Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Existence, concentration and multiplicity of positive solutions for an elliptic system in $\mathbb{R}^N$

por

Segundo Manuel Argomedo Salirrosas

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Segundo Manuel Argomedo Salirrosas

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Prof. Dr. Giovany de Jesus Malcher Figueiredo- MAT/UnB (Orientador)

____

Prof. Dr. Ricardo Ruviaro– MAT/UnB (Membro)

____

Prof. Dr. Hamilton Prado Bueno– UFMG (Membro)

____

Prof. Dr. Lucas Catão de Freitas Ferreira– UNICAMP (Membro)

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Resumo

Neste trabalho estamos interessados na existência, concentração e multiplicidade de soluções para os sistemas

$$\begin{cases} -\varepsilon^2 \text{div}(a(x)\nabla u) + u = Q_u(u,v) + \frac{\gamma}{2^*}K_u(u,v) \;\; \text{em } \mathbb{R}^N, \\ -\varepsilon^2 \text{div}(b(x)\nabla v) + v = Q_v(u,v) + \frac{\gamma}{2^*}K_v(u,v) \;\; \text{em } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) > 0 \quad \text{para cada } x \in \mathbb{R}^N, \end{cases}$$

e

$$\begin{cases} -\varepsilon^2 \text{div}(a(x)\nabla u) + u = Q_u(u,v) + \frac{\gamma}{2^*}K_u(u,v) \;\; \text{em} \mathbb{R}^N, \\ \quad -\varepsilon^2 \Delta v + b(x)v \quad = Q_v(u,v) + \frac{\gamma}{2^*}K_v(u,v) \;\; \text{em } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) > 0 \quad \text{para cada } x \in \mathbb{R}^N, \end{cases}$$

onde $2^* = 2N/(N-2)$, $N \geq 3$, $\varepsilon > 0$, Q e K são funções homogêneas com K tendo crescimento crítico, a e b são potenciais continuous positivos tais que existem $a_0, b_0 > 0$ com

$$a_0 \leq a(x), \quad b_0 \leq b(x) \;\; \text{para todo} \;\; x \in \mathbb{R}^N$$

e existe um domínio limitado $\Lambda \subset \mathbb{R}^N$ tal que

$$0 < a_0 = \inf_{x \in \Lambda} a(x) < \inf_{x \in \partial\Lambda} a(x) \quad \text{e} \quad 0 < b_0 = \inf_{x \in \Lambda} b(x) < \inf_{x \in \partial\Lambda} b(x).$$

Palavras-chave: Sistemas elípticos; equação de Schrödinger; Teoria de Ljusternick-Schnirelman; Soluções positivas.

Abstract

In this work we are interested in the existence, concentration and multiplicity of solutions for the systems

$$\begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) + \frac{\gamma}{2^*}K_u(u,v) \ \text{ em } \mathbb{R}^N, \\ -\varepsilon^2 \mathrm{div}(b(x)\nabla v) + v = Q_v(u,v) + \frac{\gamma}{2^*}K_v(u,v) \ \text{ em } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) > 0 \quad \text{para cada } x \in \mathbb{R}^N, \end{cases}$$

and

$$\begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) + \frac{\gamma}{2^*}K_u(u,v) \ \text{ em} \mathbb{R}^N, \\ \quad -\varepsilon^2 \Delta v + b(x)v \ = Q_v(u,v) + \frac{\gamma}{2^*}K_v(u,v) \ \text{ em } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) > 0 \quad \text{para cada } x \in \mathbb{R}^N, \end{cases}$$

where $2^* = 2N/(N-2)$, $N \geq 3$, $\varepsilon > 0$, Q and K are homogeneous function with K having critical growth, a and b are positive continuous potentials such that there exist $a_0, b_0 > 0$ with

$$a_0 \leq a(x), \quad b_0 \leq b(x) \ \text{ for all } \ x \in \mathbb{R}^N$$

and there exist a bounded domain $\Lambda \subset \mathbb{R}^N$ such that

$$0 < a_0 = \inf_{x \in \Lambda} a(x) < \inf_{x \in \partial\Lambda} a(x) \quad \text{ and } \quad 0 < b_0 = \inf_{x \in \Lambda} b(x) < \inf_{x \in \partial\Lambda} b(x).$$

Key words: Elliptic systems; Schrödinger equation; Ljusternik-Schnirelmann theory; positive solutions.

Contents

Introduction

Several physical phenomena related to the equilibrium of continuous media are modeled by the problem

$$(P_1) \qquad \begin{cases} -\mathrm{div}(a(x)\nabla u) &= g(x,u) \text{ in } \Omega, \\ u &= 0 \quad \text{ on } \partial\Omega, \end{cases}$$

where Ω is a domain of $\mathbb{R}^N$, g is a regular function and a is a nonnegative weight. For example, equations like (P_1) are introduced in [18] by Dautray and Lions as models for several physical phenomena related to equilibrium of anisotropic media which possibly are somewhere perfect insulators or perfect conductors. In order to be able to deal with these problems we allow the coefficient a to vanish somewhere or to be unbounded.

Caldiroli and Musina [12] used variational methods to prove the existence of solutions to problem (P_1) under suitable assumptions on the data. They assumed that Ω is a given domain in $\mathbb{R}^N$ with $N \geq 2$, which can be either bounded or unbounded. The coefficient a is a measurable and non-negative weight on Ω, with at most a finite number of (essential) zeroes. Here g is a given regular function.

In [38], Passaseo considers problem (P_1), where g has a powerlike behaviour, and a is bounded. Here the case $\inf a = 0$ is considered so that the equation is degenerate, and standard variational techniques do not apply; on the contrary, some concentration phenomena arise, similar to those occurring with critical exponent. The author proves first a very general identity (similar to the Pohozaev identity), from which he deduces a nonexistence result in star-shaped domains. Then he gives a condition on a, sufficient to ensure the existence and multiplicity of nonnegative solutions, and shows that this condition is optimal. Finally, he studies the effect of the topology of the vanishing set of the function a on the number of positive solutions of uniformly elliptic problems, which approximate the one given.

In [39], Pomponio and Secchi considered a problem of the form

$$(P_2) \qquad \begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + V(x)u = u^q \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N \end{cases}$$

where $N \geq 3$, $1 < q < 2^* - 1$, and V is a positive potential, possibly unbounded from above. They studied the existence and concentrating behavior of the single-peaked solutions for problem (P_2) by considering a to be a symmetric uniformly elliptic matrix depending on x.

In [15], Chabrowski studied the problem

$$(P_3) \qquad -\mathrm{div}(a(x)\nabla u) + \lambda u = K(x)|u|^{q-2}u \text{ in } \mathbb{R}^N$$

with $N \geq 3$, $\lambda > 0$, $2 < q < 2^*$ and $a \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying

$$0 \leq a(x) \leq \lim_{|x|\to\infty} a(x),$$

supposing additionaly that a is positive in the exterior of some ball $B_R(0)$. The author showed an existence result by assuming an integrability condition for a and requiring that $K \in L^\infty(\mathbb{R}^N)$ verifies either a periodicity condition or $K(x) \geq \lim_{|x|\to\infty} K(x)$.

In [30], Lazzo considered the problem (P_3) with $K \equiv 1$ and the function a satisfying

$$0 < a_0 := \inf_{x\in\mathbb{R}^N} a(x) < a_\infty := \liminf_{|x|\to\infty} a(x). \tag{0.0.1}$$

It was proved that for λ sufficiently large, the number of solutions of (P_3) is bounded below by the Ljusternik-Schnirelmann category $\mathrm{cat}_M(M)$, where $M := \{x \in \mathbb{R}^N : a(x) = \inf_{\mathbb{R}^N} a\}$.

Another result involving the Ljusternik-Schnirelmann theory were treated by Figueiredo and Furtado in [22]. They studied the problem

$$(P_4) \qquad -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = f(u) \text{ in } \mathbb{R}^N$$

with f being a superlinear function and a satisfying (0.0.1). They show the existence of a ground state solution using minimax theorems and a result on the existence of multiple solutions.

In [23], the same authors dealt with problem (P_4) by considering a weaker condition than (0.0.1), namely

$$0 < a_0 = \inf_{x\in\Lambda} a(x) < \inf_{x\in\partial\Lambda} a(x), \tag{0.0.2}$$

where Λ is a bounded domain in $\mathbb{R}^N$ and f with subcritical growth. The critical version of (P_4) was studied in [24], another work by the same authors.

Motivated by the aforementioned results, we study in this thesis some classes of elliptical systems, which we now describe. In Chapter 1, we consider a version for systems of (P_4) with f being a homogeneous function. More precisely, we studied the system

$$(S_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \mathrm{div}(b(x)\nabla v) + v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, $N \geq 3$. The continuous potentials a and b satisfy the following conditions:

(ab_1) there are $a_0 > 0$ and $b_0 > 0$ such that

$$a_0 \leq a(x) \quad \text{and} \quad b_0 \leq b(x) \quad \text{for all} \quad x \in \mathbb{R}^N;$$

(ab_2) there exists a bounded domain $\Lambda \subset \mathbb{R}^N$ such that

$$0 < a_0 = \inf_{x\in\Lambda} a(x) < \inf_{x\in\partial\Lambda} a(x)$$

and

$$0 < b_0 = \inf_{x\in\Lambda} b(x) < \inf_{x\in\partial\Lambda} b(x).$$

We suppose that $Q \in C^2(\mathbb{R}^2_+, \mathbb{R})$ where $\mathbb{R}^2_+ := [0,\infty) \times [0,\infty)$. In addition, the nonlinearity Q satisfies the following properties:

(Q_0) there exits $2 < p < 2^* := 2N/(N-2)$ such that

$$Q(tu, tv) = t^p Q(u,v), \quad \text{for each} \quad t > 0, \ (u,v) \in \mathbb{R}^2_+;$$

(Q_1) there exists $c_1 > 0$ such that

$$|Q_u(u,v)| + |Q_v(u,v)| \le c_1 \left(u^{p-1} + v^{p-1}\right), \quad \text{for each } (u,v) \in \mathbb{R}^2_+;$$

(Q_2) $Q_u(0,1) = 0$, $Q_v(1,0) = 0$;

(Q_3) $Q_u(1,0) = 0$, $Q_v(0,1) = 0$;

(Q_4) $Q(u,v) > 0$, for each $u, v > 0$;

(Q_5) $Q_u(u,v), Q_v(u,v) \ge 0$, for each $(u,v) \in \mathbb{R}^2_+$.

We also introduce the following set:

$$M = \{x \in \mathbb{R}^N : a(x) = a_0 \ \text{ and } \ b(x) = b_0\}.$$

We notice that the lack of compactness originated by the unboundedness of $\mathbb{R}^N$ is one of our issues. We have adapted a penalization method used by Alves in [1] while studying the system

$$(C_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u + W(x)u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + V(x)v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, $N \ge 3$, and W, V are Hölder continuous potentials. The Penalization method was introduced by Del Pino and Felmer for the scalar case [20]. It consists in modifying appropriately the function Q outside the set Λ so that the energy functional of the modified system satisfies the Palais-Smale condition. After that, it is proved that the solution of modified system is in fact solution of the original system by obtaining uniform convergence of the solution on compact sets. However, the arguments used in [1] are different from that [20]. While in the scalar case each solution concentrates around the global minimum of V as the parameter, ε tends to zero, in the case of the system studied in [1], each solution is concentrated around the function $\xi \to C(\xi)$, where $C(\xi)$ is minimum value of the functional restricted to the Nehari manifold associated to system

$$(C_\xi) \qquad \begin{cases} -\ \Delta u + W(\xi)u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\Delta v + V(\xi)v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

Multiplicity results for the system (C_ε) involving Ljusternik-Schnirelmann theory and the topology of the set of minimum points of the functions W and V were studied in [3] by Alves, Figueiredo and Furtado in the subcritical case. The existence of positive radial solutions concentrating on spheres was studied in [13] by Carrião, Lisboa and Miyagaki.

The present work is strongly influenced by the above articles. Below we list what we believe that are the main contributions of Chapter 1.

(1) To the best of our knowledge, there are no concentration and multiplicity involving Ljusternik-Schnirelmann theory and the topology of the set of minimum points of functions a and b for the sistem (S_ε). The results in this chapter extend or complement the results in [12], [15], [22], [23], [24], [30], [38], [39] in the sense that we are working with elliptic systems.

(2) Since in (S_ε) the potentials a and b appear in divergence term, we cannot apply the same argument used in [1] to show that the solution of the modified system is in fact solution of the original system. We overcome this difficult using a Moser's iteration argument to estimate the L^∞ norm of the solution (see Lemma 1.5.1).

(3) The concentration result is also different from the result found in [1]. Moser's iteration allowed to show that each solution is concentrates around the global minimum of the potentials a and b when the parameter ε tends to zero.

The main result of Chapter 1 is:

Theorem 1. *Suppose that a and b are continuous potentials satisfying $(ab_1)-(ab_2)$ and $M \neq \emptyset$. Suppose also that Q satisfies $(Q_0)-(Q_5)$. Then,*

(i) for all $\varepsilon > 0$, the system (S_ε) has a positive ground state solution.

(ii) for any $\delta > 0$ verifying

$$M_\delta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) < \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system (S_ε) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions.

(iii) if $(u_\varepsilon, v_\varepsilon)$ is a solution for (S_ε) and if $\Pi_{\varepsilon,a}$ and $\Pi_{\varepsilon,b}$ are maximum points of u_ε and v_ε respectively, then $\Pi_{\varepsilon,a}, \Pi_{\varepsilon,b} \in \Lambda$, $\lim\limits_{\varepsilon\to 0^+} a(\Pi_{\varepsilon,a}) = a_0$ and $\lim\limits_{\varepsilon\to 0^+} b(\Pi_\varepsilon) = b_0$, furthermore, each solution $(u_\varepsilon, v_\varepsilon) \in C^{2,\lambda}(\mathbb{R}^N)$, for some $\lambda \in (0,1)$.

For the reader's convenience, the hypotheses in the previous theorem will be stated again in the corresponding chapter.

Chapter 1 of this thesis was published in the following article,
G. M. Figueiredo and S. M. A. Salirrosas, *Concentration, existence of ground state and multiplicity of solutions for a subcritical elliptic system via penalization method*, SN Partial Differential Equations and Applications, 2, 6 (2021).
https://doi.org/10.1007/s42985-020-00064-6

We now consider some results for system involving critical growth. In [19], Morais and Souto show existence of solution for this system

$$\begin{cases} -\Delta_p u = Q_u(u,v) + K_u(u,v) \text{ in } \Omega, \\ -\Delta_p v = Q_v(u,v) + K_v(u,v) \text{ in } \Omega. \end{cases}$$

The version of this system in an unbounded cylinder or a domain between two infinite cylinders was studied in [14] by Carrião and Miyagaki. Infinitely many solutions were obtained in a bounded domain in [21] by Demarque and Lisboa in the case of radial functions and biharmonic operators. A multiplicity result with critical growth and Laplacian operators involving the topology of the domain was studied in [26] by Furtado and Silva.

Multiplicity and concentration results for fractional Schrödinger system appeared in [7], [8], [9] and [16].

A critical version of the system (C_ε) was studied in [4] by Alves, Figueiredo and Furtado. The authors applied Ljusternik-Schnirelmann theory to relate the number of solutions to the topology of the set where W and V attain their minimum values. This motivates us to consider in Chapter 2 a critical version of the system (S_ε), namely the system

$$(CS_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) + \frac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \mathrm{div}(b(x)\nabla v) + v = Q_v(u,v) + \frac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) > 0 \ \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, $N \geq 3$ and $2^* = \frac{2N}{N-2}$. The conditions on the potentials a and b are the same as in Chapter 1. Due to the similarities of the conditions that we would be imposing on Q and K, for any given $q \geq 1$ we denote by $\mathcal{H}^q$ the collection of all functions $F \in C^2(\mathbb{R}^2_+, \mathbb{R})$ satisfying the following properties:

$(\mathcal{H}^q_0)$ F is q-homogeneous; that is

$$F(\lambda s, \lambda t) = \lambda^q F(s,t), \quad \text{for each} \quad \lambda > 0 \quad \text{and} \quad (s,t) \in \mathbb{R}^2_+;$$

$(\mathcal{H}^q_1)$ there exists $c_1 > 0$ such that

$$|F_s(s,t)| + |F_t(s,t))| \leq c_1 \left(s^{q-1} + t^{q-1}\right) \quad \text{for each } (s,t) \in \mathbb{R}^2_+;$$

$(\mathcal{H}_2)$ $F(s,t) > 0$ for each $s, t > 0$;

$(\mathcal{H}_3)$ $\nabla F(1,0) = \nabla F(0,1) = (0,0)$;

$(\mathcal{H}_4)$ $F_s(s,t), F_t(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}^2_+$.

The hypotheses on the functions Q and K are the following:

(A_1) $K \in \mathcal{H}^{2^*}$ and $Q \in \mathcal{H}^p$ for some $2 < p < 2^*$;

(A_2) the 1-homogeneous function $G : \mathbb{R}^2_+ \to \mathbb{R}$ given by $G(s^{2^*}, t^{2^*}) := K(s,t)$ is concave;

(A_3)

$$Q(s,t) \geq \frac{\sigma}{p_1} s^\lambda t^\beta, \quad \text{for all} \quad (s,t) \in \mathbb{R}^2_+,$$

where $\lambda, \beta > 1$, $\lambda + \beta =: p_1 \in (2, 2^*)$ and

$$\sigma > \sigma^* := \left(\frac{C(a_0, b_0)}{\frac{1}{N}\left(\min\{a_0, b_0\}\widetilde{S}_K\right)^{N/2}} \right)^{\frac{p_1 - 2}{2}}.$$

The constants that define σ^* will appear naturally in Proposition 2.3.1 and the definition of $\widetilde{S}_K$ will be given below.

Following the same ideas used in Chapter 1, we obtain an equivalent system to (CS_ε). Consequently, taking into account the term K and applying a penalization method, we obtain the modified system. Due to the presence of the terms a and b, the energy levels corresponding to functional associated to the autonomous system (namely c_0) and modified system (namely c_ε) are different. In [4], these values have a common bound, that is, are below $\frac{1}{N}\widetilde{S}_K^{N/2}$, but this not happens in our case. We solved this issue by obtaining bounds for c_0 and c_ε, as $\varepsilon \to 0$. Furthermore, we cannot argue as in [4] to show that a solution of the modified system is a solution of the original system. Once again we had to apply Moser's iteration techinique, see Lemma 2.5.1. As in the subcritical case, each solution of the system (CS_ε) concentrates around the global minimum of the potentials a and b as $\varepsilon \to 0$ and the problem (CS_ε) has at least $\text{cat}_{M_\delta}(M)$ positive solutions.

Since the nonlinearity K has critical growth, we apply the ideas of Brezis and Nirenberg [11] and also Morais and Souto [19]. In that paper it is proved that the number

$$\widetilde{S}_K := \inf \left\{ \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right) dx : \ u, v \in H^1(\mathbb{R}^N), \ \int_{\mathbb{R}^N} K(u^+, v^+) dx = 1 \right\}$$

plays an important role when dealing with a critical system. This constant was used to obtain the energy levels where the Palais-Smale condition fails.

Below we list what we believe that are the main contributions of our Chapter 2.

(1) To the best of our knowledge, there are no concentration and multiplicity involving Ljusternik-Schnirelmann theory and the topology of the set of minimum points of functions a and b for the sistem (S_ε). The results in this paper extend or complement the results in [12], [15], [22], [23], [24], [30], [38], [39] in the sense that we are working with elliptic systems.

(2) Here we also use the penalization method introduced in [1] and our result is similar to the result found in [4]. It is worthwhile to mention that, since in our case the potentials a and b appear in divergence term, we cannot apply the same argument found in [1] to show that the solution of the modified system is in fact solution of the original system. We overcome this difficult using a Moser's iteration argument to estimate the L^∞ norm of the solution, as can be seen in section 2.5.

(3) The concentration result is also different from that obtained in [1]. Moser's iteration allowed to show that each solution concentrates around the global minimum of the potentials a and b when the parameter ε tends to zero.

The main result of Chapter 2 is the following.

Theorem 2. *Suppose that a and b are continuous potentials satisfying $(ab_1)-(ab_2)$ and $M \neq \emptyset$. Suppose also that Q and K satisfy $(A_1)-(A_3)$. Then,*

(i) there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ the system (CS_ε) has a positive ground state solution.

(ii) for any $\delta > 0$ verifying

$$M_\delta := \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \leq \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system (CS_ε) has at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions.

(iii) if $(u_\varepsilon, v_\varepsilon)$ is a solution for (CS_ε) and if $\Pi_{\varepsilon,a}$ and $\Pi_{\varepsilon,b}$ are maximum points of u_ε and v_ε respectively, then $\Pi_{\varepsilon,a}, \Pi_{\varepsilon,b} \in \Lambda$, $\lim\limits_{\varepsilon\to 0^+} a(\Pi_{\varepsilon,a}) = a_0$ and $\lim\limits_{\varepsilon\to 0^+} b(\Pi_\varepsilon) = b_0$, furthermore, each solution $(u_\varepsilon, v_\varepsilon) \in C^{2,\lambda}(\mathbb{R}^N)$, for some $\lambda \in (0,1)$.

The hypotheses of the previous theorem will be stated again throughout the thesis.

The following article is a consequence of Chapter 2.
Giovany M. Figueiredo, Segundo Manuel A. Salirrosas, On multiplicity and concentration behavior of solutions for a critical system with equations in divergence form, Journal of Mathematical Analysis and Applications, v. 494, p. 124446, 2021.
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Inspired by systems (S_ε), (CS_ε), (C_ε), and their critical versions [4], we deal in Chapter 3 with the system

$$(P_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \operatorname{div}(a(x)\nabla u) + u = Q_u(u,v) + \frac{\gamma}{2^*} K_u(u,v) \text{ in } \mathbb{R}^N, \\ \quad -\varepsilon^2 \Delta v + b(x) v \;\; = Q_v(u,v) + \frac{\gamma}{2^*} K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, $N \geq 3$ and $2^* = \frac{2N}{N-2}$. We consider the subcritical case for $\gamma = 0$ and the critical case when $\gamma = 1$. The hypotheses made for a, b, Q and K are as in Chapter 2.

We now state the main results of Chapter 3,

Theorem 3 ($\gamma = 0$)**.** *Suppose that a and b are continuous potentials and satisfy $(ab_1)-(ab_2)$. Suppose also that $Q \in \mathcal{H}^p$ for any $2 < p < 2^*$. Then,*

(i) for all $\varepsilon > 0$, system (P_ε) has a ground state positive solution.

(ii) for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system (P_ε) has at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions.

(iii) if $(u_\varepsilon, v_\varepsilon)$ is a solution for (S_ε) and if $\Pi_{\varepsilon,a}$ and $\Pi_{\varepsilon,b}$ are maximum points of u_ε and v_ε respectively, then $\Pi_{\varepsilon,a}, \Pi_{\varepsilon,b} \in \Lambda$, $\lim\limits_{\varepsilon \to 0^+} a(\Pi_{\varepsilon,a}) = a_0$ and $\lim\limits_{\varepsilon \to 0^+} b(\Pi_\varepsilon) = b_0$, furthermore, each solution $(u_\varepsilon, v_\varepsilon) \in C^{2,\lambda}(\mathbb{R}^N)$, for some $\lambda \in (0,1)$.

The main novelty of the above theorem is that we give results of concentration and multiplicity for a new class of system that, to our knowledge there have not been studied. For achieving those results, we adapted again a penalization method given in [1] such that the energy functional of the modified system satisfies the Palais-Smale condition. Next, we proceed as in Chapter 1.

A critical version of Theorem 3 ($\gamma = 1$) can be obtained with Q and K satisfying $(A_1) - (A_3)$. But in this case

$$\sigma > \sigma^* := \left(\frac{C(a_0, b_0)}{\frac{1}{N} \left(\min\{a_0, 1\} \widetilde{S}_K \right)^{N/2}} \right)^{\frac{p_1 - 2}{2}}.$$

To obtain these results we do as in Chapter 2.

Chapter 3 of this thesis gave rise to an article entitled "On concentration behavior and multiplicity of solutions for a system in $\mathbb{R}^N$", which is submitted.

After the completion of this work we achieved other results:
Continuing with the study of critical systems started in Chapter 2, we study the following system

$$\begin{cases} -\varepsilon^2 \operatorname{div}(a(x)\nabla u) + u = f(x) Q_u(u,v) + \frac{1}{2^*} g(x) K_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \operatorname{div}(b(x)\nabla v) + v = f(x) Q_v(u,v) + \frac{1}{2^*} g(x) K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where the potentials f and g also are continuous.

The major novelty of the paper relies on the fact that nonconstant coefficients appear in the nonlinearities, which means that a competition occurs in the concentration among the different potentials. In this case solutions reveal to concentrate where the $x-$depending energy minimizes, i.e. in the set M of points of minima of a and b (in the divergence operators) and of f and g (in the nonlinearities).

The presence of variable coefficients increases the difficulties and the technicalities of the procedure, based on careful comparisons with different limiting problems and the analysis of corresponding Nehari least energy levels. The presence of the critical growth reduces moreover the applicability of Palais-Smale arguments to some particular sets of levels.

This work gave rise an article:
Giovany M. Figueiredo, Segundo Manuel A. Salirrosas, Multiplicity of Semiclassical States Solutions for a Weakly Coupled Schrödinger System with Critical Growth in Divergent Form. Potential Anal (2022).
https://doi.org/10.1007/s11118-021-09966-5

When we tried to solve a system with supercritical growth, our main difficulty was that the nonlinearity that appeared in our truncated problem was no longer $p-$homogeneous, which made impossible to apply the result obtained in Chapter 1. So we begin by studying a system where nonlinearity is not p-homogeneous. More specifically we study the following system

$$\begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) + \lambda K_u(u,v) & \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \mathrm{div}(b(x)\nabla v) + v = Q_v(u,v) + \lambda K_v(u,v) & \text{in } \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, $N \geq 3$, $\lambda \geq 0$, a, b are continuous potentials. The nonlinearity Q satisfies the following properties:

(Q_0) $Q \in C^1(\mathbb{R}^2, \mathbb{R})$ such that $Q(s,t) > 0$ if $(s,t) \neq (0,0)$, $Q(0,0) = 0$, $Q_s(s,t) = 0$ if $s \leq 0$ and $Q_t(s,t) = 0$ if $t \leq 0$;

(Q_1) there exist $p_1, p_2 \in (2, 2^*)$ and $c_1 > 0$ such that

$$|Q_s(s,t)| + |Q_t(s,t)| \leq c_1(|s|^{p_1-1} + |t|^{p_2-1}) \quad \text{for all} \quad (s,t) \in \mathbb{R}^2;$$

(Q_2) there exists $2 < \mu < p_1, p_2$ such that

$$0 < \mu Q(s,t) \leq sQ_s(s,t) + tQ_t(s,t) \quad \text{for all} \quad (s,t) \in \mathbb{R}^2 \setminus \{(0,0)\};$$

(Q_3) $\Upsilon \to \dfrac{sQ_s(\Upsilon s, \Upsilon t) + tQ_t(\Upsilon s, \Upsilon t)}{\Upsilon}$ is an increasing functions of $s, t > 0$;

(Q_4) there exists $\sigma^* > 0$ such that $Q(s,t) \geq \frac{\sigma}{p_5} s^\beta t^\nu$ for all $s, t \geq 0$, $\beta, \nu \geq 1$, $p_5 \in (2, 2^*)$ with $\beta + \nu = p_5$, for all $\sigma > \sigma^*$ and σ^* to be fixed later;

$(\widetilde{Q}_4)$ there exists $\sigma > 0$ such that $Q(s,t) \geq \frac{\sigma}{p_5} s^\beta t^\nu$ for all $s, t \geq 0$, $\beta, \nu \geq 1$, $p_5 \in (2, 2^*)$ with $\beta + \nu = p_5$.

We obtain results of existence and concentration of solutions for the subcritical case $\lambda = 0$. For $\lambda = \frac{1}{2^*}$ we get the same results by considering K as in Chapter 2. In the supercritical case, we consider $K(u,v) = |u|^{q_1} + |v|^{q_2}$, where $q_1, q_2 > 2^*$.

This work gave rise to an article entitled "Local Mountain Pass for a class of elliptic systems without homogeinity on the nonlinearity", which is submitted.

Notation

In this work we use the following notation:

$\nabla u = \left(\dfrac{\partial u}{\partial x_1}, \cdots \dfrac{\partial u}{\partial x_N} \right)$	gradient of the function u;
$\Delta u = \sum_{i=1}^{N} \dfrac{\partial^2 u}{\partial x_i^2} = \mathrm{div}(\nabla u)$	Laplacian of u;
$\rightharpoonup$	weak convergence;
$\to$	strong convergence;
$a.e.$	almost everywhere;
$\mathrm{supp}\ f$	support of the function f;
B_R	open ball of radius R centered at 0;
c_i and C_i with $i = 0, 1, 2, \ldots$	(possibly different) positive constants;
X'	dual space of the Banach space X;
$\mathcal{V}$	C^1-manifold;
$L^s_{\mathrm{loc}}(\mathbb{R}^N)$	space of all classes of functions which are in L^s on every compact subset of $\mathbb{R}^N$;
$\|\cdot\|_\varepsilon$	norm in the normed space X_ε;
$\|\cdot\|$	norm in the space $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$;
$\|I'(u)\|_*$	norm of the derivative of I restricted to $\mathcal{V}$ at the point u;
$\mathcal{M}_0$	Nehari manifold of I_0;
$\mathcal{N}_\varepsilon$	Nehari manifold of J_ε;
$\mathrm{cat}_X(Y)$	Ljusternik-Schnirelmann category of Y in X.

Chapter 1

Concentration, existence of a ground state and multiplicity of solutions for a subcritical elliptic system via penalization method

1.1 Introduction

In this chapter we show concentration, existence and multiple positive solutions for the following system given by

$$(S_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \mathrm{div}(b(x)\nabla v) + v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, a and b are positive continuous potentials and Q is a p-homogeneous function with subcritical growth.

More precisely, the hypotheses on functions a and b are the following:

(ab_1) there are $a_0 > 0$ and $b_0 > 0$ such that

$$0 < a_0 \leq a(x)$$

and

$$0 < b_0 \leq b(x) \text{ for all } x \in \mathbb{R}^N;$$

(ab_2) there exists a bounded domain $\Lambda \subset \mathbb{R}^N$ such that

$$0 < a_0 = \inf_{x \in \Lambda} a(x) < \inf_{x \in \partial\Lambda} a(x)$$

and

$$0 < b_0 = \inf_{x \in \Lambda} b(x) < \inf_{x \in \partial\Lambda} b(x).$$

Setting $\mathbb{R}^2_+ := [0,\infty) \times [0,\infty)$, we can state our hypothesis on $Q \in C^2(\mathbb{R}^2_+, \mathbb{R})$ in the following way:

(Q_0) there exits $2 < p < 2^* := 2N/(N-2)$ such that

$$Q(tu, tv) = t^p Q(u,v) \text{ for each } t > 0, \ (u,v) \in \mathbb{R}^2_+;$$

(Q_1) there exists $c_1 > 0$ such that

$$|Q_u(u,v)| + |Q_v(u,v)| \le c_1 \left(u^{p-1} + v^{p-1}\right) \quad \text{for each } (u,v) \in \mathbb{R}^2_+;$$

(Q_2) $Q_u(0,1) = 0$, $Q_v(1,0) = 0$;

(Q_3) $Q_u(1,0) = 0$, $Q_v(0,1) = 0$;

(Q_4) $Q(u,v) > 0$ for each $u, v > 0$;

(Q_5) $Q_u(u,v), Q_v(u,v) \ge 0$ for each $(u,v) \in \mathbb{R}^2_+$.

For each $\varepsilon > 0$, a pair $(u_\varepsilon, v_\varepsilon) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ is a positive solution of system (S_ε) if $u_\varepsilon > 0$ and $v_\varepsilon > 0$ a.e. in $\mathbb{R}^N$ and

$$\begin{aligned} &\varepsilon^2 \int_{\mathbb{R}^N} a(x) \nabla u_\varepsilon \nabla \phi dx + \varepsilon^2 \int_{\mathbb{R}^N} b(x) \nabla v_\varepsilon \nabla \psi dx + \int_{\mathbb{R}^N} u_\varepsilon \phi dx + \int_{\mathbb{R}^N} v_\varepsilon \psi dx \\ = &\int_{\mathbb{R}^N} [\phi Q_u(u_\varepsilon, v_\varepsilon) + \psi Q_v(u_\varepsilon, v_\varepsilon)] dx, \end{aligned}$$

for all $(\phi, \psi) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

A solution (u,v) of system (S_ε) is said to be ground state if

$$I(u,v) = \inf \left\{ I(w,z) : (w,z) \text{ is a solution of } (S_\varepsilon) \right\},$$

where $I : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ is the functional associated to (S_ε).

In this paper we also relate the number of solutions of (S_ε) with the topology of the set of minima of the potentials a and b. In order to present our result we introduce the following set:

$$M = \{x \in \mathbb{R}^N : a(x) = a_0 \text{ and } b(x) = b_0\}.$$

Our main result is as follows:

Theorem 1. *Suppose that a and b are continuous potentials satisfying $(ab_1) - (ab_2)$ and $M \neq \emptyset$. Suppose also that Q satisfies $(Q_0) - (Q_5)$. Then,*

(i) for all $\varepsilon > 0$, the system (S_ε) has a positive ground state solution.

(ii) for any $\delta > 0$ verifying

$$M_\delta := \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) < \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system (S_ε) has at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions.

(iii) if $(u_\varepsilon, v_\varepsilon)$ is a solution for (S_ε) and if $\Pi_{\varepsilon,a}$ and $\Pi_{\varepsilon,b}$ are maximum points of u_ε and v_ε respectively, then $\Pi_{\varepsilon,a}, \Pi_{\varepsilon,b} \in \Lambda$, $\lim\limits_{\varepsilon \to 0^+} a(\Pi_{\varepsilon,a}) = a_0$ and $\lim\limits_{\varepsilon \to 0^+} b(\Pi_\varepsilon) = b_0$, furthermore, each solution $(u_\varepsilon, v_\varepsilon) \in C^{2,\lambda}(\mathbb{R}^N)$, for some $\lambda \in (0,1)$.

We recall that, if Y is a closed set of a topological space X, $\operatorname{cat}_X(Y)$ is the Ljusternik-Schnirelmann category of Y in X, namely the least number of closed and contractible set in X which cover Y.

Now we give two examples of the potentials a and b that satisfy the hypotheses $(ab_1) - (ab_2)$.

$$a(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ |x|^2 & \text{if } |x| \ge 1, \end{cases} \qquad b(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ |x|^4 & \text{if } |x| \ge 1. \end{cases}$$

For this example, we can take $\Lambda = B_2(0)$ and $M = B_1(0)$.

Consider now

$$a(x) = b(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1 + |x| \sin\left(\frac{1}{|x|}\right) & \text{if } x \neq 0. \end{cases}$$

For this example, we can take $\Lambda = B_1(0)$.

Concerning the class of nonlinearities we are dealing, we have the following examples from [19]. Let $q \geq 1$ and

$$P_q(s,t) = \sum_{\alpha_i + \beta_i = q} a_i s^{\alpha_i} t^{\beta_i},$$

where $i \in \{1, \ldots, k\}$, $\alpha_i, \beta_i \geq 1$ and $a_i \in \mathbb{R}$. The following functions and their possible combinations, satisfy our hypothesis on Q

$$Q_1(s,t) = P_p(s,t), \quad Q_2(s,t) = \sqrt[r]{P_l(s,t)} \quad \text{and} \quad Q_3(s,t) = \frac{P_{l_1}(s,t)}{P_{l_2}(s,t)},$$

with $r = pl$ and $l_1 - l_2 = p$, under appropriate choices of the coefficients a_i.

The chapter is organized as follows. In Section 1.2 we present the variational framework and a modified system. In Section 1.3 we prove the existence of a ground sate solution for the modified system $(S_{\varepsilon,aux})$. Multiplicity result for the modified system $(S_{\varepsilon,aux})$ involving Ljusternik-Scrinirelmann theory is section 1.4. In Section 1.5 we prove that each solution of the modified system $(S_{\varepsilon,aux})$ is a solution of the original system. We also prove a concentration result.

1.2 Variational framework and a modified system

Since we are interested in positive solutions we extend the function Q to the whole $\mathbb{R}^2$ by setting $Q(u,v) = 0$ if $u \leq 0$ or $v \leq 0$. We also note that, since Q is p-homogeneous, for each $(s,t) \in \mathbb{R}^2$ we have

$$pQ(s,t) = sQ_s(s,t) + tQ_t(s,t) \tag{1.2.1}$$

and

$$p(p-1)Q(s,t) = s^2 Q_{ss}(s,t) + t^2 Q_{tt}(s,t) + 2st Q_{st}(s,t). \tag{1.2.2}$$

Hereafter, we will work with the following system equivalent to (S_ε).

$$(\widehat{S}_\varepsilon) \qquad \begin{cases} -\text{div}(a(\varepsilon x)\nabla u) + u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\text{div}(b(\varepsilon x)\nabla v) + v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \ \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In order to overcome the lack of compactness originated by the unboundedness of $\mathbb{R}^N$ we use a penalization method. Such kind of idea has first appeared in the paper of Del Pino and Felmer [20]. Here we use an adaptation of this method for systems, as introduced in [1].

We start by choosing $\alpha > 0$ and considering $\eta : \mathbb{R} \to \mathbb{R}$ a non-increasing function of class C^2 such that

$$\eta \equiv 1 \text{ on } (-\infty, \alpha], \quad \eta \equiv 0 \text{ on } [5\alpha, +\infty), \quad |\eta'(s)| \leq \frac{C}{\alpha} \quad \text{and} \quad |\eta''(s)| \leq \frac{C}{\alpha^2} \tag{1.2.3}$$

for each $s \in \mathbb{R}$ and for some positive constant $C > 0$. Using the function η, we define $\widehat{Q} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\widehat{Q}(s,t) := \eta(|(s,t)|)Q(s,t) + (1 - \eta(|(s,t)|))A(s^2 + t^2),$$

where

$$A := \max\left\{\frac{Q(s,t)}{s^2+t^2} : (s,t) \in \mathbb{R}^2,\ \alpha \le |(s,t)| \le 5\alpha\right\}.$$

Notice that, since $A > 0$ tends to zero as $\alpha \to 0^+$, we may suppose that $A < 1$.

Now we give an example of a function η. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a function of C^∞ class given by

$$\beta(s) = \begin{cases} \exp(\frac{-1}{1-s^2}) & \text{if } |s| < 1, \\ 0 & \text{if } |s| \ge 1. \end{cases}$$

Then

$$\int_{-1}^{1} \beta(t)dt = 1.$$

Consider

$$h(s) = \int_{\frac{s}{2\alpha}}^{1} \beta(t)dt$$

and $\eta(s) = h(s - 3\alpha)$. Note that

$$\eta'(s) = -\frac{1}{2\alpha}\beta\left(\frac{s-3\alpha}{2\alpha}\right) \quad \text{and} \quad \eta''(s) = -\frac{1}{2\alpha}\frac{1}{2\alpha}\beta'\left(\frac{s-3\alpha}{2\alpha}\right).$$

If $s \le \alpha$ or $s \ge 5\alpha$, $\eta'(s) = 0 = \eta''(s)$. For $\alpha < s < 5\alpha$ we have

$$|\eta'(s)| = \frac{1}{2\alpha}\beta\left(\frac{s-3\alpha}{2\alpha}\right) \le \frac{1}{2\alpha}\max\beta(s) = \frac{1}{2\alpha}\frac{1}{e} = \frac{1}{\alpha}C_1$$

and

$$|\eta''(s)| = \frac{1}{4\alpha^2}\left|\beta'\left(\frac{s-3\alpha}{2\alpha}\right)\right| \le \frac{1}{4\alpha^2}\theta = \frac{1}{\alpha^2}C_2.$$

Finally, denoting by I_Λ the characteristic function of the set Λ, we define $H : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ by setting

$$H(x,s,t) := I_\Lambda(x)Q(s,t) + (1 - I_\Lambda(x))\widehat{Q}(s,t). \tag{1.2.4}$$

For future reference we note that arguing as in [1, Lemma 2.2], for any $\alpha > 0$ small and $(s,t) \in \mathbb{R}^2$ we have the following result:

Lemma 1.2.1. *The function H satisfies the following estimates:*

(H_1) $pH(x,s,t) = sH_s(x,s,t) + tH_t(x,s,t)$, for each $x \in \Lambda$;

(H_2) $2H(x,s,t) \le sH_s(x,s,t) + tH_t(x,s,t)$, for each $x \in \mathbb{R}^N \setminus \Lambda$;

(H_3) for α small we have $sH_s(x,s,t) + tH_t(x,s,t) \le \frac{1}{4}\left(s^2+t^2\right)$ for each $x \in \mathbb{R}^N \setminus \Lambda$;

(H_4) for α small we have $\frac{|H_s(x,s,t)|}{\alpha}, \frac{|H_t(x,s,t)|}{\alpha} \le \frac{1}{4}$ for each $x \in \mathbb{R}^N \setminus \Lambda$.

From now on we assume that α is chosen in such way that the last inequality above holds. In view of definition (1.2.4), we deal in the sequel with the modified system

$$(S_{\varepsilon,aux}) \qquad \begin{cases} -\text{div}(a(\varepsilon x)\nabla u) + u = H_u(\varepsilon x, u, v) \text{ in } \mathbb{R}^N, \\ -\text{div}(b(\varepsilon x)\nabla v) + v = H_v(\varepsilon x, u, v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N) \end{cases}$$

and we will look for solutions $(u_\varepsilon, v_\varepsilon)$ verifying

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \text{ for each } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon,$$

where $\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$.

For each $\varepsilon > 0$ we denote by X_ε the Hilbert space

$$X_\varepsilon := \left\{(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|\nabla v|^2) dx < \infty\right\}$$

endowed with the norm

$$\|(u,v)\|_\varepsilon^2 := \int_{\mathbb{R}^N} (a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|\nabla v|^2 + |u|^2 + |v|^2) dx.$$

Conditions (H_3) and (Q_1) imply that the critical points of the C^1-functional $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$J_\varepsilon(u,v) := \frac{1}{2} \int_{\mathbb{R}^N} \left(a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|\nabla v|^2 + |u|^2 + |v|^2\right) dx - \int_{\mathbb{R}^N} H(\varepsilon x, u, v) dx$$

are weak solutions of $(S_{\varepsilon,aux})$. We recall that these critical points belong to the Nehari manifold of J_ε, namely

$$\mathcal{N}_\varepsilon := \{(u,v) \in X_\varepsilon \setminus \{(0,0)\} : J_\varepsilon'(u,v)(u,v) = 0\}.$$

Arguing as [40, Lemma 4.1], for any nontrivial element $(u,v) \in X_\varepsilon$ the function $t \mapsto J_\varepsilon(tu, tv)$, for $t \geq 0$, achieves its maximum value at a unique point $t_{u,v} > 0$ such that $t_{u,v}(u,v) \in \mathcal{N}_\varepsilon$. We define the number b_ε by setting

$$b_\varepsilon := \inf_{(u,v) \in \mathcal{N}_\varepsilon} J_\varepsilon(u,v). \tag{1.2.5}$$

The main result in this section is:

Theorem 1.2.2. *Suppose that a and b are continuous potentials and satisfy $(ab_1) - (ab_2)$ and $M \neq \emptyset$. Suppose also that Q satisfies $(Q_0) - (Q_5)$. Then,*

(i) for all $\varepsilon > 0$, the system $(S_{\varepsilon,aux})$ has a positive ground state solution.

(ii) for any $\delta > 0$ verifying

$$M_\delta := \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) < \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system $(S_{\varepsilon,aux})$ has at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions.

1.3 Existence of a ground state solution for the modified system $(S_{\varepsilon,aux})$

We start defining the Palais-Smale compactness condition. A sequence $((u_n, v_n)) \subset X_\varepsilon$ is a Palais-Smale sequence at level c_ε for the functional J_ε if

$$J_\varepsilon(u_n, v_n) \to c_\varepsilon$$

and

$$J_\varepsilon'(u_n, v_n) \to 0 \text{ in } (X_\varepsilon)',$$

where

$$c_\varepsilon = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} J_\varepsilon(\eta(t)) > 0$$

and

$$\Gamma := \{\eta \in C([0,1], X_\varepsilon) : \eta(0) = (0,0),\ J_\varepsilon(\eta(1)) < 0\}.$$

If every Palais-Smale sequence of J_ε has a strong convergent subsequence, then one says that J_ε satisfies the Palais-Smale condition ((PS) for short).

In order to show existence of a ground state solution for the modified system $(S_{\varepsilon,aux})$, we use the Mountain Pass Theorem [6].

Lemma 1.3.1. *The functional J_ε satisfies the following conditions*

(i) there is $C, \rho > 0$, such that

$$J_\varepsilon(u,v) \geq C, \quad \textit{if} \ \ \|(u,v)\|_\varepsilon = \rho.$$

(ii) for any $(\phi, \psi) \in C_0^\infty(\Lambda_\varepsilon) \times C_0^\infty(\Lambda_\varepsilon)$ with $\phi, \psi \geq 0$, we have

$$\lim_{t \to \infty} J_\varepsilon(t\phi, t\psi) = -\infty.$$

Proof. Using (Q_1), (1.2.1), (H_1), (H_2) and (H_3), we have

$$J_\varepsilon(u,v) \geq \frac{1}{2}\|(u,v)\|_\varepsilon^2 - \frac{2c_1}{p}\int_{\Lambda_\varepsilon}(|u|^p + |v|^p)dx - \frac{1}{8}\int_{\mathbb{R}^N \setminus \Lambda_\varepsilon}(|u|^2 + |v|^2)dx.$$

By Sobolev embeddings, there exists $C > 0$ such that

$$J_\varepsilon(u,v) \geq \frac{3}{8}\|(u,v)\|_\varepsilon^2 - \frac{C}{p}\|(u,v)\|_\varepsilon^p$$

and the proof of item (i) is finished. Now, by definition of H and (Q_0), we get

$$J_\varepsilon(t\phi, t\psi) = \frac{t^2}{2}\|(\phi,\psi)\|_\varepsilon^2 - t^p \int_{\Lambda_\varepsilon} Q(\phi,\psi)dx$$

and the proof of item (ii) is also finished. □

Hence, there exists a Palais-Smale sequence $((u_n, v_n)) \subset X_\varepsilon$ at level c_ε. Using (Q_0), it is possible to prove that

$$c_\varepsilon = b_\varepsilon = \inf_{(u,v) \in X_\varepsilon \setminus \{(0,0)\}} \sup_{t \geq 0} J_\varepsilon(tu, tv), \tag{1.3.1}$$

where b_ε was defined in (1.2.5).

In order to prove the Palais-Smale condition, we need to prove the next lemma.

Lemma 1.3.2. *Let $((u_n, v_n))$ be a $(PS)_d$ sequence for J_ε. Then for each $\xi > 0$, there exists $R = R(\xi) > 0$ such that*

$$\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} \Big[a(\varepsilon x)|\nabla u_n|^2 + b(\varepsilon x)|\nabla v_n|^2 + |u_n|^2 + |v_n|^2\Big]dx < \xi.$$

Proof. Let $\eta_R \in C^\infty(\mathbb{R}^N)$ such that $\eta_R(x) = 0$ if $x \in B_{R/2}(0)$ and $\eta_R(x) = 1$ if $x \notin B_R(0)$, with $0 \le \eta_R(x) \le 1$ and $|\nabla \eta_R| \le \dfrac{C}{R}$, where C is a constant independent of R. Since that the sequence $((\eta_R u_n, \eta_R v_n))$ is bounded in X_ε, fixing $R > 0$ such that $\Lambda_\varepsilon \subset B_{R/2}(0)$ and by definition of the functional J_ε, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_R(0)} \Big[a(\varepsilon x)|\nabla u_n|^2 + b(\varepsilon x)|\nabla v_n|^2 + |u_n|^2 + |v_n|^2\Big] dx \\
\le\ & J_\varepsilon'(u_n, v_n)(u_n \eta_R, v_n \eta_R) + \int_{\mathbb{R}^N} \left[u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)\right] \eta_R dx \\
-\ & \int_{\mathbb{R}^N} \left[a(\varepsilon x) u_n \nabla u_n + b(\varepsilon x) v_n \nabla v_n\right] \nabla \eta_R dx.
\end{aligned}$$

Using (H_3), we get the estimate

$$\begin{aligned}
&\frac{3}{4} \int_{\mathbb{R}^N \setminus B_R} \Big[a(\varepsilon x)|\nabla u_n|^2 + b(\varepsilon x)|\nabla v_n|^2 + |u_n|^2 + |v_n|^2\Big] dx \\
\le\ & \int_{\mathbb{R}^N} \Big[a(\varepsilon x)|u_n||\nabla u_n| + b(\varepsilon x)|v_n||\nabla v_n|\Big] |\nabla \eta_R| dx + o_n(1).
\end{aligned}$$

Since $((u_n, v_n))$ is bounded in X_ε and $|\nabla \eta_R| \le \dfrac{C}{R}$ and passing to the limit in the last estimate, it follows that

$$\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R} \Big[a(\varepsilon x)|\nabla u_n|^2 + b(\varepsilon x)|\nabla v_n|^2 + |u_n|^2 + |v_n|^2\Big] dx < \xi.$$

for some R sufficiently large and for some fixed $\xi > 0$. □

Lemma 1.3.3. *The functional J_ε satisfies the Palais-Smale condition at any level c.*

Proof. Let $((u_n, v_n)) \subset X_\varepsilon$ such that $J_\varepsilon(u_n, v_n) \to c$ and $J_\varepsilon'(u_n, v_n) = o_n(1)$. Then, from (H_1), we get

$$\begin{aligned}
c + o_n(1) + o(\|(u_n, v_n)\|_\varepsilon) &= J_\varepsilon(u_n, v_n) - \frac{1}{p} J_\varepsilon'(u_n, v_n)(u_n, v_n) = \left(\frac{1}{2} - \frac{1}{p}\right) \|(u_n, v_n)\|_\varepsilon^2 \\
- \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} \Big[H(\varepsilon x, u_n, v_n) &- \frac{1}{p}\big[u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)\big]\Big] dx.
\end{aligned}$$

From (H_2), we have

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{p}\right) \|(u_n, v_n)\|_\varepsilon^2 \le c + o_n(1) + o(\|(u_n, v_n)\|_\varepsilon) \\
+\ & \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} \Big[u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)\Big] dx.
\end{aligned}$$

Using (H_3) we obtain

$$\frac{3}{4}\left(\frac{1}{2} - \frac{1}{p}\right) \|(u_n, v_n)\|_\varepsilon^2 \le c + o_n(1) + o(\|(u_n, v_n)\|_\varepsilon),$$

which implies that $((u_n, v_n))$ is bounded in X_ε. Then, up to a subsequence, we may suppose that,

$$\begin{aligned}
&(u_n, v_n) \rightharpoonup (u, v) \ \text{ weakly in } \ X_\varepsilon, \\
&u_n \to u, v_n \to v \ \text{ strongly in } \ L^s_{loc}(\mathbb{R}^N), \ \text{ for any } \ 2 \le s < 2^*, \\
&u_n(x) \to u(x), v_n(x) \to v(x) \ \text{ for a.e. } \ x \in \mathbb{R}^N.
\end{aligned} \tag{1.3.2}$$

Now using a density argument, we can conclude that (u, v) is a critical point of J_ε. From Lemma 1.3.2, for any $\xi >$ given, there exists $R > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R} \left[a(\varepsilon x)|\nabla u_n|^2 + b(\varepsilon x)|\nabla v_n|^2 + |u_n|^2 + |v_n|^2\right] dx < \xi.$$

This inequality, (H_3) and the Sobolev embeddings imply that, for n large enough, there holds

$$\int_{\mathbb{R}^N\setminus B_R} [u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)] dx \leq \frac{1}{4}\xi C_1, \tag{1.3.3}$$

where C_1 is positive constant. On the other hand, taking R large enough, we can suppose that

$$\left|\int_{\mathbb{R}^N\setminus B_R} [u H_u(\varepsilon x, u, v) + v H_v(\varepsilon x, u, v)] dx\right| < \xi. \tag{1.3.4}$$

Then, by (1.3.3) and (1.3.4), we can conclude

$$\begin{aligned} &\int_{\mathbb{R}^N\setminus B_R} [u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)] dx \\ = &\int_{\mathbb{R}^N\setminus B_R} [u H_u(\varepsilon x, u, v) + v H_v(\varepsilon x, u, v)] dx + o_n(1). \end{aligned}$$

Then,

$$\begin{aligned} &\int_{\mathbb{R}^N} [u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)] dx \\ = &\int_{\mathbb{R}^N} [u H_u(\varepsilon x, u, v) + v H_v(\varepsilon x, u, v)] dx + o_n(1). \end{aligned}$$

The last equality implies

$$\|(u_n, v_n)\|_\varepsilon^2 = \|(u, v)\|_\varepsilon^2 + o_n(1).$$

□

1.3.1 Proof of the item (i) of Theorem 1.2.2

Proof. The proof is a consequence of Lemma 1.3.1, Lemma 1.3.3, Mountain Pass Theorem [6] and of the characterization of minimax level c_ε given in (1.3.1). □

1.4 Multiple solutions for the modified system $(S_{\varepsilon,aux})$

In order to prove the item (ii) of Theorem 1.2.2, we consider the following autonomous system:

$$(S_0) \qquad \begin{cases} -a_0\Delta u + u = Q_u(u, v) \text{ in } \mathbb{R}^N, \\ -b_0\Delta v + v = Q_v(u, v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In view of conditions (ab_1) and (Q_1), the above system has a variational structure and the associated functional given by

$$I_0(u, v) := \frac{1}{2}\int_{\mathbb{R}^N} \left(a_0|\nabla u|^2 + b_0|\nabla v|^2 + |u|^2 + |v|^2\right) dx - \int_{\mathbb{R}^N} Q(u, v) dx,$$

well defined for $(u,v) \in E_0 := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. We denote the norm in E_0 by

$$\|(u,v)\|^2 = a_0 \int_{\mathbb{R}^N} |\nabla u|^2 dx + b_0 \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |v|^2 dx.$$

Arguing as in Lemma 1.3.1, we can show that I_0 has the mountain pass geometry and therefore we can set the the minimax level c_0 in the following way

$$c_0 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_0(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], E_0) : \gamma(0) = (0,0),\ I_0(\gamma(1)) < 0\}$. Moreover, c_0 can be further characterized as

$$c_0 = \inf_{(u,v) \in \mathcal{M}_0} I_0(u,v), \tag{1.4.1}$$

with $\mathcal{M}_0$ being the Nehari manifold of I_0, that is

$$\mathcal{M}_0 := \{(u,v) \in E_0 \setminus \{(0,0)\} : I_0'(u,v)(u,v) = 0\}.$$

The next result allows to show that system (S_0) has a solution that reaches c_0.

Lemma 1.4.1. *Let $((u_n, v_n)) \subset \mathcal{M}_0$ be a sequence such that $I_0(u_n, v_n) \to c_0$. Then there are a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R, \eta > 0$ such that*

$$\liminf_{n \to \infty} \int_{B_R(y_n)} \left(|u_n|^2 + |v_n|^2\right) dx \geq \eta. \tag{1.4.2}$$

Proof. Suppose that (1.4.2) is not satisfied. Since $((u_n, v_n))$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, then, from [32, Lemma 1.1], we get

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^s dx = 0$$

and

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |v_n|^s dx = 0,$$

for all $s \in (2, 2^*)$. Thus, from (Q_1), we conclude

$$\int_{\mathbb{R}^N} [u_n Q_u(u_n, v_n) + v_n Q_v(u_n, v_n)] dx = o_n(1).$$

Since $I_0'(u_n, v_n)(u_n, v_n) = 0$, we obtain $\|(u_n, v_n)\| = o_n(1)$, which implies $c_0 = 0$, which is a contradiction. □

Now we are ready to show that system (S_0) has a solution that reaches c_0.

Lemma 1.4.2. *(A Compactness Lemma) Let $((u_n, v_n)) \subset \mathcal{M}_0$ be a sequence satisfying $I_0(u_n, v_n) \to c_0$. Then, there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $(w_n(x), z_n(x)) = (u_n(x + \widetilde{y}_n), v_n(x + \widetilde{y}_n))$ converges strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. In particular, there exists a minimizer for c_0.*

Proof. Applying Ekeland's Variational Principle [40, Theorem 8.5], we may suppose that $((u_n, v_n))$ is a $(PS)_{c_0}$ for I_0. Since $((u_n, v_n))$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, we have that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Then, $\|(u, v)\|^2 \leq \liminf\limits_{n\to\infty} \|(u_n, v_n)\|^2$. We are going to prove that

$$\|(u, v)\|^2 = \lim_{n\to\infty} \|(u_n, v_n)\|^2. \tag{1.4.3}$$

Suppose, by contradiction, that (1.4.3) does not hold. Then, by $(Q_2) - (Q_3)$, we can consider $(u, v) \neq (0, 0)$. Using a density argument we have that $I_0'(u, v)(u, v) = 0$, where we conclude that $(u, v) \in \mathcal{M}_0$. Using (1.2.1), we obtain

$$\begin{aligned} c_0 \leq I_0(u, v) - \frac{1}{p} I_0'(u, v)(u, v) &< \left(\frac{1}{2} - \frac{1}{p}\right) \liminf_{n\to+\infty} \|(u_n, v_n)\|^2 \\ = \liminf_{n\to+\infty} \left[I_0(u_n, v_n) - \frac{1}{p} I_0'(u_n, v_n)(u_n, v_n) \right] &= c_0, \end{aligned}$$

which is a contradiction. Hence, $(u_n, v_n) \to (u, v)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Consequently, $I_0(u, v) = c_0$ and $\widetilde{y}_n = 0$, for all $n \in \mathbb{N}$.

If $(u, v) \equiv (0, 0)$, then in this case we cannot have $(u_n, v_n) \to (u, v)$ strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ because $c_0 > 0$. Hence, using the Lemma 1.4.1, there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that

$$(w_n, z_n) \rightharpoonup (w, z) \ \ in \ \ H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N),$$

where $w_n(x) = u_n(x + \widetilde{y}_n)$ and $z_n(x) = v_n(x + \widetilde{y}_n)$. Therefore, $((w_n, z_n))$ is also a $(PS)_{c_0}$ sequence of I_0 and $(w, z) \not\equiv (0, 0)$. It follows from above arguments that, up to a subsequence, $((w_n, z_n))$ converges strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and the proof of lemma is finished. □

The proof of the (ii) of Theorem 1.2.2 is rather long and will be done by applying the following Ljusternik-Schnirelmann abstract result. The proof of this result can be found in [27, Corollary 4.17]:

Theorem 1.4.3. *Let I be a C^1-functional defined on a C^1-Finsler manifold $\mathcal{V}$. If I is bounded from below and satisfies the Palais-Smale condition, then I has at least $\mathrm{cat}_{\mathcal{V}}(\mathcal{V})$ distinct critical points.*

The following result, which has a proof similar to that presented in [10, Lemma 4.3], will be used.

Lemma 1.4.4. *Let Γ, Ω^+, Ω^- be closed sets with $\Omega^- \subset \Omega^+$. Let $\beta : \Gamma \to \Omega^+$, $\Phi : \Omega^- \to \Gamma$ be two continuous maps such that $\beta \circ \Phi$ is homotopically equivalent to the embedding $\iota : \Omega^- \to \Omega^+$. Then $\mathrm{cat}_\Gamma(\Gamma) \geq \mathrm{cat}_{\Omega^+}(\Omega^-)$.*

1.4.1 The Palais-Smale condition in the Nehari manifold associated to J_ε

Since we are intending to apply critical point theory, we need to introduce some compactness property. So, let V be a Banach space, $\mathcal{V}$ be a C^1-manifold of V and $I : V \to \mathbb{R}$ a C^1-functional. We say that $I|_{\mathcal{V}}$ satisfies the Palais-Smale condition at level c ($(\mathrm{PS})_c$ for short) if any sequence $(u_n) \subset \mathcal{V}$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_* \to 0$ contains a convergent subsequence. Here, we are denoting by $\|I'(u)\|_*$ the norm of the derivative of I restricted to $\mathcal{V}$ at the point u.

From Lemma 1.3.3, the functional J_ε satisfies $(\mathrm{PS})_c$ for each $c \in \mathbb{R}$. Nevertheless, to get multiple critical points, we need to work with the functional J_ε constrained to $\mathcal{N}_\varepsilon$. In order to prove the desired compactness result, we shall first present some properties of $\mathcal{N}_\varepsilon$, which the proofs of the next three results follow by using the same arguments employed in [3, Lemma 2.2, Lemma 2.3 and Proposition 2.4] for other classes of systems. For the sake of completeness, we sketch here.

Lemma 1.4.5. *There exist positive constants α_1, δ_1, C such that, for each $\alpha \in (0, \alpha_1)$, $(u, v) \in \mathcal{N}_\varepsilon$, there hold*

$$\int_{\Lambda_\varepsilon} Q(u, v)dx \geq \delta_1 \tag{1.4.4}$$

and

$$\int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (u^2 + v^2)dx \leq C \int_{\Lambda_\varepsilon} Q(u, v)dx. \tag{1.4.5}$$

Proof. Since H has subcritical growth, it is easy to obtain $\widehat{\delta} > 0$ such that

$$\|(u, v)\|_\varepsilon \geq \widehat{\delta} \ \text{ for each } (u, v) \in \mathcal{N}_\varepsilon.$$

Thus, we can use (1.2.1) and (H_3) to get

$$\begin{aligned} \widehat{\delta}^2 \leq \|(u, v)\|_\varepsilon^2 &= \int_{\Lambda_\varepsilon} (uQu + vQ_v)dx + \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (uH_u + vH_v)dx \\ &\leq p \int_{\Lambda_\varepsilon} Q(u, v)dx + \frac{1}{4} \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (u^2 + v^2)dx \end{aligned}$$

and therefore

$$\frac{3\widehat{\delta}^2}{4} \leq \frac{3}{4}\|(u, v)\|_\varepsilon^2 \leq p \int_{\Lambda_\varepsilon} Q(u, v)dx,$$

which implies (1.4.4) with $\delta_1 = \frac{3\widehat{\delta}^2}{4p}$.

By using (H_3) and (1.2.1) again, we obtain

$$\begin{aligned} \frac{1}{4} \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (u^2 + v^2)dx &\leq \|(u, v)\|_\varepsilon^2 + \int_{\Lambda_\varepsilon} (uQ_u + vQ_v)dx \\ &\leq \frac{4p}{3} \int_{\Lambda_\varepsilon} Q(u, v)dx + p \int_{\Lambda_\varepsilon} Q(u, v)dx, \end{aligned}$$

from which follows (1.4.5). The lemma is proved. □

The following technical results are central to the compactness result.

Lemma 1.4.6. *Let $\phi_\varepsilon : X_\varepsilon \to \mathbb{R}$ be given by*

$$\phi_\varepsilon(u, v) := \|(u, v)\|_\varepsilon^2 - \int_{\mathbb{R}^N} \Big(uH_u(\varepsilon x, u, v) + vH_v(\varepsilon x, u, v) \Big) dx.$$

Then there exist $\alpha_2, K > 0$ such that, for each $\alpha \in (0, \alpha_2)$,

$$\phi'_\varepsilon(u, v)(u, v) \leq -K < 0 \ \text{ for each } (u, v) \in \mathcal{N}_\varepsilon. \tag{1.4.6}$$

Proof. Given $(u, v) \in \mathcal{N}_\varepsilon$, we can use the definition of H, (1.2.1) and (1.2.2) to get

$$\begin{aligned} \phi'_\varepsilon(u, v)(u, v) &= \int_{\Lambda_\varepsilon} \Big((uQ_u + vQ_v) - (u^2 Q_{uu} + v^2 Q_{vv} + 2uvQ_{uv}) \Big) dx \\ &+ \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (uH_u + vH_v)\, dx - \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (u^2 H_{uu} + v^2 H_{vv} + 2uvH_{uv})\, dx \\ &= -p(p-2) \int_{\Lambda_\varepsilon} Q(u, v)dx + \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} D_1 dx - \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} D_2 dx \end{aligned} \tag{1.4.7}$$

with

$$D_1 := uH_u + vH_v \quad \text{and} \quad D_2 := u^2 H_{uu} + v^2 H_{vv} + 2uvH_{uv}.$$

In what follows we denote $|z| := \sqrt{u^2+v^2}$. By using the definition of $\widehat{Q}$, η and (1.2.1) again, we obtain

$$\begin{aligned}
|D_1| &= \left| \eta' \frac{Q}{|z|} + p\eta \frac{Q}{|z|^2} - A\eta'|z| + 2A(1-\eta) \right| |z|^2 \\
&\leq \left(\frac{C}{\alpha} A 5\alpha + pA + A\frac{C}{\alpha} 5\alpha + 4A \right) |z|^2 \\
&\leq C_1 A |z|^2.
\end{aligned}$$

Since $A \to 0$ as $\alpha \to 0^+$, the last inequality leads to

$$\int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (uH_u + vH_v)\, dx \leq o(1) \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (u^2+v^2) dx, \tag{1.4.8}$$

where $o(1) \to 0$ as $\alpha \to 0^+$.

In order to estimate the last integral in (1.4.7), we first compute

$$D_2 = -A\eta'(|z|^2 + 4|z|)|z|^2 + 2A(1-\eta)|z|^2 + \eta'' Q|z||z|^2 + D_3 + D_4, \tag{1.4.9}$$

with

$$D_3 := \frac{2\eta'}{|z|} \left(u^3 Q_u + v^3 Q_v + u^2 v Q_v + uv^2 Q_u \right)$$

and

$$D_4 := \eta(u^2 Q_{uu} + v^2 Q_{vv} + 2uvQ_{uv}).$$

In view of (1.2.3) we have that

$$\left| A\eta'(|z|^2 + 4|z|)|z|^2 \right| \leq A\frac{C}{\alpha}(25\alpha^2 + 20\alpha)|z|^2 = o(1)|z|^2.$$

By using the definition of A, we also obtain

$$2A(1-\eta)|z|^2 = o(1)|z|^2 \quad \text{and} \quad \eta'' Q|z||z|^2 = o(1)|z|^2.$$

Moreover, we infer from (1.2.1) that

$$|D_3| = |4p\eta' Q||z| \leq 4p\frac{C}{\alpha} A|z|^2 5\alpha = 20pCA|z|^2 = o(1)|z|^2.$$

Finally, (1.2.2) implies that

$$D_4 = \eta(u^2 Q_{uu} + v^2 Q_{vv} + 2uvQ_{uv}) = \eta p(p-1)Q \geq 0.$$

From these estimates, we derive that

$$\int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} \left(u^2 H_{uu} + v^2 H_{vv} + 2uvH_{uv} \right) dx \leq o(1) \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (u^2+v^2) dx.$$

Thus, it follows from (1.4.8) and (1.4.7) that

$$\phi'_\varepsilon(u,v)(u,v) \leq -p(p-2) \int_{\Lambda_\varepsilon} Q(u,v) dx + o(1) \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (u^2+v^2) dx.$$

Now we can use Lemma 1.4.5 to obtain, for α small enough,

$$\phi'_\varepsilon(u,v)(u,v) \leq (-p(p-2) + o(1)) \int_{\Lambda_\varepsilon} Q(u,v) dx \leq -\frac{p(p-2)}{2}\delta_1 = -K < 0.$$

and the lemma is proved. □

Proposition 1.4.7. *The functional J_ε restricted to $\mathcal{N}_\varepsilon$ satisfies* $(\mathrm{PS})_c$ *for each* $c \in \mathbb{R}$.

Proof. Let $((u_n, v_n)) \subset \mathcal{N}_\varepsilon$ be such that

$$J_\varepsilon(u_n, v_n) \to c \text{ and } \|J'_\varepsilon(u_n, v_n)\|_* = o_n(1),$$

where $o_n(1)$ approaches zero as $n \to \infty$. Then there exists $(\lambda_n) \subset \mathbb{R}$ satisfying

$$J'_\varepsilon(u_n, v_n) = \lambda_n \phi'_\varepsilon(u_n, v_n) + o_n(1), \tag{1.4.10}$$

with ϕ_ε as in Lemma 1.4.6. Since $(u_n, v_n) \in \mathcal{N}_\varepsilon$ we have that

$$0 = J'_\varepsilon(u_n, v_n)(u_n, v_n) = \lambda_n \phi'_\varepsilon(u_n, v_n)(u_n, v_n) + o_n(1)\|(u_n, v_n)\|_\varepsilon.$$

Straightforward calculations show that $((u_n, v_n))$ is bounded. Moreover, in view of Lemma 1.4.6, we may suppose that $\phi'_\varepsilon(u_n, v_n)(u_n, v_n) \to l < 0$. Hence, the above expression shows that $\lambda_n \to 0$ and therefore we conclude that $J'_\varepsilon(u_n, v_n) \to 0$ in the dual space of X_ε. It follows from Lemma 1.3.3 that $((u_n, v_n))$ has a convergent subsequence. □

From now on we will denote by (w_1, w_2) the solution for the system (S_0) given at the beginning of this section.

Let us consider $\delta > 0$ such that $M_\delta \subset \Lambda$ and $\psi \in C^\infty(\mathbb{R}^+, [0, 1])$ a non-increasing function such that $\psi \equiv 1$ on $[0, \delta/2]$ and $\psi \equiv 0$ on $[\delta, \infty)$. For any $y \in M$, we define the function $\Psi_{i,\varepsilon,y} \in X_\varepsilon$ by setting

$$\Psi_{i,\varepsilon,y}(x) := \psi(|\varepsilon x - y|) w_i\left(\frac{\varepsilon x - y}{\varepsilon}\right), \quad i = 1, 2,$$

and denote by $t_\varepsilon > 0$ the unique positive number verifying

$$J_\varepsilon(t_\varepsilon(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y})) = \max_{t \geq 0} J_\varepsilon(t(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y})).$$

In view of the above remarks, it is well defined the function $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ given by

$$\Phi_\varepsilon(y) := t_\varepsilon(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y}).$$

In next lemma we prove an important relationship between Φ_ε and the set M.

Lemma 1.4.8. *Uniformly for $y \in M$, we have*

$$\lim_{\varepsilon \to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = c_0,$$

where c_0 was given in (1.4.1).

Proof. Suppose, by contradiction, that the lemma is false. Then there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0^+$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_0| \geq \delta_0 > 0. \tag{1.4.11}$$

We notice that, if $z \in B_{\delta/\varepsilon_n}(0)$ then $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$. Thus, recalling that $H \equiv Q$ in Λ and $\psi(s) = 0$ for $s \geq \delta$, we can use the change of variables $z \mapsto (\varepsilon_n x - y_n)/\varepsilon_n$ to write

$$J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = \frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^N} a(\varepsilon_n z + y_n)|\nabla(\psi(|\varepsilon_n z|)w_1(z))|^2 dz$$
$$+ \frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^N} |\psi(|\varepsilon_n z|)w_1(z)|^2 dz + \frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^N} b(\varepsilon_n z + y_n)|\nabla(\psi(|\varepsilon_n z|)w_2(z))|^2 dz$$
$$+ \frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^N} |\psi(|\varepsilon_n z|)w_2(z)|^2 dz - \int_{\mathbb{R}^N} Q(t_{\varepsilon_n}\psi(|\varepsilon_n z|)w_1(z), t_{\varepsilon_n}\psi(|\varepsilon_n z|)w_2(z))dz.$$

Since Q is homogeneous, we have that $t_{\varepsilon_n} \to 1$. This and Lebesgue's theorem imply that

$$\lim_{n\to\infty} \|(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})\|_{\varepsilon_n}^2 = \|(w_1, w_2)\|^2$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} Q(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})dz = \int_{\mathbb{R}^N} Q(w_1, w_2)dz.$$

Therefore

$$\lim_{n\to\infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_0(w_1, w_2) = c_0$$

which contradicts (1.4.11). The lemma is proved. □

Proposition 1.4.9. *Let $\varepsilon_n \to 0$ and $(u_n, v_n) \in \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n, v_n) \to c_0$. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $(w_n(x), z_n(x)) := (u_n(x + \widetilde{y}_n), v_n(x + \widetilde{y}_n))$ has a convergent subsequence in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n \to y \in M$, where $y_n = \varepsilon_n \widetilde{y}_n$.*

Proof. Since $a_0 \le a(x)$ and $b_0 \le b(x)$ for $x \in \mathbb{R}^N$ and $c_0 > 0$, we can repeat the same arguments in Lemma 1.4.1 to conclude that there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ and positive constants R and η such that

$$\liminf_{n\to\infty} \int_{B_R(\widetilde{y}_n)} \left(|u_n|^2 + |v_n|^2\right) dx \ge \eta.$$

Thus, since $((u_n, v_n))$ is bounded in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, considering $(w_n(x), z_n(x)) = (u_n(x+\widetilde{y}_n), v_n(x+\widetilde{y}_n))$, up to a subsequence, we have that $w_n \rightharpoonup w \not\equiv 0$ in $H^1(\mathbb{R}^N)$ and $z_n \rightharpoonup z \not\equiv 0$ in $H^1(\mathbb{R}^N)$. Let $t_n > 0$ be such that

$$(\widetilde{w}_n, \widetilde{z}_n) = t_n(w_n, z_n) \in \mathcal{M}_0. \tag{1.4.12}$$

Then,

$$c_0 \le I_0(\widetilde{w}_n, \widetilde{z}_n) \le J_{\varepsilon_n}(t_{\varepsilon_n}(u_n, v_n)) \le J_{\varepsilon_n}(u_n, v_n) = c_0 + o_n(1) \tag{1.4.13}$$

which implies

$$I_0(\widetilde{w}_n, \widetilde{z}_n) \to c_0 \text{ and } ((\widetilde{w}_n, \widetilde{z}_n)) \subset \mathcal{M}_0.$$

From boundedness of $((w_n, z_n))$ and (1.4.13), we get that (t_n) is bounded. As a consequence, the sequence $((\widetilde{w}_n, \widetilde{z}_n))$ is also bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, which implies, for some subsequence, $(\widetilde{w}_n, \widetilde{z}_n) \rightharpoonup (\widetilde{w}, \widetilde{z})$ weakly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Note that we can assume that $t_n \to t_0 > 0$. Then, this limit and $(Q_2) - (Q_3)$ imply that $(\widetilde{w}, \widetilde{z}) \not\equiv (0, 0)$. From Lemma 1.4.2, we conclude that $(\widetilde{w}_n, \widetilde{z}_n) \to (\widetilde{w}, \widetilde{z})$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ and, as a consequence, $(w_n, z_n) \to (w, z)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

To conclude the proof of the proposition, we consider $y_n = \varepsilon_n \widetilde{y}_n$. Our goal is to show that (y_n) has a subsequence, still denoted by (y_n), satisfying $y_n \to y$ for $y \in M$. First of all, we claim that (y_n) is bounded. Indeed, suppose that there exists a subsequence, still denoted by (y_n), verifying $|y_n| \to \infty$. Note that from (ab_1) we have

$$\begin{aligned} & a_0 \int_{\mathbb{R}^N} |\nabla w_n|^2 dx + b_0 \int_{\mathbb{R}^N} |\nabla z_n|^2 dx + \int_{\mathbb{R}^N} |w_n|^2 dx + \int_{\mathbb{R}^N} |z_n|^2 dx \\ \leq & \int_{\mathbb{R}^N} [w_n H_w(\varepsilon_n x + y_n, w_n, z_n) + z_n H_z(\varepsilon_n x + y_n, w_n, z_n)] dx. \end{aligned}$$

Fixing $R > 0$ such that $B_R(0) \supset \Lambda$, since $|\varepsilon_n x + y_n| \geq R$ and (H_3), we have

$$\begin{aligned} & a_0 \int_{\mathbb{R}^N} |\nabla w_n|^2 dx + b_0 \int_{\mathbb{R}^N} |\nabla z_n|^2 dx + \int_{\mathbb{R}^N} |w_n|^2 dx + \int_{\mathbb{R}^N} |z_n|^2 dx \\ \leq & \frac{1}{4} \int_{B_{R/\varepsilon_n}(0)} (w_n^2 + z_n^2) dx + o_n(1). \end{aligned}$$

It follows that $(w_n, z_n) \to (0,0)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, which is a contradiction because $c_0 > 0$.

Hence (y_n) is bounded and, up to a subsequence,

$$y_n \to \overline{y} \in \mathbb{R}^N.$$

Arguing as above, if $\overline{y} \notin \overline{\Lambda}$, we will obtain again $(w_n, z_n) \to (0,0)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, thus $\overline{y} \in \overline{\Lambda}$.

Now we are going to show that $\overline{y} \in M$. It is sufficient to show that $a(\overline{y}) = a_0$ and $b(\overline{y}) = b_0$. Supposing, by contradiction, that $a(\overline{y}) > a_0$ or $b(\overline{y}) > b_0$, we have

$$\begin{aligned} c_0 = I_0(\widetilde{w}, \widetilde{z}) < & \frac{1}{2} \int_{\mathbb{R}^N} a(\overline{y}) |\nabla \widetilde{w}|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} b(\overline{y}) |\nabla \widetilde{z}|^2 dx \\ + & \frac{1}{2} \int_{\mathbb{R}^N} \widetilde{w}^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \widetilde{z}^2 dx - \int_{\mathbb{R}^N} Q(\widetilde{w}, \widetilde{z}) dx. \end{aligned}$$

Using again the fact that $(\widetilde{w}_n, \widetilde{z}_n) \to (\widetilde{w}, \widetilde{z})$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, from Fatou's Lemma

$$\begin{aligned} c_0 & < \liminf_{n\to\infty} \left[\frac{1}{2} \int_{\mathbb{R}^N} a(\varepsilon_n x + y_n) |\nabla \widetilde{w}_n|^2) dx + \frac{1}{2} \int_{\mathbb{R}^N} b(\varepsilon_n x + y_n) |\nabla \widetilde{z}_n|^2 dx \right] \\ & + \liminf_{n\to\infty} \left[\frac{1}{2} \int_{\mathbb{R}^N} |\widetilde{w}_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |\widetilde{z}_n|^2 dx \right] \\ & - \lim_{n\to\infty} \left[\int_{\mathbb{R}^N} Q(\widetilde{w}_n, \widetilde{z}_n) dx \right], \end{aligned}$$

that is,

$$c_0 < \liminf_{n\to\infty} J_{\varepsilon_n}(t_n(u_n, v_n)) \leq \liminf_{n\to\infty} J_{\varepsilon_n}(u_n, v_n) = c_0,$$

obtaining a contradiction. Then, we conclude that $\overline{y} \in M$. □

Corollary 1.4.10. *Assume the same hypotheses of Proposition 1.4.9. Then, for any given $\gamma > 0$, there exists $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\int_{B_R(\widetilde{y}_n)^c} \left(|\nabla u_n|^2 + |u_n|^2\right) dx + \int_{B_R(\widetilde{y}_n)^c} \left(|\nabla v_n|^2 + |v_n|^2\right) dx < \gamma, \quad \textit{for all } n \geq n_0.$$

Proof. By using the same notation of the proof of Proposition 1.4.9, we have for any $R > 0$

$$\begin{aligned} &\int_{B_R(\widetilde{y}_n)^c} \left(|\nabla u_n|^2 + |u_n|^2\right) dx + \int_{B_R(\widetilde{y}_n)^c} \left(|\nabla v_n|^2 + |v_n|^2\right) dx \\ = &\int_{B_R(0)^c} \left(|\nabla w_n|^2 + |w_n|^2\right) dx + \int_{B_R(0)^c} \left(|\nabla z_n|^2 + |z_n|^2\right) dx. \end{aligned}$$

Since $((w_n, z_n))$ strongly converges in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, the result follows. □

Let us consider $\rho = \rho_\delta > 0$ in such way that $M_\delta \subset B_\rho(0)$ and define $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ by setting $\Upsilon(x) := x$ for $|x| < \rho$ and $\Upsilon(x) := \rho x/|x|$ for $|x| \geq \rho$. We also consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u, v) := \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)\left(|u(x)|^2 + |v(x)|^2\right) dx}{\displaystyle\int_{\mathbb{R}^N} \left(|u(x)|^2 + |v(x)|^2\right) dx}.$$

Since $M \subset B_\rho(0)$, the definition of Υ and Lebesgue's theorem imply that

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \text{uniformly for } y \in M. \tag{1.4.14}$$

Following [17], we introduce the set

$$\Sigma_\varepsilon := \left\{(u, v) \in \mathcal{N}_\varepsilon : J_\varepsilon(u, v) \leq c_0 + h(\varepsilon)\right\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Given $y \in M$, we can use Lemma 1.4.8 to conclude that $h(\varepsilon) = |J_\varepsilon(\Phi_\varepsilon(y)) - c_0|$ satisfies $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Thus, $\Phi_\varepsilon(y) \in \Sigma_\varepsilon$ and therefore $\Sigma_\varepsilon \neq \emptyset$, for any $\varepsilon > 0$ small.

Lemma 1.4.11. *For any $\delta > 0$ we have*

$$\lim_{\varepsilon\to 0^+} \sup_{(u,v)\in\Sigma_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u, v), M_\delta) = 0. \tag{1.4.15}$$

Proof. Let $(\varepsilon_n) \subset \mathbb{R}$ be such that $\varepsilon_n \to 0^+$. By definition, there exists $((u_n, v_n)) \subset \Sigma_{\varepsilon_n}$ such that

$$\operatorname{dist}(\beta_{\varepsilon_n}(u_n, v_n), M_\delta) = \sup_{(u,v)\in\Sigma_{\varepsilon_n}} \operatorname{dist}(\beta_{\varepsilon_n}(u, v), M_\delta) + o_n(1).$$

Thus, it suffices to find a sequence $(y_n) \subset M_\delta$ such that

$$|\beta_{\varepsilon_n}(u_n, v_n) - y_n| = o_n(1). \tag{1.4.16}$$

Thus, recalling that $(u_n, v_n) \in \Sigma_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we obtain

$$c_0 \leq \max_{t\geq 0} I_0(tu_n, tv_n) \leq \max_{t\geq 0} J_{\varepsilon_n}(tu_n, tv_n) = J_{\varepsilon_n}(u_n, v_n) \leq c_0 + h(\varepsilon_n), \tag{1.4.17}$$

from which follows that $J_{\varepsilon_n}(u_n, v_n) \to c_0$. Thus, we may invoke Proposition 1.4.9 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $(y_n) := (\varepsilon_n \widetilde{y}_n) \subset M_\delta$, for n large. Hence,

$$\begin{aligned}
\beta_{\varepsilon_n}(u_n,v_n) &= \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon_n x)\left(|u_n|^2+|v_n|^2\right)dx}{\displaystyle\int_{\mathbb{R}^N}\left(|u_n|^2+|v_n|^2\right)dx} \\
&= \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon_n z+y_n)\left(|u_n(z+\widetilde{y}_n)|^2+|v_n(z+\widetilde{y}_n)|^2\right)dz}{\displaystyle\int_{\mathbb{R}^N}\left(|u_n(z+\widetilde{y}_n)|^2+|v_n(z+\widetilde{y}_n)|^2\right)dz} \\
&= y_n + \frac{\displaystyle\int_{\mathbb{R}^N} \left(\Upsilon(\varepsilon_n z+y_n)-y_n\right)\left(|u_n(z+\widetilde{y}_n)|^2+|v_n(z+\widetilde{y}_n)|^2\right)dz}{\displaystyle\int_{\mathbb{R}^N}\left(|u_n(z+\widetilde{y}_n)|^2+|v_n(z+\widetilde{y}_n)|^2\right)dz}.
\end{aligned}$$

Since $\varepsilon_n z + y_n \to y_0 \in M$ and from strong convergence of $(u_n(\cdot+\widetilde{y}_n), v_n(\cdot+\widetilde{y}_n))$, we have that $\beta_{\varepsilon_n}(u_n,v_n) = y_n + o_n(1)$ and therefore the sequence (y_n) satisfies (1.4.16). The lemma is proved. □

We finalize the section presenting a relation between the topology of M and the number of solutions of the modified system $(S_{\varepsilon,aux})$, which is the proof of the item (ii) of Theorem 1.2.2.

Proof. Given $\delta > 0$ such that $M_\delta \subset \Lambda$, we can use (1.4.14), Lemma 1.4.8, (1.4.15) and argue as in [17, Section 6] to obtain $\widehat{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \widehat{\varepsilon}_\delta)$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \Sigma_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Thus,

$$\operatorname{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon) \geq \operatorname{cat}_{M_\delta}(M).$$

From Proposition 1.4.7 and Theorem 1.4.3 that J_ε possesses at least $\operatorname{cat}_{M_\delta}(M)$ critical points on $\mathcal{N}_\varepsilon$. The same argument employed in the proof of Proposition 1.4.7 shows that each of these critical points is also a critical point of the unconstrained functional J_ε. Thus, we obtain $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions for $(S_{\varepsilon,aux})$. □

1.5 Proof of Theorem 1

In this section we prove our main theorem. The idea is to show that the solutions obtained in Theorem 1.2.2 verify the following estimate $|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \ \forall x \in \mathbb{R}^N \setminus \Lambda_\varepsilon$ as ε is small enough. This fact implies that these solutions are in fact solutions of the original system $(\widehat{S}_\varepsilon)$. The key ingredient is the following result, whose proof uses an adaptation of the arguments found in [31], which are related with the Moser's iteration method [34] .

Lemma 1.5.1. *Let (ε_n) be a sequence such that $\varepsilon_n \to 0^+$ and for each $n \in \mathbb{N}$, let $(u_n, v_n) \in \Sigma_{\varepsilon_n}$ be a solution of system $(S_{\varepsilon_n,aux})$. Then $J_{\varepsilon_n}(u_n,v_n) \to c_0$ and $(u_n,v_n) \in L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$. Moreover, given $\xi > 0$, there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$|w_n|_{L^\infty(\mathbb{R}^N \setminus B_R(0))} < \xi, \quad \textit{for all} \quad n \geq n_0,$$

$$|z_n|_{L^\infty(\mathbb{R}^N \setminus B_R(0))} < \xi, \quad \textit{for all} \quad n \geq n_0,$$

where $w_n(x) = u_n(x+\widetilde{y}_n)$, $z_n(x) = v_n(x+\widetilde{y}_n)$ and $(\widetilde{y}_n)$ is the sequences of Proposition 1.4.9.

Proof. Since $J_{\varepsilon_n}(u_n, v_n) \leq c_0 + h(\varepsilon_n)$ with $\lim_{n\to\infty} h(\varepsilon_n) = 0$, we can argue as in (1.4.17) to conclude that $J_{\varepsilon_n}(u_n, v_n) \to c_0$. Thus, we may invoke Proposition 1.4.9 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ satisfying the conclusions of that Proposition.

Fix $R := R_1 > R_2 > ... > R_k > R_{k-1} > ... > R_0$ and consider $\eta_{R_k} \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \eta_{R_k} \leq 1$, $\eta_{R_k} \equiv 0$ in $B_{R/2}(0)$, $\eta_{R_k} \equiv 1$ in $B_R(0)^c$ and $|\nabla \eta_{R_k}| \leq C/R_0$. For each $n \in \mathbb{N}$ and $L > 0$, we define $\eta_n(x) := \eta_{R_k}(x - \widetilde{y}_n)$, $w_{L,n}, z_{L,n} \in X_\varepsilon$ by setting

$$w_{L,n}(x) := \min\{w_n(x), L\}, \qquad \Upsilon_{w,L,n} := \eta_n^2 w_{L,n}^{2(\beta-1)} w_n$$

and

$$z_{L,n}(x) := \min\{z_n(x), L\}, \qquad \Upsilon_{z,L,n} := \eta_n^2 z_{L,n}^{2(\beta-1)} z_n,$$

with $\beta > 1$ to be determined later.

By definition of $(\Upsilon_{w,L,n}, \Upsilon_{z,L,n})$, $J'_{\varepsilon_n}(w_n, z_n)(\Upsilon_{w,L,n}, \Upsilon_{z,L,n}) = 0$ and since

$$2a_0(\beta - 1) \int_{\mathbb{R}^N} \eta_n^2 w_n w_{L,n}^{2(\beta-1)-1} \nabla w_n \nabla w_{L,n} dx \geq 0$$

and

$$2b_0(\beta - 1) \int_{\mathbb{R}^N} \eta_n^2 z_n z_{L,n}^{2(\beta-1)-1} \nabla z_n \nabla z_{L,n} dx \geq 0,$$

we have that

$$\begin{aligned} & a_0 \int_{\mathbb{R}^N} \eta_n^2 w_{L,n}^{2(\beta-1)} |\nabla w_n|^2 dx + 2a_0 \int_{\mathbb{R}^N} \eta_n w_n w_{L,n}^{2(\beta-1)} \nabla \eta_n \cdot \nabla w_n dx \\ & + b_0 \int_{\mathbb{R}^N} \eta_n^2 z_{L,n}^{2(\beta-1)} |\nabla z_n|^2 dx + 2b_0 \int_{\mathbb{R}^N} \eta_n z_n z_{L,n}^{2(\beta-1)} \nabla \eta_n \cdot \nabla z_n dx \\ \leq & \int_{\mathbb{R}^N} H_w(\varepsilon_n x + y_n, w_n, z_n) \eta_n^2 w_n w_{L,n}^{2(\beta-1)} dx \\ & + \int_{\mathbb{R}^N} H_z(\varepsilon_n x + y_n, w_n, z_n) \eta_n^2 z_n z_{L,n}^{2(\beta-1)} dx. \end{aligned} \tag{1.5.1}$$

In view of (Q_1) and (H_4) we can obtain $C_1 > 0$ such that

$$H_s(x, s, t) + H_t(x, s, t) \leq \frac{1}{4}|s| + \frac{1}{4}|t| + C_1[|s|^{(2^*-1)} + |t|^{(2^*-1)}], \quad \text{for any } (x, s, t) \in \mathbb{R}^{N+2}.$$

Using the last inequality in (1.5.1), we obtain

$$\begin{aligned} & a_0 \int_{\mathbb{R}^N} \eta_n^2 w_{L,n}^{2(\beta-1)} |\nabla w_n|^2 dx + b_0 \int_{\mathbb{R}^N} \eta_n^2 z_{L,n}^{2(\beta-1)} |\nabla z_n|^2 dx \\ \leq \; & 2a_0 \int_{\mathbb{R}^N} \eta_n w_n w_{L,n}^{2(\beta-1)} \nabla \eta_n \cdot \nabla w_n dx + 2b_0 \int_{\mathbb{R}^N} \eta_n z_n z_{L,n}^{2(\beta-1)} \nabla \eta_n \cdot \nabla z_n dx \\ & + \int_{\mathbb{R}^N} \eta_n^2 w_n^{2^*} w_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^N} \eta_n^2 z_n^{2^*} z_{L,n}^{2(\beta-1)} dx. \end{aligned}$$

For any $\widetilde{\gamma} > 0$ we can use Young's inequality to obtain

$$\begin{aligned} & a_0 \int_{\mathbb{R}^N} \eta_n^2 w_{L,n}^{2(\beta-1)} |\nabla w_n|^2 dx + b_0 \int_{\mathbb{R}^N} \eta_n^2 z_{L,n}^{2(\beta-1)} |\nabla z_n|^2 dx \\ \leq \; & 2a_0 \int_{\mathbb{R}^N} [\widetilde{\gamma} \eta_n^2 |\nabla w_n|^2 + C_{\widetilde{\gamma}} |w_n|^2 |\nabla \eta_n|^2] w_{L,n}^{2(\beta-1)} dx \\ & + 2b_0 \int_{\mathbb{R}^N} [\widetilde{\gamma} \eta_n^2 |\nabla z_n|^2 + C_{\widetilde{\gamma}} |z_n|^2 |\nabla \eta_n|^2] z_{L,n}^{2(\beta-1)} dx \\ & + \int_{\mathbb{R}^N} \eta_n^2 w_n^{2^*} w_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^N} \eta_n^2 z_n^{2^*} z_{L,n}^{2(\beta-1)} dx. \end{aligned}$$

By choosing $\widetilde{\gamma} \leq 1/4$ we get, there exists $C_2 > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \eta_n^2 w_{L,n}^{2(\beta-1)} |\nabla w_n|^2 dx + \int_{\mathbb{R}^N} \eta_n^2 z_{L,n}^{2(\beta-1)} |\nabla z_n|^2 dx \\
\leq C_2 \Big(&\int_{\mathbb{R}^N} |w_n|^2 |\nabla \eta_n|^2 w_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^N} |z_n|^2 |\nabla \eta_n|^2 z_{L,n}^{2(\beta-1)} dx \\
&+ \int_{\mathbb{R}^N} \eta_n^2 w_n^{2^*} w_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^N} \eta_n^2 z_n^{2^*} z_{L,n}^{2(\beta-1)} dx \Big).
\end{aligned} \tag{1.5.2}$$

Let S be the best constant of the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and define $\widehat{w}_{L,n} := \eta_n w_n w_{L,n}^{\beta-1}$ and $\widehat{z}_{L,n} := \eta_n z_n z_{L,n}^{\beta-1}$. Since $w_{L,n} \leq w_n$ and $z_{L,n} \leq z_n$, we have that

$$\begin{aligned}
S^{-1}\left[\|\widehat{w}_{L,n}\|_{L^{2^*}}^2 + \|\widehat{z}_{L,n}\|_{L^{2^*}}^2\right] &\leq \int_{\mathbb{R}^N} \left|\nabla \left(\eta_n w_n w_{L,n}^{\beta-1}\right)\right|^2 dx + \int_{\mathbb{R}^N} \left|\nabla \left(\eta_n z_n z_{L,n}^{\beta-1}\right)\right|^2 dx \\
&\leq 2 \int_{\mathbb{R}^N} |w_n|^2 w_{L,n}^{2(\beta-1)} |\nabla \eta_n|^2 dx + 2 \int_{\mathbb{R}^N} |z_n|^2 z_{L,n}^{2(\beta-1)} |\nabla \eta_n|^2 dx \\
&\quad + 2\beta^2 \int_{\mathbb{R}^N} \eta_n^2 w_{L,n}^{2(\beta-1)} |\nabla w_n|^2 dx + 2\beta^2 \int_{\mathbb{R}^N} \eta_n^2 z_{L,n}^{2(\beta-1)} |\nabla z_n|^2 dx.
\end{aligned}$$

The last inequality and (1.5.2) provide

$$\begin{aligned}
S^{-1}\left[\|\widehat{w}_{L,n}\|_{L^{2^*}}^2 + \|\widehat{z}_{L,n}\|_{L^{2^*}}^2\right] &\leq C_4 \beta^2 \Big(\int_{\mathbb{R}^N} |w_n|^2 w_{L,n}^{2(\beta-1)} |\nabla \eta_n|^2 dx \\
+ \int_{\mathbb{R}^N} |z_n|^2 z_{L,n}^{2(\beta-1)} |\nabla \eta_n|^2 dx &+ \int_{\mathbb{R}^N} \eta_n^2 |w_n|^{2^*} w_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^N} \eta_n^2 |z_n|^{2^*} z_{L,n}^{2(\beta-1)} dx \Big),
\end{aligned} \tag{1.5.3}$$

for all $\beta > 1$.

The above expression, the properties of η_n and $w_{L,n} \leq |w_n|$, $z_{L,n} \leq |z_n|$, imply that

$$\begin{aligned}
&S^{-1}\left[\|\widehat{w}_{L,n}\|_{L^{2^*}}^2 + \|\widehat{z}_{L,n}\|_{L^{2^*}}^2\right] \\
\leq\ & C_4 \beta^2 \int_{B_{R/2}(\widetilde{y}_n)^c} \left(|w_n|^{2\beta} |\nabla \eta_n|^2 + |w_n|^{2^*-2} |w_n|^{2\beta} \right) dx \\
&+ C_4 \beta^2 \int_{B_{R/2}(\widetilde{y}_n)^c} \left(|z_n|^{2\beta} |\nabla \eta_n|^2 + |z_n|^{2^*-2} |z_n|^{2\beta} \right) dx.
\end{aligned} \tag{1.5.4}$$

If we now set

$$t := \frac{2^* 2^*}{2(2^*-2)} > 1, \qquad \zeta := \frac{2t}{t-1} < 2^*, \tag{1.5.5}$$

we can apply Hölder's inequality with exponents $t/(t-1)$ and t in (1.5.4), to get

$$\begin{aligned}
S^{-1}\left[\|\widehat{w}_{L,n}\|_{L^{2^*}}^2 + \|\widehat{z}_{L,n}\|_{L^{2^*}}^2\right] &\leq C_4 \beta^2 \|w_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta} \left(\int_{B_{R/2}(\widetilde{y}_n)^c} |\nabla \eta_n|^{2t} dx \right)^{1/t} \\
&+ C_4 \beta^2 \|z_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta} \left(\int_{B_{R/2}(\widetilde{y}_n)^c} |\nabla \eta_n|^{2t} dx \right)^{1/t} \\
&+ C_4 \beta^2 \|w_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta} \left(\int_{B_{R/2}(\widetilde{y}_n)^c} |w_n|^{2^*(2^*/2)} dx \right)^{1/t} \\
&+ C_4 \beta^2 \|z_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta} \left(\int_{B_{R/2}(\widetilde{y}_n)^c} |z_n|^{2^*(2^*/2)} dx \right)^{1/t}.
\end{aligned} \tag{1.5.6}$$

Since η_n is constant on $B_{R/2}(\widetilde{y}_n) \cup B_R(\widetilde{y}_n)^c$ and $|\nabla \eta_n| \leq C/R_0$, we have that

$$\int_{B_{R/2}(\widetilde{y}_n)^c} |\nabla \eta_n|^{2t} dx = \int_{R/2 \leq |x-\widetilde{y}_n| \leq R} |\nabla \eta_n|^{2t} dx \leq \frac{C_5}{R_0^{2t-N}} \leq C_5, \tag{1.5.7}$$

where we have used, without of generality, that $R_0 > 1$ and $2t = \frac{2^*}{2}N > N$ in the last inequality.

Claim. There exists $n_0 \in \mathbb{N}$ and $K > 0$ such that , for any $n \geq n_0$, there holds

$$\int_{B_{R/2}(\widetilde{y}_n)^c} |w_n|^{2^*(2^*/2)} dx \leq K$$

and

$$\int_{B_{R/2}(\widetilde{y}_n)^c} |z_n|^{2^*(2^*/2)} dx \leq K.$$

Assuming the claim, we can use (1.5.6) and (1.5.7) to conclude that

$$S^{-1}\left[\|\widehat{w}_{L,n}\|_{L^{2^*}}^2 + \|\widehat{z}_{L,n}\|_{L^{2^*}}^2\right] \leq C_6\beta^2\|w_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta} + C_6\beta^2\|z_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta}.$$

Since

$$\begin{aligned}
\|w_{L,n}\|_{L^{\beta 2^*}(B_R(\widetilde{y}_n)^c)}^{2\beta} &= \left(\int_{B_R(\widetilde{y}_n)^c} w_{L,n}^{\beta 2^*} dx\right)^{2/2^*} \\
&\leq \left(\int_{\mathbb{R}^N} \eta_n^{2^*}|w_n|^{2^*} w_{L,n}^{2^*(\beta-1)} dx\right)^{2/2^*} \\
&= \|\widehat{w}_{L,n}\|_{L^{2^*}}^2 \leq C_6\beta^2\|w_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta}
\end{aligned}$$

and

$$\begin{aligned}
\|z_{L,n}\|_{L^{\beta 2^*}(B_R(\widetilde{y}_n)^c)}^{2\beta} &= \left(\int_{B_R(\widetilde{y}_n)^c} z_{L,n}^{\beta 2^*} dx\right)^{2/2^*} \\
&\leq \left(\int_{\mathbb{R}^N} \eta_n^{2^*}|z_n|^{2^*} z_{L,n}^{2^*(\beta-1)} dx\right)^{2/2^*} \\
&= \|\widehat{z}_{L,n}\|_{L^{2^*}}^2 \leq C_6\beta^2\|z_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta},
\end{aligned}$$

we can apply Fatou's lemma in the variable L to obtain

$$\begin{aligned}
\|w_n\|_{L^{\beta 2^*}(B_R(\widetilde{y}_n)^c)} + \|z_n\|_{L^{\beta 2^*}(B_R(\widetilde{y}_n)^c)} &\leq C_7^{1/\beta}\beta^{1/\beta}\|w_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)} \\
&\quad + C_7^{1/\beta}\beta^{1/\beta}\|z_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)},
\end{aligned}$$

whenever $w_n^{\beta\zeta}, z_n^{\beta\zeta} \in L^1(B_{R/2}(\widetilde{y}_n)^c)$.

We now set $\beta := 2^*/\zeta > 1$ and note that, since $w_n, z_n \in L^{2^*}(\mathbb{R}^N)$, the above inequality holds for this choice of β. Moreover, since $\beta^2\zeta = \beta 2^*$, it follows that the inequality also holds with β replaced by β^2.

Hence,

$$\|(w_n, z_n)\|_{L^{\beta^2 2^*}(B_R(\widetilde{y}_n)^c)} \leq C_7^{1/\beta^2}\beta^{2/\beta^2}\|(w_n, z_n)\|_{L^{\beta^2\zeta}(B_{R/2}(\widetilde{y}_n)^c)}.$$

By iterating this process and recalling that $\beta\zeta = 2^*$ we obtain, for $k \in \mathbb{N}$,

$$\|(w_n, z_n)\|_{L^{\beta^k 2^*}(B_R(\widetilde{y}_n)^c)} \leq C_7^{\sum_{i=1}^k \beta^{-i}}\beta^{\sum_{i=1}^k i\beta^{-i}}\|(w_n, z_n)\|_{L^{2^*}(B_{R/2}(\widetilde{y}_n)^c)}.$$

Since $\beta > 1$ we can take the limit as $k \to \infty$ to get

$$\|(w_n, z_n)\|_{L^\infty(B_R(\widetilde{y}_n)^c)} \le C_8 \|(w_n, z_n)\|_{L^{2^*}(B_{R/2}(\widetilde{y}_n)^c)}.$$

By using the change of variables $z \mapsto x - \widetilde{y}_n$ we obtain

$$\begin{aligned} \|(w_n, z_n)\|_{L^\infty(B_R(\widetilde{y}_n)^c)} \le\ & C_8 \Big(\int_{B_{R/2}(0)^c} |u_n(z + \widetilde{y}_n)|^{2^*} \mathrm{d}z \Big)^{\frac{1}{2^*}} \\ & + C_8 \Big(\int_{B_{R/2}(0)^c} |v_n(z + \widetilde{y}_n)|^{2^*} \mathrm{d}z \Big)^{\frac{1}{2^*}}, \end{aligned}$$

where $(w_n(x), z_n(x)) = (u_n(x + \widetilde{y}_n), v_n(x + \widetilde{y}_n))$. By Proposition 1.4.9 we have that (w_n, z_n) strongly converges in $L^{2^*}(\mathbb{R}^N) \times L^{2^*}(\mathbb{R}^N)$. Thus, for $R > 0$ sufficiently large, there holds

$$\|(w_n, z_n)\|_{L^\infty(B_R(\widetilde{y}_n)^c)} < \gamma,$$

for large n, which prove this lemma.

It remains to prove the claim. Of course, it is sufficient to prove that the first integral is finite. For that purpose we consider a new cut-off function given by $\widetilde{\eta}_n(x) := \eta_n(2x)$, in such way that $\widetilde{\eta}_n \equiv 0$ on $B_{R/4}(\widetilde{y}_n)$ and $\widetilde{\eta}_n \equiv 1$ on $B_{R/2}(\widetilde{y}_n)^c$. If $\widetilde{w}_{L,n} := \widetilde{\eta}_n |w_n| w_{L,n}^{\beta-1}$, we can proceed as before to prove the following version of (1.5.3)

$$\|\widetilde{w}_{L,n}\|_{L^{2^*}}^2 \le C_9 \beta^2 \left(\int_{\mathbb{R}^N} |w_n|^2 w_{L,n}^{2(\beta-1)} |\nabla \widetilde{\eta}_n|^2 dx + \int_{\mathbb{R}^N} \widetilde{\eta}_n^2 |w_n|^{2^*} w_{L,n}^{2(\beta-1)} dx \right), \tag{1.5.8}$$

We set $\beta := 2^*/2$ to obtain

$$\|\widetilde{w}_{L,n}\|_{L^{2^*}}^2 \le C_{10} \left(\int_{\mathbb{R}^N} |w_n|^2 w_{L,n}^{(2^*-2)} |\nabla \widetilde{\eta}_n|^2 dx + \int_{B_{R/4}(\widetilde{y}_n)^c} \widetilde{\eta}_n^2 |w_n|^2 w_{L,n}^{(2^*-2)} |w_n|^{(2^*-2)} dx \right).$$

By Hölder's inequality with exponents $2^*/2$ and $2^*/(2^*-2)$, we get

$$\begin{aligned} \|\widetilde{w}_{L,n}\|_{L^{2^*}}^2 \le\ & C_{10} \int_{\mathbb{R}^N} |w_n|^2 w_{L,n}^{(2^*-2)} |\nabla \widetilde{\eta}_n|^2 dx \\ & + C_{10} \left(\int_{B_{R/4}(\widetilde{y}_n)^c} \left(\widetilde{\eta}_n |w_n| w_{L,n}^{(2^*-2)/2} \right)^{2^*} dx \right)^{2/2^*} \|w_n\|_{L^{2^*}(B_{R/4}(\widetilde{y}_n)^c)}^{2^*-2}. \end{aligned}$$

From Proposition 1.4.9 we obtain $n_0 \in \mathbb{N}$ and $R > 1$ such that

$$\int_{B_{R/4}(\widetilde{y}_n)^c} |w_n|^{2^*} dx \le \left(\frac{1}{2C_{10}} \right)^{2^*/(2^*-2)},$$

for all $n \ge n_0$. Then

$$\begin{aligned} \|\widetilde{w}_{L,n}\|_{L^{2^*}}^2 \le\ & C_{10} \int_{\mathbb{R}^N} |w_n|^2 w_{L,n}^{(2^*-2)} |\nabla \widetilde{\eta}_n|^2 dx \\ & + \frac{1}{2} \left(\int_{B_{R/4}(\widetilde{y}_n)^c} \left(\widetilde{\eta}_n |w_n| w_{L,n}^{(2^*-2)/2} \right)^{2^*} dx \right)^{2/2^*}. \end{aligned}$$

Thus, recalling that $\widetilde{\eta}_n |w_n| w_{L,n}^{(2^*-2)/2} = \widetilde{w}_{L,n}$, $w_{L,n} \le |w_n|$ and $\nabla \eta_n$ is bounded, we obtain

$$\|\widetilde{w}_{L,n}\|_{L^{2^*}}^2 \le C_{12}.$$

The definition of $\widetilde{\eta}_n$ and the above inequality imply that

$$\int_{B_{R/2}(\widetilde{y}_n)^c} (|w_n| w_{L,n}^{\beta-1})^{2^*} dx \leq C_{12}^{2^*/2},$$

for all $n \geq n_0$. Using Fatou's lemma in the variable L, we have

$$\int_{B_{R/2}(\widetilde{y}_n)^c} |w_n|^{2^*(2^*/2)} dx \leq K := C_{12}^{2^*/2},$$

for all $n \geq n_0$, and therefore the claim holds. □

We are now ready to prove the main result of this chapter.
Proof of Theorem 1. Suppose that $\delta > 0$ is such that $M_\delta \subset \Lambda$. We first claim that there exists $\widetilde{\varepsilon}_\delta > 0$ such that, for any $0 < \varepsilon < \widetilde{\varepsilon}_\delta$ and any solution $(u_\varepsilon, v_\varepsilon) \in \Sigma_\varepsilon$ of the system $(S_{\varepsilon,aux})$, there holds

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \ \text{ for each } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon. \tag{1.5.9}$$

In order to prove the claim we argue by contradiction. So, suppose that for some sequence $\varepsilon_n \to 0^+$ we can obtain $(u_n, v_n) \in \Sigma_{\varepsilon_n}$ such that $J'_{\varepsilon_n}(u_n, v_n) = 0$ and

$$\|(u_n, v_n)\|_{L^\infty(\mathbb{R}^N \setminus \Lambda_{\varepsilon_n})} > \alpha. \tag{1.5.10}$$

As in Lemma 1.5.1, we have that $J_{\varepsilon_n}(u_n, v_n) \to c_0$ and therefore we can use Proposition 1.4.9 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $\varepsilon_n \widetilde{y}_n \to y_0 \in M$.

If we take $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Lambda$ we have that

$$B_{r/\varepsilon_n}(y_0/\varepsilon_n) = \frac{1}{\varepsilon_n} B_r(y_0) \subset \Lambda_{\varepsilon_n}.$$

Moreover, for any $z \in B_{r/\varepsilon_n}(\widetilde{y}_n)$, there holds

$$\left| z - \frac{y_0}{\varepsilon_n} \right| \leq |z - \widetilde{y}_n| + \left| \widetilde{y}_n - \frac{y_0}{\varepsilon_n} \right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n},$$

for n large. For this values of n we have that $B_{r/\varepsilon_n}(\widetilde{y}_n) \subset \Lambda_{\varepsilon_n}$ or, equivalently, $\mathbb{R}^N \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^N \setminus B_{r/\varepsilon_n}(\widetilde{y}_n)$. On the other hand, it follows from Lemma 1.5.1 with $\xi = \alpha$ that, for any $n \geq n_0$ such that $r/\varepsilon_n > R$, there holds

$$\|u_n\|_{L^\infty(\mathbb{R}^N \setminus \Lambda_{\varepsilon_n})} \leq \|u_n\|_{L^\infty(\mathbb{R}^N \setminus B_{r/\varepsilon_n}(\widetilde{y}_n))} \leq \|u_n\|_{L^\infty(\mathbb{R}^N \setminus B_R(\widetilde{y}_n))} < \alpha$$

and

$$\|v_n\|_{L^\infty(\mathbb{R}^N \setminus \Lambda_{\varepsilon_n})} \leq \|v_n\|_{L^\infty(\mathbb{R}^N \setminus B_{r/\varepsilon_n}(\widetilde{y}_n))} \leq \|v_n\|_{L^\infty(\mathbb{R}^N \setminus B_R(\widetilde{y}_n))} < \alpha,$$

which contradicts (1.5.10) and proves the claim.

Considering $0 < \varepsilon_\delta < \widetilde{\varepsilon}_\delta$, we shall prove the main theorem for this choice of ε_δ. Let $0 < \varepsilon < \varepsilon_\delta$ be fixed. By applying Theorem 1.2.2, we obtain $\text{cat}_{M_\delta}(M)$ nontrivial solutions of the system $(S_{\varepsilon,aux})$. If $(u, v) \in X_\varepsilon$ is one of these solutions we have that $(u, v) \in \Sigma_\varepsilon$, and therefore we can use (1.5.9) and the definition of H to conclude that $H(\cdot, u, v) \equiv Q(u, v)$. Hence, (u, v) is also a solution of the system $(\widehat{S}_\varepsilon)$. An easy calculation shows that $(\widehat{u}(x), \widehat{v}(x)) := (u(x/\varepsilon), v(x/\varepsilon))$ is a solution of the original system (S_ε). Then, (S_ε) has at least $\text{cat}_{M_\delta}(M)$ nontrivial solutions.

We now consider $\varepsilon_n \to 0^+$ and take a sequence $(u_n, v_n) \in X_{\varepsilon_n}$ of solutions of the system $(\widehat{S}_{\varepsilon_n})$ as above. By applying Lemma 1.5.1 we obtain $R > 0$ and $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that

$$\|u_n\|_{L^\infty(B_R(\widetilde{y}_n))^c} < \gamma \tag{1.5.11}$$

and

$$\|v_n\|_{L^\infty(B_R(\widetilde{y}_n))^c} < \gamma. \tag{1.5.12}$$

Up to a subsequence, we may also assume that

$$\|u_n\|_{L^\infty(B_R(\widetilde{y}_n))} \geq \gamma. \tag{1.5.13}$$

and

$$\|v_n\|_{L^\infty(B_R(\widetilde{y}_n))} \geq \gamma. \tag{1.5.14}$$

Indeed, if this is not the case, we have $\|u_n\|_{L^\infty(\mathbb{R}^N)} < \gamma$ or $\|v_n\|_{L^\infty(\mathbb{R}^N)} < \gamma$ which is a contradiction with (1.4.4). Thus (1.5.13) and (1.5.14) hold.

By using (1.5.13) and (1.5.14) we conclude that the maximum point $\pi_{n,a} \in \mathbb{R}^N$ of u_n and the maximum point $\pi_{n,b} \in \mathbb{R}^N$ of v_n belong to $B_R(\widetilde{y}_n)$. Hence $\pi_{n,a} = \widetilde{y}_n + q_{n,a}$, for some $q_{n,a} \in B_R(0)$ and $\pi_{n,b} = \widetilde{y}_n + q_{n,b}$, for some $q_{n,b} \in B_R(0)$. Recalling that the associated solution of (S_{ε_n}) is of the form $(\widehat{u}_n(x), \widehat{v}_n(x)) = (u_n(x/\varepsilon_n), v_n(x/\varepsilon_n))$, we conclude that the maximum point $\Pi_{\varepsilon_n,a}$ of $\widehat{u}_n$ and the maximum point $\Pi_{\varepsilon_n,b}$ of $\widehat{v}_n$ are $\Pi_{\varepsilon_n,a} := \varepsilon_n\widetilde{y}_n + \varepsilon_n q_{n,a}$ and $\Pi_{\varepsilon_n,b} := \varepsilon_n\widetilde{y}_n + \varepsilon_n q_{n,b}$. Since $(q_{n,a}), (q_{n,b}) \subset B_R(0)$ are bounded and $\varepsilon_n\widetilde{y}_n \to y_0 \in M$ (according to Proposition 1.4.9), we obtain

$$\lim_{n\to\infty} a(\Pi_{\varepsilon_n,a}) = a(y_0) = a_0$$

and

$$\lim_{n\to\infty} b(\Pi_{\varepsilon_n,b}) = b(y_0) = b_0.$$

Now we prove the regularity of the solution. Note that from Lema 1.5.1, (1.5.13) and (1.5.14), we have that $u_\varepsilon, v_\varepsilon \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. From interpolation inequality, we get $(u_\varepsilon, v_\varepsilon) \in L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$, $\forall\, q \geq 2$, that implies $Q_u(u_\varepsilon, v_\varepsilon), Q_v(u_\varepsilon, v_\varepsilon) \in L^q(\mathbb{R}^N)$, $\forall\, q \geq 2$. From regularity elliptic theory, we get $(u_\varepsilon, v_\varepsilon) \in W^{2,q}(\mathbb{R}^N) \times W^{2,q}(\mathbb{R}^N)$, $\forall\, q \geq 2$. For q sufficiently large, we obtain $W^{2,q}(\mathbb{R}^N) \hookrightarrow C^{1,\lambda}(\mathbb{R}^N)$, for some $0 < \lambda < 1$. Then, $u_\varepsilon, v_\varepsilon \in C^{1,\lambda}(\mathbb{R}^N)$. Since $Q \in C^2(\mathbb{R}^N)$, we obtain that $u_\varepsilon, v_\varepsilon \in C^{2,\lambda}(\mathbb{R}^N)$, which concludes the proof of the theorem. □

Chapter 2

On multiplicity and concentration behavior of solutions for a critical system with equations in divergence form

2.1 Introduction

In this chapter we show existence, multiplicity and concentration of positive solutions for the following system given by

$$(S_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) + \frac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \mathrm{div}(b(x)\nabla v) + v = Q_v(u,v) + \frac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, $N \geq 3$, $2^* = \frac{2N}{N-2}$, a and b are positive continuous potentials, and Q and K are homogeneous function with K having critical growth.

The hypotheses on functions a and b are the following:

(ab_1) there are $a_0 > 0$ and $b_0 > 0$ such that

$$0 < a_0 \leq a(x)$$

and

$$0 < b_0 \leq b(x) \quad \text{for all} \quad x \in \mathbb{R}^N;$$

(ab_2) there exists a bounded domain $\Lambda \subset \mathbb{R}^N$ such that

$$a_0 = \inf_{x\in\Lambda} a(x) < \inf_{x\in\partial\Lambda} a(x)$$

and

$$b_0 = \inf_{x\in\Lambda} b(x) < \inf_{x\in\partial\Lambda} b(x).$$

Setting $\mathbb{R}^2_+ := [0,\infty) \times [0,\infty)$, for any given $q \geq 1$ we denote by $\mathcal{H}^q$ the collection of all functions $F \in C^2(\mathbb{R}^2_+, \mathbb{R})$ satisfying the following properties:

$(\mathcal{H}^q_0)$ F is q-homogeneous; that is

$$F(\lambda s, \lambda t) = \lambda^q F(s,t), \quad \text{for each} \quad \lambda > 0 \quad \text{and} \quad (s,t) \in \mathbb{R}^2_+;$$

$(\mathcal{H}_1^q)$ there exists $c_1 > 0$ such that

$$|F_s(s,t)| + |F_t(s,t))| \leq c_1 \left(s^{q-1} + t^{q-1}\right) \quad \text{for each } (s,t) \in \mathbb{R}_+^2;$$

$(\mathcal{H}_2)$ $F(s,t) > 0$ for each $s, t > 0$;

$(\mathcal{H}_3)$ $\nabla F(1,0) = \nabla F(0,1) = (0,0)$;

$(\mathcal{H}_4)$ $F_s(s,t), F_t(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}_+^2$.

The hypotheses on the functions Q and K are the following:

(A_1) $K \in \mathcal{H}^{2^*}$ and $Q \in \mathcal{H}^p$ for some $2 < p < 2^*$;

(A_2) the 1-homogeneous function $G : \mathbb{R}_+^2 \to \mathbb{R}$ given by $G(s^{2^*}, t^{2^*}) := K(s,t)$ is concave;

(A_3)

$$Q(s,t) \geq \frac{\sigma}{p_1} s^\lambda t^\beta, \quad \text{for all} \quad (s,t) \in \mathbb{R}_+^2,$$

where $\lambda, \beta > 1$, $\lambda + \beta =: p_1 \in (2, 2^*)$ and

$$\sigma > \sigma^* := \left(\frac{C(a_0, b_0)}{\frac{1}{N}\left(\min\{a_0, b_0\}\widetilde{S}_K\right)^{N/2}} \right)^{\frac{p_1-2}{2}}.$$

The hypothesis (A_2) appeared in the first time in [19] and will be used in Proposition 2.3.1. The constant that define σ^* also will appear naturally in Proposition 2.3.1

For each $\varepsilon > 0$, a pair $(u_\varepsilon, v_\varepsilon) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ is a positive solution of system (S_ε) if $u_\varepsilon > 0$ and $v_\varepsilon > 0$ a.e. in $\mathbb{R}^N$ and

$$\begin{aligned} &\varepsilon^2 \int_{\mathbb{R}^N} a(x)\nabla u_\varepsilon \nabla \phi dx + \varepsilon^2 \int_{\mathbb{R}^N} b(x)\nabla v_\varepsilon \nabla \psi dx + \int_{\mathbb{R}^N} u_\varepsilon \phi dx + \int_{\mathbb{R}^N} v_\varepsilon \psi dx \\ = \; &\int_{\mathbb{R}^N} [\phi Q_u(u_\varepsilon, v_\varepsilon) + \psi Q_v(u_\varepsilon, v_\varepsilon)]dx + \frac{1}{2^*}\int_{\mathbb{R}^N} [\phi K_u(u_\varepsilon, v_\varepsilon) + \psi K_v(u_\varepsilon, v_\varepsilon)]dx, \end{aligned}$$

for all $(\phi, \psi) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

A solution (u, v) of system (S_ε) is said to be ground state if

$$I(u,v) = \inf\left\{ I(w,z) : \; (w,z) \text{ is a solution of } \; (S_\varepsilon) \right\},$$

where $I : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ is the functional associated to (S_ε).

In this chapter we also relate the number of solutions of (S_ε) with the topology of the set of minima of the potentials a and b. In order to present our result we introduce the following set:

$$M = \{x \in \mathbb{R}^N : a(x) = a_0 \;\text{ and }\; b(x) = b_0\}.$$

Our main result is as follows:

Theorem 2. *Suppose that a and b are continuous potentials satisfying $(ab_1) - (ab_2)$ and $M \neq \emptyset$. Suppose also that Q and K satisfy $(A_1) - (A_3)$. Then,*

(i) there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ the system (S_ε) has a positive ground state solution.

(*ii*) *for any* $\delta > 0$ *verifying*

$$M_\delta := \{x \in \mathbb{R}^N : \text{dist}(x, M) \le \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_\delta)$, *the system* (S_ε) *has at least* $\text{cat}_{M_\delta}(M)$ *positive solutions.*

(*iii*) *if* $(u_\varepsilon, v_\varepsilon)$ *is a solution for* (S_ε) *and if* $\Pi_{\varepsilon,a}$ *and* $\Pi_{\varepsilon,b}$ *are maximum points of* u_ε *and* v_ε *respectively, then* $\Pi_{\varepsilon,a}, \Pi_{\varepsilon,b} \in \Lambda$, $\lim\limits_{\varepsilon\to 0^+} a(\Pi_{\varepsilon,a}) = a_0$ *and* $\lim\limits_{\varepsilon\to 0^+} b(\Pi_\varepsilon) = b_0$, *furthermore, each solution* $(u_\varepsilon, v_\varepsilon) \in C^{2,\lambda}(\mathbb{R}^N)$, *for some* $\lambda \in (0, 1)$.

We recall that, if Y is a closed set of a topological space X, $\text{cat}_X(Y)$ is the Ljusternik-Schnirelmann category of Y in X, namely the least number of closed and contractible set in X which cover Y.

Concerning the class of nonlinearities we are dealing, we have the following examples from [19]. Let $q \ge 1$ and

$$P_q(s,t) = \sum_{\alpha_i+\beta_i=q} a_i s^{\alpha_i} t^{\beta_i},$$

where $i \in \{1, \ldots, k\}$, $\alpha_i, \beta_i \ge 1$ and $a_i \in \mathbb{R}$. The following functions and their possible combinations, satisfy our hypothesis on Q

$$Q_1(s,t) = P_p(s,t), \quad Q_2(s,t) = \sqrt[r]{P_l(s,t)} \text{ and } Q_3(s,t) = \frac{P_{l_1}(s,t)}{P_{l_2}(s,t)},$$

with $r = pl$ and $l_1 - l_2 = p$, under appropriate choices of the coefficients a_i. Condition (A_2) restricts the expression of the critical function K. However, it can have the polynomial form $K(s,t) = P_{2^*}(s,t)$.

The chapter is organized as follows. In section 2.2 we present the variational framework and a modified system. In section 2.3 we give some information on the autonomous system. Existence of a ground state solution and multiplicity result for the modified system $(S_{\varepsilon,aux})$ involving Ljusternik-Scrinirelmann theory is section 2.4. In section 2.5 we prove that each solution of the modified system $(S_{\varepsilon,aux})$ is a solution of the original system. We also prove a concetration result.

2.2 Variational framework and a modified system

Since we are interested in positive solutions we extend the function Q and K to the whole $\mathbb{R}^2$ by setting $Q(u,v) = K(u,v) = 0$ if $u \le 0$ or $v \le 0$. We also note that for any function $F \in \mathcal{H}^q$, we can use the homogeneity condition $(\mathcal{H}_0^q)$ to conclude that

$$qF(s,t) = sF_s(s,t) + tF_t(s,t) \tag{2.2.1}$$

and

$$q(q-1)F(s,t) = s^2F_{ss}(s,t) + t^2F_{tt}(s,t) + 2stF_{st}(s,t) \tag{2.2.2}$$

for any $(s,t) \in \mathbb{R}^2$.

Hereafter, we will work with the following system equivalent to (S_ε).

$$(\widehat{S}_\varepsilon) \qquad \begin{cases} -\text{div}(a(\varepsilon x)\nabla u) + u = Q_u(u,v) + \frac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N, \\ -\text{div}(b(\varepsilon x)\nabla v) + v = Q_v(u,v) + \frac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In order to overcome the lack of compactness originated by the unboundedness of $\mathbb{R}^N$ we use a penalization method. Such kind of idea has first appeared in the paper of Del Pino and Felmer [20]. Here we use an adaptation of this method for systems, as introduced in [1].

We start by choosing $\alpha > 0$ and considering $\eta : \mathbb{R} \to \mathbb{R}$ a non-increasing function of class C^2 such that

$$\eta \equiv 1 \text{ on } (-\infty, \alpha], \quad \eta \equiv 0 \text{ on } [5\alpha, +\infty), \quad |\eta'(s)| \leq \frac{C}{\alpha} \quad \text{and} \quad |\eta''(s)| \leq \frac{C}{\alpha^2} \tag{2.2.3}$$

for each $s \in \mathbb{R}$ and for some positive constant $C > 0$. Using the function η, we define $\widehat{Q} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\widehat{Q}(s,t) := \eta(|(s,t)|)\left(Q(s,t) + \frac{1}{2^*}K(s,t)\right) + (1 - \eta(|(s,t)|))A(s^2 + t^2)$$

where

$$A := \max\left\{\frac{Q(s,t) + \frac{1}{2^*}K(s,t)}{s^2 + t^2} : (s,t) \in \mathbb{R}^2, \ \alpha \leq |(s,t)| \leq 5\alpha\right\}.$$

Notice that, since $A > 0$ tends to zero as $\alpha \to 0^+$, we may suppose that $A < 1$.

Finally, denoting by I_Λ the characteristic function of the set Λ, we define $H : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ by setting

$$H(x,s,t) := I_\Lambda(x)\left(Q(s,t) + \frac{1}{2^*}K(s,t)\right) + (1 - I_\Lambda(x))\widehat{Q}(s,t). \tag{2.2.4}$$

For any $\alpha > 0$ small and $(s,t) \in \mathbb{R}^2$ we have the following result.

Lemma 2.2.1. *The function H satisfies the following estimates:*

(H_1) $pH(x,s,t) \leq sH_s(x,s,t) + tH_t(x,s,t)$, for each $x \in \Lambda$;

(H_2) $2H(x,s,t) \leq sH_s(x,s,t) + tH_t(x,s,t)$, for each $x \in \mathbb{R}^N \setminus \Lambda$;

(H_3) for α small we have $sH_s(x,s,t) + tH_t(x,s,t) \leq \frac{1}{4}\left(s^2 + t^2\right)$ for each $x \in \mathbb{R}^N \setminus \Lambda$;

(H_4) for α small we have $\dfrac{|H_s(x,s,t)|}{\alpha}, \dfrac{|H_t(x,s,t)|}{\alpha} \leq \dfrac{1}{4}$ for each $x \in \mathbb{R}^N \setminus \Lambda$.

Proof. We note that arguing as in [4, Lemma 2.4], it is possible to prove the items $(H_1) - (H_3)$. Then, we prove (H_4). Since $H(x,u,v) = \widehat{Q}(u,v)$ for all $x \in \mathbb{R} \setminus \Lambda$, from definition of $\widehat{Q}$, we obtain

$$\begin{aligned} H_u(x,u,v) &= \frac{\eta'(|(u,v)|)u\left(Q(u,v) + \frac{1}{2^*}K(u,v)\right)}{\sqrt{u^2+v^2}} + \eta(|(u,v)|)\left(Q_u(u,v) + \frac{1}{2^*}K_u(u,v)\right) \\ &\quad - \frac{\eta'(|(u,v)|)uA(u^2+v^2)}{\sqrt{u^2+v^2}} + (1 - \eta(|(u,v)|))2uA \end{aligned}$$

Then, using $(\mathcal{H}_1^p)$, $(\mathcal{H}_1^{2^*})$ and (2.2.3) we have

$$\begin{aligned} |H_u(x,u,v)| &\leq |\eta'|\frac{Q(u,v) + \frac{1}{2^*}K(u,v)}{u^2+v^2}|(u,v)|^2 + 2|\eta|c_1\left(|(u,v)|^{p-1} + \frac{1}{2^*}|(u,v)|^{2^*-1}\right) \\ &\quad |\eta'|A|(u,v)|^2 + 4A|(u,v)| \\ &\leq \frac{C}{\alpha}A \cdot 25 \cdot \alpha^2 + 2c_1\left((5\alpha)^{p-1} + (5\alpha)^{2^*-1}\right) + \frac{C}{\alpha}A \cdot 25 \cdot \alpha^2 + 20 \cdot \alpha \cdot A \end{aligned}$$

Then, for α sufficiently small we have that

$$\frac{|H_u(x,u,v)|}{\alpha} \leq \frac{1}{4}.$$

Using similar arguments, it is possible to prove that

$$\frac{|H_v(x,u,v)|}{\alpha} \leq \frac{1}{4}.$$

□

From now on we assume that α is chosen in such way that the last inequality above holds. In view of definition (2.2.4), we deal in the sequel with the modified system

$$(S_{\varepsilon,aux}) \qquad \begin{cases} -\text{div}(a(\varepsilon x)\nabla u) + u = H_u(\varepsilon x, u, v) \text{ in } \mathbb{R}^N, \\ -\text{div}(b(\varepsilon x)\nabla v) + v = H_v(\varepsilon x, u, v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N) \end{cases}$$

and we will look for solutions $(u_\varepsilon, v_\varepsilon)$ verifying

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \ \text{ for each } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon,$$

where $\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$.

For each $\varepsilon > 0$ we denote by X_ε the Hilbert space

$$X_\varepsilon := \left\{ (u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|\nabla v|^2) dx < \infty \right\}$$

endowed with the norm

$$\|(u,v)\|_\varepsilon^2 := \int_{\mathbb{R}^N} (a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|\nabla v|^2 + |u|^2 + |v|^2) dx.$$

Conditions (H_3) and (A_1) imply that the critical points of the C^1-functional $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$J_\varepsilon(u,v) := \frac{1}{2} \int_{\mathbb{R}^N} \left(a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|\nabla v|^2 + |u|^2 + |v|^2 \right) dx - \int_{\mathbb{R}^N} H(\varepsilon x, u, v) dx$$

are weak solutions of $(S_{\varepsilon,aux})$. We recall that these critical points belong to the Nehari manifold of J_ε, namely on the set

$$\mathcal{N}_\varepsilon := \{(u,v) \in X_\varepsilon \setminus \{(0,0)\} : J'_\varepsilon(u,v)(u,v) = 0\}.$$

It is well known that, for any nontrivial element $(u,v) \in X_\varepsilon$ the function $t \mapsto J_\varepsilon(tu, tv)$, for $t \geq 0$, achieves its maximum value at a unique point $t_{u,v} > 0$ such that $t_{u,v}(u,v) \in \mathcal{N}_\varepsilon$. We define the number b_ε by setting

$$b_\varepsilon := \inf_{(u,v) \in \mathcal{N}_\varepsilon} J_\varepsilon(u,v). \tag{2.2.5}$$

We define the Palais-Smale compactness condition for the functional J_ε. A sequence $((u_n, v_n)) \subset X_\varepsilon$ is a Palais-Smale sequence at level c_ε for the functional J_ε if

$$J_\varepsilon(u_n, v_n) \to c_\varepsilon$$

and

$$\|J'_\varepsilon(u_n,v_n)\| \to 0 \text{ in } (X_\varepsilon)',$$

where

$$c_\varepsilon = \inf_{\eta\in\Gamma}\max_{t\in[0,1]} J_\varepsilon(\eta(t)) > 0$$

and

$$\Gamma := \{\eta \in C([0,1],X_\varepsilon) : \eta(0)=(0,0),\ J_\varepsilon(\eta(1))<0\}.$$

If every Palais-Smale sequence of J_ε has a strong convergent subsequence, then one says that J_ε satisfies the Palais-Smale condition.

2.3 On the autonomous problem

In order to prove the multiplicity result for the system $(\widehat{S}_\varepsilon)$, we consider the autonomous system (S_0), namely

$$(S_0) \qquad \begin{cases} -a_0\Delta u + u = Q_u(u,v) + \frac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N, \\ -b_0\Delta v + v = Q_v(u,v) + \frac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N, \\ u,v \in H^1(\mathbb{R}^N), \quad u(x),v(x)>0 \text{ for each } x\in\mathbb{R}^N. \end{cases}$$

In view of conditions (ab_1), $(\mathcal{H}_1^p)$ and $(\mathcal{H}_1^{2^*})$, the above system has a variational structure and the associated functional is given by

$$\begin{aligned} I_0(u,v) \ &:= \ \frac{1}{2}\int_{\mathbb{R}^N}\left(a_0|\nabla u|^2 + b_0|\nabla v|^2 + |u|^2 + |v|^2\right)dx - \int_{\mathbb{R}^N} Q(u,v)dx \\ &\quad -\frac{1}{2^*}\int_{\mathbb{R}^N} K(u,v)dx, \end{aligned}$$

is well defined for $(u,v)\in E_0 := H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. We denote the norm in E_0 by

$$\|(u,v)\|^2 = a_0\int_{\mathbb{R}^N}|\nabla u|^2dx + b_0\int_{\mathbb{R}^N}|\nabla v|^2dx + \int_{\mathbb{R}^N}|u|^2dx + \int_{\mathbb{R}^N}|v|^2dx.$$

Standard calculations show that I_0 has the Mountain Pass geometry and therefore we can set the minimax level c_0 in the following way

$$c_0 := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_0(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1],E_0) : \gamma(0)=(0,0),\ I_0(\gamma(1))\le 0\}$. Moreover, c_0 can be further characterized as

$$c_0 = \inf_{(u,v)\in\mathcal{M}_0} I_0(u,v), \tag{2.3.1}$$

with $\mathcal{M}_0$ being the Nehari manifold of I_0, that is

$$\mathcal{M}_0 := \{(u,v)\in E_0\setminus\{(0,0)\} : I_0'(u,v)(u,v)=0\}.$$

As usual, we denote by S the best constant of the embedding $D^{1,2}(\mathbb{R}^N)\hookrightarrow L^{2^*}(\mathbb{R}^N)$. To state the next result we need to define $\widetilde{S}_K$ the best constant of the immerson $D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)\hookrightarrow L^{2^*}(\mathbb{R}^N)\times L^{2^*}(\mathbb{R}^N)$, that is

$$\widetilde{S}_K := \inf_{\substack{u,v\in D^{1,2}(\mathbb{R}^N)\\ u,v\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx}{\left(\displaystyle\int_{\mathbb{R}^N}K(u,v)dx\right)^{2/2^*}}.$$

Proposition 2.3.1. *There exists $\sigma^* > 0$ such that for all $\sigma > \sigma^*$*

$$c_0 < \frac{1}{N}\left(\min\{a_0, b_0\}\widetilde{S}_K\right)^{N/2}.$$

Proof. By using $(\mathcal{H}_0^p)$ and $(\mathcal{H}_0^{2^*})$, and arguing as [40, Theorem 4.2], it is possible to prove that

$$c_0 = \inf_{(u,v)\in E_0\setminus\{(0,0)\}} \max_{t\geq 0} I_0(tu, tv) > 0.$$

Thus, it suffices to obtain $(u, v) \in E_0$ such that

$$\max_{t\geq 0} I_0(tu, tv) < \frac{1}{N}\left(\min\{a_0, b_0\}\widetilde{S}_K\right)^{N/2}.$$

We first recall that, for any $\delta > 0$ the function

$$w_\delta(x) := [\delta N(N-2)]^{(N-2)/4}(\delta + |x|^2)^{(2-N)/2}$$

satisfies

$$\int_{\mathbb{R}^N} |\nabla w_\delta|^2 dx = \int_{\mathbb{R}^N} |w_\delta|^{2^*} dx = S^{N/2}.$$

By [19, Lemma 3], there exist $A, B \in \mathbb{R}$ such that $\widetilde{S}_K$ is attained by

$$\widetilde{S}_K = \frac{\int_{\mathbb{R}^N}\left(|\nabla(Aw_\delta)|^2 + |\nabla(Bw_\delta)|^2\right)dx}{\left(\int_{\mathbb{R}^N} K(Aw_\delta, Bw_\delta)dx\right)^{2/2^*}} = \frac{S^{N/2}(A^2 + B^2)}{\left(\int_{\mathbb{R}^N} K(Aw_\delta, Bw_\delta)dx\right)^{2/2^*}}.$$

Let $\eta \in C_0^\infty(\mathbb{R}^N, [0, 1])$ be such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$. Consider

$$\psi_\delta(x) := \frac{\eta(x)w_\delta(x)}{|\eta w_\delta|_{2^*}}.$$

By using the definition of ψ_δ, (A_3) and $(\mathcal{H}_0^{2^*})$ we get

$$\begin{aligned} I_0(tA\psi_\delta, tB\psi_\delta) \leq\ & \frac{t^2}{2}D_\delta(A^2 + B^2) - \frac{\sigma}{p_1}t^{p_1}A^\lambda B^\beta \int_{B_2(0)} |\psi_\delta|^{p_1} dx \\ & -\frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N} K(A\psi_\delta, B\psi_\delta)dx \end{aligned}$$

where $p_1 \in (2, 2^*)$ is given by condition (A_3) and

$$D_\delta = \int_{\mathbb{R}^N} \max\{a_0, b_0, 1\}\left(|\nabla\psi_\delta|^2 + |\psi_\delta|^2\right) dx.$$

Thus

$$\max_{t\geq 0}\left\{\frac{t^2}{2}D_\delta(A^2 + B^2) - \frac{\sigma}{p_1}t^{p_1}A^\lambda B^\beta \int_{B_2(0)} |\psi_\delta|^{p_1} dx - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N} K(A\psi_\delta, B\psi_\delta)dx\right\}$$
$$\geq I_0(tA\psi_\delta, tB\psi_\delta).$$

Straightforward calculations show that

$$\begin{aligned} I_0(tA\psi_\delta, tB\psi_\delta) &\leq \frac{1}{\sigma^{2/(p_1-2)}}\left(\frac{1}{2} - \frac{1}{p_1}\right)\frac{(D_\delta(A^2 + B^2))^{p_1/(p_1-2)}}{\left(A^\lambda B^\beta \int_{B_2(0)} |\psi_\delta|^{p_1} dx\right)^{2/(p_1-2)}} \\ &= \frac{1}{\sigma^{2/(p_1-2)}}C(a_0, b_0). \end{aligned}$$

Thus, $\max_{t\geq 0} I_0(tA\psi_\delta, tB\psi_\delta) < \frac{1}{N}\left(\min\{a_0,b_0\}\widetilde{S}_K\right)^{N/2}$, for all $\sigma > \sigma^*$ where

$$\sigma^* := \left(\frac{C(a_0,b_0)}{\frac{1}{N}\left(\min\{a_0,b_0\}\widetilde{S}_K\right)^{N/2}}\right)^{\frac{p_1-2}{2}}.$$

The proof is finished. □

The proof of the next result is in the same spirit of [4, Lemma 2.2]. We omit the details.

Lemma 2.3.2. *Let $((u_n,v_n)) \subset \mathcal{M}_0$ be a sequence such that $I_0(u_n,v_n) \to c_0$. Then there are a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R, \eta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2)dx \geq \eta. \tag{2.3.2}$$

Now we are ready to show that system (S_0) has a solution that reaches c_0.

Lemma 2.3.3. *(A Compactness Lemma) Let $((u_n,v_n)) \subset \mathcal{M}_0$ be a sequence satisfying $I_0(u_n,v_n) \to c_0$. Then, there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $(w_n(x), z_n(x)) = (u_n(x+\widetilde{y}_n), v_n(x+\widetilde{y}_n))$ converges strongly in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. In particular, there exists a minimizer for c_0.*

Proof. Applying Ekeland's Variational Principle [40, Theorem 8.5], we may suppose that $((u_n,v_n))$ is a $(PS)_{c_0}$ for I_0. Since $((u_n,v_n))$ is bounded in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, we have that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$. From weak convergence, we obtain

$$\|(u,v)\| \leq \liminf_{n\to\infty} \|(u_n,v_n)\|.$$

We are going to prove that

$$\|(u,v)\| = \lim_{n\to\infty} \|(u_n,v_n)\|. \tag{2.3.3}$$

Suppose, by contradiction, that (2.3.3) does not hold. Then, by $(\mathcal{H}_3)$, we can consider $(u,v) \neq (0,0)$, using a density argument we have that $I_0'(u,v)(u,v) = 0$, where we conclude that $(u,v) \in \mathcal{M}_0$ and

$$\begin{aligned} c_0 &\leq I_0(u,v) - \frac{1}{p}I_0'(u,v)(u,v) \\ &< \liminf_{n\to+\infty}\left[\left(\frac{1}{2}-\frac{1}{p}\right)\|(u_n,v_n)\|^2 + \left(\frac{1}{p}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N} K(u_n,v_n)dx\right] \\ &= \liminf_{n\to+\infty}\left[I_0(u_n,v_n) - \frac{1}{p}I_0'(u_n,v_n)(u_n,v_n)\right] = c_0, \end{aligned}$$

which is a contradiction. Hence, $(u_n,v_n) \to (u,v)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, consequently, $I_0(u,v) = c_0$ and the sequence $(\widetilde{y}_n)$ is the sequence null.

If $(u,v) \equiv (0,0)$, then in this case we cannot have $(u_n,v_n) \to (u,v)$ strongly in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ because $c_0 > 0$. Hence, using the Lemma 2.3.2, there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that

$$(w_n, z_n) \rightharpoonup (w,z) \ \ in \ \ H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N),$$

where $w_n = u_n(\cdot + \widetilde{y}_n)$ and $z_n = v_n(\cdot+\widetilde{y}_n)$. Therefore, $((w_n,z_n))$ is also a $(PS)_{c_0}$ sequence of I_0 and $(w,z) \not\equiv (0,0)$. It follows form above arguments that, up to a subsequence, (w_n,z_n) converges strongly in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ and the proof of lemma is over. □

2.4 Existence of a ground state and multiple solutions for the modified system $(S_{\varepsilon,aux})$

In this section we show existence of a ground state and multiple solutions for the modified system $(S_{\varepsilon,aux})$. The main result in this section is:

Theorem 2.4.1. *Suppose that a and b are continuous potentials satisfying $(ab_1)-(ab_2)$ and $M \neq \emptyset$. Suppose also $(A_1)-(A_3)$. Then,*

(i) there exists $\varepsilon_1 > 0$ such that, for any $\varepsilon \in (0,\varepsilon_1)$ the system $(S_{\varepsilon,aux})$ has a positive ground state solution.

(ii) for any $\delta > 0$ verifying

$$M_\delta := \{x \in \mathbb{R}^N : \operatorname{dist}(x,M) \le \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0,\varepsilon_\delta)$, the system $(S_{\varepsilon,aux})$ has at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions.

In order to show existence of a ground state solution for the modified system $(S_{\varepsilon,aux})$, we use the Mountain Pass Theorem [6].

Lemma 2.4.2. *The functional J_ε satisfies the following conditions*

(i) there is $C, \rho > 0$, such that

$$J_\varepsilon(u,v) \ge C, \quad \text{if} \ \ \|(u,v)\|_\varepsilon = \rho.$$

(ii) for any $(\phi,\psi) \in C_0^\infty(\Lambda_\varepsilon) \times C_0^\infty(\Lambda_\varepsilon)$ with $\phi,\psi \ge 0$, we have

$$\lim_{t\to\infty} J_\varepsilon(t\phi,t\psi) = -\infty.$$

Proof. Using (H_1), (H_2), (H_3), $(\mathcal{H}_1^p)$ and $(\mathcal{H}_1^{2^*})$, we have

$$\begin{aligned} J_\varepsilon(u,v) \ \ge \ & \frac{1}{2}\|(u,v)\|_\varepsilon^2 - \frac{2c_1}{p}\int_{\Lambda_\varepsilon}(|u|^p+|v|^p)dx - \frac{2c_1}{2^*p}\int_{\Lambda_\varepsilon}(|u|^{2^*}+|v|^{2^*})dx \\ & -\frac{1}{8}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(|u|^2+|v|^2)dx. \end{aligned}$$

By Sobolev embeddings, there exists $C > 0$ such that

$$J_\varepsilon(u,v) \ge \frac{3}{8}\|(u,v)\|_\varepsilon^2 - \frac{C}{p}\|(u,v)\|_\varepsilon^p - \frac{C}{2^*p}\|(u,v)\|_\varepsilon^{2^*}$$

and the proof of item (i) is over. Now, by definition of H, $(\mathcal{H}_0^p)$ and $(\mathcal{H}_0^{2^*})$, we get

$$J_\varepsilon(t\phi,t\psi) = \frac{t^2}{2}\|(\phi,\psi)\|_\varepsilon^2 - t^p\int_{\Lambda_\varepsilon}Q(\phi,\psi)dx - \frac{t^{2^*}}{2^*}\int_{\Lambda_\varepsilon}K(\phi,\psi)dx$$

and the proof of item (ii) is over. □

Hence, there exists a Palais-Smale sequence $((u_n,v_n)) \subset X_\varepsilon$ at level c_ε. Using $(\mathcal{H}_0^p)$ and $(\mathcal{H}_0^{2^*})$, it is possible to prove that

$$c_\varepsilon = b_\varepsilon = \inf_{(u,v)\in X_\varepsilon\setminus\{(0,0)\}} \sup_{t\ge0} J_\varepsilon(tu,tv), \tag{2.4.1}$$

where b_ε was defined in (2.2.5)

In order to prove the Palais-Smale condition, we need to prove the next lemma.

Lemma 2.4.3. *Let $((u_n,v_n))$ be a $(PS)_d$ sequence for J_ε. Then for each $\xi>0$, there exists $R=R(\xi)>0$ such that*

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)}\Big[a(\varepsilon x)|\nabla u_n|^2+b(\varepsilon x)|\nabla v_n|^2+|u_n|^2+|v_n|^2\Big]dx<\xi.$$

Proof. Straight forward calculations show that $((u_n,v_n))$ is bounded in X_ε. Let $\eta_R\in C^\infty(\mathbb{R}^N)$ such that $\eta_R(x)=0$ if $x\in B_{R/2}(0)$ and $\eta_R(x)=1$ if $x\notin B_R(0)$, with $0\le\eta_R(x)\le 1$ and $|\nabla\eta_R|\le\dfrac{C}{R}$, where C is a constant independent of R. Since that the sequence $((u_n\eta_R,v_n\eta_R))$ is bounded in X_ε, fixing $R>0$ such that $\Lambda_\varepsilon\subset B_{R/2}(0)$ and by definition of the functional J_ε, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus B_R(0)}\Big[a(\varepsilon x)|\nabla u_n|^2+b(\varepsilon x)|\nabla v_n|^2+|u_n|^2+|v_n|^2\Big]dx\\ \le\ & J_\varepsilon'(u_n,v_n)(u_n\eta_R,v_n\eta_R)+\int_{\mathbb{R}^N}[u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)]\eta_R dx\\ -\ &\int_{\mathbb{R}^N}[a(\varepsilon x)u_n\nabla u_n+b(\varepsilon x)v_n\nabla v_n]\nabla\eta_R dx.\end{aligned}$$

Using (H_3), we get the estimate

$$\begin{aligned}&\frac{3}{4}\int_{\mathbb{R}^N\setminus B_R(0)}\Big[a(\varepsilon x)|\nabla u_n|^2+b(\varepsilon x)|\nabla v_n|^2+|u_n|^2+|v_n|^2\Big]dx\\ \le\ &\int_{\mathbb{R}^N}[a(\varepsilon x)|u_n||\nabla u_n|+b(\varepsilon x)|v_n||\nabla v_n|]|\nabla\eta_R|dx+o_n(1).\end{aligned}$$

Since $((u_n,v_n))$ is bounded in X_ε and $|\nabla\eta_R|\le\dfrac{C}{R}$, we get

$$\int_{\mathbb{R}^N\setminus B_R(0)}\Big[a(\varepsilon x)|\nabla u_n|^2+b(\varepsilon x)|\nabla v_n|^2+|u_n|^2+|v_n|^2\Big]dx\le\frac{C_1}{R}+o_n(1).$$

proving the lemma. □

Lemma 2.4.4. *Any sequence $((u_n,v_n))\subset X_\varepsilon$ such that*

$$J_\varepsilon(u_n,v_n)\to c<\frac{1}{N}\Big(\min\{a_0,b_0\}\widetilde{S}_K\Big)^{N/2}\quad\textit{and}\quad J_\varepsilon'(u_n,v_n)\to 0$$

possesses a convergent subsequence.

Proof. Standart calculations show that $((u_n,v_n))$ is bounded in X_ε. Then, up to a subsequence, we may suppose that

$$\begin{aligned}&(u_n,v_n)\rightharpoonup(u,v)\ \text{ weakly in }\ X_\varepsilon,\\ &u_n\to u,v_n\to v\ \text{ strongly in }\ L^s_{loc}(\mathbb{R}^N),\ \text{ for any }\ 2\le s<2^*,\\ &u_n(x)\to u(x),v_n(x)\to v(x)\ \text{ for a.e. }\ x\in\mathbb{R}^N.\end{aligned}\tag{2.4.2}$$

Now using a density argument, we can conclude that (u,v) is a critical point of J_ε. Hence

$$\|(u,v)\|_\varepsilon^2=\int_{\mathbb{R}^N}[uH_u(\varepsilon x,u,v)+vH_v(\varepsilon x,u,v)]dx.\tag{2.4.3}$$

On the other hand, we have

$$\|(u_n,v_n)\|_\varepsilon^2=\int_{\mathbb{R}^N}[u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)]dx+o_n(1).\tag{2.4.4}$$

Claim 1. $\lim_{n\to\infty}\int_{\Lambda_\varepsilon} K(u_n,v_n)dx = \int_{\Lambda_\varepsilon} K(u,v)dx.$

Since $((u_n,v_n))$ is bounded, we may suppose that

$$|\nabla u_n|^2 \rightharpoonup \mu,\quad |\nabla v_n|^2 \rightharpoonup \sigma \quad\text{and}\quad K(u_n,v_n)\rightharpoonup \nu \quad \text{(weak*-sense of measures)}.$$

From [19, Lemma 6], we obtain an at most countable index set Γ, sequences $(x_i)\in\mathbb{R}^N$, (μ_i), (σ_i), $(\nu_i)\subset(0,\infty)$ such that

$$\begin{aligned}\mu \geq |\nabla u|^2 + \sum_{i\in\Gamma}\mu_i\delta_{x_i},\quad \sigma\geq|\nabla v|^2+\sum_{i\in\Gamma}\sigma_i\delta_{x_i}\\ \nu = K(u,v)+\sum_{i\in\Gamma}\nu_i\delta_{x_i} \quad\text{and}\quad \widetilde{S}_K\nu_i^{2/2^*}\leq\mu_i+\sigma_i\end{aligned}\tag{2.4.5}$$

for all $i\in\Gamma$, where δ_{x_i} is the Dirac mass at the point $x_i\in\mathbb{R}^N$.

Suppose that $\{x_i\}_{i\in\Gamma}\cap\Lambda_\varepsilon\neq\emptyset$, then exists $x_i\in\Lambda_\varepsilon$ for some $i\in\Gamma$. Define, for $\varrho>0$, the function $\psi_\varrho(x):=\psi((x-x_i)/\varrho)$ where $\psi\in C_0^\infty(\mathbb{R}^N,[0,1])$ is such that $\psi\equiv1$ on $B_1(0)$, $\psi\equiv0$ on $\mathbb{R}^N\setminus B_2(0)$ and $|\nabla\psi|_\infty\leq2$. We suppose that ϱ is chosen in such a way that the support of ψ_ϱ is contained in Λ_ε. Since $((\psi_\varrho u_n,\psi_\varrho v_n))$ is bounded, $J'_\varepsilon(u_n,v_n)(\psi_\varrho u_n,\psi_\varrho v_n)=o_n(1)$. Then

$$\begin{aligned}&\int_{\mathbb{R}^N}[a(\varepsilon x)\psi_\varrho|\nabla u_n|^2+b(\varepsilon x)\psi_\varrho|\nabla v_n|^2]dx\\ &+\int_{\mathbb{R}^N}[a(\varepsilon x)u_n\nabla u_n\nabla\psi_\varrho+b(\varepsilon x)v_n\nabla v_n\nabla\psi_\varrho]dx+\int_{\mathbb{R}^N}[\psi_\varrho u_n^2+\psi_\varrho v_n^2]dx\\ &=\int_{\mathbb{R}^N}[u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)]\psi_\varrho dx+o_n(1).\end{aligned}$$

Since $\text{supp}(\psi_\varrho)\subset\Lambda_\varepsilon$, we can use definition of H, (2.2.1) and (ab_1) to get

$$\begin{aligned}&\min\{a_0,b_0\}\int_{\mathbb{R}^N}[\psi_\varrho|\nabla u_n|^2+\psi_\varrho|\nabla v_n|^2]dx\\ \leq\ &-\int_{\mathbb{R}^N}[a(\varepsilon x)u_n\nabla u_n\nabla\psi_\varrho+b(\varepsilon x)v_n\nabla v_n\nabla\psi_\varrho]dx\\ &+p\int_{\mathbb{R}^N}Q(u_n,v_n)\psi_\varrho dx+\int_{\mathbb{R}^N}K(u_n,v_n)\psi_\varrho dx+o_n(1).\end{aligned}$$

Since Q has subcritical growth and ψ_ϱ has compact support, we can let $n\to\infty$, $\varrho\to0$ and use (2.4.5) to conclude that

$$\min\{a_0,b_0\}(\mu_i+\sigma_i)\leq\nu_i$$

As $\widetilde{S}_K\nu_i^{2/2^*}\leq\mu_i+\sigma_i$, we get

$$\nu_i\geq\left(\min\{a_0,b_0\}\widetilde{S}_K\right)^{N/2}.$$

By using Lemma 2.2.1, $p>2$ and (2.2.1) we get

$$\begin{aligned}c\ &=\ J_\varepsilon(u_n,v_n)-\frac{1}{2}J'_\varepsilon(u_n,v_n)(u_n,v_n)+o_n(1)\\ &=\ \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}\left(\frac{1}{2}[u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)]-H(\varepsilon x,u_n,v_n)\right)dx\\ &+\ \int_{\Lambda_\varepsilon}\left(\frac{1}{2}[u_nQ_u(u_n,v_n)+v_nQ_v(u_n,v_n)]-Q(u_n,v_n)\right)dx\\ &+\ \frac{1}{2^*}\int_{\Lambda_\varepsilon}\left(\frac{1}{2}[u_nK_u(u_n,v_n)+v_nK_v(u_n,v_n)]-K(u_n,v_n)\right)dx+o_n(1)\\ &\geq\ \frac{1}{N}\int_{\Lambda_\varepsilon}K(u_n,v_n)dx+o_n(1)\geq\frac{1}{N}\int_{\Lambda_\varepsilon}\psi_\varrho K(u_n,v_n)dx+o_n(1).\end{aligned}$$

By taking the limit and using (2.4.5) we get

$$c \geq \frac{1}{N} \sum_{\{i\in\Gamma: x_i\in\Lambda_\varepsilon\}} \psi_\varrho(x_i)\nu_i = \frac{1}{N} \sum_{\{i\in\Gamma: x_i\in\Lambda_\varepsilon\}} \nu_i \geq \frac{1}{N}\left(\min\{a_0, b_0\}\widetilde{S}_K\right)^{N/2}$$

which does not make sense. Therefore $\{x_i\}_{i\in\Gamma} \cap \Lambda_\varepsilon = \emptyset$, this conclude the proof of the claim 1.

Claim 2.

$$\int_{\mathbb{R}^N} [u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)]dx \to \int_{\mathbb{R}^N} [u H_u(\varepsilon x, u, v) + v H_v(\varepsilon x, u, v)]dx.$$

From Lemma 2.4.3, for any $\xi > 0$ given, there exists $R > 0$ such that $\Lambda_\varepsilon \subset B_R(0)$ and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} \left[a(\varepsilon x)|\nabla u_n|^2 + b(\varepsilon x)|\nabla v_n|^2 + |u_n|^2 + |v_n|^2\right] dx < \xi.$$

This inequality, (H_3) and the Sobolev embeddings imply that, for n large enough, there holds

$$\int_{\mathbb{R}^N\setminus B_R(0)} [u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)]dx \leq C_1 \frac{1}{4}\xi, \tag{2.4.6}$$

where C_1 is positive constant. On the other hand, taking R large enough, we can suppose that

$$\left|\int_{\mathbb{R}^N\setminus B_R(0)} [u H_u(\varepsilon x, u, v) + v H_v(\varepsilon x, u, v)]dx\right| < \xi. \tag{2.4.7}$$

Then, by (2.4.6) and (2.4.7), we can conclude

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus B_R(0)} [u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)]dx \\ = &\int_{\mathbb{R}^N\setminus B_R(0)} [u H_u(\varepsilon x, u, v) + v H_v(\varepsilon x, u, v)]dx + o_n(1).\end{aligned} \tag{2.4.8}$$

On the other hand, since the set $B_R(0) \cap (\mathbb{R}^N \setminus \Lambda_\varepsilon)$ is bounded, we can use (H_3), (2.4.2) and Lebesgue's theorem to conclude that

$$\begin{aligned}&\lim_{n\to\infty} \int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)} [u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)]dx \\ = &\int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)} [u H_u(\varepsilon x, u, v) + v H_v(\varepsilon x, u, v)]dx.\end{aligned} \tag{2.4.9}$$

By using Claim 1, $(\mathcal{H}_1^p)$, (2.4.2) and Lebesgue's theorem again, we obtain

$$\begin{aligned}&\lim_{n\to\infty} \int_{\Lambda_\varepsilon} [u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)]dx \\ = &\int_{\Lambda_\varepsilon} [u H_u(\varepsilon x, u, v) + v H_v(\varepsilon x, u, v)]dx.\end{aligned} \tag{2.4.10}$$

From (2.4.8), (2.4.9) and (2.4.10) the claim 2 is proved.

By using (2.4.3), claim 2 and (2.4.4), we have $\|(u_n, v_n)\|_\varepsilon^2 \to \|(u, v)\|_\varepsilon^2$. Then $(u_n, v_n) \to (u, v)$ in X_ε. □

The multiplicity result for system $(S_{\varepsilon,aux})$ is rather long and will be done by applying the following Ljusternik-Schnirelmann abstract result. The proof of this result can be found in [27, Corollary 4.17].

Theorem 2.4.5. *Let I be a C^1-functional defined on a C^1-Finsler manifold $\mathcal{V}$. If I is bounded from below and satisfies the Palais-Smale condition, then I has at least $\mathrm{cat}_{\mathcal{V}}(\mathcal{V})$ distinct critical points.*

The following result, which has a proof similar to that presented in [10, Lemma 4.3], will be used.

Lemma 2.4.6. *Let Γ, Ω^+, Ω^- be closed sets with $\Omega^- \subset \Omega^+$. Let $\beta : \Gamma \to \Omega^+$, $\Phi : \Omega^- \to \Gamma$ be two continuous maps such that $\beta \circ \Phi$ is homotopically equivalent to the embedding $\iota : \Omega^- \to \Omega^+$. Then $\mathrm{cat}_\Gamma(\Gamma) \geq \mathrm{cat}_{\Omega^+}(\Omega^-)$.*

2.4.1 The Palais-Smale condition in the Nehari manifold associated to J_ε

Since we are intending to apply critical point theory we need to introduce some compactness property. So, let V be a Banach space, $\mathcal{V}$ be a C^1-manifold of V and $I : V \to \mathbb{R}$ a C^1-functional. We say that $I|_{\mathcal{V}}$ satisfies the Palais-Smale condition at level c ($(\mathrm{PS})_c$ for short) if any sequence $(u_n) \subset \mathcal{V}$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_* \to 0$ contains a convergent subsequence. Here, we are denoting by $\|I'(u)\|_*$ the norm of the derivative of I restricted to $\mathcal{V}$ at the point u.

From Lemma 2.4.4, the unconstrained functional satisfies $(\mathrm{PS})_c$ for $c < \frac{1}{N}(\min\{a_0, b_0\}\widetilde{S}_K)^{N/2}$. Nevertheless, to get multiple critical points, we need to work with the functional J_ε constrained to $\mathcal{N}_\varepsilon$. In order to prove the desired compactness result we shall first present some properties of $\mathcal{N}_\varepsilon$, which the proofs of the next three results follows by using the same arguments employed in [3, Lemma 2.2, Lemma 2.3 and Proposition 2.4] for other class of system. For the sake of completeness, we sketch here.

Lemma 2.4.7. *There exist positive constants α_1, δ_1, C such that, for each $\alpha \in (0, \alpha_1)$, $(u,v) \in \mathcal{N}_\varepsilon$, there hold*

$$\int_{\Lambda_\varepsilon} [pQ(u,v) + K(u,v)]dx \geq \delta_1 \tag{2.4.11}$$

and

$$\int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (u^2 + v^2)dx \leq C \int_{\Lambda_\varepsilon} [pQ(u,v) + K(u,v)]dx. \tag{2.4.12}$$

Proof. From the growth conditions on Q and K, it is easy to obtain $\widehat{\delta} > 0$ such that

$$\|(u,v)\|_\varepsilon \geq \widehat{\delta} \ \text{ for each } (u,v) \in \mathcal{N}_\varepsilon.$$

Thus, we can use (2.2.1) and (H_3) to get

$$\begin{aligned} \widehat{\delta}^2 \leq \|(u,v)\|_\varepsilon^2 &= \int_{\Lambda_\varepsilon} \left[uQu + vQ_v + \frac{1}{2^*}[uK_u + vK_v] \right] dx \\ &\quad + \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} [uH_u + vH_v]dx \\ &\leq \int_{\Lambda_\varepsilon} [pQ(u,v) + K(u,v)]dx + \frac{1}{4} \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (u^2 + v^2)dx \end{aligned}$$

and therefore

$$\frac{3\widehat{\delta}^2}{4} \leq \frac{3}{4}\|(u,v)\|_\varepsilon^2 \leq \int_{\Lambda_\varepsilon} [pQ(u,v) + K(u,v)]dx,$$

which implies (2.4.11) with $\delta_1 = \frac{3\widehat{\delta}^2}{4}$.

By using (H_3) and (2.2.1) again, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(u^2+v^2)dx &\leq \|(u,v)\|_\varepsilon^2 \\ &\leq \int_{\Lambda_\varepsilon}[pQ(u,v)+K(u,v)]dx + \frac{1}{4}\int_{\mathbb{R}\setminus\Lambda_\varepsilon}(u^2+v^2)dx,\end{aligned}$$

from which follows (2.4.12). The lemma is proved. □

The following technical results is the key stone in our compactness result.

Lemma 2.4.8. *Let $\phi_\varepsilon : X_\varepsilon \to \mathbb{R}$ be given by*

$$\phi_\varepsilon(u,v) := \|(u,v)\|_\varepsilon^2 - \int_{\mathbb{R}^N}[uH_u(\varepsilon x,u,v)+vH_v(\varepsilon x,u,v)]dx.$$

Then there exist $\alpha_2, \widetilde{M} > 0$ such that, for each $\alpha \in (0,\alpha_2)$,

$$\phi_\varepsilon'(u,v)(u,v) \leq -\widetilde{M} < 0 \quad \text{for each } (u,v) \in \mathcal{N}_\varepsilon. \tag{2.4.13}$$

Proof. Given $(u,v) \in \mathcal{N}_\varepsilon$, we can use the definition of H, (2.2.1) and (2.2.2) to get

$$\begin{aligned}\phi_\varepsilon'(u,v)(u,v) &= \int_{\Lambda_\varepsilon}[uQ_u+vQ_v]dx - \int_{\Lambda_\varepsilon}[u^2Q_{uu}+v^2Q_{vv}+2uvQ_{uv}]dx \\ &+\frac{1}{2^*}\int_{\Lambda_\varepsilon}[uK_u+vK_v]dx - \frac{1}{2^*}\int_{\Lambda_\varepsilon}[u^2K_{uu}+v^2K_{vv}+2uvK_{uv}] \\ &+\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}[uH_u+vH_v]dx - \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}[u^2H_{uu}+v^2H_{vv}+2uvH_{uv}]dx \\ &= -p(p-2)\int_{\Lambda_\varepsilon}Q(u,v)dx - (2^*-2)\int_{\Lambda_\varepsilon}K(u,v)dx + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}[D_1-D_2]dx\end{aligned}$$

with

$$D_1 := uH_u + vH_v \quad \text{and} \quad D_2 := u^2H_{uu}+v^2H_{vv}+2uvH_{uv}.$$

Since $p < 2^*$, we get

$$\phi_\varepsilon'(u,v)(u,v) \leq -(p-2)\int_{\Lambda_\varepsilon}[pQ(u,v)+K(u,v)]dx + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}[D_1-D_2]dx.$$

Arguing as in the proof of [3, Lemma 2.3], we have

$$\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}[D_1-D_2]dx \leq o(1)\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(u^2+v^2)dx$$

where $o(1) \to 0$ as $\alpha \to 0^+$.

Now we can use Lemma 2.4.7 to obtain, for α small enough

$$\phi_\varepsilon'(u,v)(u,v) \leq (-(p-2)+o(1))\int_{\Lambda_\varepsilon}[pQ(u,v)+K(u,v)]dx \leq -\frac{(p-2)}{2}\delta_1 = -\widetilde{M} < 0$$

and the proof is over. □

Proposition 2.4.9. *The functional J_ε restricted to $\mathcal{N}_\varepsilon$ satisfies $(\mathrm{PS})_c$ at any level $c < \frac{1}{N}\left(\min\{a_0,b_0\}\widetilde{S}_K\right)^{N/2}$.*

Proof. Let $((u_n,v_n))\subset\mathcal{N}_\varepsilon$ be such that

$$J_\varepsilon(u_n,v_n)\to c \text{ and } \|J'_\varepsilon(u_n,v_n)\|_*=o_n(1),$$

where $o_n(1)$ approaches zero as $n\to\infty$. Then there exists $(\lambda_n)\subset\mathbb{R}$ satisfying

$$J'_\varepsilon(u_n,v_n)=\lambda_n\phi'_\varepsilon(u_n,v_n)+o_n(1), \tag{2.4.14}$$

with ϕ_ε as in Lemma 2.4.8. Since $(u_n,v_n)\in\mathcal{N}_\varepsilon$ we have that

$$0=J'_\varepsilon(u_n,v_n)(u_n,v_n)=\lambda_n\phi'_\varepsilon(u_n,v_n)(u_n,v_n)+o_n(1)\|(u_n,v_n)\|_\varepsilon.$$

Straightforward calculations show that $((u_n,v_n))$ is bounded. Moreover, in view of Lemma 2.4.8, we may suppose that $\phi'_\varepsilon(u_n,v_n)(u_n,v_n)\to l<0$. Hence, the above expression shows that $\lambda_n\to 0$ and therefore we conclude that $J'_\varepsilon(u_n,v_n)\to 0$ in the dual space of X_ε. It follows from Lemma 2.4.4 that $((u_n,v_n))$ has a convergent subsequence. □

From now on we will denote by (w_1,w_2) the solution for the system (S_0) given in Lemma 2.3.3 in section 3.

Let us consider $\delta>0$ such that $M_\delta\subset\Lambda$ and $\psi\in C^\infty(\mathbb{R}^+,[0,1])$ a non-increasing function such that $\psi\equiv 1$ on $[0,\delta/2]$ and $\psi\equiv 0$ on $[\delta,\infty)$. For any $y\in M$, we define the function $\Psi_{i,\varepsilon,y}\in X_\varepsilon$ by setting

$$\Psi_{i,\varepsilon,y}(x):=\psi(|\varepsilon x-y|)w_i\left(\frac{\varepsilon x-y}{\varepsilon}\right),\quad i=1,2,$$

and denote by $t_\varepsilon>0$ the unique positive number verifying

$$J_\varepsilon(t_\varepsilon(\Psi_{1,\varepsilon,y},\Psi_{2,\varepsilon,y}))=\max_{t\geq 0}J_\varepsilon(t(\Psi_{1,\varepsilon,y},\Psi_{2,\varepsilon,y})).$$

In view of the above remarks, it is well defined the function $\Phi_\varepsilon:M\to\mathcal{N}_\varepsilon$ given by

$$\Phi_\varepsilon(y):=t_\varepsilon(\Psi_{1,\varepsilon,y},\Psi_{2,\varepsilon,y}).$$

In next lemma we prove an important relationship between Φ_ε and the set M.

Lemma 2.4.10. *Uniformly for $y\in M$, we have*

$$\lim_{\varepsilon\to 0^+}J_\varepsilon(\Phi_\varepsilon(y))=c_0,$$

where c_0 was given in (2.3.1).

Proof. Suppose, by contradiction, that the lemma is false. Then there exist $\delta>0$, $(y_n)\subset M$ and $\varepsilon_n\to 0^+$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))-c_0|\geq\delta>0. \tag{2.4.15}$$

We notice that, if $z\in B_{\delta/\varepsilon_n}(0)$ then $\varepsilon_n z+y_n\in B_\delta(y_n)\subset M_\delta\subset\Lambda$. Thus, recalling that $H\equiv Q+\frac{1}{2^*}K$ in Λ and $\psi(s)=0$ for $s\geq\delta$, we can use the change of variables $z\mapsto(\varepsilon_n x-y_n)/\varepsilon_n$ to write

$$
\begin{aligned}
J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^N} a(\varepsilon_n z+y_n)|\nabla(\psi(|\varepsilon_n z|)w_1(z))|^2dz + \frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^N}|\psi(|\varepsilon_n z|)w_1(z)|^2dz \\
&+\frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^N} b(\varepsilon_n z+y_n)|\nabla(\psi(|\varepsilon_n z|)w_2(z))|^2dz + \frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^N}|\psi(|\varepsilon_n z|)w_2(z)|^2dz \\
&-\int_{\mathbb{R}^N} Q(t_{\varepsilon_n}\psi(|\varepsilon_n z|)w_1(z), t_{\varepsilon_n}\psi(|\varepsilon_n z|)w_2(z))\,dz \\
&-\frac{1}{2^*}\int_{\mathbb{R}^N} K(t_{\varepsilon_n}\psi(|\varepsilon_n z|)w_1(z), t_{\varepsilon_n}\psi(|\varepsilon_n z|)w_2(z))\,dz.
\end{aligned}
$$

Since Q and K are homogeneous, we have that $t_{\varepsilon_n} \to 1$. This and Lebesgue's theorem imply that

$$\lim_{n\to\infty} \|(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})\|_{\varepsilon_n}^2 = \|(w_1,w_2)\|^2,$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} Q(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})dz = \int_{\mathbb{R}^N} Q(w_1,w_2)dz,$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})dz = \int_{\mathbb{R}^N} K(w_1,w_2)dz.$$

Therefore

$$\lim_{n\to\infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_0(w_1,w_2) = c_0$$

which contradicts (2.4.15). The lemma is proved. □

Proposition 2.4.11. *Let $\varepsilon_n \to 0^+$ and $((u_n,v_n)) \subset \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n,v_n) \to c_0$. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $(w_n(x),z_n(x)) := (u_n(x+\widetilde{y}_n), v_n(x+\widetilde{y}_n))$ has a convergent subsequence in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n \to y \in M$, where $y_n = \varepsilon_n\widetilde{y}_n$.*

Proof. Since $a_0 \le a(x)$ and $b_0 \le b(x)$ for $x \in \mathbb{R}^N$ and $c_0 > 0$, we can repeat the same arguments in Lemma 2.3.2 to conclude that there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ and positive constants R and η such that

$$\liminf_{n\to\infty}\int_{B_R(\widetilde{y}_n)} (|u_n|^2+|v_n|^2)dx \ge \eta.$$

Thus, since $((u_n,v_n))$ is bounded in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, considering $(w_n(x),z_n(x)) = (u_n(x+\widetilde{y}_n), v_n(x+\widetilde{y}_n))$, up to a subsequence, we have that $w_n \rightharpoonup w \not\equiv 0$ in $H^1(\mathbb{R}^N)$ and $z_n \rightharpoonup z \not\equiv 0$ in $H^1(\mathbb{R}^N)$. Let $t_n > 0$ be such that

$$(\widetilde{w}_n, \widetilde{z}_n) = t_n(w_n,z_n) \in \mathcal{M}_0. \tag{2.4.16}$$

Then,

$$c_0 \le I_0(\widetilde{w}_n,\widetilde{z}_n) \le J_{\varepsilon_n}(t_n(u_n,v_n)) \le J_{\varepsilon_n}(u_n,v_n) = c_0 + o_n(1), \tag{2.4.17}$$

which implies

$$I_0(\widetilde{w}_n,\widetilde{z}_n) \to c_0 \text{ and } ((\widetilde{w}_n,\widetilde{z}_n)) \subset \mathcal{M}_0.$$

From boundedness of $((w_n,z_n))$ and (2.4.17), we get that (t_n) is bounded. As a consequence, the sequence $((\widetilde{w}_n,\widetilde{z}_n))$ is also bounded in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, which implies, for some subsequence, $(\widetilde{w}_n,\widetilde{z}_n) \rightharpoonup (\widetilde{w},\widetilde{z})$ weakly in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.

Note that we can assume that $t_n \to t_0 > 0$. Then, this limit implies that $(\widetilde{w}, \widetilde{z}) \not\equiv (0,0)$. From Lemma 2.3.3, we conclude that $(\widetilde{w}_n, \widetilde{z}_n) \to (\widetilde{w}, \widetilde{z})$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, and as a consequence, $(w_n, z_n) \to (w, z)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

To conclude the proof of the Proposition, we consider $y_n = \varepsilon_n \widetilde{y}_n$. Our goal is to show that (y_n) has a subsequence, still denoted by (y_n), satisfying $y_n \to y$ for $y \in M$. First of all, we claim that (y_n) is bounded. Indeed, suppose that there exists a subsequence, still denote by (y_n), verifying $|y_n| \to \infty$. Note that from (ab_1) we have

$$\begin{aligned} & a_0 \int_{\mathbb{R}^N} |\nabla w_n|^2 dx + b_0 \int_{\mathbb{R}^N} |\nabla z_n|^2 dx + \int_{\mathbb{R}^N} |w_n|^2 dx + \int_{\mathbb{R}^N} |z_n|^2 dx \\ \leq & \int_{\mathbb{R}^N} [w_n H_w(\varepsilon_n x + y_n, w_n, z_n) + z_n H_z(\varepsilon_n x + y_n, w_n, z_n)] dx. \end{aligned}$$

Fixing $R > 0$ such that $B_R(0) \supset \Lambda$, since $|\varepsilon_n x + y_n| \geq R$ and (H_3), we have

$$\begin{aligned} & a_0 \int_{\mathbb{R}^N} |\nabla w_n|^2 dx + b_0 \int_{\mathbb{R}^N} |\nabla z_n|^2 dx + \int_{\mathbb{R}^N} |w_n|^2 dx + \int_{\mathbb{R}^N} |z_n|^2 dx \\ \leq & \frac{1}{4} \int_{B_{R/\varepsilon_n}(0)} (w_n^2 + z_n^2) dx + o_n(1). \end{aligned}$$

It follows that $(w_n, z_n) \to (0,0)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, obtain this way a contradiction because $c_0 > 0$.

Hence (y_n) is bounded and, up to a subsequence,

$$y_n \to y \in \mathbb{R}^N.$$

Arguing as above, if $y \notin \overline{\Lambda}$, we will obtain again $(w_n, z_n) \to (0,0)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, thus $y \in \overline{\Lambda}$.

Now we are going to show that $y \in M$. It is sufficient to show that $a(y) = a_0$ and $b(y) = b_0$. Supposing, by contradiction, that $a(y) > a_0$ or $b(y) > b_0$, we have

$$\begin{aligned} c_0 = I_0(\widetilde{w}, \widetilde{z}) < & \frac{1}{2} \int_{\mathbb{R}^N} a(y) |\nabla \widetilde{w}|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} b(y) |\nabla \widetilde{z}|^2 dx \\ + & \frac{1}{2} \int_{\mathbb{R}^N} |\widetilde{w}|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |\widetilde{z}|^2 dx - \int_{\mathbb{R}^N} Q(\widetilde{w}, \widetilde{z}) dx - \frac{1}{2^*} \int_{\mathbb{R}^N} K(\widetilde{w}, \widetilde{z}) dx. \end{aligned}$$

Using again the fact that $(\widetilde{w}_n, \widetilde{z}_n) \to (\widetilde{w}, \widetilde{z})$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, from Fatou's lemma

$$\begin{aligned} c_0 & < \liminf_{n\to\infty} \left[\frac{1}{2} \int_{\mathbb{R}^N} a(\varepsilon_n x + y_n) |\nabla \widetilde{w}_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} b(\varepsilon_n x + y_n) |\nabla \widetilde{z}_n|^2 dx \right] \\ & + \lim_{n\to\infty} \left[\frac{1}{2} \int_{\mathbb{R}^N} |\widetilde{w}_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |\widetilde{z}_n|^2 dx \right] \\ & - \lim_{n\to\infty} \left[\int_{\mathbb{R}^N} Q(\widetilde{w}_n, \widetilde{z}_n) dx + \frac{1}{2^*} \int_{\mathbb{R}^N} K(\widetilde{w}_n, \widetilde{z}_n) dx \right], \end{aligned}$$

that is,

$$c_0 < \liminf_{n\to\infty} J_{\varepsilon_n}(t_n(u_n, v_n)) \leq \liminf_{n\to\infty} J_{\varepsilon_n}(u_n, v_n) = c_0,$$

obtaining a contradiction. Then, we conclude that $y \in M$. □

Corollary 2.4.12. *Assume the same hypotheses of Proposition 2.4.11. Then, for any given $\gamma > 0$, there exists $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\int_{B_R(\widetilde{y}_n)^c} \left(|\nabla u_n|^2 + |u_n|^2\right) dx + \int_{B_R(\widetilde{y}_n)^c} \left(|\nabla v_n|^2 + |v_n|^2\right) dx < \gamma, \quad \textit{for all } n \geq n_0.$$

Proof. By using the same notation of the proof of Proposition 2.4.11, we have for any $R > 0$

$$\begin{aligned} &\int_{B_R(\widetilde{y}_n)^c} \left(|\nabla u_n|^2 + |u_n|^2\right) dx + \int_{B_R(\widetilde{y}_n)^c} \left(|\nabla v_n|^2 + |v_n|^2\right) dx \\ = \ &\int_{B_R(0)^c} \left(|\nabla w_n|^2 + |w_n|^2\right) dx + \int_{B_R(0)^c} \left(|\nabla z_n|^2 + |z_n|^2\right) dx. \end{aligned}$$

Since $((w_n, z_n))$ strongly converges in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, the result follows. □

Let us consider $\rho = \rho(\delta) > 0$ in such way that $M_\delta \subset B_\rho(0)$ and define $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ by setting $\Upsilon(x) := x$ for $|x| < \rho$ and $\Upsilon(x) := \rho x/|x|$ for $|x| \geq \rho$. We also consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u, v) := \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)\left(|u(x)|^2 + |v(x)|^2\right) dx}{\displaystyle\int_{\mathbb{R}^N} \left(|u(x)|^2 + |v(x)|^2\right) dx}.$$

Since $M \subset B_\rho(0)$, the definition of Υ and Lebesgue's theorem imply that

$$\lim_{\varepsilon \to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \ \text{ uniformly for } y \in M. \tag{2.4.18}$$

Following [17], we introduce the set

$$\Sigma_\varepsilon := \left\{(u, v) \in \mathcal{N}_\varepsilon : J_\varepsilon(u, v) \leq c_0 + h(\varepsilon)\right\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Given $y \in M$, we can use Lemma 2.4.10 to conclude that $h(\varepsilon) = |J_\varepsilon(\Phi_\varepsilon(y)) - c_0|$ satisfies $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Thus, $\Phi_\varepsilon(y) \in \Sigma_\varepsilon$ and therefore $\Sigma_\varepsilon \neq \emptyset$, for any $\varepsilon > 0$ small.

Lemma 2.4.13. *For any $\delta > 0$ we have*

$$\lim_{\varepsilon \to 0^+} \sup_{(u,v) \in \Sigma_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u, v), M_\delta) = 0. \tag{2.4.19}$$

Proof. Let $(\varepsilon_n) \subset \mathbb{R}$ be such that $\varepsilon_n \to 0^+$. By definition, there exists $((u_n, v_n)) \subset \Sigma_{\varepsilon_n}$ such that

$$\operatorname{dist}(\beta_{\varepsilon_n}(u_n, v_n), M_\delta) = \sup_{(u,v) \in \Sigma_{\varepsilon_n}} \operatorname{dist}(\beta_{\varepsilon_n}(u, v), M_\delta) + o_n(1).$$

Thus, it suffices to find a sequence $(y_n) \subset M_\delta$ such that

$$|\beta_{\varepsilon_n}(u_n, v_n) - y_n| = o_n(1). \tag{2.4.20}$$

Thus, recalling that $((u_n, v_n)) \subset \Sigma_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we obtain

$$c_0 \leq \max_{t \geq 0} I_0(tu_n, tv_n) \leq \max_{t \geq 0} J_{\varepsilon_n}(tu_n, tv_n) = J_{\varepsilon_n}(u_n, v_n) \leq c_0 + h(\varepsilon_n), \tag{2.4.21}$$

from which follows that $J_{\varepsilon_n}(u_n, v_n) \to c_0$. Thus, we may invoke Proposition 2.4.11 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $(y_n) := (\varepsilon_n \widetilde{y}_n) \subset M_\delta$, for n large. Hence,

$$
\begin{aligned}
\beta_{\varepsilon_n}(u_n,v_n) &= \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon_n x)\left(|u_n|^2+|v_n|^2\right)dx}{\displaystyle\int_{\mathbb{R}^N}\left(|u_n|^2+|v_n|^2\right)dx} \\
&= \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon_n z+y_n)\left(|u_n(z+\widetilde{y}_n)|^2+|v_n(z+\widetilde{y}_n)|^2\right)dz}{\displaystyle\int_{\mathbb{R}^N}\left(|u_n(z+\widetilde{y}_n)|^2+|v_n(z+\widetilde{y}_n)|^2\right)dz} \\
&= y_n + \frac{\displaystyle\int_{\mathbb{R}^N} \left(\Upsilon(\varepsilon_n z+y_n)-y_n\right)\left(|u_n(z+\widetilde{y}_n)|^2+|v_n(z+\widetilde{y}_n)|^2\right)dz}{\displaystyle\int_{\mathbb{R}^N}\left(|u_n(z+\widetilde{y}_n)|^2+|v_n(z+\widetilde{y}_n)|^2\right)dz}.
\end{aligned}
$$

Since $\varepsilon_n z + y_n \to y_0 \in M$ and from strong convergence of $(u_n(\cdot+\widetilde{y}_n), v_n(\cdot+\widetilde{y}_n))$, we have that $\beta_{\varepsilon_n}(u_n,v_n) = y_n + o_n(1)$ and therefore the sequence (y_n) satisfies (2.4.20). The lemma is proved. □

Lemma 2.4.14. *The minimax level c_ε satisfies*

$$
\limsup_{\varepsilon\to 0^+} c_\varepsilon \le c_0.
$$

Proof. Let $\eta \in C_0^\infty(\mathbb{R}^N,[0,1])$ be such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$. For any given $r>0$ we define $(v_{1,r}(x), v_{2,r}(x)) := (\eta(x/r)w_1(x), \eta(x/r)w_2(x))$, where (w_1,w_2) is a ground state solution of the system (S_0).

Let $t_{\varepsilon,r}>0$ be such that $t_{\varepsilon,r}(v_{1,r},v_{2,r}) \in \mathcal{N}_\varepsilon$ and note that

$$
\begin{aligned}
c_\varepsilon &\le J_\varepsilon(t_{\varepsilon,r}v_{1,r}, t_{\varepsilon,r}v_{2,r}) = \frac{t_{\varepsilon,r}^2}{2}\int_{\mathbb{R}^N}\left(a(\varepsilon x)|\nabla v_{1,r}|^2+|v_{1,r}|^2\right)dx \\
&+\frac{t_{\varepsilon,r}^2}{2}\int_{\mathbb{R}^N}\left(b(\varepsilon x)|\nabla v_{2,r}|^2+|v_{2,r}|^2\right)dx - \int_{\mathbb{R}^N} H(\varepsilon x, t_{\varepsilon,r}v_{1,r}, t_{\varepsilon,r}v_{2,r})dx.
\end{aligned}
$$

It is easy to check that, for r fixed, $t_{\varepsilon,r} \to t_r > 0$ as $\varepsilon \to 0$. Moreover, without loss of generality, we may suppose that $a(0) = a_0$ and $b(0) = b_0$. Hence, since $(v_{1,r}, v_{2,r})$ has compact support, we can use Lebesgue's theorem to get

$$
\begin{aligned}
\limsup_{\varepsilon\to 0^+} c_\varepsilon &\le \frac{t_r^2}{2}\int_{\mathbb{R}^N}\left(a_0|\nabla v_{1,r}|^2+|v_{1,r}|^2\right)dx + \frac{t_r^2}{2}\int_{\mathbb{R}^N}\left(b_0|\nabla v_{2,r}|^2+|v_{2.r}|^2\right)dx \\
&-\int_{\mathbb{R}^N} Q(t_r v_{1,r}, t_r v_{2,r})dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(t_r v_{1,r}, t_r v_{2,r})dx.
\end{aligned}
$$

Since $(w_1,w_2) \in \mathcal{M}_0$ and $(v_{1,r},v_{2,r}) \to (w_1,w_2)$ in E_0 as $r\to\infty$, we can check that $t_r \to 1$ as $r \to \infty$. Thus, it follows from the above expression that

$$
\limsup_{\varepsilon\to 0^+} c_\varepsilon \le \lim_{r\to\infty} I_0(t_r v_{1,r}, t_r v_{2,r}) = I_0(w_1,w_2) = c_0
$$

and the proof is over. □

We are now ready to present the proof of Theorem 2.4.1.

2.4.2 Proof of Theorem 2.4.1

Proof. (i) From Lemma 2.4.14, we obtain $\varepsilon_1 > 0$ such that $c_\varepsilon < c_0$ for any $\varepsilon \in (0, \varepsilon_1)$. For these values of ε, since J_ε has the mountain pass geometry, we can take a sequence $((u_n, v_n)) \subset X_\varepsilon$ such that

$$J_\varepsilon(u_n, v_n) \to c_\varepsilon \quad \text{and} \quad J'_\varepsilon(u_n, v_n) \to 0.$$

By using Proposition 2.3.1, we garantee that $c_\varepsilon < \frac{1}{N}\left(\min\{a_0, b_0\}\widetilde{S}_K\right)^{N/2}$. Thus, from Lemma 2.4.4 we get that, along a subsequence $(u_n, v_n) \to (u_\varepsilon, v_\varepsilon)$ with $(u_\varepsilon, v_\varepsilon)$ being such that $J_\varepsilon(u_\varepsilon, v_\varepsilon) = c_\varepsilon$ and $J'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$.

Now we prove the item (ii). Given $\delta > 0$ such that $M_\delta \subset \Lambda$, we can use (2.4.18), Lemma 2.4.10, (2.4.13) and argue as in [17, Section 6] to obtain $\widehat{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \widehat{\varepsilon}_\delta)$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \Sigma_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Thus

$$\operatorname{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon) \geq \operatorname{cat}_{M_\delta}(M).$$

It follows from Proposition 2.4.9 and Theorem 2.4.5 that J_ε possesses at least $\operatorname{cat}_{M_\delta}(M)$ critical points on $\mathcal{N}_\varepsilon$. The same argument employed in the proof of Proposition 2.4.9 shows that each of these critical points is also a critical point of the unconstrained functional J_ε. Thus, we obtain $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions for $(S_{\varepsilon,aux})$. □

2.5 Proof of Theorem 2

In this section we prove our main theorem. The idea is to show that the solutions obtained in Theorem 2.4.1 verify the following estimate $|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \; \forall x \in \mathbb{R}^N \setminus \Lambda_\varepsilon$ as ε is small enough. This fact implies that these solutions are in fact solutions of the system $(\widehat{S}_\varepsilon)$. The key ingredient is the following result, whose proof uses an adaptation of the arguments found in [31], which are related with the Moser's iteration method [34] .

Lemma 2.5.1. *Let Let (ε_n) be a sequence such that $\varepsilon_n \to 0^+$ and for each $n \in \mathbb{N}$, let $(u_n, v_n) \in \Sigma_{\varepsilon_n}$ be a solution of system $(S_{\varepsilon_n,aux})$. Then $J_{\varepsilon_n}(u_n, v_n) \to c_0$ and $(u_n, v_n) \in L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$. Moreover, given $\xi > 0$, there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that, for $w_n(x) = u_n(x + \widetilde{y}_n)$ and $z_n(x) = v_n(x + \widetilde{y}_n)$, we have*

$$|w_n|_{L^\infty(\mathbb{R}^N \setminus B_R(0))} < \xi, \quad \textit{for all} \;\; n \geq n_0,$$

$$|z_n|_{L^\infty(\mathbb{R}^N \setminus B_R(0))} < \xi, \quad \textit{for all} \;\; n \geq n_0,$$

where $(\widetilde{y}_n)$ is the sequences of Proposition 2.4.11.

Proof. Since $J_{\varepsilon_n}(u_n, v_n) \leq c_0 + h(\varepsilon_n)$ with $\lim_{n\to\infty} h(\varepsilon_n) = 0$, we can argue as in (2.4.21) to conclude that $J_{\varepsilon_n}(u_n, v_n) \to c_0$. Thus, we may invoke Proposition 2.4.11 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ satisfying the conclusions of that Proposition.

Fix $R := R_1 > R_2 > ... > R_k > R_{k-1} > ... > R_0$ and consider $\eta_{R_k} \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \eta_{R_k} \leq 1$, $\eta_{R_k} \equiv 0$ in $B_{R/2}(0)$, $\eta_{R_k} \equiv 1$ in $B_R(0)^c$ and $|\nabla \eta_{R_k}| \leq C/R_0$. For each $n \in \mathbb{N}$ and $L > 0$, we define $\eta_n(x) := \eta_{R_k}(x - \widetilde{y}_n)$, $w_{L,n}, z_{L,n} \in X_\varepsilon$ by setting

$$w_{L,n}(x) := \min\{w_n(x), L\}, \qquad \Upsilon_{w,L,n} := \eta_n^2 w_{L,n}^{2(\beta-1)} w_n$$

and

$$z_{L,n}(x) := \min\{z_n(x), L\}, \qquad \Upsilon_{z,L,n} := \eta_n^2 z_{L,n}^{2(\beta-1)} z_n,$$

with $\beta > 1$ to be determined later.

By definition of $(\Upsilon_{w,L,n}, \Upsilon_{z,L,n})$, $J'_{\varepsilon_n}(w_n, z_n)(\Upsilon_{w,L,n}, \Upsilon_{z,L,n}) = 0$ and since

$$2a_0(\beta-1)\int_{\mathbb{R}^N} \eta_n^2 w_n w_{L,n}^{2(\beta-1)-1} \nabla w_n \nabla w_{L,n} dx \geq 0$$

and

$$2b_0(\beta-1)\int_{\mathbb{R}^N} \eta_n^2 z_n z_{L,n}^{2(\beta-1)-1} \nabla z_n \nabla z_{L,n} dx \geq 0,$$

we have that

$$\begin{aligned} &a_0\int_{\mathbb{R}^N} \eta_n^2 w_{L,n}^{2(\beta-1)}|\nabla w_n|^2 dx + 2a_0\int_{\mathbb{R}^N} \eta_n w_n w_{L,n}^{2(\beta-1)} \nabla\eta_n \cdot \nabla w_n dx \\ &+b_0\int_{\mathbb{R}^N} \eta_n^2 z_{L,n}^{2(\beta-1)}|\nabla z_n|^2 dx + 2b_0\int_{\mathbb{R}^N} \eta_n z_n z_{L,n}^{2(\beta-1)} \nabla\eta_n \cdot \nabla z_n dx \\ &\leq \int_{\mathbb{R}^N} H_w(\varepsilon x + y_n, w_n, z_n)\eta_n^2 w_n w_{L,n}^{2(\beta-1)} dx \\ &+\int_{\mathbb{R}^N} H_z(\varepsilon x + y_n, w_n, z_n)\eta_n^2 z_n z_{L,n}^{2(\beta-1)} dx. \end{aligned} \tag{2.5.1}$$

In view of $(\mathcal{H}_1^p)$, $(\mathcal{H}_1^{2^*})$ and (H_4) we can obtain $C_1 > 0$ such that

$$H_s(x,s,t) + H_t(x,s,t) \leq \frac{1}{4}|s| + \frac{1}{4}|t| + C_1[|s|^{(2^*-1)} + |t|^{(2^*-1)}], \quad \text{for any } (x,s,t) \in \mathbb{R}^{N+2}.$$

Using the last inequality in (2.5.1), we obtain

$$\begin{aligned} &a_0\int_{\mathbb{R}^N} \eta_n^2 w_{L,n}^{2(\beta-1)}|\nabla w_n|^2 dx + b_0\int_{\mathbb{R}^N} \eta_n^2 z_{L,n}^{2(\beta-1)}|\nabla z_n|^2 dx \\ &\leq 2a_0\int_{\mathbb{R}^N} \eta_n w_n w_{L,n}^{2(\beta-1)} \nabla\eta_n \cdot \nabla w_n dx + 2b_0\int_{\mathbb{R}^N} \eta_n z_n z_{L,n}^{2(\beta-1)} \nabla\eta_n \cdot \nabla z_n dx \\ &+\int_{\mathbb{R}^N} \eta_n^2 w_n^{2^*} w_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^N} \eta_n^2 z_n^{2^*} z_{L,n}^{2(\beta-1)} dx. \end{aligned}$$

For any $\widetilde{\gamma} > 0$ we can use Young's inequality to obtain

$$\begin{aligned} &a_0\int_{\mathbb{R}^N} \eta_n^2 w_{L,n}^{2(\beta-1)}|\nabla w_n|^2 dx + b_0\int_{\mathbb{R}^N} \eta_n^2 z_{L,n}^{2(\beta-1)}|\nabla z_n|^2 dx \\ &\leq 2a_0\int_{\mathbb{R}^N} [\widetilde{\gamma}\eta_n^2|\nabla w_n|^2 + C_{\widetilde{\gamma}}|w_n|^2|\nabla\eta_n|^2] w_{L,n}^{2(\beta-1)} dx \\ &+2b_0\int_{\mathbb{R}^N} [\widetilde{\gamma}\eta_n^2|\nabla z_n|^2 + C_{\widetilde{\gamma}}|z_n|^2|\nabla\eta_n|^2] z_{L,n}^{2(\beta-1)} dx \\ &+\int_{\mathbb{R}^N} \eta_n^2 w_n^{2^*} w_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^N} \eta_n^2 z_n^{2^*} z_{L,n}^{2(\beta-1)} dx. \end{aligned}$$

By choosing $\widetilde{\gamma} \leq 1/4$ we get, there exists $C_2 > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^N} \eta_n^2 w_{L,n}^{2(\beta-1)}|\nabla w_n|^2 dx + \int_{\mathbb{R}^N} \eta_n^2 z_{L,n}^{2(\beta-1)}|\nabla z_n|^2 dx \\ &\leq C_2\Bigg(\int_{\mathbb{R}^N} |w_n|^2|\nabla\eta_n|^2 w_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^N} |z_n|^2|\nabla\eta_n|^2 z_{L,n}^{2(\beta-1)} dx \\ &\quad+\int_{\mathbb{R}^N} \eta_n^2 w_n^{2^*} w_{L,n}^{2(\beta-1)} dx + \int_{\mathbb{R}^N} \eta_n^2 z_n^{2^*} z_{L,n}^{2(\beta-1)} dx\Bigg). \end{aligned} \tag{2.5.2}$$

Let S be the best constant of the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and define $\widehat{w}_{L,n} := \eta_n w_n w_{L,n}^{\beta-1}$ and $\widehat{z}_{L,n} := \eta_n z_n z_{L,n}^{\beta-1}$. Since $w_{L,n} \leq w_n$ and $z_{L,n} \leq z_n$, we have that

$$\begin{aligned} S^{-1}\left[\|\widehat{w}_{L,n}\|_{L^{2^*}}^2 + \|\widehat{z}_{L,n}\|_{L^{2^*}}^2\right] \leq & \int_{\mathbb{R}^N} \left|\nabla\left(\eta_n w_n w_{L,n}^{\beta-1}\right)\right|^2 dx + \int_{\mathbb{R}^N} \left|\nabla\left(\eta_n z_n z_{L,n}^{\beta-1}\right)\right|^2 dx \\ & \leq 2\int_{\mathbb{R}^N} |w_n|^2 w_{L,n}^{2(\beta-1)} |\nabla\eta_n|^2 dx + 2\int_{\mathbb{R}^N} |z_n|^2 z_{L,n}^{2(\beta-1)} |\nabla\eta_n|^2 dx + \\ & 2\beta^2 \int_{\mathbb{R}^N} \eta_n^2 w_{L,n}^{2(\beta-1)} |\nabla w_n|^2 dx + 2\beta^2 \int_{\mathbb{R}^N} \eta_n^2 z_{L,n}^{2(\beta-1)} |\nabla z_n|^2 dx. \end{aligned}$$

The last inequality and (2.5.2) provide

$$\begin{aligned} S^{-1}\left[\|\widehat{w}_{L,n}\|_{L^{2^*}}^2 + \|\widehat{z}_{L,n}\|_{L^{2^*}}^2\right] \leq C_4\beta^2 \Bigg(& \int_{\mathbb{R}^N} |w_n|^2 w_{L,n}^{2(\beta-1)} |\nabla\eta_n|^2 dx \\ + \int_{\mathbb{R}^N} |z_n|^2 z_{L,n}^{2(\beta-1)} |\nabla\eta_n|^2 dx + \int_{\mathbb{R}^N} \eta_n^2 |w_n|^{2^*} w_{L,n}^{2(\beta-1)} dx & + \int_{\mathbb{R}^N} \eta_n^2 |z_n|^{2^*} z_{L,n}^{2(\beta-1)} dx \Bigg), \end{aligned} \tag{2.5.3}$$

for all $\beta > 1$.

The above expression, the properties of η_n and $w_{L,n} \leq |w_n|$, $z_{L,n} \leq |z_n|$, imply that

$$\begin{aligned} S^{-1}\left[\|\widehat{w}_{L,n}\|_{L^{2^*}}^2 + \|\widehat{z}_{L,n}\|_{L^{2^*}}^2\right] \leq C_4\beta^2 & \int_{B_{R/2}(\widetilde{y}_n)^c} \left(|w_n|^{2\beta}|\nabla\eta_n|^2 + |w_n|^{2^*-2}|w_n|^{2\beta}\right) dx \\ + C_4\beta^2 & \int_{B_{R/2}(\widetilde{y}_n)^c} \left(|z_n|^{2\beta}|\nabla\eta_n|^2 + |z_n|^{2^*-2}|z_n|^{2\beta}\right) dx. \end{aligned} \tag{2.5.4}$$

If we now set

$$t := \frac{2^* 2^*}{2(2^*-2)} > 1, \qquad \zeta := \frac{2t}{t-1} < 2^*, \tag{2.5.5}$$

we can apply Hölder's inequality with exponents $t/(t-1)$ and t in (2.5.4), to get

$$\begin{aligned} S^{-1}\left[\|\widehat{w}_{L,n}\|_{L^{2^*}}^2 + \|\widehat{z}_{L,n}\|_{L^{2^*}}^2\right] \leq C_4\beta^2 \|w_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta} & \left(\int_{B_{R/2}(\widetilde{y}_n)^c} |\nabla\eta_n|^{2t} dx\right)^{1/t} \\ + C_4\beta^2 \|z_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta} & \left(\int_{B_{R/2}(\widetilde{y}_n)^c} |\nabla\eta_n|^{2t} dx\right)^{1/t} \\ + C_4\beta^2 \|w_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta} & \left(\int_{B_{R/2}(\widetilde{y}_n)^c} |w_n|^{2^*(2^*/2)} dx\right)^{1/t} \\ + C_4\beta^2 \|z_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta} & \left(\int_{B_{R/2}(\widetilde{y}_n)^c} |z_n|^{2^*(2^*/2)} dx\right)^{1/t}. \end{aligned} \tag{2.5.6}$$

Since η_n is constant on $B_{R/2}(\widetilde{y}_n) \cup B_R(\widetilde{y}_n)^c$ and $|\nabla\eta_n| \leq C/R_0$, we have that

$$\int_{B_{R/2}(\widetilde{y}_n)^c} |\nabla\eta_n|^{2t} dx = \int_{R/2 \leq |x-\widetilde{y}_n| \leq R} |\nabla\eta_n|^{2t} dx \leq \frac{C_5}{R_0^{2t-N}} \leq C_5, \tag{2.5.7}$$

where we have used, without of generality, that $R_0 > 1$ and $2t = \frac{2^*}{2}N > N$ in the last inequality.

Claim. There exists $n_0 \in \mathbb{N}$ and $\widetilde{K} > 0$ such that , for any $n \geq n_0$, there holds

$$\int_{B_{R/2}(\widetilde{y}_n)^c} |w_n|^{2^*(2^*/2)} dx \leq \widetilde{K}$$

and
$$\int_{B_{R/2}(\widetilde{y}_n)^c} |z_n|^{2^*(2^*/2)} dx \le \widetilde{K}.$$

Assuming the claim, we can use (2.5.6) and (2.5.7) to conclude that

$$S^{-1}\left[\|\widehat{w}_{L,n}\|_{L^{2^*}}^2 + \|\widehat{z}_{L,n}\|_{L^{2^*}}^2\right] \le C_6\beta^2\|w_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta} + C_6\beta^2\|z_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta}.$$

Since

$$\begin{aligned}
\|w_{L,n}\|_{L^{\beta 2^*}(B_R(\widetilde{y}_n)^c)}^{2\beta} &= \left(\int_{B_R(\widetilde{y}_n)^c} w_{L,n}^{\beta 2^*} dx\right)^{2/2^*} \\
&\le \left(\int_{\mathbb{R}^N} \eta_n^{2^*}|w_n|^{2^*} w_{L,n}^{2^*(\beta-1)} dx\right)^{2/2^*} \\
&= \|\widehat{w}_{L,n}\|_{L^{2^*}}^2 \le C_6\beta^2\|w_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta}
\end{aligned}$$

and

$$\begin{aligned}
\|z_{L,n}\|_{L^{\beta 2^*}(B_R(\widetilde{y}_n)^c)}^{2\beta} &= \left(\int_{B_R(\widetilde{y}_n)^c} z_{L,n}^{\beta 2^*} dx\right)^{2/2^*} \\
&\le \left(\int_{\mathbb{R}^N} \eta_n^{2^*}|z_n|^{2^*} z_{L,n}^{2^*(\beta-1)} dx\right)^{2/2^*} \\
&= \|\widehat{z}_{L,n}\|_{L^{2^*}}^2 \le C_6\beta^2\|z_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)}^{2\beta},
\end{aligned}$$

we can apply Fatou's lemma in the variable L to obtain

$$\begin{aligned}
\|w_n\|_{L^{\beta 2^*}(B_R(\widetilde{y}_n)^c)} + \|z_n\|_{L^{\beta 2^*}(B_R(\widetilde{y}_n)^c)} \le\; & C_7^{1/\beta}\beta^{1/\beta}\|w_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)} \\
&+ C_7^{1/\beta}\beta^{1/\beta}\|z_n\|_{L^{\beta\zeta}(B_{R/2}(\widetilde{y}_n)^c)},
\end{aligned}$$

whenever $w_n^{\beta\zeta}, z_n^{\beta\zeta} \in L^1(B_{R/2}(\widetilde{y}_n)^c)$.

We now set $\beta := 2^*/\zeta > 1$ and note that, since $w_n, z_n \in L^{2^*}(\mathbb{R}^N)$, the above inequality holds for this choice of β. Moreover, since $\beta^2\zeta = \beta 2^*$, it follows that the inequality also holds with β replaced by β^2.

Hence,

$$\|(w_n, z_n)\|_{L^{\beta^2 2^*}(B_R(\widetilde{y}_n)^c)} \le C_7^{1/\beta^2}\beta^{2/\beta^2}\|(w_n, z_n)\|_{L^{\beta^2\zeta}(B_{R/2}(\widetilde{y}_n)^c)}.$$

By iterating this process and recalling that $\beta\zeta = 2^*$ we obtain, for $k \in \mathbb{N}$,

$$\|(w_n, z_n)\|_{L^{\beta^k 2^*}(B_R(\widetilde{y}_n)^c)} \le C_7^{\sum_{i=1}^k \beta^{-i}}\beta^{\sum_{i=1}^k i\beta^{-i}}\|(w_n, z_n)\|_{L^{2^*}(B_{R/2}(\widetilde{y}_n)^c)}.$$

Since $\beta > 1$ we can take the limit as $k \to \infty$ to get

$$\|(w_n, z_n)\|_{L^\infty(B_R(\widetilde{y}_n)^c)} \le C_8\|(w_n, z_n)\|_{L^{2^*}(B_{R/2}(\widetilde{y}_n)^c)}.$$

By using the change of variables $z \mapsto x - \widetilde{y}_n$ we obtain

$$\begin{aligned}
\|(w_n, z_n)\|_{L^\infty(B_R(\widetilde{y}_n)^c)} \le\; & C_8\Big(\int_{B_{R/2}(0)^c} |u_n(z + \widetilde{y}_n)|^{2^*} dz\Big)^{\frac{1}{2^*}} \\
&+ C_8\Big(\int_{B_{R/2}(0)^c} |v_n(z + \widetilde{y}_n)|^{2^*} dz\Big)^{\frac{1}{2^*}},
\end{aligned}$$

where $(w_n(x), z_n(x)) = (u_n(x+\widetilde{y}_n), v_n(x+\widetilde{y}_n))$. By Proposition 2.4.11 we have that $((w_n, z_n))$ strongly converges in $L^{2^*}(\mathbb{R}^N) \times L^{2^*}(\mathbb{R}^N)$. Thus, for $R > 0$ sufficiently large, there holds

$$\|(w_n, z_n)\|_{L^\infty(B_R(\widetilde{y}_n)^c)} < \gamma,$$

for large n, which prove this lemma.

It remains to prove the claim. Of course, it is sufficient to prove that the first integral is finite. For that purpose we consider a new cut-off function given by $\widetilde{\eta}_n(x) := \eta_n(2x)$, in such way that $\widetilde{\eta}_n \equiv 0$ on $B_{R/4}(\widetilde{y}_n)$ and $\widetilde{\eta}_n \equiv 1$ on $B_{R/2}(\widetilde{y}_n)^c$. If $\widetilde{w}_{L,n} := \widetilde{\eta}_n|w_n|w_{L,n}^{\beta-1}$, we can proceed as before to prove the following version of (2.5.3)

$$\|\widetilde{w}_{L,n}\|_{L^{2^*}}^2 \le C_9\beta^2\left(\int_{\mathbb{R}^N} |w_n|^2 w_{L,n}^{2(\beta-1)}|\nabla\widetilde{\eta}_n|^2 dx + \int_{\mathbb{R}^N} \widetilde{\eta}_n^2|w_n|^{2^*} w_{L,n}^{2(\beta-1)} dx\right), \tag{2.5.8}$$

We set $\beta := 2^*/2$ to obtain

$$\|\widetilde{w}_{L,n}\|_{L^{2^*}}^2 \le C_{10}\left(\int_{\mathbb{R}^N} |w_n|^2 w_{L,n}^{(2^*-2)}|\nabla\widetilde{\eta}_n|^2 dx + \int_{B_{R/4}(\widetilde{y}_n)^c} \widetilde{\eta}_n^2|w_n|^2 w_{L,n}^{(2^*-2)}|w_n|^{(2^*-2)} dx\right).$$

By Hölder's inequality with exponents $2^*/2$ and $2^*/(2^*-2)$ we get

$$\begin{aligned}\|\widetilde{w}_{L,n}\|_{L^{2^*}}^2 &\le C_{10}\int_{\mathbb{R}^N} |w_n|^2 w_{L,n}^{(2^*-2)}|\nabla\widetilde{\eta}_n|^2 dx \\ &+ C_{10}\left(\int_{B_{R/4}(\widetilde{y}_n)^c} \left(\widetilde{\eta}_n|w_n|w_{L,n}^{(2^*-2)/2}\right)^{2^*} dx\right)^{2/2^*} \|w_n\|_{L^{2^*}(B_{R/4}(\widetilde{y}_n)^c)}^{2^*-2}.\end{aligned}$$

From Proposition 2.4.11 we obtain $n_0 \in \mathbb{N}$ and $R > 1$ such that

$$\int_{B_{R/4}(\widetilde{y}_n)^c} |w_n|^{2^*} dx \le \left(\frac{1}{2C_{10}}\right)^{2^*/(2^*-2)},$$

for all $n \ge n_0$.

Then

$$\begin{aligned}\|\widetilde{w}_{L,n}\|_{L^{2^*}}^2 &\le C_{10}\int_{\mathbb{R}^N} |w_n|^2 w_{L,n}^{(2^*-2)}|\nabla\widetilde{\eta}_n|^2 dx \\ &+ \frac{1}{2}\left(\int_{B_{R/4}(\widetilde{y}_n)^c} \left(\widetilde{\eta}_n|w_n|w_{L,n}^{(2^*-2)/2}\right)^{2^*} dx\right)^{2/2^*}.\end{aligned}$$

Thus, recalling that $\widetilde{\eta}_n|w_n|w_{L,n}^{(2^*-2)/2} = \widetilde{w}_{L,n}$, $w_{L,n} \le |w_n|$ and $\nabla\eta_n$ is bounded, we obtain

$$\|\widetilde{w}_{L,n}\|_{L^{2^*}}^2 \le C_{12}.$$

The definition of $\widetilde{\eta}_n$ and the above inequality imply that

$$\int_{B_{R/2}(\widetilde{y}_n)^c} (|w_n|w_{L,n}^{\beta-1})^{2^*} dx \le C_{12}^{2^*/2},$$

for all $n \ge n_0$. Using Fatou's lemma in the variable L, we have

$$\int_{B_{R/2}(\widetilde{y}_n)^c} |w_n|^{2^*(2^*/2)} dx \le \widetilde{K} := C_{12}^{2^*/2},$$

for all $n \ge n_0$, and therefore the claim holds. □

We are now ready to prove the main result of this chapter.

Proof of Theorem 2. Suppose that $\delta > 0$ is such that $M_\delta \subset \Lambda$. We first claim that there exists $\widetilde{\varepsilon}_\delta > 0$ such that, for any $0 < \varepsilon < \widetilde{\varepsilon}_\delta$ and any solution $(u_\varepsilon, v_\varepsilon) \in \Sigma_\varepsilon$ of the system $(S_{\varepsilon,aux})$, there holds

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \quad \text{for each } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon. \tag{2.5.9}$$

In order to prove the claim we argue by contradiction. So, suppose that for some sequence $\varepsilon_n \to 0^+$ we can obtain $(u_n, v_n) \in \Sigma_{\varepsilon_n}$ such that $J'_{\varepsilon_n}(u_n, v_n) = 0$ and

$$\|(u_n, v_n)\|_{L^\infty(\mathbb{R}^N \setminus \Lambda_{\varepsilon_n})} > \alpha. \tag{2.5.10}$$

As in Lemma 2.5.1, we have that $J_{\varepsilon_n}(u_n, v_n) \to c_0$ and therefore we can use Proposition 2.4.11 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $\varepsilon_n \widetilde{y}_n \to y_0 \in M$.

If we take $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Lambda$ we have that

$$B_{r/\varepsilon_n}(y_0/\varepsilon_n) = \frac{1}{\varepsilon_n} B_r(y_0) \subset \Lambda_{\varepsilon_n}.$$

Moreover, for any $z \in B_{r/\varepsilon_n}(\widetilde{y}_n)$, there holds

$$\left| z - \frac{y_0}{\varepsilon_n} \right| \leq |z - \widetilde{y}_n| + \left| \widetilde{y}_n - \frac{y_0}{\varepsilon_n} \right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n},$$

for n large. For this values of n we have that $B_{r/\varepsilon_n}(\widetilde{y}_n) \subset \Lambda_{\varepsilon_n}$ or, equivalently, $\mathbb{R}^N \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^N \setminus B_{r/\varepsilon_n}(\widetilde{y}_n)$. On the other hand, it follows from Lemma 2.5.1 with $\xi = \alpha$ that, for any $n \geq n_0$ such that $r/\varepsilon_n > R$, there holds

$$\|u_n\|_{L^\infty(\mathbb{R}^N \setminus \Lambda_{\varepsilon_n})} \leq \|u_n\|_{L^\infty(\mathbb{R}^N \setminus B_{r/\varepsilon_n}(\widetilde{y}_n))} \leq \|u_n\|_{L^\infty(\mathbb{R}^N \setminus B_R(\widetilde{y}_n))} < \alpha$$

and

$$\|v_n\|_{L^\infty(\mathbb{R}^N \setminus \Lambda_{\varepsilon_n})} \leq \|v_n\|_{L^\infty(\mathbb{R}^N \setminus B_{r/\varepsilon_n}(\widetilde{y}_n))} \leq \|v_n\|_{L^\infty(\mathbb{R}^N \setminus B_R(\widetilde{y}_n))} < \alpha,$$

which contradicts (2.5.10) and proves the claim.

Considereing $0 < \varepsilon_\delta < \widetilde{\varepsilon}_\delta$, we shall prove the main theorem for this choice of ε_δ. Let $0 < \varepsilon < \varepsilon_\delta$ be fixed. By applying Theorem 2.4.1, we obtain $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions of the system $(S_{\varepsilon,aux})$. If $(u, v) \in X_\varepsilon$ is one of these solutions we have that $(u, v) \in \Sigma_\varepsilon$, and therefore we can use (2.5.9) and the definition of H to conclude that $H(\cdot, u, v) \equiv Q(u, v) + \frac{1}{2^*} K(u, v)$. Hence, (u, v) is also a solution of the system $(\widehat{S}_\varepsilon)$. An easy calculation shows that $(\widehat{u}(x), \widehat{v}(x)) := (u(x/\varepsilon), v(x/\varepsilon))$ is a solution of the original system (S_ε). Then, (S_ε) has at least $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions.

We now consider $\varepsilon_n \to 0^+$ and take a sequence $(u_n, v_n) \in X_{\varepsilon_n}$ of solutions of the system $(\widehat{S}_{\varepsilon_n})$ as above. By applying Lemma 2.5.1, we obtain $R > 0$ and $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that

$$\|u_n\|_{L^\infty(B_R(\widetilde{y}_n))^c} < \gamma \tag{2.5.11}$$

and

$$\|v_n\|_{L^\infty(B_R(\widetilde{y}_n))^c} < \gamma. \tag{2.5.12}$$

Up to a subsequence, we may also assume that

$$\|u_n\|_{L^\infty(B_R(\widetilde{y}_n))} \geq \gamma. \tag{2.5.13}$$

and

$$\|v_n\|_{L^\infty(B_R(\widetilde{y}_n))} \geq \gamma. \tag{2.5.14}$$

Indeed, if this is not the case, we have $\|u_n\|_{L^\infty(\mathbb{R}^N)} < \gamma$ or $\|v_n\|_{L^\infty(\mathbb{R}^N)} < \gamma$, which is a contradiction with (2.4.11). Thus (2.5.13) and (2.5.14) hold.

By using (2.5.13) and (2.5.14) we conclude that the maximum point $\pi_{n,a} \in \mathbb{R}^N$ of u_n and the maximum point $\pi_{n,b} \in \mathbb{R}^N$ of v_n belong to $B_R(\widetilde{y}_n)$. Hence $\pi_{n,a} = \widetilde{y}_n + q_{n,a}$, for some $q_{n,a} \in B_R(0)$ and $\pi_{n,b} = \widetilde{y}_n + q_{n,b}$, for some $q_{n,b} \in B_R(0)$. Recalling that the associated solution of (S_{ε_n}) is of the form $(\widehat{u}_n(x), \widehat{v}_n(x)) = (u_n(x/\varepsilon_n), v_n(x/\varepsilon_n))$, we conclude that the maximum point $\Pi_{\varepsilon_n,a}$ of $\widehat{u}_n$ and the maximum point $\Pi_{\varepsilon_n,b}$ of $\widehat{v}_n$ are $\Pi_{\varepsilon_n,a} := \varepsilon_n\widetilde{y}_n + \varepsilon_n q_{n,a}$ and $\Pi_{\varepsilon_n,b} := \varepsilon_n\widetilde{y}_n + \varepsilon_n q_{n,b}$. Since $(q_{n,a}), (q_{n,b}) \subset B_R(0)$ are bounded and $\varepsilon_n\widetilde{y}_n \to y_0 \in M$ (according to Proposition 2.4.11), we obtain

$$\lim_{n\to\infty} a(\Pi_{\varepsilon_n,a}) = a(y_0) = a_0$$

and

$$\lim_{n\to\infty} b(\Pi_{\varepsilon_n,b}) = b(y_0) = b_0.$$

Now we prove the regularity of the solution. Note that from Lema 2.5.1, (2.5.13) and (2.5.14), we have that $u_\varepsilon, v_\varepsilon \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. From interpolation inequality, we get $(u_\varepsilon, v_\varepsilon) \in L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$, $\forall\, q \geq 2$, that implies $Q_u(u_\varepsilon, v_\varepsilon) + \frac{1}{2^*}K_u(u_\varepsilon, v_\varepsilon)$, $Q_v(u_\varepsilon, v_\varepsilon) + \frac{1}{2^*}K_v(u_\varepsilon, v_\varepsilon) \in L^q(\mathbb{R}^N)$, $\forall\, q \geq 2$. From regularity elliptic theory, we get $(u_\varepsilon, v_\varepsilon) \in W^{2,q}(\mathbb{R}^N) \times W^{2,q}(\mathbb{R}^N)$, $\forall\, q \geq 2$. For q sufficiently large, we obtain $W^{2,q}(\mathbb{R}^N) \hookrightarrow C^{1,\lambda}(\mathbb{R}^N)$, for some $0 < \lambda < 1$. Then, $u_\varepsilon, v_\varepsilon \in C^{1,\lambda}(\mathbb{R}^N)$. Since $Q, K \in C^2(\mathbb{R}^N)$, we obtain that $u_\varepsilon, v_\varepsilon \in C^{2,\lambda}(\mathbb{R}^N)$, which concludes the proof of the theorem. □

Chapter 3

On concentration behavior and multiplicity of solutions for a system in $\mathbb{R}^N$

3.1 Introduction

In this chapter we will describe a result on the behavior asymptotic of the solutions of a system with two elliptic equations in the $\mathbb{R}^N$ involving a small parameter. More precisely, we study the system

$$\begin{cases} -\varepsilon^2 \text{div}(a(x)\nabla u) + u = Q_u(u,v) + \frac{\gamma}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N, \\ \quad -\varepsilon^2 \Delta v + b(x)v \quad = Q_v(u,v) + \frac{\gamma}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $2^* = 2N/(N-2)$, $N \geq 3$, $\varepsilon > 0$, a and b are positive continuous potentials, and Q and K are homogeneous function with K having critical growth.

In the first part of this chapter we are concerned with the existence, multiplicity and concentration of positive solutions for the following system given by

$$(S_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \text{div}(a(x)\nabla u) + u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ \quad -\varepsilon^2 \Delta v + b(x)v \quad = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, $N \geq 3$, $2^* = \frac{2N}{N-2}$ and a, b are continuous potentials.

The hypotheses on functions a and b are the following:

(ab_1) there are $a_0 > 0$ and $b_0 > 0$ such that

$$a_0 \leq a(x)$$

and

$$b_0 \leq b(x) \text{ for all } x \in \mathbb{R}^N;$$

(ab_2) there exists a bounded domain $\Lambda \subset \mathbb{R}^N$ such that

$$a_0 = \inf_{x \in \Lambda} a(x) < \inf_{x \in \partial\Lambda} a(x)$$

and

$$b_0 = \inf_{x \in \Lambda} b(x) < \inf_{x \in \partial\Lambda} b(x).$$

Setting $\mathbb{R}^2_+ := [0,\infty)\times[0,\infty)$, for any given $q \geq 1$ we denote by $\mathcal{H}^q$ the collection of all functions $F \in C^2(\mathbb{R}^2_+, \mathbb{R})$ satisfying the following properties:

($\mathcal{H}^q_0$) F is q-homogeneous; that is

$$F(\lambda s, \lambda t) = \lambda^q F(s,t), \quad \text{for each} \quad \lambda > 0 \ \text{ and } \ (s,t) \in \mathbb{R}^2_+;$$

($\mathcal{H}^q_1$) there exists $c_1 > 0$ such that

$$|F_s(s,t)| + |F_t(s,t)| \leq c_1 \left(s^{q-1} + t^{q-1}\right) \quad \text{for each } (s,t) \in \mathbb{R}^2_+;$$

($\mathcal{H}_2$) $F(s,t) > 0$ for each $s, t > 0$;

($\mathcal{H}_3$) $\nabla F(1,0) = \nabla F(0,1) = (0,0)$;

($\mathcal{H}_4$) $F_s(s,t), F_t(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}^2_+$.

We relate the number of solutions of (S_ε) with the topology of the set of minima of the potentials a and b. In order to present our result we introduce the following set:

$$M = \{x \in \mathbb{R}^N : a(x) = a_0 \ \text{ and } \ b(x) = b_0\}.$$

We recall that, if Y is a closed set of a topological space X, $\text{cat}_X(Y)$ is the Ljusternik-Schnirelmann category of Y in X, namely the least number of closed and contractible set in X which cover Y. We denote by

$$M_\delta := \{x \in \mathbb{R}^N : \text{dist}(x, M) < \delta\} \subset \Lambda,$$

the closed δ-neighborhood of M, and we shall prove the following result.

Theorem 3. *Suppose that a and b are continuous potentials satisfying $(ab_1) - (ab_2)$ and $M \neq \emptyset$. Suppose also that $Q \in \mathcal{H}^p$ for any $2 < p < 2^*$. Then,*

(i) for all $\varepsilon > 0$, the system (S_ε) has a positive ground state solution.

(ii) for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system (S_ε) has at least $\text{cat}_{M_\delta}(M)$ positive solutions.

(iii) if $(u_\varepsilon, v_\varepsilon)$ is a solution for (S_ε) and if $\Pi_{\varepsilon,a}$ and $\Pi_{\varepsilon,b}$ are maximum points of u_ε and v_ε respectively, then $\Pi_{\varepsilon,a}, \Pi_{\varepsilon,b} \in \Lambda$, $\lim\limits_{\varepsilon\to 0^+} a(\Pi_{\varepsilon,a}) = a_0$ and $\lim\limits_{\varepsilon\to 0^+} b(\Pi_\varepsilon) = b_0$, furthermore, each solution $(u_\varepsilon, v_\varepsilon) \in C^{2,\lambda}(\mathbb{R}^N)$, for some $\lambda \in (0,1)$.

In the second part of the chapter we deal with a critical version of (S_ε), namely the system

$$(CS_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \text{div}(a(x)\nabla u) + u = Q_u(u,v) + \frac{1}{2^*} K_u(u,v) \text{ in } \mathbb{R}^N, \\ \quad -\varepsilon^2 \Delta v + b(x) v \ = Q_v(u,v) + \frac{1}{2^*} K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In order to deal with the critical growth of the nonlinearity we assume the following hypotheses on the functions Q and K:

(A_1) $K \in \mathcal{H}^{2^*}$ and $Q \in \mathcal{H}^p$ for some $2 < p < 2^*$;

(A_2) the 1-homogeneous function $G : \mathbb{R}^2_+ \to \mathbb{R}$ given by $G(s^{2^*}, t^{2^*}) := K(s,t)$ is concave;

(A_3)

$$Q(s,t) \geq \frac{\sigma}{p_1} s^{\lambda} t^{\beta}, \quad \text{for all} \quad (s,t) \in \mathbb{R}^2_+,$$

where $\lambda, \beta > 1$, $\lambda + \beta =: p_1 \in (2, 2^*)$ and

$$\sigma > \sigma^* := \left(\frac{C(a_0, b_0)}{\frac{1}{N}\left(\min\{a_0, 1\}\widetilde{S}_K\right)^{N/2}} \right)^{\frac{p_1-2}{2}}.$$

The hypothesis (A_2) appeared in the first time in [19] and will be used in Proposition 3.6.2. The constants that define σ^* will appear naturally in Proposition 3.6.2.

The critical version of Theorem 3 can be stated as follows.

Theorem 4. *Suppose that a and b are continuous potentials satisfying $(ab_1) - (ab_2)$ and $M \neq \emptyset$. Suppose also that Q and K satisfy $(A_1) - (A_3)$. Then,*

- (i) *there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ the system (CS_ε) has a positive ground state solution.*
- (ii) *for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system (CS_ε) has at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions.*
- (iii) *if $(u_\varepsilon, v_\varepsilon)$ is a solution for (CS_ε) and if $\Pi_{\varepsilon,a}$ and $\Pi_{\varepsilon,b}$ are maximum points of u_ε and v_ε respectively, then $\Pi_{\varepsilon,a}, \Pi_{\varepsilon,b} \in \Lambda$, $\lim\limits_{\varepsilon \to 0^+} a(\Pi_{\varepsilon,a}) = a_0$ and $\lim\limits_{\varepsilon \to 0^+} b(\Pi_\varepsilon) = b_0$, furthermore, each solution $(u_\varepsilon, v_\varepsilon) \in C^{2,\lambda}(\mathbb{R}^N)$, for some $\lambda \in (0,1)$.*

The chapter is organized as follows. In order to overcome the lack of compactness, in section 3.2 we make a penalization of the nonlinearity using arguments that can be found in [1]. In section 3.3 we show existence of solution for the auxiliary system introduced in section 3.2. In section 3.4 we obtain uniform estimates in order to show that the solution of the auxiliary system is a solution of the original system. The proof of the main result in the subcritical case is in section 3.5. The critical case is studied in section 3.6.

3.2 Variational framework and a modified system

Since we are interested in positive solutions we extend the function Q and K to the whole $\mathbb{R}^2$ by setting $Q(u,v) = K(u,v) = 0$ if $u \leq 0$ or $v \leq 0$. We also note that for any function $F \in \mathcal{H}^q$, we can use the homogeneity condition $(\mathcal{H}_0^q)$ to conclude that

$$qF(s,t) = sF_s(s,t) + tF_t(s,t) \tag{3.2.1}$$

and

$$q(q-1)F(s,t) = s^2 F_{ss}(s,t) + t^2 F_{tt}(s,t) + 2stF_{st}(s,t) \tag{3.2.2}$$

for any $(s,t) \in \mathbb{R}^2$.

Hereafter, we will work with the following system equivalent to (S_ε).

$$(\widehat{S}_\varepsilon) \qquad \begin{cases} -\operatorname{div}(a(\varepsilon x)\nabla u) + u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ \quad -\Delta v + b(\varepsilon x)v \ \ = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In order to overcome the lack of compactness originated by the unboundedness of $\mathbb{R}^N$ we use a penalization method. Such kind of idea has first appeared in the paper of Del Pino and Felmer [20]. Here we use an adaptation of this method for systems, as introduced in [1].

We start by choosing $\alpha > 0$ and considering $\eta : \mathbb{R} \to \mathbb{R}$ a non-increasing function of class C^2 such that

$$\eta \equiv 1 \text{ on } (-\infty, \alpha], \quad \eta \equiv 0 \text{ on } [5\alpha, +\infty), \quad |\eta'(s)| \le \frac{C}{\alpha} \text{ and } |\eta''(s)| \le \frac{C}{\alpha^2} \tag{3.2.3}$$

for each $s \in \mathbb{R}$ and for some positive constant $C > 0$. Using the function η, we define $\widehat{Q} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\widehat{Q}(s,t) := \eta(|(s,t)|)Q(s,t) + (1 - \eta(|(s,t)|))A(s^2 + t^2)$$

where

$$A := \max\left\{ \frac{Q(s,t)}{s^2 + t^2} : (s,t) \in \mathbb{R}^2, \ \alpha \le |(s,t)| \le 5\alpha \right\}.$$

Notice that, since $A > 0$ tends to zero as $\alpha \to 0^+$, we may suppose that $A \in (0, \mu/4)$ where $\mu = \max\{1, 1/b_0\}^{-1}$.

Finally, denoting by I_Λ the characteristic function of the set Λ, we define $H : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ by setting

$$H(x,s,t) := I_\Lambda(x)Q(s,t) + (1 - I_\Lambda(x))\widehat{Q}(s,t). \tag{3.2.4}$$

For any $\alpha > 0$ small and $(s,t) \in \mathbb{R}^2$ we have the following result.

Lemma 3.2.1. *The function H satisfies the following estimates:*

(H_1) $pH(x,s,t) = sH_s(x,s,t) + tH_t(x,s,t)$, for each $x \in \Lambda$;

(H_2) $2H(x,s,t) \le sH_s(x,s,t) + tH_t(x,s,t)$, for each $x \in \mathbb{R}^N \setminus \Lambda$;

(H_3) for α small we have $sH_s(x,s,t) + tH_t(x,s,t) \le \frac{1}{4}\left(s^2 + b(x)t^2\right)$ for each $x \in \mathbb{R}^N \setminus \Lambda$;

(H_4) for α small we have $\frac{|H_s(x,s,t)|}{\alpha}, \frac{|H_t(x,s,t)|}{\alpha} \le \frac{\mu}{4}$ for each $x \in \mathbb{R}^N \setminus \Lambda$.

Proof. Since $H(x,s,t) = Q(s,t)$ on the set Λ, we can use (3.2.1) to get

$$pH(x,s,t) = sH_s(x,s,t) + tH_t(x,s,t)$$

for all $x \in \Lambda$. This proves (H_1).

In what follows we denote $|z| := \sqrt{s^2 + t^2}$. Notice that $H(x,s,t) = \widehat{Q}(s,t)$ for all $x \in \mathbb{R}^N \setminus \Lambda$, consequently

$$H_s = \eta' \frac{s}{|z|} Q + \eta Q_s - \eta' \frac{s}{|z|} A(s^2 + t^2) + 2A(1 - \eta)s$$

and

$$H_t = \eta' \frac{t}{|z|} Q + \eta Q_t - \eta' \frac{t}{|z|} A(s^2 + t^2) + 2A(1 - \eta)t$$

so,

$$sH_s + tH_t = \eta'|z| \left[Q - A(s^2 + t^2)\right] + \eta \left[sQ_s + tQ_t\right] + 2A(1 - \eta)(s^2 + t^2) \tag{3.2.5}$$

Notice that, in view of the definition of A, we have that

$$Q(s,t) - A|z|^2 \le 0,$$

for all x belonging in the support of η'. Hence recalling that $\eta' \le 0$, we can use the above estimate, (3.2.5), (3.2.1) and the fact $2 < p < 2^*$, to obtain

$$
\begin{aligned}
sH_s + tH_t &\ge p\eta Q + 2A(1-\eta)|z|^2 \\
&\ge 2\left[\eta Q + A(1-\eta)|z|^2\right] = 2H
\end{aligned}
$$

for all $x \in \mathbb{R}^N \setminus \Lambda$. Thus, (H_2) holds.

Since η is smooth and supp $\eta' \subset [\alpha, 5\alpha]$, we can use (3.2.5), $(\mathcal{H}_1^p)$, (3.2.3) and the definition of A, to get

$$
\begin{aligned}
\frac{sH_s + tH_t}{s^2+t^2} &= \eta'|z|\left[\frac{Q(s,t)}{s^2+t^2} - A\right] + \frac{\eta}{|z|^2}\left[sQ_s + tQ_t\right] + 2A(1-\eta) \\
&\le \eta'|z|\left[\frac{Q(s,t)}{s^2+t^2} - A\right] + 2c_1\frac{\eta}{|z|^2}(s^p + t^p) + 2A(1-\eta) \\
&\le |\eta'||z|\left|\frac{Q(s,t)}{s^2+t^2} - A\right| + 4c_1|z|^{p-2} + 4A \\
&\le \frac{C}{\alpha}\cdot 5\alpha \cdot 2A + 4c_1(5\alpha)^{p-2} + 4A.
\end{aligned}
$$

Then, for α sufficiently small we have that

$$
sH_s + tH_t \le \frac{\mu}{4}(s^2+t^2).
$$

Thus, for this choice of α, we can use the above estimate and (ab_1) to obtain, for each $x \in \mathbb{R}^N \setminus \Lambda$,

$$
sH_s + tH_t \le \frac{1}{4}\left(s^2 + b(x)t^2\right)
$$

showing that (H_3) holds.

Since $H(x,s,t) = \widehat{Q}(s,t)$ for all $x \in \mathbb{R}^N \setminus \Lambda$, from definition of $\widehat{Q}$, supp $\eta' \subset [\alpha, 5\alpha]$, $\mathcal{H}_1^p$ and (3.2.3)

$$
\begin{aligned}
|H_s(x,s,t)| &= \left|\eta'\frac{s}{|z|}Q + \eta Q_s - \eta'\frac{s}{|z|}A(s^2+t^2) + 2A(1-\eta)s\right| \\
&\le |\eta'|\frac{Q(s,t)}{s^2+t^2}|z|^2 + |\eta|c_1(s^{p-1} + t^{p-1}) + |\eta'|A|z|^2 + 4A|z| \\
&\le |\eta'|\frac{Q(s,t)}{s^2+t^2}|z|^2 + |\eta|2c_1|z|^{p-1} + |\eta'|A|z|^2 + 4A|z| \\
&\le \frac{C}{\alpha}\cdot A\cdot 25\alpha^2 + 2c_1(5\alpha)^{p-1} + \frac{C}{\alpha}\cdot A \cdot 25\alpha^2 + 20\alpha A.
\end{aligned}
$$

Then, for α sufficiently small we have that

$$
\frac{|H_s(x,s,t)|}{\alpha} \le \frac{\mu}{4}.
$$

Using similar arguments, it is possible to prove that

$$
\frac{|H_t(x,s,t)|}{\alpha} \le \frac{\mu}{4}
$$

proving (H_4). □

In view by definition (3.2.4), we deal in the sequel with the modified system

$$(S_{\varepsilon,aux}) \qquad \begin{cases} -\mathrm{div}(a(\varepsilon x)\nabla u) + u = H_u(\varepsilon x, u, v) \text{ in } \mathbb{R}^N, \\ \quad -\Delta v + b(\varepsilon x)v \quad = H_v(\varepsilon x, u, v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

and we will look for solutions $(u_\varepsilon, v_\varepsilon)$ verifying

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \ \text{ for each } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon,$$

where $\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$.

For each $\varepsilon > 0$ we denote by X_ε the Hilbert space

$$X_\varepsilon := \left\{(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|v|^2\right) dx < \infty\right\}$$

endowed with the norm

$$\|(u, v)\|_\varepsilon^2 := \int_{\mathbb{R}^N} \left[a(\varepsilon x)|\nabla u|^2 + |\nabla v|^2 + |u|^2 + b(\varepsilon x)|v|^2\right] dx.$$

Conditions (H_3) and $(\mathcal{H}_1^p)$ imply that the critical points of the C^1-functional $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$J_\varepsilon(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} \left[a(\varepsilon x)|\nabla u|^2 + |\nabla v|^2 + |u|^2 + b(\varepsilon x)|v|^2\right] dx - \int_{\mathbb{R}^N} H(\varepsilon x, u, v) dx$$

are weak solutions of $(S_{\varepsilon,aux})$. We recall that these critical points belong to the Nehari manifold of J_ε, namely on the set

$$\mathcal{N}_\varepsilon := \left\{(u, v) \in X_\varepsilon \setminus \{(0, 0)\} : J_\varepsilon'(u, v)(u, v) = 0\right\}.$$

It is well known that, for any nontrivial element $(u, v) \in X_\varepsilon$ the function $t \mapsto J_\varepsilon(tu, tv)$, for $t \geq 0$, achieves its maximum value at a unique point $t_{u,v}(u, v) \in \mathcal{N}_\varepsilon$. We define the number b_ε by setting

$$b_\varepsilon := \inf_{(u,v) \in \mathcal{N}_\varepsilon} J_\varepsilon(u, v). \tag{3.2.6}$$

3.3 Existence of a ground state solution for the modified system $(S_{\varepsilon,aux})$

We start defining the Palais-Smale compactness condition. A sequence $((u_n, v_n)) \subset X_\varepsilon$ is a Palais-Smale sequence at level c_ε for the functional J_ε if

$$J_\varepsilon(u_n, v_n) \to c_\varepsilon$$

and

$$\|J_\varepsilon'(u_n, v_n)\| \to 0 \quad \text{in} \quad (X_\varepsilon)',$$

where

$$c_\varepsilon = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} J_\varepsilon(\eta(t)) > 0$$

and

$$\Gamma := \{\eta \in C([0, 1], X_\varepsilon) : \eta(0) = (0, 0),\ J_\varepsilon(\eta(1)) < 0\}.$$

If every Palais-Smale sequence of J_ε has a strong convergent subsequence, then one says that J_ε satisfies the Palais-Smale condition ((PS) for short).

In order to show existence of a ground state solution for the modified system $(S_{\varepsilon,aux})$, we use the Mountain Pass Theorem [6].

Lemma 3.3.1. *The functional J_ε satisfies the following conditions*

(i) There exists $C, \rho > 0$, such that

$$J_\varepsilon(u,v) \geq C, \quad \textit{if} \quad \|(u,v)\|_\varepsilon = \rho.$$

(ii) For any $(\phi,\psi) \in C_0^\infty(\Lambda_\varepsilon) \times C_0^\infty(\Lambda_\varepsilon)$ with $\phi, \psi \geq 0$, we have

$$\lim_{t\to+\infty} J_\varepsilon(t\phi, t\psi) = -\infty.$$

Proof. By using Lemma 3.2.1 and $(\mathcal{H}_1^p)$, we have

$$J_\varepsilon(u,v) \geq \frac{1}{2}\|(u,v)\|_\varepsilon^2 - \frac{2c_1}{p}\int_{\Lambda_\varepsilon} (|u|^p + |v|^p)\, dx - \frac{1}{8}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left(|u|^2 + b(\varepsilon x)|v|^2\right) dx.$$

By Sobolev embeddings, there exists $C > 0$ such that

$$J_\varepsilon(u,v) \geq \frac{3}{8}\|(u,v)\|_\varepsilon^2 - \frac{C}{p}\|(u,v)\|_\varepsilon^p$$

and the proof of item (i) is over. Now, by definition of H and $(\mathcal{H}_0^p)$, we get

$$J_\varepsilon(t\phi, t\psi) = \frac{t^2}{2}\|(\phi,\psi)\|_\varepsilon^2 - t^p \int_{\Lambda_\varepsilon} Q(\phi,\psi) dx,$$

and the proof of item (ii) is over. □

Hence, there exists a Palais-Smale sequence $((u_n, v_n)) \subset X_\varepsilon$ at level c_ε. Using $(\mathcal{H}_0^p)$, it is possible to prove that

$$c_\varepsilon = b_\varepsilon = \inf_{(u,v)\in X_\varepsilon\setminus\{(0,0)\}} \sup_{t\geq 0} J_\varepsilon(tu, tv) \tag{3.3.1}$$

where b_ε was defined in (3.2.6).

In order to prove the Palais-Smale condition, we need to prove the next lemma.

Lemma 3.3.2. *Let $((u_n, v_n))$ be a $(PS)_d$ sequence for J_ε. Then for each $\xi > 0$, there exists $R = R(\xi)$ such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} \left[a(\varepsilon x)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon x)|v_n|^2\right] dx < \xi$$

Proof. Let $\eta_R \in C^\infty(\mathbb{R}^N)$ such that $\eta_R(x) = 0$ if $x \in B_{R/2}(0)$ and $\eta_R(x) = 1$ if $x \notin B_R(0)$, with $0 \leq \eta_R(x) \leq 1$ and $|\nabla \eta_R| \leq \frac{C}{R}$, where C is constant independent of R. Since that the sequence $((u_n\eta_R, v_n\eta_R))$ is bounded in X_ε, fixing $R > 0$ such that $\Lambda_\varepsilon \subset B_{R/2}(0)$ and by definition of the functional J_ε, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[a(\varepsilon x)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon x)|v_n|^2\right] \eta_R dx \\
= \ & J_\varepsilon'(u_n, v_n)(u_n\eta_R, v_n\eta_R) + \int_{\mathbb{R}^N} \left[u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)\right] \eta_R dx \\
&- \int_{\mathbb{R}^N} \left[a(\varepsilon x) u_n \nabla u_n + v_n \nabla v_n\right] \nabla \eta_R dx.
\end{aligned}$$

Using (H_3), we get the estimate

$$\begin{aligned} &\frac{3}{4}\int_{\mathbb{R}^N\setminus B_R(0)} \left[a(\varepsilon x)|\nabla u_n|^2+|\nabla v_n|^2+|u_n|^2+b(\varepsilon x)|v_n|^2\right]dx \\ \le &\int_{\mathbb{R}^N}\Big[a(\varepsilon x)|u_n||\nabla u_n|+|v_n||\nabla v_n|\Big]|\nabla\eta_R|dx+o_n(1). \end{aligned}$$

Since $((u_n,v_n))$ is bounded in X_ε and $|\nabla\eta_R|\le\frac{C}{R}$, exists $C_1>0$ such that

$$\int_{\mathbb{R}^N\setminus B_R(0)}\Big[a(\varepsilon x)|\nabla u_n|^2+|\nabla v_n|^2+|u_n|^2+b(\varepsilon x)|v_n|^2\Big]dx\le\frac{C_1}{R}+o_n(1)$$

proving the lemma. □

Lemma 3.3.3. *The functional J_ε satisfies the Palais-Smale condition at any level c.*

Proof. Let $((u_n,v_n))\subset X_\varepsilon$ such that $J_\varepsilon(u_n,v_n)\to c$ and $J'_\varepsilon(u_n,v_n)\to 0$. Standart calculations show that $((u_n,v_n))$ is bounded in X_ε. Then, up to a subsequence, we may suppose that,

$$\begin{array}{lll} (u_n,v_n)\rightharpoonup(u,v) & \text{weakly in} & X_\varepsilon, \\ u_n\to u,\ v_n\to v & \text{strongly in} & L^s_{\text{loc}}(\mathbb{R}^N),\quad \text{for any}\quad 2\le s<2^*, \\ u_n(x)\to u(x),\ v_n(x)\to v(x) & \text{for a.e.} & x\in\mathbb{R}^N. \end{array} \tag{3.3.2}$$

Now using a density argument, we can conclude that (u,v) is a criticl point of J_ε. Hence

$$\|(u,v)\|^2_\varepsilon=\int_{\mathbb{R}^N}\left[uH_u(\varepsilon x,u,v)+vH_v(\varepsilon x,u,v)\right]dx. \tag{3.3.3}$$

On the other hand, we have

$$\|(u_n,v_n)\|^2_\varepsilon=\int_{\mathbb{R}^N}\left[u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)\right]dx+o_n(1). \tag{3.3.4}$$

From Lemma 3.3.2, for any $\xi>0$ given, there exists $R>0$ such that $\Lambda_\varepsilon\subset B_R(0)$ and

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R(0)}\left[a(\varepsilon x)|\nabla u_n|^2+|\nabla v_n|^2+|u_n|^2+b(\varepsilon x)|v_n|^2\right]dx<\xi.$$

This inequality, (H_3) and the Sobolev embedding imply that, for n large enough, there holds

$$\int_{\mathbb{R}^N\setminus B_R(0)}\left[u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)\right]dx\le C_1\frac{1}{4}\xi \tag{3.3.5}$$

where C_1 is positive constant. On the other hand, taking R large enough, we can suppose that

$$\left|\int_{\mathbb{R}^N\setminus B_R(0)}\left[uH_u(\varepsilon x,u,v)+vH_v(\varepsilon x,u,v)\right]dx\right|<\xi. \tag{3.3.6}$$

Then, by (3.3.5) and (3.3.6), we conclude that

$$\begin{aligned} &\int_{\mathbb{R}^N\setminus B_R(0)}\left[u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)\right]dx \\ =&\int_{\mathbb{R}^N\setminus B_R(0)}\left[uH_u(\varepsilon x,u,v)+vH_v(\varepsilon x,u,v)\right]dx+o_n(1). \end{aligned} \tag{3.3.7}$$

Since the set $B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)$ is bounded, we can use (H_3), (3.3.2) and Lebesgue's theorem to conclude that

$$\begin{aligned}&\lim_{n\to\infty}\int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)}[u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)]\,dx\\ =&\int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)}[uH_u(\varepsilon x,u,v)+vH_v(\varepsilon x,u,v)]\,dx. \end{aligned}\tag{3.3.8}$$

Using $(\mathcal{H}_1^p)$, (3.3.2) and Lebesgue's theorem again, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\int_{\Lambda_\varepsilon}[u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)]\,dx\\ =&\int_{\Lambda_\varepsilon}[uH_u(\varepsilon x,u,v)+vH_v(\varepsilon x,u,v)]\,dx. \end{aligned}\tag{3.3.9}$$

From (3.3.7), (3.3.8) and (3.3.9) we get

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}[u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)]\,dx\\ =&\int_{\mathbb{R}^N}[uH_u(\varepsilon x,u,v)+vH_v(\varepsilon x,u,v)]\,dx. \end{aligned}$$

This, (3.3.3) and (3.3.4) implies that $\|(u_n,v_n)\|_\varepsilon^2\to\|(u,v)\|_\varepsilon^2$. Then $(u_n,v_n)\to(u,v)$ in X_ε. □

3.4 Multiple solutions for the modified system $(S_{\varepsilon,aux})$

In order to prove the multiplicity result, we consider the following autonomous system associated to (S_0), namely

$$(S_0)\qquad\begin{cases}-a_0\Delta u+u=Q_u(u,v) \text{ in } \mathbb{R}^N,\\ -\Delta v+b_0v\ =Q_v(u,v) \text{ in } \mathbb{R}^N,\\ u,v\in H^1(\mathbb{R}^N),\quad u(x),v(x)>0 \text{ for each } x\in\mathbb{R}^N.\end{cases}$$

In view of conditions (ab_1) and $(\mathcal{H}_1^p)$, the above system has a variational structure and the associated functional is given by

$$I_0(u,v):=\frac{1}{2}\int_{\mathbb{R}^N}\left[a_0|\nabla u|^2+|\nabla v|^2+|u|^2+b_0|v|^2\right]dx-\int_{\mathbb{R}^N}Q(u,v)dx$$

is well defined for $(u,v)\in E_0:=H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. We denote the norm in E_0 by

$$\|(u,v)\|^2:=\int_{\mathbb{R}^N}\left[a_0|\nabla u|^2+|\nabla v|^2+|u|^2+b_0|v|^2\right]dx.$$

We can show that I_0 has the Mountain Pass geometry and therefore we can set the minimax level c_0 in the following way

$$c_0:=\inf_{\eta\in\Gamma}\max_{t\in[0,1]}I_0(\gamma(t)),$$

where $\Gamma:=\{\gamma\in C([0,1],E_0):\gamma(0)=(0,0),\ I_0(\gamma(1))<0\}$. Moreover, c_0 can be further characterized as

$$c_0=\inf_{(u,v)\in\mathcal{M}_0}I_0(u,v)\tag{3.4.1}$$

with $\mathcal{M}_0$ being the Nehari manifold of I_0, that is

$$\mathcal{M}_0:=\left\{(u,v)\in E_0\setminus\{(0,0)\}:I_0'(u,v)(u,v)=0\right\}.$$

Lemma 3.4.1. *Let $((u_n, v_n)) \subset \mathcal{M}_0$ be a sequence such that $I_0(u_n, v_n) \to c_0$. Then there are a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R, \eta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} \left(|u_n|^2 + |v_n|^2\right) dx \geq \eta. \tag{3.4.2}$$

Proof. Suppose that (3.4.2) is not satisfied. Since $((u_n, v_n))$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, then, from [32, Lemma 1.1], we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^s dx = 0$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^s dx = 0,$$

for all $s \in (2, 2^*)$. Thus, from $(\mathcal{H}_1^p)$, we conclude

$$\int_{\mathbb{R}^N} [u_n Q_u(u_n, v_n) + v_n Q_v(u_n, v_n)]\, dx = o_n(1).$$

Since $I_0'(u_n, v_n)(u_n, v_n) = 0$, we obtain $\|(u_n, v_n)\| = o_n(1)$, which implies $c_0 = 0$, which is a contradiction. □

The next result allows to show that system (S_0) has a solution that reaches c_0.

Lemma 3.4.2. *(A Compactness Lemma) Let $((u_n, v_n)) \subset \mathcal{M}_0$ be a sequence satisfying $I_0(u_n, v_n) \to c_0$. Then, there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $(w_n(x), z_n(x)) = (u_n(x + \widetilde{y}_n), v_n(x + \widetilde{y}_n))$ converges strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. In particular, there exists a minimizer for c_0.*

Proof. Applying Ekeland's Variational Principle [40, Theorem 8.5], we may suppose that $((u_n, v_n))$ is a $(PS)_{c_0}$ for I_0. Since $((u_n, v_n))$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, we have that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$.

Then, $\|(u, v)\|^2 \leq \liminf_{n\to\infty} \|(u_n, v_n)\|^2$. We are going to prove that

$$\|(u, v)\|^2 = \lim_{n\to\infty} \|(u_n, v_n)\|^2. \tag{3.4.3}$$

Suppose, by contradiction, that (3.4.3) does not hold. Then, by $(\mathcal{H}_3)$, we can consider $(u, v) \neq (0, 0)$, using a density argument we have that $I_0'(u, v)(u, v) = 0$, where we conclude that $(u, v) \in \mathcal{M}_0$. Using (3.2.1), we obtain

$$\begin{aligned}
c_0 \leq I_0(u, v) &= I_0(u, v) - \frac{1}{p} I_0'(u, v)(u, v) \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) \|(u, v)\|^2 \\
&< \left(\frac{1}{2} - \frac{1}{p}\right) \liminf_{n\to\infty} \|(u_n, v_n)\|^2 \\
&= \liminf_{n\to\infty} \left[I_0(u_n, v_n) - \frac{1}{p} I_0'(u_n, v_n)(u_n, v_n)\right] = c_0
\end{aligned}$$

which is a contradiction. Hence, $(u_n, v_n) \to (u, v)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Consequently, $I_0(u, v) = c_0$ and the sequence $(\widetilde{y}_n)$ is the sequence null.

If $(u,v) \equiv (0,0)$, then in this case we cannot have $(u_n,v_n) \to (u,v)$ strongly in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ because $c_0>0$. Hence, using the Lemma 3.4.1, there exists a sequence $(\widetilde{y}_n)\subset \mathbb{R}^N$ such that

$$(w_n,z_n) \rightharpoonup (w,z) \quad \text{in} \quad H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N),$$

where $w_n(x)=u_n(x+\widetilde{y}_n)$ and $z_n(x)=v_n(x+\widetilde{y}_n)$. Therefore, $((w_n,z_n))$ is also $(PS)_{c_0}$ sequence of I_0 and $(w,z)\not\equiv(0,0)$. It follows form above arguments that, up to a subsequence, $((w_n,z_n))$ converges strongly in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ and the proof of lemma is over. □

In order to prove the multiplicity results, we need of the following the abstract results that involve category theory.

Theorem 3.4.3. *Let I be a C^1-functional defined on a C^1-Finsler manifold $\mathcal{V}$. If I is bounded from below and satisfies the Palais-Smale condition, then I has at least $\operatorname{cat}_{\mathcal{V}}(\mathcal{V})$ distinct critical points.*

The following result, which has a proof similar to that presented in [10, Lemma 4.3], will be used.

Lemma 3.4.4. *Let Γ, Ω^+, Ω^- be closed sets with $\Omega^-\subset\Omega^+$. Let $\beta:\Gamma\to\Omega^+$, $\Phi:\Omega^-\to\Gamma$ be two continuous maps such that $\beta\circ\Phi$ is homotopically equivalent to the embedding $\iota:\Omega^-\to\Omega^+$. Then $\operatorname{cat}_\Gamma(\Gamma)\geq\operatorname{cat}_{\Omega^+}(\Omega^-)$.*

3.4.1 The Palais-Smale condition in the Nehari manifold associated to J_ε

From Lemma 3.3.3, the unconstrained functional satisfies $(PS)_c$ for each $c\in\mathbb{R}$. Nevertheless, to get multiple critical points, we need to work with the functional J_ε constrained to $\mathcal{N}_\varepsilon$. We denote by $\|J_\varepsilon'(u)\|_*$ the norm of the derivative of J_ε restricted to $\mathcal{N}_\varepsilon$ at the point u. In order to prove the desired compactness result we shall first present some properties of $\mathcal{N}_\varepsilon$, which the proofs of the next three results follows by using the same arguments employed in [3, Lemma 2.2, Lemma 2.3 and Proposition 2.4] for other class of system. For the sake of completeness, we sketch here.

Lemma 3.4.5. *There exist positive constants α_1,δ_1,C such that, for each $\alpha\in(0,\alpha_1)$, $(u,v)\in\mathcal{N}_\varepsilon$, there hold*

$$\int_{\Lambda_\varepsilon} Q(u,v)dx \geq \delta_1 \tag{3.4.4}$$

and

$$\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}\left(u^2+b(\varepsilon x)v^2\right)dx \leq C\int_{\Lambda_\varepsilon} Q(u,v)dx. \tag{3.4.5}$$

Proof. Since H has subcritical growth, it is easy to obtain $\widehat{\delta}>0$ such that

$$\|(u,v)\|_\varepsilon \geq \widehat{\delta} \quad \text{for each} \quad (u,v)\in\mathcal{N}_\varepsilon.$$

Thus, we can use (3.2.1) and (H_3) to get

$$\begin{aligned}\widehat{\delta}^2 \leq \|(u,v)\|_\varepsilon^2 &\leq \int_{\Lambda_\varepsilon}\left[uQ_u(u,v)+vQ_v(u,v)\right]dx+\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}\left[uH_u+vH_v\right]dx\\ &\leq p\int_{\Lambda_\varepsilon}Q(u,v)dx+\frac{1}{4}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(u^2+b(\varepsilon x)v^2)dx\end{aligned}$$

and therefore

$$\frac{3}{4}\widehat{\delta}^2 \leq \frac{3}{4}\|(u,v)\|_\varepsilon^2 \leq p\int_{\Lambda_\varepsilon}Q(u,v)dx$$

which implies (3.4.4) with $\delta_1 = \frac{3\widehat{\delta}^2}{4p}$.

By using (3.2.1) and (H_3) again, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left(u^2 + b(\varepsilon x)v^2\right) dx &\leq \|(u,v)\|_\varepsilon^2 \\ &\leq p\int_{\Lambda_\varepsilon} Q(u,v)dx + \frac{1}{4}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left(u^2 + b(\varepsilon x)v^2\right) dx\end{aligned}$$

from which follows (3.4.5). The lemma is proved. □

The following technical results is the key stone in our compactness result.

Lemma 3.4.6. *Let $\phi_\varepsilon : X_\varepsilon \to \mathbb{R}$ be given by*

$$\phi_\varepsilon(u,v) := \|(u,v)\|_\varepsilon^2 - \int_{\mathbb{R}^N} \left[uH_u(\varepsilon x,u,v) + vH_v(\varepsilon x,u,v)\right] dx.$$

Then there exist $\alpha_2, \widetilde{M} > 0$ such that, for each $\alpha \in (0,\alpha_2)$,

$$\phi_\varepsilon'(u,v)(u,v) \leq -\widetilde{M} < 0 \quad \text{for each} \quad (u,v) \in \mathcal{N}_\varepsilon. \tag{3.4.6}$$

Proof. Given $(u,v) \in \mathcal{N}_\varepsilon$, we can use the definition of H, (3.2.1) and (3.2.2) to get

$$\begin{aligned}\phi_\varepsilon'(u,v)(u,v) &= \int_{\Lambda_\varepsilon} \left[(uQ_u + vQ_v) - (u^2Q_{uu} + v^2Q_{vv} + 2uvQ_{uv})\right] dx \\ &\quad + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left[uH_u + vH_v\right] dx - \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left[u^2H_{uu} + v^2H_{vv} + 2uvH_{uv}\right] dx \\ &= -p(p-2)\int_{\Lambda_\varepsilon} Q(u,v)dx + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left[D_1 - D_2\right] dx\end{aligned}$$

with

$$D_1 := uH_u + vH_v \quad \text{and} \quad D_2 := u^2H_{uu} + v^2H_{vv} + 2uvH_{uv}.$$

Arguing as in the proof of [3, Lemma 2.3], and using (ab_1), we have

$$\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left[D_1 - D_2\right] dx \leq o(1)\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} (u^2+v^2)dx \leq o(1)\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} (u^2 + b(\varepsilon x)v^2)dx$$

where $o(1) \to 0$ as $\alpha \to 0^+$.

Now we can use Lemma 3.4.5 to obtain, for α small enough

$$\phi_\varepsilon'(u,v)(u,v) \leq \left[-p(p-2) + o(1)\right]\int_{\Lambda_\varepsilon} Q(u,v)dx \leq -\frac{p(p-2)}{2}\delta_1 = -\widetilde{M} < 0.$$

The lemma is proved. □

Proposition 3.4.7. *The functional J_ε restricted to $\mathcal{N}_\varepsilon$ satisfies $(PS)_c$ for each $c \in \mathbb{R}$.*

Proof. Let $((u_n,v_n)) \subset \mathcal{N}_\varepsilon$ be such that

$$J_\varepsilon(u_n,v_n) \to c \quad \text{and} \quad \|J_\varepsilon'(u_n,v_n)\|_* = o_n(1),$$

where $o_n(1)$ approaches zero as $n \to \infty$. Then there exists $(\lambda_n) \subset \mathbb{R}$ satisfying

$$J_\varepsilon'(u_n,v_n) = \lambda_n\phi_\varepsilon'(u_n,v_n) + o_n(1) \tag{3.4.7}$$

with ϕ_ε as in Lemma 3.4.6. Since $(u_n, v_n) \in \mathcal{N}_\varepsilon$ we have that

$$0 = J'_\varepsilon(u_n, v_n)(u_n, v_n) = \lambda_n \phi'_\varepsilon(u_n, v_n)(u_n, v_n) + o_n(1)\|(u_n, v_n)\|_\varepsilon.$$

Straightforward calculations show that $((u_n, v_n))$ is bounded. Moreover, in view of Lemma 3.4.6, we may suppose that $\phi'_\varepsilon(u_n, v_n)(u_n, v_n) \to l < 0$. Hence, the above expression shows that $\lambda_n \to 0$ and therefore we conclude that $J'_\varepsilon(u_n, v_n) \to 0$ in the dual space of X_ε. It follows from Lemma 3.3.3 that $((u_n, v_n))$ has a convergent subsequence. □

From now on we will denote by (w_1, w_2) the solution for the system (S_0) given by Lemma 3.4.2.

Let us consider $\delta > 0$ such that $M_\delta \subset \Lambda$ and $\psi \in C^\infty(\mathbb{R}^+, [0, 1])$ a non-increasing function such that $\psi \equiv 1$ on $[0, \delta/2]$ and $\psi \equiv 0$ on $[\delta, \infty)$. For any $y \in M$, we define the function $\Psi_{i,\varepsilon,y} \in X_\varepsilon$ by setting

$$\Psi_{i,\varepsilon,y}(x) := \psi(|\varepsilon x - y|)w_i\left(\frac{\varepsilon x - y}{\varepsilon}\right), \quad i = 1, 2,$$

and denote by $t_\varepsilon > 0$ the unique positive number verifying

$$J_\varepsilon(t_\varepsilon(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y})) = \max_{t\geq 0} J_\varepsilon(t(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y})).$$

In view of the above remarks, it is well defined the function $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ given by

$$\Phi_\varepsilon(y) := t_\varepsilon(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y}).$$

In next lemma we prove an important relationship between Φ_ε and the set M.

Lemma 3.4.8. *Uniformly for $y \in M$, we have*

$$\lim_{\varepsilon\to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = c_0$$

where c_0 was given in (3.4.1).

Proof. Suppose, by contradiction, that the lemma is false. Then there exist $\delta > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0^+$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_0| \geq \delta > 0. \tag{3.4.8}$$

We notice that, if $z \in B_{\delta/\varepsilon_n}(0)$ then $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$. Thus recalling that $H \equiv Q$ in Λ and $\psi(s) = 0$ for $s \geq \delta$, we can use the change of variables $z \mapsto \frac{\varepsilon_n x - y_n}{\varepsilon_n}$ to write

$$\begin{aligned}
J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y)) &= \frac{1}{2}\|(t_{\varepsilon_n}\Psi_{1,\varepsilon_n,y}, t_{\varepsilon_n}\Psi_{2,\varepsilon_n,y})\|^2_{\varepsilon_n} - \int_{\mathbb{R}^N} H(\varepsilon_n x, t_{\varepsilon_n}\Psi_{1,\varepsilon_n,y}, t_{\varepsilon_n}\Psi_{2,\varepsilon_n,y})dx \\
&= \frac{t^2_{\varepsilon_n}}{2}\int_{\mathbb{R}^N} a(\varepsilon_n z + y_n)|\nabla(\psi(|\varepsilon_n z|)w_1(z))|^2 dz + \frac{t^2_{\varepsilon_n}}{2}\int_{\mathbb{R}^N} |\nabla(\psi(|\varepsilon_n z|)w_2(z))|^2 dz \\
&\quad + \frac{t^2_{\varepsilon_n}}{2}\int_{\mathbb{R}^N} |\psi(|\varepsilon_n z|)w_1(z)|^2 dz + \frac{t^2_{\varepsilon_n}}{2}\int_{\mathbb{R}^N} b(\varepsilon_n z + y_n)|\psi(|\varepsilon_n z|)w_2(z)|^2 dz \\
&\quad - \int_{\mathbb{R}^N} Q(t_{\varepsilon_n}\psi(|\varepsilon_n z|)w_1(z), t_{\varepsilon_n}\psi(|\varepsilon_n z|)w_2(z))dz.
\end{aligned}$$

Since Q is homogeneous, we have that $t_{\varepsilon_n} \to 1$. This and Lebesgue's theorem imply that

$$\lim_{n\to\infty} \|(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})\|^2_{\varepsilon_n} = \|(w_1, w_2)\|^2$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} Q(\Psi_{1,\varepsilon_n,y_n},\Psi_{2,\varepsilon_n,y_n})dx=\int_{\mathbb{R}^N} Q(w_1,w_2)dx.$$

Therefore

$$\lim_{n\to\infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))=I_0(w_1,w_2)=c_0$$

which contradicts (3.4.8). The lemma is proved. □

Proposition 3.4.9. *Let $\varepsilon_n\to 0$ and $((u_n,v_n))\subset\mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n,v_n)\to c_0$. Then there exists a sequence $(\widetilde{y}_n)\subset\mathbb{R}^N$ such that $(w_n(x),z_n(x)):=(u_n(x+\widetilde{y}_n),v_n(x+\widetilde{y}_n))$ has a convergent subsequence in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Moreover up to a subsequence, $y_n\to y\in M$, where $y_n=\varepsilon_n\widetilde{y}_n$.*

Proof. Since $a_0\le a(x)$ and $b_0\le b(x)$ for $x\in\mathbb{R}^N$ and $c_0>0$, we can use $(\mathcal{H}_1^p)$, (H_3) and repeat the same arguments in Lemma 3.4.1 to conclude that there exists a sequence $(\widetilde{y}_n)\subset\mathbb{R}^N$ and constants $R,\eta>0$ such that

$$\liminf_{n\to\infty}\int_{B_R(\widetilde{y}_n)}\left(|u_n|^2+|v_n|^2\right)dx\ge\eta.$$

Thus, since $((u_n,v_n))$ is bounded in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, considering $(w_n(x),z_n(x))=(u_n(x+\widetilde{y}_n),v_n(x+\widetilde{y}_n))$, up to a subsequence, we have that $w_n\rightharpoonup w\not\equiv 0$ in $H^1(\mathbb{R}^N)$ and $z_n\rightharpoonup z\not\equiv 0$ in $H^1(\mathbb{R}^N)$. Let $t_n>0$ be such that

$$(\widetilde{w}_n,\widetilde{z}_n)=t_n(w_n,z_n)\in\mathcal{M}_0. \tag{3.4.9}$$

Then,

$$c_0\le I_0(\widetilde{w}_n,\widetilde{z}_n)\le J_{\varepsilon_n}(t_{\varepsilon_n}(u_n,v_n))\le J_{\varepsilon_n}(u_n,v_n)=c_0+o_n(1) \tag{3.4.10}$$

which implies

$$I_0(\widetilde{w}_n,\widetilde{z}_n)\to c_0\quad\text{and}\quad((\widetilde{w}_n,\widetilde{z}_n))\subset\mathcal{M}_0.$$

From boundedness of $((w_n,z_n))$ and (3.4.10), we get that (t_n) is bounded. As consequence, the sequence $((\widetilde{w}_n,\widetilde{z}_n))$ is also bounded in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, which implies, for some subsequence, $(\widetilde{w}_n,\widetilde{z}_n)\rightharpoonup(\widetilde{w},\widetilde{z})$ weakly in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.

Note that we can assume that $t_n\to t_0>0$. Then, this limit implies that $(\widetilde{w},\widetilde{z})=t_0(w,z)\not\equiv(0,0)$. From Lemma 3.4.2, we conclude that $(\widetilde{w}_n,\widetilde{z}_n)\to(\widetilde{w},\widetilde{z})$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, and as a consequence $(w_n,z_n)\to(w,z)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.

Now, we consider $y_n=\varepsilon_n\widetilde{y}_n$. Our goal is to show that (y_n) has a subsequence, still denoted by (y_n), satisfying $y_n\to y$ for $y\in M$. First of all, we claim that (y_n) is bounded. Indeed, suppose that there exists a subsequence, still denoted by (y_n), verifying $|y_n|\to\infty$. Note that from (ab_1)

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left[a_0|\nabla w_n|^2+|\nabla z_n|^2+|w_n|^2+b_0|z_n|^2\right]dx\\
\le&\int_{\mathbb{R}^N}\left[a(\varepsilon_n x+y_n)|\nabla w_n|^2+|\nabla z_n|^2+|w_n|^2+b(\varepsilon_n x+y_n)|z_n|^2\right]dx\\
=&\int_{\mathbb{R}^N}\left[a(\varepsilon_n z)|\nabla u_n(z)|^2+|\nabla v_n(z)|^2+|u_n|^2+b(\varepsilon_n z)|v_n(z)|^2\right]dz\\
=&\int_{\mathbb{R}^N}\left[w_nH_w(\varepsilon_n x+y_n,w_n,z_n)+z_nH_z(\varepsilon_n x+y_n,w_n,z_n)\right]dx.
\end{aligned}$$

Fixing $R > 0$ such that $\Lambda \subset B_R(0)$, since $|\varepsilon_n x + y_n| \geq R$ and (H_3), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} [w_n H_w(\varepsilon_n x + y_n, w_n, z_n) + z_n H_z(\varepsilon_n x + y_n, w_n, z_n)]\, dx \\ \leq\ & \frac{1}{4}\int_{B_{R/\varepsilon_n}(0)} \left(|w_n|^2 + b(\varepsilon_n x + y_n)|z_n|^2\right) dx + o_n(1). \end{aligned}$$

This implies that,

$$\frac{3}{4}\|(w_n, z_n)\|^2 \leq o_n(1).$$

It follows that $(w_n, z_n) \to (0,0)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, obtain this way a contradiction because $c_0 > 0$.

Hence (y_n) is bounded and, up to a subsequence,

$$y_n \to y \in \mathbb{R}^N.$$

Arguing as above, if $y \notin \overline{\Lambda}$, we will obtain again $(w_n, z_n) \to (0,0)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, thus $y \in \overline{\Lambda}$.

Now we are going to show that $y \in M$. It is sufficient to show that $a(y) = a_0$ and $b(y) = b_0$. Supposing, by contradiction, that $a(y) > a_0$ or $b(y) > b_0$, we have

$$c_0 = I_0(\widetilde{w}, \widetilde{z}) < \frac{1}{2}\int_{\mathbb{R}^N} \left[a(y)|\nabla \widetilde{w}|^2 + |\nabla \widetilde{z}|^2 + |\widetilde{w}|^2 + b(y)|\widetilde{z}|^2\right] dx - \int_{\mathbb{R}^N} Q(\widetilde{w}, \widetilde{z}) dx.$$

Using again the fact that $(\widetilde{w}_n, \widetilde{z}_n) \to (\widetilde{w}, \widetilde{z})$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, from Fatou's lemma we get

$$\begin{aligned} c_0 \ &< \ \liminf_{n\to\infty} \left\{ \frac{t_n^2}{2}\int_{\mathbb{R}^N} \left[a(\varepsilon_n z)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon_n z)|v_n|^2\right] dz \right. \\ &\qquad \left. - \int_{\mathbb{R}^N} Q(t_n u_n, t_n v_n) dz \right\} \\ &\leq \ \liminf_{n\to\infty} \left\{ \frac{t_n^2}{2}\int_{\mathbb{R}^N} \left[a(\varepsilon_n z)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon_n z)|v_n|^2\right] dz \right. \\ &\qquad \left. - \int_{\mathbb{R}^N} H(\varepsilon_n z, t_n u_n, t_n v_n) dz \right\} \\ &= \ \liminf_{n\to\infty} J_{\varepsilon_n}(t_n(u_n, v_n)) \leq \liminf_{n\to\infty} J_{\varepsilon_n}(u_n, v_n) = c_0 \end{aligned}$$

obtaining a contradiction. Then, we conclude that $y \in M$. □

Let us consider $\rho = \rho_\delta > 0$ in such way that $M_\delta \subset B_\rho(0)$ and define $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ by setting $\Upsilon(x) := x$ for $|x| < \rho$ and $\Upsilon(x) := \rho x/|x|$ for $|x| \geq \rho$. We also consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u, v) := \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)\left(|u(x)|^2 + |v(x)|^2\right) dx}{\displaystyle\int_{\mathbb{R}^N} \left(|u(x)|^2 + |v(x)|^2\right) dx}.$$

Since $M \subset B_\rho(0)$, the definition of Υ and Lebesgue's theorem imply that

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \text{uniformly for} \quad y \in M. \tag{3.4.11}$$

Following [17], we introduce the set

$$\Sigma_\varepsilon := \{(u,v) \in \mathcal{N}_\varepsilon : J_\varepsilon(u,v) \leq c_0 + h(\varepsilon)\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Given $y \in M$, we can use Lemma 3.4.8 to conclude that $h(\varepsilon) = |J_\varepsilon(\Phi_\varepsilon(y)) - c_0|$ satisfies $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Thus, $\Phi_\varepsilon(y) \in \Sigma_\varepsilon$ and therefore $\Sigma_\varepsilon \neq \emptyset$, for any $\varepsilon > 0$ small.

Lemma 3.4.10. *For any $\delta > 0$ we have*

$$\lim_{\varepsilon \to 0^+} \sup_{(u,v)\in\Sigma_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u,v), M_\delta) = 0. \tag{3.4.12}$$

Proof. Let $(\varepsilon_n) \subset \mathbb{R}$ be such that $\varepsilon_n \to 0^+$. By definition, there exists $((u_n,v_n)) \subset \Sigma_{\varepsilon_n}$ such that

$$\operatorname{dist}(\beta_{\varepsilon_n}(u_n,v_n), M_\delta) = \sup_{(u,v)\in\Sigma_{\varepsilon_n}} \operatorname{dist}(\beta_{\varepsilon_n}(u,v), M_\delta) + o_n(1).$$

Thus, it suffices to find a sequence $(y_n) \subset M_\delta$ such that

$$|\beta_{\varepsilon_n}(u_n,v_n) - y_n| = o_n(1). \tag{3.4.13}$$

Thus, recalling that $((u_n,v_n)) \subset \Sigma_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we obtain

$$c_0 \leq \max_{t\geq 0} I_0(tu_n, tv_n) \leq \max_{t\geq 0} J_{\varepsilon_n}(tu_n,tv_n) = J_{\varepsilon_n}(u_n,v_n) \leq c_0 + h(\varepsilon_n)$$

from which follows that $J_{\varepsilon_n}(u_n,v_n) \to c_0$. Thus, we may invoke Proposition 3.4.9 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $(y_n) := (\varepsilon_n \widetilde{y}_n) \subset M_\delta$, for n large. Hence,

$$\begin{aligned}
\beta_{\varepsilon_n}(u_n,v_n) &= \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon_n x)\left(|u_n(x)|^2 + |v_n(x)|^2\right) dx}{\displaystyle\int_{\mathbb{R}^N} \left(|u_n(x)|^2 + |v_n(x)|^2\right) dx} \\
&= \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon_n z + y_n)\left(|u_n(z+\widetilde{y}_n)|^2 + |v_n(z+\widetilde{y}_n)|^2\right) dz}{\displaystyle\int_{\mathbb{R}^N} \left(|u(z+\widetilde{y}_n)|^2 + |v_n(z+\widetilde{y}_n)|^2\right) dz} \\
&= y_n + \frac{\displaystyle\int_{\mathbb{R}^N} \left(\Upsilon(\varepsilon_n z + y_n) - y_n\right)\left(|u_n(z+\widetilde{y}_n)|^2 + |v_n(z+\widetilde{y}_n)|^2\right) dz}{\displaystyle\int_{\mathbb{R}^N} \left(|u(z+\widetilde{y}_n)|^2 + |v_n(z+\widetilde{y}_n)|^2\right) dz}.
\end{aligned}$$

Since $\varepsilon_n z + y_n \to y \in M$ and from strong convergence of $((u_n(\cdot + \widetilde{y}_n), v_n(\cdot + \widetilde{y}_n)))$, we have that $\beta_{\varepsilon_n}(u_n,v_n) = y_n + o_n(1)$ and therefore the sequence (y_n) satisfies (3.4.13). The lemma is proved. □

Theorem 3.4.11. *Suppose that a and b are continuous potentials satisfying $(ab_1) - (ab_2)$ and $M \neq \emptyset$. Suppose also that Q satisfies $(Q_0) - (Q_5)$. Then,*

(i) for all $\varepsilon > 0$, the system $(S_{\varepsilon,aux})$ has a positive ground state solution.

(ii) for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system $(S_{\varepsilon,aux})$ has at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions.

Proof. By using Lemma 3.3.1, Lemma 3.3.3, Mountain Pass Theorem [6] and of the characterization of minimax level c_ε given in (3.3.1) we conclude that the system $(S_{\varepsilon,aux})$ has a ground state positive solution.

Now, given $\delta > 0$ such that $M_\delta \subset \Lambda$, we can use (3.4.11), Lemma 3.4.8, (3.4.12) and argue as in [17, Section 6] to obtain $\widehat{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \widehat{\varepsilon}_\delta)$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \Sigma_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Thus

$$\text{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon) \geq \text{cat}_{M_\delta}(M).$$

It follows from Proposition 3.4.7 and Theorem 3.4.3 that J_ε possesses at least $\text{cat}_{M_\delta}(M)$ critical points on $\mathcal{N}_\varepsilon$. The same argument employed in the proof of Proposition 3.4.7 shows that each of these critical points is also a critical point of the unconstrained functional J_ε. Thus, we obtain $\text{cat}_{M_\delta}(M)$ nontrivial solutions for $(S_{\varepsilon,aux})$. □

3.5 Proof of Theorem 3

Proof. Suppose that $\delta > 0$ is such that $M_\delta \subset \Lambda$. Arguing by contradiction we can use Lemma 1.5.1 given in Chapter 1 to get $\widetilde{\varepsilon}_\delta > 0$ such that, for any $0 < \varepsilon < \widetilde{\varepsilon}_\delta$ and any solution $(u_\varepsilon, v_\varepsilon) \in \Sigma_\varepsilon$ of the system $(S_{\varepsilon,aux})$ there holds

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \quad \text{for each} \quad x \in \mathbb{R}^N \setminus \Lambda_\varepsilon. \tag{3.5.1}$$

Considering $0 < \varepsilon_\delta < \widetilde{\varepsilon}_\delta$, we shall prove the theorem for this choice of ε_δ. Let $0 < \varepsilon < \varepsilon_\delta$ be fixed. By applying Theorem 3.4.11, we obtain $\text{cat}_{M_\delta}(M)$ nontrivial solutions of the system $(S_{\varepsilon,aux})$. If $(u,v) \in X_\varepsilon$ is one of these solutions we have that $(u,v) \in \Sigma_\varepsilon$, and therefore we can use (3.5.1) and the definition of H to conclude that $H(\cdot, u, v) \equiv Q(u,v)$. Hence, (u,v) is also a solution of the system $(\widehat{S}_\varepsilon)$. An easy calculation shows that $(\widehat{u}(x), \widehat{v}(x)) := (u(x/\varepsilon), v(x/\varepsilon))$ is a solution of the original system (S_ε). Then, (S_ε) has at least $\text{cat}_{M_\delta}(M)$ nontrivial solutions.

We now consider $\varepsilon_n \to 0^+$ and take a sequence $(u_n, v_n) \in X_{\varepsilon_n}$ of solutions of the system $(\widehat{S}_{\varepsilon_n})$ as above. By applying Lemma 1.5.1, we obtain $R > 0$ and $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that

$$\|u_n\|_{L^\infty(\mathbb{R}^N \setminus B_R(\widetilde{y}_n))} < \gamma$$

and

$$\|v_n\|_{L^\infty(\mathbb{R}^N \setminus B_R(\widetilde{y}_n))} < \gamma.$$

Up to a subsequence, we may assume that

$$\|u_n\|_{L^\infty(B_R(\widetilde{y}_n))} \geq \gamma \tag{3.5.2}$$

and

$$\|v_n\|_{L^\infty(B_R(\widetilde{y}_n))} \geq \gamma. \tag{3.5.3}$$

Indeed, if this is not the case, we have $\|u_n\|_{L^\infty(\mathbb{R}^N)} < \gamma$ or $\|v_n\|_{L^\infty(\mathbb{R}^N)} < \gamma$, which is a contradiction with (3.4.4). Thus, (3.5.2) and (3.5.3) hold.

By using (3.5.2) and (3.5.3) we conclude that the maximum point $\pi_{n,a} \in \mathbb{R}^N$ of u_n and the maximum point $\pi_{n,b} \in \mathbb{R}^N$ of v_n belong to $B_R(\widetilde{y}_n)$. Hence $\pi_{n,a} = \widetilde{y}_n + q_{n,a}$, for some $q_{n,a} \in B_R(0)$ and $\pi_{n,b} = \widetilde{y}_n + q_{n,b}$, for some $q_{n,b} \in B_R(0)$. Recalling that the associated solution of (S_{ε_n}) is of the form $(\widehat{u}_n(x), \widehat{v}_n(x)) = (u_n(x/\varepsilon_n), v_n(x/\varepsilon_n))$, we conclude that the maximum point $\Pi_{\varepsilon_n,a}$ of $\widehat{u}_n$ and the maximum point $\Pi_{\varepsilon_n,b}$ of v_n are $\Pi_{\varepsilon_n,a} := \varepsilon_n\widetilde{y}_n + \varepsilon_n q_{n,a}$ and $\Pi_{\varepsilon_n,b} := \varepsilon_n\widetilde{y}_n + \varepsilon_n q_{n,b}$. Since $(q_{n,a}), (q_{n,b}) \subset B_R(0)$ are bounded and $\varepsilon_n\widetilde{y}_n \to y \in M$ (according to Proposition 3.4.9), we obtain

$$\lim_{n\to\infty} a(\Pi_{\varepsilon_n,a}) = a(y) = a_0$$

and

$$\lim_{n\to\infty} b(\Pi_{\varepsilon_n,a}) = b(y) = b_0.$$

Now we prove the regularity of the solution. By using Lemma 1.5.1, (3.5.2) and (3.5.3), we have that $u_\varepsilon, v_\varepsilon \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. From interpolation inequality, we get $(u_\varepsilon, v_\varepsilon) \in L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$, $\forall q \geq 2$. That implies $Q_u(u_\varepsilon, v_\varepsilon), Q_v(u_\varepsilon, v_\varepsilon) \in L^q(\mathbb{R}^N)$, $\forall q \geq 2$. From regularity elliptic theory, we get $(u_\varepsilon, v_\varepsilon) \in W^{2,q}(\mathbb{R}^N) \times W^{2,q}(\mathbb{R}^N)$, $\forall q \geq 2$. For q sufficiently large, we obtain $W^{2,q}(\mathbb{R}^N) \hookrightarrow C^{1,\lambda}(\mathbb{R}^N)$, for some $0 < \lambda < 1$. Then $u_\varepsilon, v_\varepsilon \in C^{1,\lambda}(\mathbb{R}^N)$. Since $Q \in C^2(\mathbb{R}^N)$, we obtain that $u_\varepsilon, v_\varepsilon \in C^{2,\lambda}(\mathbb{R}^N)$, which concludes the proof of the theorem. □

3.6 The critical case

In this section we present the proof of Theorem 4. Since many calculations are adaptations to that presented in the early section, we will emphasize only the differences between the subcritical and the critical case.

Hereafter, we will work with the following system equivalent to (CS_ε)

$$(C\widehat{S}_\varepsilon) \qquad \begin{cases} -\text{div}(a(\varepsilon x)\nabla u) + u = Q_u(u,v) + \frac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N, \\ \quad -\Delta v + b(\varepsilon x)v \ = Q_v(u,v) + \frac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N, \\ u,v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

Using a function η given in (3.2.3), we define $\widehat{K} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\widehat{K}(s,t) := \eta(|(s,t)|)\left(Q(s,t) + \frac{1}{2^*}K(s,t)\right) + (1 - \eta(|(s,t)|))\widetilde{A}(s^2+t^2)$$

where

$$\widetilde{A} := \max\left\{\frac{Q(s,t) + \frac{1}{2^*}K(s,t)}{s^2+t^2} : (s,t) \in \mathbb{R}^2,\ \alpha \leq |(s,t)| \leq 5\alpha\right\}.$$

Notice that, since $\widetilde{A} > 0$ tends to zero as $\alpha \to 0^+$, we may suppose that $\widetilde{A} \in (0, \mu/4)$.

We define $\widetilde{H} : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ by setting

$$\widetilde{H}(x,s,t) := I_\Lambda(x)\left(Q(s,t) + \frac{1}{2^*}K(s,t)\right) + (1 - I_\Lambda(x))\widehat{K}(s,t). \tag{3.6.1}$$

Using the fact that $Q \in \mathcal{H}^p$ and $K \in \mathcal{H}^{2^*}$, we can arguing as the proof of Lemma 3.2.1 to get

Lemma 3.6.1. *The function $\widetilde{H}$ satisfies the following estimates:*

$(\widetilde{H}_1)$ $p\widetilde{H}(x,s,t) \leq s\widetilde{H}_s(x,s,t) + t\widetilde{H}_t(x,s,t)$, for each $x \in \Lambda$;

$(\widetilde{H}_2)$ $2\widetilde{H}(x,s,t) \leq s\widetilde{H}_s(x,s,t) + t\widetilde{H}_t(x,s,t)$, for each $x \in \mathbb{R}^N \setminus \Lambda$;

$(\widetilde{H}_3)$ for α small we have $s\widetilde{H}_s(x,s,t) + t\widetilde{H}_t(x,s,t) \leq \frac{1}{4}\left(s^2 + b(x)t^2\right)$ for each $x \in \mathbb{R}^N \setminus \Lambda$;

$(\widetilde{H}_4)$ *for* α *small we have* $\dfrac{|\widetilde{H}_s(x,s,t)|}{\alpha}, \dfrac{|\widetilde{H}_t(x,s,t)|}{\alpha} \leq \dfrac{\mu}{4}$ *for each* $x \in \mathbb{R}^N \setminus \Lambda$.

Using the definition (3.6.1), we deal in the sequel with the modified system

$$(CS_{\varepsilon,aux}) \qquad \begin{cases} -\text{div}(a(\varepsilon x)\nabla u) + u = \widetilde{H}_u(\varepsilon x, u, v) \text{ in } \mathbb{R}^N, \\ \quad -\Delta v + b(\varepsilon x) v \quad = \widetilde{H}_v(\varepsilon x, u, v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

and we will look for solutions $(u_\varepsilon, v_\varepsilon)$ verifying

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \text{ for each } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon.$$

Conditions $(\widetilde{H}_3)$ and (A_1) imply that the critical points of the C^1-functional $\widetilde{J}_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$\widetilde{J}_\varepsilon(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} \left[a(\varepsilon x)|\nabla u|^2 + |\nabla v|^2 + |u|^2 + b(\varepsilon x)|v|^2\right] dx - \int_{\mathbb{R}^N} \widetilde{H}(\varepsilon x, u, v) dx$$

are weak solutions of $(CS_{\varepsilon,aux})$. We recall that these critical points belong to the Nehari manifold of $\widetilde{J}_\varepsilon$, namely on the set

$$\widetilde{\mathcal{N}}_\varepsilon := \left\{(u,v) \in X_\varepsilon \setminus \{(0,0)\} : \widetilde{J}'_\varepsilon(u,v)(u,v) = 0\right\}$$

and we define the number $\widetilde{b}_\varepsilon$ by setting

$$\widetilde{b}_\varepsilon := \inf_{(u,v)\in\widetilde{\mathcal{N}}_\varepsilon} \widetilde{J}_\varepsilon(u,v). \tag{3.6.2}$$

In order to prove the multiplicity result for the system $(C\widehat{S}_\varepsilon)$, we consider the critical version of the problem (S_0), namely

$$(CS_0) \qquad \begin{cases} -a_0 \Delta u + u = Q_u(u,v) + \frac{1}{2^*} K_u(u,v) \text{ in } \mathbb{R}^N, \\ -\Delta v + b_0 v \quad = Q_v(u,v) + \frac{1}{2^*} K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In view of conditions (ab_1), $(\mathcal{H}_1^p)$ and $(\mathcal{H}_1^{2^*})$, the above system has a variational structure and the associated functional is given by

$$\begin{aligned} \widetilde{I}_0(u,v) \quad &:= \quad \frac{1}{2}\int_{\mathbb{R}^N} \left[a_0|\nabla u|^2 + |\nabla v|^2 + |u|^2 + b_0|v|^2\right] dx - \int_{\mathbb{R}^N} Q(u,v) dx \\ & \qquad -\frac{1}{2^*}\int_{\mathbb{R}^N} K(u,v) dx, \end{aligned}$$

is well defined for $(u,v) \in E_0$.

Standard calculations show that $\widetilde{I}_0$ has the Mountain Pass geometry and therefore we can set the the minimax level $\widetilde{c}_0$ in the following way

$$\widetilde{c}_0 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \widetilde{I}_0(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], E_0) : \gamma(0) = (0,0), \ \widetilde{I}_0(\gamma(1)) < 0\}$. Moreover, $\widetilde{c}_0$ can be further characterized as

$$\widetilde{c}_0 = \inf_{(u,v)\in\widetilde{\mathcal{M}}_0} \widetilde{I}_0(u,v), \tag{3.6.3}$$

with $\widetilde{\mathcal{M}}_0$ being the Nehari manifold of $\widetilde{I}_0$, that is

$$\widetilde{\mathcal{M}}_0 := \{(u,v) \in E_0 \setminus \{(0,0)\} : \widetilde{I}_0'(u,v)(u,v) = 0\}.$$

As usual, we denote by S the best constant of the embedding $W^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. To state the next result we need to define $\widetilde{S}_K$ the best constant of the immersion $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \times L^{2^*}(\mathbb{R}^N)$, that is,

$$\widetilde{S}_K := \inf_{\substack{u,v \in D^{1,2}(\mathbb{R}^N) \\ u,v \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx}{\left(\displaystyle\int_{\mathbb{R}^N} K(u,v) dx\right)^{2/2^*}}.$$

Proposition 3.6.2. *There exists $\sigma^* > 0$ such that for all $\sigma > \sigma^*$*

$$\widetilde{c}_0 < \frac{1}{N}\left(\min\{a_0, 1\}\widetilde{S}_K\right)^{N/2}.$$

Proof. By using $(\mathcal{H}_0^p)$ and $(\mathcal{H}_0^{2^*})$, and arguing as [40, Theorem 4.2], it is possible to prove that

$$\widetilde{c}_0 = \inf_{(u,v) \in E_0 \setminus \{(0,0)\}} \max_{t \geq 0} \widetilde{I}_0(tu, tv) > 0.$$

Thus, it suffices to obtain $(u,v) \in E_0$ such that

$$\max_{t \geq 0} \widetilde{I}_0(tu, tv) < \frac{1}{N}\left(\min\{a_0, 1\}\widetilde{S}_K\right)^{N/2}.$$

We first recall that, for any $\delta > 0$ the function

$$w_\delta(x) := [\delta N(N-2)]^{(N-2)/4}(\delta + |x|^2)^{(2-N)/2}$$

satisfies

$$\int_{\mathbb{R}^N} |\nabla w_\delta|^2 dx = \int_{\mathbb{R}^N} |w_\delta|^{2^*} dx = S^{N/2}.$$

By [19, Lemma 3], there exist $A, B \in \mathbb{R}$ such that $\widetilde{S}_K$ is attained by

$$\widetilde{S}_K = \frac{\int_{\mathbb{R}^N} \left(|\nabla(Aw_\delta)|^2 + |\nabla(Bw_\delta)|^2\right) dx}{\left(\int_{\mathbb{R}^N} K(Aw_\delta, Bw_\delta) dx\right)^{2/2^*}} = \frac{S^{N/2}(A^2 + B^2)}{\left(\int_{\mathbb{R}^N} K(Aw_\delta, Bw_\delta) dx\right)^{2/2^*}}.$$

Let $\eta \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$. Consider

$$\psi_\delta(x) := \frac{\eta(x) w_\delta(x)}{|\eta w_\delta|_{2^*}}.$$

By using the definition of ψ_δ, (A_3) and $(\mathcal{H}_0^{2^*})$ we get

$$\begin{aligned} \widetilde{I}_0(tA\psi_\delta, tB\psi_\delta) \quad &\leq \quad \frac{t^2}{2} D_\delta(A^2 + B^2) - \frac{\sigma}{p_1} t^{p_1} A^\lambda B^\beta \int_{B_2(0)} |\psi_\delta|^{p_1} dx \\ &\quad - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^N} K(A\psi_\delta, B\psi_\delta) dx, \end{aligned}$$

where $p_1 \in (2, 2^*)$ is given by condition (A_3) and

$$D_\delta = \int_{\mathbb{R}^N} \max\{a_0, b_0, 1\} \left(|\nabla \psi_\delta|^2 + |\psi_\delta|^2\right) dx.$$

Thus

$$\max_{t\geq 0}\left\{\frac{t^2}{2}D_\delta(A^2+B^2)-\frac{\sigma}{p_1}t^{p_1}A^\lambda B^\beta\int_{B_2(0)}|\psi_\delta|^{p_1}dx-\frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}K(A\psi_\delta,B\psi_\delta)dx\right\}$$
$$\geq \widetilde{I}_0(tA\psi_\delta,tB\psi_\delta).$$

Straightforward calculations show that

$$\begin{aligned}\widetilde{I}_0(tA\psi_\delta,tB\psi_\delta) &\leq \frac{1}{\sigma^{2/(p_1-2)}}\left(\frac{1}{2}-\frac{1}{p_1}\right)\frac{(D_\delta(A^2+B^2))^{p_1/(p_1-2)}}{\left(A^\lambda B^\beta\int_{B_2(0)}|\psi_\delta|^{p_1}dx\right)^{2/(p_1-2)}}\\ &= \frac{1}{\sigma^{2/(p_1-2)}}C(a_0,b_0).\end{aligned}$$

Thus, $\max_{t\geq 0}\widetilde{I}_0(tA\psi_\delta,tB\psi_\delta)<\frac{1}{N}\left(\min\{a_0,1\}\widetilde{S}_K\right)^{N/2}$, for all $\sigma>\sigma^*$ where

$$\sigma^*:=\left(\frac{C(a_0,b_0)}{\frac{1}{N}\left(\min\{a_0,1\}\widetilde{S}_K\right)^{N/2}}\right)^{\frac{p_1-2}{2}}.$$

The proof is finished. □

Lemma 3.6.3. *Let $((u_n,v_n))\subset\widetilde{\mathcal{M}}_0$ be a sequence such that $\widetilde{I}_0(u_n,v_n)\to\widetilde{c}_0$ with $\widetilde{c}_0<\frac{1}{N}\left(\min\{a_0,1\}\widetilde{S}_K\right)^{N/2}$. Then we have either*

(i) $\|(u_n,v_n)\|\to 0$, *or*

(ii) there exists a sequence $(y_n)\subset\mathbb{R}^N$ and constants $R,\eta>0$ such that

$$\liminf_{n\to\infty}\int_{B_R(y_n)}(|u_n|^2+|v_n|^2)dx\geq\eta.$$

Proof. Suppose that *(ii)* does not hold. Since $((u_n,v_n))$ is bounded in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, then, by in [32, Lemma I.1], we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^s dx=0$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^s dx=0,$$

for all $s\in(2,2^*)$. Thus, from $(\mathcal{H}_1^p)$, we conclude

$$\int_{\mathbb{R}^N}[u_nQ_u(u_n,v_n)+v_nQ_v(u_n,v_n)]dx=o_n(1).$$

Since $\widetilde{I}_0'(u_n,v_n)(u_n,v_n)=0$, taking a subsequence, we obtain $l\geq 0$ such that

$$\|(u_n,v_n)\|^2\to l\ \text{ and }\ \int_{\mathbb{R}^N}K(u_n,v_n)dx\to l. \tag{3.6.4}$$

Since $\widetilde{I}_0(u_n, v_n) \to \widetilde{c}_0$, we can use (3.6.4) to conclude that $l = N\widetilde{c}_0$. Recalling the definition of $\widetilde{S}_K$ we get

$$\begin{aligned}
\|(u_n, v_n)\|^2 &= \int_{\mathbb{R}^N} \left[a_0|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b_0|v_n|^2\right] dx \\
&\geq \min\{a_0, 1\} \int_{\mathbb{R}^N} \left[|\nabla u_n|^2 + |\nabla v_n|^2\right] dx \\
&\geq \min\{a_0, 1\}\widetilde{S}_K \left(\int_{\mathbb{R}^N} K(u_n, v_n) dx\right)^{2/2^*}.
\end{aligned}$$

Taking the limit we conclude that $l \geq \min\{a_0, 1\}\widetilde{S}_K l^{2/2^*}$. If $l > 0$ we obtain $N\widetilde{c}_0 = l \geq \left(\min\{a_0, 1\}\widetilde{S}_K\right)^{N/2}$, which does not make sense. Hence $l = 0$ and therefore (i) holds. □

By using Lemma 3.6.3, we can arguing as the proof of Lemma 3.4.2 and show that the system (CS_0) has a solution that reaches $\widetilde{c}_0$.

Lemma 3.6.4. *(A Compactness Lemma) Let $((u_n, v_n)) \subset \widetilde{\mathcal{M}}_0$ be a sequence satisfying $\widetilde{I}_0(u_n, v_n) \to \widetilde{c}_0$. Then, there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $(w_n(x), z_n(x)) = (u_n(x + \widetilde{y}_n), v_n(x + \widetilde{y}_n))$ converges strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.*
In particular, there exists a minimizer for $\widetilde{c}_0$.

As in Lemma 3.3.1, the functional $\widetilde{J}_\varepsilon$ satisfies the mountain Pass Geometry. Hence there exists a Palais-Smale sequence $((u_n, v_n)) \subset X_\varepsilon$ at level $\widetilde{c}_\varepsilon$. Using $(\mathcal{H}_0^p)$ and $(\mathcal{H}_0^{2^*})$, it is possible to prove that

$$\widetilde{c}_\varepsilon = \widetilde{b}_\varepsilon = \inf_{(u,v)\in X_\varepsilon\setminus\{0\}} \sup_{t\geq 0} \widetilde{J}_\varepsilon(tu, tv), \tag{3.6.5}$$

where $\widetilde{b}_\varepsilon$ was defined in (3.6.2).

Lemma 3.6.5. *Any sequence $((u_n, v_n)) \subset X_\varepsilon$ such that*

$$\widetilde{J}_\varepsilon(u_n, v_n) \to c < \frac{1}{N}\left(min\{a_0, 1\}\widetilde{S}_K\right)^{N/2} \quad and \quad \widetilde{J}'_\varepsilon(u_n, v_n) \to 0$$

possesses a convergent subsequence.

Proof. Standart calculations show that $((u_n, v_n))$ is bounded in X_ε. Then, up to a subsequence, we may suppose that

$$\begin{array}{l}
(u_n, v_n) \rightharpoonup (u, v) \text{ weakly in } X_\varepsilon, \\
u_n \to u, v_n \to v \text{ strongly in } L^s_{loc}(\mathbb{R}^N), \text{ for any } 2 \leq s < 2^*, \\
u_n(x) \to u(x), v_n(x) \to v(x) \text{ for a.e. } x \in \mathbb{R}^N.
\end{array} \tag{3.6.6}$$

Now using a density argument, we can conclude that (u, v) is a critical point of $\widetilde{J}_\varepsilon$. Hence

$$\|(u, v)\|_\varepsilon^2 = \int_{\mathbb{R}^N} [u\widetilde{H}_u(\varepsilon x, u, v) + v\widetilde{H}_v(\varepsilon x, u, v)]dx. \tag{3.6.7}$$

On the other hand, we have

$$\|(u_n, v_n)\|_\varepsilon^2 = \int_{\mathbb{R}^N} [u_n\widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n\widetilde{H}_v(\varepsilon x, u_n, v_n)]dx + o_n(1). \tag{3.6.8}$$

Claim 1. $\displaystyle\lim_{n\to\infty} \int_{\Lambda_\varepsilon} K(u_n, v_n)dx = \int_{\Lambda_\varepsilon} K(u, v)dx.$

Since $((u_n, v_n))$ is bounded, we may suppose that

$$|\nabla u_n|^2 \rightharpoonup \mu, \quad |\nabla v_n|^2 \rightharpoonup \sigma \quad \text{and} \quad K(u_n, v_n) \rightharpoonup \nu \quad \text{(weak*-sense of measures)}.$$

From [19, Lemma 6], we obtain an at most countable index set Γ, sequences $(x_i) \in \mathbb{R}^N$, (μ_i), (σ_i), $(\nu_i) \subset (0, \infty)$ such that

$$\begin{aligned} \mu \geq |\nabla u|^2 + \sum_{i\in\Gamma} \mu_i \delta_{x_i}, \quad \sigma \geq |\nabla v|^2 + \sum_{i\in\Gamma} \sigma_i \delta_{x_i} \\ \nu = K(u, v) + \sum_{i\in\Gamma} \nu_i \delta_{x_i} \quad \text{and} \quad \widetilde{S}_K \nu_i^{2/2^*} \leq \mu_i + \sigma_i \end{aligned} \tag{3.6.9}$$

for all $i \in \Gamma$, where δ_{x_i} is the Dirac mass at the point $x_i \in \mathbb{R}^N$.

Suppose that $\{x_i\}_{i\in\Gamma} \cap \Lambda_\varepsilon \neq \emptyset$, then exists $x_i \in \Lambda_\varepsilon$ for some $i \in \Gamma$. Define, for $\varrho > 0$, the function $\psi_\varrho(x) := \psi((x - x_i)/\varrho)$ where $\psi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ is such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$ and $|\nabla\psi|_\infty \leq 2$. We suppose that ϱ is chosen in such a way that the support of ψ_ϱ is contained in Λ_ε. Since $((\psi_\varrho u_n, \psi_\varrho v_n))$ is bounded, $\widetilde{J}'_\varepsilon(u_n, v_n)(\psi_\varrho u_n, \psi_\varrho v_n) = o_n(1)$. Then

$$\begin{aligned} &\int_{\mathbb{R}^N} [a(\varepsilon x)\psi_\varrho |\nabla u_n|^2 + \psi_\varrho |\nabla v_n|^2] dx \\ &+ \int_{\mathbb{R}^N} [a(\varepsilon x) u_n \nabla u_n \nabla \psi_\varrho + v_n \nabla v_n \nabla \psi_\varrho] dx + \int_{\mathbb{R}^N} [\psi_\varrho u_n^2 + b(\varepsilon x)\psi_\varrho v_n^2] dx \\ &= \int_{\mathbb{R}^N} [u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)] \psi_\varrho dx + o_n(1). \end{aligned}$$

Since $\operatorname{supp}(\psi_\varrho) \subset \Lambda_\varepsilon$, we can use definition of $\widetilde{H}$, (3.2.1) and (ab_1) to get

$$\begin{aligned} &\min\{a_0, 1\} \int_{\mathbb{R}^N} [\psi_\varrho |\nabla u_n|^2 + \psi_\varrho |\nabla v_n|^2] dx \\ \leq\ & - \int_{\mathbb{R}^N} [a(\varepsilon x) u_n \nabla u_n \nabla \psi_\varrho + v_n \nabla v_n \nabla \psi_\varrho] dx \\ & + p \int_{\mathbb{R}^N} Q(u_n, v_n)\psi_\varrho dx + \int_{\mathbb{R}^N} K(u_n, v_n)\psi_\varrho dx + o_n(1). \end{aligned}$$

Since Q has subcritical growth and ψ_ϱ has compact support, we can let $n \to \infty$, $\varrho \to 0$ and use (3.6.9) to conclude that

$$\min\{a_0, 1\}(\mu_i + \sigma_i) \leq \nu_i.$$

As $\widetilde{S}_K \nu_i^{2/2^*} \leq \mu_i + \sigma_i$, we get

$$\nu_i \geq \left(\min\{a_0, 1\}\widetilde{S}_K\right)^{N/2}.$$

By using Lemma 3.6.1, $p > 2$ and (3.2.1) we get

$$\begin{aligned} c &= \widetilde{J}_\varepsilon(u_n, v_n) - \frac{1}{2}\widetilde{J}'_\varepsilon(u_n, v_n)(u_n, v_n) + o_n(1) \\ &= \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left(\frac{1}{2}[u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)] - \widetilde{H}(\varepsilon x, u_n, v_n)\right) dx \\ &+ \int_{\Lambda_\varepsilon} \left(\frac{1}{2}[u_n Q_u(u_n, v_n) + v_n Q_v(u_n, v_n)] - Q(u_n, v_n)\right) dx \\ &+ \frac{1}{2^*}\int_{\Lambda_\varepsilon} \left(\frac{1}{2}[u_n K_u(u_n, v_n) + v_n K_v(u_n, v_n)] - K(u_n, v_n)\right) dx + o_n(1) \\ &\geq \frac{1}{N}\int_{\Lambda_\varepsilon} K(u_n, v_n) dx + o_n(1) \geq \frac{1}{N}\int_{\Lambda_\varepsilon} \psi_\varrho K(u_n, v_n) dx + o_n(1). \end{aligned}$$

By taking the limit and using (3.6.9) we get

$$c \geq \frac{1}{N} \sum_{\{i\in\Gamma : x_i \in \Lambda_\varepsilon\}} \psi_\varrho(x_i)\nu_i = \frac{1}{N} \sum_{\{i\in\Gamma : x_i \in \Lambda_\varepsilon\}} \nu_i \geq \frac{1}{N}\Big(\min\{a_0, 1\}\widetilde{S}_K\Big)^{N/2}$$

which does not make sense. Therefore $\{x_i\}_{i\in\Gamma} \cap \Lambda_\varepsilon = \emptyset$, this conclude the proof of the claim 1.

Claim 2.

$$\int_{\mathbb{R}^N} [u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)]dx \to \int_{\mathbb{R}^N} [u\widetilde{H}_u(\varepsilon x, u, v) + v\widetilde{H}_v(\varepsilon x, u, v)]dx.$$

Arguing as in the Lemma 3.3.2, for any $\xi >$ given, there exists $R > 0$ such that $\Lambda_\varepsilon \subset B_R(0)$ and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} \Big[a(\varepsilon x)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon x)|v_n|^2\Big]dx < \xi.$$

This inequality, $(\widetilde{H}_3)$ and the Sobolev embeddings imply that, for n large enough, there holds

$$\int_{\mathbb{R}^N\setminus B_R(0)} [u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)]dx \leq C_1 \frac{1}{4}\xi, \tag{3.6.10}$$

where C_1 is positive constant. On the other hand, taking R large enough, we can suppose that

$$\left| \int_{\mathbb{R}^N\setminus B_R(0)} [u\widetilde{H}_u(\varepsilon x, u, v) + v\widetilde{H}_v(\varepsilon x, u, v)]dx \right| < \xi. \tag{3.6.11}$$

Then, by (3.6.10) and (3.6.11), we can conclude

$$\begin{aligned} &\int_{\mathbb{R}^N\setminus B_R(0)} [u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)]dx \\ = &\int_{\mathbb{R}^N\setminus B_R(0)} [u\widetilde{H}_u(\varepsilon x, u, v) + v\widetilde{H}_v(\varepsilon x, u, v)]dx + o_n(1). \end{aligned} \tag{3.6.12}$$

On the other hand, since the set $B_R(0) \cap (\mathbb{R}^N \setminus \Lambda_\varepsilon)$ is bounded, we can use $(\widetilde{H}_3)$, (3.6.6) and Lebesgue's theorem to conclude that

$$\begin{aligned} &\lim_{n\to\infty} \int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)} [u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)]dx \\ = &\int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)} [u\widetilde{H}_u(\varepsilon x, u, v) + v\widetilde{H}_v(\varepsilon x, u, v)]dx. \end{aligned} \tag{3.6.13}$$

By using Claim 1, $(\mathcal{H}_1^p)$, (3.6.6) and Lebesgue's theorem again, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \int_{\Lambda_\varepsilon} [u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)]dx \\ = &\int_{\Lambda_\varepsilon} [u\widetilde{H}_u(\varepsilon x, u, v) + v\widetilde{H}_v(\varepsilon x, u, v)]dx. \end{aligned} \tag{3.6.14}$$

From (3.6.12), (3.6.13) and (3.6.14) the claim 2 is proved.

By using (3.6.7), claim 2 and (3.6.8), we have $\|(u_n, v_n)\|_\varepsilon^2 \to \|(u, v)\|_\varepsilon^2$. Then $(u_n, v_n) \to (u, v)$ in X_ε. □

From Lemma 3.6.5, the unconstrained functional satisfies $(\mathrm{PS})_c$ for $c < \frac{1}{N}(\min\{a_0, 1\}\widetilde{S}_K)^{N/2}$. Nevertheless, to get multiple critical points, we need to work with the functional $\widetilde{J}_\varepsilon$ constrained to $\widetilde{\mathcal{N}}_\varepsilon$. The proof the next three results follows by using the same arguments employed in Lemma 3.4.5, Lemma 3.4.6 and Proposition 3.4.7

Lemma 3.6.6. *There exist positive constants $\widetilde{\alpha}_1, \widetilde{\delta}_1, \widetilde{C}$ such that, for each $\alpha \in (0, \widetilde{\alpha}_1)$, $(u,v) \in \widetilde{\mathcal{N}}_\varepsilon$, there hold*

$$\int_{\Lambda_\varepsilon} [pQ(u,v) + K(u,v)]dx \geq \widetilde{\delta}_1$$

and

$$\int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} \left(u^2 + b(\varepsilon x)v^2\right) dx \leq \widetilde{C} \int_{\Lambda_\varepsilon} [pQ(u,v) + K(u,v)]dx.$$

Lemma 3.6.7. *Let $\widetilde{\phi}_\varepsilon : X_\varepsilon \to \mathbb{R}$ be given by*

$$\widetilde{\phi}_\varepsilon(u,v) := \|(u,v)\|_\varepsilon^2 - \int_{\mathbb{R}^N} \left[u\widetilde{H}_u(\varepsilon x, u, v) + v\widetilde{H}_v(\varepsilon x, u, v)\right] dx.$$

Then there exist $\widetilde{\alpha}_2, \widetilde{M} > 0$ such that, for each $\alpha \in (0, \widetilde{\alpha}_2)$,

$$\widetilde{\phi}'_\varepsilon(u,v)(u,v) \leq -\widetilde{M} < 0 \quad \text{for each} \quad (u,v) \in \widetilde{\mathcal{N}}_\varepsilon.$$

Proposition 3.6.8. *The functional $\widetilde{J}_\varepsilon$ restricted to $\widetilde{\mathcal{N}}_\varepsilon$ satisfies $(PS)_c$ at any level $c < \frac{1}{N}\left(\min\{a_0, 1\}\widetilde{S}_K\right)^{N/2}$.*

We also have the critical version of Proposition 3.4.9 and her proof is similar.

Proposition 3.6.9. *Let $\varepsilon_n \to 0^+$ and $((u_n, v_n)) \subset \widetilde{\mathcal{N}}_{\varepsilon_n}$ be such that $\widetilde{J}_{\varepsilon_n}(u_n, v_n) \to \widetilde{c}_0$. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $(w_n(x), z_n(x)) := (u_n(x + \widetilde{y}_n), v_n(x + \widetilde{y}_n))$ has a convergent subsequence in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n \to y \in M$, where $y_n = \varepsilon_n \widetilde{y}_n$.*

The proof of the next result is in the same spirit of Lemma 2.4.14. We omit the details.

Lemma 3.6.10. *The minimax level $\widetilde{c}_\varepsilon$ satisfies*

$$\limsup_{\varepsilon \to 0^+} \widetilde{c}_\varepsilon \leq \widetilde{c}_0.$$

Theorem 3.6.11. *Suppose that a and b are continuous potentials and satisfy $(ab_1) - (ab_2)$. Suppose also $(A_1) - (A_3)$. Then,*

- (i) *there exists $\varepsilon_1 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_1)$ the system $(CS_{\varepsilon,aux})$ has a positive ground state solution.*
- (ii) *for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system $(CS_{\varepsilon,aux})$ has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions.*

Proof. The demonstration of item (i) follows from the fact that $\widetilde{J}_\varepsilon$ satisfies the mountain Pass Geometry, Lemma 3.6.5, Mountain Pass Theorem [6], the characterization of minimax level c_ε given in (3.6.5) and Lemma 3.6.10.

Now we prove the item (ii). As in the Section 3.4. Fix $\delta > 0$ such that $M_\delta \subset \Lambda$ and $\psi \in C^\infty(\mathbb{R}^+, [0,1])$ a non-increasing function such that $\psi(s) = 1$ if $0 \leq s \leq \delta/2$ and $\psi(s) = 0$ if $s \geq \delta$. Let $(\widetilde{w}_1, \widetilde{w}_2) \in E_0$ be a solution of (CS_0) given by Lemma 3.6.4 and define, for any $y \in M$

$$\widetilde{\Psi}_{i,\varepsilon,y}(x) := \psi(|\varepsilon x - y|)\widetilde{w}_i\left(\frac{\varepsilon x - y}{\varepsilon}\right), \quad i = 1, 2.$$

We introduce the map $\widetilde{\Phi}_\varepsilon : M \to \widetilde{\mathcal{N}}_\varepsilon$ by setting

$$\widetilde{\Phi}_\varepsilon(y) := \widetilde{t}_\varepsilon(\widetilde{\Psi}_{1,\varepsilon,y}, \widetilde{\Psi}_{2,\varepsilon,y}),$$

where $\widetilde{t}_\varepsilon$ is the unique positive number satisfying

$$\widetilde{J}_\varepsilon(\widetilde{t}_\varepsilon(\widetilde{\Psi}_{1,\varepsilon,y}, \widetilde{\Psi}_{2,\varepsilon,y})) = \max_{t\geq 0} \widetilde{J}_\varepsilon(t(\widetilde{\Psi}_{1,\varepsilon,y}, \widetilde{\Psi}_{2,\varepsilon,y})).$$

The following holds

$$\lim_{\varepsilon\to 0^+} \widetilde{J}_\varepsilon(\widetilde{\Phi}_\varepsilon(y)) = \widetilde{c}_0,$$

Let $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ be a function defined in Section 3.4 and consider the barycenter map $\widetilde{\beta}_\varepsilon : \widetilde{\mathcal{N}}_\varepsilon \to \mathbb{R}^N$ given by

$$\widetilde{\beta}_\varepsilon(u, v) := \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon x) \left(|u(x)|^2 + |v(x)|^2\right) dx}{\displaystyle\int_{\mathbb{R}^N} \left(|u(x)|^2 + |v(x)|^2\right) dx}.$$

As before we can to check that

$$\lim_{\varepsilon\to 0^+} \widetilde{\beta}_\varepsilon(\widetilde{\Phi}_\varepsilon(y)) = y \quad \text{uniformly for} \quad y \in M$$

and

$$\lim_{\varepsilon\to 0^+} \sup_{(u,v)\in\widetilde{\Sigma}_\varepsilon} \operatorname{dist}\left(\widetilde{\beta}_\varepsilon(u, v), M_\delta\right) = 0,$$

where

$$\widetilde{\Sigma}_\varepsilon := \left\{(u, v) \in \widetilde{\mathcal{N}}_\varepsilon : \widetilde{J}_\varepsilon(u, v) \leq \widetilde{c}_0 + \widetilde{h}(\varepsilon)\right\},$$

and $\widetilde{h} : [0, \infty) \to [0, \infty)$ satisfies $\widetilde{h}(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

The above equations provide $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram

$$M \xrightarrow{\widetilde{\Phi}_\varepsilon} \widetilde{\Sigma}_\varepsilon \xrightarrow{\widetilde{\beta}_\varepsilon} M_\delta$$

is well defined and $\widetilde{\beta}_\varepsilon \circ \widetilde{\Phi}_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Hence we conclude that $\operatorname{cat}_{\widetilde{\Sigma}_\varepsilon}(\widetilde{\Sigma}_\varepsilon) \geq \operatorname{cat}_{M_\delta}(M)$. It follows from Proposition 3.6.8 and Theorem 3.4.3 that $\widetilde{J}_\varepsilon$ possesses at least $\operatorname{cat}_{M_\delta}(M)$ critical points on $\widetilde{\mathcal{N}}_\varepsilon$. The same argument employed in the proof of Proposition 3.6.8 shows that each of these critical points is also a critical point of the unconstrained functional $\widetilde{J}_\varepsilon$. Thus, we obtain $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions for $(CS_{\varepsilon,aux})$. □

3.6.1 Proof of Theorem 4

Proof. By using Lemma 2.5.1 and repeating the same arguments that Theorem 3 we get the result. □

Appendix A

The Ljusternick-Schnirelmann category

In this appendix we briefly define the notion of category and state some of its properties, according to [5] and [40]. We have used category to obtain multiplicity results of critical points of functionals.

Definition A.0.1. *A closed subset A is contractible in a topological space X if there exist $v \in X$ and a continuous map $h : [0,1] \times A \to X$, such that*

$$h(0,u) = u \quad \textit{and} \quad h(1,u) = v, \ \textit{for all } u \in A.$$

Definition A.0.2. *Let X be a topological space. The Ljusternick-Schnirelmann category of A with respect to X, denoted by $\mathrm{cat}_X(A)$, is the least integer k such that $A \subset A_1 \cup \ldots \cup A_k$, with A_i $(i = 1, \ldots, k)$ closed and contractible in X. We set $\mathrm{cat}_X(\emptyset) = 0$ and $\mathrm{cat}_X(A) = +\infty$ if there are no integers with the above property.*

The essential idea of the Lusternik-Schnirelmann method is the following one: *The number of critical point of a C^1-functional I defined on a compact manifold X is greater than or equal to $\mathrm{cat}_X(X)$. The corresponding critical values are given by*

$$c_k := \inf_{A \in \mathcal{A}_k} \sup_{u \in A} I(u) \quad \textit{where} \quad \mathcal{A}_k := \{A \subset X : A \textit{ closed}, \mathrm{cat}_X(A) \geq k\}.$$

From the definition it holds:

1. if $A \subset B$ are subsets of X, $\mathrm{cat}_X(A) \leq \mathrm{cat}_X(B)$;
2. $\mathrm{cat}_X(A) = \mathrm{cat}_X(\bar{A})$;
3. if $A \subset X \subset Y$ with X closed in Y, $\mathrm{cat}_Y(A) \leq \mathrm{cat}_X(A)$.

We will set $\mathrm{cat}(X) := \mathrm{cat}_X(X)$.

Example A.0.12.

(i) Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Since S^{n-1} is not contractible in itself but can be covered by two closed hemispheres, then $\mathrm{cat}(S^{n-1}) = 2$. Note that, $\mathrm{cat}_{\mathbb{R}^n}(S^{n-1}) = 1$.

(ii) If $T^2 = S^1 \times S^1$ denotes the two-dimensional torus in $\mathbb{R}^3$ then $\mathrm{cat}(T^2) = 3$. In general, for the k dimensional torus $T^k = \mathbb{R}^k / \mathbb{Z}^k$ one has $\mathrm{cat}(T^k) = k + 1$.

Definition A.0.3. *Let X be a topological space. A deformation of $A \subset X$ in X is a continuous map $\eta : A \to X$ homotopic to the inclusion $A \hookrightarrow X$, i.e. there exists a continuous map $h : [0,1] \times A \to X$ such that*

$$h(0,u) = u \quad \text{and} \quad h(1,u) = \eta(u), \text{ for all } u \in A.$$

Let us now state the main properties of the category

Lemma A.0.13. *Let $A, B \subset X$.*

(i) If $A \subset B$ then $\operatorname{cat}_X(A) \leq \operatorname{cat}_X(B)$;

(ii) $\operatorname{cat}_X(A \cup B) \leq \operatorname{cat}_X(A) + \operatorname{cat}_X(B)$;

(iii) Let A be a closed in X, η a deformation of A in X. Then $\operatorname{cat}_X(A) \leq \operatorname{cat}_X(\overline{\eta(A)})$.

Proof. See [5]. □

Definition A.0.4. *Let X be a metric space. X satisfies the extension property if for every metric space Y, every subset S closed in Y and every continuous map $f : S \to X$, there are U a neighborhood of S in Y and a map $\widetilde{f} \in C(U; X)$ such that $\widetilde{f}|_S = f$.*

Lemma A.0.14. *Let X me a metric space with the extension property and let $A \subset X$ be a compact subset. Then*

(i) $\operatorname{cat}_X(A) < +\infty$.

(ii) There exists U_A neighborhood of A in X such that $\operatorname{cat}_X(\overline{U}_A) = \operatorname{cat}_X(A)$.

Proof. See [5]. □

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