Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Existence of positive solutions for a class of elliptic systems

por

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Aos meus pais,

Ao meu irmão.

"A tarefa não é tanto ver aquilo que ninguém viu, mas pensar o que ninguém ainda pensou sobre aquilo que todo mundo vê". (Arthur Schopenhauer)

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Resumo

Os capítulos 1 e 2 deste trabalho tratam respectivamente do estudo de existência de solução dos seguintes sistemas:

$$\begin{cases} -\Delta u + a(x)u = \dfrac{1}{2^*}K_u(u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + b(x)v = \dfrac{1}{2^*}K_v(u,v) & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \\ u, v \in D^{1,2}(\mathbb{R}^N), & N \geq 3, \end{cases}$$

e

$$\begin{cases} -\Delta u + a(x)u = \dfrac{1}{2^*}K_u(u,v) & \text{in } \mathbb{R}^N_+, \\ -\Delta v + b(x)v = \dfrac{1}{2^*}K_v(u,v) & \text{in } \mathbb{R}^N_+, \\ u > 0, v > 0 & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

com as hipóteses sobre as funções $K \in C^2(\mathbb{R}^2_+, \mathbb{R})$ e a, b a serem apresentadas.

No capítulo 3 é estudada a multiplicidade de solução usando resultados de categoria de Ljusternick-Schnirelmann no seguinte sistema

$$\begin{cases} -\Delta u = \frac{2\alpha_\epsilon}{\alpha_\epsilon+\beta_\epsilon}|u|^{\alpha_\epsilon-2}u|v|^{\beta_\epsilon} & \text{in } \Omega, \\ -\Delta v = \frac{2\beta_\epsilon}{\alpha_\epsilon+\beta_\epsilon}|u|^{\alpha_\epsilon}|v|^{\beta_\epsilon-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

onde Ω é domínio regular limitado em $\mathbb{R}^N, N \geq 3$, $\alpha_\epsilon, \beta_\epsilon > 1$, $\alpha_\epsilon = \alpha - \epsilon/2$, $\beta_\epsilon = \beta - \epsilon/2$ e $\alpha + \beta = 2^*$.

Abstract

In the chapters 1 and 2 we study respectively the existence of solutions of the following systems:

$$\begin{cases} -\Delta u + a(x)u = \dfrac{1}{2^*}K_u(u,v) \;\text{ in }\; \mathbb{R}^N, \\ -\Delta v + b(x)v = \dfrac{1}{2^*}K_v(u,v) \;\text{ in }\; \mathbb{R}^N, \\ u, v > 0 \;\text{ in }\; \mathbb{R}^N, \\ u, v \in D^{1,2}(\mathbb{R}^N), \quad N \geq 3, \end{cases}$$

and

$$\begin{cases} -\Delta u + a(x)u = \dfrac{1}{2^*}K_u(u,v) \;\text{ in }\; \mathbb{R}^N_+, \\ -\Delta v + b(x)v = \dfrac{1}{2^*}K_v(u,v) \;\text{ in }\; \mathbb{R}^N_+, \\ u > 0, v > 0 \;\text{ in }\; \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \;\text{ on }\; \partial\mathbb{R}^N_+, \end{cases}$$

where the hypotheses about the functions $K \in C^2(\mathbb{R}^2_+, \mathbb{R})$ and a, b will be defined in the related chapter.

In Chapter 3 we study multiplicity of solutions using Ljusternick-Schnirelmann category results in the following system

$$\begin{cases} -\Delta u = \frac{2\alpha_\epsilon}{\alpha_\epsilon+\beta_\epsilon}|u|^{\alpha_\epsilon-2}u|v|^{\beta_\epsilon} & \text{in } \Omega, \\ -\Delta v = \frac{2\beta_\epsilon}{\alpha_\epsilon+\beta_\epsilon}|u|^{\alpha_\epsilon}|v|^{\beta_\epsilon-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N, N \geq 3$, $\alpha_\epsilon, \beta_\epsilon > 1$, $\alpha_\epsilon = \alpha - \epsilon/2$, $\beta_\epsilon = \beta - \epsilon/2$ and $\alpha + \beta = 2^*$.

Notations

$2^* = \frac{2N}{N-2}$;

$\mathbb{R}^N_+ = \{x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N : x_N \geq 0\}$;

$\mathrm{cat}_X(A)$ it is the Ljusternik-Schnirelmann category of A with respect to X;

$\mathrm{cat}_{X,Y}(A)$ it is the category of A in X relative to Y;

$H^1_0(A) = W^{1,2}_0(A) = \overline{C^\infty_0(A)}^{\|\cdot\|_{W^{1,2}}}$;

$D^{1,2}(A) = \{u \in L^{2^*}(A) : |\nabla u| \in L^2(A)\}$;

$S = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2 dx;\quad u \in D^{1,2}(\mathbb{R}^N),\quad \int_{\mathbb{R}^N} |u|^{2^*} dx = 1\right\}$;

$S_K = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 dx;\quad (u,v) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N),\quad \int_{\mathbb{R}^N} K(u,v) dx = 1\right\}$;

$\Sigma_K = \inf\left\{\int_{\mathbb{R}^N_+} |\nabla u|^2 + |\nabla v|^2 dx;\quad (u,v) \in D^{1,2}(\mathbb{R}^N_+) \times D^{1,2}(\mathbb{R}^N_+),\quad \int_{\mathbb{R}^N_+} K(u,v) dx = 1\right\}$;

$\Omega^+_r = \{x \in \mathbb{R}^N : \mathrm{d}(x, \partial\Omega) \leq r\}$;

$\Omega^-_r = \{x \in \mathbb{R}^N : \mathrm{d}(x, \partial\Omega) \geq r\}$.

Contents

Introduction

Consider the elliptic problems given by

$$(BC_1) \qquad \left\{ \; -\Delta u + a(x)u = |u|^{2^*-2}u \;\; \text{in} \;\; \mathbb{R}^N \right.$$

and

$$(BC_2) \qquad \left\{ \begin{array}{l} -\Delta u + a(x)u = |u|^{2^*-2}u \;\; \text{in} \;\; \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = 0 \;\; \text{on} \;\; \partial\mathbb{R}^N_+, \end{array} \right.$$

where $N \geq 3$ and $2^* = 2N/(2N-2)$. In the last years the main interest in this general class of problems has been due to the fact that they arise from applications in physics and related sciences, such as biophysics, plasma physics and chemical reaction, as it can be seen for example in [19], [23], [24] and [26].

An interesting fact about this kind of problem is that Pohozaev's identity [25] shows that problems (BC_1) and (BC_2) have no solution if $a(x)$ is a positive constant. But in the celebrated paper [6], Benci and Cerami studied the semilinear elliptic problem (BC_1) and proved existence of positive solutions with the following hypotheses about the function $a(x)$:

$(a)_1$ $a(x) \geq 0$ and $a(x) \geq a_0 > 0$ for all x in a neighborhood of a point x_0.

$(a)_2$ $a \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$ with $1 < p_1 < \frac{N}{2} < p_2$ and $p_2 < \dfrac{N}{4-N}$ if $N = 3$.

$(a)_3$ $|a|_{L^{N/2}(\mathbb{R}^N)} < S(2^{2/N} - 1)$, where $S = \displaystyle\inf_{u \in D^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}}$.

This conditions on $a(x)$ were sufficient to guarantee existence and multiplicity of positive solutions for problem (BC_1). It was used properties of the solution of the Limit problem, where $a \equiv 0$, the version to $\mathbb{R}^N$ of Struwe's Global Compactness result [29], Lions's Concentration and Compactness result [22] and arguments of Brouwer degree theory. This paper motivated other works as follows.

The version of [6] for $p-$Laplacian operator was studied in [1], where in this case there is some technical difficulties with the lack of linearity and homogeneity. The version of bi-Laplacian operator was studied by Alves and do Ó in [3]. A multiplicity result involving category theory was studied in [11] by Chabrowski and Yang. More recently, in [32] Xie, Ma and Xu proved a version for [6] considering the Kirchhoff operator. Nascimento and Figueiredo showed the same result of [6] in [10] considering the fractional Laplacian. In [8] it was studied existence and positive solutions for a Schrödinger-Poisson system. Recently, a version for Choquard equation using variational methods combined with degree theory was proved in [4].

A natural extension of problem (BC_1) consists in studying elliptic systems such as

$$(S_1) \qquad \begin{cases} -\Delta u + a(x)u = \dfrac{1}{2^*}K_u(u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + b(x)v = \dfrac{1}{2^*}K_v(u,v) & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \\ u, v \in D^{1,2}(\mathbb{R}^N), & N \geq 3. \end{cases}$$

The main difficult of this class of systems is a double lack of compactness due to the unboundedness of the domain and the presence of the critical Sobolev exponent, since K is 2^*-homogeneous. Then in Chapter 1 we shall focus our attention on questions of existence and positivity of solutions for the system (S_1).

We state our main hypotheses on the function $K \in C^2(\mathbb{R}^2_+;\mathbb{R})$ as follows:

$(\mathcal{K}_0)$ K is 2^*-homogeneous, that is,

$$K(\lambda s, \lambda t) = \lambda^{2^*} K(s,t) \quad \text{for each } \lambda > 0,\ (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{K}_1)$ there exists $c_1 > 0$ such that

$$|K_s(s,t)| + |K_t(s,t)| \leq c_1 \left(s^{2^*-1} + t^{2^*-1} \right) \quad \text{for each } (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{K}_2)$ $K(s,t) > 0$ for each $s, t > 0$;

$(\mathcal{K}_3)$ $\nabla K(0,1) = \nabla K(1,0) = (0,0)$;

$(\mathcal{K}_4)$ $K_s(s,t), K_t(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}^2_+$.

$(\mathcal{K}_5)$ the 1-homogeneous function $G : \mathbb{R}^2_+ \to \mathbb{R}$ given by $G(s^{2^*}, t^{2^*}) := K(s,t)$ is concave.

Let us denote

$$P_q(s,t) = \sum_{p_i+q_i=q} C_i |s|^{p_i} |t|^{q_i}.$$

where $p_i \geq 1$, $q_i \geq 1$ and $i \in J$, with $J \subset \mathbb{N}$ finite set. With appropriate choices of coefficients C_i and exponents p_i and q_i, we have the following examples of functions that satisfy hypotheses $(\mathcal{K}_0)$ - $(\mathcal{K}_5)$:

$$K_1(s,t) = P_{2^*}(s,t),$$

$$K_2(s,t) = \sqrt[r]{P_q(s,t)} \text{ with } q/r = 2^*,$$

$$K_3(s,t) = \frac{P_{q_1}(s,t)}{P_{q_2}(s,t)}, \text{ with } q_1 - q_2 = 2^*.$$

Hypothesis $(\mathcal{K}_3)$ allow us to give a C^1 extension of K to the whole plan as

$$H(s,t) = H(s^+, t^+),$$

with $s, t \in \mathbb{R}$ and $u^+ := \max\{u, 0\}$.

Hypothesis $(\mathcal{K}_5)$ provides a Hölder type inequality for all $u, v \in L^{2^*}(\Omega)$

$$\int_\Omega K(u,v)dx \geq K(|u|_{2^*}, |v|_{2^*}),$$

which is used in [13] to prove some lemmas that we used in Chapter 1 and 2. It is important to remark that in those lemmas we used, the domain Ω is not necessarily bounded.

The hypotheses on the functions $a; b : \mathbb{R}^N \longmapsto \mathbb{R}^+$ are given by:

$(a,b)_1$ The functions a, b are positive in a same set of positive measure.

$(a,b)_2$ $a, b \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$ with $1 < p_1 < \frac{N}{2} < p_2$ and $p_2 < \dfrac{N}{4-N}$ if $N = 3$.

$(a,b)_3$ $s_o^N |a|_{L^{N/2}(\mathbb{R}^N)} + t_o^N |b|_{L^{N/2}(\mathbb{R}^N)} < S_K(2^{2/N} - 1)$, where

$$S_K = \inf_{u,v \in D^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 dx}{\left(\int_{\mathbb{R}^N} K(u,v)dx\right)^{2/2^*}}$$

and s_0 and t_0 will be defined in Chapter 1 and 2.

Using the above notation about the functions K, a and b we are able to state our main result of Chapter 1:

Theorem 0.0.1. *Assume that $(a,b)_1 - (a,b)_3$ and $(\mathcal{K}_0) - (\mathcal{K}_5)$ hold. Then, (S_1) has a positive solution $(u_0, v_0) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ with*

$$\frac{1}{N} S_K^{N/2} < I(u_0, v_0) < \frac{2}{N} S_K^{N/2},$$

where the associated functional I will be defined in Chapter 1.

In order to prove this main result, Chapter 1 is organized as follows. In Section 1 we study the limit system associated to (S_1). In Section 2 we are interested in a compactness result and we obtain some properties about Palais-Smale sequences. In Section 3 we start showing some technical lemmas and we finalize this section proving our main result.

The work studied in Chapter 1 was published in [14].

In Chapter 2 we are interested in the same kind of problem defined in the half-space. In the paper [9], Cerami and Passaseo gave sufficient conditions on function $a(x)$ to guarantee existence and multiplicity of positive solutions for problem (BC_2). Also, in [2] the authors studied the $p-$ laplacian problem defined in half-space involving a critical exponent. Then, motivated by these papers, in Chapter 2 we study a natural extension of the problem (BC_2) consisting in the following elliptic systems defined in the half-space:

$$(S_2) \qquad \begin{cases} -\Delta u + a(x)u = \dfrac{1}{2^*} K_u(u,v) \text{ in } \mathbb{R}^N_+, \\ -\Delta v + b(x)v = \dfrac{1}{2^*} K_v(u,v) \text{ in } \mathbb{R}^N_+, \\ u, v > 0 \text{ in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\mathbb{R}^N_+, \quad N \geq 3. \end{cases}$$

Using the above notation about functions a, b and K we are able to state our main result of Chapter 2:

Theorem 0.0.2. *Assume that* $(a,b)_1 - (a,b)_2$ *and* $(\mathcal{K}_0) - (\mathcal{K}_5)$ *hold. Then,* (S_2) *has a positive solution* $(u_0, v_0) \in D^{1,2}(\mathbb{R}^N_+) \times D^{1,2}(\mathbb{R}^N_+)$.

As in Chapter 1, we have observed that there is not a version of the paper [9] for systems. Motivated by this fact, we have decided to study this class of systems. However, we would like point out that some estimates made in [9] or [2] are not immediate for systems. For example, in Lemma 2.1.3, Lemma 2.1.4 and Proposition 2.1.5 was necessary to use a Global Compactness Lemma for system that was proved in Chapter 1 and can be found published in [14]. In other words, some results that were proved in Chapter 1, were also important to obtain the second main theorem.

This chapter is organized as follows. In Section 1 we show a nonexistence result of solution for a minimization problem and some properties. In Section 2 we prove some technical lemmas. Then, finally in Section 3 we prove the main result.

Chapter 3 was inspired by the following problem studied by Benci, Cerami and Passaseo in [7]

$$\begin{cases} -\Delta u = u^{2^*-\epsilon-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain.

The interesting fact about this problem is that, by Pohozaev's identity [25], we have nonexistence of positive solutions for $\epsilon = 0$ for certain class of domain Ω, but when we take $\epsilon > 0$ small, we can prove that we have multiplicity of solutions influenced by the topology of the domain.

There are other papers motivated by this class of problems. In [18] it was studied a problem involving the fractional Laplacian, obtaining a lower bound on the number of positive solutions when the exponent of the non-linearity is near to the critical Sobolev exponent $2^*_s = 2N/(N-2s)$. This lower bound is also given by the topology of the domain. In [28] it was studied the same kind of problem for a Schrödinger-Poisson-Slater system.

Also motivated by [7], in [17] it was studied the following problem:

$$\begin{cases} -\Delta u - u\left(\Delta u^2\right) = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where p is taken near to the exponent $22^* = 4N/(N-2)$. In [17] the number of positive solutions is estimated from below by values related to topological properties of the domain Ω, in this case the Ljusternick-Schnirelmann category and the Poincaré polynomial. In [5] it was studied a case with a discontinuous non-linearity, where using an auxiliary problem, the authours proved the multiplicity of positive solutions using Ljusternick-Schnirelmann category.

In Chapter 3, we are interested in the search of positive solutions for the following problem

$$(S_3) \qquad \begin{cases} -\Delta u = \frac{2\alpha_\epsilon}{\alpha_\epsilon+\beta_\epsilon}|u|^{\alpha_\epsilon-2}u|v|^{\beta_\epsilon} & \text{in } \Omega \\ -\Delta v = \frac{2\beta_\epsilon}{\alpha_\epsilon+\beta_\epsilon}|u|^{\alpha_\epsilon}|v|^{\beta_\epsilon-2}v & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N, N \geq 3$, $\alpha_\epsilon, \beta_\epsilon > 1$, $\alpha_\epsilon = \alpha - \epsilon/2$, $\beta_\epsilon = \beta - \epsilon/2$ and $\alpha + \beta = 2^*$. Then we have the following multiplicity result.

Theorem 0.0.3. *There exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0)$, *problem* (S_3) *has at least* $\mathrm{cat}\,\Omega$ *positive weak solutions. Moreover if* Ω *is not contractible in itself then* (S_3) *has at least* $\mathrm{cat}\,\Omega + 1$ *positive weak solutions.*

As in the problem studied in [7], for $\epsilon = 0$, we prove that the system only has the trivial solution, but for $\epsilon > 0$ small enough we have a multiplicity result associated to the topology of the domain.

Our approach to study the system case and prove Theorem 0.0.3 is variational, finding its solutions as critical points of a C^1 functional on the Nehari manifold. We show that the functional on the Nehari manifold is bounded from below, achieves the *ground state level* $\mathfrak{m}_\epsilon$, for $\epsilon \in (0, \epsilon_0)$,and by means of the Ljusternik-Schnirelmann we prove the multiplicity result.

In the first section of this chapter, we prove some Nehari manifold and compactness results. Then in section 2, we prove some barycentre map results. In the final section we prove the main theorem using Ljusternik-Schnirelmann category.

The results obtained in Chapter 2 and 3 are submitted in [15] and [16], respectively.

The hypotheses about the functions and the definitions presented in this Introduction will be recovered in the related chapters.

Chapter 1

Existence of positive solutions of a critical system in $\mathbb{R}^N$

In this chapter we will show existence of positive solution to the following system

$$
(S_1) \qquad \begin{cases} -\Delta u + a(x)u = \dfrac{1}{2^*}K_u(u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + b(x)v = \dfrac{1}{2^*}K_v(u,v) & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \\ u, v \in D^{1,2}(\mathbb{R}^N), & N \geq 3. \end{cases}
$$

Let $\mathbb{R}^2_+ := [0,\infty) \times [0,\infty)$ and set $2^* := 2N/(N-2)$. We state our main hypothesis on the function $K \in C^2(\mathbb{R}^2_+, \mathbb{R})$ as follows.

$(\mathcal{K}_0)$ K is 2^*-homogeneous, that is,

$$K(\lambda s, \lambda t) = \lambda^{2^*} K(s,t) \quad \text{for each } \lambda > 0,\ (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{K}_1)$ there exists $c_1 > 0$ such that

$$|K_s(s,t)| + |K_t(s,t)| \leq c_1 \left(s^{2^*-1} + t^{2^*-1}\right) \quad \text{for each } (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{K}_2)$ $K(s,t) > 0$ for each $s, t > 0$;

$(\mathcal{K}_3)$ $\nabla K(0,1) = \nabla K(1,0) = (0,0)$;

$(\mathcal{K}_4)$ $K_s(s,t), K_t(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}^2_+$.

$(\mathcal{K}_5)$ the 1-homogeneous function $G : \mathbb{R}^2_+ \to \mathbb{R}$ given by $G(s^{2^*}, t^{2^*}) := K(s,t)$ is concave.

To state our main result we need some previous definitions and notations. Let us denote by S_K the following constant

$$S_K := \inf_{u,v \in D^{1,2}(\mathbb{R}^N), u,v \neq 0} \frac{\displaystyle\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx}{\left(\displaystyle\int_{\mathbb{R}^N} K(u,v)dx\right)^{2/2^*}}.$$

From now on, we consider the function $\Phi_{\delta,y} \in D^{1,2}(\mathbb{R}^N)$ given by

$$\Phi_{\delta,y}(x) = c\left(\frac{\delta}{\delta^2 + |x-y|^2}\right)^{(N-2)/2}, \quad x, y \in \mathbb{R}^N \text{ and } \delta > 0, \tag{1.0.1}$$

where c is a positive constant. In [30] we can see that every positive solution of

$$(P_\infty) \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \quad N \geq 3. \end{cases}$$

is as (1.0.1). Moreover, it satisfies for a suitable constant c

$$\|\Phi_{\delta,y}\|^2 = S \text{ and } |\Phi_{\delta,y}|_{2^*} = 1, \tag{1.0.2}$$

where

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N), u \neq 0} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}}.$$

By [13, Lemma 3], there exist $s_o, t_o > 0$ such that S_K is attained by $(s_o\Phi_{\delta,y}, t_o\Phi_{\delta,y})$. Moreover,

$$M_K S_K = S, \tag{1.0.3}$$

where $M_K = \max_{s^2+t^2=1} K(s,t)^{2/2^*} = K(s_o, t_o)^{2/2^*}$.

The hypotheses on the functions $a, b : \mathbb{R}^N \mapsto \mathbb{R}^+$ are given by:

$((a,b)_1)$ The functions a, b are positive in a same set of positive measure.

$((a,b)_2)$ $a, b \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$ with $1 < p_1 < \frac{N}{2} < p_2$ and $p_2 < \dfrac{N}{4-N}$ if $N = 3$.

$((a,b)_3)$ $s_o^N |a|_{L^{N/2}(\mathbb{R}^N)} + t_o^N |b|_{L^{N/2}(\mathbb{R}^N)} < S_K(2^{2/N} - 1)$.

We say that $(u, v) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}$ is a positive weak solution of (S_1) if $u, v > 0$ in $D^{1,2}(\mathbb{R}^N)$ and for all $\varphi, \psi \in D^{1,2}(\mathbb{R}^N)$ we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla u \nabla \varphi dx + \int_{\mathbb{R}^N} \nabla v \nabla \psi dx + \int_{\mathbb{R}^N} a(x) u \varphi dx + \int_{\mathbb{R}^N} b(x) v \psi dx \\ = \;& \frac{1}{2^*} \int_{\mathbb{R}^N} K_u(u,v) \varphi dx + \frac{1}{2^*} \int_{\mathbb{R}^N} K_v(u,v) \psi dx. \end{aligned}$$

In order to state the main result, we consider the C^1 functional $I : D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \mapsto \mathbb{R}$ associated to system (S_1) given by

$$I(u,v) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} a(x) u^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} b(x) v^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(u,v) dx,$$

where $\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx$, $\|v\|^2 = \int_{\mathbb{R}^N} |\nabla v|^2 dx$. Note that

$$\begin{aligned} I'(u,v)(\varphi,\psi) &= \int_{\mathbb{R}^N} \nabla u \nabla \varphi dx + \int_{\mathbb{R}^N} \nabla v \nabla \psi dx + \int_{\mathbb{R}^N} a(x) u \varphi dx + \int_{\mathbb{R}^N} b(x) v \psi dx \\ &- \frac{1}{2^*}\int_{\mathbb{R}^N} K_u(u,v) \varphi dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K_v(u,v) \psi dx, \end{aligned}$$

for all $(\varphi, \psi) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$.

Using the above notation we are able to state our main result.

Theorem 1.0.1. *Assume that* $((a,b)_1) - ((a,b)_3)$ *and* $(\mathcal{K}_0) - (\mathcal{K}_5)$ *hold. Then,* (S_1) *has a positive solution* $(u_0, v_0) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ *with*

$$\frac{1}{N} S_K^{N/2} < I(u_0, v_0) < \frac{2}{N} S_K^{N/2}.$$

1.1 Limit problem

We notice that we can use the homogeneity condition $(\mathcal{K}_0)$ to conclude that

$$K(s,t) = \frac{1}{2^*} s K_s(s,t) + \frac{1}{2^*} t K_t(s,t), \tag{1.1.1}$$

since by $(\mathcal{K}_0)$, we have

$$\frac{d}{d\lambda} K(\lambda s, \lambda t) = \frac{d}{d\lambda} \left(\lambda^{2^*} K(s,t) \right) 2^* = 2^* \lambda^{2^*-1} K(s,t), \tag{1.1.2}$$

and

$$\begin{aligned} \frac{d}{d\lambda} K(\lambda s, \lambda t) &= s K_s(\lambda s, \lambda t) + t K_t(\lambda s, \lambda t) \\ &= s \lambda^{2^*-1} K_s(\lambda s, \lambda t) + t \lambda^{2^*-1} K_t(\lambda s, \lambda t) \end{aligned} \tag{1.1.3}$$

Then, by equations (1.1.2) and (1.1.3) we got

$$2^* K(s,t) = s K_s(s,t) + t K_t(s,t).$$

In this section we study the limit problem given by

$$(S_\infty) \qquad \begin{cases} -\Delta u = \dfrac{1}{2^*} K_u(u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v = \dfrac{1}{2^*} K_v(u,v) & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \\ u, v \in D^{1,2}(\mathbb{R}^N), & N \geq 3, \end{cases}$$

which the functional associated $I_\infty : D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \mapsto \mathbb{R}$ given by

$$I_\infty(u,v) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} K(u,v) dx.$$

Lemma 1.1.1. *Let* (u_n, v_n) *be a sequence* $(PS)_c$ *for* I_∞*. Then*

(i) The sequence (u_n, v_n) *is bounded in* $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$*.*

(ii) If $u_n \rightharpoonup u$ *in* $D^{1,2}(\mathbb{R}^N)$ *and* $v_n \rightharpoonup v$ *in* $D^{1,2}(\mathbb{R}^N)$*, then* $I'_\infty(u,v) = 0$*.*

(iii) If $c \in (-\infty, \frac{1}{N} S_K^{N/2})$*, then* I_∞ *satisfies the* $(PS)_c$ *condition, i.e, up to a subsequence,*

$$(u_n, v_n) \to (u,v) \quad \text{in} \quad D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N).$$

Proof. Since $I_\infty(u_n, v_n) \to c$ and $I'_\infty(u_n, v_n) \to 0$ and from (1.1.1), we conclude that there exists $C > 0$ such that

$$C + \|u_n\| + \|v_n\| \geq I_\infty(u_n v_n) - \frac{1}{2^*} I'_\infty(u_n, v_n)(u_n, v_n) = \frac{1}{N}\|u_n\|^2 + \frac{1}{N}\|v_n\|^2 + o_n(1)$$

and the proof of part (i) is over. Now we prove (ii). Since $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$ and $v_n \rightharpoonup v$ in $D^{1,2}(\mathbb{R}^N)$, up to a subsequence, we get

$$u_n \to u \ \ in \ \ L^q_{loc}(\mathbb{R}^N), \ \ v_n \to v \ \ in \ \ L^q_{loc}(\mathbb{R}^N),$$

and

$$u_n(x) \to u(x) \ \ a.e \ \ in \ \ \mathbb{R}^N, \ \ v_n(x) \to v(x) \ \ a.e \ \ in \ \ \mathbb{R}^N.$$

Using a density argument we obtain

$$\int_{\mathbb{R}^N} K_u(u_n, v_n)\varphi dx + \int_{\mathbb{R}^N} K_v(u_n, v_n)\psi dx \to \int_{\mathbb{R}^N} K_u(u, v)\varphi dx + \int_{\mathbb{R}^N} K_v(u, v)\psi dx.$$

for all $\varphi, \psi \in D^{1,2}(\mathbb{R}^N)$, which implies (ii).

In order to prove (iii), consider $w_n = u_n - u$ and $z_n = v_n - v$. Note that applying [20, Lemma 4.6], we get

$$\begin{aligned} o_n(1) &= I'_\infty(u_n, v_n)(u_n, v_n) = \|u_n\|^2 + \|v_n\|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K_u(u_n, v_n)u_n dx \qquad (1.1.4) \\ &- \frac{1}{2^*}\int_{\mathbb{R}^N} K_v(u_n, v_n)v_n dx \\ &= \|w_n\|^2 + \|u\|^2 + \|z_n\|^2 + \|v\|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K_u(w_n + u, z_n + v)(w_n + u)dx \\ &- \frac{1}{2^*}\int_{\mathbb{R}^N} K_v(w_n + u, z_n + v)(z_n + v)dx. \qquad (1.1.5) \end{aligned}$$

From [13, Lemma 8], we have

$$\begin{aligned} o_n(1) &= \|w_n\|^2 + \|u\|^2 + \|z_n\|^2 + \|v\|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K_u(w_n, z_n)w_n dx \\ &- \frac{1}{2^*}\int_{\mathbb{R}^N} K_v(w_n z_n)z_n dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K_u(u, v)u dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K_v(u, v)v dx. \end{aligned}$$

Using the item (ii) and (1.1.1) we obtain

$$o_n(1) \ = \ \|w_n\|^2 + \|z_n\|^2 - \int_{\mathbb{R}^N} K(w_n, z_n)dx.$$

Up to a subsequence, we conclude that there exists $\rho \geq 0$ such that

$$0 \leq \rho = \lim_{n\to\infty}\Big[\|w_n\|^2 + \|z_n\|^2\Big] = \lim_{n\to\infty}\int_{\mathbb{R}^N} K(w_n, z_n)dx.$$

Suppose, by contradiction, that $\rho > 0$. From the inequality

$$S_K\left(\int_{\mathbb{R}^N} K(w_n, z_n)dx\right)^{2/2^*} \leq \|w_n\|^2 + \|z_n\|^2,$$

we get

$$\rho \geq S_K \rho^{2/2^*} \Rightarrow \rho \geq S_K^{N/2}. \qquad (1.1.6)$$

Since

$$I_\infty(u, v) = \left(\frac{1}{2} - \frac{1}{2^*}\right)[\|u\|^2 + \|v\|^2] = \frac{1}{N}[\|u\|^2 + \|v\|^2] \geq 0$$

and

$$c = \frac{1}{N}[\|w_n\|^2 + \|z_n\|^2] + I_\infty(u,v) + o_n(1), \tag{1.1.7}$$

we conclude

$$c = \frac{1}{N}[\|w_n\|^2 + \|z_n\|^2] + I_\infty(u,v) + o_n(1) \geq \frac{1}{N}[\|w_n\|^2 + \|z_n\|^2] + o_n(1) = \frac{1}{N}\rho \geq \frac{1}{N}S_K^{N/2},$$

which is a contradiction. Hence $\rho = 0$ and

$$\|w_n\|^2 = \|u_n - u\|^2 \to 0 \ \ \textit{and} \ \ \|z_n\|^2 = \|v_n - v\|^2 \to 0.$$

□

1.2 A compactness result

Now, we establish the following lemma which will be useful to prove a compactness result.

Lemma 1.2.1. *Let (u_n, v_n) be a $(PS)_c$ sequence for the functional I_∞ with $u_n \rightharpoonup 0$, $v_n \rightharpoonup 0$ and $u_n \nrightarrow 0$, $v_n \nrightarrow 0$. Then, there are sequences $(R_n) \subset \mathbb{R}$, $(x_n) \subset \mathbb{R}^N$ and $(\Upsilon_0, \Upsilon_1) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ nontrivial solution of (P_∞) and a sequence (τ_n, ζ_n) which is a $(PS)_{\tilde{c}}$ for I_∞ such that, up to a subsequence of (u_n, v_n),*

$$\tau_n(x) = u_n(x) - R_n^{(N-2)/2}\Upsilon_0(R_n(x - x_n)) + o_n(1)$$

and

$$\zeta_n(x) = v_n(x) - R_n^{(N-2)/2}\Upsilon_1(R_n(x - x_n)) + o_n(1).$$

Proof. Let $(u_n, v_n) \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for the functional I_∞, i.e,

$$I_\infty(u_n, v_n) \to c \ \text{ and } \ I'_\infty(u_n, v_n) \to 0. \tag{1.2.1}$$

From Lemma 1.1.1, (i), we get that (u_n, v_n) is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$. Since $u_n \rightharpoonup 0$, $v_n \rightharpoonup 0$ by hypotheses and $u_n \nrightarrow 0$, $v_n \nrightarrow 0$ it follows from Lemma 1.1.1 (iii) that

$$c \geq \frac{1}{N}S_K^{N/2}.$$

Note that from (1.1.1) we obtain

$$c + o_n(1) = I_\infty(u_n, v_n) - \frac{1}{2^*}I'_\infty(u_n, v_n)(u_n, v_n) = \frac{1}{N}\int_{\mathbb{R}^N}[|\nabla u_n|^2 + |\nabla v_n|^2]dx,$$

which implies

$$\int_{\mathbb{R}^N}[|\nabla u_n|^2 + |\nabla v_n|^2]dx = S_K^{N/2}. \tag{1.2.2}$$

Let L be a integer such that $B_2(0)$ is covered by L balls of radius 1, $(R_n) \subset \mathbb{R}$, $(x_n) \subset \mathbb{R}^N$ such that

$$\sup_{y\in\mathbb{R}^N}\int_{B_{R_n^{-1}}(y)}[|\nabla u_n|^2 + |\nabla v_n|^2]dx = \int_{B_{R_n^{-1}}(x_n)}[|\nabla u_n|^2 + |\nabla v_n|^2]dx = \frac{S_K^{N/2}}{2L}$$

and define the vectors

$$(w_n(x), z_n(x)) = \left(R_n^{(2-N)/2} u_n\left(\frac{x}{R_n} + x_n\right), R_n^{(2-N)/2} v_n\left(\frac{x}{R_n} + x_n\right)\right).$$

Using a change of variables, we can prove that

$$\int_{B_1(0)} [|\nabla w_n|^2 + \nabla z_n|^2] dx = \frac{S_K^{N/2}}{2L} = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} [|\nabla w_n|^2 + |\nabla z_n|^2] dx.$$

Now, for each $(\Phi_1, \Phi_2) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, we define

$$(\tilde{\Phi}_{1,n}, \tilde{\Phi}_{2,n})(x) = (R_n^{(N-2)/2} \Phi_1(R_n(x - x_n)), R_n^{(N-2)/2} \Phi_2(R_n(x - x_n)))$$

which satisfies

$$\int_{\mathbb{R}^N} [\nabla u_n \nabla \tilde{\Phi}_{1,n} + \nabla v_n \nabla \tilde{\Phi}_{2,n} dx] dx = \int_{\mathbb{R}^N} [\nabla w_n \nabla \Phi_1 + \nabla z_n \nabla \Phi_2] dx \tag{1.2.3}$$

and

$$\int_{\mathbb{R}^N} [K_u(u_n, v_n) \tilde{\Phi}_{1,n} + K_v(u_n, v_n) \tilde{\Phi}_{2,n}] dx = \int_{\mathbb{R}^N} [K_w(w_n, z_n) \Phi_1 + K_z(w_n, z_n) \Phi_2] dx, \tag{1.2.4}$$

where we conclude that

$$I_\infty(w_n, z_n) \to c \quad \text{and} \quad I'_\infty(w_n, z_n) \to 0. \tag{1.2.5}$$

From Lemma 1.1.1, there exists $(\Upsilon_0, \Upsilon_1) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ such that, up to a subsequence, $(w_n, z_n) \rightharpoonup (\Upsilon_0, \Upsilon_1)$ in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ and $I'_\infty(\Upsilon_0, \Upsilon_1) = 0$.

As a consequence of [13, Lemma 6], we get

$$\int_{\mathbb{R}^N} K(w_n, z_n) \phi dx \to \int_{\mathbb{R}^N} K(\Upsilon_0, \Upsilon_1) \phi dx + \sum_{j \in J} \phi(x_j) \nu_j, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N) \tag{1.2.6}$$

and

$$|\nabla w_n|^2 + |\nabla z_n|^2 \rightharpoonup \mu + \sigma \geq |\nabla \Upsilon_0|^2 + |\nabla \Upsilon_1|^2 + \sum_{j \in J} \phi(x_j) \mu_j + \sum_{j \in J} \phi(x_j) \sigma_j, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N),$$

for some $\{x_j\}_{j \in J} \subset \mathbb{R}^N$ and for some $\{\nu_j\}_{j \in J}$, $\{\mu_j\}_{j \in J}$, $\{\sigma_j\}_{j \in J} \subset \mathbb{R}^+$.

Since $S_K \nu_j^{2/2^*} \leq \mu_j + \sigma_j$, we can conclude that J is finite. From now on, we denote by $J = \{1, 2, ..., m\}$ and $\Gamma \subset \mathbb{R}^N$ the set given by

$$\Gamma = \{x_j \in \{x_j\}_{j \in J}; |x_j| > 1\}, \quad (x_j \text{ given by } (1.2.6)).$$

We are going to show that $(\Upsilon_0, \Upsilon_1) \neq (0, 0)$. Suppose, by contradiction, that $(\Upsilon_0, \Upsilon_1) = (0, 0)$. Then, by (1.2.6) we have

$$\int_{\mathbb{R}^N} K(w_n, z_n) \phi dx \to 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N \setminus \{x_1, x_2, ..., x_m\}). \tag{1.2.7}$$

Since $(\phi_{1,n}, \phi_{2,n}) = (\phi w_n, \phi z_n)$, with $\phi \in C_0^\infty(\mathbb{R}^N \setminus \{x_1, x_2, ..., x_m\})$, is bounded, we obtain

$$I'_\infty(w_n, z_n)(\phi_{1,n}, \phi_{2,n}) = o_n(1),$$

that is,

$$\int_{\mathbb{R}^N}[\nabla w_n \nabla \phi_{1,n} + \nabla z_n \nabla \phi_{2,n} dx] - \frac{1}{2^*}\int_{\mathbb{R}^N}[K_w(w_n,z_n)\phi_{1,n} + K_z(w_n,z_n)\phi_{2,n} dx = o_n(1).$$

Using the definition of $(\phi_{1,n},\phi_{2,n})$ and (1.1.1) , we have

$$\int_{\mathbb{R}^N}[|\nabla w_n|^2 + |\nabla z_n|^2]\phi dx + \int_{\mathbb{R}^N}[w_n \nabla w_n \nabla \phi + z_n \nabla z_n \nabla \phi] dx - \int_{\mathbb{R}^N} K(w_n,z_n)\phi dx = o_n(1).$$

Then,

$$\int_{\mathbb{R}^N}[|\nabla w_n|^2 + |\nabla z_n|^2]\phi dx \leq \int_{\mathbb{R}^N}[|w_n||\nabla w_n||\nabla \phi| + |z_n||\nabla z_n||\nabla \phi|] dx + \int_{\mathbb{R}^N} K(w_n,z_n)\phi dx.$$

Using Hölder inequality we get

$$\int_{\mathbb{R}^N}[|\nabla w_n|^2 + |\nabla z_n|^2]\phi dx \leq |\nabla w_n|_2\Big(\int_{\mathbb{R}^N}|w_n|^2|\nabla\phi|^2 dx\Big)^{1/2}$$
$$+ \quad |\nabla z_n|_2\Big(\int_{\mathbb{R}^N}|z_n|^2|\nabla\phi|^2 dx\Big)^{1/2} + \int_{\mathbb{R}^N} K(w_n,z_n)\phi dx.$$

Since there exists $R>0$ such that $supp\phi \subset B_R(0)$, we have

$$\int_{\mathbb{R}^N}[|\nabla w_n|^2 + |\nabla z_n|^2]\phi dx \leq C|\nabla w_n|_2\Big(\int_{B_R(0)}|w_n|^2 dx\Big)^{1/2}$$
$$+ \quad C|\nabla z_n|_2\Big(\int_{B_R(0)}|z_n|^2 dx\Big)^{1/2} + \int_{\mathbb{R}^N} K(w_n,z_n)\phi dx = o_n(1).$$

Since (w_n,z_n) is bounded in $D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$, from compact embedding in $L^2(\mathbb{R}^N)$ and (1.2.7), we obtain

$$\int_{\mathbb{R}^N}[|\nabla w_n|^2 + |\nabla z_n|^2]\phi dx \to 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N \setminus \{x_1,x_2,...,x_m\}). \tag{1.2.8}$$

Let $\rho \in \mathbb{R}$ be a number that satisfies $0<\rho<\min\{dist(\Gamma,\bar{B}_1(0)),1\}$. We will show that

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}[|\nabla w_n|^2 + |\nabla z_n|^2]\phi dx \to 0. \tag{1.2.9}$$

We consider $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \phi(x) \leq 1$ and $\phi(x)=1$ if $x \in B_{1+\rho}(0)$. If $\tilde{\phi} = \phi|_{\mathbb{R}^N\setminus\{x_1,\ldots,x_m\}}$, follows by (1.2.8) that

$$\int_{\mathbb{R}^N}[|\nabla w_n|^2 + |\nabla z_n|^2]\tilde{\phi} dx \to 0.$$

Since

$$0 \leq \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}[|\nabla w_n|^2 + |\nabla z_n|^2] dx \leq \int_{B_{1+\rho}(0)}[|\nabla w_n|^2 + |\nabla z_n|^2] dx$$
$$= \int_{B_{1+\rho}(0)}[|\nabla w_n|^2 + |\nabla z_n|^2]\tilde{\phi} dx \leq \int_{\mathbb{R}^N}[|\nabla w_n|^2 + |\nabla z_n|^2]\tilde{\phi} dx,$$

we have that (1.2.9) is true.

Let $\Psi \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \leq \Psi(x) \leq 1$ for all $x \in \mathbb{R}^N$ and

$$\Psi(x) = \begin{cases} 1, & x \in B_{1+\frac{\rho}{3}}(0), \\ 0, & x \in B^c_{1+\frac{2\rho}{3}}(0) \end{cases}$$

and consider the sequence $(\Psi_{1,n}, \Psi_{2,n})$ given by $(\Psi_{1,n}, \Psi_{2,n})(x) = (\Psi(x)w_n(x), \Psi(x)z_n(x))$.

Note that

$$\begin{aligned} & \int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} [|\nabla \Psi_{1,n}|^2 + |\nabla \Psi_{2,n}|^2] dx \\ \leq \; & 4\int_{[B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)]^2} |\Psi|^2 |\nabla w_n|^2 dx + 4\int_{[B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)]^2} |\Psi|^2 |\nabla z_n|^2 dx \\ + \; & 4\int_{[B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)]^2} |w_n|^2 |\nabla \Psi|^2 dx + 4\int_{[B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)]^2} |z_n|^2 |\nabla \Psi|^2 dx \end{aligned}$$

From (1.2.9) we obtain

$$\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} [|\nabla \Psi_{1,n}|^2 + |\nabla \Psi_{2,n}|^2] dx \to 0. \tag{1.2.10}$$

Since $(\Psi_{1,n}, \Psi_{2,n})$ is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, we derive that

$$\begin{aligned} & \int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \nabla w_n \nabla \Psi_{1,n} dx + \int_{B_{1+\frac{\rho}{3}}(0)} \nabla w_n \nabla \Psi_{1,n} dx \\ + & \int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \nabla z_n \nabla \Psi_{2,n} dx + \int_{B_{1+\frac{\rho}{3}}(0)} \nabla z_n \nabla \Psi_{2,n} dx \\ - & \frac{1}{2^*}\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \Psi_{1,n} K_w(w_n, z_n) dx - \frac{1}{2^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Psi_{1,n} K_w(w_n, z_n) dx \\ - & \frac{1}{2^*}\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \Psi_{2,n} K_z(w_n, z_n) dx - \frac{1}{2^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Psi_{2,n} K_z(w_n, z_n) dx = o_n(1). \end{aligned}$$

From definition of Ψ we have

$$\begin{aligned} o_n(1) = & \int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \nabla w_n \nabla \Psi_{1,n} dx + \int_{B_{1+\frac{\rho}{3}}(0)} |\nabla \Psi_{1,n}|^2 dx \\ + & \int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \nabla z_n \nabla \Psi_{2,n} dx + \int_{B_{1+\frac{\rho}{3}}(0)} |\nabla \Psi_{2,n}|^2 dx \\ - & \frac{1}{2^*}\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \Psi_{1,n} K_w(w_n, z_n) dx - \frac{1}{2^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Psi_{1,n} K_w(\Psi_{1,n}, \Psi_{2,n}) dx \\ - & \frac{1}{2^*}\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \Psi_{2,n} K_z(w_n, z_n) dx - \frac{1}{2^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Psi_{2,n} K_z(\Psi_{1,n}, \Psi_{2,n}) dx. \end{aligned} \tag{1.2.11}$$

Note that from Hölder inequality and (1.2.10) we get

$$\int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \nabla w_n \nabla \Psi_{1,n} dx + \int_{B_{1+\rho}(0)\backslash B_{1+\frac{\rho}{3}}(0)} \nabla z_n \nabla \Psi_{2,n} dx \to 0, \tag{1.2.12}$$

when $n \to \infty$.

Moreover, from direct calculations we have

$$\frac{1}{2^*}\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} \Psi_{1,n}K_w(w_n,z_n)dx + \frac{1}{2^*}\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} \Psi_{2,n}K_z(w_n,z_n)dx = o_n(1). \quad (1.2.13)$$

From (1.2.11), (1.2.12) and (1.2.13) we obtain

$$\begin{aligned} &\int_{B_{1+\frac{\rho}{3}}(0)} |\nabla\Psi_{1,n}|^2dx + \int_{B_{1+\frac{\rho}{3}}(0)} |\nabla\Psi_{2,n}|^2dx - \frac{1}{2^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Psi_{1,n}K_w(\Psi_{1,n},\Psi_{2,n})dx \\ - &\frac{1}{2^*}\int_{B_{1+\frac{\rho}{3}}(0)} \Psi_{2,n}K_z(\Psi_{1,n},\Psi_{2,n})dx = o_n(1). \end{aligned} \quad (1.2.14)$$

Note that

$$\begin{aligned} \int_{\mathbb{R}^N}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx &= \int_{B_{1+\rho}(0)}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx \\ &= \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2] \\ &+ \int_{B_{1+\frac{\rho}{3}}(0)}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx \\ &= o_n(1) + \int_{B_{1+\frac{\rho}{3}}(0)}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx \end{aligned}$$

and using (1.1.1), we get

$$\begin{aligned} \int_{\mathbb{R}^N} K(\Psi_{1,n},\Psi_{2,n})dx &= \int_{B_{1+\rho}(0)} K(\Psi_{1,n},\Psi_{2,n})dx \\ &= \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} K(\Psi_{1,n},\Psi_{2,n})dx + \int_{B_{1+\frac{\rho}{3}}(0)} K(\Psi_{1,n},\Psi_{2,n})dx \end{aligned}$$

Then we conclude that

$$\int_{\mathbb{R}^N}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx - \int_{\mathbb{R}^N} K(\Psi_{1,n},\Psi_{2,n})dx = o_n(1).$$

From definition of S_K, we have

$$\begin{aligned} &\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2\left[1-\left(\frac{1}{S_K^{2^*/2}}\right)[\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]^{2^*-2}\right] \\ = &[\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2] - \frac{1}{S_K^{2^*/2}}[\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]^{2^*} \\ \leq &\int_{\mathbb{R}^N}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx - \int_{\mathbb{R}^N} K(\Psi_{1,n},\Psi_{2,n})dx = o_n(1). \end{aligned} \quad (1.2.15)$$

Note that

$$\begin{aligned} \|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2 &= \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx + \int_{B_{1+\frac{\rho}{3}}(0)}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx \\ &= o_n(1) + \int_{B_{1+\frac{\rho}{3}}(0)}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx. \end{aligned}$$

Since $\Phi_{1,n} = w_n$, $\Phi_{2,n} = z_n$ in $B_{1+\frac{\rho}{3}(0)}$ and that $B_{1+\frac{\rho}{3}(0)} \subset B_2(0)$, we obtain

$$\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2 \leq o_n(1) + \int_{B_2(0)}[|\nabla\Psi_{1,n}|^2+|\nabla\Psi_{2,n}|^2]dx,$$

which implies

$$\begin{aligned}\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2 &\leq o_n(1)+\int_{\bigcup_{k=1}^{L}B_1(y_k)}[|\nabla w_n|^2+|\nabla z_n|^2]dx\\ &\leq o_n(1)+\sum_{k=1}^{L}\int_{B_1(y_k)}[|\nabla w_n^2+|\nabla z_n|^2]dx\\ &\leq o_n(1)+L\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}[|\nabla w_n|^2+|\nabla z_n|^2]dx\leq o_n(1)+\frac{S_K^{N/2}}{2}.\end{aligned}$$

Then,

$$\left(\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2\right)^{1/2}\leq o_n(1)+\frac{S_K^{N/4}}{2^{1/2}}$$

implies

$$\left(\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2\right)^{(2^*-2)/2}\leq o_n(1)+\left(\frac{S_K^{N/4}}{2^{1/2}}\right)^{2^*-2}. \tag{1.2.16}$$

Using (1.2.15) and (1.2.16), we have that

$$\begin{aligned}&[\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]\left[1+o_n(1)-\frac{1}{S_K^{2^*/2}}\left(\frac{S_K^{N/4}}{2^{1/2}}\right)^{2^*-2}\right]\\ =\;&[\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]\left\{1+\frac{1}{S_K^{2^*/2}}\left[o_n(1)-\left(\frac{S_K^{N/4}}{2^{1/2}}\right)^{2^*-2}\right]\right\}\\ \leq\;&[\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]\left[1-\frac{1}{S_K^{2^*/2}}[\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]^{2^*-2}\right]=o_n(1).\end{aligned}$$

Using the equality

$$\frac{N}{4}(2^*-2)-\frac{2^*}{2}=\frac{N}{4}\left(\frac{4}{N-2}\right)-\frac{N}{N-2}=0,$$

implies

$$[\|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2]\left[1-\left(\frac{1}{2}\right)^{(2^*-2)/2}\right]\leq o_n(1),$$

and we conclude that $(\Psi_{1,n},\Psi_{2,n})\to(0,0)$ in $D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$.

Since $w_n=\Psi_{1,n}$, $z_n=\Psi_{2,n}$ in $B_1(0)$, we obtain

$$0\leq\int_{B_1(0)}[|\nabla w_n|^2+|\nabla z_n|^2]dx\leq|\Psi_{1,n}\|^2+\|\Psi_{2,n}\|^2,$$

which implies

$$\int_{B_1(0)}[|\nabla w_n|^2+|\nabla z_n|^2]dx\to 0\;\text{ when }\; n\to\infty.$$

But this last convergence is a contradiction with

$$\int_{B_1(0)}[|\nabla w_n|^2+|\nabla z_n|^2]dx=\frac{S_K^{N/2}}{2L},\quad\forall n\in\mathbb{N}.$$

Then, $(\Upsilon_0,\Upsilon_1)\neq(0,0)$. Now we are going to show that there is (τ_n,ζ_n) in $D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$ such that (τ_n,ζ_n) is a $(PS)_{\tilde{c}}$ sequence for I_∞ satisfying

$$\tau_n(x)=u_n(x)-R_n^{(N-2)/2}\Upsilon_0(R_n(x-x_n))+o_n(1),$$

$$\zeta_n(x) = v_n(x) - R_n^{(N-2)/2}\Upsilon_1(R_n(x-x_n)) + o_n(1),$$

for some subsequence of (u_n, v_n) that still will be denoted by (u_n, v_n). For this, we consider $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \psi(x) \le 1$ for all $x \in \mathbb{R}^N$ and

$$\psi(x) = \begin{cases} 1, & \text{if } \ x \in B_1(0), \\ 0, & \text{if } \ x \in B_2^c(0) \end{cases}$$

and consider (τ_n, ζ_n) a sequence defined by

$$\tau_n(x) = u_n(x) - R_n^{(N-2)/2}\Upsilon_0(R_n(x-x_n))\psi(\bar{R}_n(x-x_n)), \tag{1.2.17}$$

$$\zeta_n(x) = v_n(x) - R_n^{(N-2)/2}\Upsilon_1(R_n(x-x_n))\psi(\bar{R}_n(x-x_n)), \tag{1.2.18}$$

where $(\bar{R}_n)$ satisfies $\tilde{R}_n = \dfrac{R_n}{\bar{R}_n} \to \infty$. From (1.2.17) and (1.2.18), we obtain

$$R_n^{(2-N)/2}\tau_n(x) = R_n^{(2-N)/2}u_n(x) - \Upsilon_0(R_n(x-x_n))\psi(\bar{R}_n(x-x_n))$$

and

$$R_n^{(2-N)/2}\zeta_n(x) = R_n^{(2-N)/2}v_n(x) - \Upsilon_1(R_n(x-x_n))\psi(\bar{R}_n(x-x_n)).$$

Making a change of variables, we conclude

$$R_n^{(2-N)/2}\tau_n\left(\frac{z}{R_n} + x_n\right) = R_n^{(2-N)/2}u_n\left(\frac{z}{R_n} + x_n\right) - \Upsilon_0\psi\left(\frac{z}{\tilde{R}_n}\right).$$

and

$$R_n^{(2-N)/2}\zeta_n\left(\frac{z}{R_n} + x_n\right) = R_n^{(2-N)/2}v_n\left(\frac{z}{R_n} + x_n\right) - \Upsilon_1\psi\left(\frac{z}{\tilde{R}_n}\right).$$

Now we define

$$\widetilde{\tau}_n = R_n^{(2-N)/2}\tau_n\left(\frac{z}{R_n} + x_n\right)$$

and

$$\widetilde{\zeta}_n = R_n^{(2-N)/2}\zeta_n\left(\frac{z}{R_n} + x_n\right).$$

Since

$$w_n(x) = R_n^{(2-N)/2}u_n\left(\frac{x}{R_n} + x_n\right)$$

and

$$z_n(x) = R_n^{(2-N)/2}v_n\left(\frac{x}{R_n} + x_n\right),$$

it holds

$$\widetilde{\tau}_n(z) = w_n(z) - \Upsilon_0(z)\psi\left(\frac{z}{\tilde{R}_n}\right) \tag{1.2.19}$$

and

$$\widetilde{\zeta}_n(z) = \zeta_n(z) - \Upsilon_1(z)\psi\left(\frac{z}{\tilde{R}_n}\right). \tag{1.2.20}$$

If

$$\psi_n(z) = \psi\left(\frac{z}{\tilde{R}_n}\right) \tag{1.2.21}$$

we have that

$$\psi_n(z) = \begin{cases} 1, & \text{if } \ z \in B_{\tilde{R}_n}(0), \\ 0, & \text{if } \ z \in B_{2\tilde{R}_n}^c(0). \end{cases}$$

From (1.2.19), (1.2.20) and (1.2.21), we derive that

$$\widetilde{\tau}_n(z) = w_n(z) - \Upsilon_0(z)\psi_n(z)$$

and

$$\widetilde{\zeta}_n(z) = z_n(z) - \Upsilon_1(z)\psi_n(z).$$

Since $\tilde{R}_n \to \infty$, we get that $\Upsilon_i\psi_n \to \Upsilon_i$ in $D^{1,2}(\mathbb{R}^N)$, $i = 0, 1$. Then

$$\widetilde{\tau}_n(z) = w_n(z) - \Upsilon_0(z) + o_n(1) \tag{1.2.22}$$

and

$$\widetilde{\zeta}_n(z) = z_n(z) - \Upsilon_1(z) + o_n(1). \tag{1.2.23}$$

To finish the proof, it is enough to show that (τ_n, ζ_n) is a $(PS)_{\tilde{c}}$ sequence for I_∞. Note that making a change of variables we get

$$I_\infty(\tau_n, \zeta_n) = I_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n)$$

Using (1.2.22) and (1.2.23) and applying [20, Lemma 4.6], [13, Lemma 8] and (1.2.5), we have

$$I_\infty(\tau_n, \zeta_n) \quad = \quad I_\infty(w_n, z_n) - I_\infty(\Upsilon_0, \Upsilon_1) + o_n(1) = \widetilde{c} + o_n(1),$$

where $\widetilde{c} = c - I_\infty(\Upsilon_0, \Upsilon_1)$.

Now, since

$$0 \leq \|I'_\infty(\tau_n, \zeta_n)\|_{D'} \leq \|I'_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n)\|_{D'},$$

it is sufficient to prove that $\|I'_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n)\|_{D'} \to 0$ which is equivalent to show that

$$\|I'_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n) - I'_\infty(w_n, z_n) + I'_\infty(\Upsilon_0, \Upsilon_1)\|_{D'} \to 0. \tag{1.2.24}$$

But the last convergence is a direct consequence of [13, Lemma 8]. □

The next result is a version for a gradient system in $\mathbb{R}^N$ of the result due to Struwe that can be found in [29].

Theorem 1.2.2. *(A global compactness result) Let (u_n, v_n) be a $(PS)_c$ sequence for I with $u_n \rightharpoonup u_0$ in $D^{1,2}(\mathbb{R}^N)$ and $v_n \rightharpoonup v_0$ in $D^{1,2}(\mathbb{R}^N)$. Then, up to a subsequence, (u_n, v_n) satisfies either,*

(a) $(u_n, v_n) \to (u_0, v_0)$ in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ or,

(b) there exist $k \in \mathbb{N}$ and nontrivial solutions $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), ..., (z_0^k, \zeta_0^k)$ of the system (S_∞), such that

$$\|u_n\|^2 + \|v_n\|^2 \to \|u_0\|^2 + \|v_0\|^2 + \sum_{j=1}^{k}[\|z_0^j\|^2 + \|\zeta_0^j\|^2]$$

and

$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j).$$

Proof. From the weak convergence and a density argument, we have that (u_0, v_0) is a critical point of I. Suppose that $u_n \nrightarrow u_0$, $v_n \nrightarrow v_0$ in $D^{1,2}(\mathbb{R}^N)$ and let $(w_n^1, z_n^1) \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be the sequence given by $w_n^1 = u_n - u_0$ and $z_n^1 = v_n - v_0$. Then, $w_n^1 \rightharpoonup 0$, $z_n^1 \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^N)$ and $w_n^1 \nrightarrow 0$, $z_n^1 \nrightarrow 0$ in $D^{1,2}(\mathbb{R}^N)$.

Applying [20, Lemma 4.6] and [13, Lemma 8], we obtain

$$I_\infty(w_n^1, z_n^1) = I(u_n, v_n) - I(u_0, v_0) + o_n(1) \tag{1.2.25}$$

and

$$I'_\infty(w_n^1, z_n^1) = I'(u_n, v_n) - I'(u_0, v_0) + o_n(1). \tag{1.2.26}$$

Then, we conclude from (1.2.25) and (1.2.26) that (w_n^1, z_n^1) is a $(PS)_{c_1}$ sequence for I_∞. Hence, by Lemma 1.2.1, there are sequences $(R_{n,1}) \subset \mathbb{R}$, $(x_{n,1}) \subset \mathbb{R}^N$, $(z_0^1, \zeta_0^1) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ nontrivial solution of the system (P_∞) and a $(PS)_{c_2}$ sequence $(w_n^2, z_n^2) \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ for I_∞ such that

$$w_n^2(x) = w_n^1(x) - R_{n,1}^{(N-2)/2} z_0^1(R_{n,1}(x - x_{n,1})) + o_n(1)$$

and

$$z_n^2(x) = z_n^1(x) - R_{n,1}^{(N-2)/2} \zeta_0^1(R_{n,1}(x - x_{n,1})) + o_n(1).$$

If we define

$$\Phi_n^1(x) = R_{n,1}^{(2-N)/2} w_n^1\left(\frac{x}{R_{n,1}} + x_{n,1}\right), \tag{1.2.27}$$

$$\Psi_n^1(x) = R_{n,1}^{(2-N)/2} z_n^1\left(\frac{x}{R_{n,1}} + x_{n,1}\right) \tag{1.2.28}$$

and

$$\widetilde{w}_n^2(x) = R_{n,1}^{(2-N)/2} w_n^2\left(\frac{x}{R_{n,1}} + x_{n,1}\right),$$
$$\widetilde{z}_n^2(x) = R_{n,1}^{(2-N)/2} z_n^2\left(\frac{x}{R_{n,1}} + x_{n,1}\right),$$

we get

$$\widetilde{w}_n^2(x) = \Phi_n^1(x) - z_0^1(x) + o_n(1), \tag{1.2.29}$$

$$\widetilde{z}_n^2(x) = \Psi_n^1(x) - \zeta_0^1(x) + o_n(1) \tag{1.2.30}$$

and

$$\|\Phi_n^1\| = \|w_n^1\|, \quad \|\Psi_n^1\| = \|z_n^1\| \quad \text{and} \quad \int_{\mathbb{R}^N} K(\Phi_n^1, \Psi_n^1)dx = \int_{\mathbb{R}^N} K(w_n^1, z_n^1)dx. \tag{1.2.31}$$

Hence,

$$I_\infty(\Phi_n^1, \Psi_n^1) = I_\infty(w_n^1, z_n^1) \tag{1.2.32}$$

and

$$I'_\infty(\Phi_n^1, \Psi_n^1) \to 0 \quad \text{in} \quad (D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N))'. \tag{1.2.33}$$

From (1.2.32), (1.2.33) and from item (a) by Lemma 1.1.1, we have that (Φ_n^1, Ψ_n^1) is a bounded sequence in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ and, up to a subsequence,

$$\Phi_n^1 \rightharpoonup z_0^1, \quad \Psi_n^1 \rightharpoonup \zeta_0^1 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N). \tag{1.2.34}$$

Applying [20, Lemma 4.6] and [13, Lemma 8] again, we obtain

$$\begin{aligned} I_\infty(\widetilde{w}_n^2, \widetilde{z}_n^2) &= I_\infty(\Phi_n^1, \Psi_n^1) - I_\infty(z_0^1, \zeta_0^1) + o_n(1) \\ &= I(u_n, v_n) - I(u_0, v_0) - I_\infty(z_0^1, \zeta_0^1) + o_n(1). \end{aligned} \tag{1.2.35}$$

and

$$I'_\infty(\widetilde{w}_n^2, \widetilde{z}_n^2) = I'_\infty(\Phi_n^1, \Psi_n^1) - I'_\infty(z_0^1, \zeta_0^1) + o_n(1). \tag{1.2.36}$$

If $\widetilde{w}_n^2, \widetilde{z}_n^2 \to 0$ in $D^{1,2}(\mathbb{R}^N)$, the proof is over for $k = 1$, because in this case, we have

$$\|u_n\|^2 + \|v_n\|^2 \to \|u_0\|^2 + \|v_0\|^2 + \|z_0^1\|^2 + \zeta_0^1\|^2.$$

Moreover, by continuity of I_∞, we get

$$I(u_n, v_n) \to I(u_0, v_0) + I_\infty(z_0^1, \zeta_0^1).$$

If $\widetilde{w}_n^2 \nrightarrow 0, \widetilde{z}_n^2 \nrightarrow 0$ in $D^{1,2}(\mathbb{R}^N)$, using (1.2.29), (1.2.30) and (1.2.34) that $\widetilde{w}_n^2, \widetilde{z}_n^2 \rightharpoonup 0$ $D^{1,2}(\mathbb{R}^N)$, by (1.2.35) and (1.2.36), we conclude that $(\widetilde{w}_n^2, \widetilde{z}_n^2)$ is a $(PS)_{c_2}$ sequence for I_∞. By Lemma 1.2.1, there are sequences $(R_{n,2}) \subset \mathbb{R}$, $(x_{n,2}) \subset \mathbb{R}^N$, $(z_0^2, \zeta_0^2) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ nontrivial solutions of (S_∞) and a $(PS)_{c_3}$ sequence $(w_n^3, z_n^3) \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ for I_∞ such that

$$w_n^3(x) = \widetilde{w}_n^2(x) - R_{n,2}^{(N-2)/2} z_0^2(R_{n,2}(x - x_{n,2})) + o_n(1),$$

$$z_n^3(x) = \widetilde{z}_n^2(x) - R_{n,2}^{(N-2)/2} \zeta_0^2(R_{n,2}(x - x_{n,2})) + o_n(1).$$

If

$$\Phi_n^2(x) = R_{n,2}^{(2-N)/2} \widetilde{w}_n^2\left(\frac{x}{R_{n,2}} + x_{n,2}\right),$$

$$\Psi_n^2(x) = R_{n,2}^{(2-N)/2} \widetilde{z}_n^2\left(\frac{x}{R_{n,2}} + x_{n,2}\right)$$

and

$$\widetilde{w}_n^3(x) = R_{n,2}^{(2-N)/2} w_n^3\left(\frac{x}{R_{n,2}} + x_{n,2}\right),$$

$$\widetilde{z}_n^3(x) = R_{n,2}^{(2-N)/2} z_n^3\left(\frac{x}{R_{n,2}} + x_{n,2}\right),$$

we have that

$$\widetilde{w}_n^3(x) = \Phi_n^2(x) - z_0^2(x) + o_n(1), \tag{1.2.37}$$

$$\widetilde{z}_n^3(x) = \Psi_n^2(x) - \zeta_0^2(x) + o_n(1). \tag{1.2.38}$$

Arguing as before, we conclude

$$\begin{aligned} \|\widetilde{w}_n^3\|^2 + \|\widetilde{z}_n^3\|^2 &= \|u_n\|^2 + \|v_n\|^2 - \|u_0\|^2 - \|v_0\|^2 - \|z_0^1\|^2 - \|\zeta_0^1\|^2 \\ &- \|z_0^2\|^2 - \|\zeta_0^2\|^2 + o_n(1), \end{aligned} \tag{1.2.39}$$

$$I_\infty(\widetilde{w}_n^3,\widetilde{z}_n^3) = I(u_n,v_n) - I(u_0,v_0) - I_\infty(z_0^1,\zeta_0^1) - I_\infty(z_0^2,\zeta_0^2) + o_n(1), \tag{1.2.40}$$

and

$$I'_\infty(\widetilde{w}_n^3,\widetilde{z}_n^3) = I'_\infty(\Phi_n^2,\Psi_n^2) - I'_\infty(z_0^2,\zeta_0^2) + o_n(1). \tag{1.2.41}$$

If $\widetilde{w}_n^3, \widetilde{z}_n^3 \to 0$ in $D^{1,2}(\mathbb{R}^N)$, the proof is over with $k=2$, because $\|\widetilde{w}_n^3\|^2 \to 0$, $\|\widetilde{z}_n^3\|^2 \to 0$ and from (1.2.39), we have

$$\|u_n\|^2 + \|v_n\|^2 \to \|u_0\|^2 + \|v_0\|^2 + \sum_{j=1}^{2}[\|z_0^j\|^2 + \|\zeta_0^j\|^2].$$

Moreover, by continuity of I_∞, we have that $I_\infty(\widetilde{w}_n^3,\widetilde{z}_n^3) \to 0$. Now using (1.2.40) we get

$$I(u_n,v_n) \to I(u_0,v_0) + \sum_{j=1}^{2} I_\infty(z_0^j,\zeta_0^j).$$

If $\widetilde{w}_n^3, \widetilde{z}_n^3 \nrightarrow 0$ in $D^{1,2}(\mathbb{R}^N)$, we can repeat the same previous arguments to find (z_0^1,ζ_0^1), (z_0^2,ζ_0^2),..., $(z_0^{k-1},\zeta_0^{k-1})$ nontrivial solutions for the system (S_∞) satisfying

$$\|\widetilde{w}_n^k\|^2 + \|\widetilde{z}_n^k\|^2 = \|u_n\|^2 + \|v_n\|^2 - \|u_0\|^2 - \|v_0\|^2 - \sum_{j=1}^{k-1}[\|z_0^j\|^2 - \|\zeta_0^j\|^2] + o_n(1), \tag{1.2.42}$$

and

$$I_\infty(\widetilde{z}_n^k,\widetilde{z}_n^k) = I(u_n,v_n) - I(u_0,v_0) - \sum_{j=1}^{k-1} I_\infty(z_0^j,\zeta_0^j) + o_n(1). \tag{1.2.43}$$

From definition of S_K, we conclude that

$$\left(\int_{\mathbb{R}^N} K(z_0^j,\zeta_0^j)dx\right)^{2/2^*} S_K \le \|z_0^j\|^2 + \|\zeta_0^j\|^2, \quad j = 1,2,...,k-1. \tag{1.2.44}$$

Since (z_0^j,ζ_0^j) is nontrivial solution of (S_∞), for all $j = 1,2,...,k-1$, we get

$$\|z_0^j\|^2 + \|\zeta_0^j\|^2 = \int_{\mathbb{R}^N} K(z_0^j,\zeta_0^j)dx$$

Hence,

$$-\|z_0^j\|^2 - \|\zeta_0^j\|^2 \le -S_K^{N/2}, \quad j = 1,2,...,k-1. \tag{1.2.45}$$

From (1.2.42) and (1.2.45), we have

$$\begin{aligned} & \|\widetilde{w}_n^k\|^2 + \|\widetilde{z}_n^k\|^2 = \|u_n\|^2 + \|v_n\|^2 - \|u_0\|^2 - \|v_0\|^2 \\ - & \sum_{j=1}^{k-1}[\|z_0^j\|^2 + \|\zeta_0^j\|^2] + o_n(1) \\ \le & \|u_n\|^2 + \|v_n\|^2 - \|u_0\|^2 - \|v_0\|^2 - (k-1)S_K^{N/2} + o_n(1). \end{aligned} \tag{1.2.46}$$

Since (u_n,v_n) is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, for k sufficient large, we conclude that $\widetilde{w}_n^k$, $\widetilde{z}_n^k \to 0$ in $D^{1,2}(\mathbb{R}^N)$ and the proof is over. □

Corollary 1.2.3. *Let (u_n, v_n) be a $(PS)_c$ sequence for I with $c \in (0, \frac{1}{N}S_K^{N/2})$. Then, up to a subsequence, (u_n, v_n) strongly converges in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$.*

Proof. We have that (u_n, v_n) is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$,

$$u_n \rightharpoonup u_0, \quad v_n \rightharpoonup v_0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N)$$

and by a density argument $I'(u_0, v_0) = 0$. Suppose, by contradiction, that

$$u_n \not\rightharpoonup u_0, \quad v_n \not\rightharpoonup v_0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N).$$

From Theorem 1.2.2, there are $k \in \mathbb{N}$ and nontrivial solutions $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), ..., (z_0^k, \zeta_0^k)$ of the system (S_∞) such that,

$$\|u_n\|^2 + \|v_n\|^2 \to \|u_0\|^2 + \|v_0\|^2 + \sum_{j=1}^{k} [|z_0^j\|^2 + |\zeta_0^j\|^2]$$

and

$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j).$$

Note that by (1.1.1) we have

$$\begin{aligned}
I(u_0, v_0) &= \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} a(x)u_0^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} b(x)v_0^2 dx \\
&- \frac{1}{2^*}\int_{\mathbb{R}^N} K(u_0, v_0)dx \\
&= \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\Big(\int_{\mathbb{R}^N} K(u_0, v_0)dx - \|u_0\|^2 - \|v_0\|^2\Big) \\
&- \frac{1}{2^*}\int_{\mathbb{R}^N} K(u_0, v_0)dx \\
&= \frac{1}{N}\int_{\mathbb{R}^N} K(u_0, v_0)dx \geq 0.
\end{aligned}$$

Then,

$$c = I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) \geq \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) \geq \frac{k}{N}S_K^{N/2} \geq \frac{1}{N}S_K^{N/2},$$

which is a contradiction with $c \in (0, \frac{1}{N}S_K^{N/2})$. □

Corollary 1.2.4. *The functional $I : D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ satisfies the Palais-Smale condition in $(\frac{1}{N}S_K^{N/2}, \frac{2}{N}S_K^{N/2})$.*

Proof. Let (u_n, v_n) be a sequence in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ that satisfies

$$I(u_n, v_n) \to c \in (\frac{1}{N}S_K^{N/2}, \frac{2}{N}S_K^{N/2}) \quad \text{and} \quad I'(u_n, v_n) \to 0.$$

Since (u_n, v_n) is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, up to a subsequence, we have

$$u_n \rightharpoonup u_0, \quad v_n \rightharpoonup v_0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N).$$

Moreover, $I(u_0, v_0) \geq 0$. Suppose, by contradiction, that

$$u_n \nrightarrow u_0, \quad v_n \nrightarrow v_0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N).$$

From Theorem 1.2.2, there are $k \in \mathbb{N}$ and nontrivial solutions $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), ..., (z_0^k, \zeta_0^k)$ of the system (S_∞) such that

$$\|u_n\|^2 + \|v_n\|^2 \to \|u_0\|^2 + \|v_0\|^2 + \sum_{j=1}^{k} [\|z_0^j\|^2 + \|\zeta_0^j\|^2]$$

and

$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) = c.$$

Since $I(u_0, v_0) \geq 0$, then $k = 1$ and z_0^1, ζ_0^1 cannot change of the sign. Hence,

$$c = I(u_0, v_0) + I_\infty(z_0^1, \zeta_0^1) = I(u_0, v_0) + \frac{1}{N} S_K^{N/2}.$$

From definition of S_K, $I'(u_0, v_0)(u_0, v_0) = 0$ and

$$I(u_0, v_0) = \frac{1}{N} \int_{\mathbb{R}^N} K(u_0, v_0) dx,$$

we have,

$$\frac{2}{N} S_K^{N/2} \leq I(u_0, v_0) + \frac{1}{N} S_K^{N/2} = c,$$

which contradicts the fact that $c \in (\frac{1}{N} S_K^{N/2}, \frac{2}{N} S_K^{N/2})$. □

Corollary 1.2.5. *Let $(u_n, v_n) \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for I with $c \in (\frac{k}{N} S_K^{N/2}, \frac{(k+1)}{N} S_K^{N/2})$, where $k \in \mathbb{N}$. Then, the weak limit (u_0, v_0) of (u_n, v_n) is not the trivial one.*

Proof. Suppose, by contradiction, that $u_0, v_0 \equiv 0$. Since $c > 0$, then $u_n, v_n \nrightarrow 0$ in $D^{1,2}(\mathbb{R}^N)$. From Theorem 1.2.2, up to subsequence, we get

$$\|u_n\|^2 + \|v_n\|^2 \to \|u_0\|^2 + \|v_0\|^2 + \sum_{j=1}^{k} [\|z_0^j\|^2 + \|\zeta_0^j\|^2] = \sum_{j=1}^{k} [\|z_0^j\|^2 + \|\zeta_0^j\|^2]$$

and

$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) = \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) = c \geq \frac{(k+1)}{N} S_K^{N/2},$$

which a contradiction with $c \in (\frac{k}{N} S_K^{N/2}, \frac{(k+1)}{N} S_K^{N/2})$. □

From now on we consider the functional $f : D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$f(u, v) = \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} a(x) u^2 dx + \int_{\mathbb{R}^N} b(x) v^2 dx$$

and the manifold $\mathcal{M} \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ given by

$$\mathcal{M} = \left\{ (u, v) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} K(u, v) dx = 1 \right\}.$$

The next results are direct consequence of the above corollaries.

Lemma 1.2.6. *Let $(u_n, v_n) \subset \mathcal{M}$ be a sequence that satisfies*

$$f(u_n, v_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0.$$

Then, the sequence $(w_n, z_n) \subset D^{1,2}(\mathbb{R}^N)$, where $(w_n, z_n) = (c^{(N-2)/4}u_n, c^{(N-2)/4}v_n)$, satisfies the following limits

$$I(w_n, z_n) \to \frac{1}{N}c^{N/2} \quad \text{and} \quad I'(w_n, z_n) \to 0.$$

Lemma 1.2.7. *Suppose that there are a sequence $(u_n, v_n) \subset \mathcal{M}$ and $c \in (S_K, 2^{2/N}S_K)$ such that*

$$f(u_n, v_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0.$$

Then, up to a subsequence, $u_n \to u$, $v_n \to v$ in $D^{1,2}(\mathbb{R}^N)$, for some $u, v \in D^{1,2}(\mathbb{R}^N)$.

Corollary 1.2.8. *Suppose that there are a sequence $(u_n, v_n) \subset \mathcal{M}$ and $c \in (S_K, 2^{2/N}S_K)$ such that*

$$f(u_n, v_n) \to c \quad \text{and} \quad f'(u_n, v_n) \to 0.$$

Then I has a critical point $(u_0, v_0) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ with $I(u_0, v_0) = \dfrac{1}{N}c^{N/2}$.

1.3 Existence of positive solution to (S_1)

Now we recall some properties on the function $\Phi_{\delta,y}$ given by in (1.0.1). Note that

$$(\Phi_{\delta,y}, \Phi_{\delta,y}) \in \Sigma = \left\{(u, v) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N); u, v \geq 0\right\}. \tag{1.3.1}$$

Moreover, making a change of variable we can prove that

$$\Phi_{\delta,y} \in L^q(\mathbb{R}^N) \quad \text{for} \quad q \in \left(\frac{N}{N-2}, 2^*\right], \quad \forall \delta > 0 \ \text{ and } \ \forall y \in \mathbb{R}^N. \tag{1.3.2}$$

The proof of next result can be seen in [1, Lemma 4].

Lemma 1.3.1. *For each $y \in \mathbb{R}^N$, we have*

(i) $\|\Phi_{\delta,y}\|_{H^{1,\infty}(\mathbb{R}^N)} \to 0$ *when* $\delta \to +\infty$,

(ii) $|\Phi_{\delta,y}|_q \to 0$ *when* $\delta \to 0$, $\forall q \in \left(\dfrac{N}{N-2}, 2^*\right)$,

(iii) $|\Phi_{\delta,y}|_q \to +\infty$ *when* $\delta \to +\infty$, $\forall q \in \left(\dfrac{N}{N-2}, 2^*\right)$.

The proof of next result can be seen in [1, Lemma 5].

Lemma 1.3.2. *For each $\varepsilon > 0$, we have*

$$\int_{\mathbb{R}^N \setminus B_\varepsilon(0)} |\nabla \Phi_{\delta,0}|^2 dx \to 0 \quad \text{when} \quad \delta \to 0.$$

1.3.1 Technical Lemmas

Lemma 1.3.3. *Suppose that $a, b \in L^q(\mathbb{R}^N)$, $\forall q \in [p_1, p_2]$, where $1 < p_1 < \dfrac{N}{2} < p_2$ with $p_2 < 3$ if $N = 3$. Then, for each $\varepsilon > 0$, there are $\underline{\delta} = \underline{\delta}(\varepsilon) > 0$ and $\bar{\delta} = \bar{\delta}(\varepsilon) > 0$ such that*

$$\sup_{y \in \mathbb{R}^N} f(s_o \Phi_{\delta,y}, t_o \Phi_{\delta,y}) < S_K + \varepsilon, \quad \delta \in (0, \underline{\delta}] \cup [\bar{\delta}, \infty).$$

Proof. Consider $y \in \mathbb{R}^N$, $q \in \left(\dfrac{N}{2}, p_2\right]$ and $t \in (1, +\infty)$ with $\dfrac{1}{q} + \dfrac{1}{t} = 1$. Making a direct calculations we have

$$\frac{N}{N-2} < 2t < 2^*. \tag{1.3.3}$$

Since $\Phi_{\delta,b} \in L^d(\mathbb{R}^N), \forall d \in \left(\dfrac{N}{N-2}, 2^*\right)$, we get $|\Phi_{\delta,b}|^2 \in L^t(\mathbb{R}^N)$. Then, using Holder inequality and change of variable, we have

$$\int_{\mathbb{R}^N} a(x) |\Phi_{\delta,b}|^2 dx \leq |a|_q |\Phi_{\delta,0}|^2_{2t}, \quad \forall y \in \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} b(x) |\Phi_{\delta,b}|^2 dx \leq |b|_q |\Phi_{\delta,0}|^2_{2t}, \quad \forall y \in \mathbb{R}^N$$

From item (iii) of Lemma 1.3.1, given $\varepsilon > 0$, there exists $\underline{\delta} = \underline{\delta}(\varepsilon) > 0$ such that

$$\sup_{y \in \mathbb{R}^N} f(s_o \Phi_{\delta,y}, t_o \Phi_{\delta,y}) \leq S_K + \frac{\varepsilon}{2} < S_K + \varepsilon, \quad \forall \delta \in (0, \underline{\delta}].$$

Suppose that $q \in \left[p_1, \dfrac{N}{2}\right)$ with $t \in (1, +\infty)$ and $\dfrac{1}{q} + \dfrac{1}{t} = 1$. Note that $2t - 2^* > 0$ and for $\delta > 1$,

$$|\Phi_{\delta,y}| \in L^\infty(\mathbb{R}^N) \tag{1.3.4}$$

and $|\Phi_{\delta,y}|^{2^*} \in L^1(\mathbb{R}^N)$. Then, $|\Phi_{\delta,y}|^2 \in L^t(\mathbb{R}^N)$. Using Holder inequality with q and t, we get

$$\begin{aligned}
s_o^2 \int_{\mathbb{R}^N} a(x) |\Phi_{\delta,y}|^2 dx &\leq s_o^2 |a|_q \left(\int_{\mathbb{R}^N} |\Phi_{\delta,0}|^{2t} dz \right)^{1/t} \\
&= s_o^2 |a|_q \left(\int_{\mathbb{R}^N} |\Phi_{\delta,0}|^{2^*_s} |\Phi_{\delta,0}|^{2t - 2^*_s} dz \right)^{1/t} \\
&\leq s_o^2 |a|_q |\Phi_{\delta,0}|_\infty^{(2t-2^*)/t} \left(\int_{\mathbb{R}^N} |\Phi_{\delta,0}|^{2^*} dz \right)^{1/t} \leq s_o^2 |a|_q |\Phi_{\delta,0}|_\infty^{(2t-2^*)/t} \\
&\leq s_o^2 |a|_q c^{(2t-2^*)/t} \delta^{((2-N)/2)((2t-2^*)/t)}, \quad \forall y \in \mathbb{R}^N.
\end{aligned}$$

Then, given $\varepsilon > 0$, there is $\bar{\delta} = \bar{\delta}(\varepsilon) > 1$ such that

$$\delta^{((2-N)/2)/2)((2t-2^*)/t)} < \frac{\varepsilon}{2 s_o^2 |a|_q c^{(2t-2^*)/t}}, \quad \forall \delta \in [\bar{\delta}, \infty).$$

Arguing in the same way, we have

$$t_o^2 \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,y}|^2 dx \quad \leq \quad t_o^2 |b|_q c^{(2t-2^*)/t} \delta^{((2-N)/2)((2t-2^*)/t)}, \quad \forall y \in \mathbb{R}^N.$$

Then

$$\begin{aligned} f(s_o\Phi_{\delta,y}, t_o\Phi_{\delta,y}) &= S_K + s_o^2 \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,y}|^2 dx + t_o^2 \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,y}|^2 dx \\ &\leq S + s_o^2 \sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,y}|^2 dx + t_o^2 \sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,y}|^2 dx \\ &\leq S_K + \frac{\varepsilon}{2} < S_K + \varepsilon, \quad \forall y \in \mathbb{R}^N \quad \text{and} \quad \forall \delta \in [\bar{\delta}, \infty). \end{aligned}$$

□

Lemma 1.3.4. *Suppose that $(a, b)_3$ is true. Then,*

$$\sup_{\substack{y\in\mathbb{R}^N \\ \delta\in(0,+\infty)}} f(s_o\Phi_{\delta,y}, t_o\Phi_{\delta,y}) < 2^{2/N} S_K.$$

Proof. Using Holder inequality with $N/2$ and $N/(N-2)$, we get

$$f(s_o\Phi_{\delta,y}, t_o\Phi_{\delta,y}) \leq S_K + s_o^N |a|_{L^{N/2}(\mathbb{R}^N)} + t_o^N |b|_{L^{N/2}(\mathbb{R}^N)}.$$

From $(a, b)_3$ we conclude

$$\sup_{\substack{y\in\mathbb{R}^N \\ \delta\in(0,\infty)}} f(s_o\Phi_{\delta,y}, t_o\Phi_{\delta,y}) \leq S_K + S_K(2^{2/N} - 1) = 2^{2/N} S_K.$$

□

Consider the function

$$\xi(x) = \begin{cases} 0, & \text{if} \quad |x| < 1 \\ 1, & \text{if} \quad |x| \geq 1 \end{cases}$$

and define $\alpha : D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \to \mathbb{R}^{N+1}$ by

$$\alpha(u, v) = \frac{s_o^2 + t_0^2}{S_K} \int_{\mathbb{R}^N} \left(\frac{x}{|x|}, \xi(x)\right)[|\nabla u|^2 + |\nabla v|^2]dx = (\beta(u, v), \gamma(u, v)),$$

where

$$\beta(u, v) = \frac{s_o^2 + t_0^2}{S_K} \int_{\mathbb{R}^N} \frac{x}{|x|}[|\nabla u|^2 + |\nabla v|^2]dx$$

and

$$\gamma(u, v) = \frac{s_o^2 + t_0^2}{S_K} \int_{\mathbb{R}^N} \xi(x)[|\nabla u|^2 + |\nabla v|^2]dx.$$

Lemma 1.3.5. *If $|y| \geq \frac{1}{2}$, then*

$$\beta(\Phi_{\delta,y}, \Phi_{\delta,y}) = \frac{y}{|y|} + o_\delta(1) \quad \text{when} \quad \delta \to 0.$$

Proof. Given $\varepsilon > 0$, from Lemma 1.3.2, there is $\hat{\delta} > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_\varepsilon(y)} |\nabla \Phi_{\delta,y}|^2 dx = \int_{\mathbb{R}^N \setminus B_\varepsilon(0)} |\nabla \Phi_{\delta,0}|^2 dz < \varepsilon, \quad \forall \delta \in (0, \hat{\delta}).$$

Then,

$$\begin{aligned} \left| \beta(\Phi_{\delta,y}, \Phi_{\delta,y}) - \frac{s_o^2 + t_0^2}{S_K} \int_{B_\varepsilon(y)} \frac{x}{|x|} |\nabla \Phi_{\delta,y}|^2 dx \right| &\leq \frac{s_o^2 + t_0^2}{S_K} \int_{\mathbb{R}^N \setminus B_\varepsilon(y)} |\nabla \Phi_{\delta,y}|^2 dx \\ &< \varepsilon, \quad \forall \delta \in (0, \hat{\delta}). \end{aligned} \tag{1.3.5}$$

Note that

$$\left| \frac{y}{|y|} - \frac{s_o^2 + t_0^2}{S_K} \int_{B_\varepsilon(y)} \frac{x}{|x|} |\nabla \Phi_{\delta,y}|^2 dx \right| < 4\varepsilon + \varepsilon = C\varepsilon, \quad \forall \delta \in (0, \hat{\delta}). \tag{1.3.6}$$

From (1.3.5) and (1.3.6), we have

$$\begin{aligned} \left| \beta(\Phi_{\delta,y}, \Phi_{\delta,y}) - \frac{y}{|y|} \right| &= \left| \beta(\Phi_{\delta,y}, \Phi_{\delta,y}) - \frac{s_o^2 + t_0^2}{S_K} \int_{B_\varepsilon(y)} \frac{x}{|x|} |\nabla \Phi_{\delta,y}|^2 dx \right. \\ &+ \left. \frac{s_o^2 + t_0^2}{S_K} \int_{B_\varepsilon(y)} \frac{x}{|x|} |\nabla \Phi_{\delta,y}|^2 dx - \frac{y}{|y|} \right| \\ &\leq \left| \beta(\Phi_{\delta,y}, \Phi_{\delta,y}) - \frac{s_o^2 + t_0^2}{S_K} \int_{B_\varepsilon(y)} \frac{x}{|x|} |\nabla \Phi_{\delta,y}|^2 dx \right| \\ &+ \left| \frac{s_o^2 + t_0^2}{S_K} \int_{B_\varepsilon(y)} \frac{x}{|x|} |\nabla \Phi_{\delta,y}|^2 dx - \frac{y}{|y|} \right| \\ &< \varepsilon + C\varepsilon \\ &= K\varepsilon, \quad \forall \delta \in (0, \hat{\delta}). \end{aligned}$$

□

Lemma 1.3.6. *Suppose that $a, b \in L^q(\mathbb{R}^N)$, $\forall q \in [p_1, p_2]$, where $1 < p_1 < \frac{N}{2} < p_2$ with $p_2 < 3$ if $N = 3$. Then, for every $\delta > 0$, we have*

$$\lim_{|y| \to \infty} f(s_o \Phi_{\delta,y}, t_o \Phi_{\delta,y}) = S_K.$$

Proof. Since

$$f(s_o \Phi_{\delta,y}, t_o \Phi_{\delta,y})) = S_K + s_o^2 \int_{\mathbb{R}^N} a(x) |\Phi_{\delta,y}|^2 dx + t_o^2 \int_{\mathbb{R}^N} b(x) |\Phi_{\delta,y}|^2 dx,$$

we need to prove that

$$\lim_{|y| \to \infty} \int_{\mathbb{R}^N} a(x) |\Phi_{\delta,y}|^2 dx = 0, \quad \forall \delta > 0 \tag{1.3.7}$$

and

$$\lim_{|y| \to \infty} \int_{\mathbb{R}^N} b(x) |\Phi_{\delta,y}|^2 dx = 0, \quad \forall \delta > 0. \tag{1.3.8}$$

Note that given $\varepsilon > 0$, there is $k_0 > 0$ such that

$$\left(\int_{\mathbb{R}^N \setminus B_\rho(0)} a(x)^{N/2} dx \right)^{2/N} < \varepsilon, \quad \forall \rho > k_0.$$

and

$$\left(\int_{\mathbb{R}^N\setminus B_\rho(y)}|\Phi_{\delta,y}|^{2^*}dx\right)^{1/2^*}=\left(\int_{\mathbb{R}^N\setminus B_\rho(0)}|\Phi_{\delta,0}|^{2^*}dz\right)^{1/2^*}<\varepsilon,\quad\forall\rho>k_0. \tag{1.3.9}$$

For ρ fixed, consider

$$k_0<2\rho<|y| \tag{1.3.10}$$

and note that

$$B_\rho(0)\cap B_\rho(y)=\emptyset. \tag{1.3.11}$$

Using Holder inequality with $N/2$ and $N/(N-2)$, we get

$$\begin{aligned}
\int_{\mathbb{R}^N}a(x)|\Phi_{\delta,y}|^2dx &\leq \left(\int_{\mathbb{R}^N\setminus(B_\rho(0)\cup B_\rho(y))}a^{N/2}dx\right)^{2/N}\left(\int_{\mathbb{R}^N\setminus(B_\rho(0)\cup B_\rho(y))}|\Phi_{\delta,y}|^{2^*}dx\right)^{(N-2)/N}\\
&+\left(\int_{B_\rho(0)}a^{N/2}dx\right)^{2/N}\left(\int_{B_\rho(0)}|\Phi_{\delta,y}|^{2^*}dx\right)^{(N-2)/N}\\
&+\left(\int_{B_\rho(y)}a^{N/2}dx\right)^{2/N}\left(\int_{B_\rho(y)}|\Phi_{\delta,y}|^{2^*}dx\right)^{(N-2)/N}\\
&\leq \left(\int_{\mathbb{R}^N\setminus B_\rho(0)}a^{N/2}dx\right)^{2/N}\left(\int_{\mathbb{R}^N\setminus B_\rho(y)}|\Phi_{\delta,y}|^{2^*}dx\right)^{(N-2)/N}\\
&+\left(\int_{\mathbb{R}^N}a^{N/2}dx\right)^{2/N}\left(\int_{\mathbb{R}^N\setminus B_\rho(y)}|\Phi_{\delta,y}|^{2^*}dx\right)^{(N-2)/N}\\
&+\left(\int_{\mathbb{R}^N\setminus B_\rho(0)}a^{N/2}dx\right)^{2/N}\left(\int_{\mathbb{R}^N}|\Phi_{\delta,y}|^{2^*}dx\right)^{(N-2)/N}\\
&=\left(\int_{\mathbb{R}^N\setminus B_\rho(0)}a^{N/2}dx\right)^{2/N}\\
&<\varepsilon\varepsilon^2+|a|_{N/2}\varepsilon^2+\varepsilon.
\end{aligned}$$

Arguing of the same way for the term (1.3.8), the proof is over. □

Now we define the set

$$\Im=\left\{(u,v)\in\mathcal{M};\alpha(u,v)=\left(0,\frac{1}{2}\right)\right\}.$$

and note that from Lemma 1.3.2 and Lemma 1.3.1, item (i), there is $\delta_1>0$ such that $(\Phi_{\delta_1,0},\Phi_{\delta_1,0})\in\Im$.

Lemma 1.3.7. *The number* $c_0=\inf_{(u,v)\in\Im}f(u,v)$ *satisfies the inequality* $c_0>S_K$.

Proof. Since $\Im\subset\mathcal{M}$, we have

$$S_K\leq c_0.$$

Suppose, by contradiction, that $S_K=c_0$. By Ekeland variational principle [31], there exists $(u_n,v_n)\subset D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N}K(u_n,v_n)dx=1,\quad\alpha(u_n,v_n)\to\left(0,\frac{1}{2}\right) \tag{1.3.12}$$

and

$$f(u_n, v_n) \to S_K, \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0. \tag{1.3.13}$$

Then, (u_n, v_n) is bounded in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ and, up to a subsequence, $u_n \rightharpoonup u_0$, $v_n \rightharpoonup v_0$ in $D^{1,2}(\mathbb{R}^N)$.

If $w_n = S^{(N-2)/4} u_n$, $z_n = S^{(N-2)/4} v_n$ and $w_0 = S^{(N-2)/4} u_0$, $z_0 = S^{(N-2)/4} v_0$, we have that $w_n \rightharpoonup w_0$, $z_n \rightharpoonup z_0$ in $D^{1,2}(\mathbb{R}^N)$. Moreover, from (1.3.13) and Lemma 1.2.6, we get

$$I(w_n, z_n) \to \frac{1}{N} S_K^{N/2} \quad \text{and} \quad I'(w_n, z_n) \to 0.$$

We are going to show that $(w_0, z_0) \equiv (0, 0)$. Note that

$$u_n \not\to u_0, \quad u_n \not\to u_0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N), \tag{1.3.14}$$

since otherwise, $(u_0, v_0) \in \mathcal{M}$ implies $u_0 \neq 0$, $v_0 \neq 0$. Then,

$$\begin{aligned} S_K &\leq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \int_{\mathbb{R}^N} |\nabla v_0|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} K(u_0, v_0) dx \right)^{2/2^*}} = \int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \int_{\mathbb{R}^N} |\nabla v_0|^2 dx \\ &< \int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \int_{\mathbb{R}^N} |\nabla v_0|^2 dx + \int_{\mathbb{R}^N} a(x)|u_0|^2 dx + \int_{\mathbb{R}^N} b(x)|v_0|^2 dx = S_K, \end{aligned}$$

which is an absurd. Hence, $w_n \not\to w_0$, $z_n \not\to z_0$ in $D^{1,2}(\mathbb{R}^N)$ and, since (w_n, z_n) is a $(PS)_c$ sequence for I, by Theorem 1.2.2 we obtain that

$$I(w_n, z_n) \to I(w_0, z_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) = \frac{1}{N} S_K^{N/2}.$$

Since $I'_\infty(z_0^j, \zeta_0^j) = 0$, we have that

$$I(w_0, z_0) = 0, \tag{1.3.15}$$

$$k = 1, \tag{1.3.16}$$

$$z_0^1, \zeta_0^1 > 0, \tag{1.3.17}$$

$$I(w_0, z_0) = \frac{1}{N} \int_{\mathbb{R}^N} K(w_0, z_0) dx$$

and from (1.3.15), we conclude that $w_0 \equiv 0$ and $z_0 \equiv 0$. Then, (w_n, z_n) is a $(PS)_c$ sequence for I such that $w_n \rightharpoonup 0$, $z_n \rightharpoonup 0$ and $w_n \not\to 0$, $z_n \not\to 0$.

Note that $\displaystyle\int_{\mathbb{R}^N} a(x)|w_n|^2 dx = o_n(1)$ and $\displaystyle\int_{\mathbb{R}^N} b(x)|z_n|^2 dx = o_n(1)$. Then,

$$\begin{aligned} \frac{1}{N} S_K^{N/2} + o_n(1) = I(w_n, z_n) &= I_\infty(w_n, z_n) + \int_{\mathbb{R}^N} a(x)|w_n|^2 dx + \int_{\mathbb{R}^N} b(x)|z_n|^2 dx \\ &= I_\infty(v_n) + o_n(1) \end{aligned} \tag{1.3.18}$$

and

$$\|I'_\infty(w_n, z_n)\|_{D'} \leq \|I'(w_n, z_n)\|_{D'} + o_n(1). \tag{1.3.19}$$

From (1.3.18) and (1.3.19) we conclude that (w_n, z_n) is a $(PS)_c$ sequence for I_∞ and by Lemma 1.2.1, there are sequences $(R_n) \subset \mathbb{R}$, $(x_n) \subset \mathbb{R}^N$, (z_0^1, ζ_0^1) nontrivial solution of (S_∞) and (Φ_n, Ψ_n) a $(PS)_c$ sequence for I_∞ such that

$$\begin{aligned} w_n(x) &= \Phi_n(w) + R_n^{(N-2)/2} z_0^1(R_n(x - x_n)) + o_n(1) \\ z_n(x) &= \Psi_n(w) + R_n^{(N-2)/2} \zeta_0^1(R_n(x - x_n)) + o_n(1). \end{aligned} \tag{1.3.20}$$

Note that if we define

$$\widetilde{\Phi}_n(x) = R_n^{(N-2)/2} z_0^1(R_n(x - x_n)), \quad \widetilde{\Psi}_n(x) = R_n^{(N-2)/2} \zeta_0^1(R_n(x - x_n)),$$

making a change of variable, we have

$$I'_\infty(\widetilde{\Phi}_n, \widetilde{\Psi}_n)(\varphi, \psi) = I'_\infty(z_0^1, \zeta_0^1)(\varphi_n, \psi_n) = 0, \quad \forall (\varphi, \psi) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N), \quad \forall n \in \mathbb{N},$$

i.e, $(\widetilde{\Phi}_n, \widetilde{\Psi}_n)$ a solution of (S_∞) , for all $n \in \mathbb{N}$.

Moreover, from the definition of $(\widetilde{\Phi}_n, \widetilde{\Psi}_n)$ and by (1.3.17), we get

$$\widetilde{\Phi}_n(x) = \widetilde{\Psi}_n(x) = c\left(\frac{\delta_n}{\delta_n^2 + |x - y_n|^2}\right)^{(N-2)/2}.$$

By (1.3.20), we obtain

$$u_n(x) = \widehat{\Phi}_n(x) + \Phi_{\delta_n, y_n}(x) + o_n(1), \quad v_n(x) = \widehat{\Psi}_n(x) + \Phi_{\delta_n, y_n}(x) + o_n(1)$$

where

$$\widehat{\Phi}_n(x) = \frac{1}{S_K^{(N-2)/4}} \Phi_n(x), \quad \widehat{\Psi}_n(x) = \frac{1}{S_K^{(N-2)/4}} \Psi_n(x).$$

Using (1.3.16), we derive that $\Phi_n \to 0$, $\Psi_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$, which implies that $\widehat{\Phi}_n \to 0$, $\widehat{\Psi}_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$. From (1.3.12) we have

$$\begin{aligned} \left(0, \frac{1}{2}\right) + o_n(1) &= \alpha(u_n, v_n) = \alpha(\widehat{\Phi}_n(x) + \Phi_{\delta_n, y_n}(x), \widehat{\Psi}_n(x) + \Phi_{\delta_n, y_n}(x)) + o_n(1)) \\ &= \alpha(\Phi_{\delta_n, y_n}, \Phi_{\delta_n, y_n}) \end{aligned}$$

which implies

$$(i) \quad \beta(\Phi_{\delta_n, y_n}, \Phi_{\delta_n, y_n}) \to 0$$

and

$$(ii) \quad \gamma(\Phi_{\delta_n, y_n}, \Phi_{\delta_n, y_n}) \to \frac{1}{2}.$$

Passing to a subsequence, one of these possibilities can occur.

(a) $\delta_n \to +\infty$ when $n \to +\infty$;

(b) $\delta_n \to \tilde{\delta} \neq 0$ when $n \to +\infty$;

(c) $\delta_n \to 0$ and $y_n \to \tilde{y}$ when $n \to +\infty$ with $|\tilde{y}| < \dfrac{1}{2}$;

(d) $\delta_n \to 0$ when $n \to +\infty$ and $|y_n| \geq \dfrac{1}{2}$ for n sufficient large.

Suppose that (a) is true. Then,

$$\gamma(\Phi_{\delta_n,y_n}) = 1 - \frac{s_o^2+t_0^2}{S_K}\int_{B_1(0)}|\nabla\Phi_{\delta_n,y_n}|^2dx,$$

which implies by Lemma 1.3.1,

$$|\gamma(\Phi_{\delta_n,y_n}) - 1| = \frac{s_o^2+t_0^2}{S_K}\int_{B_1(0)}|\nabla\Phi_{\delta_n,y_n}|^2dx \leq \frac{s_o^2+t_0^2}{S_K}\int_{\mathbb{R}^N}|\nabla\Phi_{\delta_n,y_n}|^2dx = o_n(1),$$

which contradicts (ii).

Suppose that (b) is true. In this case we can suppose that $|y_n| \to +\infty$, because if $y_n \to \tilde{y}$, we can prove that

$$\Phi_{\delta_n,y_n} \to \Phi_{\tilde{\delta},\tilde{y}} \ \text{ in } \ D^{1,2}(\mathbb{R}^N).$$

Since $\widehat{\Phi}_n, \widehat{\Psi}_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$ and $u_n = \widehat{\Phi}_n + \Phi_{\delta_n,y_n} + o_n(1)$, $v_n = \widehat{\Psi}_n + \Phi_{\delta_n,y_n} + o_n(1)$, we have that (u_n, v_n) converges in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ but this is a contradiction with (1.3.14).

Then,

$$\begin{aligned}\gamma(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) &= \frac{s_o^2+t_0^2}{S_K}\int_{\mathbb{R}^N}\xi(x)|\nabla\Phi_{\delta_n,y_n}|^2dx = \frac{s_o^2+t_0^2}{S_K}\int_{\mathbb{R}^N\setminus B_1(0)}|\nabla\Phi_{\delta_n,y_n}|^2dx \\ &= 1 - \frac{s_o^2+t_0^2}{S_K}\int_{B_1(-y_n)}|\nabla\Phi_{\delta_n,0}|^2dx. \qquad (1.3.21)\end{aligned}$$

From Lebesgue Theorem we can prove that

$$\int_{B_1(-y_n)}|\nabla\Phi_{\delta_n,0}|^2dx \to 0$$

and from (1.3.21), we obtain

$$\gamma(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) \to 1 \quad \text{when} \quad n \to +\infty,$$

which is a contradiction with (ii).

Suppose that (c) is true. We have that

$$\begin{aligned}\gamma(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) &= \frac{s_o^2+t_0^2}{S_K}\int_{\mathbb{R}^N}\xi(x)|\nabla\Phi_{\delta_n,y_n}|^2dx = \frac{s_o^2+t_0^2}{S_K}\int_{\mathbb{R}^N\setminus B_1(0)}|\nabla\Phi_{\delta_n,y_n}|^2dx \\ &= \frac{s_o^2+t_0^2}{S_K}\int_{\mathbb{R}^N}|\nabla\Phi_{\delta_n,y_n}|^2dx - \frac{s_o^2+t_0^2}{S_K}\int_{B_1(-y_n)}|\nabla\Phi_{\delta_n,0}|^2dz \\ &= 1 - \frac{s_o^2+t_0^2}{S_K}\int_{B_1(-y_n)}|\nabla\Phi_{\delta_n,0}|^2dz. \qquad (1.3.22)\end{aligned}$$

Note that using Lebesgue Theorem again, we can prove that

$$\lim_{n\to+\infty}\frac{s_o^2+t_0^2}{S_K}\int_{B_1(-y_n)}|\nabla\Phi_{\delta_n,0}|^2dz = 1.$$

Then, by (1.3.22) we have that

$$\gamma(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) \to 0,$$

which is a contradiction with (ii).

Suppose that (d) is true. Since $|b_n| \geq \dfrac{1}{2}$ for n large, then $b_n \nrightarrow 0$ in $\mathbb{R}^N$. From Lemma 1.3.5, we get

$$\beta(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) = \frac{y_n}{|y_n|} + o_n(1).$$

Hence,

$$\beta(\Phi_{\delta_n,y_n}, \Phi_{\delta_n,y_n}) \nrightarrow 0,$$

which is a contradiction with (i). The, we conclude that $S_K < c_0$ and the proof is over. ☐

Lemma 1.3.8. *There is $\delta_1 \in (0, 1/2)$ such that*

(a) $f(s_0\Phi_{\delta_1,y}, t_0\Phi_{\delta_1,y}) < \dfrac{S_K + c_0}{2}, \quad \forall y \in \mathbb{R}^N$;

(b) $\gamma(\Phi_{\delta_1,y}, \Phi_{\delta_1,y}) < \dfrac{1}{2}, \quad \forall y \in \mathbb{R}^N$ *such that* $|y| < \dfrac{1}{2}$;

(c) $\left|\beta(\Phi_{\delta_1,y}, \Phi_{\delta_1,y}) - \dfrac{y}{|y|}\right| < \dfrac{1}{4}, \quad \forall y \in \mathbb{R}^N$ *such that* $|y| \geq \dfrac{1}{2}$.

Proof. From Lemma 1.3.3, we can choose $\varepsilon = \dfrac{c_0 - S}{2} > 0$ and $\delta_2 < \min\{\underline{\delta}, 1/2\}$ and conclude that for all $y \in \mathbb{R}^N$

$$f(s_0\Phi_{\delta,y}, t_0\Phi_{\delta,y}) \leq \sup_{y\in\mathbb{R}^N} f(s_0\Phi_{\delta,y}, t_0\Phi_{\delta,y}) < S_K + \frac{c_0 - S_K}{2} = \frac{S_K + c_0}{2}. \tag{1.3.23}$$

Now by definition of ξ, we have

$$\gamma(\Phi_{\delta,y}, \Phi_{\delta,y}) = 1 - \frac{s_o^2 + t_0^2}{S_K} \int_{B_1(-y)} |\nabla \Phi_{\delta,0}|^2 dz.$$

From Lebesgue Theorem

$$\frac{s_o^2 + t_0^2}{S_K} \int_{B_1(-y)} |\nabla \Phi_{\delta,0}|^2 dz = 1$$

and the proof of this item is over.

Note that from Lemma 1.3.5, we conclude that

$$\beta(\Phi_{\delta,y}, \Phi_{\delta,y}) = \frac{y}{|y|} + o_\delta(1) \quad \text{when} \quad \delta \to 0, \quad \forall y \in \mathbb{R}^N; \;\; |y| \geq \frac{1}{2}$$

and the proof is finished. ☐

Lemma 1.3.9. *There is $\delta_2 > 1$ such that*

(a) $f(s_0\Phi_{\delta_2,y}, t_0\Phi_{\delta_2,y}) < \dfrac{S_K + c_0}{2}, \quad \forall y \in \mathbb{R}^N$,

(b) $\gamma(\Phi_{\delta_2,y}, \Phi_{\delta_2,y}) > \dfrac{1}{2}, \quad \forall y \in \mathbb{R}^N$.

Proof. From Lemma 1.3.3, we can choose $\varepsilon = \dfrac{c_0 - S_K}{2} > 0$ and $\delta_3 > \max\{\bar{\delta}, 1\}$ we have

$$f(s_0\Phi_{\delta,y}, t_0\Phi_{\delta,y}) \leq \sup_{y\in\mathbb{R}^N} f(s_0\Phi_{\delta,y}, t_0\Phi_{\delta,y}) < S_K + \frac{c_0 - S_K}{2} = \frac{S_K + c_0}{2}, \quad \forall y \in \mathbb{R}^N \tag{1.3.24}$$

Moreover, from definition of ξ and Lemma 1.3.1, we can conclude that

$$\gamma(\Phi_{\delta,y}, \Phi_{\delta,y}) \to 1 \quad \text{when} \quad \delta \to +\infty$$

and the proof is over. □

Lemma 1.3.10. *There is $R > 0$ such that*

(a) $f(s_0\Phi_{\delta,y}, t_0\Phi_{\delta,y}) < \dfrac{S_K + c_0}{2}, \quad \forall y; \ \ |y| \geq R$ *and* $\delta \in [\delta_1, \delta_2]$,

(b) $(\beta(\Phi_{\delta,y}, \Phi_{\delta,y})|y))_{\mathbb{R}^N} > 0 \ \ \forall y; \ \ |y| \geq R$ *and* $\delta \in [\delta_1, \delta_2]$.

Proof. The first item follows by Lemma 1.3.3 and the choose of $\varepsilon = \dfrac{c_0 - S}{2} > 0$. The second item follows of the definition of β and $\Phi_{\delta,y}$ and adaptations the same arguments explored in [6] □

Consider the set

$$\mathcal{V} = \{(y, \delta) \in \mathbb{R}^{\mathbb{N}} \times (0, \infty); |y| < R \ \text{ and } \ \delta \in (\delta_1, \delta_2)\},$$

where δ_1, δ_2 and R are given by Lemmas 1.3.8, 1.3.9 and 1.3.10, respectively.

Let $Q : \mathbb{R}^N \times (0, +\infty) \to D^{1,2}(\mathbb{R}^N)$ be the continuous function given by

$$Q(y, \delta) = \Phi_{\delta,y}.$$

Consider now the sets

$$\Theta = \{(Q(y, \delta), Q(y, \delta)); (y, \delta) \in \overline{\mathcal{V}}\},$$

$$\mathcal{H} = \left\{h \in C(\Sigma \cap \mathcal{M}); h(u, v) = (u, v), \forall (u, v) \in \Sigma \cap \mathcal{M}; f(s_o u, t_o v) < \frac{S_K + c_0}{2}\right\}$$

and

$$\Gamma = \{\mathcal{A} \subset \Sigma \cap \mathcal{M}; \mathcal{A} = h(\Theta), h \in \mathcal{H}\}.$$

Note that $\Theta \subset \Sigma \cap \mathcal{M}$, $\Theta = Q(\overline{\mathcal{V}}) \times Q(\overline{\mathcal{V}})$ is compact and $\mathcal{H} \neq \emptyset$, because the identity function is in $\mathcal{H}$.

Lemma 1.3.11. *Let $\mathcal{F} : \overline{\mathcal{V}} \to \mathbb{R}^{N+1}$ be a function given by*

$$\mathcal{F}(y, \delta) = (\alpha \circ (Q, Q))(y, \delta) = \frac{s_o^2 + t_0^2}{S_K} \int_{\mathbb{R}^N} \left(\frac{x}{|x|}, \xi(x)\right) |\nabla \Phi_{\delta,y}|^2 dx.$$

Then,

$$d(\mathcal{F}, \mathcal{V}, (0, 1/2)) = 1.$$

Proof. Let
$$\mathcal{Z} : [0,1] \times \overline{\mathcal{V}} \to \mathbb{R}^{N+1}$$
be the homotopy given by
$$\mathcal{Z}(t,(y,\delta)) = t\mathcal{F}(y,\delta) + (1-t)I_{\overline{\mathcal{V}}}(y,\delta),$$
where $I_{\overline{\mathcal{V}}}$ is the identity operator. Using lemma 1.3.8 and Lemma 1.3.9, we can show that $(0,1/2) \notin \mathcal{Z}([0,1]\times(\partial\mathcal{V}))$, i.e,
$$t\beta(\Phi_{\delta,y},\Phi_{\delta,y}) + (1-t)y \neq 0, \quad \forall t \in [0,1] \text{ and } \forall (y,\delta) \in \partial\mathcal{V} \tag{1.3.25}$$
or
$$t\gamma(\Phi_{\delta,y},\Phi_{\delta,y}) + (1-t)\delta \neq \frac{1}{2}, \quad \forall t \in [0,1] \text{ and } \forall (y,\delta) \in \partial\mathcal{V}. \tag{1.3.26}$$

Hence $(0,1/2) \notin \mathcal{Z}([0,1]\times\partial\mathcal{V})$ where we conclude that $d(\mathcal{F},\mathcal{V},(0,1/2))$, $d(i_{\overline{\mathcal{V}}},\mathcal{V},(0,1/2))$ and $d(\mathcal{Z}(t,\cdot),\mathcal{V},(0,1/2))$ are well defined and
$$d(\mathcal{F},\mathcal{V},(0,1/2)) = d(i_{\overline{\mathcal{V}}},\mathcal{V},(0,1/2)) = 1.$$
□

Lemma 1.3.12. *If $\mathcal{A} \in \Gamma$, then $\mathcal{A} \cap \Im \neq \emptyset$.*

Proof. It is sufficient to prove that for all $h \in \mathcal{H}$, there exists $(y_0,\delta_0) \in \overline{\mathcal{V}}$ such that
$$(\alpha \circ \mathcal{H} \circ (Q,Q))(y_0,\delta_0) = \left(0,\frac{1}{2}\right).$$

Given $h \in \mathcal{H}$, let
$$\mathcal{F}_h : \overline{\mathcal{V}} \to \mathbb{R}^{N+1}$$
be the continuous function given by
$$\mathcal{F}_h(y,\delta) = (\alpha \circ h \circ (Q,Q))(y,\delta).$$

We are going to show that $\mathcal{F}_h = \mathcal{F}$ in $\partial\mathcal{V}$. Note that
$$\partial\mathcal{V} = \Pi_1 \cup \Pi_2 \cup \Pi_3, \tag{1.3.27}$$
where
$$\Pi_1 = \{(y,\delta_1); |y| \le R\},$$
$$\Pi_2 = \{(y,\delta_2); |y| \le R\}$$
and
$$\Pi_3 = \{(y,\delta); |y| = R \text{ and } \delta \in [\delta_1,\delta_2]\}.$$
If $(y,\delta) \in \Pi_1$, then $(y,\delta) = (y,\delta_1)$ and by item (a) from Lemma 1.3.8, we have
$$\begin{aligned} f(s_oQ(y,\delta),t_oQ(y,\delta)) &= f(s_oQ(y,\delta_1),t_oQ(y,\delta_1)) = f(s_0\Phi_{\delta_1,y},t_0\Phi_{\delta_1,y}) \\ &< \frac{S_K + c_0}{2}, \quad \forall (y,\delta) \in \Pi_1. \end{aligned} \tag{1.3.28}$$
If $(y,\delta) \in \Pi_2$, then $(y,\delta) = (y,\delta_2)$ and by item(a) from Lemma 1.3.9, we get
$$\begin{aligned} f(s_oQ(y,\delta),t_oQ(y,\delta)) &= f(s_oQ(y,\delta_2),t_oQ(y,\delta_2)) = f(s_0\Phi_{\delta_2,y},t_0\Phi_{\delta_2,y}) \\ &< \frac{S_K + c_0}{2}, \quad \forall (y,\delta) \in \Pi_2. \end{aligned} \tag{1.3.29}$$

If $(y,\delta) \in \Pi_3$, then $|y| = R$ and $\delta \in [\delta_1, \delta_2]$ and by item (a) from Lemma 1.3.10, we obtain

$$
\begin{aligned}
f(s_oQ(y,\delta), t_oQ(y,\delta)) &= f(s_0\Phi_{\delta,y}, t_0\Phi_{\delta,y}) \\
&< \frac{S_K + c_0}{2}, \quad \forall (y,\delta) \in \Pi_3.
\end{aligned} \tag{1.3.30}
$$

From (1.3.27), (1.3.28), (1.3.29) and (1.3.30) we conclude that

$$
f(s_oQ(y,\delta), t_oQ(y,\delta)) < \frac{S_K + c_0}{2}, \quad \forall (y,\delta) \in \partial\mathcal{V}.
$$

Hence,

$$
\begin{aligned}
\mathcal{F}_h(y,\delta) &= (\alpha \circ h \circ (Q,Q))(y,\delta) = (\alpha \circ h)(Q(y,\delta), Q(y,\delta)) \\
&= \alpha(h((Q(y,\delta), Q(y,\delta)))) = \alpha((Q(y,\delta), Q(y,\delta))) \\
&= (\alpha \circ (Q,Q))(y,\delta) = \mathcal{F}(y,\delta), \quad \forall (y,\delta) \in \partial\mathcal{V}.
\end{aligned}
$$

Since $(0, 1/2) \notin \mathcal{F}(\partial\mathcal{V})$, we have

$$
d(\mathcal{F}, \mathcal{V}, (0,1/2)) = d(\mathcal{F}_h, \mathcal{V}, (0,1/2)).
$$

From Lemma 1.3.11, we get

$$
d(\mathcal{F}_h, \mathcal{V}, (0,1/2)) = d(\mathcal{F}, \mathcal{V}, (0,1/2)) = 1,
$$

and there exists $(y_0, \delta_0) \in \mathcal{V}$ such that

$$
\mathcal{F}_h(y_0,\delta_0) = (\alpha \circ h \circ (Q,Q))(y_0,\delta_0) = \left(0, \frac{1}{2}\right)
$$

and the proof is over. □

1.3.2 Proof of the main theorem

Consider the number

$$
c = \inf_{\mathcal{A}\in\Gamma} \max_{(u,v)\in\mathcal{A}} f(u,v)
$$

and for each $q \in \mathbb{R}$,

$$
f^q = \{(u,v) \in \Sigma \cap \mathcal{M}; f(u,v) \le q\}.
$$

We are going to show that

$$
S_K < c < 2^{2/N} S_K. \tag{1.3.31}
$$

Note that

$$
c = \inf_{\mathcal{A}\in\Gamma} \max_{(u,v)\in\mathcal{A}} f(u,v) \le \max_{(u,v)\in\Theta} f(u,v) \le \sup_{\substack{y\in\mathbb{R}^N \\ \delta\in(0,+\infty)}} f(s_o\Phi_{\delta,y}, t_o\Phi_{\delta,y}) < 2^{2/N} S_K.
$$

On the other hand, from Lemma 1.3.12, we have that

$$\begin{aligned} c_0 &= \inf_{u\in\Im} f(u,v) \le c = \inf_{\mathcal{A}\in\Gamma}\max_{u\in\mathcal{A}} f(s_o u, t_o v) \\ &\le \sup_{\substack{y\in\mathbb{R}^N \\ \delta\in(0,+\infty)}} f(s_o\Phi_{\delta,y}, t_o\Phi_{\delta,y}) < 2^{2/N} S_K. \end{aligned} \tag{1.3.32}$$

From Lemma 1.3.7, we have that $S_K < c_0$ and the proof of (1.3.31) is over. Using the definition of c, there exists $(u_n, v_n) \subset \Sigma \cap \mathcal{M}$ such that

$$f(u_n, v_n) \to c. \tag{1.3.33}$$

Suppose, by contradiction, that

$$f'|_{\mathcal{M}}(u_n, v_n) \nrightarrow 0.$$

Then, there exists $(u_{nj}, v_{nj}) \subset (u_n, v_n)$ such that

$$\|f'|_{\mathcal{M}}(u_{nj}, v_{nj})\|_* \ge C > 0, \quad \forall j \in \mathbb{N}.$$

Using a Deformation Lemma [31], there exists a continuous application $\eta : [0,1] \times (\Sigma \cap \mathcal{M}) \to (\Sigma \cap \mathcal{M})$, $\varepsilon_0 > 0$ such that

(1) $\eta(0, u, v) = (u, v)$;

(2) $\eta(t, u, v) = (u, v)$, $\forall (u, v) \in f^{c-\varepsilon_0} \cup \{(\Sigma \cap \mathcal{M}) \setminus f^{c+\varepsilon_0}\}$, $\forall t \in [0, 1]$;

(3) $\eta(1, f^{c+\frac{\varepsilon_0}{2}}) \subset f^{c-\frac{\varepsilon_0}{2}}$.

From the definition of c, there exists $\tilde{\mathcal{A}} \in \Gamma$ such that

$$c \le \max_{(u,v)\in\tilde{\mathcal{A}}} f(u, v) < c + \frac{\varepsilon_0}{2},$$

where

$$\tilde{\mathcal{A}} \subset f^{c+\frac{\varepsilon_0}{2}}. \tag{1.3.34}$$

Since $\tilde{\mathcal{A}} \in \Gamma$, we have $\tilde{\mathcal{A}} \subset (\Sigma \cap \mathcal{M})$ and there exists $\bar{h} \in \mathcal{H}$ such that

$$\bar{h}(\Theta) = \tilde{\mathcal{A}}. \tag{1.3.35}$$

From the definition of η, we have

$$\eta(1, \tilde{\mathcal{A}}) \subset (\Sigma \cap \mathcal{M}). \tag{1.3.36}$$

Let $\hat{h} : (\Sigma \cap \mathcal{M}) \to (\Sigma \cap \mathcal{M})$ be the function given by $\hat{h}(u, v) = \eta(1, \bar{h}(u, v))$ and note that $\hat{h} \in C(\Sigma \cap \mathcal{M})$. We are going to show that

$$f^{c+\varepsilon_0} \setminus f^{c-\varepsilon_0} \subset f^{2^{2s/N}S} \setminus f^{(S+c_0)/2}. \tag{1.3.37}$$

Considering $(u, v) \in f^{c+\varepsilon_0} \setminus f^{c-\varepsilon_0}$, we have

$$c - \varepsilon_0 < f(u, v) \le c + \varepsilon_0$$

and by (1.3.31), for ε_0 sufficiently small, we get

$$c - \varepsilon_0 < f(u,v) \le c + \varepsilon_0 < 2^{2/N} \widetilde{S}_K. \tag{1.3.38}$$

Now from Lemma 1.3.7 and (1.3.32), we obtain

$$\frac{S_K + c_0}{2} < c_0 - \varepsilon_0 < c - \varepsilon_0 < 2^{2/N} \widetilde{S}_K$$

and

$$\frac{S_K + c_0}{2} < c_0 - \varepsilon_0 \le c - \varepsilon_0 < f(u,v), \tag{1.3.39}$$

which implies

$$(u,v) \in f^{2^{2/N} S_K} \setminus f^{(S_K + c_0)/2}.$$

Consider $(u,v) \in (\Sigma \cap \mathcal{M})$ such that

$$f(u,v) < \frac{S_K + c_0}{2}. \tag{1.3.40}$$

Then,

$$\bar{h}(u,v) = (u,v)$$

and from (1.3.40), we have that $(u,v) \notin f^{2^{2/N} S_K} \setminus f^{(S_K + c_0)/2}$ and by (1.3.37), we get

$$(u,v) \notin f^{c+\varepsilon_0} \setminus f^{c-\varepsilon_0}.$$

Then,

$$(u,v) \in f^{c-\varepsilon_0} \cup \{(\Sigma \cap \mathcal{M}) \setminus f^{c+\varepsilon_0}\}$$

and from Deformation Lemma, we obtain

$$\eta(1,u,v) = (u,v).$$

Hence,

$$\hat{h}(u,v) = \eta(1, \bar{h}(u,v)) = \eta(1,u,v) = (u,v)$$

where we conclude that $\hat{h} \in \mathcal{H}$, which implies

$$\hat{h}(\Theta) = \eta(1, \bar{h}(\Theta))$$

and from (1.3.35), we conclude that

$$\hat{h}(\Theta) = \eta(1, \bar{h}(\Theta)) = \eta(1, \tilde{\mathcal{A}}). \tag{1.3.41}$$

From (1.3.36), we have $\eta(1, \tilde{\mathcal{A}}) \in \Gamma$, which implies

$$c = \inf_{\mathcal{A} \in \Gamma} \max_{u \in \mathcal{A}} f(u,v) \le \max_{u \in \eta(1,\tilde{\mathcal{A}})} f(u,v). \tag{1.3.42}$$

From Deformation Lemma again and by (1.3.34), we get

$$\eta(1, \tilde{\mathcal{A}}) \subset \eta(1, f^{c+\frac{\varepsilon_0}{2}}) \subset f^{c-\frac{\varepsilon_0}{2}}.$$

Then,

$$f(u,v) \le c - \frac{\varepsilon_0}{2}, \quad \forall (u,v) \in \eta(1, \tilde{\mathcal{A}}),$$

which implies

$$\max_{u\in\eta(1,\tilde{\mathcal{A}})} f(u,v) \leq c - \frac{\varepsilon_0}{2}$$

and using (1.3.42), we conclude that

$$c \leq \max_{u\in\eta(1,\tilde{\mathcal{A}})} f(u,v) \leq c - \frac{\varepsilon_0}{2},$$

which is an absurd.

Then,

$$f(u_n, v_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0$$

and from Lemma 1.2.7, up to a subsequence, $u_n \to \widetilde{u}_0$, $v_n \to \widetilde{v}_0$ in $D^{1,2}(\mathbb{R}^N)$, which implies that $\widetilde{u}_0, \widetilde{v}_0 \geq 0$,

$$f(\widetilde{u}_0, \widetilde{v}_0) = c \quad \text{and} \quad f'|_{\mathcal{M}}(\widetilde{u}_0, \widetilde{v}_0) = 0$$

and from(1.3.31)

$$S_K < f(\widetilde{u}_0, \widetilde{v}_0) < 2^{2/N} S_K.$$

The positivity of $\widetilde{u}_0$ and $\widetilde{v}_0$ is a consequence of the classical maximum principle.

Chapter 2

Existence of positive solution for a critical system in $\mathbb{R}^N_+$

In this chapter we will deal with the following system

$$(S_2) \qquad \begin{cases} -\Delta u + a(x)u = \dfrac{1}{2^*}K_u(u,v) \ \text{ in } \ \mathbb{R}^N_+, \\ -\Delta v + b(x)v = \dfrac{1}{2^*}K_v(u,v) \ \text{ in } \ \mathbb{R}^N_+, \\ u > 0, v > 0 \text{ in } \ \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \ \text{ on } \ \partial\mathbb{R}^N_+. \end{cases}$$

Let $\mathbb{R}^2_+ := [0,\infty) \times [0,\infty)$ and set $2^* := 2N/(N-2)$. We state our main hypothesis on the function $K \in C^2(\mathbb{R}^2_+, \mathbb{R})$ as follows.

$(\mathcal{K}_0)$ K is 2^*-homogeneous, that is,

$$K(\lambda s, \lambda t) = \lambda^{2^*} K(s,t) \quad \text{for each } \lambda > 0, \ (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{K}_1)$ there exists $c_1 > 0$ such that

$$|K_s(s,t)| + |K_t(s,t)| \le c_1 \left(s^{2^*-1} + t^{2^*-1} \right) \quad \text{for each } (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{K}_2)$ $K(s,t) > 0$ for each $s, t > 0$;

$(\mathcal{K}_3)$ $\nabla K(0,1) = \nabla K(1,0) = (0,0)$;

$(\mathcal{K}_4)$ $K_s(s,t), K_t(s,t) \ge 0$ for each $(s,t) \in \mathbb{R}^2_+$.

$(\mathcal{K}_5)$ the 1-homogeneous function $G : \mathbb{R}^2_+ \to \mathbb{R}$ given by $G(s^{2^*}, t^{2^*}) := K(s,t)$ is concave.

In the sequel, we denote by S_K and Σ_K, respectively,

$$S_K = \inf\left\{ \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 dx; \ \ (u,v) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N), \ \ \int_{\mathbb{R}^N} K(u,v)dx = 1 \right\} \tag{2.0.1}$$

and

$$\Sigma_K = \inf\left\{ \int_{\mathbb{R}^N_+} |\nabla u|^2 + |\nabla v|^2 dx; \ \ (u,v) \in D^{1,2}(\mathbb{R}^N_+) \times D^{1,2}(\mathbb{R}^N_+), \ \ \int_{\mathbb{R}^N_+} K(u,v)dx = 1 \right\}. \tag{2.0.2}$$

The hypotheses on the functions $a, b : \mathbb{R}^N \mapsto \mathbb{R}^+$ are given by:

$(a,b)_1$ The functions $a(x) \geq 0$ and $b(x) \geq 0$, for all $x \in \mathbb{R}^N_+$.

$(a,b)_2$ $a, b \in L^{N/2}(\mathbb{R}^N_+)$ and $|a|_{L^{N/2}(\mathbb{R}^N_+)} \neq 0$ and $|b|_{L^{N/2}(\mathbb{R}^N_+)} \neq 0$.

Using the above notation we are able to state our main result.

Theorem 2.0.1. *Assume $(a,b)_1$ and $(a,b)_2$, $(\mathcal{K}_0) - (\mathcal{K}_5)$ and*

$$|a|_{L^{N/2}(\mathbb{R}^N_+)} + |b|_{L^{N/2}(\mathbb{R}^N_+)} < S_K - \Sigma_K. \tag{2.0.3}$$

Then, system (S_2) has a positive solution $(u,v) \in D^{1,2}(\mathbb{R}^N_+) \times D^{1,2}(\mathbb{R}^N_+)$.

We denote by $J : D^{1,2}(\mathbb{R}^N_+) \times D^{1,2}(\mathbb{R}^N_+) \to \mathbb{R}$ the functional given by

$$J(u,v) = \int_{\mathbb{R}^N_+} |\nabla u|^2 + |\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2 dx$$

and by $\mathcal{M}$ the manifold

$$\mathcal{M} = \left\{ (u,v) \in D^{1,2}(\mathbb{R}^N_+) \times D^{1,2}(\mathbb{R}^N_+); \int_{\mathbb{R}^N_+} K(u,v)dx = 1 \right\}.$$

The solutions of (S_2) correspond to the positive functions that are critical points of the energy functional J constrained on the manifold $\mathcal{M}$.

Let us denote by S the following number

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N), u \neq 0} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{2/2^*}}.$$

It is well known (see for example [12, 27]) that all the minimizers for S are of the type

$$\Phi_{\delta,y}(x) = c\left(\frac{\delta}{\delta^2 + |x-y|^2} \right)^{(N-2)/2}, \quad x, y \in \mathbb{R}^N \text{ and } \delta > 0, \tag{2.0.4}$$

Moreover, it satisfies for a suitable choice of c

$$\|\Phi_{\delta,y}\|^2 = S \quad \text{and} \quad |\Phi_{\delta,y}|_{2^*} = 1.$$

By [13, Lemma 3], there exist $s_o, t_o > 0$ such that S_K is attained by $(s_o \Phi_{\delta,y}, t_o \Phi_{\delta,y})$. Moreover,

$$M_K S_K = S, \tag{2.0.5}$$

where $M_K = \max_{s^2+t^2=1} K(s,t)^{2/2^*} = K(s_o,t_o)^{2/2^*}$.

If we consider the definition and properties of S_K, we can check that $\Sigma_K = 2^{-2/N} S_K$ and the constant Σ_K is achieved by the function $(s_0 \tilde{\Phi}_{1,0}, t_0 \tilde{\Phi}_{1,0})$ where

$$\tilde{\Phi}_{1,0}(x) = 2^{1/2^*} \Phi_{1,0}(x), \quad \forall x \in \mathbb{R}^N_+$$

and all the minimizers for Σ_K are of the type $(s_0 \tilde{\Phi}_{\delta,y}, t_0 \tilde{\Phi}_{\delta,y})$ where

$$\tilde{\Phi}_{\delta,y}(x) = \sigma^{-\frac{N-2}{2}} \tilde{\Phi}_{1,0}\left(\frac{x-y}{\delta} \right), \quad \delta > 0, \text{ and } y \in \partial\mathbb{R}^N_+.$$

2.1 Preliminaries

We notice that we can use the homogeneity condition $(\mathcal{K}_0)$ to conclude that

$$K(s,t) = \frac{1}{2^*} sK_s(s,t) + \frac{1}{2^*} tK_t(s,t),$$

since by $(\mathcal{K}_0)$, we have

$$\frac{d}{d\lambda}K(\lambda s, \lambda t) = \frac{d}{d\lambda}\left(\lambda^{2^*}K(s,t)\right) 2^* = 2^*\lambda^{2^*-1}K(s,t), \tag{2.1.1}$$

and

$$\begin{aligned} \frac{d}{d\lambda}K(\lambda s, \lambda t) &= sK_s(\lambda s, \lambda t) + tK_t(\lambda s, \lambda t) \\ &= s\lambda^{2^*-1}K_s(\lambda s, \lambda t) + t\lambda^{2^*-1}K_t(\lambda s, \lambda t) \end{aligned} \tag{2.1.2}$$

Then, by equations (2.1.1) and (2.1.2) we got

$$2^*K(s,t) = sK_s(s,t) + tK_t(s,t).$$

We started showing a result of non-existence.

Proposition 2.1.1. *Assume that $(a,b)_1$ - $(a,b)_2$ holds and consider*

$$\Sigma_K^* = \inf\left\{J(u,v); \quad (u,v) \in \mathcal{M}\right\}. \tag{2.1.3}$$

Then, $\Sigma_K^* = \Sigma_K$ and the minimization problem (2.1.3) has no solution.

Proof. Since $a(x) \geq 0$ and $b(x) \geq 0$ in $\mathbb{R}^N_+$, we have $\Sigma^* \geq \Sigma$. To show that the equality holds, let us consider the sequence

$$(\psi_\epsilon(x), \phi_\epsilon(x)) = \xi(|x|)(s_0\Phi_{\epsilon,0}(x), t_0\Phi_{\epsilon,0}(x))$$

where $\xi \in C^\infty(0,+\infty)$ is a non increasing cut-off such that

$$\xi(t) = \begin{cases} 1, & \text{if} \quad t \in [0, 1/2], \\ 0, & \text{if} \quad t \geq 1. \end{cases}$$

We have

$$\begin{aligned} \int_{\mathbb{R}^N_+} |\nabla\psi_\epsilon|^2 dx &= \int_{\mathbb{R}^N_+} |\xi\nabla s_0\Phi_{\epsilon,0} + s_0\Phi_{\epsilon,0}\nabla\xi|^2 dx \leq \int_{\mathbb{R}^N_+} |\nabla s_0\Phi_{\epsilon,0}|^2 dx \\ &+ 2\left(\int_{\mathbb{R}^N_+\backslash B_{\frac{1}{2}}(0)} |s_0\Phi_{\epsilon,0}\xi|^2|\nabla\xi|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N_+\backslash B_{\frac{1}{2}}(0)} |\nabla s_0\Phi_{\epsilon,0}|^2 dx\right)^{\frac{1}{2}} dx \\ &+ \int_{\mathbb{R}^N_+} |\nabla\xi|^2|s_0\Phi_{\epsilon,0}|^2 dx \leq \int_{\mathbb{R}^N_+} |\nabla s_0\Phi_{\epsilon,0}|^2 dx \\ &+ C\left(\int_{\mathbb{R}^N_+\backslash B_{\frac{1}{2}}(0)} |\Phi_{\epsilon,0}|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N_+\backslash B_{\frac{1}{2}}(0)} |\nabla\Phi_{\epsilon,0}|^2 dx\right)^{\frac{1}{2}} dx \\ &+ C\int_{\mathbb{R}^N_+} |\Phi_{\epsilon,0}|^2 dx. \end{aligned}$$

By properties of $\Phi_{\epsilon,0}$ we have $|\Phi_{\epsilon,0}|_2 \longrightarrow 0$ and

$$\int_{\mathbb{R}^N_+\setminus B_{\frac{1}{2}}(0)} |\nabla s_0 \Phi_{\epsilon,0}|^2 dx \longrightarrow 0,$$

as $\epsilon \longrightarrow 0$. Then

$$\int_{\mathbb{R}^N_+} |\nabla \psi_\epsilon|^2 dx \leq \int_{\mathbb{R}^N_+} |\nabla s_0 \Phi_{\epsilon,0}|^2 + o_\epsilon(1).$$

Similarly

$$\int_{\mathbb{R}^N_+} |\nabla \phi_\epsilon|^2 dx \leq \int_{\mathbb{R}^N_+} |\nabla t_0 \Phi_{\epsilon,0}|^2 + o_\epsilon(1).$$

Then,

$$\begin{aligned}\int_{\mathbb{R}^N_+} |\nabla \psi_\epsilon|^2 + |\nabla \phi_\epsilon|^2 dx &\leq \int_{\mathbb{R}^N_+} |\nabla s_0 \Phi_{\epsilon,0}|^2 + |\nabla t_0 \Phi_{\epsilon,0}|^2 + o_\epsilon(1) = \frac{1}{2} S_K + o_\epsilon(1) \\ &\leq \Sigma_K + o_\epsilon(1). \end{aligned} \tag{2.1.4}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^N_+} a(x)\psi_\epsilon^2(x)dx &= \int_{\mathbb{R}^N_+\cap B_\rho(0)} a(x)\psi_\epsilon^2(x)dx + \int_{\mathbb{R}^N_+\setminus B_\rho(0)} a(x)\psi_\epsilon^2(x)dx \\ &\leq |\psi_\epsilon|^2_{L^{2^*}(\mathbb{R}^N_+)} \left(\int_{\mathbb{R}^N_+\cap B_\rho(0)} |a(x)|^{N/2} dx\right)^{2/N} \\ &+ |a|_{L^{N/2}(\mathbb{R}^N_+)} \left(\int_{\mathbb{R}^N_+\setminus B_\rho(0)} |\psi_\epsilon^2(x)|^{2^*} dx\right)^{2/2^*}. \end{aligned}$$

Now note that

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^N_+\setminus B_\rho(0)} |\psi_\epsilon(x)|^{2^*} dx = 0$$

and

$$\lim_{\epsilon\to 0} |\psi_\epsilon|_{L^{2^*}(\mathbb{R}^N_+)} = 1.$$

Then, for all $\rho > 0$, we have

$$\int_{\mathbb{R}^N_+} a(x)\psi_\epsilon^2(x)dx \leq \left(\int_{\mathbb{R}^N_+\cap B_\rho(0)} |a(x)|^{N/2} dx\right)^{2/N} + o_\epsilon(1).$$

Since $a \in L^{N/2}(\mathbb{R}^N_+)$, we get

$$\lim_{\rho\to 0} \left(\int_{\mathbb{R}^N_+\cap B_\rho(0)} |a(x)|^{N/2} dx\right)^{2/N} = 0$$

and then

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^N_+} a(x)\psi_\epsilon^2(x)dx = 0. \tag{2.1.5}$$

Similarly,

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^N_+} b(x)\phi_\epsilon^2(x)dx = 0. \tag{2.1.6}$$

Therefore, from (2.1.4), (2.1.5) and (2.1.6), we obtain

$$\Sigma_K^* \le \lim_{\epsilon\to 0} J(\psi_\epsilon(x),\phi_\epsilon(x)) \le \Sigma_K,$$

and we conclude $\Sigma_K^* = \Sigma_K$.

Now, suppose that the minimization problem (2.1.3) has a solution (u,v) and without loss of generality that $u,v\ge 0$. Let us denote by u^*, v^*, a^* and b^* the extension by reflection to all of $\mathbb{R}^N$ of u, v, a and b, respectively. Then

$$\frac{\displaystyle\int_{\mathbb{R}^N}[|\nabla u^*|^2+|\nabla v^*|^2+a^*|u^*|^2+b^*|u^*|^2]dx}{\left(\displaystyle\int_{\mathbb{R}^N}K(u^*,v^*)dx\right)^{\frac{2}{2^*}}} = S_K.$$

Since $a^*,b^*\ge 0$ and by definition of S_K we have

$$S_K \le \frac{\displaystyle\int_{\mathbb{R}^N}[|\nabla u^*|^2+|\nabla v^*|^2]dx}{\left(\displaystyle\int_{\mathbb{R}^N}K(u^*,v^*)dx\right)^{\frac{2}{2^*}}} \le \frac{\displaystyle\int_{\mathbb{R}^N}[|\nabla u^*|^2+|\nabla v^*|^2+a^*|u^*|^2+b^*|u^*|^2]dx}{\left(\displaystyle\int_{\mathbb{R}^N}K(u^*,v^*)dx\right)^{\frac{2}{2^*}}} = S_K,$$

which implies that $\int_{\mathbb{R}^N} a^*|u^*|^2dx = \int_{\mathbb{R}^N} b^*|v^*|^2dx = 0$ and $(u^*,v^*) = (s_0\Phi_{\delta,y}, t_0\Phi_{\delta,y})$, for some $\delta>0$ and $y\in\mathbb{R}^N$. Thus, using the assumptions on a and b and the fact that $\Phi_{\delta,y}>0$ for all $x\in\mathbb{R}^N$, we deduce

$$0 = \int_{\mathbb{R}^N} a^*|u^*|^2+b^*|v^*|^2dx = \int_{\mathbb{R}^N} a^*|s_0\Phi_{\delta,y}|^2+b^*|t_0\Phi_{\delta,y}|^2dx > 0,$$

which is an absurd. □

Lemma 2.1.2. *Let a and b be functions verifying $(a,b)_1$ - $(a,b)_2$. If (u,v) is a critical point of J on $\mathcal{M}$ such that $J(u,v)\le S_K$, then u and v do not change sign.*

Proof. Assume that $u = u^+ + u^-$, $v = v^+ + v^-$ with $u^+,u^-\neq 0$ or $v^+,v^-\neq 0$. By Proposition 2.1.1,

$$\Sigma_K\left(\int_{\mathbb{R}^N_+}K(u^\pm,v^\pm)dx\right)^{\frac{2}{2^*}} < \int_{\mathbb{R}^N_+}[|\nabla u^\pm|^2+|\nabla v^\pm|+a|u^\pm|^2+b|v^\pm|^2]dx$$

and since (u,v) is a critical point of J on $\mathcal{M}$,

$$\int_{\mathbb{R}^N_+}[|\nabla u^\pm|^2+|\nabla v^\pm|+a|u^\pm|^2+b|v^\pm|^2]dx \le J(u,v)\int_{\mathbb{R}^N_+}K(u^\pm,v^\pm)dx. \tag{2.1.7}$$

Then

$$\int_{\mathbb{R}^N_+}K(u^\pm,v^\pm)dx \ge \left(\frac{\Sigma_K}{J(u,v)}\right)^{N/2},$$

which, considering that $\int_{\mathbb{R}^N_+} K(u,v)dx = 1$, gives

$$J(u,v) > 2^{2/N}\Sigma_K = S_K,$$

which contradicts our assumption. ☐

The next proposition guarantees us the existence of an interval where the functional J verifies the Palais-Smale conditions on $\mathcal{M}$.

Proposition 2.1.3. *Assume that a and b satisfies $(a,b)_1$ - $(a,b)_2$ and let $(u_n, v_n) \subset \mathcal{M}$ be a sequence verifying*

$$J(u_n, v_n) \to c \quad \textit{and} \quad J'|_{\mathcal{M}}(u_n, v_n) \to 0,$$

with $c \in (\Sigma_K, S_K)$. Then (u_n, v_n) has a strongly convergent subsequence in $D^{1,2}(\mathbb{R}^N_+) \times D^{1,2}(\mathbb{R}^N_+)$.

Proof. If (u_n^*, v_n^*), a^* and b^* denote the functions obtained by (u_n, v_n), a and b extended to $\mathbb{R}^N$ by reflection, we have that $u_n^*, v_n^* \in D^{1,2}(\mathbb{R}^N)$, $\forall n \in \mathbb{N}$.

Moreover, using the definition of the reflection, we obtain

$$\int_{\mathbb{R}^N} K\left(\frac{1}{2^{1/2^*}}(u_n^*, v_n^*)\right) dx = 1, \quad \frac{1}{2^{2/2^*}} \int_{\mathbb{R}^N} [|\nabla u_n^*|^2 + |\nabla v_n^*|^2 + a^*|u_n^*|^2 + b^*|v_n^*|^2]dx \to 2^{2/N}c$$

and

$$\int_{\mathbb{R}^N} [\nabla u_n^* \nabla u + \nabla v_n^* \nabla v + a^* u_n^* v + b^* v_n^* v]dx + (2^{2/N} + o_n(1)) \int_{\mathbb{R}^N} K_u(u_n^*, v_n^*)u + K_v(u_n^*, v_n^*)v dx = o_n(1),$$

for all $u, v \in D^{1,2}(\mathbb{R}^N)$.

Since $c \in (\Sigma_K, S_K)$ we have

$$2^{2/N}c \in (2^{2/N}\Sigma_K, 2^{2/N}S_K) = (S_K, 2^{2/N}S_K)$$

and from [14, Lemma 3.3], $(u_n^*/2^{1/2^*}, v_n^*/2^{1/2^*})$ has a strongly convergent subsequence in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, and thus (u_n, v_n) has a strongly convergent subsequence in $D^{1,2}(\mathbb{R}^N_+) \times D^{1,2}(\mathbb{R}^N_+)$. ☐

Let $\Pi : \mathbb{R}^N_+ \to \partial\mathbb{R}^N_+$ denote the projection

$$\Pi(x_1, x_2, ..., x_N) = (x_1, x_2, ..., x_{N-1}, 0).$$

We consider the functions $\beta : D^{1,2}(\mathbb{R}^N_+) \times D^{1,2}(\mathbb{R}^N_+) \to \partial\mathbb{R}^N_+$ and $\gamma : D^{1,2}(\mathbb{R}^N_+) \times D^{1,2}(\mathbb{R}^N_+) \to \mathbb{R}$ defined by

$$\beta(u,v) = \frac{\displaystyle\int_{\mathbb{R}^N_+} \frac{\Pi(x)}{1 + |\Pi(x)|} K(u,v)dx}{\displaystyle\int_{\mathbb{R}^N_+} K(u,v)dx}$$

and

$$\gamma(u,v) = \frac{\displaystyle\int_{\mathbb{R}^N_+} \left|\frac{\Pi(x)}{1 + |\Pi(x)|} - \beta(u,v)\right| K(u,v)}{\displaystyle\int_{\mathbb{R}^N_+} K(u,v)dx}.$$

For all $\rho > 0$ and $y \in \mathbb{R}^N$, let us denote by $A_\rho(y)$ the following set:

$$A_\rho(y) = \left\{x \in \mathbb{R}^N_+ ; |\Pi(x) - \Pi(y)| < \rho\right\}.$$

Lemma 2.1.4. *Let (u_n, v_n) be a sequence in $D^{1,2}(\mathbb{R}^N_+) \times D^{1,2}(\mathbb{R}^N_+)$ verifying*

$$(u_n, v_n) \subset \mathcal{M}, \quad \lim_{n\to+\infty} \int_{\mathbb{R}^N_+} |\nabla u_n|^2 + |\nabla v_n|^2 dx = \Sigma_K, \quad \beta(u_n, v_n) = 0 \quad \text{and} \quad \gamma(u_n, v_n) = \frac{1}{3}.$$

Then, up to subsequences, there are sequences $(\delta_n) \subset \mathbb{R}_+$, $(y_n) \subset \partial\mathbb{R}^N_+$ and $w_n, \zeta_n \subset D^{1,2}(\mathbb{R}^N_+)$ such that

(i) $u_n = \tilde{\Phi}_{\delta_n, y_n} + w_n$, $v_n = \tilde{\Phi}_{\delta_n, y_n} + \zeta_n$

(ii) (δ_n) *and* (y_n) *are bounded, and*

(iii) $w_n, \zeta_n \to 0$ *in* $D^{1,2}(\mathbb{R}^N_+)$.

Proof. From [14, Lemma 3.1], we deduce

$$u_n(x) = s_0 \tilde{\Phi}_{\delta_n, y_n}(x) + w_n(x), \quad \forall x \in \mathbb{R}^N_+,$$

$$v_n(x) = t_0 \tilde{\Phi}_{\delta_n, y_n}(x) + \zeta_n(x), \quad \forall x \in \mathbb{R}^N_+,$$

where $\delta_n \in \mathbb{R}^+ \setminus \{0\}$, $y_n \in \partial\mathbb{R}^N_+$ and w_n, ζ are sequences that goes strongly to zero in $D^{1,2}(\mathbb{R}^N_+)$. Consequently, by Brezis Lieb, for all $\rho > 0$ holds

$$\int_{A_\rho(0)} K(u_n, v_n) dx = \int_{A_\rho(0)} K(s_0 \tilde{\Phi}_{\delta_n, y_n}, t_0 \tilde{\Phi}_{\delta_n, y_n}) dx + o_n(1). \tag{2.1.8}$$

Therefore, in order to complete the proof of the lemma, it is enough to show that, up to subsequences,

$$\begin{cases} (a) & \lim_{n\to+\infty} \delta_n = \overline{\delta} > 0; \\ (b) & \lim_{n\to+\infty} y_n = \overline{y} \in \partial\mathbb{R}^N_+. \end{cases} \tag{2.1.9}$$

To prove (2.1.9) (a), let us first show that (δ_n) is bounded. In fact, if for some subsequence, still denoted by (δ_n), $\lim_{n\to+\infty} \delta_n = +\infty$ occurs, then using (2.1.8), for all $\rho > 0$, we have

$$\begin{aligned} \lim_{n\to+\infty} \int_{A_\rho(0)} K(u_n, v_n) dx &= \lim_{n\to+\infty} \int_{A_\rho(0)} K(s_0 \tilde{\Phi}_{\delta_n, y_n}(x), t_0 \tilde{\Phi}_{\delta_n, y_n}(x)) dx \\ &= \lim_{n\to+\infty} \int_{A_{\frac{\rho}{\delta_n}}(0)} K\left(s_0 \tilde{\Phi}_{1,0}\left(x - \frac{y_n}{\delta_n}\right), t_0 \tilde{\Phi}_{1,0}\left(x - \frac{y_n}{\delta_n}\right)\right) dx = 0. \end{aligned}$$

Since $\beta(u_n, v_n) = 0$ and $\int_{\mathbb{R}^N_+} K(u_n, v_n) dx = 1$, for all $\rho > 0$, we deduce

$$\begin{aligned} \gamma(u_n, v_n) &= \int_{\mathbb{R}^N_+} \frac{|\Pi(x)|}{1 + |\Pi(x)|} K(u_n, v_n) dx \geq \int_{\mathbb{R}^N_+ \setminus A_\rho(0)} \frac{|\Pi(x)|}{1 + |\Pi(x)|} K(u_n, v_n) dx \\ &\geq \frac{\rho}{1+\rho} \int_{\mathbb{R}^N_+ \setminus A_\rho(0)} K(u_n, v_n) dx. \end{aligned}$$

Since $\lim_{n\to+\infty} \int_{A_\rho(0)} K(u_n, v_n) dx = 0$, we have $\lim_{n\to+\infty} \int_{\mathbb{R}^N_+ \setminus A_\rho(0)} K(u_n, v_n) dx = 1$. Then,

$$\liminf_{n\to+\infty} \gamma(u_n, v_n) \geq \frac{\rho}{1+\rho}, \quad \forall \rho > 0,$$

and then

$$\liminf_{n\to+\infty} \gamma(u_n, v_n) \geq 1. \tag{2.1.10}$$

Since $\gamma(u_n, v_n) = 1/3$, we have a contradiction. Thus, (δ_n) is bounded and we can assume that

$$\lim_{n\to+\infty} \delta_n = \overline{\delta} \text{ with } \overline{\delta} \geq 0.$$

We claim that $\overline{\delta}$ is positive. In fact, if $\overline{\delta} = 0$, using again (2.1.8), for all $\rho > 0$, we have that

$$\begin{aligned} \lim_{n\to+\infty} \int_{\mathbb{R}^N_+ \setminus A_\rho(y_n)} K(u_n, v_n)dx &= \lim_{n\to+\infty} \int_{\mathbb{R}^N_+ \setminus A_\rho(y_n)} K(s_0\tilde{\Phi}_{\delta_n,y_n}(x), t_0\tilde{\Phi}_{\delta_n,y_n}(x))dx \\ = \lim_{n\to+\infty} \int_{\mathbb{R}^N_+ \setminus A_{\frac{\rho}{\delta_n}}(\frac{y_n}{\delta_n})} & K\left(s_0\tilde{\Phi}_{1,0}\left(x - \frac{y_n}{\delta_n}\right), t_0\tilde{\Phi}_{1,0}\left(x - \frac{y_n}{\delta_n}\right)\right) dx = 0. \end{aligned} \tag{2.1.11}$$

Since $\beta(u_n, v_n) = 0$, $\displaystyle\int_{\mathbb{R}^N_+} K(u_n, v_n)dx = 1$ and from (2.1.11), there is $K > 0$ verifying

$$\begin{aligned} \frac{|y_n|}{1+|y_n|} &= \left| \int_{\mathbb{R}^N_+} \left(\frac{y_n}{1+|y_n|} - \frac{\Pi}{1+|\Pi|} \right) K(u_n, v_n)dx \right| \\ &\leq \left| \int_{\mathbb{R}^N_+ \setminus A_\rho(y_n)} \left(\frac{y_n}{1+|y_n|} - \frac{\Pi}{1+|\Pi|} \right) K(u_n, v_n)dx \right| \\ &+ \left| \int_{A_\rho(y_n)} \left(\frac{y_n}{1+|y_n|} - \frac{\Pi}{1+|\Pi|} \right) K(u_n, v_n) \right| \leq K\rho + o_n(1). \end{aligned}$$

Hence,

$$\limsup_{n\to+\infty} \frac{|y_n|}{1+|y_n|} \leq K\rho, \quad \forall \rho > 0,$$

from where it follows

$$\lim_{n\to+\infty} |y_n| = 0.$$

On the other hand,

$$\begin{aligned} \lim_{n\to+\infty} \gamma(u_n, v_n) &= \lim_{n\to+\infty} \int_{\mathbb{R}^N_+} \left| \frac{\Pi(x)}{1+|\Pi(x)|} - \beta(u_n, v_n) \right| K(u_n, v_n)dx \\ &= \lim_{n\to+\infty} \int_{\mathbb{R}^N_+} \left| \frac{\Pi(x)}{1+|\Pi(x)|} - \frac{y_n}{1+|y_n|} \right| K(u_n, v_n)dx = 0, \end{aligned}$$

which is a contradiction.

Now, we are able to prove that (y_n) is bounded. For this, suppose by contradiction, that there is a subsequence, still denoted by (y_n), such that

$$\lim_{n\to+\infty} |y_n| = +\infty.$$

Then, for all $\epsilon > 0$, there is $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$|\Pi(x) - y_n| < R \Rightarrow \left| \frac{\Pi(x)}{1+|\Pi(x)|} - \frac{y_n}{1+|y_n|} \right| < \epsilon, \quad \forall n \geq n_0 \tag{2.1.12}$$

and

$$\int_{\mathbb{R}^N_+\backslash A_R(y_n)} K(s_0\tilde{\Phi}_{\bar{\delta},y_n}, t_0\tilde{\Phi}_{\bar{\delta},y_n})dx = \int_{\mathbb{R}^N_+\backslash A_R(0)} K(s_0\tilde{\Phi}_{\bar{\delta},0}, t_0\tilde{\Phi}_{\bar{\delta},0})dx < \epsilon. \tag{2.1.13}$$

From (2.1.12) and (2.1.13),

$$\begin{aligned}
\left|\beta(u_n,v_n) - \frac{y_n}{1+|y_n|}\right| &\leq \int_{\mathbb{R}^N_+}\left|\frac{\Pi(x)}{1+|\Pi(x)|} - \frac{y_n}{1+|y_n|}\right| K(u_n,v_n)dx \\
&= \int_{\mathbb{R}^N_+\backslash A_R(y_n)}\left|\frac{\Pi(x)}{1+|\Pi(x)|} - \frac{y_n}{1+|y_n|}\right| K(s_0\tilde{\Phi}_{\bar{\delta},y_n}, t_0\tilde{\Phi}_{\bar{\delta},y_n})dx \\
&+ \int_{A_R(y_n)}\left|\frac{\Pi(x)}{1+|\Pi(x)|} - \frac{y_n}{1+|y_n|}\right| K(s_0\tilde{\Phi}_{\bar{\delta},y_n}, t_0\tilde{\Phi}_{\bar{\delta},y_n})dx + o_n(1) \\
&\leq \epsilon + 2\epsilon + o_n(1) = 3\epsilon + o_n(1),
\end{aligned}$$

where we conclude

$$\lim_{n\to+\infty} |\beta(u_n,v_n)| = 1,$$

which is an absurd. Therefore, (y_n) is bounded. □

We will present below some important properties involving the functions β, γ and the constant Σ_K. Hereafter, we assume that a, b verifies $(a,b)_1$ - $(a,b)_2$. Moreover let us denote by C_{ab} the following real number:

$$C_{ab} = \inf\left\{\int_{\mathbb{R}^N_+}[|\nabla u|^2 + |\nabla v|^2 + a|u|^2 + b|v|^2]dx; (u,v)\in\mathcal{M}, \beta(u,v)=0, \gamma(u,v)=\frac{1}{3}\right\}.$$

Proposition 2.1.5. *Let $a, b \in L^{N/2}(\mathbb{R}^N_+)$ be a non-negative functions with $|a|_{L^{N/2}(\mathbb{R}^N_+)} \neq 0$,$|b|_{L^{N/2}(\mathbb{R}^N_+)} \neq 0$. Then, $\Sigma_K < C_{ab}$.*

Proof. By definition of Σ_K we have $\Sigma_K \leq C_{ab}$. Then, suppose by contradiction that equality holds in the above relation. Thus, there is a sequence $(u_n, v_n) \subset D^{1,2}(\mathbb{R}^N_+) \times D^{1,2}(\mathbb{R}^N_+)$ verifying

$$\begin{cases} (a) \quad \int_{\mathbb{R}^N_+} K(u_n,v_n)dx = 1, \quad \beta(u_n,v_n) = 0, \quad \gamma(u_n,v_n) = \frac{1}{3}; \\ (b) \quad \lim_{n\to+\infty}\int_{\mathbb{R}^N_+}[|\nabla u_n|^2 + |\nabla v_n|^2 + a|u_n|^2 + b|u_n|^2]dx = \Sigma_K. \end{cases} \tag{2.1.14}$$

Since $a(x), b(x) \geq 0$ for all $x \in \mathbb{R}^N_+$, from (2.1.14)

$$\Sigma_K = \lim_{n\to+\infty}\int_{\mathbb{R}^N_+}[|\nabla u_n|^2 + |\nabla v_n|^2 + a|u_n|^2 + b|u_n|^2]dx \tag{2.1.15}$$

$$\geq \lim_{n\to+\infty}\int_{\mathbb{R}^N_+}[|\nabla u_n|^2 + |\nabla v_n|^2]dx \geq \Sigma_K, \tag{2.1.16}$$

we obtain

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N_+}[|\nabla u_n|^2 + |\nabla v_n|^2]dx = \Sigma_K.$$

From Lemma 2.1.4, we have

$$\begin{aligned} u_n(x) &= s_0\tilde{\Phi}_{\delta_n,y_n}(x)+w_n(x), \quad \forall x\in\mathbb{R}^N_+, \\ v_n(x) &= t_0\tilde{\Phi}_{\delta_n,y_n}(x)+\zeta_n(x), \quad \forall x\in\mathbb{R}^N_+, \end{aligned}$$

where $\delta_n \in \mathbb{R}^+\setminus\{0\}$, $y_n \in \partial\mathbb{R}^N_+$ and w_n, ζ_n are sequences that go strongly to zero in $D^{1,2}(\mathbb{R}^N_+)$.

Also from Lemma 2.1.4, we can assume that

$$\lim_{n\to+\infty}\delta_n=\overline{\delta}>0, \quad \lim_{n\to+\infty}y_n=\overline{y}\in\partial\mathbb{R}^N_+$$

and so by Lebesgue's Theorem we have

$$\tilde{\Phi}_{\delta_n,y_n}\to\tilde{\Phi}_{\overline{\delta},\overline{y}} \text{ in } D^{1,2}(\mathbb{R}^N_+) \text{ and } L^{2^*}(\mathbb{R}^N_+). \tag{2.1.17}$$

Thus, from (2.1.14)

$$\begin{aligned} \Sigma_K &= \lim_{n\to+\infty}\int_{\mathbb{R}^N_+}[|\nabla u_n|^2+|\nabla v_n|^2+a|u_n|^2+b|u_n|^2]dx \\ &= \lim_{n\to+\infty}\int_{\mathbb{R}^N_+}[|\nabla s_0\tilde{\Phi}_{\delta_n,y_n}|^2+|\nabla t_0\tilde{\Phi}_{\delta_n,y_n}|^2+a|s_0\tilde{\Phi}_{\delta_n,y_n}|^2+b|t_0\tilde{\Phi}_{\delta_n,y_n}|^2]dx \\ &= \lim_{n\to+\infty}\int_{\mathbb{R}^N_+}[|\nabla s_0\tilde{\Phi}_{\overline{\delta},\overline{y}}|^2+|\nabla t_0\tilde{\Phi}_{\overline{\delta},\overline{y}}|^2+a|s_0\tilde{\Phi}_{\overline{\delta},\overline{y}}|^2+b|t_0\tilde{\Phi}_{\overline{\delta},\overline{y}}|^2]dx \\ &= \Sigma_K+\int_{\mathbb{R}^N_+}[a|s_0\tilde{\Phi}_{\overline{\delta},\overline{y}}|^2+b|t_0\tilde{\Phi}_{\overline{\delta},\overline{y}}|^2]dx, \end{aligned}$$

from where it follows that

$$\int_{\mathbb{R}^N_+}[a|s_0\tilde{\Phi}_{\overline{\delta},\overline{y}}|^2+b|t_0\tilde{\Phi}_{\overline{\delta},\overline{y}}|^2]dx=0,$$

which is an absurd, because $\tilde{\Phi}_{\overline{\delta},\overline{y}}$ is positive. Thus, the proposition is proved. □

2.2 Technical results

From $(a,b)_1$ and $(a,b)_2$ and Proposition 2.1.5 we derive that

$$C_{ab}>\Sigma_K.$$

Using the numbers C_{ab} and Σ_K, we consider a new number $\overline{C}$ given by

$$\overline{C}=\frac{C_{ab}+\Sigma_K}{2}$$

and remark that the following inequality holds:

$$\Sigma_K<\overline{C}<C_{ab}. \tag{2.2.1}$$

We denote by φ, ϕ functions that belong to $W^{1,2}_0(B_1(0))$ and has the following properties:

$$\begin{cases} \varphi,\phi\in C^\infty_0(B_1(0)), \quad \varphi(x),\phi(x)>0 \quad \forall x\in B_1(0), \\ \varphi,\phi \text{ are symmetric and } |x_1|<|x_2|\Rightarrow\varphi(x_1)>\varphi(x_2) \text{ and } \phi(x_1)>\phi(x_2), \\ \displaystyle\int_{\mathbb{R}^N_+\cap B_1(0)}K(\varphi,\phi)dx=1, |\varphi|_{L^{2^*}(\mathbb{R}^N_+\cap B_1(0))}\le 1, |\phi|_{L^{2^*}(\mathbb{R}^N_+\cap B_1(0))}\le 1 \\ \displaystyle\Sigma_K<\int_{\mathbb{R}^N_+\cap B_1(0)}|\nabla\varphi|^2+|\nabla\phi|^2dx\equiv\overline{\Sigma_K}<\min\left\{\overline{C}, S_K-|a|_{L^{N/2}(\mathbb{R}^N_+)}-|b|_{L^{N/2}(\mathbb{R}^N_+)}\right\}. \end{cases} \tag{2.2.2}$$

For every $\sigma > 0$ and $b \in \mathbb{R}^N$, we set

$$\varphi_{\delta,y}(x) = \begin{cases} \delta^{-\frac{N-2}{2}} \varphi\left(\frac{x-y}{\delta}\right), & x \in B_\delta(y), \\ 0, & x \notin B_\delta(y). \end{cases} \tag{2.2.3}$$

$$\phi_{\delta,y}(x) = \begin{cases} \delta^{-\frac{N-2}{2}} \phi\left(\frac{x-y}{\delta}\right), & x \in B_\delta(y), \\ 0, & x \notin B_\delta(y). \end{cases} \tag{2.2.4}$$

We remark that by the definition of $\varphi_{\sigma,b}$ and $\phi_{\sigma,b}$ we have

$$\int_{\mathbb{R}^N_+} K(\varphi_{\delta,y}, \phi_{\delta,y})dx = \int_{B_\delta(y)} K(\varphi_{\delta,y}, \phi_{\delta,y})dx = \int_{B_1(0)} K(\varphi, \phi)dx$$

and

$$|\nabla\varphi_{\delta,y}|_{L^2(\mathbb{R}^N_+)} = |\nabla\varphi_{\delta,y}|_{L^2(B_\delta(y))} = |\nabla\varphi|_{L^2(B_10)},$$
$$|\nabla\phi_{\delta,y}|_{L^2(\mathbb{R}^N_+)} = |\nabla\phi_{\delta,y}|_{L^2(B_\delta(y))} = |\nabla\phi|_{L^2(B_10)}.$$

Lemma 2.2.1. *Let $a, b \in L^{N/2}(\mathbb{R}^N_+)$ be non-negative functions. Then,*

$$\begin{cases} (a) & \lim_{\delta\to 0}\sup\left\{\int_{\mathbb{R}^N_+} [a\varphi^2_{\delta,y} + b\phi^2_{\delta,y}]dx; \quad y \in \partial\mathbb{R}^N_+\right\} = 0; \\ (b) & \lim_{\delta\to+\infty}\sup\left\{\int_{\mathbb{R}^N_+} [a\varphi^2_{\delta,y} + b\phi^2_{\delta,y}]dx; \quad y \in \partial\mathbb{R}^N_+\right\} = 0; \\ (c) & \lim_{r\to+\infty}\sup\left\{\int_{\mathbb{R}^N_+} [a\varphi^2_{\delta,y} + b\phi^2_{\delta,y}]dx; \quad |y| = r, \quad \delta > 0, \quad y \in \partial\mathbb{R}^N_+\right\} = 0. \end{cases} \tag{2.2.5}$$

Proof. Let $y \in \partial\mathbb{R}^N_+$ be chosen arbitrarily. Then, by the Hölder inequality, we get

$$\begin{aligned} \int_{\mathbb{R}^N_+} a\varphi^2_{\delta,y}dx &= \int_{\mathbb{R}^N_+\cap B_\delta(y)} a\varphi^2_{\delta,y}dx \le |a|_{L^{N/2}(\mathbb{R}^N_+\cap B_\delta(y))}|\varphi_{\delta,y}|^2_{L^{2^*}(\mathbb{R}^N_+\cap B_\delta(y))} \\ &= |a|_{L^{N/2}(\mathbb{R}^N_+\cap B_\delta(y))}|\varphi|^2_{L^{2^*}(\mathbb{R}^N_+\cap B_1(y))} = |a|_{L^{N/2}(\mathbb{R}^N_+\cap B_\delta(y))}, \quad \forall\delta > 0, \end{aligned}$$

Similarly

$$\int_{\mathbb{R}^N_+} b\phi^2_{\delta,y}dx \le |b|_{L^{N/2}(\mathbb{R}^N_+\cap B_\delta(y))}, \quad \forall\delta > 0,$$

Then

$$\sup\left\{\int_{\mathbb{R}^N_+} [a\varphi^2_{\delta,y} + b\phi^2_{\delta,y}]dx; \quad y \in \partial\mathbb{R}^N_+\right\} \le \sup\left\{|a|_{L^{N/2}(\mathbb{R}^N_+\cap B_\delta(y))} + |b|_{L^{N/2}(\mathbb{R}^N_+\cap B_\delta(y))}; \quad y \in \partial\mathbb{R}^N_+\right\}. \tag{2.2.6}$$

We note that

$$\lim_{\delta\to 0}|a|_{L^{N/2}(\mathbb{R}^N_+\cap B_\delta(y))} = \lim_{\delta\to 0}|b|_{L^{N/2}(\mathbb{R}^N_+\cap B_\delta(y))} = 0, \quad \forall y \in \partial\mathbb{R}^N_+,$$

so (a) follows from (2.2.6).

To prove (b), we fix arbitrarily $y \in \partial\mathbb{R}^N_+$ and note that by the Hölder inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N_+} a\varphi^2_{\delta,y}dx &= \int_{\mathbb{R}^N_+\cap B_\rho(0)} a\varphi^2_{\delta,y}dx + \int_{\mathbb{R}^N_+\setminus B_\rho(0)} a\varphi^2_{\delta,y}dx \\ &\leq |a|_{L^{N/2}(\mathbb{R}^N_+\cap B_\rho(0))}|\varphi_{\delta,y}|^2_{L^{2^*}(B_\rho(0))} + |a|_{L^{N/2}(\mathbb{R}^N_+\setminus B_\rho(0))}|\varphi_{\delta,y}|^2_{L^{2^*}(\mathbb{R}^N_+\setminus B_\rho(0))} \\ &\leq |a|_{L^{N/2}(\mathbb{R}^N_+\cap B_\rho(0))}\sup_{y\in\partial\mathbb{R}^N_+}|\varphi_{\delta,y}|^2_{L^{2^*}(B_\rho(0))} + |a|_{L^{N/2}(\mathbb{R}^N_+\setminus B_\rho(0))}, \quad \forall \rho,\delta > 0.\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^N_+} b\phi^2_{\delta,y}dx \leq |b|_{L^{N/2}(\mathbb{R}^N_+\cap B_\rho(0))}\sup_{y\in\partial\mathbb{R}^N_+}|\phi_{\delta,y}|^2_{L^{2^*}(B_\rho(0))} + |b|_{L^{N/2}(\mathbb{R}^N_+\setminus B_\rho(0))}, \quad \forall \rho,\delta > 0.$$

Moreover,

$$\lim_{\delta\to+\infty}|\varphi_{\delta,y}|_{L^{2^*}(B_\rho(0))} = \lim_{\delta\to+\infty}|\phi_{\delta,y}|_{L^{2^*}(B_\rho(0))} = 0, \quad \forall y \in \mathbb{R}^N,$$

hence

$$\lim_{\sigma\to+\infty}\sup\left\{\int_{\mathbb{R}^N_+}[a\varphi^2_{\delta,y} + b\phi^2_{\delta,y}dx; \quad y \in \partial\mathbb{R}^N_+\right\} \leq |a|_{L^{N/2}(\mathbb{R}^N_+\setminus B_\rho(0))} + |b|_{L^{N/2}(\mathbb{R}^N_+\setminus B_\rho(0))}.$$

Passing the limit of $\rho \to +\infty$ in the last inequality, we obtain (b).

To prove (c), we will assume by contradiction that there are sequences $(y_n) \subset \partial\mathbb{R}^N_+$ and $(\delta_n) \subset \mathbb{R}_+$ such that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N_+}[a\varphi^2_{\delta_n,y_n} + b\phi^2_{\delta_n,y_n}]dx = L > 0 \quad \text{and} \quad |y_n| \to +\infty. \tag{2.2.7}$$

From (a) and (b), we can suppose that

$$\lim_{n\to+\infty}\delta_n = \overline{\delta} > 0.$$

Using the hypotheses that $|y_n| \to +\infty$ and $a, b \in L^{N/2}(\mathbb{R}^N_+)$ together with Lebesgue's Theorem, we have

$$\lim_{n\to+\infty}|a|_{L^{N/2}(\mathbb{R}^N_+\cap B_{\delta_n}(b_n))} = \lim_{n\to+\infty}|b|_{L^{N/2}(\mathbb{R}^N_+\cap B_{\delta_n}(b_n))} = 0.$$

Then

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N_+}[a\varphi^2_{\delta_n,y_n} + b\phi^2_{\delta_n,y_n}]dx \leq \lim_{n\to+\infty}[|a|_{L^{N/2}(\mathbb{R}^N_+\cap B_{\delta_n}(y_n))} + |b|_{L^{N/2}(\mathbb{R}^N_+\cap B_{\delta_n}(y_n))}] = 0,$$

which contradicts (2.2.7). Therefore (c) occurs. □

Lemma 2.2.2. *The following relations hold:*

$$\begin{cases}(a) & \displaystyle\lim_{\delta\to 0}\sup\left\{\gamma(\varphi_{\delta,y},\phi_{\delta,y}); \quad y \in \partial\mathbb{R}^N_+\right\} = 0; \\ (b) & \displaystyle\lim_{\delta\to+\infty}\inf\left\{\gamma(\varphi_{\delta,y},\phi_{\delta,y}); \quad y \in \partial\mathbb{R}^N_+, \quad |y| \leq r\right\} = 1, \quad \forall r > 0; \\ (c) & (\beta(\varphi_{\delta,y},\phi_{\delta,y})|y)_{\mathbb{R}^N} > 0; \quad \forall y \in \partial\mathbb{R}^N_+ \setminus \{0\}, \quad \forall \delta > 0.\end{cases} \tag{2.2.8}$$

Proof. Let $y \in \partial\mathbb{R}^N_+$ be chosen arbitrarily. For any $\delta > 0$, we have

$$\begin{aligned}0 \leq \gamma(\varphi_{\delta,y},\phi_{\delta,y}) &= \int_{\mathbb{R}^N_+}\left|\frac{\Pi(x)}{1+|\Pi(x)|} - \beta(\varphi_{\delta,y},\phi_{\delta,y})\right|K(\varphi_{\delta,y},\phi_{\delta,y})dx \\ &\leq \int_{\mathbb{R}^N_+\cap B_\delta(y)}\left|\frac{\Pi(x)}{1+|\Pi(x)|} - \beta(\varphi_{\delta,y},\phi_{\delta,y})\right|K(\varphi_{\delta,y},\phi_{\delta,y})dx \\ &+ \left|\frac{y}{1+|y|} - \beta(\varphi_{\delta,y},\phi_{\delta,y})\right|.\end{aligned} \tag{2.2.9}$$

We remark that by (2.2.3) and (2.2.2) we can write

$$\begin{aligned}\left|\frac{y}{1+|y|}-\beta(\varphi_{\delta,y},\phi_{\delta,y})\right| &= \left|\int_{\mathbb{R}^N_+}\left(\frac{y}{1+|y|}-\frac{\Pi(x)}{1+|\Pi(x)|}\right)K(\varphi_{\delta,y},\phi_{\delta,y})dx\right| \\ &\leq \int_{\mathbb{R}^N_+\cap B_\delta(y)}\left|\frac{y}{1+|y|}-\frac{\Pi(x)}{1+|\Pi(x)|}\right|K(\varphi_{\delta,y},\phi_{\delta,y})dx. \quad (2.2.10)\end{aligned}$$

Combining (2.2.9) with (2.2.10) and taking into account that $x \in \mathbb{R}^N_+ \cap B_\sigma(b)$, we have

$$\begin{aligned}0 \leq \gamma(\varphi_{\delta,y},\phi_{\delta,y}) &\leq \int_{\mathbb{R}^N_+\cap B_\delta(y)}\left|\frac{\Pi(x)}{1+|\Pi(x)|}-\beta(\varphi_{\delta,y},\phi_{\delta,y})\right|K(\varphi_{\delta,y},\phi_{\delta,y})dx \\ &+ \int_{\mathbb{R}^N_+\cap B_\delta(y)}\left|\frac{y}{1+|y|}-\frac{\Pi(x)}{1+|\Pi(x)|}\right|K(\varphi_{\delta,y},\phi_{\delta,y})dx \leq 2\delta.\end{aligned}$$

Hence

$$0 \leq \sup\left\{\gamma(\varphi_{\delta,y},\phi_{\delta,y});\quad y \in \partial\mathbb{R}^N_+\right\} \leq 2\delta,$$

which letting $\delta \to 0$, we obtain (a).

To prove (b), let us first show that for all $y \in \partial\mathbb{R}^N_+$,

$$\lim_{\delta\to+\infty}|\beta(\varphi_{\delta,y},\phi_{\delta,y})| = 0. \quad (2.2.11)$$

Since $\beta(\varphi_{\delta,0},\phi_{\delta,0}) = 0$ because of symmetry, we have

$$\begin{aligned}|\beta(\varphi_{\delta,y},\phi_{\delta,y})| &= |\beta(\varphi_{\delta,y},\phi_{\delta,y})-\beta(\varphi_{\delta,y},\phi_{\delta,0})| \\ &= \left|\int_{\mathbb{R}^N_+}\frac{\Pi(x)}{1+|\Pi(x)|}\left(K(\varphi_{\delta,y},\phi_{\delta,y})-K(\varphi_{\delta,0},\phi_{\delta,0})\right)dx\right| \\ &\leq \int_{\mathbb{R}^N_+}\frac{|\Pi(x)|}{1+|\Pi(x)|}\left|K(\varphi_{\delta,y},\phi_{\delta,y})-K(\varphi_{\delta,0},\phi_{\delta,0})\right|dx \\ &\leq \int_{\mathbb{R}^N_+}\left|K(\varphi_{\delta,y},\phi_{\delta,y})-K(\varphi_{\delta,0},\phi_{\delta,0})\right|dx \\ &= \int_{\mathbb{R}^N_+}\left|K(\varphi_{1,\frac{y}{\delta}},\phi_{1,\frac{y}{\delta}})-K(\varphi_{1,0},\phi_{1,0})\right|dx \to 0,\quad \sigma\to+\infty,\end{aligned}$$

showing that (2.2.11) occurs. Now, fix $r > 0$ arbitrarily and let $y \in \partial\mathbb{R}^N_+$ such that $|y| \leq r$. For any $\delta > 0$, we have

$$\begin{aligned}\gamma(\varphi_{\delta,y},\phi_{\delta,y}) &= \int_{\mathbb{R}^N_+}\left|\frac{\Pi(x)}{1+|\Pi(x)|}-\beta(\varphi_{\delta,y},\phi_{\delta,y})\right|K(\varphi_{\delta,y},\phi_{\delta,y})dx \\ &\leq \int_{\mathbb{R}^N_+}\frac{|\Pi(x)|}{1+|\Pi(x)|}K(\varphi_{\delta,y},\phi_{\delta,y})dx + |\beta(\varphi_{\delta,y},\phi_{\delta,y})| \\ &\leq 1+|\beta(\varphi_{\delta,y},\phi_{\delta,y})|,\end{aligned}$$

which together with (2.2.11) leads us to

$$\limsup_{\delta\to+\infty}\left[\inf\left\{\gamma(\varphi_{\delta,y},\phi_{\delta,y});\quad y\in\partial\mathbb{R}^N_+,\quad |y|\leq r\right\}\right] \leq 1. \quad (2.2.12)$$

If

$$\limsup_{\delta\to+\infty}\left[\inf\left\{\gamma(\varphi_{\delta,y},\phi_{\delta,y});\quad y\in\partial\mathbb{R}^N_+,\quad |y|\leq r\right\}\right] < 1,$$

there are sequences $(\delta_n) \subset (0,+\infty)$ and $(y_n) \subset \partial\mathbb{R}^N_+$ such that $\delta_n \to +\infty$, $|y_n| \leq r$ and

$$\lim_{n\to+\infty}\gamma(\varphi_{\delta_n,y_n},\phi_{\delta_n,y_n}) < 1. \quad (2.2.13)$$

On the other hand, considering (2.2.11), for all $\rho > 0$ we deduce that

$$\begin{aligned}
\gamma(\varphi_{\delta_n,y_n},\phi_{\delta_n,y_n}) &= \int_{\mathbb{R}^N_+}\left|\frac{\Pi(x)}{1+|\Pi(x)|}-\beta(\varphi_{\delta_n,y_n},\phi_{\delta_n,y_n})\right|K(\varphi_{\delta_n,y_n},\phi_{\delta_n,y_n})dx \\
&\geq \int_{\mathbb{R}^N_+}\frac{|\Pi(x)|}{1+|\Pi(x)|}K(\varphi_{\delta_n,y_n},\phi_{\delta_n,y_n})dx - |\beta(\varphi_{\delta_n,y_n},\phi_{\delta_n,y_n})| \\
&\geq \int_{\mathbb{R}^N_+\setminus A_\rho(0)}\frac{|\Pi(x)|}{1+|\Pi(x)|}K(\varphi_{\delta_n,y_n},\phi_{\delta_n,y_n})dx - o_n(1) \\
&\geq \frac{\rho}{1+\rho}\int_{\mathbb{R}^N_+\setminus A_\rho(0)}K(\varphi_{\delta_n,y_n},\phi_{\delta_n,y_n})dx - o_n(1) \\
&\geq \frac{\rho}{1+\rho}\int_{\mathbb{R}^N_+\setminus A_{\frac{\rho}{\delta_n}}(0)}K(\varphi_{\delta_n,y_n},\phi_{\delta_n,y_n})dx - o_n(1),
\end{aligned}$$

hence

$$\lim_{n\to+\infty}\gamma(\varphi_{\delta_n,y_n},\phi_{\delta_n,y_n}) \geq \frac{\rho}{1+\rho}, \quad \forall \rho > 0.$$

From this, since $\rho > 0$ is arbitrarily, we have that

$$\lim_{n\to+\infty}\gamma(\varphi_{\sigma_n,b_n}) \geq 1,$$

which contradicts (2.2.13). Thus, the equality in (2.2.12) holds and the proof of (b) is finished.

Now, we will prove (c). We note that if $0 \notin B_\delta(y)$, we have $(\Pi(x)|y) > 0$ and thus

$$(\beta(\varphi_{\delta,y},\phi_{\delta,y})|y) = \int_{\mathbb{R}^N_+}\frac{(\Pi(x)|y)}{1+|\Pi(x)|}K(\varphi_{\delta,y},\phi_{\delta,y})dx > 0.$$

If $0 \in B_\delta(y)$, for each $x \in B_\delta(y)\cap\mathbb{R}^N_+$ such that $(\Pi(x)|y) < 0$, the point $\overline{x}$, symmetrical to $-x$ with respect to $\partial\mathbb{R}^N_+$, belongs to $B_\delta(y)\cap\mathbb{R}^N_+$ and $(\Pi(\overline{x})|y) > 0$ which leads to

$$(\beta(\varphi_{\delta,y},\phi_{\delta,y})|y) = \int_{\mathbb{R}^N_+}\frac{(\Pi(x)|y)}{1+|\Pi(x)|}K(\varphi_{\delta,y},\phi_{\delta,y})dx > 0,$$

as desired. □

Corollary 2.2.3. *There is δ_1, δ_2 whit $0 < \delta_1 < \frac{1}{3} < \delta_2$ such that*

(a) $\gamma(\varphi_{\delta_1,y},\phi_{\delta_1,y}) < \frac{1}{3}$ *to any* $y \in \partial\mathbb{R}^N_+$;

(b) $\gamma(\varphi_{\delta_2,y},\phi_{\delta_2,y}) > \frac{1}{3}$ *to any* $y \in \partial\mathbb{R}^N_+$.

Proof. By Lemma 2.2.2 (a), we have that

$$\gamma(\varphi_{\delta,y},\phi_{\delta,y}) \to 0 \quad \text{as} \quad \delta \to 0 \quad \forall y \in \partial\mathbb{R}^N_+.$$

So there is $\hat{\sigma} > 0$ such that

$$\gamma(\varphi_{\delta,y},\phi_{\delta,y}) < \frac{1}{3} \quad \forall \delta \in (0,\hat{\sigma}) \quad \text{and} \quad \forall y \in \partial\mathbb{R}^N_+.$$

Choosing $\delta_1 < \min\{\hat{\sigma}, 1/3\}$ we deduce that

$$\gamma(\varphi_{\delta_1,y},\phi_{\delta_1,y}) < \frac{1}{3} \quad \forall y \in \partial\mathbb{R}^N_+,$$

proving item (a).

Now, we will prove (b). We note that by Lemma 2.2.2 (b), there is $\overline{\sigma} > 0$ such that

$$\gamma(\varphi_{\delta,y}, \phi_{\delta,y}) > \frac{1}{3} \quad \forall \delta \in (\overline{\sigma}, +\infty) \quad \text{and} \quad \forall y \in \partial\mathbb{R}^N_+.$$

Choosing $\delta_2 > \max\{\overline{\sigma}, 1/3\}$ we deduce that

$$\gamma(\varphi_{\delta_2,y}, \phi_{\delta_2,y}) > \frac{1}{3} \quad \forall y \in \partial\mathbb{R}^N_+$$

proving item (b). □

Now, consider the set

$$\Upsilon = \left\{(y,\delta) \in \partial\mathbb{R}^N_+ \times \mathbb{R}_+; |y| \leq r, \delta \in [\delta_1, \delta_2]\right\}, \tag{2.2.14}$$

with δ_1, δ_2 as chosen before, so we have the following result:

Corollary 2.2.4. *Let a, b satisfy $(a,b)_1$ - $(a,b)_2$, (2.0.3) and $\epsilon > 0$ verify*

$$\overline{\Sigma_K} + \epsilon < \min\{\overline{C}, S_K - |a|_{L^{N/2}(\mathbb{R}^N_+)} - |b|_{L^{N/2}(\mathbb{R}^N_+)}\}.$$

Then, there are $r, \delta_1, \delta_2 > 0$ with

$$\sup\left\{\int_{\mathbb{R}^N_+} [|\nabla\varphi_{\delta,y}|^2 + |\nabla\phi_{\delta,y}|^2 + a|\varphi_{\delta,y}|^2 + b|\phi_{\delta,y}|^2]dx; \quad (y,\delta) \in \partial\Upsilon\right\} < \overline{\Sigma_K} + \frac{\epsilon}{2}.$$

Proof. The existence of δ_1 and δ_2 is given by the Corollary 2.2.3. Now, note that by the Lemma 2.2.1 (a) and (2.2.2) it follows that

$$\int_{\mathbb{R}^N_+} [|\nabla\varphi_{\delta,y}|^2 + |\nabla\phi_{\delta,y}|^2 + a|\varphi_{\delta,y}|^2 + b|\phi_{\delta,y}|^2]dx < \overline{\Sigma_K} + \frac{\epsilon}{2}, \quad \forall y \in \partial\mathbb{R}^N_+ \text{ and } \delta = \delta_1. \tag{2.2.15}$$

Furthermore, by the Lemma 2.2.1, we can chose $r > 0$ such that if $|y| = r$ and $b \in \partial\mathbb{R}^N_+$ so

$$\int_{\mathbb{R}^N_+} [|\nabla\varphi_{\delta,y}|^2 + |\nabla\phi_{\delta,y}|^2 + a|\varphi_{\delta,y}|^2 + b|\phi_{\delta,y}|^2]dx < \overline{\Sigma_K} + \frac{\epsilon}{2}, \quad \forall \delta > 0. \tag{2.2.16}$$

Lastly, fixing $r > 0$ as chosen before, by the Lemma 2.2.1 (b) together with Corollary 2.2.3, we can find δ_2 such that

$$\int_{\mathbb{R}^N_+} [|\nabla\varphi_{\delta,y}|^2 + |\nabla\phi_{\delta,y}|^2 + a|\varphi_{\delta,y}|^2 + b|\phi_{\delta,y}|^2]dx < \overline{\Sigma_K} + \frac{\epsilon}{2}, \quad \forall y \in \partial\mathbb{R}^N_+, \quad |y| \leq r \text{ and } \forall \delta = \delta_2. \tag{2.2.17}$$

Combining (2.2.15), (2.2.16) and (2.2.17) the result follows. □

Corollary 2.2.5. *Assume that a and b satisfy $(a,b)_1$ - $(a,b)_2$, (2.0.3) and let $\epsilon, \delta_1, \delta_2$ and r be the numbers given in Corollary 2.2.3, Corollary 2.2.4 and Υ defined in (2.2.14). Then,*

$$\sup\left\{\int_{\mathbb{R}^N_+} [|\nabla\varphi_{\delta,y}|^2 + |\nabla\phi_{\delta,y}|^2 + a|\varphi_{\delta,y}|^2 + b|\phi_{\delta,y}|^2]dx; \quad (y,\delta) \in \partial\Upsilon\right\} < S_K.$$

Proof. For all $y \in \partial\mathbb{R}^N_+$ and $\delta > 0$, by Hölder inequality we get

$$\begin{aligned}\int_{\mathbb{R}^N_+}[a|\varphi_{\delta,y}|^2 + b|\phi_{\delta,y}|^2]dx &\leq |a|_{L^{N/2}(\mathbb{R}^N_+)}|\varphi_{\delta,y}|^2_{L^{2^*}(\mathbb{R}^N_+)} + |b|_{L^{N/2}(\mathbb{R}^N_+)}|\phi_{\delta,y}|^2_{L^{2^*}(\mathbb{R}^N_+)} \\ &\leq |a|_{L^{N/2}(\mathbb{R}^N_+)} + |b|_{L^{N/2}(\mathbb{R}^N_+)}.\end{aligned}$$

From the last inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N_+}[|\nabla\varphi_{\delta,y}|^2 + |\nabla\phi_{\delta,y}|^2 + a|\varphi_{\delta,y}|^2 + b|\phi_{\delta,y}|^2]dx \\ \leq\ &\int_{\mathbb{R}^N_+}[|\nabla\varphi_{\delta,y}|^2 + |\nabla\phi_{\delta,y}|^2]dx + |a|_{L^{N/2}(\mathbb{R}^N_+)} + |b|_{L^{N/2}(\mathbb{R}^N_+)} \\ =\ &\int_{\mathbb{R}^N_+\cap B_1(0)}[|\nabla\varphi|^2 + |\nabla\phi|^2]dx + |a|_{L^{N/2}(\mathbb{R}^N_+)} + |b|_{L^{N/2}(\mathbb{R}^N_+)},\end{aligned}$$

which combined with (2.2.2) and (2.0.3) give us

$$\int_{\mathbb{R}^N_+}[|\nabla\varphi_{\delta,y}|^2 + |\nabla\phi_{\delta,y}|^2 + a|\varphi_{\delta,y}|^2 + b|\phi_{\delta,y}|^2]dx \leq \overline{\Sigma_K} + |a|_{L^{N/2}(\mathbb{R}^N_+)} + |b|_{L^{N/2}(\mathbb{R}^N_+)} < S_K,$$

for all $(y, \delta) \in \partial\Upsilon$. Therefore,

$$\sup\left\{\int_{\mathbb{R}^N_+}[|\nabla\varphi_{\delta,y}|^2 + |\nabla\phi_{\delta,y}|^2 + a|\varphi_{\delta,y}|^2 + b|\phi_{\delta,y}|^2]dx;\quad (y,\delta) \in \partial\Upsilon\right\} < S_K$$

as we wanted. □

Lemma 2.2.6. *Let Υ be the set defined in (2.2.14) with δ_1, δ_2 and r be the numbers given in Corollary 2.2.3 and Corollary 2.2.4. Then, there is $(\hat{y}, \hat{\delta}) \in \Upsilon$ satisfying*

$$\beta(\varphi_{\hat{y},\hat{\delta}}, \phi_{\hat{y},\hat{\delta}}) = 0 \quad \text{and} \quad \gamma(\varphi_{\hat{y},\hat{\delta}}, \phi_{\hat{y},\hat{\delta}}) = \frac{1}{3}.$$

Proof. To prove the lemma, define the map $g : \partial\Upsilon \to \mathbb{R}^{N-1} \times \mathbb{R}$ by

$$g(y, \delta) = (\beta(\varphi_{\delta,y}, \phi_{\delta,y}), \gamma(\varphi_{\delta,y}, \phi_{\delta,y}));$$

it is sufficient to show that its restriction to $\partial\Upsilon$ is homotopically equivalent to the identity map in $\mathbb{R}^{N-1} \times \mathbb{R} \setminus \{(0, 1/3)\}$.

Therefore, let us consider the homotopy $G : [0,1] \times \partial\Upsilon \to \mathbb{R}^{N-1} \times \mathbb{R}$ given by

$$G(t, y, \delta) = (1-t)(y, \delta) + t(\beta(\varphi_{\delta,y}, \phi_{\delta,y}), \gamma(\varphi_{\delta,y}, \phi_{\delta,y})).$$

We remark G is continuous and that

$$G(0, y, \delta) = (y, \delta)$$

and

$$G(1, y, \delta) = (\beta(\varphi_{\delta,y}, \phi_{\delta,y}), \gamma(\varphi_{\delta,y}, \phi_{\delta,y})) = g(y, \delta).$$

So it remains to show that

$$\left(0, \frac{1}{3}\right) \notin G(t, \partial\Upsilon) \quad \forall t \in [0,1] \tag{2.2.18}$$

or equivalently

$$G(t, y, \delta) \neq \left(0, \frac{1}{3}\right) \quad \forall (y, \delta) \quad \text{and} \quad \forall t \in [0, 1].$$

In fact, set $\partial\Upsilon = \Upsilon_1 \cup \Upsilon_2 \cup \Upsilon_3$ with

$$\begin{cases} \Upsilon_1 = \{(y, \delta); \quad |y| \leq r, \quad \delta = \delta 1\}, \\ \Upsilon_2 = \{(y, \delta); \quad |y| \leq r, \quad \delta = \delta_2\}, \\ \Upsilon_3 = \{(y, \delta); \quad |y| = r, \quad \delta \in [\delta_1, \delta_2]\}. \end{cases}$$

If $(y, \delta) \in \Upsilon_1$, then $\delta = \delta_1$ and by the Corollary 2.2.3 (a)

$$(1 - t)\delta_1 + t\gamma(\varphi_{\delta_1,y}, \phi_{\delta_1,y}) < (1 - t)\frac{1}{3} + t\frac{1}{3} = \frac{1}{3}, \quad \forall t \in [0, 1].$$

Analogously, if $(y, \delta) \in \Upsilon_2$, then $\delta = \delta_2$ and again by the Corollary 2.2.3 (b)

$$(1 - t)\delta_2 + t\gamma(\varphi_{\delta_2,y}, \phi_{\delta_2,y}) > (1 - t)\frac{1}{3} + t\frac{1}{3} = \frac{1}{3}, \quad \forall t \in [0, 1].$$

If $(y, \delta) \in \Upsilon_3$, then $|y| = r$ and $0 < \delta_1 \leq \delta \leq \delta_2$, so using Lemma 2.2.2 (c), we obtain

$$((1 - t)b + t\beta(\varphi_{\delta,y}, \phi_{\delta,y})|y) = (1 - t)|y|^2 + t(\beta(\varphi_{\delta,y}, \phi_{\delta,y})|y) > 0.$$

□

Finally, with the help of the previous lemmas we are ready to prove our main result.

2.3 Existence of positive solution of (S_2)

Firstly we consider

$$d = \sup \{J(\varphi_{\delta,y}, \phi_{\delta,y}); \quad (y, \delta) \in \Upsilon\},$$
$$J^l = \{(u, v) \in \mathcal{M}; \quad J(u, v) \leq l\}$$

and fix $\epsilon > 0$ verifying

$$\overline{\Sigma_K} + \epsilon < \min \left\{ \overline{C}, S_K - |a|_{L^{N/2}(\mathbb{R}^N_+)} - |b|_{L^{N/2}(\mathbb{R}^N_+)} \right\}.$$

Combining the definition of C_{ab} with (2.2.1), Corollary 2.2.5 and Lemma 2.2.6, we have

$$\Sigma_K < \overline{C} < C_{ab} \leq J(\varphi_{\hat{\delta},\hat{y}}, \phi_{\hat{\delta},\hat{y}}) \leq d < S_K.$$

We will prove that functional J constrained to $\mathcal{M}$ has a critical level in the interval $(\overline{C}, S_K)$. For this, we fix $\sigma > 0$ such that

$$\overline{C} < C_{ab} - \sigma < d + \sigma < S \tag{2.3.1}$$

and we define

$$H = \left\{(u, v) \in \mathcal{M}; \quad C_{ab} - \sigma \leq J(u, v) \leq d + \sigma; \quad J'|_{\mathcal{M}}(u, v) = 0\right\}.$$

To prove the theorem, it remains to show that $H \neq \emptyset$. In order to achieved this goal, we will suppose by contradiction, that $H = \emptyset$. From (2.3.1) and Proposition 2.1.3, the pair $(J, \mathcal{M})$ satisfies the Palais-Smale condition in interval $(C_{ab} - \sigma, d + \sigma)$. Thus, using a variant of the Deformation Lemma (see [29]) we can find a continuous map $\eta : [0,1] \times \mathcal{M} \to \mathcal{M}$ and a positive number $\epsilon_0 < \delta$ such that

$$\eta(0,u,v) = (u,v), \quad \forall (u,v) \in \mathcal{M},$$

$$\eta(t,u,v) = (u,v), \quad \forall (u,v) \in J^{C_{ab}-\epsilon_0} \cup (\mathcal{M} \setminus J^{d+\epsilon_0}), \quad \forall t \in [0,1],$$
$$(J \circ \eta)(t,u,v) \leq J(u,v), \quad \forall t \in [0,1],$$

and

$$\eta(1, J^{d+\epsilon_0}) \subset J^{C_{ab}-\epsilon_0}.$$

By the definition of d and Deformation Lemma, we have in particular that

$$\forall (y,\delta) \in \Upsilon \Rightarrow J(\varphi_{\delta,y}, \phi_{\delta,y}) < d \Rightarrow J(\eta(1,\varphi_{\delta,y}, \phi_{\delta,y})) < C_{ab} - \epsilon_0. \tag{2.3.2}$$

Now, we define for all $t \in [0,1]$ and for all $(y,\delta) \in \Upsilon$ the map

$$\tilde{\Gamma}(t,y,\delta) = \begin{cases} G(2t-1,y,\delta), & t \in [0,1/2], \\ (\beta \circ \eta(2t-1,\varphi_{\delta,y},\phi_{\delta,y}), \gamma \circ \eta(2t-1,\varphi_{\delta,y},\phi_{\delta,y})), & t \in [1/2,1], \end{cases}$$

where G is the map defined in Lemma 2.2.6. Clearly $\tilde{\Gamma}$ is continuous and as a consequence of (2.2.18), we have

$$\left(0, \frac{1}{3}\right) \neq \tilde{\Gamma}(t,y,\delta), \quad \forall (y,\delta) \in \partial\Upsilon \quad \text{and} \quad \forall t \in [0,1/2].$$

Moreover, since

$$\begin{aligned} (y,\delta) \in \partial\Upsilon \quad &\Rightarrow \quad J(\varphi_{\delta,y},\phi_{\delta,y}) \leq \overline{\Sigma_K} + \epsilon < \overline{C} < C_{ab} - \sigma < C_{ab} - \epsilon_0 \\ &\Rightarrow \quad \eta(2t-1,\varphi_{\delta,y},\phi_{\delta,y}) = (\varphi_{\delta,y},\phi_{\delta,y}), \quad \forall t \in [1/2,1], \end{aligned}$$

we have

$$\begin{aligned} \tilde{\Gamma}(t,y,\delta) &= (\beta \circ \eta(2t-1,\varphi_{\sigma,b}), \gamma \circ \eta(2t-1,\varphi_{\delta,y},\phi_{\delta,y})) = (\beta(\varphi_{\delta,y},\phi_{\delta,y}), \gamma(\varphi_{\delta,y},\phi_{\delta,y})) \\ &= \tilde{\Gamma}\left(\frac{1}{2},y,\delta\right) = G(1,y,\delta), \quad \forall t \in [1/2,1], \quad \forall (y,\delta) \in \partial\Upsilon. \end{aligned}$$

Therefore, using again (2.2.18), we have

$$\left(0, \frac{1}{3}\right) \neq \tilde{\Gamma}(t,y,\delta), \quad \forall (y,\delta) \in \partial\Upsilon \quad \text{and} \quad \forall t \in [1/2,1].$$

Hence, there is $(y^*,\delta^*) \in \Upsilon$ such that

$$\beta \circ \eta(1,\varphi_{\delta^*,y^*},\phi_{\delta^*,y^*}) = 0, \quad \gamma \circ \eta(1,\varphi_{\delta^*,y^*},\phi_{\delta^*,y^*}) = \frac{1}{3},$$

and so

$$\begin{aligned} J(\eta(1,\varphi_{\delta^*,y^*},\phi_{\delta^*,y^*})) &\geq \inf\left\{J(u,v); \quad (u,v) \in \mathcal{M}, \quad \beta(u,v) = 0, \quad \gamma(u,v) = \frac{1}{3}\right\} \\ &= C_{ab} > C_{ab} - \epsilon_0, \end{aligned}$$

which contradicts (2.3.2) and so $H \neq \emptyset$. Therefore, the functional J constrained on $\mathcal{M}$ has at least one critical point $(u,v) \in \mathcal{M}$ such that $\Sigma_K < \overline{C} < J(u) < S$. Moreover, by Lemma 2.1.2, we deduce $u, v > 0$, concluding the proof.

Chapter 3

Multiplicity of positive solutions for an elliptic system

We are now interested in the search of positive solutions for the problem

$$\begin{cases} -\Delta u = \frac{2\alpha_\epsilon}{\alpha_\epsilon+\beta_\epsilon}|u|^{\alpha_\epsilon-2}u|v|^{\beta_\epsilon} & \text{in } \Omega, \\ -\Delta v = \frac{2\beta_\epsilon}{\alpha_\epsilon+\beta_\epsilon}|u|^{\alpha_\epsilon}|v|^{\beta_\epsilon-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.0.1}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N, N \geq 3$, $\alpha_\epsilon, \beta_\epsilon > 1$, $\alpha_\epsilon = \alpha - \epsilon/2$, $\beta_\epsilon = \beta - \epsilon/2$ and $\alpha + \beta = 2^*$.

The main goal of this chapter is to show that for ϵ small, the topology of the domain influences the number of positive solutions in the sense of Theorem 3.0.1 below.

Before stating our main results we recall that if Y is a closed set of a topological space X, we denote the Ljusternik-Schnirelmann category of Y in X by $\operatorname{cat}_X(Y)$, which is the least number of closed and contractible sets in X that cover Y. Moreover, $\operatorname{cat} X$ denotes $\operatorname{cat}_X(X)$. Then we have the first multiplicity result.

Theorem 3.0.1. *There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, problem (3.0.1) has at least $\operatorname{cat}\Omega$ positive weak solutions. Moreover if Ω is not contractible in itself then (3.0.1) has at least $\operatorname{cat}\Omega + 1$ positive weak solutions.*

The functional I_ϵ associated to problem (3.0.1) is defined as

$$I_\epsilon(u,v) := \frac{1}{2}\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx - \frac{2}{\alpha_\epsilon + \beta_\epsilon}\int_\Omega |u|^{\alpha_\epsilon}|v|^{\beta_\epsilon} dx \tag{3.0.2}$$

which is well defined on the space $H_0^1(\Omega) \times H_0^1(\Omega)$ endowed with the usual norm

$$\|(u,v)\|^2 = \int_\Omega |\nabla u|^2 + |\nabla v|^2 dx.$$

A straightforward computation shows that the functional (3.0.2) is of class C^1 with

$$\begin{aligned} I'_\epsilon(u,v)[\phi,\psi] &= \int_\Omega \nabla u \nabla \phi + \nabla v \nabla \psi dx - \frac{2\alpha_\epsilon}{\alpha_\epsilon+\beta_\epsilon}\int_\Omega |u|^{\alpha_\epsilon-2}u|v|^{\beta_\epsilon}\phi dx \\ &- \frac{2\beta_\epsilon}{\alpha_\epsilon+\beta_\epsilon}\int_\Omega |u|^{\alpha_\epsilon}|v|^{\beta_\epsilon-2}v\psi dx \end{aligned}$$

for $u, v, \phi, \psi \in H_0^1(\Omega)$. Thus, the critical points of I_ϵ correspond exactly to the weak solutions of the problem (3.0.1).

3.1 The Nehari manifolds and compactness results

In this section we study the Nehari manifolds which appear in relation to problem that involves problem (3.0.1). We have the set, usually called the *Nehari manifold* associated to (3.0.1),

$$\mathcal{N}_\epsilon = \left\{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{(0,0)\} : I'_\epsilon(u,v)(u,v) = 0\right\}.$$

In particular all the critical points of I_ϵ lie in $\mathcal{N}_\epsilon$. In the next Lemma we show the basic properties of $\mathcal{N}_\epsilon$.

Lemma 3.1.1. *For all $0 < \epsilon < 1$, we have:*

(i) $\mathcal{N}_\epsilon$ is a C^1 manifold;

(ii) there exists $c_\epsilon > 0$ such that $\|(u,v)\| \geq c_\epsilon$ for every $(u,v) \in \mathcal{N}_\epsilon$;

(iii) it holds $\inf\limits_{(u,v)\in\mathcal{N}_\epsilon} I_\epsilon(u,v) > 0$;

(iv) for every $v \neq 0$, $u \neq 0$ there exists a unique $t_\epsilon = t_\epsilon[u,v] > 0$ such that $t_\epsilon(u,v) \in \mathcal{N}_\epsilon$;

(v) $\mathcal{N}_\epsilon$ is homeomorphic to the unit sphere $\mathbb{S} = \{(u,v) \in H_0^1(\Omega)\times H_0^1(\Omega) : \|(u,v)\|_{1,2} = 1\}$;

(vi) the following equalities are true

$$\inf_{(u,v)\in\mathcal{N}_\epsilon} I_\epsilon(u,v) = \inf_{u\neq 0, v\neq 0} \max_{t>0} I_\epsilon(tu,tv) = \inf_{(g,h)\in\Gamma_\epsilon} \max_{t\in[0,1]} I_\epsilon(g(t),h(t)),$$

where

$$\begin{aligned}\Gamma_\epsilon = \{ \ & (g,h) \in C([0,1]; H_0^1(\Omega) \times H_0^1(\Omega)) : g(0) = h(0) = 0, I_\epsilon(g(1),h(1)) \leq 0, \\ & g(1) \neq 0, h(1) \neq 0\}.\end{aligned}$$

Proof. Let $G_\epsilon(u,v) := I'_\epsilon(u,v)(u,v)$ Since

$$G'_\epsilon(u,v)[u,v] = 2\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx - 2(\alpha_\epsilon + \beta_\epsilon)\int_\Omega |u|^{\alpha_\epsilon}|v|^{\beta_\epsilon} dx$$

and $G_\epsilon(u,v) = 0$ if $(u,v) \in \mathcal{N}_\epsilon$, we obtain

$$G'_\epsilon(v)[v] = -(\alpha_\epsilon + \beta_\epsilon - 2)\|(u,v)\|^2 < 0,$$

which proves (i).

Let $(u,v) \in \mathcal{N}_\epsilon$. Since $G_\epsilon(u,v) = 0$, we have

$$\|(u,v)\|^2 = 2\int_\Omega |u|^{\alpha_\epsilon}|v|^{\beta_\epsilon} dx \leq 2\left(\int_\Omega |u|^{\alpha_\epsilon+\beta_\epsilon} dx\right)^{\frac{\alpha_\epsilon}{\alpha_\epsilon+\beta_\epsilon}} \left(\int_\Omega |v|^{\alpha_\epsilon+\beta_\epsilon} dx\right)^{\frac{\beta_\epsilon}{\alpha_\epsilon+\beta_\epsilon}}$$

Since

$$\int_\Omega |u|^{\alpha_\epsilon+\beta_\epsilon} dx \leq \left(\int_\Omega |u|^{2^*} dx\right)^{\frac{\alpha_\epsilon+\beta_\epsilon}{2^*}} |\Omega|^{\frac{\epsilon}{2^*}} = |u|_{2^*}^{\alpha_\epsilon+\beta_\epsilon}|\Omega|^{\frac{\epsilon}{2^*}}$$

Similarly $\int_\Omega |v|^{\alpha_\epsilon+\beta_\epsilon} dx \leq |v|_{2^*}^{\alpha_\epsilon+\beta_\epsilon}|\Omega|^{\frac{\epsilon}{2^*}}$ and hence we infer

$$\begin{aligned}\|(u,v)\|^2 &\leq 2\left(|u|_{2^*}^{\alpha_\epsilon+\beta_\epsilon}|\Omega|^{\frac{\epsilon}{2^*}}\right)^{\frac{\alpha_\epsilon}{\alpha_\epsilon+\beta_\epsilon}} \left(|v|_{2^*}^{\alpha_\epsilon+\beta_\epsilon}|\Omega|^{\frac{\epsilon}{2^*}}\right)^{\frac{\beta_\epsilon}{\alpha_\epsilon+\beta_\epsilon}} = 2|\Omega|^{\frac{\epsilon}{2^*}}|u|_{2^*}^{\alpha_\epsilon}|v|_{2^*}^{\beta_\epsilon} \\ &\leq 2|\Omega|^{\frac{\epsilon}{2^*}}(|u|_{2^*} + |v|_{2^*})^{\alpha_\epsilon+\beta_\epsilon} \leq 2|\Omega|^{\frac{\epsilon}{2^*}} C^{\alpha_\epsilon+\beta_\epsilon}(\|(u,v)\|)^{\alpha_\epsilon+\beta_\epsilon}\end{aligned}$$

Then,

$$\|(u,v)\| \geq \left(\frac{1}{2|\Omega|^{\frac{\epsilon}{2^*}} C^{\alpha_\epsilon+\beta_\epsilon}}\right)^{\frac{1}{\alpha_\epsilon+\beta_\epsilon-2}} =: c_\epsilon$$

which shows (ii).

On $\mathcal{N}_\epsilon$, since $G_\epsilon(u,v)=0$ we have

$$I_\epsilon(u,v) = \left(\frac{1}{2} - \frac{1}{\alpha_\epsilon+\beta_\epsilon}\right)\|(u,v)\|^2 \geq \left(\frac{1}{2} - \frac{1}{\alpha_\epsilon+\beta_\epsilon}\right) c_\epsilon^2 > 0$$

and concludes the proof of (iii).

Let $u,v \neq 0$ and, for $t \geq 0$ define the map

$$g(t) := I_\epsilon(tv,tu) = \frac{t^2}{2}\|(u,v)\| - \frac{2t^{\alpha_\epsilon+\beta_\epsilon}}{\alpha_\epsilon+\beta_\epsilon}\int_\Omega |u|^{\alpha_\epsilon}|v|^{\beta_\epsilon}dx.$$

Since $\alpha_\epsilon+\beta_\epsilon > 2$, we have $g(0)=0$, $g(t)>0$ for small t and $g(t) =< 0$ for suitably large t. Then there is a $t_\epsilon = t_\epsilon(u,v) > 0$ such that $g'(t_\epsilon)=0$ and $g(t_\epsilon) = \max_{t>0} g(t)$, i.e. $t_\epsilon(u,v) \in \mathcal{N}_\epsilon$, proving (iv). It is easy to verify that t_ϵ is unique.

The proof of (v) and (vi) follows by standard arguments. □

Remark 1. *Actually in* (ii) *of Lemma 3.1.1 the constant c_ϵ can be made independent on ϵ. Indeed it is easily seen that* $\lim_{\epsilon\to0} c_\epsilon = \left(\frac{1}{2C^{2^*}}\right)^{\frac{1}{2^*-2}} =: c_0 > 0$. *Then, it is possible to take a small $\epsilon_0 > 0$ such that*

$$c_\epsilon > \xi = \frac{1}{2}c_0 > 0,$$

for all $\epsilon \in (0,\epsilon_0)$.

In other words, all the Nehari manifolds $\mathcal{N}_\epsilon$ are bounded away from zero, independently on ϵ, i.e. there exists $\xi > 0$ such that, for all $\epsilon \in (0,\epsilon_0)$

$$(u,v) \in \mathcal{N}_\epsilon \Longrightarrow \|(u,v)\| \geq \xi.$$

The Nehari manifold well-behaves with respect to the (PS) sequences. Again, since at this stage no compactness condition is involved, we can even state the result for $\epsilon \geq 0$.

Lemma 3.1.2. *Let $\epsilon \geq 0$ be fixed and $\{(u_n,v_n)\} \subset \mathcal{N}_\epsilon$ be a (PS) sequence for $I_\epsilon|_{\mathcal{N}_\epsilon}$. Then $\{u_n,v_n\}$ is a (PS) sequence for the free functional I_ϵ on $H_0^1(\Omega)\times H_0^1(\Omega)$.*

Now for $\epsilon > 0$ it is known that the free functional I_ϵ satisfies the (PS) condition on $H_0^1(\Omega)\times H_0^1(\Omega)$ and also when restricted to $\mathcal{N}_\epsilon$. In addition to the properties listed in Corollary 3.1.1, the manifold $\mathcal{N}_\epsilon$ is a natural constraint for I_ϵ in the sense that any $(u,v)\in\mathcal{N}_\epsilon$ critical point of $I_\epsilon|_{\mathcal{N}_\epsilon}$ is also a critical point for the free functional I_ϵ. Hence the (constraint) critical points we find are solutions of our problem since no Lagrange multipliers appear.

In particular, as a consequence of the (PS) condition we have

$$\forall\,\epsilon > 0 : \quad \mathfrak{m}_\epsilon := \min_{(u,v)\in\mathcal{N}_\epsilon} I_\epsilon(u,v) = I_\epsilon(\mathfrak{g}_\epsilon,\mathfrak{h}_\epsilon) > 0\,,$$

i.e. $\mathfrak{m}_\epsilon$ is achieved on functions, hereafter denoted with $(\mathfrak{g}_\epsilon,\mathfrak{h}_\epsilon)$. Since $(\mathfrak{g}_\epsilon,\mathfrak{h}_\epsilon)$ minimizes the energy I_ϵ, it will be called a *ground state*. Observe that $\mathfrak{g}_\epsilon,\mathfrak{h}_\epsilon \geq 0$ and are indeed positive by the maximum principle.

Remark 2. *We note that, for all $\epsilon \in (0, \epsilon_0)$, if $(w_\epsilon, z_\epsilon) \in \mathcal{N}_\epsilon$, then*

$$0 < \xi \leq \|(w_\epsilon, z_\epsilon)\|^2 = 2\int_\Omega |w_\epsilon|^{\alpha_\epsilon}|z_\epsilon|^{\beta_\epsilon} \leq 2|\Omega|^{\frac{\epsilon}{2^*}}|w_\epsilon|_{2^*}^{\alpha_\epsilon}|z_\epsilon|_{2^*}^{\beta_\epsilon} < 2|\Omega|\left(|w_\epsilon|_{2^*}^{\alpha_\epsilon} + |z_\epsilon|_{2^*}^{\beta_\epsilon}\right).$$

We deduce that the sequences $\{\|(w_\epsilon, z_\epsilon)\|\}$, $\{|(w_\epsilon, z_\epsilon)|_{2^*}\}$ and $\left\{\int_\Omega |\mathfrak{g}_\epsilon|^{\alpha_\epsilon}|\mathfrak{h}_\epsilon|^{\beta_\epsilon}dx\right\}$ are bounded away from zero.

In particular, this is true for the family of ground states $\{(\mathfrak{g}_\epsilon, \mathfrak{h}_\epsilon)\}$. This last fact will be used in the next sections and in particular in Proposition 3.1.7.

We address now two limit cases related to our equation involving the Laplacian operator. They involve the critical problems both in the domain Ω and in the whole space $\mathbb{R}^N$.

3.1.1 Behavior of the family of ground state levels $\{\mathfrak{m}_\epsilon\}$

We introduce the critical problem in the domain Ω. This is done in order to evaluate the limit of the ground state levels $\{\mathfrak{m}_\epsilon\}$ when $\epsilon \to 0$. The main theorem in this subsection is Theorem 3.1.9, which requires first some preliminary work.

Let us introduce the C^1 functional associated to $\epsilon = 0$,

$$I_0(u, v) := \frac{1}{2}\int_\Omega |\nabla u|^2 + |\nabla u|^2 dx - \frac{2}{2^*}\int_\Omega |u|^\alpha |v|^\beta dx, \qquad (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$$

whose critical points are the solutions of

$$\begin{cases} -\Delta u = \frac{2\alpha}{2^*}|u|^{\alpha-2}u|v|^\beta & \text{in } \Omega \\ -\Delta v = \frac{2\beta}{2^*}|u|^\alpha|v|^{\beta-2}v & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.1}$$

It is known that the lack of compactness of the embedding of $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$ implies that I_0 does not satisfies the (PS) condition at every level. This is due to the invariance with respect to the conformal scaling

$$u(\cdot) \longmapsto v_R(\cdot) := R^{N/2^*}v(R(\cdot)) \qquad (R > 1)$$

which leaves invariant the L^2-norm of the gradient as well as the $L^{2^*}-$norm, i.e. $|\nabla v_R|_2^2 = |\nabla v|_2^2$ and $|v_R|_{2^*}^{2^*} = |v|_{2^*}^{2^*}$.

Related to the critical problem we have the following:

Lemma 3.1.3. *If Ω is a star-shaped domain then there exists only the trivial solution to* (3.1.1).

Proof. Let $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be a solution to (3.1.1). According to elliptic regularity theory, we have $u, v \in C^1(\overline{\Omega})$. Thus, by using the Pohozaev identity (see e.g. [21]) we obtain

$$\frac{1}{2}\int_{\partial\Omega}(|\nabla u|^2 + |\nabla v|^2)\sigma.\nu d\sigma = N\frac{2}{2^*}\int_\Omega |u|^\alpha|v|^\beta dx - \frac{N-2}{2}\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx$$

where ν denotes the unit outward normal to $\partial\Omega$. Since (u, v) is a solution, one also has

$$\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx = \int_\Omega |u|^\alpha|v|^\beta dx$$

Now, combining the last two equalities we reach that $\int_{\partial\Omega}(|\nabla u|^2 + |\nabla v|^2)\sigma.\nu d\sigma \leq 0$ and we must have $u = v = 0$ since $\sigma.\nu > 0$ on $\partial\Omega$. □

Let
$$\mathcal{N}_0 = \left\{(u,v) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \setminus \{(0,0)\} : I_0'(u,v)(u,v) = 0\right\}$$
be the Nehari manifold associated to the critical problem (3.1.1). By Lemma 3.1.1 it results

$$\mathfrak{m}_0 := \inf_{(u,v)\in\mathcal{N}_0} I_0(u,v) > 0. \tag{3.1.2}$$

In contrast to the case $\epsilon > 0$, now $\mathfrak{m}_0$ is not achieved.

The value $\mathfrak{m}_0$ turns out to be an upper bound for the sequence of ground states levels $\{\mathfrak{m}_\epsilon\}$, as we will prove below. First we need a lemma.

Lemma 3.1.4. *Let $(w_1, w_2) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{(0,0)\}$ be fixed. For every $0 < \epsilon < 1$, let $t_\epsilon = t_\epsilon[w_1, w_2] > 0$ given in (iv) of Lemma 3.1.1, i.e. such that $t_\epsilon(w_1, w_2) \in \mathcal{N}_\epsilon$. Then*

$$\lim_{\epsilon\to 0} t_\epsilon = t_0 > 0 \quad \text{and} \quad t_0(w_1, w_2) \in \mathcal{N}_0.$$

Moreover if $(w_1, w_2) \in \mathcal{N}_0$, then $\lim_{\epsilon\to 0} t_\epsilon = 1$.

Proof. By definition

$$t_\epsilon^2 \int_\Omega |\nabla w_1|^2 + |\nabla w_2|^2 dx = 2t_\epsilon^{\alpha_\epsilon+\beta_\epsilon} \int_\Omega |w_1|^{\alpha_\epsilon} |w_2|^{\beta_\epsilon} dx. \tag{3.1.3}$$

Then

$$t_\epsilon = \left(\frac{\displaystyle\int_\Omega |\nabla w_1|^2 + |\nabla w_2|^2 dx}{2\displaystyle\int_\Omega |w_1|^{\alpha_\epsilon} |w_2|^{\beta_\epsilon} dx} \right)^{\frac{1}{\alpha_\epsilon+\beta_\epsilon-2}}.$$

Then the result follows by

$$\lim_{\epsilon\to 0} t_\epsilon = \left(\frac{\displaystyle\int_\Omega |\nabla w_1|^2 + |\nabla w_2|^2 dx}{2\displaystyle\int_\Omega |w_1|^{\alpha} |w_2|^{\beta} dx} \right)^{\frac{1}{2^*-2}} = t_0.$$

If $(w_1, w_2) \in \mathcal{N}_0$, then $t_0 = 1$. □

Proposition 3.1.5. *We have*
$$\limsup_{\epsilon\to 0} \mathfrak{m}_\epsilon \le \mathfrak{m}_0.$$

Proof. Fix $\delta > 0$. By definition of m_0 there exists $(\overline{u}, \overline{v}) \in \mathcal{N}_0$ such that

$$I_0(\overline{u}, \overline{v}) = \frac{1}{2}\|(\overline{u}, \overline{v}\|^2 - \frac{2}{2^*}\int_\Omega |\overline{u}|^\alpha |\overline{v}|^\beta < \mathfrak{m}_0 + \delta.$$

For every $0 < \epsilon < 1$, there exists a unique $t_\epsilon = t_\epsilon(\overline{u}, \overline{v}) > 0$ such that $t_\epsilon(\overline{u}, \overline{v}) \in \mathcal{N}_\epsilon$ and by Lemma 3.1.4 we know that $\lim_{\epsilon\to 0} t_\epsilon = 1$, since $(\overline{u}, \overline{v}) \in \mathcal{N}_0$.

Then
$$\mathfrak{m}_\epsilon \le I_\epsilon(t_\epsilon\overline{u}, t_\epsilon\overline{v}) = \frac{t_\epsilon^2}{2}\|(\overline{u}, \overline{v})\|^2 - \frac{2t_\epsilon^{\alpha_\epsilon+\beta_\epsilon}}{\alpha_\epsilon + \beta_\epsilon}\int_\Omega |\overline{u}|^{\alpha_\epsilon} |\overline{v}|^{\beta_\epsilon} dx$$
and so $\limsup_{\epsilon\to 0} \mathfrak{m}_\epsilon \le I_0(\overline{u}, \overline{v}) < \mathfrak{m}_0 + \delta$ concluding the proof.

□

In particular we deduce the following:

Corollary 3.1.6. *The family of minimizers* $\{(\mathfrak{g}_\epsilon, \mathfrak{h}_\epsilon)\}_{\epsilon>0}$ *is bounded in* $H_0^1(\Omega) \times H_0^1(\Omega)$.

Proof. By a direct calculation we get

$$\mathfrak{m}_\epsilon = I_\epsilon(\mathfrak{g}_\epsilon, \mathfrak{h}_\epsilon) - \frac{1}{\alpha_\epsilon + \beta_\epsilon} I'_\epsilon(\mathfrak{g}_\epsilon, \mathfrak{h}_\epsilon)[\mathfrak{g}_\epsilon, \mathfrak{h}_\epsilon] = \left(\frac{1}{2} - \frac{1}{\alpha_\epsilon + \beta_\epsilon}\right) \|(\mathfrak{g}_\epsilon \mathfrak{h}_\epsilon)\|^2.$$

Since $\lim_{\epsilon \to 0} \alpha_\epsilon + \beta_\epsilon = 2^*$ and by from Proposition 3.1.5, the result follows. □

It will be useful the next result:

Remark 3. *Corollary 3.1.6 can be generalized to arbitrary functions in the Nehari manifolds* $\mathcal{N}_\epsilon$, *not necessary the ground states, as long as the functionals converge.*

In other words, let $\epsilon_n \to 0$ *as* $n \to +\infty$. *If* $\{(w_n, z_n)\} \subset H_0^1(\Omega) \times H_0^1(\Omega)$ *is such that* $(w_n, z_n) \in \mathcal{N}_{\epsilon_n}$ *for every* n, *and* $I_{\epsilon_n}(w_n, z_n) \to l \in (0, +\infty)$ *as* $n \to \infty$, *then* $\{w_n, z_n\}$ *is bounded in* $H_0^1(\Omega) \times H_0^1(\Omega)$.

Indeed, similarly to the proof of Corollary 3.1.6, this easily follows from

$$l = I_{\epsilon_n}(w_n, z_n) - \frac{1}{\alpha_{\epsilon_n} + \beta_{\epsilon_n}} I'_{\epsilon_n}(w_n, z_n)[w_n, z_n] + o_n(1) = \left(\frac{1}{2} - \frac{1}{\alpha_{\epsilon_n} + \beta_{\epsilon_n}}\right) \|(w_n, z_n)\|^2 + o_n(1).$$

We need now a technical lemma about the "projections" of the minimizers $(\mathfrak{g}_\epsilon, \mathfrak{h}_\epsilon)$ on the Nehari manifold of the critical problem $\mathcal{N}_0$. Let us first observe the following remark which generalizes Lemma 3.1.4.

Remark 4. *If* $\{(w_\epsilon, z_\epsilon)\} \subset H_0^1(\Omega) \times H_0^1(\Omega)$ *is such that*

(a) for every $0 < \epsilon < 1 : (w_\epsilon, z_\epsilon) \in \mathcal{N}_\epsilon$,

(b) there exist $C_1, C_2 > 0$ *such that for every* $0 < \epsilon < 1 : 0 < C_1 \leq \int_\Omega |w_\epsilon|^{\alpha_\epsilon} |z_\epsilon|^{\beta_\epsilon} dx$ *and* $\|(w_\epsilon, z_\epsilon)\| \leq C_2$,

then setting $t_{0,\epsilon} > 0$ *such that* $t_{0,\epsilon}(w_\epsilon, z_\epsilon) \in \mathcal{N}_0$ *(see (iv) of Lemma 3.1.1), it holds*

$$0 < \lim_{\epsilon \to 0} t_{0,\epsilon} < +\infty. \tag{3.1.4}$$

By (a), the sequence (w_ϵ, z_ϵ) *is bounded in* $H_0^1(\Omega) \times H_0^1(\Omega)$ *and since* $I'_0(t_{0,\epsilon} w_\epsilon, t_{0,\epsilon} z_\epsilon)[(t_{0,\epsilon} w_\epsilon, t_{0,\epsilon} z_\epsilon)] = 0$ *we have*

$$t_{0,\epsilon}^{2^*-2} = \frac{\|(w_\epsilon, z_\epsilon)\|^2}{\displaystyle\int_\Omega |w_\epsilon|^\alpha |z_\epsilon|^\beta dx}.$$

proving (3.1.4).

Proposition 3.1.7. *Assume that* $\{(w_\epsilon, z_\epsilon)\} \subset H_0^1(\Omega) \times H_0^1(\Omega)$ *is such that*

(a) for every $0 < \epsilon < 1 : (w_\epsilon, z_\epsilon) \in \mathcal{N}_\epsilon$,

(b) there exist $C_1, C_2 > 0$ *such that*

$$0 < \epsilon < 1 : 0 < C_1 \leq \int_\Omega |w_\epsilon|^\alpha |z_\epsilon|^\beta dx \quad \textit{and} \quad \|(w_\epsilon, z_\epsilon)\| \leq C_2,$$

(c) $w_\epsilon \geq 0$ *and* $z_\epsilon \geq 0$ *for every* $0 < \epsilon < 1$.

Let $t_0[w_\epsilon, z_\epsilon] > 0$ *the unique value such that* $t_0[w_\epsilon, z_\epsilon](w_\epsilon, z_\epsilon) \in \mathcal{N}_0$. *Then*

$$\lim_{\epsilon\to 0} t_0[w_\epsilon, z_\epsilon] = 1.$$

In particular

$$\lim_{\epsilon\to 0} t_0[\mathfrak{g}_\epsilon, \mathfrak{h}_\epsilon] = 1.$$

Proof. We assume that $\epsilon_n \to 0$ as $n \to +\infty$, $(w_n, z_n) := (w_{\epsilon_n}, z_{\epsilon_n}) \in \mathcal{N}_{\epsilon_n}$ and $t_{0,n} = t_0[w_{\epsilon_n}, z_{\epsilon_n}]$. By Remark 4 we can assume that

$$\lim_{n\to+\infty} t_{0,n} = \tilde{t}_0 > 0.$$

Observe now that, since $(w_n, z_n) \in \mathcal{N}_{\epsilon_n}$, up to subsequences, we have

$$\|(w_n, z_n)\|^2 = 2\int_\Omega |w_n|^{\alpha_{\epsilon_n}} |z_n|^{\beta_n} dx \to L \geq 0. \tag{3.1.5}$$

Since Nehari manifold is uniformly bounded away from zero (see Remark 1), we have $L > 0$.

By the definition of $t_{0,n}$,

$$t_{0,n}^2 \|(w_n, z_n)\|^2 = 2t_{0,n}^{2^*} \int_\Omega |w_n|^\alpha |z_n|^\beta dx$$

Then,

$$\tilde{t}_0^{2^*-2} = \lim_{n\to+\infty} t_{0,n}^{2^*-2} = \lim_{n\to+\infty} \frac{\|(w_n, z_n)\|^2}{2\displaystyle\int_\Omega |w_n|^\alpha |z_n|^\beta dx} = 1$$

and the conclusion follows

Finally, since $\left\{\displaystyle\int_\Omega |\mathfrak{g}_\epsilon|^{\alpha_\epsilon} |\mathfrak{h}_\epsilon|^{\beta_\epsilon} dx\right\}$ is bounded away from zero by Remark 2, and $\{(\mathfrak{g}_\epsilon, \mathfrak{h}_\epsilon)\}$ is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$ by Corollary 3.1.6, we have that $\{(\mathfrak{g}_\epsilon, \mathfrak{h}_\epsilon)\}$ satisfy (b), and also (a) and (c). □

Thanks to the previous result we get the next:

Proposition 3.1.8. *We have*

$$\mathfrak{m}_0 \leq \liminf_{\epsilon\to 0} \mathfrak{m}_\epsilon.$$

Proof. For $\epsilon_n \to 0$ and $(\mathfrak{g}_n, \mathfrak{h}_n) := (\mathfrak{g}_{\epsilon_n}, \mathfrak{h}_{\epsilon_n}) \in \mathcal{N}_{\epsilon_n}$ by Corollary 3.1.6 we have $(\mathfrak{g}_n, \mathfrak{h}_n) \rightharpoonup (u, v)$ in $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$. With $t_{0,n}(\mathfrak{g}_n, \mathfrak{h}_n) \in \mathcal{N}_0$, we get

$$2\int_\Omega |\mathfrak{g}_n|^{\alpha_{\epsilon_n}} |\mathfrak{h}_n|^{\beta_{\epsilon_n}} dx = \|(\mathfrak{g}_n, \mathfrak{h}_n)\|^2, \qquad 2t_{0,n}^{2^*} \int_\Omega |\mathfrak{g}_n|^\alpha |\mathfrak{h}_n|^\beta dx = t_{0,n}^2 \|(\mathfrak{g}_n, \mathfrak{h}_n)\|^2$$

By Proposition 3.1.7 we have $t_{0,n} \to 1$ and since $(\mathfrak{g}_n, \mathfrak{h}_n)$ is bounded

$$\int_\Omega |\mathfrak{g}_n|^{\alpha_{\epsilon_n}} |\mathfrak{h}_n|^{\beta_{\epsilon_n}} dx - t_{0,n}^{2^*} \int_\Omega |\mathfrak{g}_n|^\alpha |\mathfrak{h}_n|^\beta dx = 1/2(1 - t_{0,n}^2)\|(\mathfrak{g}_n, \mathfrak{h}_n)\|^2 = o_n(1)$$

Also by definition

$$\mathfrak{m}_{\epsilon_n} = \frac{1}{2}\|(\mathfrak{g}_n, \mathfrak{h}_n)\|^2 - \frac{2}{\alpha_{\epsilon_n} + \beta_{\epsilon_n}} \int_\Omega |\mathfrak{g}_n|^{\alpha_{\epsilon_n}} |\mathfrak{h}_n|^{\beta_{\epsilon_n}} dx.$$

Then we get

$$\begin{aligned}
\mathfrak{m}_0 &\leq I_0(t_{0,n}\mathfrak{g}_n, t_{0,n}\mathfrak{h}_n) = \frac{t_{0,n}^2}{2}\|(\mathfrak{g}_n, \mathfrak{h}_n)\|^2 - \frac{2t_{0,n}{}^{2^*}}{2^*}\int_\Omega |\mathfrak{g}_n|^\alpha |\mathfrak{h}_n|^\beta \\
&= t_{0,n}^2 \mathfrak{m}_{\epsilon_n} + \frac{2t_{0,n}{}^2}{\alpha_{\epsilon_n} + \beta_{\epsilon_n}} \int_\Omega |\mathfrak{g}_n|^{\alpha_{\epsilon_n}} |\mathfrak{h}_n|^{\beta_{\epsilon_n}} dx - \frac{2t_{0,n}^{2*}}{2^*}\int_\Omega |\mathfrak{g}_n|^{2^*} |\mathfrak{h}_n|^{2^*} dx \\
&= t_{0,n}^2 \mathfrak{m}_{\epsilon_n} + o_n(1).
\end{aligned}$$

and passing to the limit we deduce $\mathfrak{m}_0 \leq \liminf_{n\to+\infty} \mathfrak{m}_{\epsilon_n}$. □

By Proposition 3.1.5 and Proposition 3.1.8 we deduce the following desired result.

Theorem 3.1.9. *For any bounded domain* Ω*, it holds*

$$\lim_{\epsilon\to 0} \mathfrak{m}_\epsilon = \mathfrak{m}_0.$$

3.1.2 A local Palais-Smale condition for I_0

To show the local Palais-Smale condition for I_0 it will be useful the next auxiliary result. The constant $S_{\alpha\beta}$ is defined as follow

$$S_{\alpha\beta} = \inf_{u,v \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\|(u,v)\|^2}{\left(2\int_\Omega |u|^\alpha |v|^\beta dx\right)^{\frac{2}{2^*}}}$$

Lemma 3.1.10. *Let* $\{(u_n, v_n)\}$ *be a* (PS) *sequence for the functional* I_0 *at level* $d \in \mathbb{R}$*. Then, up to subsequences*

1. $(u_n, v_n) \rightharpoonup (u, v)$ *in* $H_0^1(\Omega) \times H_0^1(\Omega)$*,*
2. $I_0'(u, v) = 0$*, i.e.* (u, v) *is a solution of* (3.1.1)*,*
3. *setting,* $w_n := u_n - u$ *and* $z_n := v_n - v$ *, then*

$$I_0(u_n, v_n) = I_0(u, v) + I_0(w_n, z_n) + o_n(1) \quad \textit{and} \quad I_0'(w_n, z_n) \to 0.$$

In particular $\{(w_n, z_n)\}$ *is a* (PS) *sequence for* I_0 *at level* $d - I_0(u, v)$*.*

Proof. If $d \in \mathbb{R}$, $I_0(u_n, v_n) \to d$ and $I_0'(u_n, v_n) \to 0$ then

$$I_0(u_n, v_n) - \frac{1}{2^*} I_0'(u_n, v_n)[u_n, v_n] \leq C(1 + \|(u_n, v_n)\|).$$

On the other hand, by the above computation

$$I_0(u_n, v_n) - \frac{1}{2^*} I_0'(u_n, v_n)[u_n, v_n] = \left(\frac{1}{2} - \frac{1}{2^*}\right) \|(u_n, v_n)\|^2$$

and the boundedness of $\{(u_n, v_n)\}$ follows.

Then we can assume that $(u_n, v_n) \rightharpoonup (u, v)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$ and $\{u_n\}$, $\{v_n\}$ have strong convergence in $L^s(\Omega), s \in [1, 2^*)$ and $u_n(x) \to u(x)$, $v_n(x) \to v(x)$ a.e. in Ω.

For all $\phi, \psi \in C_0^\infty(\Omega)$, we have that $I_0'(u_n, v_n)[\phi, \psi] \to 0$. Then we conclude that $I_0'(u,v)[\phi,\psi] = 0$, for all $\phi \in C_0^\infty(\Omega)$. By density, we get that $I_0'(u,v)(\phi,\psi) = 0$ for all $\phi, \psi \in W_0^{1,2}(\Omega)$.

The last item is a consequence of Brezis-Lieb splitting Lemma. □

Then we have the local (PS) condition for the functional I_0.

Proposition 3.1.11. *The functional I_0 satisfies the (PS) condition at level $d \in \mathbb{R}$, for*

$$d < \frac{1}{N} S_{\alpha\beta}^{N/2}.$$

Proof. Let $\{(u_n, v_n)\}$ be a $(PS)_d$ sequence for I_0. We know that $(u_n, v_n) \rightharpoonup (u,v)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$, $I_0'(u,v) = 0$ and $I_0(u,v) \geq 0$. By defining $w_n := u_n - u$ and $z_n := v_n - v$, we have (w_n, z_n) (PS) sequence for I_0, then

$$\int_\Omega |\nabla w_n|^2 + |\nabla z_n|^2 dx \to A \geq 0, \qquad \int_\Omega |w_n|^\alpha |z_n|^\beta dx \to A \geq 0. \tag{3.1.6}$$

All that we need to show is that $A = 0$. By contradiction, suppose $A > 0$. Note that

$$S_{\alpha\beta} \leq \frac{\int_\Omega |\nabla w_n|^2 + |\nabla z_n|^2 dx}{\left(\int_\Omega |w_n|^\alpha |z_n|^\beta dx\right)^{\frac{2}{2^*}}} = \frac{A}{A^{\frac{2}{2^*}}} + o_n(1)$$

implies that $S_{\alpha\beta}^{N/2} \leq A$. By using the Brezis-Lieb splitting we have

$$\begin{aligned} d + o_n(1) &= I_0(u_n, v_n) - \frac{1}{2^*} I_0'(u_n, v_n)[u_n, v_n] = \frac{1}{N} \|(u_n, v_n)\|^2 \\ &= \frac{1}{N} \|(w_n, z_n)\|^2 + \frac{1}{N} \|(u,v)\|^2 \geq \frac{1}{N} A \geq \frac{1}{N} S_{\alpha\beta}^{N/2} \end{aligned}$$

and this contradiction implies that $A = 0$, concluding the proof. □

3.1.3 A global compactness result

In order to prove our multiplicity results we need to deal with another "limit" functional, now related to the critical problem in the whole $\mathbb{R}^N$.

Let us introduce the space $D^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N) \right\}$ which can also be characterized as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the (squared) norm

$$\|u\|^2_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

A function in $H_0^1(\Omega)$ can be thought as an element of $D^{1,2}(\mathbb{R}^N)$.

Let us define the functional

$$\widehat{I}(u,v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 dx - \frac{2}{2^*} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx$$

whose critical points are the weak solutions of

$$\begin{cases} -\Delta u = \frac{2\alpha}{2^*} |u|^{\alpha-2} u |v|^\beta & \text{in } \mathbb{R}^N \\ -\Delta v = \frac{2\beta}{2^*} |u|^\alpha |v|^{\beta-2} v & \text{in } \mathbb{R}^N \\ (u,v) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N). \end{cases} \tag{3.1.7}$$

Setting as usual

$$\widehat{\mathcal{N}} = \left\{(u,v) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \setminus \{(0,0)\} : \widehat{I}'(u,v)(u,v) = 0\right\},$$

all the solutions of (3.1.7) are in $\widehat{\mathcal{N}}$; it is a differentiable manifold, bounded away from zero, and

$$\widehat{\mathfrak{m}} := \inf_{(u,v)\in\widehat{\mathcal{N}}} \widehat{I}(u,v) > 0.$$

The proof of these facts is exactly as in (i)-(iii) of Lemma 3.1.1.

As a matter of notation, in the rest of the paper given a function $z \in D^{1,2}(\mathbb{R}^N), \xi \in \mathbb{R}^N$ and $R > 0$, we define the *conformal rescaling* $z_{R,\xi}$ as

$$z_{R,\xi}(x) := R^{N/2^*} z(R(x-\xi)). \tag{3.1.8}$$

Of course $\|z\|_{D^{1,2}(\mathbb{R}^N)} = \|z_{R,\xi}\|_{D^{1,2}(\mathbb{R}^N)}$.

We need the following important Lemma.

Lemma 3.1.12. *Let $\{(w_n, z_n)\}$ be a $(PS)_c$ sequence for I_0 such that $(w_n, z_n) \rightharpoonup (0,0)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$. Then there exist sequences $\{x_n\} \subset \Omega, \{R_n\} \subset (0,+\infty)$ with $R_n \to +\infty$, and a nontrivial solution $(\widehat{u}, \widehat{v})$ of (3.1.7) such that, up to subsequences,*

- *(a) $\widehat{w}_n := w_n - \widehat{u}_{R_n,x_n} + o_n(1)$ and $\widehat{z}_n := z_n - \widehat{v}_{R_n,x_n} + o_n(1)$ is a (PS) sequence for I_0 in $H_0^1(\Omega) \times H_0^1(\Omega)$,*
- *(b) $(\widehat{w}_n, \widehat{z}_n) \rightharpoonup (0,0)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$,*
- *(c) $I_0(\widehat{w}_n, \widehat{z}_n) = I_0(w_n, z_n) - \widehat{I}(\widehat{u}, \widehat{v}) + o_n(1)$,*
- *(d) $R_n d(x_n, \partial\Omega) \to +\infty$,*
- *(e) if $c < c^* := \frac{1}{N} S_{\alpha\beta}^{N/2}$ then $\{(w_n, z_n)\}$ is relatively compact; in particular $(w_n, z_n) \to (0,0)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$ and $I_0(w_n, z_n) \to \beta = 0$.*

Proof. If $c \in (0, \frac{1}{N} S_{\alpha\beta}^{N/2})$, by Proposition 3.1.11(*iii*), we have (w_n, z_n) strongly convergent. Then suppose that $c \geq \frac{1}{N} S_{\alpha\beta}^{N/2}$. Let the Lévy concentration function be

$$Q_n(\lambda) := \sup_{y\in\mathbb{R}^N} \int_{B_\lambda(y)} |w_n|^\alpha |z_n|^\beta dx$$

Note that there exists $(x_n, \lambda_n) \in \mathbb{R}^N \times (0,\infty)$ such that

$$Q_n(\lambda_n) := \int_{B_{\lambda_n}(x_n)} |w_n|^\alpha |z_n|^\beta dx = \frac{1}{2} S_{\alpha\beta}^{N/2}.$$

Setting

$$\overline{w}_n(x) = \lambda_n^{\frac{N-2}{2}} w_n(\lambda_n(x+x_n)),$$
$$\overline{z}_n(x) = \lambda_n^{\frac{N-2}{2}} z_n(\lambda_n(x+x_n)),$$

we have

$$\sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\overline{w}_n|^\alpha |\overline{z}_n|^\beta dx = \int_{B_1} |\overline{w}_n|^\alpha |\overline{z}_n|^\beta dx = \frac{1}{2} S_{\alpha\beta}^{N/2}.$$

Moreover,

$$\int_{\Omega_n} |\overline{w}_n|^\alpha |\overline{z}_n|^\beta dx = \int_\Omega |w_n|^\alpha |z_n|^\beta dx,$$
$$\int_{\Omega_n} [|\nabla \overline{w}_n|^2 + |\nabla \overline{z}_n|^2] dx = \int_\Omega [|\nabla w_n|^2 + |\nabla z_n|^2] dx.$$

where $\Omega_n = \frac{1}{\lambda_n}(\Omega - x_n)$. In what follows, Ω_∞ is the limit set of Ω_n. For each $\{(\Phi_n, \Psi_n)\} \subset H_0^1(\Omega_n) \times H_0^1(\Omega_n)$ with bounded norm in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, we get

$$\begin{aligned} o_n(1) &= \int_{\mathbb{R}^N} [\nabla \overline{w}_n \nabla \Phi_n + \nabla \overline{z}_n \nabla \Psi_n] dx \\ &- \frac{2}{2^*} \int_{\mathbb{R}^N} |\overline{w}_n|^{\alpha-2} \overline{w}_n |\overline{z}_n|^\beta \Phi_n + |\overline{w}_n|^\alpha |\overline{z}_n|^{\beta-2} \overline{z}_n \Psi_n] dx. \end{aligned} \tag{3.1.9}$$

since, setting $\overline{\Phi}_n(x) = \lambda_n^{\frac{2-N}{2}} \Phi_n(\frac{1}{\lambda_n}(x - x_n))$ and $\overline{\Psi}_n(x) = \lambda_n^{\frac{2-N}{2}} \Psi_n(\frac{1}{\lambda_n}(x - x_n))$, we have that (3.1.9) is equivalent to

$$I_0'(w_n, z_n)(\overline{\Phi}_n, \overline{\Psi}_n) = o_n(1)$$

Let $(\widehat{u}, \widehat{v})$ be the weak limit of $\{\overline{w}_n, \overline{z}_n\} \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$. Now, we wil show that $\widehat{u}, \widehat{v} \neq 0$.

Suppose by contradiction $\widehat{u} = \widehat{v} = 0$. Applying Lemma 6 from [13] there exists $(x_j) \subset \mathbb{R}^N$, (μ_j), (σ_j) and $(\nu_j) \subset (0, \infty)$, where J is at most a countable set, such that

$$\|\overline{w}_n\|^2 + \|\overline{z}_n\|^2 \rightharpoonup \mu + \sigma \geq \sum_{j=1}^k (\mu_j + \sigma_j) \delta_{x_j}$$

in the sense of measures and

$$\int_{\mathbb{R}^N} |\overline{w}_n|^\alpha |\overline{z}_n|^\beta \phi dx \rightharpoonup \nu = \sum_{j=1}^k \nu_j \phi(x_j)$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$. Moreover, $\mu_j + \sigma_j \geq \nu_j S_{\alpha\beta}^{\frac{2}{2^*}}$. We can conclude that $x_j \in \overline{\Omega_\infty}$ and J is finite or empty. If we suppose $\nu_j > 0$ for some j, by well known arguments we get $\nu_j \geq S_{\alpha\beta}^{N/2}$. By properties of $(\overline{w}_n, \overline{z}_n)$ we get

$$\frac{1}{2} S_{\alpha\beta}^{N/2} = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\overline{w}_n|^\alpha |\overline{z}_n|^\beta dx \geq \int_{B_1(x_j)} |\overline{w}_n|^\alpha |\overline{z}_n|^\beta dx \geq \int_{B_1(x_j)} |\overline{w}_n|^\alpha |\overline{z}_n|^\beta \phi_\epsilon(x) dx$$

where $\phi_\epsilon \in C_0^\infty(\mathbb{R}^N)$, $\phi_\epsilon = 1$ in $B_\epsilon(x_j)$ and $\phi_\epsilon = 0$ in $B_{3\epsilon}^c(x_j)$. Then passing to the limit $n \to \infty$ and $\epsilon \to 0$ we get

$$\frac{1}{2} S_{\alpha\beta}^{N/2} \geq \nu_j$$

which is a contradiction. Then J is empty and for all $\phi \in C_0^\infty(\mathbb{R}^N)$ we get

$$\int_{\mathbb{R}^N} |\overline{w}_n|^\alpha |\overline{z}_n|^\beta \phi dx \rightharpoonup \nu = 0$$

which is an absurd since

$$0 < \frac{1}{2} S_{\alpha\beta}^{N/2} = \int_{B_1} |\overline{w}_n|^\alpha |\overline{z}_n|^\beta dx \leq \int_{\mathbb{R}^N} |\overline{w}_n|^\alpha |\overline{z}_n|^\beta dx.$$

Consequently, $\widehat{u}, \widehat{v} \neq 0$. Using the fact the $\widehat{u}, \widehat{v} \neq 0$ we have that $\lambda_n \longrightarrow 0$, because if there exists $\delta > 0$ such that $\lambda_n \geq \delta$, we have the following inequality

$$\int_{\mathbb{R}^N} [|\overline{w}_n|^2 + |\overline{z}_n|^2] dx = \frac{1}{\lambda_n^2} \int_{\mathbb{R}^N} [|w_n|^2 + |z_n|^2] dx \leq \frac{1}{\delta^2} \int_{\mathbb{R}^N} [|w_n|^2 + |z_n|^2] dx$$

Since $(w_n, z_n) \longrightarrow (0,0)$ in $L^2(\Omega) \times L^2(\Omega)$ it follows that

$$\int_{\mathbb{R}^N} [|\widehat{u}|^2 + |\widehat{v}|^2] dx = 0,$$

which is a contradiction. Then $\lambda_n \longrightarrow 0$. Since Ω is bounded, we may assume that there exists $x_0 \in \overline{\Omega}$ such that $x_n \longrightarrow x_0$. By weak continuity of $(\overline{w}_n, \overline{z}_n)$ and (3.1.9) the function $(\widehat{u}, \widehat{v})$ is a solution of the problem

$$\begin{cases} -\Delta u = \dfrac{2\alpha}{2^*} |u|^{\alpha-2} u |v|^{\beta} \ \text{ in } \ \Omega_\infty, \\ -\Delta v = \dfrac{2\beta}{2^*} |u|^{\alpha} |v|^{\beta-2} v \ \text{ in } \ \Omega_\infty, \\ u, v \geq 0, u, v \neq 0 \ \text{ in } \ \Omega_\infty, \\ u = v = 0 \text{ in } \ \partial\Omega_\infty. \end{cases}$$

Then we have to consider two cases:

(A) $\frac{1}{\lambda_n} dist(x_n, \partial\Omega) \to \infty$ as $n \to \infty$,

(B) $\frac{1}{\lambda_n} dist(x_n, \partial\Omega) \leq \alpha$ for all $n \in \mathbb{N}$ for some $\alpha > 0$.

Assume by contradiction that (B) holds and without loss of generality that $x_n \longrightarrow 0 \in \partial\Omega$.

Moreover there exists $\delta > 0$, an open neighborhood $\mathcal{N}$ of 0 and a diffeomorphism $\Psi : B_\delta(0) \longrightarrow \mathcal{N}$ which has a jacobian determinant at 0 equal to 1, with $\Psi(B_\delta^+) = \mathcal{N} \cap \Omega$ where $B_\delta^+ = B_\delta(0) \cap \{x^N > 0\}$.

Now let us define the functions $(\xi_n, \zeta_n) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ given by

$$\xi_n(x) = \begin{cases} \lambda_n^{\frac{N-2}{2}} w_n(\Psi(\lambda_n x + P_n)) \chi(\Psi(\lambda_n x + P_n)), \quad x \in B_{\delta/\lambda_n}(-P_n/\lambda_n) \\ 0, \quad x \in \mathbb{R}^N \setminus B_{\delta/\lambda_n}(-P_n/\lambda_n) \end{cases}$$

$$\zeta_n(x) = \begin{cases} \lambda_n^{\frac{N-2}{2}} z_n(\Psi(\lambda_n x + P_n)) \chi(\Psi(\lambda_n x + P_n)), \quad x \in B_{\delta/\lambda_n}(-P_n/\lambda_n) \\ 0, \quad x \in \mathbb{R}^N \setminus B_{\delta/\lambda_n}(-P_n/\lambda_n) \end{cases}$$

where $P_n = \Psi(x_n)$, $\chi \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \chi \leq 1$, $\chi \equiv 1$ in $\Psi(B_{\frac{\delta}{2}})$ and $\chi \equiv 0$ in $\Psi(B_{\frac{3\delta}{4}})^c$. It is possible to show that for some subsequence

$$\frac{P_n^N}{\lambda_n} \longrightarrow \alpha_0 \quad \text{for some } \alpha_0 \geq 0 \text{ as } n \to \infty$$

and there exists $\xi, \zeta \in D_0^{1,2}(\{x^N > -\alpha_0\})$ such that $\xi_n \to \xi$ and $\zeta_n \to \zeta$ in $D^{1,2}(\mathbb{R}^N)$ wich satisfies

$$\begin{cases} -\Delta \xi = \dfrac{2}{2^*} |\xi|^{\alpha-2} \xi |\zeta|^{\beta} \ \text{ in } \ \{x^N > -\alpha_0\}, \\ -\Delta \zeta = \dfrac{2}{2^*} |\xi|^{\alpha} |\zeta|^{\beta-2} \zeta \ \text{ in } \ \{x^N > -\alpha_0\}, \\ \zeta \geq 0, \xi \geq 0 \text{ in } \ \{x^N > -\alpha_0\}, \\ \xi = \zeta = 0 \ \text{ on } \{x^N = -\alpha_0\}. \end{cases}$$

From Proposition 3.1.3, we have $\xi = \zeta \equiv 0$. On the other hand,

$$\int_{B_1} [\overline{w_n}^2 + \overline{z_n}^2] dx \leq C \int_{\mathcal{A}} [\xi_n^2 + \zeta_n^2] dx$$

where $\mathcal{A} \subset \{x^N > \alpha_0\}$ is a bounded domain. Since $\{\xi_n\}$ is a bounded sequence in $W^{1,2}(\mathcal{A})$ by Sobolev embedding

$$\int_{\mathcal{A}} [\xi_n^2 + \zeta_n^2] dx \longrightarrow 0.$$

Then,

$$\int_{B_1} [\overline{w_n}^2 + \overline{z_n}^2] dx \longrightarrow 0.$$

and so $\widehat{u} = \widehat{v} \equiv 0$ in B_1 which is a contradiction. Thus Case (A) holds, so that $\Omega_\infty = \mathbb{R}^N$ and $(\widehat{u}, \widehat{v})$ is a solution of 3.1.7.

To conclude, we consider $\Phi \in C_0^\infty(\mathbb{R}^N)$ verifying $0 \leq \Phi \leq 1$, $\Phi \equiv 1$ in B_1 and $\Phi \equiv 0$ in B_2^c. Let

$$\tilde{w}_n(x) = w_n(x) - \lambda_n^{\frac{2-N}{2}} \widehat{u}(\frac{1}{\lambda_n}(x - x_n))\Phi(\frac{1}{\overline{\lambda}_n}(x - x_n)),$$
$$\tilde{z}_n(x) = z_n(x) - \lambda_n^{\frac{2-N}{2}} \widehat{v}(\frac{1}{\lambda_n}(x - x_n))\Phi(\frac{1}{\overline{\lambda}_n}(x - x_n)).$$

where we choose $\overline{\lambda}_n$ verifying $\widetilde{\lambda}_n = \frac{\lambda_n}{\overline{\lambda}_n} \longrightarrow 0$. Considering

$$\widehat{w}_n(x) = \lambda_n^{\frac{N-2}{2}} \tilde{w}_n(\lambda_n x + x_n) = \overline{w}_n(x) - \widehat{u}(x)\Phi(\widetilde{\lambda}_n x),$$
$$\widehat{z}_n(x) = \lambda_n^{\frac{N-2}{2}} \tilde{z}_n(\lambda_n x + x_n) = \overline{z}_n(x) - \widehat{v}(x)\Phi(\widetilde{\lambda}_n x)$$

and by repeating the same arguments explored in [14], we conclude the proof. □

Now we can prove the following "splitting lemma", which is useful to study the behaviour of the (PS) sequences for the limit functional I_0 related to the critical problem in the domain Ω.

In particular it says that, if the (PS) sequences does not converges strongly to their weak limit, then this is due to the solutions of the problem in the whole $\mathbb{R}^N$.

Lemma 3.1.13 (Splitting)**.** *Let $\{u_n, v_n\} \subset H_0^1(\Omega) \times H_0^1(\Omega)$ be a (PS) sequence for the functional I_0. Then either $\{u_n, v_n\}$ is convergent in $H_0^1(\Omega) \times H_0^1(\Omega)$, or there exist*

i. a solution $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega) \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ of problem (3.1.1),

ii. a number $k \in \mathbb{N}, k$ sequences of points $\{x_n^j\} \subset \Omega$ and k sequences $\{R_n^j\}$ with $R_n^j \to +\infty$, where $j = 1, \ldots, k$,

iii. nontrivial solutions $\{(u^j, v^j\}_{j=1,\ldots,k} \subset D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ of problem (3.1.7)

such that, up to subsequences,

$$u_n - u_0 = \sum_{j=1}^{k} u^j_{R_n^j, x_n^j} + o_n(1) \quad in \quad D^{1,2}(\mathbb{R}^N) \tag{3.1.10}$$

$$v_n - v_0 = \sum_{j=1}^{k} v^j_{R_n^j, x_n^j} + o_n(1) \quad in \quad D^{1,2}(\mathbb{R}^N) \tag{3.1.11}$$

$$I_0(u_n, v_n) = I_0(u_0, v_0) + \sum_{j=1}^{k} \widehat{I}(u^j, v^j) + o_n(1). \tag{3.1.12}$$

Proof. We already know (see Lemma 3.1.10) that $\{(u_n, v_n)\}$ is bounded and then we can assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$,and $(u_0, v_0$ is a weak solution of (3.1.1) and $|I_0(v_n)| \leq C$. Assume that $\{u_n\}$ and $\{v_n\}$ does not converges strongly to u_0 and v_0.

Let $(w_n^1, z_n^1) := (u_n, v_n) - (u_0, v_0) \rightharpoonup 0$. Then by Lemma 3.1.10, $\{(w_n^1, z_n^1)\}$ is a (PS) sequence for I_0 and

$$I_0(u_n, v_n) = I_0(u_0, v_0) + I_0(w_n^1, z_n^1) + o_n(1). \tag{3.1.13}$$

By Lemma 3.1.12 applied to $\{(w_n^1, z_n^1)\}$, we get the existence of sequences $\{x_n^1\} \subset \Omega, \{R_n^1\} \subset (0, +\infty)$ with $R_n^1 \to +\infty$ and $(u^1, v^1) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ solution of (3.1.1), such that

(1a) defining $(w_n^2, z_n^2) := (w_n^1, z_n^1) - (u^1_{R_n^1, x_n^1}, v^1_{R_n^1, x_n^1}) + o_n(1)$ with $o_n(1) \to 0$ in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, and $\{(w_n^2, z_n^2)\}$ is a (PS) sequence for I_0,

(1b) $(w_n^2, z_n^2) \rightharpoonup (0, 0)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$,

(1c) $I_0(w_n^2, z_n^2) = I_0(w_n^1, z_n^1) - \widehat{I}(u^1, v^1) + o_n(1)$,

(1d) $R_n^1 d(x_n^1, \partial\Omega) \to +\infty$,

(1e) if $I_0(w_n^1, z_n^1) \to \beta < \beta^*$, then $\{(w_n^1, z_n^1)\}$ is relatively compact; in particular $(w_n^1, z_n^1) \to (0, 0)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$ and $I_0(w_n^1, z_n^1) \to 0$.

Then by (1c) equation (3.1.13) becomes

$$I_0(u_n, v_n) = I_0(u_0, v_0) + I_0(w_n^2, z_n^2) + \widehat{I}(u^1, v^1) + o_n(1). \tag{3.1.14}$$

Note that, by definitions, $w_n^2 = u_n - u_0 - u^1_{R_n^1, x_n^1} + o_n(1)$ and $z_n^2 = v_n - v_0 - v^1_{R_n^1, x_n^1} + o_n(1)$ Hence, if $\{(w_n^2, z_n^2)\}$ is strongly convergent to zero, the Theorem is proved with $k = 1$. Otherwise, in virtue of $(1a)$ and $(1b)$, we can apply Lemma 3.1.12 to the sequence $\{(w_n^2, z_n^2)\}$: then we get the existence of sequences $\{x_n^2\} \subset \Omega, \{R_n^2\} \subset (0, +\infty)$ with $R_n^2 \to +\infty$ and $(u^2, v^2) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ solution of (3.1.1), such that

(2a) $(w_n^3, z_n^3) := (w_n^2, z_n^2) - (u^2_{R_n^2, x_n^2}, v^2_{R_n^2, x_n^2}) + o_n(1)$ with $o_n(1) \to 0$ in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, and $\{(w_n^3, z_n^3)\}$ is a (PS) sequence for I_0,

(2b) $(w_n^3, z_n^3) \rightharpoonup (0, 0)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$,

(2c) $I_0(w_n^3, z_n^3) = I_0(w_n^2, z_n^2) - \widehat{I}(u^2, v^2) + o_n(1)$,

(2d) $R_n^2 d(x_n^2, \partial\Omega) \to +\infty$,

(2e) if $I_0(w_n^2, z_n^2) \to \beta < \beta^*$, then $\{(w_n^2, z_n^2)\}$ is relatively compact; in particular $(w_n^2, z_n^2) \to (0, 0)$ in $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ and $I_0(w_n^2, z_n^2) \to 0$.

Then by (3.1.14) and (2c):

$$I_0(u_n, v_n) = I_0(u_0, v_0) + I_0(w_n^3, z_n^3) + \widehat{I}(u^1, v^1) + \widehat{I}(u^2, v^2) + o_n(1).$$

Let $w_n^3 = u_n - u_0 - u^1_{R^1_n,x^1_n} + u^2_{R^2_n,x^2_n} + o_n(1)$ and $z_n^3 = v_n - v_0 - v^1_{R^1_n,x^1_n} + v^2_{R^2_n,x^2_n} + o_n(1)$. If $\{(w_n^3, z_n^3)\}$ is strongly convergent to $(0,0)$, the theorem is proved with $k = 2$, otherwise we can repeat the arguments.

By arguing in this way, at the $j-$th stage ($j > 1$) we have: $(w_n^{j-1}, z_n^{j-1}) \rightharpoonup (0,0)$ in $H^1_0(\Omega) \times H^1_0(\Omega)$ and we get the existence of sequences $\{x_n^{j-1}\} \subset \Omega, \{R_n^{j-1}\} \subset (0,+\infty)$ with $R_n^{j-1} \to +\infty$ and $(u^{j-1}, v^{j-1}) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ solution of (3.1.1), such that

(ja) $w_n^j := w_n^{j-1} - v^{j-1}_{R_n^{j-1},x_n^{j-1}} + o_n(1)$ with $o_n(1) \to 0$ in $D^{1,p}(\mathbb{R}^N)$, and $\{w_n^j\}$ is a (PS) sequence for I_0,

(jb) $(w_n^j, z_n^j) \rightharpoonup (0,0)$ in $H^1_0(\Omega) \times H^1_0(\Omega)$,

(jc) $I_0(w_n^j, z_n^j) = I_0(w_n^{j-1}, z_n^{j-1}) - \widehat{I}(u^{j-1}, v^{j-1}) + o_n(1)$,

(jd) $R_n^{j-1} d(x_n^{j-1}, \partial\Omega) \to +\infty$,

(je) if $I_0(w_n^{j-1}, z_n^{j-1}) \to \beta < \beta^*$, then $\{(w_n^{j-1}, z_n^{j-1})\}$ is relatively compact; in particular $(w_n^{j-1}, z_n^{j-1}) \to (0,0)$ in $H^1_0(\Omega) \times H^1_0(\Omega)$ and $I_0(w_n^{j-1}, z_n^{j-1}) \to 0$.

As before it is

$$w_n^j = u_n - u_0 - \sum_{i=1}^{j-1} u^i_{R^i_n,x^i_n}, \tag{3.1.15}$$

$$z_n^j = v_n - v_0 - \sum_{i=1}^{j-1} v^i_{R^i_n,x^i_n}, \tag{3.1.16}$$

and by (jc) we have

$$I_0(u_n, v_n) = I_0(u_0, v_0) + I_0(w_n^j, z_n^j) + \sum_{i=1}^{j-1} \widehat{I}(u^i, v^i) + o_n(1). \tag{3.1.17}$$

Recalling that $I_0(u_0, v_0) \geq 0$ the previous identity gives

$$C \geq I_0(u_n, v_n) \geq I_0(w_n^j, z_n^j) + (j-1)\widehat{m} + o_n(1). \tag{3.1.18}$$

On the other hand, since $\{(w_n^j, z_n^j)\}$ is a bounded (PS) sequence for I_0,

$$\begin{aligned} I_0(w_n^j, z_n^j) &= I_0(w_n^j, z_n^j) - \frac{1}{2^*} I_0'(w_n^j, z_n^j)[w_n^j, z_n^j] + o_n(1) \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \|(w_n^j, z_n^j)\| + o_n(1) \geq o_n(1) \end{aligned}$$

Then,

$$C \geq I_0(w_n^j, z_n^j) + (j-1)\widehat{m} + o_n(1) \geq (j-1)\widehat{m} + o_n(1)$$

so that, since $\widehat{m} > 0$, we deduce that the process has to finish after a finite number of steps, let us say at some index k. This means, by (3.1.15), that

$$w_n^{k+1} = u_n - u_0 - \sum_{i=1}^{k} u^i_{R^i_n,x^i_n} \to 0,$$

$$z_n^{k+1} = v_n - v_0 - \sum_{i=1}^{k} v^i_{R_n^i, x_n^i} \to 0,$$

giving (3.1.10). Moreover as in (3.1.17) it is

$$I_0(u_n, v_n) = I_0(u_0, v_0) + I_0(w_n^{k+1}, z_n^{k+1}) + \sum_{i=1}^{k} \widehat{I}(u^i, v^i) + o_n(1)$$

and we deduce (3.1.12), concluding the proof. □

Now, there exists (U, V) solution of

$$\begin{cases} -\Delta u = \frac{2\alpha}{2^*}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \mathbb{R}^N \\ -\Delta v = \frac{2\beta}{2^*}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \mathbb{R}^N \\ u, v \in D^{1,2}(\mathbb{R}^N) \end{cases}$$

such that $\widehat{I}(U_{R,\xi}, V_{R,\xi}) = m_*$ (recall the definitions in (3.1.2) adapted to the case $\Omega = \mathbb{R}^N$ and (3.1.8)) and moreover for any other solution (W, Z) which is not of this type, one has $\widehat{I}(W, Z) \geq 2m_*$.

By this observation, we deduce that if $\{(u_n, v_n)\}$ is a (PS) sequence for I_0 at level $\mathfrak{m}_0$ and $(u_n, v_n) \rightharpoonup (u_0, v_0)$. By Lemma 3.1.13 we have $(u_n, v_n) \to (u, v)$ in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, and in this case we have compactness, or equivalently, the Lemma holds with $k = 1$. In this case

$$m_0 = I_0(u_0, v_0) + \widehat{I}(u^1, v^1) + o_n(1)$$

and since $I_0'(u_0, v_0) = 0$, it has to be necessarily $(u_0, v_0) = (0, 0)$, and denoting $u^1 = U$ and $v^1 = V$, we have

$$u_n = U_{R_n, x_n} + o_n(1)$$

$$v_n = V_{R_n, x_n} + o_n(1)$$

in $D^{1,2}(\mathbb{R}^N)$. This final observation will be used below.

3.2 The barycenter map

We begin by introducing the barycenter map that will allow us to compare the topology of Ω with the topology of suitable sublevels of I_ϵ; precisely sublevels with energy near the minimum level $\mathfrak{m}_\epsilon$.

For $u \in W^{1,2}(\mathbb{R}^N)$ with compact support, let us denote with the same symbol u its trivial extension out of $\operatorname{supp} u$. In particular a function in $H_0^1(\Omega)$ can be thought also as an element of $D^{1,2}(\mathbb{R}^N)$.

The barycenter of (u, v) (see [6]) is defined as

$$\Upsilon(u, v) = \frac{\displaystyle\int_{\mathbb{R}^N} x(|\nabla u|^2 + |\nabla v|^2)dx}{\displaystyle\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx} \in \mathbb{R}^N.$$

From now on, we fix $r > 0$ a radius sufficiently small such that $B_r \subset \Omega$ and the sets

$$\Omega_r^+ = \{x \in \mathbb{R}^N : d(x, \Omega) \leq r\}$$

$$\Omega_r^- = \{x \in \Omega : d(x, \partial\Omega) \geq r\}$$

are homotopically equivalent to Ω. B_r stands for the ball of radius $r > 0$ centred in 0. We denote by

$$h : \Omega_r^+ \to \Omega_r^- \tag{3.2.1}$$

the homotopic equivalence map such that $h|_{\Omega_r^-}$ is the identity.

Now we have the following:

Proposition 3.2.1. *There exists $\epsilon_0 > 0$ such that if $\epsilon \in (0, \epsilon_0)$, it follows*

$$(u, v) \in \mathcal{N}_\epsilon \quad \text{and} \quad I_\epsilon(u, v) < \mathfrak{m}_\epsilon + \varepsilon \implies \Upsilon(u, v) \in \Omega_r^+.$$

Proof. We argue by contradiction. Assume that there exist sequences $\varepsilon_n \to 0$ and $(w_n, z_n) \in \mathcal{N}_{\epsilon_n}$ such that

$$\mathfrak{m}_{\epsilon_n} \le I_{\epsilon_n}(w_n, z_n) \le \mathfrak{m}_{\epsilon_n} + \varepsilon_n \quad \text{and} \quad \Upsilon(w_n, z_n) \notin \Omega_r^+. \tag{3.2.2}$$

Then by Theorem 3.1.9 we deduce

$$I_{\epsilon_n}(w_n, z_n) \to \mathfrak{m}_0 \tag{3.2.3}$$

and then by Remark 3, $\{(w_n, z_n)\}$ is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$. We can suppose that $(w_n, z_n) \rightharpoonup (w, z)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$. Since all the Nehari manifolds $\mathcal{N}_\epsilon$ are bounded away from zero (see Lemma 3.1.1 and Remark 1) we know that $w_n \not\to 0$ and $z_n \not\to 0$ in $H_0^1(\Omega)$ and then, by Remark 2, we deduce $\int_\Omega |w_n|^\alpha |z_n|^\beta dx \not\to 0$. We can assume, without loss of generality, that $w_n, z_n \ge 0$.

Let $t_{0,n} := t_0(w_n, z_n) > 0$ such that $t_0(w_n, z_n)(w_n, z_n) \in \mathcal{N}_0$. By Proposition 3.1.7 we have $\lim_{n\to+\infty} t_{0,n} = 1$.

The proof now consists in

- **STEP 1:** prove that $\{t_{0,n}(w_n, z_n)\} \subset \mathcal{N}_0$ is a minimizing sequence for I_0 on $\mathcal{N}_0$;
- **STEP 2:** use the Ekeland Variational Principle and write

$$t_{0,n}(w_n, z_n) = (U_{R_n,x_n}, V_{R_n,x_n}) + (\phi_n, \psi_n)$$

  where U_{R_n,x_n} and V_{R_n,x_n} are introduced at the end of Section 3.1 and $(\phi_n, \psi_n) \to (0, 0)$ in $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$;
- **STEP 3:** compute the barycentre of $t_{0,n}(w_n, z_n)$ by using the representation obtained in STEP 2 and contradict (3.2.2), finishing the proof of the proposition.

STEP 1: $\lim_{n\to+\infty} I_0(t_{0,n}(w_n, z_n)) = m_0$.

Observe that by the Hölder inequality and since $\lim_{n\to+\infty} t_{0,n} = 1$, we have:

$$\begin{aligned} I_0(t_{0,n}(w_n, z_n)) - I_{\epsilon_n}(w_n, z_n) &= \frac{t_{0,n}^2 - 1}{2} \|(w_n, z_n)\|^2 - \frac{2t_{0,n}^{2^*}}{2^*} \int_\Omega |w_n|^\alpha |z_n|^\beta dx \\ &+ \frac{2}{\alpha_{\epsilon_n} + \beta_{\epsilon_n}} \int_\Omega |w_n|^{\alpha_{\epsilon_n}} |z_n|^{\beta_{\epsilon_n}} dx \\ &\le \frac{t_{0,n}^2 - 1}{2} \|(w_n, z_n)\|^2 + o_n(1) \end{aligned}$$

By using that $\{(w_n, z_n)\}$ is bounded, we infer

$$I_0(t_{0,n}(w_n, z_n)) - I_{\epsilon_n}(w_n, z_n) \le o_n(1).$$

Then
$$0 < m_0 \le I_0(t_{0,n}(w_n,z_n)) \le I_{\epsilon_n}(w_n,z_n) + o_n(1)$$
and by (3.2.3) we conclude $I_0(t_{0,n}(w_n,z_n)) \to m_0$ for $n \to +\infty$.

STEP 2: Representation of the minimizing sequence $\{t_{0,n}(w_n,z_n)\}$.

Since $\{t_{0,n}(w_n,z_n)\}$ is a minimizing sequence for I_0, the Ekeland's Variational Principle implies that there exist $\{(u_n,v_n)\} \subset \mathcal{N}_0$ and $\{\mu_n\} \subset \mathbb{R}$, a sequence of Lagrange multipliers, such that
$$\begin{aligned}&\|t_{0,n}(w_n,z_n) - (u_n,v_n)\| \to 0\\ &I_0(u_n,v_n) \to m_0\\ &I_0'(u_n,v_n) - \mu_n G_0'(u_n,v_n) \to 0\end{aligned}$$
and Lemma 3.1.2 ensures that $\{(u_n,v_n)\}$ is a (PS) sequence for the free functional I_0 on the whole space $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ at level m_0. By the arguments at the end of Section 3.1 we have
$$u_n - U_{R_n,x_n} \to 0 \quad \text{and } v_n - V_{R_n,x_n} \to 0$$
in $D^{1,2}(\mathbb{R}^N)$ where $\{x_n\} \subset \Omega$, $R_n \to +\infty$. Then we can write
$$u_n = U_{R_n,x_n} + \phi_n$$
$$v_n = V_{R_n,x_n} + \psi_n$$
with a remainder (ϕ_n,ψ_n) such that $\|(\phi_n,\psi_n)\|_{D^{1,2}(\mathbb{R}^N)} \to 0$. It is clear that
$$t_{0,n}(w_n,z_n) = (u_n,v_n) + t_{0,n}(w_n,z_n) - (u_n,v_n) = (u_n,v_n) + o_n(1);$$
so, renaming the remainder again ϕ_n and ψ_n, we have
$$t_{0,n}(w_n,z_n) = (U_{R_n,x_n}, V_{R_n,x_n}) + (\phi_n,\psi_n).$$

STEP 3: Computing the barycenter and finishing the proof.

By using the representation obtained in STEP 2, the $i-$th coordinate of the barycenter of $t_{0,n}(w_n,z_n)$ satisfies
$$\begin{aligned}\Upsilon(t_{0,n}(w_n,z_n))^i \|t_{0,n}(w_n,z_n)\|^2_{D^{1,2}(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} x^i(|\nabla U_{R_n,x_n}|^2 + |\nabla V_{R_n,x_n}|^2)dx\\ &+ \int_{\mathbb{R}^N} x^i(|\nabla\phi_n|^2 + |\nabla\psi_n|^2)dx\\ &+ 2\int_{\mathbb{R}^N} x^i(\nabla U_{R_n,x_n}\nabla\phi_n + \nabla V_{R_n,x_n}\nabla\psi_n)dx\end{aligned}$$
where x^i is the $i-$th coordinate of $x \in \mathbb{R}^N$. In order to localize the barycenters we need to pass to the limit in each term in the above expression; By computation of each using changes of variables in the integrals, we get
$$\begin{aligned}\|t_{0,n}(w_n,z_n)\|^2_{D^{1,2}(\mathbb{R}^N)} &= \|(U,V)\|^2_{D^{1,2}(\mathbb{R}^N)} + o_n(1),\\ \int_{\mathbb{R}^N} x^i|\nabla U_{R_n,x_n}|^2dx &= x_n^i\int_{\mathbb{R}^N}|\nabla U|^2dx + o_n(1),\\ \int_{\mathbb{R}^N} x^i|\nabla V_{R_n,x_n}|^2dx &= x_n^i\int_{\mathbb{R}^N}|\nabla V|^2dx + on(1),\\ \int_{\mathbb{R}^N} x^i|\nabla\phi_n|^2dx &= \int_{\mathbb{R}^N} x^i\nabla U_{R_n,x_n}\nabla\phi_n dx = o_n(1)\\ \int_{\mathbb{R}^N} x^i|\nabla\psi_n|^2dx &= \int_{\mathbb{R}^N} x^i\nabla V_{R_n,x_n}\nabla\psi_n dx = o_n(1).\end{aligned}$$

Then we have the $i-th$ coordinate of the barycenter,

$$\Upsilon(t_{0,n}(w_n,z_n))^i = \frac{x_n^i \displaystyle\int_{\mathbb{R}^N} |\nabla U|^2 + |\nabla V|^2 dx + o_n(1)}{\|(U,V)\|^2_{D^{1,2}(\mathbb{R}^N)} + o_n(1)} = x_n^i + o_n(1), .$$

Since $\{x_n\} \subset \Omega$ the above equation implies that $\Upsilon(w_n,z_n) = \Upsilon(t_{0,n}(w_n,z_n)) \to x_0 \in \overline{\Omega}$, when $n \to +\infty$ and this is in contrast with (3.2.2) and proves the proposition. $\square$

3.3 Proof of Theorem 3.0.1

Here we complete the proof of our theorem but first we need a slight modification to the previous notations. Let $r > 0$ be the one fixed at the beginning of Section 3.2, that is in such a way that $\Omega_r^+ = \{x \in \mathbb{R}^N : d(x,\Omega) \leq r\}$ and $\Omega_r^- = \{x \in \Omega : d(x,\partial\Omega) \geq r\}$ are homotopically equivalent to Ω. We add a subscript r, to denote the same quantities defined in the previous sections when the domain Ω is replaced by B_r; namely integrals are taken on B_r and norms are taken for functional spaces defined on B_r. Hence for example, for all $\epsilon > 0$ we set:

$$\mathcal{N}_{\epsilon,r} = \left\{(u,v) \in H_0^1(B_r) \times H_0^1(B_r) : \|(u,v)\|^2_{W_0^{1,2}(B_r)\times W_0^{1,2}(B_r)} = 2\int_{B_r} |u|^{\alpha_\epsilon}|v|^{\beta_\epsilon} dx\right\},$$

$$I_{\epsilon,r}(u,v) = \frac{1}{2}\|(u,v)\|^2_{H_0^1(B_r)\times H_0^1(B_r)} - \frac{2}{\alpha_\epsilon + \beta_\epsilon}\int_{B_r} |u|^{\alpha_\epsilon}|v|^{\beta_\epsilon} dx,$$

$$\mathfrak{m}_{\epsilon,r} = \min_{v \in \mathcal{N}_{\epsilon,r}} I_{\epsilon,r}(u,v) = I_{\epsilon,r}(\mathfrak{g}_{\epsilon,r},\mathfrak{h}_{\epsilon,r}).$$

Observe that, by means of the Palais Symmetric Criticality Principle, the ground state $(\mathfrak{g}_{\epsilon,r},\mathfrak{h}_{\epsilon,r})$ is radial. Moreover let

$$I_\epsilon^{\mathfrak{m}_{\epsilon,r}} = \{(u,v) \in \mathcal{N}_\epsilon : I_\epsilon(u,v) \leq \mathfrak{m}_{\epsilon,r}\}$$

which is non vacuous since $\mathfrak{m}_\epsilon < \mathfrak{m}_{\epsilon,r}$.

Define also, for $\epsilon > 0$ the map $(\Psi_{\epsilon,r},\Phi_{\epsilon,r}) : \Omega_r^- \to \mathcal{N}_\epsilon$ such that

$$(\Psi_{\epsilon,r}(y)(x),\Phi_{\epsilon,r}(y)(x)) = \begin{cases} (\mathfrak{g}_{\epsilon,r}(|x-y|),\mathfrak{h}_{\epsilon,r}(|x-y|)) & \text{if} \quad x \in B_r(y) \\ (0,0) & \text{if} \quad x \in \Omega \setminus B_r(y) \end{cases}$$

and note that we have

$$\Upsilon(\Psi_{\epsilon,r}(y),\Phi_{\epsilon,r}(y)) = y \text{ and } (\Psi_{\epsilon,r}(y),\Phi_{\epsilon,r}(y)) \in I_\epsilon^{\mathfrak{m}_{\epsilon,r}}.$$

Moreover, since $\mathfrak{m}_\epsilon + k_\epsilon = \mathfrak{m}_{\epsilon,r}$ where $k_\epsilon > 0$ and tends to zero if $\epsilon \to 0$ (see Theorem 3.1.9), in correspondence of $\varepsilon > 0$ provided by Proposition 3.2.1, there exists a $\epsilon_0 > 0$ such that $\epsilon \in (0,\epsilon_0)$ such that it results $k_\epsilon < \varepsilon$; so if $(u,v) \in I_\epsilon^{\mathfrak{m}_{\epsilon,r}}$ we have

$$I_\epsilon(u,v) \leq \mathfrak{m}_{\epsilon,r} < \mathfrak{m}_\epsilon + \varepsilon,$$

at least for ϵ near 0. Hence we can define the following maps:

$$\Omega_r^- \xrightarrow{(\Psi_{\epsilon,r},\Phi_{\epsilon,r})} I_\epsilon^{\mathfrak{m}_{\epsilon,r}} \xrightarrow{h\circ\beta} \Omega_r^-$$

with h given by (3.2.1). Since the composite map $h \circ \beta \circ \Psi_{\epsilon,r}$ is homotopic to the identity of Ω_r^- by a property of the category we have

$$\mathrm{cat}_{I_\epsilon^{\mathfrak{m}_{\epsilon,r}}}(I_\epsilon^{\mathfrak{m}_{\epsilon,r}}) \geq \mathrm{cat}_{\Omega_r^-}(\Omega_r^-)$$

and due to our choice of r, it follows $\mathrm{cat}_{\Omega_r^-}(\Omega_r^-) = \mathrm{cat}_{\overline{\Omega}}(\overline{\Omega})$.

Then, we have found a sublevel of I_ϵ on $\mathcal{N}_\epsilon$ with category greater than $\mathrm{cat}_{\overline{\Omega}}(\overline{\Omega})$ and since the (PS) condition is verified on $\mathcal{N}_\epsilon$, the Lusternik-Schnirelmann theory guarantees the existence of at least $\mathrm{cat}_{\overline{\Omega}}(\overline{\Omega})$ critical points for I_ϵ on the manifold $\mathcal{N}_\epsilon$ which give rise to solutions of (3.0.1).

The existence of another solution is obtained with the same arguments of Benci, Cerami and Passaseo [7]. We recall here the arguments for the reader convenience. Since by assumption Ω is not contractible in itself, by the choice of r it results $\mathrm{cat}_{\Omega_r^+}(\Omega_r^-) > 1$, namely Ω_r^- is not contractible in Ω_r^+.

Claim: the set $(\Psi_{\epsilon,r}(\Omega_r^-), \Phi_{\epsilon,r}(\Omega_r^-))$ is not contractible in $I_\epsilon^{\mathfrak{m}_{\epsilon,r}}$.

Indeed, assume by contradiction that $\mathrm{cat}_{I_\epsilon^{\mathfrak{m}_{\epsilon,r}}}(\Psi_{\epsilon,r}(\Omega_r^-), \Phi_{\epsilon,r}(\Omega_r^-)) = 1$: this means that there exists a map $\mathcal{H} \in C([0,1] \times (\Psi_{\epsilon,r}(\Omega_r^-), \Phi_{\epsilon,r}(\Omega_r^-)); I_\epsilon^{\mathfrak{m}_{\epsilon,r}})$ such that

$$\mathcal{H}(0,u,v) = (u,v) \quad \forall (u,v) \in (\Psi_{\epsilon,r}(\Omega_r^-), \Phi_{\epsilon,r}(\Omega_r^-)) \quad \text{and}$$

$$\exists\, (w,z) \in I_\epsilon^{\mathfrak{m}_{\epsilon,r}} : \mathcal{H}(1,u,v) = (w,z) \quad \forall (u,v) \in (\Psi_{\epsilon,r}(\Omega_r^-), \Phi_{\epsilon,r}(\Omega_r^-)).$$

Then $F = (\Psi_{\epsilon,r}^{-1}(\Psi_{\epsilon,r}(\Omega_r^-)), \Phi_{\epsilon,r}^{-1}(\Phi_{\epsilon,r}(\Omega_r^-)))$ is closed, contains Ω_r^- and is contractible in Ω_r^+ since one can define the map

$$\mathcal{G}(t,x) = \begin{cases} \Upsilon(\Psi_{\epsilon,r}(x)), \Phi_{\epsilon,r}(x)) & \text{if } 0 \leq t \leq 1/2, \\ \Upsilon(\mathcal{H}(2t-1, \Psi_{\epsilon,r}(x), \Phi_{\epsilon,r}(x)))) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Then also Ω_r^- is contractible in Ω_r^+ and this gives a contradiction.

On the other hand we can choose a function $(w_0, z_0) \in \mathcal{N}_\epsilon \setminus (\Psi_{\epsilon,r}(\Omega_r^-), \Phi_{\epsilon,r}(\Omega_r^-))$ so that the cone

$$\mathcal{C} = \left\{\theta(w_0,z_0) + (1-\theta)(u,v) : (u,v) \in (\Psi_{\epsilon,r}(\Omega_r^-), \Phi_{\epsilon,r}(\Omega_r^-)), \theta \in [0,1]\right\}$$

is compact and contractible in $H_0^1(\Omega) \times H_0^1(\Omega)$ and $(0,0) \notin \mathcal{C}$. For every $u, v \neq 0$ let $t_{\epsilon,u,v}$ be the unique positive number provided by (iv) in Lemma 3.1.1; it follows that if we set

$$\widehat{\mathcal{C}} := \{t_{\epsilon,u,v}(u,v) : (u,v) \in \mathcal{C}\}, \qquad M_\epsilon := \max_{\widehat{\mathcal{C}}} I_\epsilon$$

then $\widehat{\mathcal{C}}$ is contractible in $I_\epsilon^{M_\epsilon}$ and $M_\epsilon > \mathfrak{m}_{\epsilon,r}$. As a consequence also $(\Psi_{\epsilon,r}(\Omega_r^-), \Phi_{\epsilon,r}(\Omega_r^-))$ is contractible in $I_\epsilon^{M_\epsilon}$.

In conclusion the set $(\Psi_{\epsilon,r}(\Omega_r^-), \Phi_{\epsilon,r}(\Omega_r^-))$ is contractible in $I_\epsilon^{M_\epsilon}$ and not in $I_\epsilon^{\mathfrak{m}_{\epsilon,r}}$ and this is possible, since the (PS) condition holds, only if there is another critical point with critical level between $\mathfrak{m}_{\epsilon,r}$ and M_ϵ.

It remains to prove that these solutions are positive. Note that we can apply all the previous machinery replacing the functional I_ϵ with

$$I_\epsilon^+(u,v) = \frac{1}{2}\|(u,v)\|^2 - \frac{1}{\alpha_\epsilon + \beta_\epsilon}\int_\Omega |u^+|^{\alpha_\epsilon}|v^+|^{\beta_\epsilon} dx$$

where $w^+ := \max\{w, 0\}$. Then we obtain again at least $\mathrm{cat}_{\overline{\Omega}}(\overline{\Omega})$ (or $\mathrm{cat}_{\overline{\Omega}}(\overline{\Omega}) + 1$) nontrivial solutions that now are positive by the maximum principle.

Appendix A

The Ljusternik-Schnirelmann category is a tool used to obtain multiplicity results of critical points of functionals, then obtaining solutions for variational equations. Let $\mathcal{M}$ be a topological space.

Definition 1: $A \subset \mathcal{M}$ is contractible if the inclusion $A \hookrightarrow \mathcal{M}$ is homotopic to a constant map defined on A with value in $\mathcal{M}$. In other words, there is $H \in C([0,1] \times A, \mathcal{M})$ such that for all $u \in A$ and for some $p \in \mathcal{M}$ fixed we have

$$H(0,u) = u$$
$$H(1,u) = p.$$

Definition 2: The Ljusternik-Schnirelmann category of A with respect to $\mathcal{M}$ is defined by

$$\text{cat}_{\mathcal{M}}(A) = \begin{cases} 0, \quad \text{if} \quad A = \emptyset, \\ \inf\{m \in \mathbb{N} : A \subset \cup_{j-1}^{m}, A_j, \ A_j \text{ contractible in } \mathcal{M}\} \\ \infty, \quad \text{if there isn't } k \in \mathbb{N} \text{ such that } A \subset \cup_{j-1}^{k}, \ A_j \text{ contractible in } \mathcal{M}. \end{cases}$$

We denote $\text{cat}\mathcal{M} = \text{cat}_{\mathcal{M}}\mathcal{M}$.

Definition 3: Let A, B, Y be closed spaces of E. Then $A \prec_Y B$ if $Y \subset A \cap B$ and there exists $h \in C([0,1] \times A, E)$

1) for all $u \in A$, $h(0,u) = u$ and $h(1,u) \in B$,

2) for all $t \in [0,1]$, $h(t,Y) \subset Y$.

In the reference that this appendix is based, [31], it is used the relative category as follows

Definition 4: Let $Y \subset A$ be closed subsets in a topological space E. The category of A in E relative to Y is the least integer n such that exists $n+1$ closed subsets $A_0, A_1, ..., A_n$ of E satisfying

1) $A = \cup_{j=1}^{n} A_j$,

2) $A_1, ..., A_n$ are contractible in E,

3) $A_0 \prec_Y Y$ in E.

We denote the category of A in E relative to Y by $\text{cat}_{X,Y}(A)$.

Observation: Note that

$$\text{cat}_X(A) = \text{cat}_{X,\emptyset}(A).$$

Examples:

1. Let $B = B_R(y) := \{x \in \mathbb{R}^N : |x-y| < R\}$, then $\text{cat}_B(B) = \text{cat}_{\mathbb{R}^N}(B)$;
2. Let $S := S_R(y) := \{x \in \mathbb{R}^N : |x-y| = R\}$, then $\text{cat}_S(S) = \text{cat}_{\mathbb{R}^N}(S) = 2$
3. Let $\mathbb{T} \subset \mathbb{R}^3$ the torus, then $\text{cat}_{\mathbb{T}^2}(\mathbb{T}^2) = \text{cat}_{\mathbb{R}^N}(\mathbb{T}^2) = 4$.

Then we have some important properties.

Lemma 3.3.1. *Let A, B, C, Y be closed subsets of X such that $Y \subset A \cap B \cap C$. If $A \prec_Y B$ and $B \prec_Y C$, then $A \prec_Y C$.*

Proof. Since $A \prec_Y B$ there exists $h \in C([0,1] \times A, X)$ such that for all $u \in A$:

$$h(0,u) = u \quad \text{and} \quad h(1,u) \in B$$

$$h(t,Y) \subset Y.$$

Since $B \prec_Y C$ there exists $g \in C([0,1] \times B, X)$ such that for all $u \in B$:

$$g(0,u) = u \quad \text{and} \quad g(1,u) \in C$$

$$g(t,Y) \subset Y.$$

Define the following continuous function $f : [0,1] \times A \to X$:

$$f(t,u) = \begin{cases} h(2t,u) & \text{if} \quad t \in [0,1/2], \\ g(2t-1, h(1,u)), & \text{if} \quad t \in (1/2,1]. \end{cases}$$

We get for all $u \in A$

$$f(0,u) = h(0,u) = u \quad \text{and} \quad f(1,u) = g(1,h(1,u)) \in C, \quad \text{since } h(1,u) \in B.$$

By definition of g and h, $f(t,Y) \subset Y$, for all t. Then $A \prec_Y C$.

□

Theorem 3.3.2. *Let A, B, Y be closed subsets of X such that $Y \subset A$. The relative category satisfies the following properties*

1) *Normalisation: $cat_{X,Y}(Y) = 0$*
2) *Subadditivity: $cat_{X,Y}(A \cup B) \leq cat_{X,Y}(A) + cat_X(B)$*
3) *Monotonicity: If $A \prec_Y B$, then $cat_{X,Y}(A) \leq cat_{X,Y}(B)$*

Proof. 1. Note that we can take $A_0 = Y$, since $Y \subset A_0 \cap Y$ and $h : [0,1] \times A_0 \to X$ defined as $h(t,u) = u$ is continuous and satisfies

$$h(0,u) = u \quad \text{for all } u \in A_0,$$

$$h(1,u) = u \in Y \quad \text{for all } u \in A_0,$$

$$h(t,y) = y \in A_0 = Y \quad \text{for all } y \in Y.$$

Then $A_0 \prec_Y Y$ and we conclude $\text{cat}_{X,Y}(Y) = 0$, by definition.

2. Let $\text{cat}_{X,Y}(A) = n$, then we have $A_j \subset X$ closed subsets with $A_1, ..., A_n$ contractible in X and $A_0 \prec_Y Y$ such that

$$A = \cup_{j=1}^{n} A_j.$$

Also we have $\text{cat}_X(B) = m$, the there exists $B_1, ..., B_m$ closed subsets and contractible in X such that

$$B = \cup_{j=1}^{n} B_j.$$

Then $A \cup B \subset A_0 \cup [A_1 \cup ... \cup A_n \cup B_1 \cup ... \cup B_m]$ and $A_0 \prec_Y Y$. By definition, we get

$$\text{cat}_{X,Y}(A) \leq n + m = \text{cat}_{X,Y}(A) + \text{cat}_X(B).$$

3. Let $\text{cat}_{X,Y}(B) = n$ and $B_0, ..., B_n$ the subsets of the definition. Define

$$A_0 = \{u \in A : h(1, u) \in B_j\}.$$

Then we get $A = \cup_{j=1}^{n} A_j$. We need to prove that $A_0 \prec_Y Y$ and $A_1, ..., A_n$ are contractible.

Since $B_0 \prec_Y Y$, we get $Y \subset B_0$. Also, if $u \in Y$, then $h(1, u) \in Y \subset B_0$. By definition, $u \in A_0$. Finally, we use $h_0 := h \mid_{A_0}$. Then $A_0 \prec_Y B_0$. Since $B_0 \prec_Y Y$, by Lemma we get $A_0 \prec_Y Y$.

In order to show that A_j is contractible, let g_j the deformation associated to B_j which is contractible and define $f_j : [0, 1] \times A_j \to X$ as

$$f_j(t, u) = \begin{cases} h_j(2t, u) & \text{if } \ t \in [0, 1/2], \\ g_j(2t - 1, h_j(1, u)), & \text{if } \ t \in (1/2, 1]. \end{cases}$$

Then we got A_j contractible, $j = 1, ..., n$, then

$$\text{cat}_{X,Y}(A) \leq n = \text{cat}_{X,Y}(B).$$

□

Now we assume that E is a Banach space, $V \subset E$ is a manifold given by $V = \psi^{-1}(1)$ with $\psi \in C^2(E, \mathbb{R})$ and $\psi'(u) \neq 0$ for all $v \in V$.

Also we define

$$\psi^d := \{v \in V : \psi(v) \leq d\}$$

and

$$K_c := \{v \in V : \psi(v) = c \text{ and } \|\psi'(u)\|_*.$$

For $j \geq 1$,

$$\mathcal{A}_j := \{A \subset \psi^d : A \text{ is closed}, \text{cat}_{\phi^d}(A) \geq j\}$$

$$c_j := \inf_{A \in \mathcal{A}_j} \sup_{u \in A} \psi(u)$$

Definition 5: The function $\psi \mid_V$ satisfies the $(PS)_c$ condition if any sequence $(u_n) \subset V$ such that $\psi(u_n) \to c$ and $\|\psi'(u_n)\|_* \to 0$ has a convergent subsequence.

Theorem 3.3.3. *If $a := sup_Y \psi < c := c_k = c_{k+1} = ... = c_{k+m} \leq d$ and if $\psi\mid_V$ satisfies the $(PS)_c$ condition, then $cat_{\psi^d}(K_c) \geq m+1$.*

Proof: See [31].

Theorem 3.3.4. *If $\sup_{u \in Y} \psi(u) < c_1$ and if $\psi\mid_V$ satisfies the $(PS)_c$ condition for all $c \in [c_1, d]$, then $\psi^{-1}([c_1, d])$ contains at least $cat_{\psi^d, Y}(\psi^d)$ critical points of $\psi\mid_V$.*

Proof. If $\text{cat}_{\psi^d, Y}(\psi^d) = n$ and by consequence of the definition of c_j, we obtain

$$\sup_{u \in Y} \psi(u) < c_1 \leq c_2 \leq ... \leq c_n \leq d.$$

We can separate

$$\psi(u) < c_1 = ... = c_{m_1} < c_{m_1+1} = ... = c_{m_2} < ... < c_{m_{j-1}+1} = ...c_{m_j}$$

where $m_0 = 0$ and $m_j = n$.

Then, applying the last theorem for $\sup_{u \in Y} \psi(u) < c_{m_{i-1}+1} = ... = c_{m_i} \leq d$, we obtain

$$\text{cat}_{\psi^d}(K_{c_{m_i}}) = m_i - m_{i-1}.$$

Since $K_{c_{m_i}}$ are disjoint sets ,

$$\text{cat}_{\psi^d}(\cup_{i=1}^{j} K_{c_{m_i}}) = \sum_{i=1}^{j} m_i - m_{i-1} = m_j = n$$

Finally, $\cup_{i=1}^{j} K_{c_{m_i}}$ has at least n points and since $\cup_{i=1}^{j} K_{c_{m_i}} \subset \psi^{-1}([c_1, d)]$, we obtain that $\psi^{-1}([c_1, d)]$ contains at least n critical points of $\psi\mid_V$. □

Theorem 3.3.5. *If $\psi\mid_V$ is bounded from below and satisfies the $(PS)_c$ condition for all $c \in [\inf_{u \in V} \psi(u), d]$, then $\psi\mid_V$ has a minimum and ψ^d has at least $cat_{\psi^d}(\psi^d)$ critical points of $\psi\mid_V$*

Proof. First, let show that $c_1 = \inf_{u \in V} \psi(u)$. Note that for all $A \in \mathcal{A}_1$ we have

$$\inf_{u \in V} \psi(u) \leq \inf_{u \in A} \psi(u) \leq \sup_{u \in A} \psi(u)$$

By taking the infimum for all $A \in \mathcal{A}_1$, we get $\inf_{u \in V} \psi(u) \leq c_1$.

Since $\inf_{u \in V} \psi(u) \leq c_1 \leq d$, we get $\psi^d \neq \emptyset$. By consequence of the definition, we obtain

$$\inf_{u \in V} \psi(u) = \inf_{u \in \psi^d} \psi(u).$$

Note that for all $u \in \psi^d$, $\{u\} \in \mathcal{A}_1$, then

$$c_1 = \inf_{A \in \mathcal{A}_1} \sup_{u \in A} \psi(u) \leq \sup_{v \in \{u\}} \psi(v) = \psi(u)$$

Taking the infimum in $u \in \psi^d$, we get

$$c_1 \leq \inf_{u \in \psi^d} \psi(u) = \inf_{u \in V} \psi(u)$$

Consequently, $c_1 = \inf_{u \in V} \psi(u)$.

By hipothesis, $c_1 = \inf_{u \in V} \psi(u) > -\infty$ and $\psi^{-1}([c_1, d]) = \psi^d$. Finally, apllying the last theorem with $y = \emptyset$, we get that ψ^d has at least $\mathrm{cat}_{\psi^d}(\psi^d)$ critical points of $\psi \mid_V$. And by Theorem 3.3.3 we get $\mathrm{cat}_{\psi^d}(K_{c_1} \geq 1$, then there exists $u \in V$ such that $psi(u) = c_1$ and $\|\psi'(u)\|_* = 0$. In other words $\psi \mid_V$ has a minimum point.

□

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