of a. The multiplication is defined analogously using distributivity, paying the same attention on carrying the coefficient during calculations.

We can visualize the p-adic integers as infinite paths on a tree, where each vertex represents the partial sum of the formal power series. In the following picture, with $p = 2$, we illustrate the first levels of such tree:



An analogous way of defining the p-adic integers is using inverse limits:

Definition 1.2.2. The *ring of p-adic integers* $\mathbb{Z}_p$ is defined as the inverse limit

$$\mathbb{Z}_p = \varprojlim \frac{\mathbb{Z}}{p\mathbb{Z}} = \{(a_n) \in \prod_{n \in \mathbb{N}} \frac{\mathbb{Z}}{p^n\mathbb{Z}} \mid a_m \equiv a_n \ (mod\ p^m), if\ \ n \geq m\},$$

with respective transition maps $\varphi_n : \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ defined by

$$\sum_{i \leq n} a_i p^i \ mod\ p^{n+1} \mapsto \sum_{i \leq n} a_i p^i \ mod\ p^n.$$

By this definition, the p-adic integers are the coherent sequences in $\prod \mathbb{Z}/p^n\mathbb{Z}$ of partial sums of the formal series $\sum_{i \geq 0} a_i p^i$, with $0 \leq a_i \leq p - 1$.

For example, to see the p-adic number $1 + 2 + 2^2 + 2^3 + \dots$ in this context, observe that

$$1 + 2 + 2^2 + 2^3 + \dots \ \text{ means } \ (1\ mod\ 2, 1 + 2\ mod\ 2^2, 1 + 2 + 2^2\ mod\ 2^3, \dots).$$

In the next section we will see that the additive group of the p-adic integers have important topological properties related to the inverse limit structure.

Now we present a fundamental structure theorem for the ring of p-adic integers:

Proposition 1.2.3. *Let p be a prime. Then*

i) The group of units of $\mathbb{Z}_p$ is

$$\mathbb{Z}_p^\times = \{a_0 + a_1 p + a_2 p^2 + ... \in \mathbb{Z}_p \mid a_0 \neq 0,\ 0 \leq a_i < p\},$$

that is, it is the subset of p-adic integers that are not divisible by p or with nonzero "constant terms".

ii) $\mathbb{Z}_p$ is a Unique Factorization Domain (in fact a Principal Ideal Domain) with a unique irreducible p, except for associated elements. Every nonzero ideal of $\mathbb{Z}_p$ is a power of $p\mathbb{Z}_p$, therefore $p\mathbb{Z}_p$ is the only maximal ideal of $\mathbb{Z}_p$ and $\mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$.

Proof. i) It follows directly from the fact that $a \bmod p^n$ is a unit in $\mathbb{Z}/p^n\mathbb{Z}$ if and only if a is not a multiple of p.
ii) Observe that every nontrivial element $f \in \mathbb{Z}_p$ admits a factorization of the form

$$f = \ p^n \ \times \ (a_n + a_{n+1}p^{n+1} + a_{n+2}p^{n+2} + ...),$$

$(0 \leq a_i < p,\ a_n \neq 0)$, where the term in parenthesis is a unit in $\mathbb{Z}_p$. This shows that p is the only irreducible, except for associates. As the coefficients a_i are in $\mathbb{Z}/p\mathbb{Z}$, which is in particular a domain, $\mathbb{Z}_p$ is also a domain. Now, given a nontrivial ideal $I \subseteq \mathbb{Z}_p$, let $f \in I$ with minimum n in the above factorization. We have that $p^n\mathbb{Z}_p = (f) \subseteq I$. To show that in fact the equality holds, choose a nontrivial $g \in I$ and write $g = p^m \cdot u$, with $u \in \mathbb{Z}_p^\times$. As $m \geq n$ by the choice of n, we have that g is a multiple of p^n and then $g \in (p^n)$, as desired.

∎

The field of fractions of $\mathbb{Z}_p$ is denoted by $\mathbb{Q}_p$ and is called the *field of p-adic numbers*. By the above proposition we see that every nonzero element $f \in \mathbb{Q}_p$ can be written uniquely as

$$f = u \cdot p^n; \quad u \in \mathbb{Z}_p^\times,\ n \in \mathbb{Z}.$$

Remark. In the case where p is not a prime, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ will have zero divisors. This case is discussed in detail in [Kat07], page 47.

1.3 Topological and Profinite Groups

Topological groups

Definition 1.3.1. A *topological group* is a set G that is a group and a topological space and for which the map from $G \times G$ (with the product topology) to G, given by

$$\begin{aligned} G \times G &\to G \\ (x, y) &\mapsto xy^{-1} \end{aligned}$$

is continuous.

We collect in the following lemma some basic properties of these groups that give some perspective on the topological aspects involved in our work:

Lemma 1.3.2. *Let G be a topological group.*

i) The map $(x, y) \mapsto xy$ from $G \times G$ to G is continuous and the map $G \to G,\ x \mapsto x^{-1}$ is a homeomorphism. For each $g \in G$ the maps $x \mapsto xg$ and $x \mapsto gx$ are homeomorphisms.

ii) If H is an open (resp. closed) subgroup of G then every coset gH or Hg of H in G is open (resp. closed).

iii) Every open subgroup of G is closed, and every closed subgroup of finite index is open. If G is compact then every open subgroup of G has finite index.

iv) If H is a subgroup containing a non-empty open subset U of G then H is open in G.

v) G is Hausdorff if and only if $\{1\}$ is a closed subset of G; if K is a normal subgroup of G then G/K is Hausdorff if and only if K is closed in G. If G is totally disconnected, then G is Hausdorff.

Proof. i) It follows immediately from the fact that a map from a space X to $G \times G$ is continuous if and only if its product with each of the projection maps is continuous; thus if $\theta : G \to G$ and $\varphi : G \to G$ are continuous, so is the map $x \mapsto (\theta(x), \varphi(x))$ from G to $G \times G$.

ii) It follows directly from the continuity of the maps on item i).

iii) We have $G \backslash H = \bigcup\{Hg \mid g \notin H\}$. Hence, if H is open then so is $G\backslash H$ by ii), and H is closed. If H has finite index then $G\backslash H$ is a union of finitely many cosets, and then if H is also closed then so is $G\backslash H$, and H is open. If H is open then the sets Hg are open and disjoint and their union is G; thus it follows from the definition of compactness that if G is compact then H must have finite index in G.

iv) This follows since by i) each set $Uh = \{uh \mid u \in U\}$ is open and since $H = \bigcup\{Uh \mid h \in H\}$.

v) Noticing that one-element subsets in Hausdorff spaces are closed, we must show that if the set $\{1\}$ is closed then G is Hausdorff. Let a, b be distinct elements of G. From i), the set $\{ab^{-1}\}$ is closed, and so there is an open set U with $1 \in U$ and $ab^{-1} \notin U$. The map $(x, y) \mapsto xy^{-1}$ is continuous and so the inverse image of U is open. It follows that there are open sets V, W containing 1 with $VW^{-1} \subseteq U$. Thus $a^{-1}b \notin VW^{-1}$, and so $aV \cap bW = \emptyset$. Since aV, bW are open the first assertion follows. The second assertion follow immediately from the definition of quotient topology. For the third, observe that if G is a totally disconnected space, then $\{g\}$ is closed in G, for each $g \in G$.

$\blacksquare$

Profinite Groups

The p-adic integers are the incarnation of a more general and useful construction, which we now present.

Definition 1.3.3. Let $\mathcal{C}$ be a nonempty class of finite groups. Define a *pro-$\mathcal{C}$ group* G as the inverse limit

$$G = \varprojlim_{i \in I} G_i$$

of a surjective inverse system $\{G_i, \varphi_{ij}, I\}$ of groups $G_i \in \mathcal{C}$, where each group G_i is assumed to have the discrete topology.

For example, if $\mathcal{C}$ is the class of *finite groups*, G is called a *profinite group*; if $\mathcal{C}$ is the class of finite p-groups, G is a *pro-p* group. Notice that all pro-$\mathcal{C}$ groups are profinite groups. We think of pro-$\mathcal{C}$ groups as topological groups, whose topology is inherited from the product topology on $\prod_{i \in I} G_i$.

Example 1.3.4. a) Any finite group is profinite, given the discrete topology;

b) The additive group of $\mathbb{Z}_p$ is a pro-p group, as seen on the previous section;

c) The upper unitriangular group over $\mathbb{Z}_p$ of degree n

$$UT_n(\mathbb{Z}_p) = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} ;\ a_{ij} \in \mathbb{Z}_p \right\}$$

is a pro-p group.

Now we state a standard result when dealing with profinite groups, which can be found on [RZ00] or [Wil98].

Theorem 1.3.5. *Let $\mathcal{C}$ be a class of finite groups that is closed under taking subgroups, quotients and finite direct products. Then the following conditions on a topological group G are equivalent:*

i) G is a pro-$\mathcal{C}$ group;

ii) G is compact, Hausdorff and totally disconnected, and for each open normal subgroup U of G, $G/U \in \mathcal{C}$;

iii) G is compact and the identity 1 of G admits a fundamental system $\mathcal{U}$ of open neighborhoods U such that $\bigcap_{U \in \mathcal{U}} U = 1$ and each U is an open normal subgroup of G with $G/U \in \mathcal{C}$;

iv) The identity 1 of G admits a fundamental system $\mathcal{U}$ of open neighborhoods U such that each U is a normal subgroup of G with $G/U \in \mathcal{C}$, and

$$G = \varprojlim_{u \in \mathcal{U}} G/U.$$

1.4 Abelian Groups

Now we introduce some definitions and results that will be fundamental in the study of torsion on abelian self-similar groups.

Definition 1.4.1. Let G be an abelian group. An element g of G is said to be *divisible* by a positive integer m if $g = mg_1$, for some g_1 in G. An abelian group G is called *divisible* if every element is divisible by every positive integer. If p^h is the largest power of the prime p dividing g, then h is called the *p-height* of g in G. If g is divisible by every power of p, then is said that g has *infinite* p-height in G.

As immediate examples we have the additive group of $\mathbb{Q}$ and the multiplicative group of $\mathbb{C}$. Now, for a positive integer n, define the subgroup $nG = \{ng \mid g \in G\}$. Then we have the following definition:

Definition 1.4.2. Let G be an abelian group. A subgroup H is called *pure* if

$$nG \cap H = nH$$

for all integers $n \geq 1$, i.e., H is pure if every element of H that is divisible by n in G is divisible by n in H.

A immediate consequence is that if G is an abelian p-group, then a subgroup H is pure if and only if $p^m G \cap H = p^m H$, for all positive integers m.

As an example, consider H a direct summand of $G = H \oplus K$. Then

$$nG \cap H = (nH + nK) \cap H = nH$$

by the modular law. It follows that every direct summand of G is pure. Pure subgroups can be seen then as generalizations of direct summands.

In the case that H is a subgroup of G such that G/H is torsion-free, then H is pure; in particular the torsion subgroup is pure. This gives us examples of pure subgroups that are not direct summands.

Example 1.4.3. ([Rob12], 4.3.10) Let C be the cartesian (unrestricted) sum of cyclic groups of order $p, p^2, p^3, \dots$. Then, $T = tor(C)$ is not a direct summand of C.

In fact, consider

$$C = \underset{i \geq 1}{Cr} \langle x_i \rangle, \text{ where } |x_i| = p^i,$$

and denote by y the element of C whose nonzero components are px_2, p^2x_4, p^4x_8, etc. Then $y \notin T$ and $y \in p^n C + T$, for all n. Therefore $y + T$ is an element of infinite p-height in C/T. Since C has no such elements, T cannot be a direct summand of C.

In the special case of bounded subgroups, that is, the elements of the subgroup have a bound for its order, the concepts of pure subgroups and direct summands coincide. For this, we have the following result:

Proposition 1.4.4. *A pure bounded subgroup H of an abelian group G is a direct summand.*

Proof. Suppose that $nH = 0$. Let $K = H + nG$ and consider the quotient $\overline{G} = G/K$. By [Rob12], Theorem 4.3.5, it follows that $\overline{G}$ is a direct sum of cyclic groups; denote them by $\langle x_i + K \rangle$, $i \in I$. If $x_i + K$ has order n_i, then $n_i x_i = h_i + n g_i$ where $h_i \in H$ and $g_i \in G$. Now, n_i divides n and then $h_i = n_i(x_i - (n/n_i)g_i) \in n_i G \cap H = n_i H$, by the purity of H. Therefore we can write $h_i = n_i h'_i$ with h'_i in H. Setting $y_i = x_i - h'_i$, we have $n_i y_i = n_i x_i - h_i = n g_i$. Also, $y_i + K = x_i + K$.

Now, define L to be the subgroup generated by nG and the elements $y_i, i \in I$. We will prove that $G = H \oplus L$. If $x = \sum_i m_i y_i + ng \in H$, then $\sum_i m_i(x_i + K) = \sum_i m_i(y_i + K) = 0_{\overline{G}}$, which implies that n_i divides m_i since $\overline{G}$ is a direct sum. But we saw that $n_i y_i = n g_i$; thus $x = \sum_i m_i y_i + ng \in nG \cap H = nH = 0$. Hence $H \cap L = 0$.

Finally, if $g \in G$ and $g + K = \sum_i l_i(y_i + K)$, one has $g - \sum_i l_i y_i \in K$ and then $g - \sum_i l_i y_i = h + n g_1$, where $h \in H$, $g_1 \in G$. Therefore,

$$g = h + n g_1 + \sum_i l_i y_i,$$

which belongs to $H + L$. Hence $G = H \oplus L$.

■

Chapter 2

Automorphism Groups of Trees

In this chapter we define the group $\mathcal{A}_m$ of automorphisms of a tree, its main properties, important subgroups and give some examples that will prepare us for the main results in the next chapter. Also, we define the virtual endomorphisms; they will provide us a method to investigate if a group can be represented as a subgroup of $\mathcal{A}_m$.

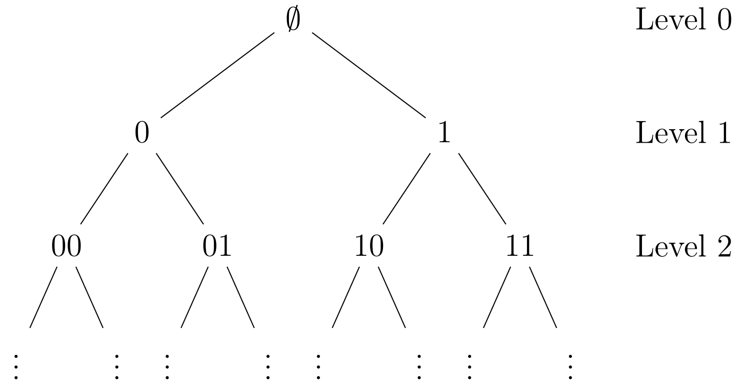
2.1 Trees and their automorphisms

Let m be a positive integer and Y be the set $\{1, ..., m\}$. Define $\mathcal{M} = \mathcal{M}(Y)$ by the semigroup consisting of all finite words on the *alphabet* Y. The operation on $\mathcal{M}$ is the concatenation of words and the identity is the empty word $\emptyset$.

Definition 2.1.1. The *one-rooted m-regular tree* is the graph $\mathcal{T}_m = (V(\mathcal{T}_m), E(\mathcal{T}_m))$, where $V(\mathcal{T}_m) = \mathcal{M}$ and for an ordered pair (u, v), we have that $(u, v) \in E(\mathcal{T}_m)$ if and only if $v = uy$, for some $y \in Y$, $u, v \in \mathcal{M}$.

In this definition we have a tree where all vertices have the same number of incident vertices (m-regular), with the exception of one vertex (one-rooted); this vertex is called the *root* of the tree.

Thus, such tree has its vertices labeled by the words in $\mathcal{M}$ with increasing length $|u|, u \in \mathcal{M}$. As an example, with $Y = \{0, 1\}$:



The set of all words of length n is called the *n-th level* of the tree $\mathcal{T}_m$. In the above example we have on level 0 only the root $\emptyset$, on level 1 the words $\{0, 1\}$, on level 2 the words $\{00, 01, 10, 11\}$ and so on; the n-th level is the set $\{u \mid |u| = n, u \in \mathcal{M}\}$.

Definition 2.1.2. An *automorphism* of the tree $\mathcal{T}_m$ is a graph bijection that preserves vertex incidence (alternatively, preserves the length $|u|$ of a vertex labeled by $u \in \mathcal{M}$). The set of all such automorphisms is a group with respect to function composition, and will be denoted by $\mathcal{A}_m$.

Example 2.1.3. Let γ be a permutation of the alphabet Y. We can extend γ to an automorphism σ of the entire tree by setting:

$$(\emptyset)\sigma = \emptyset$$
$$(yu)\sigma = y^{\gamma}u, \text{ for all } y \in Y, u \in \mathcal{M}.$$

On the other side, every automorphism α of $\mathcal{T}_m$ induces a permutation $\sigma(\alpha)$ on Y just by considering $\sigma(\alpha)$ to be the restriction $\alpha\big|_{Y} : Y \to Y$.

Now, considering $\sigma(\alpha)$ to be the restriction on the above example (that is, we can see the permutation as an automorphism of the tree), we have that the composition $\alpha\sigma(\alpha)^{-1}$ has trivial action on the first level of the tree, i.e.,

$$(y)\alpha\sigma(\alpha)^{-1} = y, \text{ for all } y \in Y.$$

In this way, we can write the composition $\alpha\sigma(\alpha)^{-1}$ as

$$\alpha\sigma(\alpha)^{-1} = (\alpha_1, ..., \alpha_m), \quad (*)$$

where each $\alpha_y,\ y = 1, ..., m$, is an automorphism of the tree rooted on y, which is

$$y\mathcal{T}_m = (yV(\mathcal{T}_m), yE(\mathcal{T}_m)),$$

where $yV(\mathcal{T}_m) = \{yu \mid u \in \mathcal{M}\}$ and $yE(\mathcal{T}_m) = \{(yu, yv) \mid (u, v) \in E(\mathcal{T}_m)\}$.

We can establish an isomorphism between $\mathcal{T}_m$ and $y\mathcal{T}_m$ by setting $yu \mapsto u$ (simply deleting the prefix y); so we can consider α_y itself an automorphism of $\mathcal{T}_m$. From this fact and $(*)$ we conclude that α can be expressed as

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)\sigma(\alpha),$$

where $\alpha_i \in \mathcal{A}_m,\ i = 1, ..., m$. As the entries α_i run over $\mathcal{A}_m$, we can identify $\mathcal{A}_m$ as the semidirect product

$$\mathcal{A}_m = (\mathcal{A}_m \times ... \times \mathcal{A}_m) \rtimes S_m,$$

where the action of S_m on $(\mathcal{A}_m \times ... \times \mathcal{A}_m)$ is given by the permutation of the indexes. Then, given $\sigma \in S_m$ (again, σ is seen as an automorphism of the tree, $\sigma = (e, e, \ldots, e)\sigma$) and $(\alpha_1, \alpha_2, ..., \alpha_m) \in (\mathcal{A}_m \times ... \times \mathcal{A}_m)$ it follows that

$$\sigma(\alpha_1, \alpha_2, ..., \alpha_m) = (\alpha_{1^\sigma}, \alpha_{2^\sigma}, ..., \alpha_{m^\sigma})\sigma$$

thus, for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)\sigma(\alpha)$ and $\beta = (\beta_1, \beta_2, ..., \beta_m)\sigma(\beta)$ in $\mathcal{A}_m$, the product and the inverses are given by

$$\begin{gathered}\alpha\beta = (\alpha_1\beta_{1^{\sigma(\alpha)}}, \alpha_2\beta_{2^{\sigma(\alpha)}}, ..., \alpha_m\beta_{m^{\sigma(\alpha)}})\sigma(\alpha)\sigma(\beta)\\ \alpha^{-1} = (\alpha^{-1}_{1^{\sigma(\alpha)^{-1}}}, \alpha^{-1}_{2^{\sigma(\alpha)^{-1}}}, ..., \alpha^{-1}_{m^{\sigma(\alpha)^{-1}}})\sigma(\alpha)^{-1}.\end{gathered}$$

The previous semidirect product has great importance in the theory, so we make its definition precise. Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of groups, where Λ is an index set. We denote the *Cartesian Product* of this family by

$$\underset{\lambda\in\Lambda}{Cr} G_\lambda = \{(g_\lambda)_{\lambda\in\Lambda} \, | g_\lambda \in G_\lambda\},$$

endowed with coordinatewise multiplication $(g_\lambda)_{\lambda\in\Lambda}(h_\lambda)_{\lambda\in\Lambda} = (g_\lambda h_\lambda)_{\lambda\in\Lambda}$. This is a group with identity denoted by $(e_\lambda)_{\lambda\in\Lambda}$, where e_λ is the identity element in G_λ. Now, the *Direct Product* $\underset{\lambda\in\Lambda}{Dr} G_\lambda$ is the subgroup of $\underset{\lambda\in\Lambda}{Cr} G_\lambda$ given by all elements $(x_\lambda)_{\lambda\in\Lambda}$ where $x_\lambda \neq e_\lambda$ for a finite number of indexes λ. Notice that if Λ is finite we have $\underset{\lambda\in\Lambda}{Cr} G_\lambda = \underset{\lambda\in\Lambda}{Dr} G_\lambda$.

Consider a group K, Λ an index set and H a group acting on Λ. Denote by $\varphi : H \to S_\Lambda$ the action of H on Λ, where S_Λ is the set of all bijections of Λ. The *Unrestricted Wreath Product* of K by H with respect to φ is defined by

$$K wr_\varphi H = (\underset{\lambda\in\Lambda}{Cr} K) \rtimes_\varphi H,$$

where $(k_\lambda)^h_{\lambda\in\Lambda} = (k_{\lambda^{h^\varphi}})_{\lambda\in\Lambda}$, for all $h \in H$ and $\lambda \in \Lambda$.
Analogously, the *Restricted Wreath Product* of K by H with respect to φ is defined by

$$K \wr_\varphi H = (\underset{\lambda\in\Lambda}{Dr} K) \rtimes_\varphi H,$$

where $(k_\lambda)^h_{\lambda\in\Lambda} = (k_{\lambda^{h^\varphi}})_{\lambda\in\Lambda}$, for all $h \in H$ and $\lambda \in \Lambda$.

The following example is classic in the theory and is known as the *binary adding machine*:

Example 2.1.4. *(Binary Adding Machine)* Let $\alpha = (e, \alpha)\sigma$ be an automorphism in $\mathcal{A}_2$, where e is the identity automorphism in $\mathcal{A}_2$ and $\sigma = (01)$, the transposition in S_2. Let $010 \in \mathcal{M}(\{0,1\})$. Then

$$(101)^\alpha = (101)^{(e,\alpha)\sigma} = 1^\sigma(01)^{\alpha_1} = 0(01)^\alpha = 0(0)^\sigma(1)^{\alpha_0} = 01(1)^e = 011$$

For our notation, we write the binary numbers/words backwards.

Definition 2.1.5. Given $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)\sigma(\alpha) \in \mathcal{A}_m$, the *set of states* of α is defined recursively by

$$Q(\alpha) = \{\alpha\} \cup Q(\alpha_1) \cup ... \cup Q(\alpha_m).$$

In the previous example we can see that the set of states of α is $\{\alpha, e\}$.
Additionally, a group $G \leq \mathcal{A}_m$ will be called *finite-state* if $Q(\alpha)$ is finite, for all $\alpha \in G$.

2.2 Some Important Subgroups

Now we proceed to define some standard subgroups that will help us to study the automorphism groups of trees.

Definition 2.2.1. Let G be a subgroup of $\mathcal{A}_m$. Then we define

$$Stab_G(n) = \{\alpha \in G \mid u^\alpha = u, \forall u \in \mathcal{M}, |u| = n\};$$
$$Fix_G(u) = \{\alpha \in G \mid u^\alpha = u, \text{ for a fixed } u \in \mathcal{M}\};$$
$$P(G) = \{\sigma(\alpha) \in S_m \mid \alpha \in G\}.$$

They are respectively the *stabilizer* of the level n, the *fixator* of the word u and the subgroup of the permutations induced by the elements of G. We say that G is *transitive* if $P(G)$ is a transitive subgroup of S_m.

The topological closure

Given $\alpha \in Stab_{\mathcal{A}_m}(1)$, it follows that $\sigma(\alpha) = e$ (α acts trivially on the first level). Then,

$$\alpha = (\alpha_1, ..., \alpha_m),\ \alpha_i \in \mathcal{A}_m.$$

Thus, given $\beta = (\beta_1, ..., \beta_m)\sigma(\beta) \in \mathcal{A}_m$, we have

$$\alpha^\beta = (\alpha_1^{\beta_1}, ..., \alpha_m^{\beta_m})^{\sigma(\beta)} \in Stab_{\mathcal{A}_m}(1),$$

where we write $(\gamma_1, ..., \gamma_m)^\sigma = (\gamma_{1^\sigma}, ..., \gamma_{m^\sigma})$ to ease the notation. Then, we have that $Stab_G(1) = G \cap Stab_{\mathcal{A}_m}(1)$ is a normal subgroup of G, for all $G \leq \mathcal{A}_m$. We can see that $Stab_G(n)$ is a normal subgroup of G in a similar way; also, we notice that $Stab_{\mathcal{A}_m}(n)$ has finite index in $\mathcal{A}_m$.

Using these facts we can express the elements of $\mathcal{A}_m$ as infinite products of the form

$$\beta = \alpha_0\alpha_1\alpha_2\ldots, \text{ where each } \alpha_i \text{ belongs to } Stab_{\mathcal{A}_m}(i),$$

which is equivalent to say that

$$\mathcal{A}_m \simeq \varprojlim \frac{\mathcal{A}_m}{Stab_{\mathcal{A}_m}(n)}.$$

With the above observation, we define a closure operation for $G \leq \mathcal{A}_m$ by taking all well defined infinite products of elements of G. Such group is called the *topological closure* of G, and will be denoted by $\overline{G}$.

The diagonal closure

Let $\alpha \in \mathcal{A}_m$ be an automorphism. We define recursively the *diagonal map* by

$$\alpha^{(0)} = \alpha,\ \alpha^{(1)} = (\alpha, \alpha, ..., \alpha),\ \alpha^{(n+1)} = (\alpha^{(n)})^{(1)}, \text{ for } i \geq 0.$$

The *diagonal closure*, denoted by $\widetilde{G}$, will be the group $\widetilde{G} = \langle G^{(i)} \mid i \geq 0 \rangle$, where $G^{(i)} = \{g^{(i)} \mid g \in G\}$, for a fixed positive integer i. Intuitively, we are taking an automorphism of the tree $\mathcal{T}_m$ and making it act on every subtree, in order to "exhaust all possibilities" for its actions.

Now, writing the diagonal map as $x : \mathcal{A}_m \to \mathcal{A}_m$, $\alpha \mapsto (\alpha, \alpha, ..., \alpha)$ and $\alpha^{(i)}$ as α^{x^i} we can write

$$\alpha^{a_0}(\alpha^{a_1})^{(1)}(\alpha^{a_2})^{(2)}...(\alpha^{a_n})^{(n)} = \alpha^{a_0+a_1x^1+a_2x^2+...+a_nx^n},$$

where $a_i \in \mathbb{Z}_m$. We remark that, despite the above identification, it is possible that these "powers" of a single element alpha do not commute; more precisely, the factors of the polynomial in the exponent, do not commute if the states of alpha also do not commute. This notation will ease our calculations in the next chapter.

The state closure

A group $G \leq \mathcal{A}_m$ is called *state-closed* if G contains all of its "possible states", i.e., $Q(\alpha)$ is a subset of G, for all $\alpha \in G$. The *state closure* of G, denoted by $\widehat{G}$, is the group generated by all states of all elements of G. We will call *recurrent* a transitive state-closed group such that the projection $\pi_1 : Fix_G(1) \to G$, defined by

$$\alpha^{\pi_1} = ((\alpha_1, ..., \alpha_m)\sigma(\alpha))^{\pi_1} = \alpha_1$$

is surjective, where $\alpha \in G$ and $1^{\sigma(\alpha)} = 1$.

The set of states of an automorphisms can be "tricky", in the sense that a simple automorphism can have infinite states. To illustrate this we have the following

Example 2.2.2. Let $\alpha = (\alpha, \alpha^2)\sigma$ be an automorphism in $\mathcal{A}_2$. Noticing that

$$\alpha^2 = (\alpha^3, \alpha^3),\ \alpha^3 = (\alpha^4, \alpha^5)\sigma, \text{ and so on,}$$

we have

$$\alpha^{2n} = (\alpha^{3n}, \alpha^{3n}) \text{ and } \alpha^{2n+1} = (\alpha^{3n+1}, \alpha^{3n+2})\sigma,$$

and thus α has infinite order and its set of states, $Q(\alpha) = \{\alpha^n \mid i \geq 1\}$ is also infinite.

Another example that the state closure can get more complicated than the original group is the following:

Example 2.2.3. Let A be the group generated by $\alpha = (e, (e, \alpha))\sigma$ in $\mathcal{A}_2$. Its state closure $\widehat{A}$ is the group $\langle \alpha, \beta \rangle$, where $\beta = (e, \alpha)$. This group is known as the *Basilica Group* and it has many interesting properties (see [GŻ02]).

Now we state important properties about the closures of abelian transitive state-closed groups, that are our main interest in this work. The *diagonal-topological closure* of A, denoted by A^*, is considered as the diagonal closure applied first and then the topological closure is taken, i.e., $A^* = \bar{\tilde{A}}$. Notice that these closure operations in general do not commute. For example, consider $\sigma = (12) = (e, e)(12) \in \mathcal{A}_2$. Applying the topological closure first there are no new elements different from σ; then the diagonal closure applied next gives us elements of the form $\sigma^{(n)}$ and their finite products $\sigma^{(i_1)}\sigma^{(i_2)} \dots \sigma^{(i_n)}$. However when we apply the diagonal and then the topological closure, we obtain also *infinite* products of the terms $\sigma^{(n)}$.

Proposition 2.2.4. *Let A be an abelian transitive state-closed group of degree m. Then $Stab_A(i) \leq A^{(i)}$ for all $i \geq 0$. The diagonal closure $\widetilde{A}$ is an abelian transitive state-closed group and is a minimal recurrent group containing A. The diagonal-topological closure A^* of A is also an abelian transitive state-closed group.*

Proof. Let $\alpha = (\alpha_1, ..., \alpha_m)\sigma$ and $\beta = (\beta_1, ..., \beta_m) \in A$. The conjugate of β by α is

$$\beta^\alpha = (\beta_1^{\alpha_1}, ..., \beta_m^{\alpha_m})^\sigma.$$

As $\alpha_i, \beta_i \in A$, and A is abelian, it follows that $\beta = (\beta_1, ..., \beta_m)^\sigma$. Furthermore, since A is transitive, $\beta = (\beta_1, ...\beta_1) = (\beta_1)^{(1)}$. Thus, $Stab_A(i) \leq A^{(i)}$, for all i. A similar verification shows that $\tilde{A} = \langle A^{(i)} \mid i \geq 0 \rangle$ is abelian.

Let G be a recurrent group such that $A \leq G \leq \tilde{A}$. Given $\alpha \in G$, as G is recurrent, there exists $\beta \in Stab_G(1)$ such that $\beta = (\beta_1, ..., \beta_m)$ with $\beta_1 = \alpha$. Since G is transitive and abelian, we have $\beta_1 = ... = \beta_m = \alpha$; that is, $\beta = \alpha^{(1)}$. Hence, $A^{(i)} \leq G$ and $G = \tilde{A}$ follows.

Now, writing the elements of A^* as products of elements of the form

$$\alpha^* = \alpha^{a_0}(\alpha^{a_1})^{(1)}(\alpha^{a_2})^{(2)} \ldots = \alpha^{a_0 + a_1 x^1 + a_2 x^2 + \ldots}$$

the last assertion is proved.

■

Definition 2.2.5. Let G be a permutation group on an alphabet X. G is said to be *regular* if it is transitive and $Stab_G(x)$ is trivial, for all $x \in X$.

Proposition 2.2.6. *i) Let A be a recurrent abelian group of degree m and let $C_{\mathcal{A}_m}(A)$ be the centralizer of A in $\mathcal{A}_m$. Then $C_{\mathcal{A}_m}(A) = \bar{A}$.*
ii) Let m be a prime number and A be an infinite abelian transitive state-closed group. Then $C_{\mathcal{A}_m}(A) = \bar{A}$.

Proof. i) Let $P = P(G)$ the permutation group on Y induced by G. Since P is an abelian transitive permutation group of degree m, we have that it is also regular; furthermore, the stabilizer in G of any $y \in Y$ is the same as the stabilizer of the first level of the tree, say $H = Stab_G(1)$. By hypothesis the representation of G is recurrent, so the projection $\pi_v : Stab_G(k) \to G$ on any of its coordinates is surjective and therefore produces the group G.

For every $\sigma \in P$, choose $\alpha_0(\sigma) = (\alpha_0(\sigma)_1, ..., \alpha_0(\sigma)_m)\sigma \in G$ which induces σ on Y. Let $h = (h_1, h_2, ..., h_m) \in H$. Then, since h_i and $a_0(\sigma)_i$ are in G, which is abelian,

$$h^{\alpha_0(\sigma)} = \left((h_1)^{\alpha_0(\sigma)_1}, (h_2)^{\alpha_0(\sigma)_2}, ..., (h_m)^{\alpha_0(\sigma)_m}\right)^{\sigma} = (h_1, h_2, ..., h_m)^{\sigma}.$$

By varying $\sigma \in P$ we find that $h = (h_1, ..., h_1)$. Now, for every $\sigma \in P$, there exists $\alpha_1(\sigma) = (\alpha_0(\sigma), ..., \alpha_0(\sigma)) \in H$, which induces $\sigma^{(1)}$ modulo $Stab_G(2)$. Thus, we produce a sequence $\alpha_i(\sigma) \in Stab_G(i)$ of elements in G such that $\alpha_i(\sigma) = \sigma^{(i)}$ modulo $Stab_G(i+1)$. Let $\gamma \in C = C_{\mathcal{A}_m}(G)$. Then,

$$\gamma = (\gamma_1, ..., \gamma_m)\sigma,$$
$$\gamma' = \gamma.\alpha_0(\sigma)^{-1} = (\gamma_1', ..., \gamma_m') \in Stab_C(1),$$

and $\gamma_1' = ... = \gamma_m'$; say γ_1' induces a permutation σ' on Y. Thus,

$$\gamma.\alpha_0(\sigma)^{-1}.\alpha_1(\sigma')^{-1} \in Stab_C(2).$$

We produce in this manner a sequence

$$\alpha_0(\sigma), \alpha_1(\sigma'), \alpha_2(\sigma''), ...$$

of elements of G such that γ is equal to the infinite product of these elements. Hence, $C_{\mathcal{A}_m}(G) = \widehat{G}$.

ii) Let $m = p$ be a prime number. The permutation group P induced on $Y = \{1, ..., p\}$ is cyclic, say generated by σ. Since G is infinite, there exists an $h = (h_1, ..., h_1) \in H$ such that $h_1 \notin H$ and therefore we may assume h_1 induces σ on Y. We produce elements $a_i \in G$ such that $a_i = \sigma^{(i)}$ modulo $Stab_{\mathcal{A}}(i+1)$ and the proof follows as in the first item.

■

2.3 Representations of groups as automorphism groups of the tree

One of the main goals of the theory of groups acting on trees is to investigate if a given group G can be represented in $\mathcal{A}_m$, that is, if G can be seen as a group of automorphisms of a tree. One of the central tools for this purpose is the notion of *virtual endomorphism*, that we will define in this section.

Definition 2.3.1. We say that a group G has a *representation of degree* m if exists an homomorphism $\varphi : G \to \mathcal{A}_m$. If φ is a monomorphism, then the representation is called *faithful.*

We will call both φ and G^φ a *representation of degree* m of G. So, if G^φ is state-closed, finite-state or transitive, we say the same about the representation.

The coset tree

Let G be a group and a chain of subgroups such that

$$G = G_0 \geq G_1 \geq G_2 \geq ... \geq G_n \geq ...$$

with $\bigcap G_i = \{1\}$. Now, we take each of these subgroups as partitions of next subgroup in the chain:

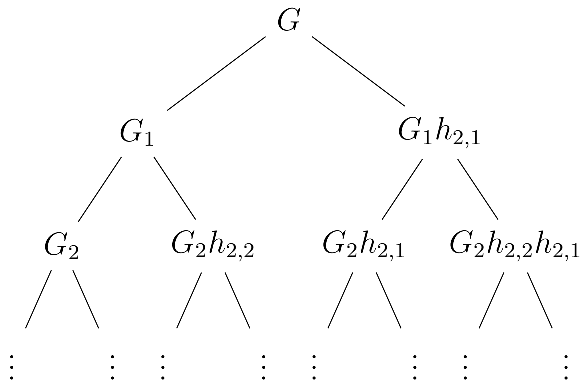
$$G = \bigcup G_1 h_{j,1}\ ,\ G_1 = \bigcup G_2 h_{j,2}, \text{ and so on.}$$

So, for some G_s, we can write its cosets in G as $G_s h_{j_s,s} h_{j_{s-1},s-1} ... h_{j_1,s}$, which will be the vertices of the tree where the edges will be determined by set inclusion. Then G acts (faithfully) on this resulting tree by right multiplication, say $h : G_i k \mapsto G_i k h$.

In this action, the set of vertices fixed by h is a subtree, although it can be irregular. But in the case G_i is a normal subgroup of G and h fixes some coset $G_i k$, we have that h fixes all such cosets of G_i on G (as in this case $h \in G_i k$), and then it fixes all the vertices of the tree down to the i-th level.

If we require a bound m for the indexes $|G_i : G_{i+1}|$, we can embed the coset tree into the m-ary tree $\mathcal{T}_m$; as the coset tree is possibly smaller than $\mathcal{T}_m$, we can extend the action of G fixing pointwise all the extra subtrees that may appear. In particular, if the indexes $|G_i : G_{i+1}|$ are constant, the coset tree is regular.

The following is an example with constant indexes $|G_i : G_{i+1}| = 2$:



Lemma 2.3.2. *Let G be a state-closed group of automorphisms of the tree $\mathcal{T}_m = \mathcal{T}(Y)$ and let X be a $P(G)$-invariant subset of Y. Then $\mathcal{T}(X)$ is G-invariant and for the resulting representation $\mu : G \to \mathcal{A}(X)$ the group G^μ is state-closed. If G is diagonally or topologically closed then so is G^μ.*

Proof. Let xu be a sequence in $\mathcal{M}(X)$ and let $\alpha \in G$. Then $(xu)^\alpha = x^{\sigma(\alpha)}u^{\alpha_x}$. As $x^{\sigma(\alpha)} \in X$ and $\alpha_x \in G$, it follows that $(xu)^\alpha$ is a sequence in $\mathcal{M}(X)$ and then $\mathcal{T}(X)$ is G-invariant. Also, for any sequence u from X, we have $(\alpha^\mu)_u = (\alpha_u)^\mu$. Thus, G^μ is state-closed.

■

Virtual Endomorphisms

Definition 2.3.3. Let G be a group and H a subgroup with finite index m. An homomorphism $f : H \to G$ is called a *virtual endomorphism.*

A subgroup U of G is called *semi-invariant* under the action of f, provided that $(U \cap H)^f \leq U$. If $U \leq H$ and $U^f \leq U$, then U is called *f-invariant.* The largest subgroup K of H which is normal in G and f-invariant is called the $f - core(H)$. If the f-core(H) is trivial then f and the triple (G, H, f) are said to be *simple.*

As a useful example for our purposes we consider G to be a transitive state-closed subgroup of $\mathcal{A}_m$ and $Fix_G(1)$ our finite index subgroup. Then $|G : Fix_G(1)| = m$ and the projection on the first coordinate produces a subgroup of G, i.e., $\pi_1 : Fix_G(1) \to G$ is a virtual endomorphism of G.

Given a triple (G, H, f) and given subgroups $V \leq G$, $U \leq H \cap V$ such that $(U)^f \leq V$, we call $(V, U, f|_U)$ a *sub-triple* of G. If N is a normal semi-invariant subgroup of G, then $\bar{f} : \frac{HN}{N} \to \frac{G}{N}$, $Nh \mapsto Nh^f$ is well defined and $(\frac{G}{N}, \frac{HN}{N}, \bar{f})$ is called a *quotient triple.*

Now, let (G, H, f) be a simple triple where G is abelian and $|G : H| = m$. Then any sub-triple of G is simple. Let $T = tor(G)$ denote the torsion subgroup of G and, for $l \geq 1$, define $G(l) = \{g \in T \mid o(g) \mid l\ \}$, $H(l) = G(l) \cap H$. Then $f : tor(H) \to tor(G)$ and $f : H(l) \to G(l)$. Then, it follows that $tor(G)$ and $G(l)$ are semi-invariant and $(tor(G), tor(H), f|_{tor(H)})$ and $(G(l), H(l), f|_{H(l)})$ are simple sub-triples.

Lemma 2.3.4. *Let (G, H, f) be a simple triple, with G abelian. The triple $(\frac{G}{G(l)}, \frac{HG(l)}{G(l)}, \bar{f})$ is also simple.*

Proof. Suppose that $K \leq H$ is such that $G(l)K^f \leq G(l)K$. Then

$$(G(l)K^f)^l = (K^f)^l = (K^l)^f \leq (G(l)K)^l = (K)^l;$$

that is, K^l is f-invariant. Since f is simple, $K^l = \{e\}$ and so $K \leq G(l)$.

∎

In [NS04] the authors establish a useful way to produce state-closed transitive representations, which we now present.

Let G be a group, H a subgroup of finite index m and $f : H \to G$ a homomorphism. Taking $T = \{t_1, t_2, ..., t_m\}$ as a tranversal of H in G, every element $g \in G$ induces a permutation on the alphabet $Y = \{1, ..., m\}$, $\sigma(g) : Y \to Y$, given in terms of such tranversal,

$$i^{\sigma(g)} = j \Leftrightarrow Ht_ig = Ht_j,\ i, j = 1, ..., m.$$

Now, having that $t_igt_j^{-1} = t_igt_{i^{\sigma}(g)}^{-1} \in H$ we define a map $\varphi : G \to \mathcal{A}_m$ by

$$g \mapsto ((t_1gt_{1^{\sigma(g)}}^{-1})^{f\varphi}, (t_2gt_{2^{\sigma(g)}}^{-1})^{f\varphi}, ..., (t_mgt_{m^{\sigma(g)}}^{-1})^{f\varphi})\sigma(g).$$

Proposition 2.3.5. *The map φ defined above is a homomorphism, where G^φ is state-closed, transitive and*

$$ker(\varphi) = \langle K \leq H \mid K \lhd G, K^f \leq K \rangle,$$

the f-core(H).

Proof. We will proceed by induction on the length $|u|$ of $u \in \mathcal{T}_m$. Let $g, h \in G$. We have that

$$(gh)^\varphi = ((t_1 gh t^{-1}_{1^{\sigma(gh)}})^{f\varphi}, (t_2 gh t^{-1}_{2^{\sigma(gh)}})^{f\varphi}, ..., (t_m gh t^{-1}_{m^{\sigma(gh)}})^{f\varphi})\sigma(gh) =$$
$$((t_1 g t^{-1}_{1^{\sigma(g)}} t_{1^{\sigma(g)}} h t^{-1}_{1^{\sigma(gh)}})^{f\varphi}, (t_2 g t^{-1}_{2^{\sigma(g)}} t_{2^{\sigma(g)}} h t^{-1}_{2^{\sigma(gh)}})^{f\varphi}, ..., (t_m g t^{-1}_{m^{\sigma(g)}} t_{m^{\sigma(g)}} h t^{-1}_{m^{\sigma(gh)}})^{f\varphi})\sigma(gh).$$

On the other hand,

$$g^\varphi h^\varphi = ((t_i g t^{-1}_{i^{\sigma(g)}})^{f\varphi})_{i \in Y}.\sigma(g).((t_i h t^{-1}_{i^{\sigma(h)}})^{f\varphi})_{i \in Y}.\sigma(h) =$$
$$((t_i g t^{-1}_{i^{\sigma(g)}})^{f\varphi}).((t_{i^{\sigma(g)}} h t^{-1}_{i^{\sigma(h)\sigma(g)}})^{f\varphi})_{i \in Y}.\sigma(g)\sigma(h).$$

Having that $\sigma(gh) = \sigma(g)\sigma(h)$, it follows, for all $i \in Y$, that

$$i^{(gh)^\varphi} = i^{\sigma(gh)} = i^{\sigma(g)\sigma(h)} = i^{g^\varphi h^\varphi}.$$

Now, suppose that the result is true for every word of length less or equal than k. Then, for every $i \in Y$ and every word u of length k it holds that

$$(iu)^{(gh)^\varphi} = i^{\sigma(gh)} u^{(gh)^\varphi_i}, \; (iu)^{g^\varphi h^\varphi} = i^{\sigma(gh)} u^{(g^\varphi h^\varphi)}.$$

By hypothesis,

$$u^{(gh)^\varphi_i} = u^{(t_i g t^{-1}_{i^{\sigma(g)}} t_{i^{\sigma(g)}} h t^{-1}_{i^{\sigma(gh)}})^{f\varphi}} = u^{(t_i g t^{-1}_{i^{\sigma(g)}})^{f\varphi} (t_{i^{\sigma(g)}} h t^{-1}_{i^{\sigma(g)\sigma(h)}})^{f\varphi}} = u^{(g^\varphi h^\varphi)_i}.$$

Thus φ is a homomorphism. The other claims follow immediately by definition.

■

If f is simple we have $G \simeq G^\varphi$ and the representation is said to be *faithful*. Despite the loaded notation of the definitions and propositions above, we illustrate the procedure on a simple example:

Example 2.3.6. Let G be the additive group of integers, $H = (2\mathbb{Z})$ and $Y = \{0, 1\}$; then $\sigma = (01)$. Define $f : 2\mathbb{Z} \to \mathbb{Z}$ by $2n \mapsto n$. Then, naming $1^{\varphi} := \alpha$, we have $1^{\varphi} = (0^{\varphi}, 1^{\varphi})\sigma = (e, \alpha)\sigma \in \mathcal{A}_2$, which is none other than the binary adding machine.

An interesting feature of this representation is that changing the transversal of H in G we obtain another representation of G, conjugate to the original one by an explicit automorphism:

Proposition 2.3.7. *Let (G, H, f) be a triple and*

$$L = \{x_1, x_2, \ldots, x_m\}, \;\; L' = \{x'_1 = h_1 x_1, x'_2 = h_2 x_2, \ldots, x'_m = h_m x_m\}$$

be right transversals of H in G, where $h_i \in H$. Let $\varphi = \varphi_{x_i}, \varphi' = \varphi_{h_i x_i} : G \to \mathcal{A}_m$ be the corresponding tree representations and define the following elements of $\mathcal{A}_m$,

$$\gamma = \gamma_{h_i,\varphi'} = ((h_i)^{f\varphi'})_{1\leq i\leq m},$$
$$\lambda = \lambda_{h_i,\varphi'} = \gamma\gamma^{(1)}...\gamma^{(n)}... \, .$$

Then

$$\varphi_{h_i x_i} = \varphi_{x_i}(\lambda_{h_i^{-1},\varphi_{x_i}}).$$

Proof. The representations $\varphi, \varphi' : G \to \mathcal{A}_m$ are defined by

$$g^{\varphi} = ((x_i g.(x_{(i)g^{\pi}})^{-1})^{f\varphi})_{1\leq i\leq m} \cdot \sigma(g)$$
$$g^{\varphi'} = ((x'_i g.(x'_{(i)g^{\pi}})^{-1})^{f\varphi'})_{1\leq i\leq m} \cdot \sigma(g).$$

The relationship between φ' and φ is established as follows,

$$\begin{aligned} g^{\varphi'} &= ((h_i x_i g.(h_{(i)g^{\pi}} x_{(i)g^{\pi}})^{-1})^{f\varphi'})_{1\leq i\leq m} \cdot \sigma(g) \\ &= ((h_i (x_i g.x^{-1}_{(i)g^{\pi}}) h^{-1}_{(i)g^{\pi}})^{f\varphi'})_{1\leq i\leq m} \cdot \sigma(g) \\ &= ((h_i)^{f\varphi'})_{1\leq i\leq m} \cdot ((x_i g.x^{-1}_{(i)g^{\pi}})^{f\varphi'})_{1\leq i\leq m} \cdot ((h^{-1}_{(i)g^{\pi}})^{f\varphi'})_{1\leq i\leq m} \cdot \sigma(g) \\ &= ((h_i)^{f\varphi'})_{1\leq i\leq m} \cdot ((x_i g.x^{-1}_{(i)g^{\pi}})^{f\varphi'})_{1\leq i\leq m} \cdot \sigma(g) \cdot ((h_i)^{f\varphi'})^{-1}_{1\leq i\leq m}. \end{aligned}$$

Therefore,

$$g^{\varphi'} = \gamma \cdot ((x_i g.x_{(i)g^\pi})^{f\varphi'})_{1\leq i\leq m} \cdot \sigma(g) \cdot \gamma^{-1},$$

where $\gamma = ((h_i)^{f\varphi'})_{1\leq i\leq m}$ is independent of g. Repeating this development for each $g_i = (x_i g \cdot x_{(i)g^\pi}^{-1})^f$, we find that

$$g^{\varphi'} = \gamma\gamma^{(1)} \cdot (((x_j g_i \cdot x_{(j)g_i^\pi})^{f\varphi'})_{1\leq i\leq m} \cdot \sigma(g) \cdot \gamma^{-(1)}\gamma^{-1}.$$

Thus in the limit we obtain $\lambda = \gamma\gamma^{(1)}...\gamma^{(n)}...$ such that

$$g^{\varphi'} = \lambda g^{\varphi}\lambda^{-1} \text{ for all } g \in G$$
$$\varphi = \varphi'\lambda.$$

Introducing the explicit dependence of φ, φ' and λ on the transversals, the previous equation becomes

$$\varphi_{x_i} = (\varphi_{h_i x_i})(\lambda_{h_i,\varphi_{h_i x_i}}).$$

On replacing h_i by h_i^{-1} and denoting $h_i^{-1}x_i$ by x_i' we obtain

$$\varphi_{h_i x_i'} = (\varphi_{x_i'})(\lambda_{h_i^{-1},\varphi_{x_i'}}).$$

■

Example 2.3.8. Let $G = C = \langle a\rangle$ be the infinite cyclic group, let $H = \langle a^2\rangle$ and let $f : H \to G$ be defined by $a^2 \mapsto a$. Given $l, k \geq 0$, $\sigma = (01)$, then on choosing the transversal $L_{k,l} = \{a^{2k}, a^{2l+1}\}$ for H in G, we obtain the representation $\varphi = \varphi_{k,l} : G \to \mathcal{A}_2$, given by

$$\begin{aligned} a \mapsto \alpha = a^{\varphi} &= ((a^{2k}aa^{-2l-1})^{f\varphi}, (a^{2l+1}aa^{-2k})^{f\varphi})\sigma \\ &= ((a^{2(k-l)})^{f\varphi}, (a^{2(-k+l+1)})^{f\varphi})\sigma \\ &= ((a^{k-l})^{\varphi}, (a^{-k+l+1})^{\varphi})\sigma \\ &= ((a^{\varphi})^{k-l}, (a^{\varphi})^{-k+l+1})\sigma = (\alpha^{k-l}, \alpha^{-k+l+1})\sigma. \end{aligned}$$

The next proposition is a simple result that gives us a glance of the importance of virtual endomorphisms.

Proposition 2.3.9. *A group G is state-closed transitive of degree m if, and only if, there exist a subgroup H of index m in G and $f : H \to G$ a simple endomorphism.*

Proof. If G is a transitive state-closed group, then

$$\pi_1 : Fix_G(1) \to G;\ \alpha \mapsto \alpha_1$$

is a simple virtual endomorphism and $|G : Fix_G(1)| = m$. The reciprocal follows from the previous proposition.

∎

Chapter 3

Abelian State-closed Representations

Now we proceed to explore the main properties of abelian state-closed subgroups of $\mathcal{A}_m$, e.g., its presentation and torsion subgroup. Then we analyse the special cases where the subgroup $A \leq \mathcal{A}_m$ is cyclic and when the induced permutation group $P(A)$ is cyclic of prime order. We also show an answer to a problem left open in [BS10], namely, whether a free abelian group of infinite rank admits a faithful transitive state-closed representation, with a proof provided in [BS20]. To conclude our work, we prove that if L is an abelian transitive state-closed group, the wreath product $L^{(\omega)} \wr C_2$ is also transitive and state-closed ([DS18]).

3.1 Presentation for A^*

In this section we explore the diagonal-topological closure A^* of an abelian transitive state-closed group A. It turns out that A^* is a finitely generated $\mathbb{Z}_m[[x]]$-module, where $\mathbb{Z}_m[[x]]$ stands for the *formal sums* in $\mathbb{Z}_m$ (meaning that we are not interested in convergence of such sums, i.e., if they in fact define a polynomial function $\mathbb{Z}_m \to \mathbb{Z}_m$).
Such module has the following properties:

i) $x\alpha = 0$ implies $\alpha = 0$;

ii) $m\alpha = x\gamma$ for some $\gamma \in A^*$.

In fact, the first says that the diagonal map x is a monomorphism. For the second, having that $\alpha = (\alpha_1, ...\alpha_m)\sigma$:

$$\alpha^m = (\alpha_1\alpha_{1^\sigma}...\alpha_{1^{\sigma^m}}, ..., \alpha_m\alpha_{m^\sigma}...\alpha_{m^{\sigma^m}})\sigma^m.$$

But, as A^* is abelian and transitive, we have that $\sigma^m = e$ and by Proposition 2.2.4, $\alpha^m \in Stab(i) \leq A^{(i)}$, that is, $\alpha_1\alpha_{1^\sigma}...\alpha_{1^{\sigma^m}} = ... = \alpha_m\alpha_{m^\sigma}...\alpha_{m^{\sigma^m}}$; defining γ as the latter the equality follows.

Before the next theorem we establish the following notation. Let A be an abelian transitive state-closed group and let

$$P(A) = \langle \sigma_i \mid \sigma_i^{m_i} = e, 1 \leq i \leq k\rangle,$$

be its abelian group of induced permutations. Choose elements $\beta_{[i]} \in A^*$ that induces σ_i on Y, and write each $\beta_{[i]}$ as $\beta(\sigma_i)$. Thus, the automorphism $\beta(\sigma_i)^{(n)}$ is an element of $\tilde{A}$ that induces the permutation $(\sigma_i)^{(n)} = (\sigma_i, ..., \sigma_i)$ on the $(n+1)$-level of the tree. That is, the index i is associating the automorphism with the permutation it induces.

Theorem A. *Let A be a transitive abelian state-closed subgroup of degree m. Then A^* is additively a $\mathbb{Z}_m[[x]]$-module generated by $\{\beta_{[i]} \mid 1 \leq i \leq k\}$, subject to the set of relations*

$$\Big\{r_i = \sum_{1\leq j\leq k} m_i\beta_{[i]} - p_{ij}\beta_{[j]}x = 0 \mid 1 \leq i \leq k\Big\},$$

for some $p_{ij} \in \mathbb{Z}_m[[x]]$. Moreover, there exist $r, q \in \mathbb{Z}_m[[x]]$ such that $r = m - xq$ and $rA^* = (0)$. The elements of A^* can be represented additively as $\sum_{1\leq i\leq k} p_i\beta_{[i]}$, where $p_i = \sum_{j\geq 0} p_{ij}x^j$, with $p_{ij} \in \mathbb{Z}$ and $0 \leq p_{ij} < m$.

Proof. Let $\alpha \in A^*$ and $\sigma(\alpha) = \prod_{1\leq i\leq k} \sigma_i^{r_{i1}}$, $0 \leq r_{i1} < m_i$. Then either $\alpha(\prod_{1\leq i\leq k} \beta_{[i]}^{r_{i1}})^{-1}$

is the identity or there exists $l_2 \geq 1$ such that

$$\alpha\Big(\prod_{1\leq i\leq k}\beta_{[i]}^{r_{i1}}\Big)^{-1} \in Stab(l_2) \setminus Stab(l_2+1),$$

that is, in this product some permutations can be cancelled up to some level of the tree. Therefore, $\alpha(\prod_{1\leq i\leq k}\beta_{[i]}^{r_{i1}})^{-1} = (\gamma)^{(l_2)}$ for some $\gamma \in A^*$. Repeating the same argument for γ and so on, we have

$$\alpha = \prod_{1\leq i\leq k}\beta_{[i]}^{r_{i1}}(\beta_{[i]}^{r_{i2}})^{(l_2)}...(\beta_{[i]}^{r_{ij}})^{(l_j)}... = \prod_{1\leq i\leq k}(\beta_{[i]})^{q_i},$$

where $0 \leq r_{ij} < m, 1 \leq l_2 < l_3 < ... < l_j < ...$, and $q_i = r_{i1} + \sum_{j\geq 2} r_{ij}x^{l_j}$ are formal power series in x. Writing additively we have then

$$\alpha = \sum_{1\leq i\leq k} q_i\beta_{[i]} \in \sum_{1\leq i\leq k}\mathbb{Z}_{m_i}[[x]]\beta_{[i]}.$$

Each relation $\sigma_i^{m_i} = e$ of the permutations in $P(A)$ produces in A^* a relation of the form

$$\beta_{[i]}^{m_i} = \prod_{1\leq j\leq k}\beta_{[j]}^{xp_{ij}},$$

where the p_{ij} are formal power series; writing additively:

$$m_i\beta_{[i]} = x\Big(\sum_{i\leq j\leq k} p_{ij}\beta_{[j]}\Big).$$

Now, let $F = \bigoplus_{1\leq i\leq k}\mathbb{Z}_m[[x]]t_i$ be a free $\mathbb{Z}_m[[x]]$-module of rank k . Define the $\mathbb{Z}_m[[x]]$-homomorphism

$$\phi : \bigoplus_{1\leq i\leq k}\mathbb{Z}_m[[x]]t_i \to A^*,\ \sum_{1\leq i\leq k} p_it_i \mapsto \prod_{1\leq i\leq k}\beta_{[i]}^{p_i},$$

and let R be the kernel of ϕ. Define J to be the $\mathbb{Z}_m[[x]]$-submodule of R generated by

$$r'_i = m_it_i - x\Big(\sum_{i\leq j\leq k} p_{ij}t_j\Big),\ \ 1 \leq i \leq k.$$

We will show that $J = R$. So let $v \in R$ and write $v = \sum_{i \leq j \leq k} v_i t_i$, where

$$v_i = \sum_{j \geq 0} v_{ij} x^j, \quad v_{ij} = v_{ij,0} + m w_{ij} \in \mathbb{Z}_m.$$

Claim: $m_i \mid v_{i0,0}$. By definition,

$$\begin{aligned} v_i &= v_{i0}x^0 + v_{i1}x + v_{i2}x^2 + ... \\ &= v_{i0} + v_{i1}x + v_{i2}x^2 + ... \\ &= (v_{i0,0} + m w_{i0}) + v_{i1}x + v_{i2}x^2 + ... \\ v_i t_i &= ((v_{i0,0} + m w_{i0}) + v_{i1}x + v_{i2}x^2 + ...)t_i \end{aligned}$$

Then,

$$v_i t_i \mapsto \beta_{[i]}^{v_i} = \beta_{[i]}^{v_{i0,0}+mw_{i0}} . \beta_{[i]}^{v_{i1}x} . \beta_{[i]}^{v_{i2}x^2} ... = 0, \text{ since } v_i t_i \in ker\phi.$$

But we have that only the term $\beta_{[i]}^{v_{i0,0}+mw_{i0}}$ can act on the first level; but the above equation gives us $\sigma(\beta_{[i]}^{v_{i0,0}+mw_{i0}}) = 0$. Thus

$$\sigma(\beta_{[i]}^{v_{i0,0}+mw_{i0}}) = 0 \Rightarrow m_i \mid (v_{i0,0} + m w_{i0}) \Rightarrow m_i \mid v_{i0,0}.$$

Now, set $v_{i0,0} = m_i v'_{i0,0}$ and factor $m = m_i m'_i$. Therefore,

$$\begin{aligned} v_i &= v_{i0} + \Big(\sum_{j \geq 1} v_{ij} x^{j-1}\Big)x, \\ v_{i0} &= m_i v'_{i0,0} + m w_{i0} = (v'_{i0,0} + m'_i w_{i0}) m_i \\ v_i t_i &= (v'_{i0,0} + m'_i w_{i0})(m_i t_i) + \Big(\sum_{j \geq 1} v_{ij} x^{j-1}\Big) x t_i \\ &\equiv (v'_{i0,0} + m'_i w_{i0})\Big(x \sum_{1 \leq j \leq k} p_{ij} t_j\Big) + \Big(\sum_{j \geq 1} v_{ij} x^{j-1}\Big) x t_i \ \ mod\ J \end{aligned}$$

Hence

$$v = \sum_{1 \leq i \leq k} v_i t_i \in x\mu + J, \ \ \mu = \sum_{1 \leq i \leq k} \mu_i t_i \in R.$$

By repeating the argument, we obtain

$$v \in \Big(\bigcap_{i\geq 1} x^i R\Big) + J = J, \quad J = R.$$

On rewriting the relations $m_i\beta_{[i]} = \sum_{1\leq j\leq k} p_{ij}x\beta_{[j]}$ in the form

$$p_{i1}x\beta_{[1]} + \ldots + (p_{ii}x - m_i)\beta_{[i]} + \ldots + p_{kk}x\beta_{[k]} = 0$$

we see that the $k \times k$ matrix of coefficients of these equations has determinant $r = m - xq$ for some $q \in \mathbb{Z}_m[[x]]$ and thus r annihilates A^*.

■

Definition 3.1.1. A group G of automorphisms of the m-ary tree is said to satisfy the *m-congruence property* provided that, given m^i there exists $l(i) \geq 1$ such that $Stab_G(l(i)) \leq G^{m^i}$, for all i.

We observe that, if G satisfies the m-congruence property, the topology of G inherited from $\mathcal{A}_m$ coincides with the pro-m topology. Also, writing A^* additively, for an abelian transitive state-closed group A, we have $Stab_G(l(i)) = x^{l(i)}A^*$, and the m-congruence property reads $x^{l(i)}A^* \leq m^iA^*$.

Theorem 3.1.2. *Let $r = m - qx^j \in \mathbb{Z}_m[[x]]$, with $q \in \mathbb{Z}_m[[x]]$ and $j \geq 1$. Let S be the quotient ring $\frac{\mathbb{Z}_m[[x]]}{(r)}$. Suppose that S is torsion-free. Then S is a finitely generated pro-m group.*

Proof. By [AF12], Proposition 6.18 and Corolary 6.19, we have the decomposition $\mathbb{Z}_m[[x]] = \bigoplus_{1\leq i\leq s} \varepsilon_i \mathbb{Z}_{p_i^{k_i}}[[x]]$ corresponding to the prime decomposition $m = \prod_{1\leq i\leq s} p_i^{k_i}$, where ε_i are the orthogonal idempotents. Thus, we obtain

$$\begin{aligned} r &= \sum_{1\leq i\leq s} r_i, \\ r_i &= \varepsilon_i r = p_i^{k_i} - q_i(x)x^j, \\ S &= \sum_{1\leq i\leq s} S_i, \quad S_i = \frac{\mathbb{Z}_{p_i^{k_i}}[[x]]}{(r_i)}, \end{aligned}$$

where each S_i is torsion-free. Thus, it is sufficient to address the case where m is a prime power p^k.

(1) First, we show that S is a pro-m group.

So let $r = p^k - qx^j$ and decompose $q = q(x) = s(x) + p \cdot t(x)$, where each non-zero coefficient of $s(x)$ is an integer relatively prime to p. If $s(x) = 0$ then $q(x) = p \cdot t(x)$ and

$$r = p^k - q(x)x^j = p^k - p \cdot t(x)x^j = p(p^{k-1} - t(x)x^j) \in (r);$$

but since S is torsion-free by assumption, we have $p^{k-1} - t(x)x^j \in (r)$, which is not possible.

Write $s(x) = x^l u(x)$, where $l \geq 0$ and $u(x)$ in invertible in $\mathbb{Z}_m[[x]]$ with inverse $u'(x)$. Then $q(x) = x^l u(x) + p.t(x)$ and

$$r = p^k - (x^l u(x)x^j + p \cdot t(x)x^j) = p(p^{k-1} - t(x)x^j) - x^{j+l}u(x).$$

Therefore, on multiplying by $u'(x)$, we obtain

$$p(p^{k-1} - t(x)x^j)u'(x) \equiv x^{j+l} \; mod \; r.$$

It follows that

$$x^{j+l}S \leq pS, \;\; x^{n(j+l)}S \leq p^n S.$$

(2) Now we show that S is finitely generated as a $\mathbb{Z}_m$-module.

By the previous step there exist $l \geq 1$ and $v(x) \in \mathbb{Z}_m[[x]]$ that

$$x^l \equiv mv(x) \; mod \; r.$$

Decompose $v(x) = v_1(x) + v_2(x)x^l$, where the degree of $v_1(x)$ is less than l. Then we deduce, modulo r:

$$\begin{aligned}
v(x) &\equiv v_1(x) + v_2(x)mv(x), \\
v_2(x)v(x) &\equiv w(x) \in \mathbb{Z}_m[[x]], \\
w(x) &= w_1(x) + w_2(x)x^l, \\
v(x) &\equiv v_1(x) + mw(x) \\
&\equiv v_1(x) + mw_1(x) + mw_2(x)x^l \\
&\vdots \\
v(x) &\equiv a_0 + a_1x + \ldots + a_{l-1}x^{l-1}, \quad a_i \in \mathbb{Z}_m.
\end{aligned}$$

We have shown that S is generated by $1, x, \ldots, x^{l-1}$ as a pro-m group.

■

Corolary 3.1.3. *Let A be an abelian transitive state-closed group of degree m. Suppose that the group A^* is torsion-free. Then A^* is a finitely generated pro-m group.*

Proof. With previous notation, the group A^* is an $\mathbb{Z}_m[[x]]$-module generated by

$$\{\beta_i = \beta(\sigma_i) \mid 1 \leq i \leq k\}$$

and is annihilated by $r = m - qx^j \in \mathbb{Z}_m[[x]]$, for some $q \in \mathbb{Z}_m[[x]]$ and $j \geq 1$.
It follows that A^* is an S-submodule, where $S = \frac{\mathbb{Z}_m[[x]]}{(r)}$. Since S satisfies the m-congruence property, it follows that A^* is a pro-m group by Theorem 3.1.2. As S is a finitely generated $\mathbb{Z}_m$-module, it follows that A^* is a finitely generated $\mathbb{Z}_m$-module.

■

3.2 Torsion in state-closed abelian groups

In this section we explore the torsion subgroup of a transitive state-closed abelian group A and infer properties about the exponent of A. The first step in this direction is the

Proposition 3.2.1. *Let A be a transitive state-closed abelian group of degree m. Then $tor(A)$ has finite exponent and is therefore a direct summand of A.*

Proof. Let $T = tor(A)$, $A_1 = Stab_A(1)$, $T_1 = T \cap A_1$ and $|T : T_1| = m'$. Then the projection on the first coordinate of T_1 is a subgroup of T and the triple $\left(T, T_1, \pi|_{T_1}\right)$ is simple of degree $m' \mid m$; let $m = m'm''$. Hence in this representation T is a torsion transitive state-closed subgroup of $\mathcal{A}_{m'}$ by Proposition 2.3.9.
Fixing this last representation of T, let $Q = P(T)$ and let $\sigma_i, (1 \leq i \leq k)$ be a minimal set of generators of Q and as before, let $\beta_i = \beta(\sigma_i) \in T$ be such that $\sigma(\beta_i) = \sigma_i$. Let r be the maximum order of the elements $\beta_1, ..., \beta_k$. As any $\alpha \in T$ can be written in the form

$$\alpha = \prod_{1 \leq i \leq k} \beta_i^{r_{i1}} (\beta_i^{r_{i2}})^{(l_2)} \dots (\beta_i^{r_{ij}})^{(l_j)} \dots,$$

by Theorem A it follows that $\alpha^r = e$. Since T has finite exponent, it is a pure bounded subgroup of A and therefore it is a direct summand of A by Proposition 1.4.4.

■

The next two lemmas will establish the exponent of $tor(A)$.

Lemma 3.2.2. *Let m be a prime number and A a transitive state-closed abelian torsion group of degree m. Then A is conjugate by a tree automorphism to a subgroup of the diagonal-topological closure of $\langle\sigma\rangle$ and so has exponent m.*

Proof. We observe that $A(m) = \{g \in A \; ; \; o(g) \mid m \}$ is not contained in $A_1 = Stab_A(1)$. For otherwise, $A(m)$ would be invariant under the projection on the first coordinate. Choose $a \in A \backslash A_1$ of order m. Therefore, $A = A_1\langle a \rangle$. On choosing $\{a^i \mid 0 \leq i \leq m-1\}$ as a transversal of A_1 in A, the image of a acquires the form $\sigma = (1 \dots m)$ in this tree representation of A. Thus, we may suppose by Proposition 2.3.7 that $\sigma \in A$. Therefore, $\tilde{A}$ contains the subgroup $\widetilde{\langle\sigma\rangle} = \langle \sigma^i \mid i \geq 0 \rangle$. By Proposition 2.2.6, we have $C_{\mathcal{A}}(\widetilde{\langle\sigma\rangle}) = \langle\sigma\rangle^*$ and thereby $A \leq C_{\mathcal{A}}(A) \leq \langle\sigma\rangle^*$.

■

Lemma 3.2.3. *Suppose that A is a transitive state-closed abelian torsion group of degree m. Then the exponent of A is equal to the exponent of $P(A)$.*

Proof. We will proceed by induction on $|P(A)| = m$. The exponent of A is a multiple of the exponent of $P(A)$. By the previous lemma, we may assume m to be composite; let p be a prime divisor of m. Then $A(p)$ is a nontrivial subgroup and $P(A(p)) \leq \{\sigma \in P \mid \sigma^p = e\}$. By Lemma 2.3.4, $\left(\frac{A}{A(p)}, \frac{A_1 A(p)}{A(p)}, \bar{\pi}_1\right)$ is simple. Also, $P\left(\frac{A}{A(p)}\right) = \frac{P(A)}{P(A(p))}$.
Now assume the claim holds for every $k \leq m$. Having that $|P(\frac{A}{A(p)})| = \frac{m}{p} < m$, by induction hypothesis $exp(\frac{A}{A(p)}) = exp(P(\frac{A}{A(p)})) \leq \frac{m}{p}$. But, as

$$exp\left(P\left(\frac{A}{A(p)}\right)\right) = exp\left(\frac{P(A)}{P(A(p)}\right) = \frac{exp(P(A))}{exp(P(A(p)))} = \frac{exp(P(A))}{p},$$

we have that

$$exp(A) = p\ .\ exp\left(\frac{A}{A(p)}\right) = exp(P(A)).$$

■

Theorem B. *Suppose that A is a transitive state-closed abelian torsion group of degree m. Then A is conjugate to a subgroup of the topological closure of*

$$\widetilde{P(A)} = \langle \sigma^{(i)} \mid \sigma \in P(A),\ i \geq 0 \rangle.$$

Proof. Let $P = P(A)$ have exponent r and let B be a maximal homogeneous subgroup of P of exponent r (that is, B is a direct sum of cyclic groups of order r), minimally generated by $\{\sigma_i \mid 1 \leq i \leq s\}$. Choose for each σ_i an element $\beta_i = \beta(\sigma_i) \in A$ and let $\dot{B} = \langle \beta_i \mid 1 \leq i \leq s \rangle$. Then, as the order of each β_i is a multiple of r, while the exponent of A is r, we conclude from the previous lemma that $o(\beta_i) = o(\sigma_i) = r$ for $1 \leq i \leq s$. Since $\beta_i \to \sigma_i$ defines a projection of $\dot{B}$ onto B we conclude that $\dot{B} \cong B$ and $\dot{B} \cap A_1 = \{e\}$, where $A_1 = Stab_A(1)$. As $\dot{B}$ is a pure bounded subgroup, by Proposition 1.4.4 it has a complement L in A, which may be chosen to contain A_1. Choose a right transversal W of A_1 in L. Then, the set $W\dot{B}$ is a right transversal of A_1 in A. With respect to this transversal, the triple (A, A_1, π_1) produces a transitive state-closed representation φ where $\dot{B}^\varphi = B$. By Proposition 2.3.7, we may write A^* as A. Then the diagonal-topological closure A^* contains B^*. Let V be a complement of B in P, which exists by Proposition 1.4.4. Each $\alpha \in A^*$ can be factored as $\alpha = \beta\gamma$, where $\beta \in B^*$ and γ is such that each of its states γ_u induces the permutation $\sigma(\gamma_u) \in V$. Therefore, the set of these $\gamma's$ is a group Γ such that $\Gamma = \Gamma^*$ and $A^* = \Gamma \oplus B^*$. Then $(\Gamma, \Gamma \cap A_1, \pi_1)$ is a simple triple with $P(\Gamma)$ having exponent smaller than r. Using the same induction argument on the exponent of the previous Lemma, the result follows.

■

Now we provide an example to show how the concepts discussed so far fit together in practice.

Example 3.2.4. Let $m = 4, Y = \{1, 2, 3, 4\}$ and let σ be the cycle (1234). Furthermore, let $\alpha = (e, e, e, \alpha^2)\sigma \in \mathcal{A}_4$ and let $A = \langle \alpha \rangle$. Then

$$
\begin{aligned}
&\alpha^2 = (\alpha^2, e, e, \alpha^2)(13)(24),\\
&\alpha^4 = (\alpha^2)^{(1)} = \alpha^{2x},\\
&(\alpha^{2-x})^2 = e.
\end{aligned}
$$

Thus A is cyclic, torsion-free, transitive and state-closed; it is, however, not diagonally closed because $\alpha^x \notin A$. Even though A is torsion-free, its diagonal closure $\tilde{A} = \langle \alpha^{x^i} \mid i \geq 0 \rangle$ is not; for $\gamma = \alpha^{2-x}$ has order 2. Let $K = \langle \gamma^{x^i} \mid i \geq 0 \rangle$. Then $K \leq tor(\tilde{A})$ and $\tilde{A} = \langle \alpha, K \rangle$. Therefore, $K = tor(\tilde{A})$ and $\tilde{A} = tor(\tilde{A}) \oplus \bar{A}$.
Now, let $Y_1 = \{1, 3\}, Y_2 = \{2, 4\}$. Then $\{Y_1, Y_2\}$ form a complete block system for the action of α on Y. Also, α^2 induces the binary adding machine on both $\mathcal{T}(Y_1)$ and $\mathcal{T}(Y_2)$. The topological closure $\bar{A}$ of A is torsion-free and

$$tor(A^*) = tor(\tilde{A}), \quad A^* = tor(A^*) \oplus \bar{A}.$$

Moreover, $tor(A^*)$ induces a faithful state-closed, diagonally and topologically closed actions on the binary tree $\mathcal{T}(y_1)$. Therefore, $tor(A^*)$ is isomorphic to $\frac{\mathbb{Z}}{2\mathbb{Z}}[[x]]$. Furthermore, α is represented as the binary adding machine on $\mathcal{T}(\{Y_1, Y_2\})$ and $\bar{A}$ is represented on this tree as the topological closure of the image of A.

3.3 Cyclic $\mathbb{Z}_m[[x]]$-modules

We now make some considerations about the special case of cyclic groups of automorphisms of the tree. Given $\langle\alpha\rangle$, for some $\alpha \in \mathcal{A}_m$, its state-diagonal-topological closure is isomorphic to a cyclic $\mathbb{Z}_m[[x]]$-module of the form $\alpha = (\alpha^{q_1}, \alpha^{q_2}, ..., \alpha^{q_m})\sigma$, where $q_i \in \mathbb{Z}_m[[x]]$ for $1 \leq i \leq m$, where

$$q_i = \sum_{j\geq 0} q_{ij}x^j, \quad q_{ij} = \sum_{u\geq 0} q_{ij,u}m^u \in \mathbb{Z}_m.$$

For that we have the

Theorem C. *i) The expression $\alpha = (\alpha^{q_1}, \alpha^{q_2}, ..., \alpha^{q_m})\sigma$ is a well-defined automorphism of the m-ary tree;*
ii) Let A be the state closure of $\langle\alpha\rangle$. Then A^* is abelian, isomorphic to the quotient ring $\mathbb{Z}_m[[x]]/(r)$, where $r = m - qx$ and $q = q_1 + ... + q_m$.

Proof. i) Let $\sigma(l)$ denote the permutation induced by α acting on the tree truncated at the l-th level. Then the expression $\alpha = (\alpha^{q_1}, ..., \alpha^{q_m})\sigma$ represents

$$\sigma(1) = \sigma, \quad \sigma(l) = (\sigma^{\overline{q_1}}, ..., \sigma^{\overline{q_m}})\sigma,$$

where $\bar{q}_i = \overline{q_{i0}} + \overline{q_{i1}}x + ... + \overline{q_{i(l-1)}}x^{l-1}$ and $\overline{q_{ij}} = q_{ij,0} + q_{ij,1}m + ... + q_{ij,l-i}m^{l-1}$; looking at $\overline{q_i}$ and $\overline{q_{ij}}$ we see that there is a finite number of permutations acting on each level, therefore the action of α is well-defined at each level.

ii.a) The states of α are words in α^p, for $p \in \mathbb{Z}_m[[x]]$. Let $v = \alpha^{l_1}...\alpha^{l_b}$, $w = \alpha^{n_1}...\alpha^{n_b}$ elements in A^*. Then $[v, w] \in Stab_A(1)$. Following this observation, we will prove that the entries of $[v, w]$ are products of conjugates of words in elements of the form $[\alpha^s, \alpha^t]$, where $s, t \in \mathbb{Z}_m[[x]]$. In this way the commutator will result in an automorphism with trivial action on every level, i.e., the identity, and therefore A^* will be abelian.
The commutator $[v, w]$ can be developed into a word in the conjugates of some $[\alpha^{l_i}, \alpha^{n_j}]$

with the usual commutator properties. Write $p = p_0 + p'x, n = n_0 + n'x$. Now, computing,

$$\begin{aligned}[\alpha^p, \alpha^n] &= ([\alpha^{p_0}, \alpha^{n'x}][\alpha^{p_0}, \alpha^{n_0}]^{\alpha^{n'x}})^{\alpha^{p'x}}[\alpha^{p'}, \alpha^{n'}]^x[\alpha^{p'x}, \alpha^{n_0}]^{\alpha^{n'x}} \\ &= [\alpha^{p_0}, \alpha^{n'x}]^{\alpha^{p'x}}[\alpha^{p'}, \alpha^{n'}]^x[\alpha^{p'x}, \alpha^{n_0}]^{\alpha^{n'x}}.\end{aligned}$$

Therefore, we have to check $[\alpha^\xi, \alpha^{nx}]$, where $\xi \in \mathbb{Z}_m,\ n \in \mathbb{Z}_m[[x]]$. Write $\xi = \xi_0 + m\xi'$. Then

$$[\alpha^\xi, \alpha^{nx}] = [\alpha^{\xi_0 + m\xi'}, \alpha^{nx}] = [\alpha^{\xi_0}, \alpha^{nx}]^{\alpha^{m\xi'}}[\alpha^{m\xi'}, \alpha^{nx}].$$

Now, with

$$\alpha^{\xi_0} = (v_1, v_2, ..., v_m)\sigma^{\xi_0}$$

where v_i are words in $\alpha^{q_1}, ..., \alpha^{q_m}$ and

$$\alpha^m = (\alpha^{q_1}...\alpha^{q_m}, \alpha^{q_2}...\alpha^{q_m}\alpha^{q_1}, ..., \alpha^{q_m}\alpha^{q_1}...\alpha^{q_{m-1}}).$$

Consequently,

$$[\alpha^{\xi_0}, \alpha^{nx}] = ([v_1, \alpha^n], ..., [v_m, \alpha^n])$$

and similarly

$$[\alpha^{m\xi'}, \alpha^{nx}] = ([(\alpha^{q_1}...\alpha^{q_m})^{\xi'}, \alpha^n], ..., [(\alpha^{q_m}\alpha^{q_1}...\alpha^{q_{m-1}})^{\xi'}, \alpha^n]).$$

Now we write $\beta = \alpha^{q_1}...\alpha^{q_m}$. Then $[\beta^{\xi'}, \alpha^n]$ can be developed further, as asserted. The same applies to the other entries.

ii.b) First, we have $r\alpha = 0$. Now let $u = u(x)$ annihilate α; write $u = u_0 + u'x$, where $u_0 = u(0)$. Then $m \mid u_0$ and so

$$u = m\tfrac{u_0}{m} + u'x = (xq)\tfrac{u_0}{m} + u'x + vr = xw_1 + vr$$

for some $v = v(x)$ and $w_1 = q\frac{u_0}{m} + u'$. Then xw_1 annihilates α and so does w_1. By repeating this argument, we find w_i such that $u \equiv x^i w_i \ mod \ r$ and w_i annihilates α for all $i \geq 1$; that is,

$$u \in \bigcap_{n \geq 1} x^n \mathbb{Z}_m + (r) = (r).$$

■

Example 3.3.1. The above theorem generalizes the direct calculations with the binary adding machine $\alpha = (e, \alpha)\sigma$. To ilustrate this, let $A = \langle\alpha\rangle$. The set of states of A are the powers of α, thus A is state-closed. Now, taking the diagonal and topological closure A^*, its elements are of the form α^q, where $q \in \mathbb{Z}_2[[x]]$, subject to the relation $\alpha^{2n} = (\alpha^n, \alpha^n) = \alpha^{nx}$, that is, $\alpha^{n(2-x)} = e$. Therefore A^* is isomorphic to the ring $\mathbb{Z}_2[[x]]/(2-x)$.

3.4 The case $P(A)$ cyclic of prime order

The group $D_m(j)$

Let $\alpha = (e, ...e, \alpha^{x^{j-1}})\sigma \in \mathcal{A}_m$. Then $\alpha^m = \alpha^{x^j}$; that is, $\alpha^r = e$ where $r = m - x^j$. The states of α are $\alpha, \alpha^x, ..., \alpha^{x^{j-1}}$ and the group

$$D_m(j) = \langle\alpha, \alpha^x, ..., \alpha^{x^{j-1}}\rangle$$

is diagonally closed. The topological closure $\overline{D_m(j)}$ is isomorphic to the quotient ring $S = \frac{\mathbb{Z}_m[[x]]}{(r)}$, which is a free $\mathbb{Z}_m$-module of rank j.

For $z \in A$, define the function $\zeta : A \to \mathbb{N}$ as $\zeta(z) = j$ if $z^m \in Stab(j)\backslash Stab(j+1)$. Intuitively, this function shows the maximum level that the m-th power of z stabilizes. And, as A is torsion-free, $\zeta(z)$ is finite for all nontrivial z and $z^m = (v)^{(j)}$, $v \in A\backslash Stab_A(1)$. Now, choose $\beta = (\beta_1, \beta_2, ..., \beta_m)\sigma \in A\backslash Stab_A(1)$ having minimum $\zeta(\beta) = j$. If $z \in Stab_A(1)$, $z \neq e$, then there exists $l \geq 0$ such that $z^m = (c)^{(l)}$ and $c \in A\backslash Stab_A(1)$. Therefore, by minimality of β we have $\zeta(c) \geq \zeta(\beta)$ and $\zeta(z) > \zeta(\beta)$.

Theorem D. *Let m be a prime number. Let A be a torsion-free abelian transitive state-closed subgroup of $\mathcal{A}_m$ and let $\beta \in A\backslash Stab_A(1)$ such that $\zeta(\beta)$ is minimum. Then $A^* = \langle\beta\rangle^*$ and is topologically finitely generated.*

Proof. We start noticing that if $z \in Stab_A(1)$, the composition with β does not stabilize a level greater than j. That is, *Claim:(Uniform Gap)* Let $z \in Stab_A(1)$. Then $\zeta(z\beta) = \zeta(\beta)$.

First notice that

$$\beta^m = (\beta_1\beta_2...\beta_m)^{(1)},$$
$$\beta_1\beta_2...\beta_m = (\gamma)^{(j-1)}, \;\; \gamma \in A \backslash Stab_A(1).$$

We have $z = c^{(1)}$ and $z\beta = (c\beta_1, c\beta_2, ..., c\beta_m)\sigma$, $(z\beta)^m = (u)^{(1)}$, where $u = c^m\beta_1\beta_2...\beta_m = c^m(\gamma)^{(j-1)}$. Now, we have two possibilities for c: if $c \in A\backslash Stab_A(1)$ then $\zeta(c) = n \geq j$, $c^m \in Stab(n)\backslash Stab(n+1)$, and so $u \in Stab_A(j-1)\backslash Stab_A(j)$. If $c \in Stab_A(1)$, then $\zeta(c) > j$ and so $c^m \in Stab_A(k)$, where $k > j$ and again $u \in Stab(j-1)\backslash Stab(j)$, and then in either way $(z\beta)^m = (u)^{(1)} \in Stab_A(j)\backslash Stab_A(j+1)$ and the claim is proved.

Now, note that

$$\beta^m = (\gamma)^{(j)}, \;\; \gamma^m = (\lambda)^{(j)},$$
$$\beta^{m^2} = (\lambda)^{(2j)},$$

where, by the uniform gap claim above, $\gamma, \lambda \in A\backslash Stab_A(1)$. Therefore, repeating this process, we find that β^{m^s} induces $\sigma^{(sj)}$ on the (sj)-th level of the tree for all $s \geq 0$. Now given a level $t \geq 0$, dividing t by j, we get $t = sj + i$ with $0 \leq i \leq j-1$, and then $(\beta^{(i)})^{m^s} = (\beta^{m^s})^{(i)}$ induces $(\sigma^{(sj)})^{(i)} = \sigma^{(sj+i)} = \sigma^{(t)}$ on the t-th level of the tree. The last equality says that, for *any* level t of the tree, the action on this level can be described in terms of a finite number of elements $\beta, \beta^{(1)}, ..., \beta^{(j-1)}$. That being said, it follows that the group A^* is a subgroup of the topological closure of $\langle \beta, \beta^{(1)}, ..., \beta^{(j-1)} \rangle$. The reverse inclusion is immediate.

■

Theorem E. *In the same conditions of the previous theorem, we have that* $A^* = \langle\beta\rangle^*$ *is conjugate to* $\overline{D_m(j)}$ *for some* $j \geq 1$.

Proof. We have for $\beta = (\beta_1, \beta_2, ...\beta_m)\sigma$,

$$\beta_i = \beta^{p_i}, \;\; p_i = r_{i0} + r_{i1}x + ... + r_{i(j-1)}x^{j-1} \in \mathbb{Z}_m[[x]],$$

and

$$\beta^m = (\beta_1\beta_2...\beta_m)^{(1)},$$
$$\beta_1\beta_2, ...\beta_m = \beta^{p_1+...+p_m},$$
$$p_1 + ... + p_m = q \ . \ x^{j-1},$$

where q is an invertible element of $\mathbb{Z}_m[[x]]$.

Claim: The element $\beta = (\beta_1, \beta_2, ...\beta_m)\sigma$ is conjugate in $\mathcal{A}_m$ to $\alpha = (e, ..., e, \alpha^{x^{j-1}})\sigma$. In fact, let $h = (h_1, ..., h_m)$ be an automorphism of the tree. Then

$$\beta^h = (h_1^{-1}\beta_1 h_2, h_2^{-1}\beta_2 h_3, ..., h_m^{-1}\beta_m h_1)\sigma.$$

Therefore $\beta^h = \alpha$ holds if and only if

$$\begin{aligned} h_2 &= \beta_1^{-1} h_1, \\ h_3 &= \beta_2^{-1} h_2, \\ &\vdots \\ h_m &= \beta_{m-1}^{-1} h_{m-1}, \\ h_1 &= \beta_m^{-1} h_m \alpha^{x^{j-1}}. \end{aligned}$$

These conditions can be rewritten as

$$\begin{aligned} h_2 &= \beta_1^{-1} h_1, \\ h_3 &= \beta_2^{-1}\beta_1^{-1} h_1, \\ &\vdots \\ h_m &= \beta_{m-1}^{-1}...\beta_1^{-1} h_1, \\ h_1 &= \beta_m^{-1}\beta_{m-1}^{-1}...\beta_1^{-1} h_1 \alpha^{x^{j-1}}, \end{aligned}$$

or as

$$h = (h_1, \beta_1^{-1}h_1, \beta_2^{-1}\beta_1^{-1}h_1, \ldots, \beta_{m-1}^{-1}...\beta_1^{-1}h_1)$$
$$= (e, \beta_1^{-1}, \beta_2^{-1}\beta_1^{-1}, \ldots, \beta_{m-1}^{-1}...\beta_1^{-1})h_1^{(1)},$$

and

$$(\beta_1\beta_2...\beta_m)^{h_1} = \alpha^{x^{j-1}}.$$

Thus, to determine h it is sufficient to determine h_1. Since

$$\beta_1\beta_2...\beta_m = \beta^{q\cdot x^{j-1}},$$

we repeat the above procedure by replacing β by β^q and by replacing h_1 by $(h_1')^{x^{j-1}}$. This leads to the conjugation equation

$$(\beta^q)^{h_1'} = \alpha.$$

In this manner we determine an automorphism h of the tree which performs the required conjugation $\beta^h = \alpha$.

Finally, by Proposition 2.2.6, we have $A \leq \langle\beta\rangle^* = (C_{\mathcal{A}_m}(\beta))^*$ and

$$A^h \leq C_{\mathcal{A}_m}(\beta^h) = C_{\mathcal{A}_m}(\alpha) = \overline{D_m(j)}.$$

This completes the proof of the theorem.

■

Example 3.4.1. Let $\beta = (e, \beta^q)\sigma$, where $q = 1 + x$. Then β is conjugate to the adding machine $\alpha = (e, \alpha)\sigma$. Note that from Example 2.3.8, β is not obtainable from α simply by choosing a different transversal. To exhibit the conjugator $h : \beta \to \alpha$ constructed in the proof, define the polynomial sequences

$$c_0 = 1,\ c_1 = q,\ c_n = 2c_{n-2} + c_{n-1};$$
$$c'_{-1} = 0,\ c'_0 = 0,\ c'_n = c_{n-1} + c'_{n-1}.$$

Then

$$h = (e,e)^{(0)}(e,\beta^{-1})^{(1)}(e,\beta^{-(1+q)})^{(2)}\dots\ (e,\beta^{-c'_n})^{(n)}\dots$$

3.5 On the self-similarity of the free abelian group of infinite rank

To finish our work, we study direct products of abelian transitive state-closed groups, in particular the case $\mathbb{Z}^{(\omega)}$. We prove a theorem about the non-abelian case $L^{(\omega)} \wr C_2$ and provide an example of an intransitive state-closed representation that is also finite-state.

Theorem F. *Let $\mathbb{Z}^{(\omega)}$ the restricted product of countably many copies of the integers. Then there exists a faithful transitive state-closed action of $\mathbb{Z}^{(\omega)}$ into the binary tree.*

Proof. First of all, notice that the multiplicative group of $\mathbb{Z}_2$ is $1+\mathbb{Z}_2$. Choose any $2\eta \in 2\mathbb{Z}_2^{\times}$; namely a 2-adic that is $\equiv 2 \ mod\ 4$. Every $a \in \mathbb{Z}_2$ admits then a unique *base-η representation*:

$$a = \sum_{i\geq 0} a_i \eta^i, \quad a_i \in \{0,1\}.$$

As we stated in the first chapter, there is an identification between the infinite paths on the binary tree and $\mathbb{Z}_2$; in this way we have a natural action $\mathbb{Z}_2 \to \mathcal{T}_2$ by translation. This is equivalent to say that, identifying the infinite paths of the tree with the representation of a above:

$$x_0x_1... \longleftrightarrow \sum_{i\geq 0} x_i\eta^i,$$

we have an action of $\mathbb{Z}_2$ over itself; in fact this is a transitive state-closed action. For this, consider the triple $(\mathbb{Z}_2, 2\mathbb{Z}_2, f)$, where $f : 2\mathbb{Z}_2 \to \mathbb{Z}_2$ is given by $a \mapsto a/\eta$.

Consider now the additive subgroup $G := \mathbb{Z}[1/\eta] \cap \mathbb{Z}_2$ of $\mathbb{Z}_2$; we claim this is a transitive state-closed subgroup. Let $H = \mathbb{Z}[1/\eta] \cap 2\mathbb{Z}_2$, with $|G : H| = 2$, and the map $\dot{f} : H \to G$ is the restriction of f to the subgroup H. We have that $H^{\dot{f}} \subseteq G$, since

$$H^{\dot{f}} = (\mathbb{Z}[1/\eta] \cap 2\mathbb{Z}_2)^{\dot{f}} = (\mathbb{Z}[1/\eta] \cap 2\mathbb{Z}_2)/\eta \subseteq \mathbb{Z}[1/\eta] \cap \mathbb{Z}_2 = G.$$

This action does not leave any nontrivial subgroup $K \leq H$ invariant: for any nontrivial element $a \in H$, applying f successively will eventually bring a nonzero entry to the first coordinate; thus $a^f \notin H$. Now, if we choose η to be transcendental, it follows that the action is faithful. ∎

The construction in the theorem may be made explicitly as follows. Starting with $G = \mathbb{Z}^{(\omega)}$, $H = 2\mathbb{Z} \times \mathbb{Z}^{(\omega - \{0\})}$ and define the virtual endomorphism $f : H \to G$ by $(2a_0, a_1, a_2, ...) \mapsto (a_0 + \alpha_1 a_1 + \alpha_2 a_2 + ..., a_0, a_1, a_2, ...)$ for appropriate choices of $\alpha_n \in \{0, 1\}$, which we define below simultaneously with the embedding $G \simeq \mathbb{Z}^{(\omega)} \hookrightarrow \mathbb{Z}_2$:

First, the 0-th basis vector of G maps to 1; then, for $n \geq 1$ the n-th basis vector of G will map to $p_n(1/\eta)$, a integral polynomial of degree at most n in $2\mathbb{Z}_2$. More precisely: $p_0(t) = 2$, and $\alpha_n \in \{0, 1\}$ chosen in such a way that $p_{n+1}(t) := tp_n(t) - \alpha_{n+1}$ belongs to $2\mathbb{Z}_2$.

Theorem G. *Let L be an abelian transitive state-closed group and $L^{(\omega)}$ an infinite countable direct sum of copies of L. Then $L^{(\omega)} \wr C_2$ is also transitive and state-closed.*

Proof. Let L be a transitive state-closed with respective simple triple (L, M, ϕ), where ϕ is then a monomorphism. Define the direct sum $B = \sum_{i \geq 1} L_i$, with $L_i = L$ for each i. Let X be the cyclic group of order 2 and $G = B \wr X$. Denote the normal closure of B in G by A; then

$$A = B^X = \Big(L_1 \oplus \sum_{i \geq 2} L_i\Big) \times B;$$

$$G = A \rtimes X.$$

Define the subgroup of G

$$H = \Big(M \oplus \sum_{i \geq 2} L_i\Big) \times B;$$

an element of H has the form

$$\beta = (\beta_1, \beta_2)$$

where

$$\beta_i = (\beta_{ij})_{j\geq 1}, \quad \beta_{ij} \in L,$$
$$\beta_1 = (\beta_{1j})_{j\geq 1}, \quad \beta_{11} \in M.$$

We note that $|G : H|$ is finite; indeed,

$$|A : H| = |L : M| \;\text{ and }\; |G : H| = 2|L : M|.$$

Define the maps

$$\phi_1' : M \oplus \Big(\sum_{i\geq 2} L_i\Big) \longrightarrow B; \quad \phi_2' : B \longrightarrow B,$$

where for $\beta = (\beta_1, \beta_2) = ((\beta_{1j}), (\beta_{2j}))_{j\geq 1}, \; \beta_{11} \in M$,

$$\phi_1' : \beta_1 \longmapsto (\beta_{11}^{\phi}, \beta_{12}, \beta_{13}, \ldots), \quad \phi_2' : \beta_2 \longmapsto (\beta_{22}, \beta_{23}, \beta_{24}, \ldots).$$

Since L is abelian, ϕ_1' and ϕ_2' are both homomorphisms.

Now, define the homomorphism

$$f : \Big(M \oplus \sum_{i\geq 2} L_i\Big) \times B \longrightarrow A$$

by

$$f : (\beta_1, \beta_2) \longmapsto ((\beta_1)^{\phi_1'}, (\beta_2)^{\phi_2'}).$$

By applying successively f and the permutation $\sigma \in X$ is clear that no subgroup of H is f-invariant and our result is proved.

■

In [BS20] the authors proved that there exist a transitive state-closed representation for $\mathbb{Z}^{(\omega)}$; however they also show that there is no such action that is also finite-state. As a final example, we apply the above theorem to present a state-closed representation of degree 4 for $\mathbb{Z}^{(\omega)}$ that is also finite-state, but intransitive.

Example 3.5.1. Let $G = \mathbb{Z}^{(\omega)} \wr C_2$, with $C_2 = \langle \sigma \rangle$, $B = \mathbb{Z}^{(\omega)} \times \mathbb{Z}^{(\omega)}$ and consider $H \leq G$ given by

$$H = \langle\ ((2n_1, n_2, n_3, \dots), e),\ (e, (m_1, m_2, m_3 \dots))\ \rangle.$$

Denote the two generators of H by x and y, respectively. Notice that $|B : H| = 2$ and $|G : H| = 4$. Then, define the endomorphism f as

$$\begin{aligned} f \colon H &\longrightarrow G \\ x &\longmapsto ((n_1, n_2, n_3, \dots), e) \\ y &\longmapsto (e, (m_2, m_3, m_4, \dots)) \end{aligned}$$

Now, consider the following group G:

$$G = \langle\ ((\underbrace{0, 0, \dots, 0}_{i-1}, 1, 0, \dots), e)\ ,\ \sigma \mid i = 1, 2, 3, \dots \rangle,$$

where we define $x_i = ((\underbrace{0, 0, \dots, 0}_{i-1}, 1, 0, \dots), e)$ and $y_i = (e, (\underbrace{0, 0, \dots, 0}_{i-1}, 1, 0, \dots))$, both with the only nonzero entry on the i-th position.

Choose a transversal $T = \{e, \sigma, x_1, x_1\sigma\}$ of H in G; observe that $H\sigma x_1 = Hy_1\sigma = H\sigma$ and $H\sigma y_1 = Hx_1\sigma$. Now, calculating $\varphi : G \to \mathcal{A}_4$:

for $i = 1$:

$$\begin{aligned} x_1^{\varphi} &= ((ex_1x_1^{-1})^{f\varphi}, (\sigma x_1 \sigma)^{f\varphi}, (x_1x_1e)^{f\varphi}, (x_1\sigma x_1 \sigma x_1^{-1})^{f\varphi})(13) \\ &= (e, e, x_1^{\varphi}, e)(13) \end{aligned}$$

$$\begin{aligned} y_1^{\varphi} &= ((ey_1e)^{f\varphi}, (\sigma y_1 \sigma x_1^{-1})^{f\varphi}, (x_1y_1x_1^{-1})^{f\varphi}, (x_1\sigma y_1 \sigma)^{f\varphi})(24) \\ &= (e, e, e, x_1^{\varphi})(24); \end{aligned}$$

for $i \geq 2$:

$$\begin{aligned} x_i^{\varphi} &= ((ex_ie)^{f\varphi}, (\sigma x_i \sigma)^{f\varphi}, (x_1 x_i x_1^{-1})^{f\varphi}, (x_1 \sigma x_i \sigma x_1^{-1})^{f\varphi}) \\ &= (x_i^{\varphi}, y_{i-1}^{\varphi}, x_i^{\varphi}, y_{i-1}^{\varphi}) \end{aligned}$$

$$\begin{aligned} y_i^{\varphi} &= ((ey_ie)^{f\varphi}, (\sigma y_i \sigma)^{f\varphi}, (x_1 y_i x_1^{-1})^{f\varphi}, (x_1 \sigma y_i \sigma x_1^{-1})^{f\varphi}) \\ &= (y_{i-1}^{\varphi}, x_i^{\varphi}, y_{i-1}^{\varphi}, x_i^{\varphi}). \end{aligned}$$

Defining $x_i^{\varphi} = \alpha_i$ and $y_i^{\varphi} = \beta_i$ we have

$$\begin{aligned} &\alpha_1 = (e, e, \alpha_1, e)(13),\ \beta_1 = (e, e, e, \alpha_1)(24) \text{ and} \\ &\alpha_i = (\alpha_i, \beta_{i-1}, \alpha_i, \beta_{i-1}),\ \beta_i = (\beta_{i-1}, \alpha_i, \beta_{i-1}, \alpha_i),\ \text{for } i \geq 2. \end{aligned}$$

Therefore we have that $\langle \alpha_i,\ \beta_j \mid i, j = 1, 2, \ldots \rangle \simeq \mathbb{Z}^{(\omega)} \times \mathbb{Z}^{(\omega)} \simeq \mathbb{Z}^{(\omega)}$ is a state-closed and finite-state representation of $\mathbb{Z}^{(\omega)}$.

Bibliography

[AF12] Frank Anderson and Kent Fuller. *Rings and categories of modules*, volume 13. Springer Verlag, 2nd edition, 2012.

[BS07] Adilson Berlatto and Said Sidki. Virtual endomorphisms of nilpotent groups. *Groups, Geometry, and Dynamics*, 1(1):21–46, 2007.

[BS10] Andrew Brunner and Said Sidki. Abelian state-closed subgroups of automorphisms of m-ary trees. *Groups, Geometry, and Dynamics*, 4(3):455–472, 2010.

[BS20] Laurent Bartholdi and Said Sidki. Self-similar products of groups. *Groups, Geometry, and Dynamics*, 14(1):107–115, 2020.

[BT15] Herivelto Borges and Eduardo Tengan. *Algebra comutativa em quatro movimentos*. Rio de Janeiro: IMPA, 2015.

[Dan16] Alex Dantas. Representações fechadas por estado de grupos metabelianos tipo entrelaçado. *PhD Thesis, Universidade de Brasília*, 2016.

[DF03] David Dummit and Richard Foote. *Abstract algebra*. Wiley, 2003.

[DS18] Alex Dantas and Said Sidki. On self-similarity of wreath products of abelian groups. *Groups, Geometry, and Dynamics*, 12(3):1061–1068, 2018.

[GŻ02] Rostislav Grigorchuk and Andrzej Żuk. On a torsion-free weakly branch group defined by a three state automaton. *International Journal of Algebra and Computation*, 12(01n02):223–246, 2002.

[Kat07] Svetlana Katok. *p-adic Analysis Compared with Real*, volume 37. American Mathematical Soc., 2007.

[NS04] Volodymyr Nekrashevych and Said Sidki. Automorphisms of the binary tree: state-closed subgroups and dynamics of 1/2-endomorphisms. *London Mathematical Society Lecture Note Series*, 311:375–404, 2004.

[Rob12] Derek Robinson. *A Course in the Theory of Groups*, volume 80. Springer-Verlag, 2nd edition, 2012.

[Rob13] Alain Robert. *A course in p-adic analysis*, volume 198. Springer Verlag, 2013.

[RZ00] Luis Ribes and Pavel Zalesskii. *Profinite groups*. Springer, 2000.

[Sid98] Said Sidki. *Regular Trees and their Automorphisms*. Rio de Janeiro: IMPA, 1998.

[Wil98] John Wilson. *Profinite Groups*. Oxford University Press, 1998.