Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

# A natural constraint for solving Schrödinger equations with $G$-symmetry and general nonlinearities

by

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# A natural constraint for solving Schrödinger equations with G-symmetry and general nonlinearities

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## DOUTOR EM MATEMÁTICA

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*This work is dedicated with all a ection to my parents, my brothers, my wife and my children.*

*"It has always seemed strange to me that all those who seriously study this science end up with a kind of passion for it. In fact, what provides maximum pleasure is not knowledge but learning; it is not possession, but acquisition; it is not presence, but the act of reaching the goal."*

GAUSS

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# Abstract

In this work, we consider two problems. In the first chapter, we establish the existence of a positive solution to the nonlinear Schrödinger equation

$$-\Delta u + V(x)u = f(u), \qquad u \in D^{1,2}(\mathbb{R}^N), \; N \geq 3, \tag{$\wp_1$}$$

with potential $V$ which is invariant under a group action $G \subset O(N)$, where $O(N)$ is the group of orthogonal transformations, and decays to zero at infinity, with an appropriate rate, approaching zero mass type limit scalar field equation, and the nonlinearity $f$, under very mild assumptions, is asymptotically linear or superlinear and subcritical at infinity, not satisfying any monotonicity condition. We deal with both finite group actions and infinite group actions.

In the second chapter, we study the existence of a positive solution for a nonlinear Schrödinger equation

$$-\Delta u + V(x)u = f(u), \qquad u \in H^1(\mathbb{R}^N), \; N \geq 3, \tag{$\wp_2$}$$

where the potential $V$ is a positive function, invariant under a group action $G \subset O(N)$, which decays to a constant positive potential $V_\infty$ at infinity. As in the first problem, the nonlinearity $f$, under very mild assumptions, is asymptotically linear or superlinear and subcritical at infinity, not satisfying any monotonicity condition.

In both problems the existence of solution is established in situations where the equation does not have a ground state solution, via a composition of two translated solitons and its projection on the so called Pohozaev manifold. However, at the end of each chapter, we justify that the method applied is also valid for any finite composition of these solitons.

**Key-Words:** Nonlinear Schrödinger equation, positive solution, Pohozaev manifold, group action, symmetry.

# Resumo

Neste trabalho, consideramos dois problemas. No primeiro capítulo, estabelecemos a existência de uma solução positiva para a equação não linear de Schrödinger

$$-\Delta u + V(x)u = f(u), \qquad u \in D^{1,2}(\mathbb{R}^N), \; N \geq 3, \tag{$\wp_1$}$$

com potencial $V$ que é invariante sob uma ação de grupo $G \subset O(N)$, onde $O(N)$ é o grupo de transformações ortogonais, e decai para zero no infinito, com uma taxa apropriada, aproximando-se da equação de campo escalar limite do tipo *massa zero*; e a não linearidade $f$, sob suposições muito suaves, é assintoticamente linear ou superlinear e subcrítica no infinito, não satisfazendo nenhuma condição de monotonicidade. Nós lidamos tanto com ações de grupos finitos quanto com ações de grupos infinitos.

No segundo capítulo, estudamos a existência de uma solução positiva para uma equação não linear de Schrödinger

$$-\Delta u + V(x)u = f(u), \qquad u \in H^1(\mathbb{R}^N), \; N \geq 3, \tag{$\wp_2$}$$

onde o potencial $V$ é uma função positiva, invariante sob uma ação de grupo $G \subset O(N)$, que decai para um potencial constante positivo $V_\infty$ no infinito. Como no primeiro problema, a não linearidade $f$, sob suposições muito suaves, é assintoticamente linear ou superlinear e subcrítica no infinito, não satisfazendo nenhuma condição de monotonicidade.

Em ambos os problemas a existência de solução da equação é estabelecida em situações onde o nível mínimo de energia não pode ser obtido, usando a composição de dois sólitons transladados e sua projeção na chamada variedade de Pohozaev. No entanto, ao final de cada capítulo, justificamos que o método aplicado também é válido para qualquer composição finita desses sólitons.

**Palavras-Chaves:** Equação não linear de Schrödinger, solução, variedade de Pohozaev, ação de grupo, simetria.

# Contents

# Introduction

In this work, we are interested in the existence of positive bound state solutions for two classes of nonlinear Schrödinger equations:

$$-\Delta u + V(x)u = f(u), \qquad u \in D^{1,2}(\mathbb{R}^N),\ N \geq 3, \tag{$\wp_1$}$$

with potential $V$ vanishing at infinity, possibly changing sign, and an appropriate rate, approaching zero mass type limit scalar field equation; and also

$$-\Delta u + V(x)u = f(u), \qquad u \in H^1(\mathbb{R}^N),\ N \geq 3, \tag{$\wp_2$}$$

where the potential $V$ is a positive function which decays to a constant positive potential $V_\infty$ at infinity, symmetric under some group action $G$. For both problems, the nonlinearity $f$, under very mild assumptions, is asymptotically linear or superlinear and subcritical at infinity, $f(s)/s$, $s > 0$, not satisfying any monotonicity condition. More precisely, we will assume that $V$ is invariant under a group action $G \subset O(N)$, that is,

$$V(gx) = V(x), \quad \text{for all } g \in G \text{ and all } x \in \mathbb{R}^N,$$

where $O(N)$ is the group of orthogonal transformations from $\mathbb{R}^N$ to $\mathbb{R}^N$. Symmetry plays a basic role in variational problems. For example, $H^1(\mathbb{R}^N)$ is not compactly embedded in $L^2(\mathbb{R}^N)$ because of the action of translations.

Let $N \geq 3$ and $2^* = 2N/(N-2)$. The Hilbert space

$$D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

will be used when $V(x) \to 0$, as $|x| \to \infty$ and the associated limit problem is $-\Delta u = f(u)$.

Given a subgroup $G$ of $O(N)$, we denote by $Gx := \{gx : g \in G\}$ the $G$-orbit of $x$ and by $\#Gx$ its cardinality. We define the *action of $G$ on $D^{1,2}(\mathbb{R}^N)$* by

$$gu(x) := u(g^{-1}x), \quad \text{for every } u \in D^{1,2}(\mathbb{R}^N),\ g \in G \text{ and } x \in \mathbb{R}^N.$$

The action of a topological group $G$ on a normed space $X$ is a continuous map

$$G \times X \to X : [g,u] \to gu$$

such that, given $g_1, g_2 \in G$ and $u \in X$,

$$(i)\ u \mapsto gu \text{ is linear}; \quad (ii)\ (g_1 g_2)u = g_1(g_2 u); \quad (iii)\ id\ u = u,$$

where $id \in G$ is the identity element of $G$. The action is *isometric* if

$$\|gu\| = \|u\|.$$

We say that a group $G$ acts effectively on $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ if, for all $x \in \mathbb{S}^{N-1}$, there exists $g \in G$ such that $gx \neq x$. This means that if $G$ is a finite or infinite group, for all $x \in \mathbb{S}^{N-1}$ the $G$-orbit of $x$ satisfies $\#Gx \in [2, \infty]$. We define

$$\ell(G) := \min\{\#Gx : x \in \mathbb{S}^{N-1}\}$$

and in this work we are going to consider only the cases for which $\ell(G) < +\infty$. Hirata in [23] also considered the case $\ell(G) = +\infty$, but assuming the condition $f(s)/s$ is increasing, for $s > 0$ small enough.
We choose $x_0 \in \mathbb{S}^{N-1}$ such that $\#\{gx_0 : g \in G\} = \ell(G)$ and define also

$$\{e_1, \dots, e_{\ell(G)}\} := \{gx_0 : g \in G\}, \tag{0.0.1}$$

$$d_G := \min_{i \neq j} |e_i - e_j| \in (0, 2]. \tag{0.0.2}$$

The space of $G$-symmetric functions in $D^{1,2}(\mathbb{R}^N)$ is defined by

$$\begin{aligned} D^{1,2}_G(\mathbb{R}^N) &:= \{u \in D^{1,2}(\mathbb{R}^N) : gu = u, \forall g \in G\} \\ &= \{u \in D^{1,2}(\mathbb{R}^N) : u(g^{-1}x) = u(x), \forall g \in G, \forall x \in \mathbb{R}^N\}. \end{aligned}$$

Similarly, we define the action of $G \subset O(N)$ on $H^1(\mathbb{R}^N)$ and the space of $G$-symmetric functions $H^1_G(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$.

In our work, we will assume that $G \subset O(N)$, where $N \geq 3$ and $\ell(G) \geq 2$. Some examples are given.

- Taking $\mathbb{R}^4$ and $G = \mathbb{Z}_5 \times \mathbb{Z}_5$, where $\mathbb{Z}_5$ is the cyclic group generated by the 5-th root of the unity, we have $\ell(G) = 5$ and $d_G = \frac{1}{2}\sqrt{10 - 2\sqrt{5}}$.
- Observe that, when $G = \{Id, -Id\}$, we have $\ell(G) = 2$ and $d_G = 2$.
- Take $\mathbb{R}^4$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then, $\ell(G) = 2$ and $d_G = 2$. Notice that $x_1 = (1, 0, 0, 0)$ is such that $\#Gx_1 = 2$, whereas $x_2 = (0, 0, 1, 0)$ has $\#Gx_2 = 3$.

## 0.1 Some known results

Bartsch-Willem in [8] considered the case $G = O(N)$, that is, the potential $V$ is radially symmetric and they showed that the corresponding functional satisfies the Palais-Smale condition and they proved the existence of a radially symmetric solution of ($\wp_2$).

Bartsch-Wang in [7], for the more general group action $G \subset O(N)$, where $G$ is an infinite group, proved that the subspace of $G$-symmetric functions $H^1_G(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$ can be compactly embedded into $L^p(\mathbb{R}^N)$, for $2 < p < 2^*$, under assumption

$$\#\{gx : g \in G\} = \infty \qquad \text{for all } x \in \mathbb{S}^{N-1}.$$

Furthermore, under the global Ambrosetti-Rabinowitz condition, they proved that problem ($\wp_2$) has a positive solution.

Hirata in [22] showed the existence of a positive solution of ($\wp_2$), under $V$ a constant potential and $f$, without Ambrosetti-Rabinowitz condition, but the monotonicity condition $f(s)/s$, for $s > 0$ increasing, restricted to a finite group $G$. In a subsequent paper, Hirata in [23] addressed the problem with a symmetric variable potential $V$ with group action $G \subset O(N)$, dealing with both finite and infinite group actions. The existence of a positive solution was shown for a wide class of nonlinearities $f$, still assuming that $f(s)/s$ is increasing, for $s > 0$ small enough.

Our goal in the first chapter is to find a positive bound state to the problem ($\wp_1$), trying to loosen the assumptions found in the literature, either in the potential or in the nonlinearity [2, 4, 5, 10, 25]. We avoid, for instance, to apply the spectral theory approach or the so called Nehari manifold constrained approach. Our purpose is to prove the existence of a positive bound state solution to the problem ($\wp_1$), when a ground state solution cannot be obtained, with potential $V$ which decays exponentially at infinity to zero and the nonlinearity $f$ does not satisfy any monotonicity condition, i. e. the function $s \mapsto f(s)/s$ is not increasing for $s > 0$. Here, we assume that the potential $V$ is invariant under a group action $G \subset O(N)$ and prove that problem $(\wp_1)$ has a positive solution, applying the symmetric mountain pass theorem of Ambrosetti-Rabinowitz [3], based on the results obtained by Jun Hirata in [22,23]. The method applied, assuming for simplicity $G = O(N-1) \times \mathbb{Z}_2 \subset O(N)$, where $\mathbb{Z}_2 := \{id, -id\}$, and $\ell(G) = 2$, allows to combine two copies of translated positive soliton solutions of the limit problem at infinity, projecting their sum onto the so called Pohozaev manifold, in order to construct a convenient path in the mountain pass theorem with $G$ symmetric functions. This was based on the important papers by Clapp and Maia [16, 17].

This new approach allows us to tackle a model problem like

$$-\Delta u + \frac{1}{(1+|x|)^k} u = \frac{2u^{11} - 4\sqrt{2}\,u^9 + 4u^7}{u^{10}+1}, \quad u > 0, \qquad u \in D^{1,2}(\mathbb{R}^3),$$

where $k > 2$ and $f(s) := (2s^{11} - 4\sqrt{2}s^9 + 4s^7)/(s^{10}+1)$ is asymptotically linear at infinity, but is such that $f(s)/s$ is not increasing for $s > 0$, for instance. Likewise, $f(s) = s^7(1 - \sin(s))/(1 + s^4)$, for $s > 0$, in $\mathbb{R}^3$ is super linear and subcritical at infinity and satisfies mild hypotheses but no monotonicity condition on $f(s)/s$.
The seminal works of Bahri and Li [6] and Cerami and Passaseo [14] presented constructions of bound state solutions, whenever the minimal action of the associated functional is not attained. They succeeded by building a convex combination of two soliton positive solutions of a limit problem (bumps) and projecting on the sphere of radius one in an $L^p$ space, for a pure power nonlinearity $f(s) = s^{p-1}$, with $2 < p < 2^*$. Their method was applied in many works that followed and in different scenarios, but it would de hard to list them all; we would refer to [15] and references therein. More recently, a similar approach was developed to construct bound state solutions by using projections of convex combinations of two positive bumps on the Nehari manifold, see [16, 19, 26, 30] and their references. The limitation, in this case, is having to assume some monotonicity on $f(s)/s$.

In a fundamental paper [17], when the nonlinearity $f$ is subcritical at infinity and supercritical near the origin, and the potential $V$ vanishes at infinity, under a suitable decay assumption on the potential, Clapp and Maia showed that the problem $(\wp_1)$ has a positive bound state.

This first chapter is organized as follows: Section 2 is devoted to presenting the variational setup and the properties of the associated Pohozaev manifold. In Section 3 we study the behaviour of constrained minimizing sequences of the operator associated with problem $(\wp_1)$. Tight estimates of interactions of two translated and dilated copies of a positive solution of the autonomous problem are obtained in Section 4. Finally, these estimates are applied in the proof of the main result of existence of a positive solution stated in main theorem.

In the second chapter, our purpose is to prove the existence of a positive bound state solution to the problem ($\wp_2$), with potential $V$ which decays exponentially at infinity to $V_\infty > 0$ and the nonlinearity $f$ does not satisfy any monotonicity condition and, furthermore, the function $s \mapsto f(s)/s$ is not increasing for $s > 0$ sufficiently small. We also assume that the potential $V$ is invariant under a group action $G \subset O(N)$, with $G = O(N-1) \times \mathbb{Z}_2 \subset O(N)$, where $\mathbb{Z}_2 := \{id, -id\}$, and $\ell(G) = 2$, from simplicity and, the method applied is also combining two copies of translated positive soliton solutions of the limit problem at infinity, projecting their sum onto the so called Pohozaev manifold. The approach used for equations of type ($\wp_2$) can be applied to the following model problem

$$-\Delta u + V(x)u = f(u), \qquad u \in H^1(\mathbb{R}^N),$$

where $V(x) := 1 + Ae^{-k|x|}$, $A, k \in \mathbb{R}$, $A > 0$ sufficiently small, $k > 2$ and $f(s) := (2s^9 - 2s^8 + 5s^7)/(s^8 + 1)$ is asymptotically linear such that $f(s)/s$ is not increasing for $s > 0$, for instance.

The primary works dealing with the existence of solutions for equations of type ($\wp_2$) via variational methods are due to Benci and Cerami in [9] in exterior domains and Bahri and Lions in [5] in unbounded domains. Using a different approach, Évéquoz and Weth in [19], Clapp and Maia in [16] and Maia and Pellacci in [30] showed the existence of a positive solution to the problem ($\wp_2$), for general non-homogeneous nonlinearities, either superlinear or asymptotically linear at infinity in an exterior domain.

In a recent paper, Jaroslaw Mederski in [32] studied the following problem

$$-\Delta u = g(u), \qquad u \in H^1(\mathbb{R}^N), \quad N \geq 3, \tag{$\wp$}$$

with a nonlinearity $g$ under the general hypotheses due to Berestycki and Lions in [10], and proved the existence and multiplicity of nonradial solutions to the problem ($\wp$). More precisely, Mederski found at least one nonradial solution for any $N \geq 4$ and, in addition, for $N \neq 5$, he showed the existence of infinitely many different nonradial solutions. These results represent an important improvement to problem ($\wp$), because they were established for the first time. Furthermore, these results give a partial positive answer to a problem which had been open for more than thirty years.

The second chapter is organized as follows: Section 2 is devoted to presenting some properties of the Pohozaev manifold associated to the problem ($\wp_2$) and preliminary results. In Section 3, we study the behaviour of constrained minimizing sequences of the operator associated to the problem ($\wp_2$). In Section 4, we obtain the estimates of interactions of two translated copies of a positive solution of the autonomous problem and, finally, these estimates are applied in the proof of the main result of existence of a positive solution stated in main theorem.

## 0.2 Our results

Motivated by important papers of Clapp and Maia [16,17] and Hirata [22,23], for both problems $(\wp_1)$ and $(\wp_2)$, we will assume that there exists a subgroup $G$ of $O(N)$ that acts effectively on $\mathbb{S}^{N-1}$, where $G$ will be considered as already mentioned, and the potential $V$ is $G$-invariant.
Let $S$ be the best constant of Gagliardo-Nirenberg-Sobolev inequality

$$S\left(\int_{\mathbb{R}^N}|u|^{2^*}\,dx\right)^{2/2^*}\le\int_{\mathbb{R}^N}|\nabla u|^2dx. \tag{0.2.1}$$

To consider problem $(\wp_1)$, we will assume the following conditions on the potential $V$:

$(V_1)$ $V\in C^2(\mathbb{R}^N)$, $V(gx)=V(x)$ for all $g\in G$ and $\displaystyle\int_{\mathbb{R}^N}|V^-|^{N/2}<S^{N/2}$, where $V^-(x):=\min\{0,V(x)\}$;

$(V_2)$ There exist constants $A_0, A_1>0$ and $k\in\mathbb{R}$, $k>\max\{2,N-2\}$ such that

$$|V(x)|\le A_0(1+|x|)^{-k} \quad\text{and}\quad |\nabla V(x)\cdot x|\le A_1(1+|x|)^{-k}, \quad\text{for all } x\in\mathbb{R}^N;$$

$(V_3)$ $\displaystyle\int_{\mathbb{R}^N}|W^+|^{N/2}<\left(\frac{S}{2}\right)^{N/2}$, where $W^+(x):=\max\{0,\nabla V(x)\cdot x\}$;

$(V_4)$ $xH(x)x\in L^{N/2}(\mathbb{R}^N)$ and $\displaystyle\lim_{|x|\to\infty}xH(x)x=0$, where $H$ denotes the Hessian matrix of $V$.

Moreover, considering $F(s)=\int_0^s f(t)dt$, we will assume the following hypotheses on the function $f$:

$(f_1)$ $f\in C^1([0,\infty))\cap C^3((0,\infty))$, $f(s)\ge 0$ for all $s>0$;

$(f_2)$ There exists a constant $A_2>0$ such that

$$\left|f^{(i)}(s)\right|\le A_2|s|^{2^*-(i+1)},$$

where $f^{(-1)}:=F$ and $f^{(i)}$ is the $i$-th derivative of $f$, $i=0,1,2,3$;

$(f_3)$ $\displaystyle\lim_{s\to 0^+}\frac{f(s)}{s^{2^*-1}}=\lim_{s\to+\infty}\frac{f(s)}{s^{2^*-1}}=0$ and $\displaystyle\lim_{s\to+\infty}\frac{f(s)}{s}\geq \ell>0$;

$(f_4)$ Setting $Q(s):=\frac{1}{2}f(s)s-F(s)$, there is a constant $D\geq 1$ such that $Q(s)\leq DQ(t)$, for all $s\in[0,t]$, $t>0$, and $\displaystyle\lim_{s\to+\infty}Q(s)=+\infty$.

Our main result in the first chapter is the following

**Theorem 1.** *Assume that* $(V_1)$–$(V_4)$ *and* $(f_1)$–$(f_4)$ *hold true. Then, problem* $(\wp_1)$ *has a positive solution* $u\in D^{1,2}(\mathbb{R}^N)$ *which satisfies*

$$u(gx)=u(x),\quad \textit{for all}\ \ g\in G\ \ \textit{and all}\ \ x\in\mathbb{R}^N.$$

To consider problem $(\wp_2)$, we will assume the following conditions on the potential $V$:

$(\widetilde{V}_1)$ $V\in C^2(\mathbb{R}^N)$, $V(gx)=V(x)$ for all $g\in G$, $\displaystyle\inf_{x\in\mathbb{R}^N}V(x)>0$ and $\displaystyle\lim_{|x|\to\infty}V(x)=V_\infty>0$;

$(\widetilde{V}_2)$ There exist constants $A_0>0$ and $k>d_G\sqrt{V_\infty}$ such that $V(x)\leq V_\infty+A_0\exp(-k|x|)$, for all $x\in\mathbb{R}^N$;

$(\widetilde{V}_3)$ $\nabla V(x)\cdot x\in L^{N/2}(\mathbb{R}^N)$, $\displaystyle\lim_{|x|\to\infty}\nabla V(x)\cdot x=0$ and $\displaystyle\int_{\mathbb{R}^N}|W^+|^{N/2}<\left(\frac{S}{2}\right)^{N/2}$, where $W^+(x):=\max\{0,\nabla V(x)\cdot x\}$;

$(\widetilde{V}_4)$ $\displaystyle\lim_{|x|\to\infty}xH(x)x=0$, where $H$ denotes the Hessian matrix of $V$.

Moreover, considering $F(s)=\int_0^s f(t)dt$, we will assume the following hypotheses on the function $f$:

$(\widetilde{f}_1)$ $f\in C^1([0,\infty))\cap C^3((0,\infty))$ and $f(s)\geq 0$ for all $s>0$;

$(\widetilde{f}_2)$ There exist $A_1>0$ and $1<p_1\leq p_2<(N+2)/(N-2)=2^*-1$ and

$$|f^{(i)}(s)|\leq A_1(|s|^{p_1-i}+|s|^{p_2-i}),$$

where $f^{(-1)}:=F$ and $f^{(i)}$ is the $i$-th derivative of $f$, $i=0,1,2,3$;

$(\widetilde{f}_3)$ $\displaystyle\lim_{s\to+\infty}\frac{f(s)}{s}\geq\ell>V_\infty>0$;

$(\widetilde{f}_4)$ Setting $Q(s):=\frac{1}{2}f(s)s-F(s)$, there is a constant $D\geq 1$ such that $Q(s)\leq DQ(t)$, for all $s\in[0,t]$, $t>0$, and $\displaystyle\lim_{s\to+\infty}Q(s)=+\infty$.

The main result of the second chapter is the following

**Theorem 2.** *Assume that* $(\widetilde{V}_1)$–$(\widetilde{V}_4)$ *and* $(\widetilde{f}_1)$–$(\widetilde{f}_4)$ *hold true. Then, problem* $(\wp_2)$ *has a positive solution* $u \in H^1(\mathbb{R}^N)$ *which satisfies*

$$u(gx) = u(x), \quad \text{for all } g \in G \text{ and all } x \in \mathbb{R}^N.$$

There are several delicate issues in dealing with the zero mass case, where the potential is vanishing at infinity. Already the variational formulation requires some care, because the energy space $D^{1,2}(\mathbb{R}^N)$ is only embedded in $L^2(\mathbb{R}^N)$. Equations of the type $(\wp_1)$, where the potential $V$ is invariant under a group action $G$ $O(N)$ and that decays to zero at infinity, is are not common in the literature. However, there are some very important works, considering equations of the type $(\wp_2)$, the positive mass case, in which the potential $V$ is invariant under a group action $G$ $O(N)$ and tends to a positive constant at infinity, for example, [7, 8, 22, 23]. Different from these fundamental roles, which inspire us to develop our work, to prove Theorems 1 and 2, we will not consider either the global Ambrosetti-Rabinowitz condition or the monotonicity $f(s)/s$ increasing, for $s > 0$ sufficiently small.

Chapter 1

# Schrödinger equations with potentials vanishing at infinity

## 1.1 Introduction

This chapter deals with the existence of a positive solution for the problem

$$-\Delta u + V(x)u = f(u), \qquad u \in D^{1,2}(\mathbb{R}^N), \ N \ge 3, \tag{$P$}$$

with a potential $V$ vanishing at infinity, possibly changing sign, and a nonlinearity $f$ under very mild hypotheses, asymptotically linear or superlinear and subcritical at infinity, not satisfying any monotonicity condition. The existence of a solution to this problem is established in situations where a ground state solution is not attained.
We will assume that the potential $V$ is invariant under a group action $G \subset O(N)$ and we try to find a positive solution in the space of $G$-symmetric functions

$$D^{1,2}_G(\mathbb{R}^N) := \{u \in D^{1,2}(\mathbb{R}^N) : u(gx) = u(x), \forall g \in G, \forall x \in \mathbb{R}^N\}.$$

We will consider the case that $G \subset O(N)$ is closed subgroup with the following property: for any $x \in \mathbb{S}^{N-1}$, there exists $g \in G$ such that $gx \neq x$. This means that $G$ acts effectively on $\mathbb{S}^{N-1}$, that is, $G$ satisfies

$$\# \{gy : g \in G\} \in [2, \infty], \qquad \text{for all } y \in \mathbb{S}^{N-1}, \tag{1.1.1}$$

where $\#\{\ \cdot\ \}$ denotes the cardinal number of sets and $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$. We will define

$$\ell(G) := \min\{\#Gx : x \in \mathbb{S}^{N-1}\}.$$

We observe that in this work we are going to consider only the case $\ell(G)$ finite and

$$\ell(G) \in [2, \infty).$$

In fact, for simplicity, our study is focused in the case $\ell(G) = 2$, but could clearly be extended to finite $\ell(G) > 2$.
Let $S$ be the best constant of Gagliardo-Nirenberg-Sobolev inequality (0.2.1).
Throughout Chapter 1, we will consider the potential $V$ under assumptions $(V_1)$–$(V_4)$ and the nonlinearity $f$ under assumptions $(f_1)$–$(f_4)$.

Note that $F(0) = 0$ and by $(f_1)$, $F(s) \geq 0$ for $s > 0$.
Under assumptions $(f_1)$–$(f_3)$, the limit problem at infinity

$$-\Delta u = f(u), \qquad u \in D^{1,2}(\mathbb{R}^N), \tag{$P_0$}$$

has a ground state solution $w$ which is positive, radially symmetric and decreasing in the radial direction, see [10] and [31].
Flucher in [20, Theorem 6.5] and more recently Vétois in [35] have shown that under $(f_1)$ and $(f_2)$ there exist constants $A_4, A_5, A_6 > 0$ such that

$$A_4(1 + |x|)^{-(N-2)} \leq w(x) \leq A_5(1 + |x|)^{-(N-2)}, \tag{1.1.2}$$

$$|\nabla w(x)| \leq A_6(1 + |x|)^{-(N-1)}. \tag{1.1.3}$$

A radial solution with decay (1.1.2) is called a fast decay solution of equation $(P_0)$.

By virtue of $G$-invariant property, we do not need the uniqueness of positive solution for the limit problem $(P_0)$. Since $D^{1,2}(\mathbb{R}^N)$ is not compactly embedded into $L^{2^*}(\mathbb{R}^N)$, then the mountain pass minimax value for corresponding functional may not be attained. However, as we are assuming that the potential $V$ and the function $f$ are invariant under the group action $G$, we will show that the symmetric mountain pass minimax value for functional restricted to the subspace $D^{1,2}_G(\mathbb{R}^N)$ is attained.

Now we can restate our main result of existence of a solution in this chapter.

**Theorem 1.1.1.** *Assume that $(V_1)$–$(V_4)$, $(f_1)$–$(f_4)$ hold true. Then, problem $(P)$ has a positive solution $u \in D^{1,2}_G(\mathbb{R}^N)$.*

**Remark 1.1.2.** The condition $(V_2)$ implies that $V \in L^{N/2}(\mathbb{R}^N)$ and $\nabla V(x)\cdot x \in L^{N/2}(\mathbb{R}^N)$, for all $x \in \mathbb{R}^N$. Moreover,

$$V(x) \to 0, \quad \nabla V(x)\cdot x \to 0, \quad \text{as } |x| \to \infty, \tag{1.1.4}$$

Note that a model potential $V$, defined by $V(x) := (1+|x|)^{-k}$, with $k > \max\{2, N-2\}$, satisfies the assumptions $(V_1)$–$(V_4)$.

Also note that assumptions $(f_1)$ and $(f_2)$ imply that $f'(0) = 0$ and extends $f'$ continuously to $0$. Furthermore, L'Hôpital's rule and $(f_3)$ give that

$$\lim_{s\to 0^+} \frac{f(s)}{s^{2^*-1}} = \lim_{s\to 0^+} \frac{f'(s)}{s^{2^*-2}} = 0 \tag{1.1.5}$$

and

$$\lim_{s\to +\infty} \frac{f(s)}{s^{2^*-1}} = \lim_{s\to +\infty} \frac{f'(s)}{s^{2^*-2}} = 0. \tag{1.1.6}$$

On the other hand, hypotheses $(f_1)$, $(f_2)$ and $(f_3)$ imply that

$$\lim_{s\to 0^+} \frac{F(s)}{s^{2^*}} = \lim_{s\to +\infty} \frac{F(s)}{s^{2^*}} = 0. \tag{1.1.7}$$

## 1.2 Pohozaev manifold and variational setting

The well know identity obtained by Pohozaev in [33] has since then been very useful as a constraint in the study of scalar field equations. We will take it as a fundamental tool for our approach. Its version for non-autonomous problems is based in the work of De Figueiredo, Lions and Nussbaum [18] which we state here for the sake of completeness.

**Proposition 1.2.1.** *Let $u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}$ be a solution of problem $-\Delta u = g(x,u)$, $x \in \Omega$, $u(x) = 0$, $x \in \partial\Omega$, where $\Omega \subseteq \mathbb{R}^N$ is a regular domain in $\mathbb{R}^N$ and $g \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$. If $G(x,u) = \int_0^u g(x,s)ds$ is such that $G(\cdot,u(\cdot))$ and $x_i G_{x_i}(\cdot,u(\cdot))$ are in $L^1(\Omega)$, then $u$ satisfies*

$$N\int_\Omega G(x,u)dx + \sum_{i=1}^N \int_\Omega x_i G_{x_i}(x,u)dx - \frac{N-2}{2}\int_\Omega |\nabla u|^2 dx = \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 x\cdot\eta(x)dS_x,$$

*where $\eta$ denotes the unitary exterior normal vector to boundary $\partial\Omega$ and $dS_x$ represents the area element $(N-1)$-dimensional of $\partial\Omega$. Moreover, if $\Omega = \mathbb{R}^N$, then*

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx = N\int_{\mathbb{R}^N} G(x,u)dx + \sum_{i=1}^N \int_{\mathbb{R}^N} x_i G_{x_i}(x,u)dx. \tag{1.2.1}$$

*Proof.* We have

$$
\begin{aligned}
\Delta u(\nabla u\cdot x) &= \mathrm{div}(\nabla u(\nabla u\cdot x)) - |\nabla u|^2 - \nabla\left(\frac{|\nabla u|^2}{2}\right)\cdot x \\
&= \mathrm{div}\left(\nabla u(\nabla u\cdot x) - x\frac{|\nabla u|^2}{2}\right) + \frac{N-2}{2}|\nabla u|^2. 
\end{aligned}
\tag{1.2.2}
$$

On the other hand, we also have that

$$
g(x,u)(\nabla u\cdot x) = \mathrm{div}(x\,G(x,u)) - NG(x,u) - \sum_{i=1}^{N} x_i G_{x_i}(x,u). \tag{1.2.3}
$$

Therefore, multiplying the equation $-\Delta u = g(x,u)$ by $\nabla u\cdot x$, it follows from (1.2.2) and (1.2.3) that

$$
\mathrm{div}\left(x\,G(x,u) + \nabla u(\nabla u\cdot x) - x\frac{|\nabla u|^2}{2}\right) = NG(x,u) + \sum_{i=1}^{N} x_i G_{x_i}(x,u) - \frac{N-2}{2}|\nabla u|^2.
$$

Thus, by the Divergence Theorem, we have

$$
\begin{aligned}
&\int_{\partial\Omega}\left(x\,G(x,u) + \nabla u\, x\cdot\nabla u - x\frac{|\nabla u|^2}{2}\right)\cdot\eta(x)\,dS_x \\
&= \int_{\Omega}\mathrm{div}\left(x\,G(x,u) + \nabla u(\nabla u\cdot x) - x\frac{|\nabla u|^2}{2}\right)dx \\
&= \int_{\Omega}\left(NG(x,u) + \sum_{i=1}^{N} x_i G_{x_i}(x,u) - \frac{N-2}{2}|\nabla u|^2\right)dx.
\end{aligned}
$$

Since $u \equiv 0$ on $\partial\Omega$ and so $G(x,u) = G(x,0) = 0$, we have $\nabla u = (\nabla u\cdot\eta)\eta$. Hence, it follows that, on $\partial\Omega$,

$$
\begin{aligned}
\left(\nabla u(\nabla u\cdot x) - x\frac{|\nabla u|^2}{2}\right)\cdot\eta &= \left[(\nabla u\cdot\eta)\eta(\nabla u\cdot x) - x\frac{|\nabla u|^2}{2}\right]\cdot\eta \\
&= \left[(\nabla u\cdot\eta)(\nabla u\cdot x)\eta - x\frac{|\nabla u|^2}{2}\right]\cdot\eta \\
&= (\nabla u\cdot\eta)((\nabla u\cdot\eta)\eta)\cdot x - \frac{|\nabla u|^2}{2}x\cdot\eta \\
&= (\nabla u\cdot\eta)^2 x\cdot\eta - \frac{|\nabla u|^2}{2}x\cdot\eta \\
&= |\nabla u|^2 x\cdot\eta - \frac{|\nabla u|^2}{2}x\cdot\eta = \frac{|\nabla u|^2}{2}x\cdot\eta,
\end{aligned}
$$

and so we conclude that

$$N\int_\Omega G(x,u)dx+\sum_{i=1}^{N}\int_\Omega x_i G_{x_i}(x,u)dx-\frac{N-2}{2}\int_\Omega|\nabla u|^2dx=\frac{1}{2}\int_{\partial\Omega}|\nabla u|^2 x\cdot\eta(x)dS_x.$$

Now let us consider $\Omega=\mathbb{R}^N$. Since $|\nabla u|\in L^2(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla u|^2dx&=\int_0^{\infty}\int_{\partial B_r(0)}|\nabla u(r,\theta)|^2dS_r r^{N-1}dr\\&=\int_0^{\infty}r^{N-2}\int_{\partial B_r(0)}|\nabla u(r,\theta)|^2 r\,dS_r dr<+\infty.\end{aligned}$$

We will show that there exists a sequence of reals numbers $(r_n)$ such that, as $n\to\infty$,

$$r_n\to+\infty,\qquad r_n\int_{\partial B_{r_n}(0)}|\nabla u(r_n,\theta)|^2dS_{r_n}\to 0. \tag{1.2.4}$$

Suppose, by contradiction, that there is no such sequence satisfying (1.2.4). Then, there exists a constant $\alpha>0$ such that

$$\liminf_{r\to+\infty} r\int_{\partial B_r(0)}|\nabla u(r,\theta)|^2dS_r\geq\alpha>0.$$

Thus, we have

$$\xi(r):=r\int_{\partial B_r(0)}|\nabla u(r,\theta)|^2dS_r\geq\alpha>0$$

and so

$$\int_{\mathbb{R}^N}|\nabla u|^2dx=\int_0^{\infty}r^{N-2}\xi(r)dr\geq\int_0^{\infty}\alpha r^{N-2}dr=+\infty,$$

which is a contradiction, using that $|\nabla u|\in L^2(\mathbb{R}^N)$. So there is a sequence of reals numbers $(r_n)$ that satisfies (1.2.4) and, furthermore, as $n\to\infty$, we have:

$$\int_{B_{r_n}(0)}|\nabla u|^2dx\to\int_{\mathbb{R}^N}|\nabla u|^2dx,\qquad\int_{B_{r_n}(0)}G(x,u)dx\to\int_{\mathbb{R}^N}G(x,u)dx$$

and

$$\sum_{i=1}^{N}\int_{B_{r_n}(0)}x_iG_{x_i}(x,u)dx\to\sum_{i=1}^{N}\int_{\mathbb{R}^N}x_iG_{x_i}(x,u)dx,$$

and so we get (1.2.1). □

In the case of problem $(P)$, by (1.2.1), we have the following Pohozaev identity

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx = N\int_{\mathbb{R}^N}\left(F(u) - V(x)\frac{u^2}{2}\right)dx - \frac{1}{2}\int_{\mathbb{R}^N}\nabla V(x)\cdot x\,u^2dx. \qquad (1.2.5)$$

Associated with problem $(P)$, we define the functional $I_V : D^{1,2}_G(\mathbb{R}^N) \to \mathbb{R}$ by

$$I_V(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^N}F(u)dx.$$

Let us define the functional $J_V : D^{1,2}_G(\mathbb{R}^N) \to \mathbb{R}$ by

$$J_V(u) = \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx + \frac{N}{2}\int_{\mathbb{R}^N}\left(\frac{\nabla V(x)\cdot x}{N} + V(x)\right)u^2dx - N\int_{\mathbb{R}^N}F(u)dx,$$

and define the Pohozaev manifold associated to the problem $(P)$ by

$$\mathcal{P}^G_V := \{u \in D^{1,2}_G(\mathbb{R}^N)\setminus\{0\} : J_V(u) = 0\}.$$

Let us also consider the Pohozaev manifold $\mathcal{P}_0$ associated to the limit problem $(P_0)$. We have

$$\mathcal{P}_0 := \{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\} : J_0(u) = 0\},$$

where

$$J_0(u) := \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx - N\int_{\mathbb{R}^N}F(u)dx.$$

We recall that solutions of $(P_0)$ are critical points of the functional $I_0 : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$,

$$I_0(u) := \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx - \int_{\mathbb{R}^N}F(u)dx, \qquad u \in D^{1,2}(\mathbb{R}^N).$$

We also recall that $w$ is a ground state solution of the limit problem $(P_0)$ if

$$I_0(w) = m_0 := \inf\{I_0(u) : u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\} \text{ is a solution of } (P_0)\}. \qquad (1.2.6)$$

We will denote

$$p_0 = \inf_{u\in\mathcal{P}_0} I_0(u). \qquad (1.2.7)$$

It was shown in [31] that $m_0 = p_0$, under more general hypotheses, which contains ours as a particular case.

We define $f(s) := -f(-s)$ for $s < 0$. Then, by condition $(f_1)$, we have $f \in C^1(\mathbb{R})$ and it is an odd function. Note that, if $u$ is a positive solution of problem $(P)$ for this new function, it is also a solution of $(P)$ for the original function $f$. Hereafter, we shall consider this extension, and establish the existence of a positive solution for $(P)$.

Recall the space of $G$-symmetric functions in $D^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, with its standard scalar product and norm

$$\langle u, v\rangle := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx, \qquad \|u\| := \int_{\mathbb{R}^N} |\nabla u|^2 dx. \tag{1.2.8}$$

Since $f \in C^1(\mathbb{R})$ and $f$ satisfies $(f_1)$–$(f_3)$, a classical result of Berestycki and Lions establishes the existence of a ground state solution $w \in C^2(\mathbb{R}^N)$ to problem $(P_0)$, which is positive, radially symmetric and decreasing in the radial direction, see [10, Theorem 4].

Let us denote $\|\cdot\|_q$ the $L^q(\mathbb{R}^N)$-norm, for all $q \in [1, \infty)$ and $C$, $C_i$ are positive constants which may vary from line to line. Given $u, v \in D^{1,2}_G(\mathbb{R}^N)$, let us define

$$\langle u, v\rangle_V := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)dx, \qquad \|u\|_V^2 := \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)dx. \tag{1.2.9}$$

By assumptions $(V_1)$ and $(V_2)$, we can see that the expressions in (1.2.9) are well defined and, using the Sobolev inequality, we conclude that $\|\cdot\|_V$ is a norm in $D^{1,2}_G(\mathbb{R}^N)$ which is equivalent to the standard one. Indeed, for all $u \in D^{1,2}_G(\mathbb{R}^N) \setminus \{0\}$, using $(V_1)$, Gagliardo-Nirenberg-Sobolev inequality (0.2.1) and Hölder inequality, there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
\|u\|_V^2 &= \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)dx \\
&\geq \int_{\mathbb{R}^N} |\nabla u|^2 dx - \left(\int_{\mathbb{R}^N} |V^-(x)|^{N/2} dx\right)^{2/N} \left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*} \\
&\geq C_1 \int_{\mathbb{R}^N} |\nabla u|^2 dx = C_1 \|u\|.
\end{aligned} \tag{1.2.10}$$

On the other hand, by condition $(V_2)$, it follows that $V \in L^{N/2}(\mathbb{R}^N)$, and so using (0.2.1) and Hölder inequality, there exists a constant $C_2 > 0$ such that

$$\begin{aligned}
\|u\|_V^2 &= \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)dx \\
&\leq \int_{\mathbb{R}^N} |\nabla u|^2 dx + \left(\int_{\mathbb{R}^N} |V(x)|^{N/2} dx\right)^{2/N} \left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*} \\
&\leq \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\|V\|_{N/2}}{S} \int_{\mathbb{R}^N} |\nabla u|^2 dx \\
&\leq C_2 \int_{\mathbb{R}^N} |\nabla u|^2 dx = C_2 \|u\|.
\end{aligned} \tag{1.2.11}$$

Hence, from (1.2.10) and (1.2.11), we conclude the statement.

**Remark 1.2.2.** Throughout this chapter, to denote an inner product or norm in the space $D^{1,2}(\mathbb{R}^N)$, we will use the same notations adopted for the subspace of $G$-symmetric functions in $D_G^{1,2}(\mathbb{R}^N)$.

Consider the following problem in the space of $G$-symmetric functions $D_G^{1,2}(\mathbb{R}^N)$, for $N \geq 3$,

$$-\Delta u + V(x)u = f(u), \qquad u \in D_G^{1,2}(\mathbb{R}^N). \tag{$P_G$}$$

We will show that solutions of ($P_G$) are also solutions of ($P$). Indeed, suppose that $u_0 \in D_G^{1,2}(\mathbb{R}^N)$ is a weak solution of problem ($P_G$), that is, $u_0$ is a critical point of the restricted functional $I_V$ restricted to $D_G^{1,2}(\mathbb{R}^N)$, and so

$$I_V'(u_0)v = 0, \quad \text{for all } v \in D_G^{1,2}(\mathbb{R}^N).$$

Set

$$\left(D_G^{1,2}(\mathbb{R}^N)\right)^{\perp} := \{u \in D^{1,2}(\mathbb{R}^N) : \langle u, \varphi \rangle_V = 0, \text{ for all } \varphi \in D_G^{1,2}(\mathbb{R}^N)\}.$$

To show that $u_0$ is a critical point of the functional $I_V$ in $D^{1,2}(\mathbb{R}^N)$, it suffices to show that $I_V'(u_0)\tilde{v} = 0$, for all $\tilde{v} \in \left(D_G^{1,2}(\mathbb{R}^N)\right)^{\perp}$, and this is a consequence of the following lemma, which holds for all $u \in D_G^{1,2}(\mathbb{R}^N)$, not only critical points of $I_V$.

**Lemma 1.2.3.** *Assume that* $(V_1)$–$(V_2)$ *and* $(f_1)$–$(f_3)$ *hold true. Then,*

$$I_V'(u)\tilde{v} = 0, \quad \textit{for any } u \in D_G^{1,2}(\mathbb{R}^N) \textit{ and } \tilde{v} \in \left(D_G^{1,2}(\mathbb{R}^N)\right)^{\perp}.$$

*Proof.* Let $u \in D_G^{1,2}(\mathbb{R}^N)$ and $h : \mathbb{R}^N \to \mathbb{R}$ defined by $h(x) = f(u(x))$, for all $x \in \mathbb{R}^N$. So, we have

$$h(gx) = f(u(gx)) = f(u(x)) = h(x), \text{ for any } g \in G \text{ and } x \in \mathbb{R}^N. \tag{1.2.12}$$

Consider the following linear problem

$$\begin{cases} -\Delta v + V(x)v = h(x), & \text{in } \mathbb{R}^N, \\ v \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.2.13}$$

By Riesz representation theorem, we can find the unique solution $v_0 \in D^{1,2}(\mathbb{R}^N)$ to the auxiliary problem (1.2.13). By (1.2.12) and $V(gx) = V(x)$, $v_0(g(\cdot))$ satisfies

$$-\Delta v_0(gx) + V(x)v_0(gx) = h(gx) = h(x)$$

for any $g \in G$ and $x \in \mathbb{R}^N$. It follows from the uniqueness of solutions that $v_0 = v_0 \circ g$ and so $v_0 \in D^{1,2}_G(\mathbb{R}^N)$. Thus, for any $\varphi \in \left(D^{1,2}_G(\mathbb{R}^N)\right)^{\perp}$, we get

$$\begin{aligned} I'_V(u)\varphi &= \langle u, \varphi\rangle_V - \int_{\mathbb{R}^N} f(u(x))\varphi(x)dx = -\int_{\mathbb{R}^N} h(x)\varphi(x)dx \\ &= -\langle v_0, \varphi\rangle_V = 0, \end{aligned}$$

which proves the lemma. □

## 1.3 Auxiliary lemmas for bounded sequences

In what follows, to find solutions to the problem $(P)$, we will try to find solutions to the problem $(P_G)$, that is, let us try to find critical points of the functional $I_V$.

Next lemma presents a new variant of Lions' lemma in $D^{1,2}(\mathbb{R}^N)$, which was proved by Mederski in [31, Lemma 1.5].

**Lemma 1.3.1.** *Suppose that $(u_n) \subset D^{1,2}(\mathbb{R}^N)$ is bounded and for some $r > 0$,*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |u_n|^2 dx = 0. \tag{1.3.1}$$

*Then, $\lim_{n\to\infty} \int_{\mathbb{R}^N} \Phi(u_n)dx = 0$, for any continuous function $\Phi : \mathbb{R} \to [0, \infty)$ satisfying*

$$\lim_{s\to 0} \frac{\Phi(s)}{|s|^{2^*}} = \lim_{|s|\to\infty} \frac{\Phi(s)}{|s|^{2^*}} = 0. \tag{1.3.2}$$

*Proof.* Let $\varepsilon > 0$ and $2 < q < 2^*$, given arbitrarily, and suppose that $\Phi : \mathbb{R} \to [0, \infty)$ is a continuous function satisfying (1.3.2). Then, we find $\delta, M \in \mathbb{R}$ with $0 < \delta < M$ and $C_\varepsilon > 0$ such that

*(i)* $\Phi(s) \le \varepsilon|s|^{2^*}$, for $|s| \le \delta$;

*(ii)* $\Phi(s) \le \varepsilon|s|^{2^*}$, for $|s| > M$;

*(iii)* $\Phi(s) \le C_\varepsilon|s|^q$, for $|s| \in (\delta, M]$.

Hence, in the view of Lions' lemma we get

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \Phi(u_n)dx \le \varepsilon \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(|u_n|^2 + |u_n|^{2^*}\right) dx.$$

Since $(u_n)$ is bounded in $L^2(\mathbb{R}^N)$ and $L^{2^*}(\mathbb{R}^N)$, we may take the limit $\varepsilon \to 0$ and conclude the proof. □

Recall that a sequence $(u_n)$ in $D_G^{1,2}(\mathbb{R}^N)$ is said to be a *$(PS)_d$-sequence for $I_V$ with $d \in \mathbb{R}$* if $I_V(u_n) \to d$ and $\nabla I_V(u_n) \to 0$ in $(D_G^{1,2}(\mathbb{R}^N))'$. A sequence $(u_n)$ in $D_G^{1,2}(\mathbb{R}^N)$ is said to be a *Cerami sequence for $I_V$ at level $d \in \mathbb{R}$*, denoted by $(Ce)_d$, if $I_V(u_n) \to d$ and $\|\nabla I_V(u_n)\|_{(D_G^{1,2}(\mathbb{R}^N))'}(1 + \|u_n\|_V) \to 0$.

**Lemma 1.3.2.** *Assume that $(f_1)$–$(f_4)$ hold true and let $(u_n)$ in $D_G^{1,2}(\mathbb{R}^N)$ be a Cerami sequence for $I_V$ at level $d \in \mathbb{R}$. Then, $(u_n)$ has a bounded subsequence.*

*Proof.* Suppose, by contradiction, that $(u_n) \subset D_G^{1,2}(\mathbb{R}^N)$ has no bounded subsequence. Then, we can assume that $u_n \not\equiv 0$ for all $n \in \mathbb{N}$ and $\|u_n\|_V \to +\infty$. Let us define $\tilde{u}_n := u_n/\|u_n\|_V$ for all $n \in \mathbb{N}$. Thus, $(\tilde{u}_n)$ is a bounded sequence and $\|\tilde{u}_n\|_V = 1$. Hence, up to a subsequence, it holds $\tilde{u}_n \rightharpoonup \tilde{u}$ in $D_G^{1,2}(\mathbb{R}^N)$. Thus, one of the two cases occurs:

Case 1: $\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\tilde{u}_n|^2 dx > 0;$

Case 2: $\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\tilde{u}_n|^2 dx = 0.$

First, let us suppose that Case 2 occurs, and let $L > 1$ be an arbitrary constant. In particular, we have

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} \left|\frac{L}{\|u_n\|_V} u_n\right|^2 dx = L^2 \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\tilde{u}_n|^2 dx = 0.$$

By hypotheses $(f_1)$–$(f_3)$ and using that $f(s) = -f(-s)$ for $s < 0$, we have $F(s) \geq 0$ for all $s \in \mathbb{R}$. Moreover, we have

$$\lim_{s\to 0} \frac{F(s)}{|s|^2} = \lim_{|s|\to\infty} \frac{F(s)}{|s|^2} = 0.$$

So, applying Lemma 1.3.1, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(L\tilde{u}_n) = \lim_{n\to\infty} \int_{\mathbb{R}^N} F\left(\frac{L}{\|u_n\|_V} u_n\right) dx = 0.$$

Hence,

$$I_V\left(\frac{L}{\|u_n\|_V} u_n\right) = \frac{L^2}{2} - \int_{\mathbb{R}^N} F\left(\frac{L}{\|u_n\|_V} u_n\right) dx \geq \frac{L^2}{4}$$

for $n$ sufficiently large. Since $\|u_n\|_V \to +\infty$, then $\frac{L}{\|u_n\|_V} \in (0,1)$, for $n$ sufficiently large. So, there exists $n_1 \in \mathbb{N}$ such that

$$\max_{t\in[0,1]} I_V(tu_n) \geq I_V\left(\frac{L}{\|u_n\|_V} u_n\right) \geq \frac{L^2}{4},$$

for all $n \geq n_1$. Let $t_n \in [0,1]$ be such that $I_V(t_n u_n) := \max_{t \in [0,1]} I_V(t u_n)$. Thus,

$$I_V(t_n u_n) \geq \frac{L^2}{4}, \tag{1.3.3}$$

for all $n \geq n_1$. Since $t_n \leq 1$, using $(f_4)$ and the fact that $f(s) = -f(-s)$ for $s < 0$, we obtain

$$\begin{aligned} I_V(t_n u_n) &= I_V(t_n u_n) - \frac{1}{2} I_V'(t_n u_n)(t_n u_n) + o_n(1) \\ &= \int_{\mathbb{R}^N} \left( \frac{1}{2} f(t_n u_n)(t_n u_n) - F(t_n u_n) \right) dx + o_n(1) \\ &\leq D \int_{\mathbb{R}^N} \left( \frac{1}{2} f(u_n) u_n - F(u_n) \right) dx + o_n(1) \\ &= D \left( I_V(u_n) - \frac{1}{2} I_V'(u_n) u_n \right) + o_n(1) \\ &= Dd + o_n(1). \end{aligned}$$

So, there exists $n_2 \in \mathbb{N}$ such that

$$I_V(t_n u_n) \leq 2Dd, \tag{1.3.4}$$

for all $n \geq n_2$. Taking $n_0 := \max\{n_1, n_2\}$, it follows from (1.3.3) and (1.3.4) that

$$\frac{L^2}{4} \leq I_V(t_n u_n) \leq 2Dd,$$

for all $n \geq n_0$. Taking $L > 3\sqrt{Dd}$, we come to a contradiction.
Now suppose that Case 1 occurs, that is, there exists $\delta > 0$ such that

$$\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\tilde{u}_n|^2 dx = \delta.$$

If $(y_n) \subset \mathbb{R}^N$ is a sequence such that $|y_n| \to \infty$ and $\int_{B_1(y_n)} |\tilde{u}_n|^2 dx > \delta/2$, whereas $\tilde{u}_n(\cdot + y_n) \rightharpoonup \tilde{u}$, we obtain

$$\int_{B_1(0)} |\tilde{u}_n(x + y_n)|^2 > \frac{\delta}{2},$$

and so

$$\int_{B_1(0)} |\tilde{u}(x)|^2 dx \geq \frac{\delta}{2},$$

showing that $\tilde{u} \not\equiv 0$. Thus, there exists a subset of positive Lebesgue measure $\Omega \subset B_1(0)$

such that

$$0 < |\tilde{u}(x)| = \lim_{n\to\infty} |\tilde{u}_n(x+y_n)| = \lim_{n\to\infty} \frac{|u_n(x+y_n)|}{\|u_n\|_V}, \quad \forall x \in \Omega.$$

Since $\|u_n\|_V \to +\infty$, it follows that

$$|u_n(x+y_n)| \to +\infty, \quad \forall x \in \Omega.$$

Then, using the hypothesis $(f_4)$ and Fatou lemma, we obtain

$$\begin{aligned}
\liminf_{n\to\infty} \int_{\mathbb{R}^N} &\left[\frac{1}{2} f(u_n(x+y_n))u_n(x+y_n) - F(u_n(x+y_n))\right] dx \\
&\geq \liminf_{n\to\infty} \int_{\Omega} \left[\frac{1}{2} f(u_n(x+y_n))u_n(x+y_n) - F(u_n(x+y_n))\right] dx \\
&\geq \int_{\Omega} \liminf_{n\to\infty} \left[\frac{1}{2} f(u_n(x+y_n))u_n(x+y_n) - F(u_n(x+y_n))\right] dx \\
&= +\infty.
\end{aligned}$$

On the other hand, we have

$$|I_V'(u_n)u_n| \leq \|I_V'(u_n)\|_{(D_G^{1,2}(\mathbb{R}^N))'}\|u_n\|_V \leq \|I_V'(u_n)\|_{(D_G^{1,2}(\mathbb{R}^N))'}(1+\|u_n\|_V) \to 0,$$

and so, $I_V'(u_n)u_n = o_n(1)$. Therefore, for $n$ sufficiently large, we have

$$\int_{\mathbb{R}^N} \left[\frac{1}{2} f(u_n(x+y_n))u_n(x+y_n) - F(u_n(x+y_n))\right] dx = I_V(u_n) - \frac{1}{2} I_V'(u_n)u_n \leq d+1,$$

which gives a contradiction.

If $(y_n)$ is bounded, then there exists $R > 1$ such that $|y_n| \leq R$ for all $n \in \mathbb{N}$ and

$$\int_{B_{2R}(0)} |\tilde{u}_n(x+y_n)|^2 dx \geq \int_{B_1(0)} |\tilde{u}_n(x+y_n)|^2 dx > \frac{\delta}{2}.$$

Since $\tilde{u}_n(\cdot + y_n) \to \tilde{u}$ in $B_{2R}(0)$, it follows that

$$\int_{B_1(0)} |\tilde{u}(x)|^2 dx \geq \frac{\delta}{2}.$$

Similarly to the previous case, there exists $\Omega_1 \subset B_1(0)$, with $|\Omega_1| > 0$ such that

$$\lim_{n\to\infty} \frac{|u_n(x+y_n)|}{\|u_n\|_V} = \lim_{n\to\infty} |\tilde{u}_n(x+y_n)| = |\tilde{u}(x)| \neq 0, \quad \forall x \in \Omega_1.$$

The argument follows as in the previous case where $|y_n| \to +\infty$ and we arrive at a contradiction. Therefore, neither Case 1 nor Case 2 can occur and lemma is proved. □

For future purposes, we need a version of Brezis-Lieb lemma [12] for $D^{1,2}(\mathbb{R}^N)$ found in [31], Lemma A.1.

**Lemma 1.3.3.** *Suppose that $(u_n) \subset D^{1,2}(\mathbb{R}^N)$ is bounded and $u_n(x) \to u_0(x)$ for a.e. $x \in \mathbb{R}^N$. Then*

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}\Phi(u_n)\,dx - \int_{\mathbb{R}^N}\Phi(u_n - u_0)\,dx\right) = \int_{\mathbb{R}^N}\Phi(u_0)\,dx \tag{1.3.5}$$

*for any function $\Phi : \mathbb{R} \to \mathbb{R}$ of class $C^1$ such that $\Phi(0) = 0$ and $|\Phi'(s)| \leq C|s|^{2^*-1}$ for any $s \in \mathbb{R}$ and some constant $C > 0$.*

*Proof.* Observe that

$$\begin{aligned}\int_{\mathbb{R}^N}[\Phi(u_n) - \Phi(u_n - u_0)]dx &= \int_{\mathbb{R}^N}\int_0^1 \frac{d}{ds}\Phi(u_n - su_0)\,ds\,dx \\ &= -\int_{\mathbb{R}^N}\int_0^1 \Phi'(u_n - su_0)u_0\,ds\,dx.\end{aligned}$$

So, by Vitali's convergence theorem, where we have to

$$|\Phi'(u_n - su_0)| \leq C|u_n - su_0|^{2^*-1} \leq C\gamma(|u_n|^{2^*-1} + |u_0|^{2^*-1}).$$

Let

$$f_n := |\Phi'(u_n - su_0)||u_0| \leq C(|u_n|^{2^*-1}|u_0| + |u_0|^{2^*-1}|u_0|) =: g_n,$$

note that

$$f_n(x) \to |\Phi'(u_0 - su_0)||u_0| := f(x), \quad \text{as } n \to +\infty$$

and

$$g_n(x) \to C(|u_0|^{2^*-1}|u_0| + |u_0|^{2^*}) := g(x), \quad \text{as } n \to +\infty.$$

Where $|f_n| \leq |g_n|$ a.e. in $\mathbb{R}^N$ and $\|g_n - g\|_{L^1(\mathbb{R}^N)} \to 0$, as $n \to +\infty$. Then $\|f_n - f\|_{L^1} \to 0$

and $|f| \leq |g|$ a.e $\mathbb{R}^N$. Thus, we get

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}^N}[\Psi(u_n) - \Psi(u_n - u_0)]dx &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\int_0^1 \Psi'(u_n - su_0)u_0\, ds dx \\
&= \int_0^1\int_{\mathbb{R}^N} \Psi'(u_0 - su_0)u_0\, dx ds \\
&= -\int_{\mathbb{R}^N}\int_0^1 \frac{d}{ds}\Psi(u_0 - su_0) ds dx \\
&= \int_{\mathbb{R}^N}\Psi(u_0)dx - \Psi(0) = \int_{\mathbb{R}^N}\Psi(u_0)dx.
\end{aligned}$$

□

The following lemma, combined with assumptions $(f_1)$ and $(f_2)$, provides the interpolation and boundedness properties that are needed to prove the next results. Its proof can be found in [17, Proposition 3.1]. Let $2 < p < 2^* < q$, in the next results.

**Lemma 1.3.4.** *Let $\alpha, \beta > 0$ and $h \in C^0(\mathbb{R}^N)$. Assume that $\frac{\alpha}{\beta} \leq \frac{p}{q}$ and $\beta \leq q$, and that there exists $M > 0$ such that*

$$|h(s)| \leq M \min\{|s|^\alpha, |s|^\beta\} \quad \textit{for every } s \in \mathbb{R}.$$

*Then, for every $r \in \left[\frac{q}{\beta}, \frac{p}{\alpha}\right]$, the map $D^{1,2}(\mathbb{R}^N) \to L^r(\mathbb{R}^N)$ given by $u \mapsto h(u)$ is well defined, continuous and bounded.*

Also, before proving the result, we will need the following versions of Brezis-Lieb lemma.

**Lemma 1.3.5.** *Assume that $(V_1)$–$(V_2)$ and $(f_1)$–$(f_3)$ hold true. Let $(u_n)$ be a bounded sequence in $D^{1,2}_G(\mathbb{R}^N)$ such that $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$. Then, the following statements hold true:*

(a) $\|u_n\|_V^2 = \|u_n - u\|^2 + \|u\|_V^2 + o_n(1)$;

(b) $\displaystyle\int_{\mathbb{R}^N} |f(u_n) - f(u)||\varphi| dx = o_n(1)$, *for every* $\varphi \in C_0^\infty(\mathbb{R}^N)$;

(c) $\displaystyle\int_{\mathbb{R}^N} F(u_n)dx - \int_{\mathbb{R}^N} F(u_n - u)dx = \int_{\mathbb{R}^N} F(u)dx + o_n(1)$;

(d) $f(u_n) - f(u_n - u) \to f(u)$ *in* $\left(D^{1,2}_G(\mathbb{R}^N)\right)'$.

*Proof.* Since $(u_n) \subset D^{1,2}_G(\mathbb{R}^N)$, it follows that $u_n(gx) = u_n(x)$ for any $g \in G$ and $x \in \mathbb{R}^N$. Thus, as $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$, we have

$$u(gx) = \lim_{n\to\infty} u_n(gx) = \lim_{n\to\infty} u_n(x) = u(x) \quad \text{a.e. } x \in \mathbb{R}^N,$$

which shows that $u \in D^{1,2}_G(\mathbb{R}^N)$.

Next, for each $n \in \mathbb{N}$, define $v_n := u_n - u$. So, we have a sequence $(v_n)$ such that $v_n \rightharpoonup 0$ in $D^{1,2}_G(\mathbb{R}^N)$.

(a) Since $u_n \rightharpoonup u$ in $D^{1,2}_G(\mathbb{R}^N)$, it follows that $\langle u_n, u\rangle_V \to \langle u, u\rangle_V = \|u\|^2_V$. So, we have

$$\begin{aligned}\|v_n\|^2_V &= \|u_n - u\|^2_V = \langle u_n - u, u_n - u\rangle_V \\ &= \langle u_n, u_n\rangle_V - \langle u_n, u\rangle_V - \langle u, u_n\rangle_V + \langle u, u\rangle_V \\ &= \|u_n\|^2_V - 2\langle u_n, u\rangle_V + \|u\|^2_V = \|u_n\|^2_V + \|u\|^2_V + o_n(1). \qquad (1.3.6)\end{aligned}$$

On the other hand, assumption $(V_2)$ implies that $V \in L^{N/2}(\mathbb{R}^N) \cap L^{\theta}(\mathbb{R}^N)$ for any $\theta > N/2$. Hence $\eta := 2\theta/(\theta-1) < 2^*$, and it follows that $v_n \to 0$ in $L^{\eta}_{\text{loc}}(\mathbb{R}^N)$. Moreover, given $\varepsilon > 0$, we may fix $R > 1$ sufficiently large such that

$$\int_{\mathbb{R}^N\setminus B_R(0)} |V(x)|^{N/2}dx \le \varepsilon^{N/2}.$$

Thus, using Hölder inequality with conjugate exponents $\theta$ and $\theta/(\theta-1)$ and also $N/2$ and $2^*/2$, we get

$$\begin{aligned}\int_{\mathbb{R}^N} |V(x)||v_n|^2dx &= \int_{B_R(0)} |V(x)||v_n|^2dx + \int_{\mathbb{R}^N\setminus B_R(0)} |V(x)||v_n|^2dx \\ &\le \left(\int_{B_R(0)} |V(x)|^{\theta}dx\right)^{\frac{1}{\theta}} \left(\int_{B_R(0)} (v_n^2)^{\frac{\theta}{\theta-1}}dx\right)^{\frac{\theta-1}{\theta}} \\ &\quad + \left(\int_{\mathbb{R}^N\setminus B_R(0)} |V(x)|^{\frac{N}{2}}dx\right)^{\frac{2}{N}} \left(\int_{\mathbb{R}^N\setminus B_R(0)} (v_n^2)^{\frac{2^*}{2}}dx\right)^{\frac{2}{2^*}} \\ &\le \left(\int_{\mathbb{R}^N} |V(x)|^{\theta}dx\right)^{\frac{1}{\theta}} \left(\int_{B_R(0)} |v_n|^{\eta}dx\right)^{\frac{2}{\eta}} \\ &\quad + \left(\int_{\mathbb{R}^N\setminus B_R(0)} |V(x)|^{\frac{N}{2}}dx\right)^{\frac{2}{N}} \left(\int_{\mathbb{R}^N} |v_n|^{2^*}dx\right)^{\frac{2}{2^*}} \\ &= \|V\|_{\theta}\|v_n\|^2_{L^{\eta}(B_R(0))} + \|V\|_{L^{N/2}(\mathbb{R}^N\setminus B_R(0))}\|v_n\|^2_{2^*}.\end{aligned}$$

Since $(v_n)$ is bounded because $(u_n)$ is bounded in $D^{1,2}_G(\mathbb{R}^N)$, $v_n \to 0$ in $L^{\eta}_{\text{loc}}(\mathbb{R}^N)$ and

$D^{1,2}_G(\mathbb{R}^N)$ is continuously embedded into $L^{2^*}(\mathbb{R}^N)$, there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} |V(x)||v_n|^2 dx \le o_n(1) + C_1\varepsilon,$$

and so, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |V(x)||v_n|^2 dx \le C\varepsilon,$$

for $n \in \mathbb{N}$ large enough. Therefore, it follows from the last inequality that

$$\begin{aligned} \|v_n\|_V^2 &= \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(x) v_n^2 dx \\ &= \|v_n\|^2 + \int_{\mathbb{R}^N} V(x) v_n^2 dx = \|v_n\|^2 + o_n(1). \end{aligned} \tag{1.3.7}$$

Substituting (1.3.7) in (1.3.6), it follows that

$$\|u_n\|_V^2 = \|v_n\|^2 + \|u\|_V^2 + o_n(1),$$

proving item (a).
(b) By hypothesis $(f_2)$, we have

$$|f'(s)| \le A_2 |s|^{2^*-2}, \quad \forall\, s \in \mathbb{R}.$$

By the mean value theorem, there exists $\xi \in (0,1)$ such that

$$\begin{aligned} |f(u_n) - f(u)| &= |f'(u + \xi(u_n - u))||u_n - u| \\ &\le A_2 |u + \xi(u_n - u)|^{2^*-2} |u_n - u| \\ &\le A_2 (|u| + |u_n - u|)^{2^*-2} |u_n - u|. \end{aligned}$$

Note that

$$(|u| + |u_n - u|)^{2^*-2} \le (2 \max\{|u|, |u_n - u|\})^{2^*-2} \le 2^{2^*-2}\left(|u|^{2^*-2} + |u_n - u|^{2^*-2}\right),$$

and so

$$\begin{aligned} |f(u_n) - f(u)| &\le A_2 (|u| + |u_n - u|)^{2^*-2} |u_n - u| \\ &\le C_1 \left(|u|^{2^*-2} + |u_n - u|^{2^*-2}\right) |u_n - u| \\ &= C_1 \left(|u|^{2^*-2} |u_n - u| + |u_n - u|^{2^*-1}\right). \end{aligned} \tag{1.3.8}$$

Next, we fix $\delta \in \left(0, \frac{1}{N-2}\right)$ and consider $q_1 := 2^* - \delta$ and $q_2 := (2^* - \delta)/(1-\delta)$. Thus, using Hölder inequality with conjugate exponents $(2^*-\delta)/(2^*-1)$ and $(2^*-\delta)/(1-\delta)$, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} |u_n - u|^{2^*-1}|\varphi| dx &= \int_{\mathrm{supp}(\varphi)} |u_n - u|^{2^*-1}|\varphi| dx \\
&\leq \left(\int_{\mathrm{supp}(\varphi)} \left(|u_n - u|^{2^*-1}\right)^{\frac{2^*-\delta}{2^*-1}} dx\right)^{\frac{2^*-1}{2^*-\delta}} \left(\int_{\mathrm{supp}(\varphi)} |\varphi|^{\frac{2^*-\delta}{1-\delta}} dx\right)^{\frac{1-\delta}{2^*-\delta}} \\
&= \left(\int_{\mathrm{supp}(\varphi)} |u_n - u|^{q_1} dx\right)^{\frac{2^*-1}{q_1}} \left(\int_{\mathrm{supp}(\varphi)} |\varphi|^{q_2} dx\right)^{\frac{1}{q_2}} \\
&\leq C\|\varphi\|_\infty \left(\int_{\mathrm{supp}(\varphi)} |u_n - u|^{q_1} dx\right)^{\frac{2^*-1}{q_1}}.
\end{aligned}
$$

As $(u_n)$ is bounded and, passing to a subsequence, $u_n \rightharpoonup u$ in $D_G^{1,2}(\mathbb{R}^N)$ and $u_n \to u$ strongly in $L^{q_1}_{\mathrm{loc}}(\mathbb{R}^N)$, it follows that

$$
\int_{\mathbb{R}^N} |u_n - u|^{2^*-1}|\varphi| dx = o_n(1), \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \tag{1.3.9}
$$

On the other hand, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} |u|^{2^*-2}|u_n - u||\varphi| dx &= \int_{\mathrm{supp}(\varphi)} |u|^{2^*-2}|u_n - u||\varphi| dx \\
&\leq \left(\int_{\mathrm{supp}(\varphi)} \left(|u|^{2^*-2}\right)^{\frac{2^*}{2^*-2}} dx\right)^{\frac{2^*-2}{2^*}} \left(\int_{\mathrm{supp}(\varphi)} (|u_n - u||\varphi|)^{\frac{2^*}{2}} dx\right)^{\frac{2}{2^*}} \\
&= \left(\int_{\mathrm{supp}(\varphi)} |u|^{2^*} dx\right)^{\frac{2^*-2}{2^*}} \left(\int_{\mathrm{supp}(\varphi)} (|u_n - u||\varphi|)^{\frac{2^*}{2}} dx\right)^{\frac{2}{2^*}},
\end{aligned}
$$

and so, using Hölder inequality with conjugate exponents $\frac{2(2^*-\delta)}{2^*}$ and $\frac{2(2^*-\delta)}{2^*-2\delta}$, we get

$$
\begin{aligned}
\left(\int_{\mathrm{supp}(\varphi)} (|u_n - u||\varphi|)^{\frac{2^*}{2}} dx\right)^{\frac{2}{2^*}} &\leq \left(\int_{\mathrm{supp}(\varphi)} |u_n - u|^{q_1} dx\right)^{\frac{1}{q_1}} \left(\int_{\mathrm{supp}(\varphi)} |\varphi|^{q_3} dx\right)^{\frac{1}{q_3}} \\
&\leq C\|\varphi\|_\infty \left(\int_{\mathrm{supp}(\varphi)} |u_n - u|^{q_1} dx\right)^{\frac{1}{q_1}},
\end{aligned}
$$

where $q_1 := 2^* - \delta$ and $q_3 := \frac{2^*(2^*-\delta)}{2^*-2\delta}$. As $u_n \to u$ strongly in $L^{q_1}_{\text{loc}}(\mathbb{R}^N)$, it follows that

$$\left(\int_{\text{supp}(\varphi)} (|u_n - u||\varphi|)^{\frac{2^*}{2}}\, dx\right)^{\frac{2}{2^*}} = o_n(1),$$

and thus,

$$\int_{\mathbb{R}^N} |u|^{2^*-2}|u_n - u||\varphi|dx = o_n(1), \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \tag{1.3.10}$$

It follows from (1.3.8), (1.3.9) and (1.3.10) that

$$\int_{\mathbb{R}^N} |f(u_n) - f(u)||\varphi|dx = o_n(1), \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N),$$

which proves item (b).

(c) Since $(u_n)$ is bounded in $D^{1,2}_G(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$, applying Lemma 1.3.3 with $\Psi = F$, (see [31, Lemma A.1]), we get

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^N} F(u_n)dx - \int_{\mathbb{R}^N} F(u_n - u)dx\right) = \int_{\mathbb{R}^N} F(u)dx,$$

which proves item (c).

(d) Hypothesis $(f_2)$ and the fact that $f(s) = -f(-s)$, for $s < 0$, imply that $|f(s)| \leq A_2|s|^{2^*-1}$ for all $s \in \mathbb{R}$. Thus, arguing as in (b), see (1.3.8), we obtain

$$|f(u_n) - f(u_n - u)| \leq C_1\left(|u_n - u|^{2^*-2}|u| + |u|^{2^*-1}\right),$$

and so

$$\begin{aligned} |f(u_n) - f(u_n - u) - f(u)| &\leq |f(u_n) - f(u_n - u)| + |f(u)| \\ &\leq C_1\left(|u_n - u|^{2^*-2}|u| + |u|^{2^*-1}\right) + A_2|u|^{2^*-1} \\ &= C_1|u_n - u|^{2^*-2}|u| + (C_1 + A_2)|u|^{2^*-1}. \end{aligned}$$

Let $\varphi \in D^{1,2}_G(\mathbb{R}^N)$ and $R > 0$ be. Since $(v_n)$ is bounded in $D^{1,2}_G(\mathbb{R}^N)$, where $v_n := u_n - u$, and $D^{1,2}_G(\mathbb{R}^N)$ is continuously embedded into $L^{2}(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\int_{|x|>R} |f(u_n) \quad f(u_n \quad u) \quad f(u)||\varphi| dx \\
&\quad C_1 \int_{|x|>R} |u_n \quad u|^{2 \quad 2} |u||\varphi| dx + (C_1 + A_2) \int_{|x|>R} |u|^{2 \quad 1} |\varphi| dx \\
&\quad C_1 \left( \int_{|x|>R} |u_n \quad u|^{2} \, dx \right)^{\frac{2 \quad 2}{2}} \left( \int_{|x|>R} |u|^{2 \quad /2} |\varphi|^{2 \quad /2} dx \right)^{2/2} \\
&\quad + (C_1 + A_2) \left( \int_{|x|>R} |u|^{2} \, dx \right)^{\frac{2 \quad 1}{2}} \left( \int_{|x|>R} |\varphi|^{2} \, dx \right)^{1/2} \\
&\quad C_1 \|u_n \quad u\|_2^{2 \quad 2} \left( \int_{|x|>R} |u|^{2} \, dx \right)^{1/2} \left( \int_{|x|>R} |\varphi|^{2} \, dx \right)^{1/2} \\
&\quad + (C_1 + A_2) \|\varphi\|_2 \left( \int_{|x|>R} |u|^{2} \, dx \right)^{\frac{2 \quad 1}{2}} \\
&\quad C \|\varphi\|_V \left[ \left( \int_{|x|>R} |u|^{2} \, dx \right)^{1/2} + \left( \int_{|x|>R} |u|^{2} \, dx \right)^{\frac{2 \quad 1}{2}} \right].
\end{aligned}$$

Thus, given $\varepsilon > 0$, we may choose $R > 1$ sufficiently large such that

$$\int_{|x|>R} |f(u_n) \quad f(u_n \quad u) \quad f(u)||\varphi| dx \quad \varepsilon \|\varphi\|_V. \tag{1.3.11}$$

On the other hand, as $f \in C^1$, by (1.1.5) and (1.1.6), for any $\varepsilon > 0$ and $2 < p < 2 \quad < q$, we find $0 < \delta < M$ and $C_\varepsilon > 0$ such that, for $i = 0, 1$,

$$|f^{(i)}(s)| < \varepsilon |s|^{2 \quad (i+1)}, \quad \text{for } 0 < |s| < \delta \text{ or } |s| > M$$

and

$$|f^{(i)}(s)| < C_\varepsilon \min\{|s|^{p \quad (i+1)}, |s|^{q \quad (i+1)}\}, \quad \text{for } \delta \quad |s| \quad M.$$

Hence,

$$|f^{(i)}(s)| \quad \varepsilon |s|^{2 \quad (i+1)} + C_\varepsilon \min\{|s|^{p \quad (i+1)}, |s|^{q \quad (i+1)}\}, \quad \forall s \in \mathbb{R}.$$

Consider $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$h_\varepsilon(s) = C_\varepsilon \min\{|s|^{p \quad 2}, |s|^{q \quad 2}\}.$$

Note that, for any $2 < p < 2^* < q$, we have

$$\frac{2^*-p}{2^*p-2^*-p} \le \frac{2^*-p}{2^*p-2^*-2^*} = \frac{2^*-p}{2^*(p-2)} = \frac{p}{p-2},$$

$$\frac{2^*-q}{2^*q-2^*-q} \ge \frac{2^*-q}{2^*q-2^*-2^*} = \frac{2^*-q}{2^*(q-2)} = \frac{q}{q-2}$$

and

$$\frac{2^*-q}{2^*q-2^*-q} < \frac{2^*-p}{2^*p-2^*-p}.$$

Hence,

$$-\frac{q}{q-2} \le \frac{2^*-q}{2^*q-2^*-q} < \frac{2^*-p}{2^*p-2^*-p} \le \frac{p}{p-2},$$

It follows from Lemma 1.3.4 with $\alpha = p-2$ and $\beta = q-2$ that, for every $r \in \left[-\frac{q}{q-2}, \frac{p}{p-2}\right]$, the map $D^{1,2}(\mathbb{R}^N) \to L^r(\mathbb{R}^N)$ given by $v \mapsto h_\varepsilon(v)$ is well defined, continuous and bounded. In particular, for $r = \frac{2^*-p}{2^*p-2^*-p}$, it follows that $h_\varepsilon(|u|+|u_n-u|)$ is bounded in $L^r(\mathbb{R}^N)$. So by the mean value theorem, there exists $\xi \in (0,1)$ such that

$$\begin{aligned}|f(u_n)-f(u)| &= |f'(u+\xi(u_n-u))||u_n-u| \\ &\le \varepsilon(|u|+|u_n-u|)^{2^*-2}|u_n-u| + h_\varepsilon(|u|+|u_n-u|)|u_n-u|.\end{aligned}$$

Thus, given $\varphi \in D^{1,2}_G(\mathbb{R}^N)$ and $R > 0$, we have

$$\begin{aligned}\int_{|x|\le R} |f(u_n)-f(u)||\varphi|dx \le\ & \varepsilon\int_{|x|\le R}(|u|+|u_n-u|)^{2^*-2}|u_n-u||\varphi|dx \\ &+ \int_{|x|\le R} h_\varepsilon(|u|+|u_n-u|)|u_n-u||\varphi|dx.\end{aligned}$$

Observe that, by Hölder inequality, we get

$$\varepsilon\int_{|x|\le R}(|u|+|u_n-u|)^{2^*-2}|u_n-u||\varphi|dx \le \varepsilon\||u|+|u_n-u|\|_{2^*}^{2^*-2}\|u_n-u\|_{2^*}\|\varphi\|_{2^*}$$

and as $D^{1,2}_G(\mathbb{R}^N)$ is continuously embedded into $L^{2^*}(\mathbb{R}^N)$, there exists $C > 0$ such that

$$\varepsilon\int_{|x|\le R}(|u|+|u_n-u|)^{2^*-2}|u_n-u||\varphi|dx \le \varepsilon C. \tag{1.3.12}$$

Using successively Hölder inequality with conjugate exponents $p$ and $p/(p-1)$ or $2^*(p$

$1)/p)$ and $2^*(p-1)/(2^*(p-1)-p)$, for any $\varphi \in D^{1,2}_G(\mathbb{R}^N)$, we obtain

$$\begin{aligned}
\int_{|x|\geq R} & h_\varepsilon(|u| + |u_n - u|)|u_n - u||\varphi| dx \\
&\leq \left(\int_{|x|\geq R} (h_\varepsilon(|u| + |u_n - u|)|\varphi|)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_{|x|\geq R} |u_n - u|^p dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_{|x|\geq R} (h_\varepsilon(|u| + |u_n - u|))^r dx\right)^{\frac{1}{r}} \left(\int_{|x|\geq R} |\varphi|^{2^*} dx\right)^{\frac{1}{2^*}} \left(\int_{|x|\geq R} |u_n - u|^p dx\right)^{\frac{1}{p}} \\
&= \|h_\varepsilon(|u| + |u_n - u|)\|_r \|\varphi\|_{2^*} \left(\int_{|x|\geq R} |u_n - u|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $(u_n)$ is bounded and, passing to a subsequence, $u_n \rightharpoonup u$ in $D^{1,2}_G(\mathbb{R}^N)$, $u_n \to u$ strongly in $L^p_{\text{loc}}(\mathbb{R}^N)$ and $D^{1,2}_G(\mathbb{R}^N)$ is continuously embedded into $L^{2^*}(\mathbb{R}^N)$, there exists $C > 0$ such that

$$\int_{|x|\geq R} h_\varepsilon(|u| + |u_n - u|)|u_n - u||\varphi| dx \leq C\|\varphi\|_V \left(\int_{|x|\geq R} |u_n - u|^p dx\right)^{\frac{1}{p}} = o_n(1). \quad (1.3.13)$$

It follows from (1.3.12) and (1.3.13) that

$$\int_{|x|\geq R} |f(u_n) - f(u)||\varphi| dx \leq \varepsilon C, \quad (1.3.14)$$

for $n \in \mathbb{N}$ sufficiently large. Moreover, we have again

$$\begin{aligned}
|f(|u_n - u|)| &\leq \varepsilon|u_n - u|^{2^*-1} + C_\varepsilon \min\{|u_n - u|^{p-1}, |u_n - u|^{q-1}\} \\
&= \varepsilon|u_n - u|^{2^*-1} + C_\varepsilon \min\{|u_n - u|^{p-2}, |u_n - u|^{q-2}\}|u_n - u| \\
&= \varepsilon|u_n - u|^{2^*-1} + h_\varepsilon(|u_n - u|)|u_n - u|,
\end{aligned}$$

and so, for any $\varphi \in D^{1,2}_G(\mathbb{R}^N)$ and $R > 0$, arguing as before, we get

$$\begin{aligned}
\int_{|x|\geq R} |f(u_n - u)||\varphi| dx &\leq \varepsilon \int_{|x|\geq R} |u_n - u|^{2^*-1}|\varphi| dx + \int_{|x|\geq R} h_\varepsilon(|u_n - u|)|u_n - u||\varphi| dx \\
&\leq \varepsilon\|u_n - u\|_{2^*}^{2^*-1}\|\varphi\|_{2^*} + \|h_\varepsilon(|u_n - u|)\|_r\|\varphi\|_{2^*} \left(\int_{|x|\geq R} |u_n - u|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

From Lemma 1.3.4 again, it follows that $h_\varepsilon(|u_n - u|)$ is bounded in $L^r(\mathbb{R}^N)$ and, moreover,

$u_n \to u$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ and $D^{1,2}_G(\mathbb{R}^N)$ is continuously embedded into $L^{2^*}(\mathbb{R}^N)$. Hence,

$$\int_{|x|\geq R} |f(u_n - u)||\varphi| dx \leq \varepsilon C, \tag{1.3.15}$$

for $n \in \mathbb{N}$ sufficiently large. From (1.3.14) and (1.3.15), we obtain

$$\begin{aligned}\int_{|x|\geq R} &|f(u_n) - f(u_n - u) - f(u)||\varphi| dx \\ &\leq \int_{|x|\geq R} |f(u_n) - f(u)||\varphi| dx + \int_{|x|\geq R} |f(u_n - u)||\varphi| dx \\ &\leq \varepsilon C \end{aligned} \tag{1.3.16}$$

for $n \in \mathbb{N}$ sufficiently large. Therefore, from (1.3.11) and (1.3.16), given $\varepsilon > 0$ and $\varphi \in D^{1,2}_G(\mathbb{R}^N)$, there exists $C > 0$ such that

$$\left| \int_{\mathbb{R}^N} [f(u_n) - f(u_n - u) - f(u)] \varphi dx \right| \leq \varepsilon C$$

for $n \in \mathbb{N}$ sufficiently large, which proves item (d). □

Next, we will present the standard result about the splitting of bounded $(PS)$ sequences. The proof follows closely the proof of [17, Lemma 3.9] using Lemmas 1.3.3 and 1.3.1 either for $\psi(u) = F(u)$ or $\psi(u) = f(u)u$, $u \in D^{1,2}_G(\mathbb{R}^N)$, wherever convenient.

**Lemma 1.3.6** (Splitting). *Assume that $(V_1)$–$(V_2)$ and $(f_1)$–$(f_3)$ hold true. Let $c \in \mathbb{R}$ and $(u_n)$ be a bounded sequence in $D^{1,2}_G(\mathbb{R}^N)$ such that*

$$I_V(u_n) \to c \quad \text{and} \quad \nabla I_V(u_n) \to 0 \quad in \ \left(D^{1,2}_G(\mathbb{R}^N)\right)'.$$

*Replacing $(u_n)$ by a subsequence, if necessary, there exist a solution $u \in D^{1,2}_G(\mathbb{R}^N)$ of problem $(P_G)$, a number $k \in \mathbb{N} \cup \{0\}$, $k$ sequences $(y_n^j) \subset \mathbb{R}^N$, $1 \leq j \leq k$ and $k$ nontrivial solutions $w^1, \dots, w^k$ of the limit problem $(P_0)$, satisfying:*

*(i) $u_n \rightharpoonup u$ weakly in $D^{1,2}_G(\mathbb{R}^N)$;*

*(ii) for any $i, j = 1, \dots, k$, $|y_n^j| \to \infty$ and $|y_n^j - y_n^i| \to \infty$, if $i \neq j$;*

*(iii) $u_n - u - \sum_{j=1}^{k} w^j(\cdot - y_n^j) \to 0$ in $D^{1,2}(\mathbb{R}^N)$;*

*(iv) $c = I_V(u) + \sum_{j=1}^{k} I_0(w^j)$,*

*for $k \in \mathbb{N}$. In the case $k = 0$, the above holds without $w^j$, $(y_n^j)$.*

*Proof.* Since $(u_n) \subset D_G^{1,2}(\mathbb{R}^N)$ is a $(PS)_c$-sequence for $I_V$ restricted to $D_G^{1,2}(\mathbb{R}^N)$, it follows from Lemma 1.2.3 that $I_V'(u_n)\tilde{v} = 0$ for any $\tilde{v} \in \left(D_G^{1,2}(\mathbb{R}^N)\right)^\perp$, and so $(u_n)$ is also $(PS)_c$-sequence for $I_V$ defined in the whole space $D^{1,2}(\mathbb{R}^N)$. As $(u_n)$ is bounded, passing to a subsequence, we get $u \in D^{1,2}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$. Let us show that $u \in D_G^{1,2}(\mathbb{R}^N)$. In fact, as $(u_n) \subset D_G^{1,2}(\mathbb{R}^N)$, we have $u_n(gx) = u_n(x)$ for any $g \in G$ and $x \in \mathbb{R}^N$, and so

$$u(gx) = \lim_{n \to \infty} u_n(gx) = \lim_{n \to \infty} u_n(x) = u(x) \quad \text{a.e. } x \in \mathbb{R}^N,$$

which shows that $u \in D_G^{1,2}(\mathbb{R}^N)$. It follows from weak convergence and Lemma 1.3.5(b) that, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} o_n(1) = I_V'(u_n)\varphi &= \int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + V(x)u_n\varphi)dx - \int_{\mathbb{R}^N} f(u_n)\varphi\, dx \\ &= \int_{\mathbb{R}^N} (\nabla u \nabla \varphi + V(x)u\varphi)dx - \int_{\mathbb{R}^N} f(u)\varphi\, dx + o_n(1) \\ &= I_V'(u)\varphi + o_n(1), \end{aligned}$$

which shows that $I_V'(u)\varphi = 0$, and so, as $C_0^\infty(\mathbb{R}^N)$ is dense in $D^{1,2}(\mathbb{R}^N)$, it follows that $I_V'(u)v = 0$ for any $v \in D^{1,2}(\mathbb{R}^N)$. Since $u \in D_G^{1,2}(\mathbb{R}^N)$ and $I_V'(u)\tilde{v} = 0$ for any $\tilde{v} \in \left(D_G^{1,2}(\mathbb{R}^N)\right)^\perp$, we conclude that $u$ is a critical point of functional $I_V$ restricted to $D_G^{1,2}(\mathbb{R}^N)$. For each $n \in \mathbb{N}$, define $u_{n,1} := u_n - u$. Thus, we have a sequence $(u_{n,1})$ such that $u_{n,1} \rightharpoonup 0$ in $D_G^{1,2}(\mathbb{R}^N)$. By Lemma 1.3.5 the following statements hold:

(a) $\|u_n\|_V^2 = \|u_{n,1}\|^2 + \|u\|_V^2 + o_n(1)$;

(b) $\displaystyle\int_{\mathbb{R}^N} |f(u_n) - f(u)||\varphi|dx = o_n(1)$, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$;

(c) $\displaystyle\int_{\mathbb{R}^N} F(u_n)dx - \int_{\mathbb{R}^N} F(u_{n,1})dx = \int_{\mathbb{R}^N} F(u)dx + o_n(1)$;

(d) $f(u_n) - f(u_{n,1}) \to f(u)$ in $\left(D_G^{1,2}(\mathbb{R}^N)\right)'$.

Therefore, it follows from (a) and (c) that

$$\begin{aligned} I_V(u_n) - I_0(u_{n,1}) - I_V(u) &= \frac{1}{2}\|u_n\|_V^2 - \int_{\mathbb{R}^N} F(u_n)dx - \frac{1}{2}\|u_{n,1}\|^2 + \int_{\mathbb{R}^N} F(u_{n,1})dx \\ &\quad - \frac{1}{2}\|u\|_V^2 + \int_{\mathbb{R}^N} F(u)dx \\ &= \frac{1}{2}\left[\|u_n\|_V^2 - \|u_{n,1}\|^2 - \|u\|_V^2\right] \\ &\quad - \int_{\mathbb{R}^N} [F(u_n) - F(u_{n,1}) - F(u)]dx \\ &= o_n(1), \end{aligned}$$

and thus,

$$I_V(u_n) = I_V(u) + I_0(u_{n,1}) + o_n(1). \tag{1.3.17}$$

Next, we will show that $\nabla I_V(u_{n,1}) \to 0$ in $\left(D_G^{1,2}(\mathbb{R}^N)\right)'$. Indeed, by hypothesis, $\nabla I_V(u_n) \to 0$ in $\left(D_G^{1,2}(\mathbb{R}^N)\right)'$ and so it follows that $\nabla I_V(u_n)v \to 0$, for any $v \in D_G^{1,2}(\mathbb{R}^N)$. So, we have

$$\begin{aligned} o_n(1) &= \nabla I_V(u_n)v = \nabla I_V(u_{n,1} + u)v \\ &= \int_{\mathbb{R}^N} (\nabla u_{n,1}\nabla v + V(x)u_{n,1}v)dx + \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx \\ &\quad - \int_{\mathbb{R}^N} f(u_{n,1} + u)vdx \\ &= \nabla I_V(u_{n,1})v + \int_{\mathbb{R}^N} f(u_{n,1})vdx + \nabla I_V(u)v + \int_{\mathbb{R}^N} f(u)vdx \\ &\quad - \int_{\mathbb{R}^N} f(u_n)vdx \\ &= \nabla I_V(u_{n,1})v + \nabla I_V(u)v - \int_{\mathbb{R}^N} [f(u_n) - f(u_{n,1}) - f(u)]vdx. \end{aligned}$$

The fact that $\nabla I_V(u) = 0$ and item (d) imply that

$$\nabla I_V(u_{n,1})v = o_n(1), \quad \text{for all } v \in D_G^{1,2}(\mathbb{R}^N),$$

which shows that, as $n \to \infty$, $\nabla I_V(u_{n,1}) \to 0$ in $\left(D_G^{1,2}(\mathbb{R}^N)\right)'$. If $u_{n,1} \to 0$ strongly in $D_G^{1,2}(\mathbb{R}^N)$, the proof is completed. So, assume that it does not. Then, as $\nabla I_V(u_{n,1})u_{n,1} \to 0$, after passing to a subsequence, there exists a constant $C_1 > 0$ such that

$$0 < C_1 \le \|u_{n,1}\|_V^2 = \int_{\mathbb{R}^N} f(u_{n,1})u_{n,1}dx + o_n(1).$$

Therefore, applying Lemma 1.3.1 with $\psi(s) = f(s)s$, there exist $\delta > 0$ and a sequence

$(y_n^1)$ in $\mathbb{R}^N$ such that

$$\int_{B_1(y_n)} |u_{n,1}(x)|^2 dx > \delta. \tag{1.3.18}$$

Let us consider a sequence $(v_n^1)$ defined by

$$v_n^1 := u_{n,1}(\cdot + y_n^1).$$

Since $(u_{n,1})$ is bounded in $D_G^{1,2}(\mathbb{R}^N)$, then $(v_n^1)$ is bounded in $D^{1,2}(\mathbb{R}^N)$, and so we have, up to a subsequence,

$$\begin{cases} v_n^1 \rightharpoonup w^1, & \text{weakly in } D^{1,2}(\mathbb{R}^N), \\ v_n^1 \to w^1, & \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^N), \\ v_n^1(x) \to w^1(x), & \text{a.e. } x \in \mathbb{R}^N. \end{cases}$$

Since $v_n^1 \to w^1$ in $L^2(B_1(0))$ and

$$\int_{B_1(0)} \left|v_n^1(x)\right|^2 dx = \int_{B_1(0)} \left|u_{n,1}(x + y_n^1)\right|^2 dx > \delta,$$

it follows that

$$\int_{B_1(0)} |w^1(x)|^2 dx \geq \delta,$$

and so $w^1 \not\equiv 0$. The fact that $u_{n,1} \rightharpoonup 0$ weakly in $D_G^{1,2}(\mathbb{R}^N)$ implies that $(y_n^1)$ is unbounded and, passing to a subsequence, we may assume that $|y_n^1| \to \infty$.

So, about the sequence $(u_{n,1})$ the following statements hold:

(a1) $\|u_n\|_V^2 = \|u_{n,1}\|^2 + \|u\|_V^2 + o_n(1)$;

(b1) $I_V(u_n) = I_V(u) + I_0(u_{n,1}) + o_n(1)$;

(c1) $\nabla I_V(u_{n,1}) \to 0$ in $\left(D_G^{1,2}(\mathbb{R}^N)\right)'$.

Next, we shall show that $w^1$ is a nontrivial solution of the limit problem ($P_0$). As $(u_{n,1}) \subset D_G^{1,2}(\mathbb{R}^N)$, by Lemma 1.2.3, we have $I_V'(u_{n,1})v = 0$ for any $v \in \left(D_G^{1,2}(\mathbb{R}^N)\right)^\perp$, and so $I_V'(u_{n,1}) \to 0$ in $\left(D^{1,2}(\mathbb{R}^N)\right)'$. Moreover, assumption ($V_2$) implies that $V \in L^{N/2}(\mathbb{R}^N) \cap L^\theta(\mathbb{R}^N)$ for every $\theta > N/2$. So, taking $\theta > N/2$, as $\eta := 2\theta/(\theta - 1) < 2^*$, it follows that

$u_{n,1} \to 0$ in $L^{\eta}_{\mathrm{loc}}(\mathbb{R}^N)$. Thus, given $\varphi \in C_0^\infty(\mathbb{R}^N)$, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |V(x)||u_{n,1}||\varphi|\,dx &= \int_{\mathrm{supp}(\varphi)} |V(x)||u_{n,1}||\varphi|\,dx \\
&\leq \left(\int_{\mathrm{supp}(\varphi)} |V(x)|^{\theta} dx\right)^{1/\theta} \left(\int_{\mathrm{supp}(\varphi)} (|u_{n,1}||\varphi|)^{\frac{\theta}{\theta-1}} dx\right)^{\frac{\theta-1}{\theta}} \\
&\leq \|V\|_{\theta} \left(\int_{\mathrm{supp}(\varphi)} |u_{n,1}|^{\frac{2\theta}{\theta-1}} dx\right)^{\frac{\theta-1}{2\theta}} \left(\int_{\mathrm{supp}(\varphi)} |\varphi|^{\frac{2\theta}{\theta-1}} dx\right)^{\frac{\theta-1}{2\theta}} \\
&= \|V\|_{\theta} \left(\int_{\mathrm{supp}(\varphi)} |u_{n,1}|^{\eta} dx\right)^{\frac{1}{\eta}} \left(\int_{\mathrm{supp}(\varphi)} |\varphi|^{\eta} dx\right)^{\frac{1}{\eta}} \\
&\leq C\|V\|_{\theta}\|\varphi\|_{\infty} \left(\int_{\mathrm{supp}(\varphi)} |u_{n,1}|^{\eta} dx\right)^{\frac{1}{\eta}} = o_n(1), \qquad (1.3.19)
\end{aligned}$$

and so,

$$\begin{aligned}
o_n(1) &= I_V'(u_{n,1})\varphi \\
&= \int_{\mathbb{R}^N} (\nabla u_{n,1} \nabla \varphi + V(x)u_{n,1}\varphi)dx - \int_{\mathbb{R}^N} f(u_{n,1})\varphi dx \\
&= \int_{\mathbb{R}^N} \nabla u_{n,1} \nabla \varphi dx - \int_{\mathbb{R}^N} f(u_{n,1})\varphi dx + \int_{\mathbb{R}^N} V(x)u_{n,1}\varphi dx \\
&= I_0'(u_{n,1})\varphi + \int_{\mathbb{R}^N} V(x)u_{n,1}\varphi dx \\
&= I_0'(u_{n,1})\varphi + o_n(1).
\end{aligned}$$

Hence,

$$I_0'(u_{n,1})\varphi = o_n(1), \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N), \qquad (1.3.20)$$

which shows that, as $n \to \infty$, $I_0'(u_{n,1}) \to 0$ in $\left(D^{1,2}(\mathbb{R}^N)\right)'$. For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that

$$\sup_{\|\varphi\|_V \leq 1} |I_0'(u_{n,1})\varphi| < \varepsilon, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

Given $\varphi \in C_0^\infty(\mathbb{R}^N)$, we define $\varphi_n^1 := \varphi(\cdot - y_n^1)$. Thus,

$$\begin{aligned}
\sup_{\|\varphi\|_V \leq 1} |I_0'(v_n^1)\varphi| = \sup_{\|\varphi\|_V \leq 1} |I_0'(u_{n,1}(\cdot + y_n^1))\varphi| &= \sup_{\|\varphi(\cdot - y_n^1)\|_V \leq 1} |I_0'(u_{n,1})\varphi(\cdot - y_n^1)| \\
&= \sup_{\|\varphi_n^1\|_V \leq 1} |I_0'(u_{n,1})\varphi_n^1| \leq \sup_{\|\phi\|_V \leq 1} |I_0'(u_{n,1})\phi| < \varepsilon, \quad \phi \in C_0^\infty(\mathbb{R}^N),
\end{aligned}$$

for $n \in \mathbb{N}$ sufficiently large. So, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, since $v_n^1 \rightharpoonup w^1$ weakly in $D^{1,2}(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N} \left[ \nabla v_n^1 \nabla \varphi + V(x) v_n^1 \varphi \right] dx = \int_{\mathbb{R}^N} \left[ \nabla w^1 \nabla \varphi + V(x) w^1 \varphi \right] dx + o_n(1)$$

and arguing as in (1.3.19), as $v_n^1 \to w^1$ in $L^{\eta}_{\text{loc}}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} V(x) v_n^1 \varphi \, dx = \int_{\mathbb{R}^N} V(x) w^1 \varphi \, dx + o_n(1).$$

Moreover, using the same ideas applied in Lemma 1.3.5(b), we can conclude that

$$\int_{\mathbb{R}^N} f(v_n^1) \varphi \, dx = \int_{\mathbb{R}^N} f(w^1) \varphi \, dx + o_n(1).$$

Therefore, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$

$$\begin{aligned}
o_n(1) &= I_0'(v_n^1)\varphi = \int_{\mathbb{R}^N} \nabla v_n^1 \nabla \varphi \, dx - \int_{\mathbb{R}^N} f(v_n^1) \varphi \, dx \\
&= \int_{\mathbb{R}^N} \left[ \nabla v_n^1 \nabla \varphi + V(x) v_n^1 \varphi \right] dx - \int_{\mathbb{R}^N} V(x) v_n^1 \varphi \, dx - \int_{\mathbb{R}^N} f(v_n^1) \varphi \, dx \\
&= \int_{\mathbb{R}^N} \left[ \nabla w^1 \nabla \varphi + V(x) w^1 \varphi \right] dx - \int_{\mathbb{R}^N} V(x) w^1 \varphi \, dx - \int_{\mathbb{R}^N} f(w^1) \varphi \, dx + o_n(1) \\
&= \int_{\mathbb{R}^N} \nabla w^1 \nabla \varphi \, dx - \int_{\mathbb{R}^N} f(w^1) \varphi \, dx + o_n(1) \\
&= I_0'(w^1)\varphi + o_n(1),
\end{aligned}$$

which shows that $I_0'(w^1)\varphi = 0$, and so, $w^1$ is a nontrivial solution of the limit problem ($P_0$).

Let us define now

$$u_{n,2} := u_{n,1} - w^1(\cdot - y_n^1).$$

So, as before, we have

(a2) $\|u_n\|_V^2 = \|u_{n,2}\|^2 + \|u\|_V^2 + \|w^1\|^2 + o_n(1)$;

(b2) $I_V(u_n) = I_V(u) + I_0(u_{n,2}) + I_0(w^1) + o_n(1)$;

(c2) $I_0'(u_{n,2}) \to 0$ in $\left(D^{1,2}(\mathbb{R}^N)\right)'$.

The verification of these items follows the same argument used previously in the analogous items for the sequence $(u_{n,1})$, with the necessary adaptations. Indeed, using (a1), we have

$$\begin{aligned}\|u_{n,2}\|^2 &= \langle u_{n,1} - w^1(\cdot - y_n^1), u_{n,1} - w^1(\cdot - y_n^1)\rangle \\ &= \|u_{n,1}\|^2 + \|w^1(\cdot - y_n^1)\|^2 - 2\langle u_{n,1}, w^1(\cdot - y_n^1)\rangle \\ &= o_n(1) + \|u_n\|_V^2 - \|u\|_V^2 + \|w^1(\cdot - y_n^1)\|^2 \\ &\quad - 2\langle u_{n,1}, w^1(\cdot - y_n^1)\rangle. \end{aligned} \tag{1.3.21}$$

Making a change of variables, we obtain

$$\begin{aligned}\|w^1(\cdot - y_n^1)\|^2 &= \int_{\mathbb{R}^N} |\nabla w^1(x - y_n^1)|^2 dx \\ &= \int_{\mathbb{R}^N} |\nabla w^1(x)|^2 dx = \|w^1\|^2. \end{aligned} \tag{1.3.22}$$

Moreover, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla v_n^1 \nabla \varphi dx &= \int_{\mathbb{R}^N} \left[\nabla v_n^1 \nabla \varphi + V(x) v_n^1 \varphi\right] dx - \int_{\mathbb{R}^N} V(x) v_n^1 \varphi dx \\ &= \int_{\mathbb{R}^N} \left[\nabla w^1 \nabla \varphi + V(x) w^1 \varphi\right] dx - \int_{\mathbb{R}^N} V(x) w^1 \varphi dx + o_n(1) \\ &= \int_{\mathbb{R}^N} \nabla w^1 \nabla \varphi dx + o_n(1) \end{aligned}$$

and so as $C_0^\infty(\mathbb{R}^N)$ is dense in $D^{1,2}(\mathbb{R}^N)$, it follows that

$$\begin{aligned}\langle u_{n,1}, w^1(\cdot - y_n^1)\rangle &= \int_{\mathbb{R}^N} \nabla u_{n,1}(x) \nabla w^1(x - y_n^1) dx \\ &= \int_{\mathbb{R}^N} \nabla u_{n,1}(x + y_n^1) \nabla w^1(x) dx \\ &= \int_{\mathbb{R}^N} \nabla v_n^1(x) \nabla w^1(x) dx \\ &= \|w^1\|^2 + o_n(1). \end{aligned} \tag{1.3.23}$$

Substituting (1.3.22) and (1.3.23) in (1.3.21), we obtain

$$\|u_n\|_V^2 = \|u_{n,2}\|^2 + \|u\|_V^2 + \|w^1\|^2 + o_n(1),$$

proving (a2).

Using the previous results obtained in (a2) and (c), we have

$$\begin{aligned}
I_V(u_n) &- I_V(u) - I_0(u_{n,2}) - I_0(w^1) \\
&= \frac{1}{2}\|u_n\|_V^2 - \int_{\mathbb{R}^N} F(u_n)dx - \frac{1}{2}\|u\|_V^2 + \int_{\mathbb{R}^N} F(u)dx \\
&\quad - \frac{1}{2}\|u_{n,2}\|^2 + \int_{\mathbb{R}^N} F(u_n)dx - \frac{1}{2}\|w^1\|^2 + \int_{\mathbb{R}^N} F(w^1)dx \\
&= \frac{1}{2}\left[\|u_n\|_V^2 - \|u\|_V^2 - \|u_{n,2}\|^2 - \|w^1\|^2\right] \\
&\quad - \int_{\mathbb{R}^N} [F(u_n) - F(u_{n,1}) - F(u)]dx \\
&\quad - \int_{\mathbb{R}^N} [F(u_{n,1}) - F(u_{n,2})]dx + \int_{\mathbb{R}^N} F(w^1)dx \\
&= o_n(1) - \int_{\mathbb{R}^N} \left[F(u_{n,1}(x + y_n^1)) - F(u_{n,2}(x + y_n^1))\right]dx + \int_{\mathbb{R}^N} F(w^1)dx \\
&= o_n(1) - \int_{\mathbb{R}^N} \left[F(u_{n,1}(x + y_n^1)) - F(u_{n,2}(x + y_n^1)) - F(w^1(x))\right]dx \\
&= o_n(1) - \int_{\mathbb{R}^N} \left[F(v_n^1) - F(v_n^1 - w^1)) - F(w^1)\right]dx.
\end{aligned}$$

Applying Lemma 1.3.3 with $\;= F$ again, (see [31, Lemma A.1]), changing $u_n$ by $v_n^1$ and $u_0$ by $w^1$, we conclude that

$$\int_{\mathbb{R}^N} \left[F(v_n^1) - F(v_n^1 - w^1)) - F(w^1)\right]dx = o_n(1),$$

and so

$$I_V(u_n) = I_V(u) + I_0(u_{n,2}) + I_0(w^1) + o_n(1),$$

which proves (b2).

Next, we will show that $I_0'(u_{n,2}) \to 0$ in $\left(D^{1,2}(\mathbb{R}^N)\right)'$. The fact that $\nabla I_V(u_{n,1}) \to 0$ in $\left(D_G^{1,2}(\mathbb{R}^N)\right)'$ implies that, by Lemma 1.2.3, $I_V'(u_{n,1}) \to 0$ in $\left(D^{1,2}(\mathbb{R}^N)\right)'$, and so

$I_V'(u_{n,1})\varphi \to 0$, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. On the other hand, as $I_0'(w^1) = 0$, we have

$$
\begin{aligned}
I_V'(u_{n,1})\varphi &= I_V'(u_{n,2} + w^1(\cdot - y_n))\varphi \\
&= \int_{\mathbb{R}^N} (\nabla u_{n,2}(x)\nabla\varphi(x) + V(x)u_{n,2}(x)\varphi(x))dx \\
&\quad + \int_{\mathbb{R}^N} \big(\nabla w^1(x - y_n^1)\nabla\varphi(x) + V(x)w^1(x - y_n^1)\varphi(x)\big)dx \\
&\quad - \int_{\mathbb{R}^N} f(u_{n,2}(x) + w^1(x - y_n^1))\varphi(x)dx \\
&= I_V'(u_{n,2})\varphi + \int_{\mathbb{R}^N} f(u_{n,2}(x))\varphi(x)dx \\
&\quad + \int_{\mathbb{R}^N} \big(\nabla w^1(x)\nabla\varphi(x + y_n^1) + V(x + y_n)w^1(x)\varphi(x + y_n^1)\big)dx \\
&\quad - \int_{\mathbb{R}^N} f(u_{n,1}(x))\varphi(x)dx \\
&= I_V'(u_{n,2})\varphi + \int_{\mathbb{R}^N} f(u_{n,2}(x))\varphi(x)dx \\
&\quad + \int_{\mathbb{R}^N} \nabla w^1(x)\nabla\varphi(x + y_n^1)dx + \int_{\mathbb{R}^N} V(x + y_n^1)w^1(x)\varphi(x + y_n^1)dx \\
&\quad - \int_{\mathbb{R}^N} f(u_{n,1}(x))\varphi(x)dx \\
&= I_V'(u_{n,2})\varphi + \int_{\mathbb{R}^N} f(u_{n,2}(x))\varphi(x)dx \\
&\quad + I_0'(w^1)\varphi(\cdot + y_n) + \int_{\mathbb{R}^N} f(w^1(x))\varphi(x + y_n^1)dx \\
&\quad + \int_{\mathbb{R}^N} V(x + y_n^1)w^1(x)\varphi(x + y_n^1)dx - \int_{\mathbb{R}^N} f(u_{n,1}(x))\varphi(x)dx \\
&= I_V'(u_{n,2})\varphi + \int_{\mathbb{R}^N} V(x + y_n^1)w^1(x)\varphi(x + y_n^1)dx \\
&\quad - \int_{\mathbb{R}^N} \big[f(u_{n,1}(x + y_n^1)) - f(u_{n,2}(x + y_n^1)) - f(w^1(x))\big]\varphi(x + y_n^1)dx.
\end{aligned}
$$

Using $(V_2)$ and applying Lebesgue dominated convergence theorem, it follows that

$$\int_{\mathbb{R}^N} V(x + y_n^1)w^1(x)\varphi(x + y_n^1)dx = o_n(1).$$

Next we will show that

$$\begin{aligned}\int_{\mathbb{R}^N} &\left[f(u_{n,1}(x+y_n^1)) - f(u_{n,2}(x+y_n^1)) - f(w^1(x))\right]\varphi(x+y_n^1)dx \\ &= \int_{\mathbb{R}^N} \left[f(v_n^1) - f(v_n^1 - w^1) - f(w^1)\right]\varphi(x+y_n^1)dx = o_n(1).\end{aligned}$$

Indeed, by hypothesis $(f_2)$, we have $|f(s)| \leq A_2|s|^{2^*-1}$ for all $s \in \mathbb{R}$. Thus, arguing as in (1.3.8), we obtain

$$|f(v_n^1) - f(v_n^1 - w^1)| \leq C_1\left(|v_n^1 - w^1|^{2^*-2}|w^1| + |w^1|^{2^*-1}\right),$$

and so

$$\begin{aligned}|f(v_n^1) - f(v_n^1 - w^1) - f(w^1)| &\leq |f(v_n^1) - f(v_n^1 - w^1)| + |f(w^1)| \\ &\leq C_1\left(|v_n^1 - w^1|^{2^*-2}|w^1| + |w^1|^{2^*-1}\right) + A_2|w^1|^{2^*-1} \\ &= C_1|v_n^1 - w^1|^{2^*-2}|w^1| + (C_1 + A_2)|w^1|^{2^*-1}.\end{aligned}$$

Let $R > 1$ be. Since $(v_n^1)$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and $D^{1,2}(\mathbb{R}^N)$ is continuously embedded into $L^{2^*}(\mathbb{R}^N)$, we have

$$\begin{aligned}&\int_{|x|>R} |f(v_n^1) - f(v_n^1 - w^1) - f(w^1)||\varphi(x+y_n)|dx \\ &\leq C_1\int_{|x|>R}|v_n^1 - w^1|^{2^*-2}|w^1||\varphi(x+y_n^1)|dx + (C_1+A_2)\int_{|x|>R}|w^1|^{2^*-1}|\varphi(x+y_n^1)|dx \\ &\leq C_1\left(\int_{|x|>R}|v_n^1 - w^1|^{2^*}\,dx\right)^{\frac{2^*-2}{2^*}}\left(\int_{|x|>R}|w^1|^{2^*/2}|\varphi(x+y_n^1)|^{2^*/2}dx\right)^{2/2^*} \\ &\quad + (C_1+A_2)\left(\int_{|x|>R}|w^1|^{2^*}\,dx\right)^{\frac{2^*-1}{2^*}}\left(\int_{|x|>R}|\varphi(x+y_n^1)|^{2^*}\,dx\right)^{1/2^*} \\ &\leq C_1\|v_n^1 - w^1\|_{2^*}^{2^*-2}\left(\int_{|x|>R}|w^1|^{2^*}\,dx\right)^{1/2^*}\left(\int_{|x|>R}|\varphi(x+y_n^1)|^{2^*}\,dx\right)^{1/2^*} \\ &\quad + (C_1+A_2)\|\varphi\|_{2^*}\left(\int_{|x|>R}|w^1|^{2^*}\,dx\right)^{\frac{2^*-1}{2^*}} \\ &\leq C\|\varphi\|_V\left[\left(\int_{|x|>R}|w^1|^{2^*}\,dx\right)^{1/2^*} + \left(\int_{|x|>R}|w^1|^{2^*}\,dx\right)^{\frac{2^*-1}{2^*}}\right].\end{aligned}$$

Thus, given $\varepsilon > 0$, we may choose $R > 1$ sufficiently large such that

$$\int_{|x|>R} |f(v_n^1) - f(v_n^1 - w^1) - f(w^1)||\varphi(x + y_n^1)| dx \leq \varepsilon \|\varphi\|_V. \tag{1.3.24}$$

On the other hand, from (1.3.8) and hypotheses $(f_1)$ and $(f_2)$, we get

$$\begin{aligned} |f(u_n) - f(u_n - u) - f(u)| &\leq |f(u_n) - f(u)| + |f(u_n - u)| \\ &\leq C_1\big(|u|^{2^*-2}|u_n - u| + |u_n - u|^{2^*-1}\big) + A_2|u_n - u|^{2^*-1} \\ &= C_1|u|^{2^*-2}|u_n - u| + (C_1 + A_2)|u_n - u|^{2^*-1}, \end{aligned}$$

and so

$$\begin{aligned} &\int_{|x|\leq R} |f(u_n) - f(u_n - u) - f(u)||\varphi| dx \\ &\qquad \leq C_1 \int_{|x|\leq R} |u|^{2^*-2}|u_n - u||\varphi| dx + (C_1 + A_2)\int_{|x|\leq R} |u_n - u|^{2^*-1}|\varphi| dx. \end{aligned}$$

We fix $\delta \in \left(0, \frac{1}{N-2}\right)$ and consider $q_1 := 2^* - \delta$ and $q_2 := (2^* - \delta)/(1 - \delta)$. Thus,

$$\begin{aligned} \int_{|x|\leq R} |u_n - u|^{2^*-1}|\varphi| dx &\leq \left(\int_{|x|\leq R} \big(|u_n - u|^{2^*-1}\big)^{\frac{2^*-\delta}{2^*-1}} dx\right)^{\frac{2^*-1}{2^*-\delta}} \left(\int_{|x|\leq R} |\varphi|^{\frac{2^*-\delta}{1-\delta}} dx\right)^{\frac{1-\delta}{2^*-\delta}} \\ &= \left(\int_{|x|\leq R} |u_n - u|^{q_1} dx\right)^{\frac{2^*-1}{q_1}} \left(\int_{|x|\leq R} |\varphi|^{q_2} dx\right)^{\frac{1}{q_2}} \\ &\leq C\|\varphi\|_{1} \left(\int_{|x|\leq R} |u_n - u|^{q_1} dx\right)^{\frac{2^*-1}{q_1}}. \end{aligned}$$

As $u_n \to u$ strongly in $L^{q_1}_{loc}(\mathbb{R}^N)$, it follows that

$$\int_{|x|\leq R} |u_n - u|^{2^*-1}|\varphi| dx = o_n(1), \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \tag{1.3.25}$$

Moreover, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} \int_{|x|\leq R} |u|^{2^*-2}|u_n - u||\varphi| dx &= \left(\int_{|x|\leq R} \big(|u|^{2^*-2}\big)^{\frac{2^*}{2^*-2}} dx\right)^{\frac{2^*-2}{2^*}} \left(\int_{|x|\leq R} (|u_n - u||\varphi|)^{\frac{2^*}{2}} dx\right)^{\frac{2}{2^*}} \\ &= \left(\int_{|x|\leq R} |u|^{2^*} dx\right)^{\frac{2^*-2}{2^*}} \left(\int_{|x|\leq R} (|u_n - u||\varphi|)^{\frac{2^*}{2}} dx\right)^{\frac{2}{2^*}}, \end{aligned}$$

and, using Hölder inequality with conjugate exponents $\frac{2(2^*-\delta)}{2^*}$ and $\frac{2(2^*-\delta)}{2^*-2\delta}$, we get

$$\left(\int_{|x|\geq R}(|u_n-u||\varphi|)^{\frac{2^*}{2}}\,dx\right)^{\frac{2}{2^*}} \leq \left(\int_{|x|\geq R}|u_n-u|^{q_1}dx\right)^{\frac{1}{q_1}}\left(\int_{|x|\geq R}|\varphi|^{q_3}dx\right)^{\frac{1}{q_3}} \leq C\|\varphi\|_\infty\left(\int_{|x|\geq R}|u_n-u|^{q_1}dx\right)^{\frac{1}{q_1}},$$

where $q_1 := 2^*-\delta$ and $q_3 := \frac{2^*(2^*-\delta)}{2^*-2\delta}$. As $u_n \to u$ strongly in $L^{q_1}_{\mathrm{loc}}(\mathbb{R}^N)$, it follows that

$$\left(\int_{|x|\geq R}(|u_n-u||\varphi|)^{\frac{2^*}{2}}\,dx\right)^{\frac{2}{2^*}} = o_n(1),$$

and thus,

$$\int_{|x|\geq R}|u|^{2^*-2}|u_n-u||\varphi|dx = o_n(1), \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^N). \tag{1.3.26}$$

It follows from (1.3.25) and (1.3.26) that

$$\int_{|x|\geq R}|f(u_n)-f(u_n-u)-f(u)||\varphi|dx = o_n(1). \tag{1.3.27}$$

From (1.3.24) and (1.3.27), we conclude that

$$\int_{\mathbb{R}^N}|f(u_n)-f(u_n-u)-f(u)||\varphi|dx = o_n(1),$$

Therefore,

$$I_V'(u_{n,1})\varphi = I_V'(u_{n,2})\varphi + o_n(1), \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N),$$

which shows that, as $n \to \infty$, $I_V'(u_{n,2}) \to 0$ in $\left(D^{1,2}(\mathbb{R}^N)\right)'$. Furthermore, arguing as in (1.3.19), as $u_{n,2} \to 0$ in $L^\eta_{\mathrm{loc}}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N}|V(x)||u_{n,2}||\varphi|\,dx = o_n(1),$$

and thus,

$$\begin{aligned} o_n(1) &= I_V'(u_{n,2})\varphi = \int_{\mathbb{R}^N} (\nabla u_{n,2} \nabla \varphi + V(x) u_{n,2} \varphi) dx - \int_{\mathbb{R}^N} f(u_{n,2}) \varphi dx \\ &= \int_{\mathbb{R}^N} \nabla u_{n,2} \nabla \varphi dx - \int_{\mathbb{R}^N} f(u_{n,2}) \varphi dx + \int_{\mathbb{R}^N} V(x) u_{n,2} \varphi dx \\ &= I_0'(u_{n,2})\varphi + \int_{\mathbb{R}^N} V(x) u_{n,2} \varphi dx \\ &= I_0'(u_{n,2})\varphi + o_n(1). \end{aligned}$$

Therefore,

$$I_0'(u_{n,2})\varphi = o_n(1), \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N),$$

and so, as $n \to \infty$, $I_0'(u_{n,2}) \to 0$ in $\left(D^{1,2}(\mathbb{R}^N)\right)'$, proving (c2).

Thus, if $\|u_{n,2}\| \to 0$ as $n \to \infty$, we have completed the proof. Otherwise, if $u_{n,2} \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^N)$ and does not converge strongly to zero, we take $u_{n,3} := u_{n,2} - w^2(\cdot - y_n^2)$ and repeat the argument. Hence, we obtain

$$I_V(u_n) = I_V(u) + I_0(w^1) + I_0(w^2) + o_n(1).$$

Continuing this way, we get a sequence of points $(y_n^j) \subset \mathbb{R}^N$ such that $|y_n^j| \to \infty$, $|y_n^j - y_n^i| \to \infty$ if $i \neq j$ and sequences of functions $u_{n,j} := u_{n,j-1} - w^{j-1}(\cdot - y_n^{j-1})$, $j \geq 2$, such that

$$u_{n,j}(\cdot + y_n^j) \rightharpoonup w^j \quad \text{in } D^{1,2}(\mathbb{R}^N),$$

where $w^j$ is a nontrivial solution of the limit problem ($P_0$). Since $I_0(w^j) \geq m_0 = p_0$ and $I_V(u_n) \to c$, there exists a positive integer $k$ such that

$$I_V(u_n) = I_V(u) + \sum_{j=1}^{k} I_0(w^j) + o_n(1),$$

and the proof of lemma is complete. □

**Remark 1.3.7.** Note that if $u \not\equiv 0$ is a solution of ($P_G$) then $u \in \mathcal{P}_V^G$ and the following statement holds

$$N \int_{\mathbb{R}^N} F(u) dx = \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} \left( \frac{\nabla V(x) \cdot x}{N} + V(x) \right) u^2 dx.$$

Then, using Hölder inequality and hypothesis $(V_3)$, we have that $I_V(u) > 0$. Indeed,

$$
\begin{aligned}
I_V(u) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)dx - \int_{\mathbb{R}^N} F(u)dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)dx \\
&\quad - \frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{\nabla V(x)\cdot x}{N} + V(x)\right)u^2 dx \\
&= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2N}\int_{\mathbb{R}^N} \nabla V(x)\cdot x\, u^2 dx \\
&\geq \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2N}\left(\int_{\mathbb{R}^N} \left|W^+(x)\right|^{N/2} dx\right)^{2/N}\left(\int_{\mathbb{R}^N} \left|u^2\right|^{2^*/2} dx\right)^{2/2^*} \\
&\geq \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{S}{4N}\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*} \\
&\geq \frac{3}{4N}\int_{\mathbb{R}^N} |\nabla u|^2 dx > 0.
\end{aligned}
$$

The next corollary is a fundamental result in order to prove strong convergence of $(PS)_c$-sequences.

**Corollary 1.3.8.** *Assume that $(V_1)$–$(V_3)$ and $(f_1)$–$(f_4)$ hold true. Let $(u_n) \subset D_G^{1,2}(\mathbb{R}^N)$ be a bounded $(PS)_c$-sequence for $I_V$. If $0 < c < \ell(G)p_0$, where $p_0$ is given in (1.2.7), then the functional $I_V$ has a nontrivial critical point $u \in D_G^{1,2}(\mathbb{R}^N)$ such that $I_V(u) = c$.*

*Proof.* By Lemma 1.3.6, passing to a subsequence, we get a solution $u \in D_G^{1,2}(\mathbb{R}^N)$ of problem $(P_G)$ such that $u_n \rightharpoonup u$ weakly in $D_G^{1,2}(\mathbb{R}^N)$. Next, let us show that $u_n \to u$ strongly in $D_G^{1,2}(\mathbb{R}^N)$. Suppose that $u_n \not\to u$. Applying Lemma 1.3.6 again, replacing $(u_n)$ by a subsequence, if necessary, there exist an integer $k \geq 1$, $k$ nontrivial solutions $w^1, \cdots, w^k$ of the limit problem $(P_0)$ and $k$ sequences $(y_n^j) \subset \mathbb{R}^N$, $1 \leq j \leq k$ such that $|y_n^j| \to \infty$ and

$$
u_n - u - \sum_{j=1}^{k} w^j(\cdot - y_n^j) \to 0 \quad \text{in } D^{1,2}(\mathbb{R}^N), \tag{1.3.28}
$$

$$
c = I_V(u) + \sum_{j=1}^{k} I_0(w^j). \tag{1.3.29}
$$

Then, up to a subsequence, we have

$$
z_n^j := \frac{y_n^j}{|y_n^j|} \in S^{N-1}, \quad \text{for } j = 1, \cdots, k
$$

and as $(z_n^j)$ is bounded in $\mathbb{R}^N$ and $S^{N-1}$ is closed, there exists $z_\infty^j \in S^{N-1}$ such that

$z_n^j \to z_\infty^j$, as $n \to +\infty$. We claim that the set $\{z_\infty^j : j = 1, \dots, k\}$ is $G$-symmetric. Indeed, for any integer $j \in \{1, \dots, k\}$ and $g \in G$, as $u_n, u \in D_G^{1,2}(\mathbb{R}^N)$, we have $u_n(gx) = u_n(x)$ and $u(gx) = u(x)$ for all $x \in \mathbb{R}^N$. In particular, $u_n(gx) = u_n(x)$ and $u(gx) = u(x)$ for all $x \in B_1(y_n^j)$. By (1.3.28) and (1.5.1) with $R = 1$,

$$\liminf_{n\to\infty} \int_{B_1(y_n^j)} |\nabla u_n(x) - \nabla u(x)|^2 dx = \liminf_{n\to\infty} \int_{B_1(y_n^j)} \left| \sum_{i=1}^{k} \nabla w^i(x - y_n^i) \right|^2 dx \geq \alpha > 0,$$

and so, we also get

$$\liminf_{n\to\infty} \int_{B_1(y_n^j)} |\nabla u_n(gx) - \nabla u(gx)|^2 dx > 0.$$

Let us show that, given $j \in \{1, \dots, k\}$, there exists an integer $\ell \in \{1, \dots, k\}$ such that $\{gy_n^j - y_n^\ell\}_{n=1}^\infty$ is bounded. Otherwise, for any $\ell \in \{1, \dots, k\}$ and $j$ and $g \in G$ fixed, $\{gy_n^j - y_n^\ell\}_{n=1}^\infty$ is not bounded. So, there exists a subsequence of $n_i \in \mathbb{N}$, for simplicity still denoted by $n$, such that $|gy_n^j - y_n^\ell| \to \infty$, as $n \to \infty$, for all $\ell \in \{1, \dots, k\}$. Hence,

$$\begin{aligned}
0 < \alpha &< \liminf_{n\to\infty} \int_{B_1(y_n^j)} |\nabla u_n(x) - \nabla u(x)|^2\, dx \\
&= \liminf_{n\to\infty} \int_{B_1(0)} \left| \nabla u_n(x + y_n^j) - \nabla u(x + y_n^j) \right|^2 dx \\
&= \liminf_{n\to\infty} \int_{B_1(0)} \left| \nabla u_n(g(x + y_n^j)) - \nabla u(g(x + y_n^j)) \right|^2 dx \\
&= \liminf_{n\to\infty} \int_{B_1(0)} \left| \nabla \sum_{i=1}^{k} w^i(g(x + y_n^j) - y_n^i) \right|^2 dx \\
&= \liminf_{n\to\infty} \int_{B_1(0)} \left| \sum_{i=1}^{k} \nabla w^i(gx + gy_n^j - y_n^i) \right|^2 dx \\
&= 0,
\end{aligned}$$

since the domain of integration is the ball $B_1(0)$ and $|\xi_n^i| = |gy_n^j - y_n^i| \to +\infty$, as $n \to +\infty$ and $|\nabla w^i| \in L^2(\mathbb{R}^N)$, for $1 \leq i \leq k$, and this gives us a contradiction. Therefore, there exists $\ell \in \{1, \dots, k\}$ such that $\{gy_n^j - y_n^\ell\}_{n=1}^\infty$ is bounded. So, there exists a constant $M > 0$ such that

$$|gy_n^j - y_n^\ell| \leq M, \quad \forall\, n \in \mathbb{R}^N.$$

In order to conclude the claim, using $|y_n^j| \to +\infty$, as $n \to +\infty$, then from above

$$\frac{1}{|y_n^j|}|gy_n^j - y_n^\ell| \le \frac{M}{|y_n^j|} \to 0, \quad n \to +\infty.$$

This yields,

$$gz_\infty^j = \lim_{n\to\infty} g\left(\frac{y_n^j}{|y_n^j|}\right) = \lim_{n\to\infty} \frac{1}{|y_n^j|} g y_n^j = \lim_{n\to\infty} \frac{y_n^\ell}{|y_n^j|} = \lim_{n\to\infty} \frac{|y_n^\ell|}{|y_n^j|}\frac{y_n^\ell}{|y_n^\ell|} = z_\infty^\ell, \tag{1.3.30}$$

if we prove that $\lim_{n\to\infty} |y_n^\ell|/|y_n^j| = 1$. In fact,

$$\frac{1}{|y_n^\ell|}|gy_n^j - y_n^\ell| \le \frac{M}{|y_n^\ell|} \to 0, \quad \text{as } n \to \infty.$$

So, $|gy_n^j/|y_n^\ell| - y_n^\ell/|y_n^\ell|| \to 0$ and hence

$$1 = \lim_{n\to\infty}\left|\frac{y_n^\ell}{|y_n^\ell|}\right| = \lim_{n\to\infty}\left|g\frac{y_n^j}{|y_n^\ell|}\right| = \lim_{n\to\infty}\left|\frac{y_n^j}{y_n^\ell}\right|.$$

Therefore by (1.3.30), $\{z_\infty^j : j = 1, \dots, k\}$ is $G$-symmetric, and so if we denote $\#Gx := \#\{gx : g \in G\}$,

$$\ell(G) = \min\{\#Gx : x \in S^{N-1}\} \le \min\{\#Gz_\infty^j : 1 \le j \le k\} \le \#\{z_\infty^\ell : 1 \le \ell \le k\} \le k.$$

Since $I_0(w^j) \ge m_0 = p_0$ for $j = 1, \dots, k$, we obtain from (1.3.29) and inequality above

$$c \ge I_V(u) + kp_0 \ge I_V(u) + \ell(G)p_0. \tag{1.3.31}$$

As $I_V(0) = 0$ and $u$ is a solution of problem ($P_G$), by Remark 1.3.7, $I_V(u) \ge 0$. It follows from (1.3.31) that

$$c \ge I_V(u) + \ell(G)p_0 \ge \ell(G)p_0,$$

which is a contradiction with the hypothesis that $c < \ell(G)p_0$. Therefore, $u_n \to u$ strongly in $D_G^{1,2}(\mathbb{R}^N)$. Since $(u_n)$ converges strongly to $u$ and $I_V$ is continuous, it follows that $I_V(u) = c > 0$, so $u \neq 0$ and the proof of corollary is complete. □

## 1.4 Existence of a positive solution

We will need the following result of [17, Lemma 4.1] and we refer to that for the proof.

**Lemma 1.4.1.** (a) *If $y_0, y \in \mathbb{R}^N$, $y_0 \neq y$, and $\alpha$ and $\beta$ are positive constants such that $\alpha + \beta > N$, then there exists $C_1 = C_1(\alpha, \beta, |y - y_0|) > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{\mathrm{d}x}{(1+|x - Ry_0|)^\alpha (1+|x - Ry|)^\beta} \le C_1 R^{-\mu}$$

*for all $R \ge 1$, where $\mu := \min\{\alpha, \beta, \alpha + \beta - N\}$.*

(b) *If $y_0, y \in \mathbb{R}^N \setminus \{0\}$, and $\theta$ and $\gamma$ are positive constants such that $\theta + 2\gamma > N$, then there exists $C_2 = C_2(\theta, \gamma, |y_0|, |y|) > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{\mathrm{d}x}{(1+|x|)^\theta (1+|x - Ry_0|)^\gamma (1+|x - Ry|)^\gamma} \le C_2 R^{-\tau},$$

*for all $R \ge 1$, where $\tau := \min\{\theta, 2\gamma, \theta + 2\gamma - N\}$.*

*Proof.* (a): Performing a suitable translation, we may assume that $y = -y_0$. Let $2\rho := |y - y_0| > 0$. In the following, $C$ will denote different positive constants which depend on $\alpha$, $\beta$ and $\rho$. If $|x - Ry_0| \le \rho R$, then $|x - Ry| \ge \rho R$. Hence

$$\begin{aligned}\int_{B_{\rho R}(Ry_0)} \frac{\mathrm{d}x}{(1+|x - Ry_0|)^\alpha (1+|x - Ry|)^\beta} &\le \int_{B_{\rho R}(Ry_0)} \frac{\mathrm{d}x}{(1+|x - Ry_0|)^\alpha (\rho R)^\beta} \\ &= CR^{-\beta} \int_{B_{\rho R}(0)} \frac{\mathrm{d}x}{(1+|x|)^\alpha} \le C\left[R^{-\beta} + R^{N-(\alpha+\beta)}\right] \le CR^{-\mu}.\end{aligned}$$

Similarly,

$$\int_{B_{\rho R}(Ry)} \frac{\mathrm{d}x}{(1+|x - Ry_0|)^\alpha (1+|x - Ry|)^\beta} \le C\left[R^{-\alpha} + R^{N-(\alpha+\beta)}\right] \le CR^{-\mu}.$$

Let

$$H^+ := \{z \in \mathbb{R}^N : |z - Ry| \ge |z - Ry_0|\} \quad \text{and} \quad H^- := \{z \in \mathbb{R}^N : |z - Ry| \le |z - Ry_0|\}.$$

Setting $x = Rz$ we obtain

$$\begin{aligned}\int_{H^+ \setminus B_{\rho R}(Ry_0)} \frac{\mathrm{d}x}{(1+|x - Ry_0|)^\alpha (1+|x - Ry|)^\beta} &\le \int_{H^+ \setminus B_{\rho R}(Ry_0)} \frac{\mathrm{d}x}{(1+|x - Ry_0|)^{\alpha+\beta}} \\ = \int_{H^+ \setminus B_\rho(y_0)} \frac{R^N \mathrm{d}z}{(1+R|z - y_0|)^{\alpha+\beta}} &= \int_{H^+ \setminus B_\rho(0)} \frac{R^N \mathrm{d}z}{(1+R|z|)^{\alpha+\beta}} \le CR^{N-(\alpha+\beta)} \le CR^{-\mu}.\end{aligned}$$

Similarly,

$$\int_{H^- \setminus B_{\rho R}(Ry)} \frac{\mathrm{d}x}{(1+|x - Ry_0|)^\alpha (1+|x - Ry|)^\beta} \le CR^{-\mu}.$$

Since $\mathbb{R}^N \setminus [B_{\rho R}(Ry_0) \cup B_{\rho R}(Ry)] = [H^+ \setminus B_{\rho R}(Ry_0)] \cup [H^- \setminus B_{\rho R}(Ry)]$, the previous estimates yield (a).

(b): From Hölder's inequality and inequality (a), we obtain

$$\int_{\mathbb{R}^N} \frac{\mathrm{d}x}{(1+|x|)^\theta (1+|x-Ry_0|)^\gamma (1+|x-Ry|)^\gamma}$$
$$\leq \left(\int_{\mathbb{R}^N} \frac{\mathrm{d}x}{(1+|x|)^\theta (1+|x-Ry_0|)^{2\gamma}}\right)^{1/2} \left(\int_{\mathbb{R}^N} \frac{\mathrm{d}x}{(1+|x|)^\theta (1+|x-Ry|)^{2\gamma}}\right)^{1/2} \leq C_2 R^{-\tau},$$

as claimed. □

In this section we will prove our main result. Its proof requires some important estimates and the previous lemmata.

In what follows, for simplicity, we will consider $G = O(N-1) \times \mathbb{Z}_2 \subset O(N)$, where $\mathbb{Z}_2 := \{id, -id\}$, and $\ell(G) = 2$. That is, for all $g \in G$, we have

$$g(x_1, \cdots, x_{N-1}, x_N) = (g_1(x_1, \cdots, x_{N-1}), \pm x_N),$$

where $g_1 \in O(N-1)$. Moreover, we will consider $y = (0, \cdots, 0, 1) \in \mathbb{R}^N$ and $w$ a ground state solution of the limit problem ($P_0$), which is positive, radially symmetric and decreasing in the radial direction, such that $I_0(w) = m_0$. Observe that, for any $g \in G$ and $x \in \mathbb{R}^N$, we have $w(gx) = w(|gx|) = w(|x|) = w(x)$ which shows that $w \in D^{1,2}_G(\mathbb{R}^N)$.

We will construct a positive solution of ($P_G$) exploiting the interaction of two translated bumps. Let us denote $B_r(x_0) := \{x \in \mathbb{R}^N : |x - x_0| \leq r\}$. For $y = (0, \cdots, 0, 1)$ and $R > 0$, we define

$$w_-^R := w(\cdot - Ry), \quad w_+^R := w(\cdot + Ry). \tag{1.4.1}$$

In the next lemmas we study the interaction of powers of these two translated solitons.

**Lemma 1.4.2.** *Let $\alpha$ and $\beta$ be constants such that $2\alpha > 2$ and $\beta \geq 1$. Then, for any $R \geq 1$, there exist constants $C_3 = C_3(N, \alpha, \beta) > 0$ and $C_4 = C_4(N, \alpha, \beta) > 0$ such that*

$$\int_{\mathbb{R}^N} (w_-^R)^\alpha (w_+^R)^\beta \leq C_3 R^{-(N-2)}, \tag{1.4.2}$$

*and*

$$\int_{\mathbb{R}^N} (w_+^R)^\alpha (w_-^R)^\beta \leq C_4 R^{-(N-2)}. \tag{1.4.3}$$

*Proof.* By definitions in (1.4.1) and inequalities in (1.1.2), there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} (w^R)^\alpha (w_+^R)^\beta \, dx = \int_{\mathbb{R}^N} (w(x - Ry))^\alpha (w(x+Ry))^\beta \, dx$$
$$\leq C \int_{\mathbb{R}^N} (1 + |x - Ry|)^{-\alpha(N-2)} (1 + |x + Ry|)^{-\beta(N-2)} \, dx.$$

Since $\alpha > 2^*/2$ and $\beta \geq 1$, then $\alpha(N-2) > N$ and $\beta(N-2) \geq N-2$. Therefore, we can apply Lemma 1.4.1(a) with $\alpha = \alpha(N-2)$ and $\beta = \beta(N-2)$, in which $\mu := \min\{\alpha, \beta, \alpha + \beta - N\} \geq N - 2$, to obtain $C_3 > 0$ such that

$$\int_{\mathbb{R}^N} (w^R)^\alpha (w_+^R)^\beta \, dx \leq C_3 R^{-(N-2)}.$$

Similarly, there exists $C_4 > 0$ such that

$$\int_{\mathbb{R}^N} (w_+^R)^\alpha (w^R)^\beta \, dx \leq C_4 R^{-(N-2)}.$$

□

Next, let us define

$$\varepsilon_R := \int_{\mathbb{R}^N} f(w_+^R) \, w^R \, dx = \int_{\mathbb{R}^N} f(w^R) \, w_+^R \, dx \tag{1.4.4}$$

and we will obtain some estimates for $\varepsilon_R$.

**Lemma 1.4.3.** *Assume that $(f_1)$–$(f_2)$ hold true. Then, for any $R \geq 1$, there exists a constant $C > 0$ such that*

$$\varepsilon_R \leq C R^{-(N-2)}. \tag{1.4.5}$$

*Proof.* Using hypotheses $(f_1)$ and $(f_2)$, we have

$$\varepsilon_R = \int_{\mathbb{R}^N} f(w_+^R) \, w^R \, dx \leq A_2 \int_{\mathbb{R}^N} (w_+^R)^{2^*-1} w^R \, dx.$$

Since $2^* - 1 > 2^*/2$, applying Lemma 1.4.2 with $\alpha = 2^* - 1$ and $\beta = 1$, for any $R \geq 1$, there exists $C > 0$ such that

$$\varepsilon_R \leq C R^{-(N-2)}.$$

□

Now observe that, since $w$ is the positive radial ground state solution of the limit problem $(P_0)$, it follows that $\int_{\mathbb{R}^N} |\nabla w|^2 dx = \int_{\mathbb{R}^N} f(w) w \, dx$. Then, there exists $x_0 \in \mathbb{R}^N$

such that $f(w(x_0)) > 0$. By continuity of function $f$, we can get $r_0 = r_0(f, w) > 0$ (which depends only on $f$ and $w$) such that $f(w(x)) \geq f(w(x_0))/2$, for all $x \in B_{r_0}(x_0)$.

**Lemma 1.4.4.** *Assume that* $(f_1)$–$(f_2)$ *hold true. Then, for any* $R \geq 1$, *there exists a constant* $C > 0$ *such that*

$$\varepsilon_R \geq CR^{-(N-2)}. \tag{1.4.6}$$

*Proof.* In the above considerations, since $x_0$ and $r_0$ are constants independent of $R$, we can assume without loss of generality that $x_0 = 0$ and $r_0 = 1$. So it follows that $f(w(z)) \geq f(w(0))/2$, for all $z \in B_1(0)$. Thus, for any $R \geq 1$, a change of variables $z = x - Ry$ and (1.1.2) yield

$$\begin{aligned} \varepsilon_R = \int_{\mathbb{R}^N} f(w(x-Ry))w(x+Ry)dx &= \int_{\mathbb{R}^N} f(w(z))w(z+2Ry)dz \\ \geq \int_{B_1(0)} f(w(z))w(z+2Ry)dz &\geq \int_{B_1(0)} \frac{f(w(0))}{2} w(z+2Ry)dz \\ \geq C\int_{B_1(0)} (1+|z+2Ry|)^{-(N-2)}dz &\geq CR^{-(N-2)}, \end{aligned}$$

which proves the lemma. □

The next lemma presents the order of interaction between the gradients of two translated solitons.

**Lemma 1.4.5.** *For any* $R \geq 1$, *there exists a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^N} \left|\nabla w^R_- \nabla w^R_+\right| dx \leq CR^{-(N-2)}. \tag{1.4.7}$$

*Proof.* Observe that, taking the derivatives and using (1.1.3), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \left|\nabla w^R_- \nabla w^R_+\right| dx &= \int_{\mathbb{R}^N} |\nabla w(x-Ry) \nabla w(x+Ry)|\, dx \\ &\leq C\int_{\mathbb{R}^N} (1+|x-Ry|)^{-(N-1)}(1+|x+Ry|)^{-(N-1)}\, dx \\ &= C\int_{\mathbb{R}^N} (1+|x|)^{-(N-1)}(1+|x+2Ry|)^{-(N-1)}\, dx. \end{aligned}$$

Since $|2Ry| = 2R$, if $|x + 2Ry| \le R$, then $|x| \ge R$. Hence

$$\begin{aligned}
&\int_{B_R(-2Ry)} (1+|x|)^{-(N-1)}(1+|x+2Ry|)^{-(N-1)}dx \\
&\qquad\le \int_{B_R(-2Ry)} R^{-(N-1)}(1+|x+2Ry|)^{-(N-1)}dx \\
&\qquad= R^{-(N-1)}\int_{B_R(0)} (1+|x|)^{-(N-1)}dx \le CR^{-(N-2)}.
\end{aligned}$$

Similarly, we have

$$\int_{B_R(0)} (1+|x|)^{-(N-1)}(1+|x+2Ry|)^{-(N-1)}dx \le CR^{-(N-2)}.$$

Let

$$H^+ := \{x \in \mathbb{R}^N : |x+2Ry| \ge |x|\} \quad \text{and} \quad H^- := \{x \in \mathbb{R}^N : |x+2Ry| \le |x|\}.$$

Setting $x = 2Rz$, we obtain

$$\begin{aligned}
&\int_{H^+ \setminus B_R(-2Ry)} (1+|x|)^{-(N-1)}(1+|x+2Ry|)^{-(N-1)}dx \\
&\qquad\le \int_{H^+ \setminus B_R(-2Ry)} (1+|x|)^{-2(N-1)}\,dx \\
&\qquad= \int_{H^+ \setminus B_{1/2}(-y)} 2^N(1+2R|z|)^{-2(N-1)}\,R^N dz \\
&\qquad\le CR^{-2(N-1)}\,R^N \int_{H^+ \setminus B_{1/2}(-y)} |z|^{-2(N-1)}dz \\
&\qquad\le CR^{-(N-2)}.
\end{aligned}$$

Similarly, we have

$$\int_{H^- \setminus B_R(0)} (1+|x|)^{-(N-1)}(1+|x+2Ry|)^{-(N-1)}dx \le CR^{-(N-2)}.$$

Since $\mathbb{R}^N \setminus [B_R(-2Ry) \cup B_R(0)] = [H^+ \setminus B_R(-2Ry)] \cup [H^- \setminus B_R(0)]$, by previous estimates, we obtain $C > 0$ such that

$$\int_{\mathbb{R}^N} \left|\nabla w^R_- \nabla w^R_+\right| dx \le CR^{-(N-2)}.$$

□

We will need the following estimates adapted from a result in [1, Lemma 2.2].

**Lemma 1.4.6.** *Assume that* $(f_1)$–$(f_2)$ *hold true. Then, there exists* $\sigma \in (1/2, 1]$ *with the following property: for any given* $C_5 \geq 1$ *there is a constant* $C_6 > 0$ *such that the inequalities*

$$|f(u+v) - f(u) - f(v)| \leq C_6 |uv|^{\sigma}$$

*and*

$$|F(u+v) - F(u) - F(v) - f(u)v - f(v)u| \leq C_6 |uv|^{2\sigma}$$

*hold true for all* $u, v \in \mathbb{R}$, *with* $|u|, |v| \leq C_5$.

*Proof.* Hypothesis $(f_2)$ implies that there exists a constant $C > 0$ such that $\left|f^{(i)}(s)\right| \leq C|s|^{2^* - (i+1)}$, for $i = 1, 2, 3$, and $|s| \leq C_5$. Set $q := 2^* - 1$ and $\sigma := \min\{2^*/4, 1\} = \min\{N/(2(N-2)), 1\} \in (1/2, 1]$. The proof of the inequalities follows by simple calculations. Indeed, given $u, v > 0$, there exists a constant $C = C(\sigma, C_5) > 0$ such that

$$\begin{aligned} |f(u+v) - f(u) - f(v)| &= \left| \int_0^u \int_r^{r+v} f''(s)\, ds\, dr \right| \leq C_1 \int_0^u \int_r^{r+v} s^{q-2}\, ds\, dr \\ &\leq C_2 \left[(u+v)^q - u^q - v^q\right] \leq C(uv)^{\sigma}, \end{aligned}$$

$$\begin{aligned} |F(u+v) - F(u) - F(v) - f(u)v - f(v)u| &= \left| \int_0^u \int_0^v \int_0^r \int_t^{s+t} f'''(z)\, dz\, dt\, dr\, ds \right| \\ &\leq C_1 \int_0^u \int_0^v \int_0^r \int_t^{s+t} z^{q-3}\, dz\, dt\, dr\, ds \\ &\leq C_3 \left[(u+v)^{q+1} - u^{q+1} - v^{q+1} - (q+1)u^q v - (q+1)v^q u\right] \leq C(uv)^{2\sigma}. \end{aligned}$$

□

Let us define the sum of the two translated solitons

$$U^R := w_-^R + w_+^R, \tag{1.4.8}$$

and present some of its properties and estimates. Next, we will show that $U^R \in D_G^{1,2}(\mathbb{R}^N)$. Indeed, as $w$ is radially symmetric and $G = O(N-1) \times \mathbb{Z}_2$, given $g \in G$ and $x \in \mathbb{R}^N$, we must consider two situations:

$(i)$ $g(x_1, \cdots, x_{N-1}, x_N) = (g_1(x_1, \cdots, x_{N-1}), x_N)$, where $g_1 \in O(N-1)$;

$(ii)$ $g(x_1, \cdots, x_{N-1}, x_N) = (g_1(x_1, \cdots, x_{N-1}), -x_N)$, where $g_1 \in O(N-1)$.

If $g(x_1, \ldots, x_{N-1}, x_N) = (g_1(x_1, \ldots, x_{N-1}), x_N)$, then

$$\begin{aligned} U^R(gx) &= w^R_-(gx) + w^R_+(gx) = w(gx - Ry) + w(gx + Ry) \\ &= w(g_1(x_1, \ldots, x_{N-1}), x_N - R) + w(g_1(x_1, \ldots, x_{N-1}), x_N + R) \\ &= w(x_1, \ldots, x_{N-1}, x_N - R) + w(x_1, \ldots, x_{N-1}, x_N + R) \\ &= w(x - Ry) + w(x + Ry) = w^R_-(x) + w^R_+(x) = U^R(x). \end{aligned}$$

If $g(x_1, \ldots, x_{N-1}, x_N) = (g_1(x_1, \ldots, x_{N-1}), -x_N)$, then

$$\begin{aligned} U^R(gx) &= w^R_-(gx) + w^R_+(gx) = w(gx - Ry) + w(gx + Ry) \\ &= w(g_1(x_1, \ldots, x_{N-1}), -x_N - R) + w(g_1(x_1, \ldots, x_{N-1}), -x_N + R) \\ &= w(x_1, \ldots, x_{N-1}, x_N + R) + w(x_1, \ldots, x_{N-1}, x_N - R) \\ &= w(x + Ry) + w(x - Ry) = w^R_+(x) + w^R_-(x) = U^R(x). \end{aligned}$$

Therefore, we conclude that $U^R \in D^{1,2}_G(\mathbb{R}^N)$.

**Corollary 1.4.7.** *Assume that* $(f_1)$–$(f_2)$ *hold true. Then, it holds*

$$\int_{\mathbb{R}^N} \left| F(U^R) - F(w^R_-) - F(w^R_+) - f(w^R_-)w^R_+ - f(w^R_+)w^R_- \right| dx = o(\varepsilon_R). \tag{1.4.9}$$

*Proof.* Set $w_- := w^R_-$, $w_+ := w^R_+$ and $U := U^R$. By Lemma 1.4.6, since $w_-$, $w_+$ and $U$ are bounded uniformly in $R$, there exist constants $C > 0$ and $\sigma \in (1/2, 1]$ such that

$$\int_{\mathbb{R}^N} |F(U) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-|\, dx \le C \int_{\mathbb{R}^N} (w_- w_+)^{2\sigma}\, dx.$$

Let us consider two cases: if $N \ge 4$, then $\sigma := \min\{2^*/4, 1\} = 2^*/4 = N/(2(N-2))$. Thus, using (1.1.2) and Lemma 1.4.1(a) with $\alpha = \beta = 2\sigma(N-2) = N$ and $\mu := \min\{2\sigma(N-2), 4\sigma(N-2) - N\} = N > N - 2$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (w_- w_+)^{2\sigma}\, dx &\le C \int_{\mathbb{R}^N} (1 + |x - Ry|)^{-2\sigma(N-2)} (1 + |x + Ry|)^{-2\sigma(N-2)}\, dx \\ &\le CR^{-\mu}. \end{aligned}$$

By Lemmas 1.4.3 and 1.4.4, it follows that

$$\int_{\mathbb{R}^N} |F(U) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-|\, dx = o(\varepsilon_R).$$

The case $N = 3$ is a little more delicate since $\sigma = 1$ and $\mu = 1$, which gives

$$\int_{\mathbb{R}^N} (w_- w_+)^{2\sigma} dx \le CR^{-1} = O(\varepsilon_R).$$

However, using hypothesis $(f_1)$ for $i = 3$ in the proof of Lemma 1.4.6, in fact we can obtain $C > 0$ such that

$$|F(U) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-| \le C\left[w_-^4 w_+^2 + w_-^3 w_+^3 + w_-^2 w_+^4\right],$$

and so, again using (1.1.2) and Lemma 1.4.1(a), we get

$$\int_{\mathbb{R}^N} |F(U) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-|\, dx \le CR^{-2} = o(\varepsilon_R),$$

which yields (1.4.9), and the proof is complete. □

**Lemma 1.4.8.** *Assume that* $(V_1)$–$(V_2)$ *and* $(f_1)$–$(f_3)$ *hold true. Then, the following statements hold:*

(a) $\displaystyle\int_{\mathbb{R}^N} |\nabla U^R|^2 dx = 2\int_{\mathbb{R}^N} |\nabla w|^2 dx + o_R(1)$;

(b) $\displaystyle\int_{\mathbb{R}^N} F(U^R) dx = 2\int_{\mathbb{R}^N} F(w) dx + o_R(1)$,

*where* $o_R(1) \to 0$ *as* $R \to +\infty$.

*Proof.* Set $w_- := w^R$, $w_+ := w_+^R$ and $U := U^R$. Thus, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla U|^2 dx &= \int_{\mathbb{R}^N} |\nabla w_- + \nabla w_+|^2 dx \\
&= \int_{\mathbb{R}^N} |\nabla w_-|^2 dx + 2\int_{\mathbb{R}^N} \nabla w_- \nabla w_+\, dx + \int_{\mathbb{R}^N} |\nabla w_+|^2 dx \\
&= \int_{\mathbb{R}^N} |\nabla w|^2 dx + 2\int_{\mathbb{R}^N} \nabla w_- \nabla w_+\, dx + \int_{\mathbb{R}^N} |\nabla w|^2 dx \\
&= 2\int_{\mathbb{R}^N} |\nabla w|^2 dx + 2\int_{\mathbb{R}^N} \nabla w_- \nabla w_+\, dx.
\end{aligned}$$

By Lemma 1.4.5, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla w_- \nabla w_+|\, dx \le CR^{-(N-2)},$$

proving item (a). We also have

$$\begin{aligned}\int_{\mathbb{R}^N} F(U)dx - 2\int_{\mathbb{R}^N} F(w)dx &= \int_{\mathbb{R}^N} F(U)dx - \int_{\mathbb{R}^N} F(w_-)dx - \int_{\mathbb{R}^N} F(w_+)dx \\ &= \int_{\mathbb{R}^N} [F(U) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-]\,dx \\ &\quad + \int_{\mathbb{R}^N} [f(w_-)w_+ + f(w_+)w_-]\,dx.\end{aligned}$$

By definition (1.4.4) and inequalities (1.4.5) and (1.4.6), it follows that

$$\int_{\mathbb{R}^N} [f(w_-)w_+ + f(w_+)w_-]\,dx = 2\varepsilon_R = o_R(1)$$

and, by Corollary 1.4.7,

$$\int_{\mathbb{R}^N} |F(U) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-|\,dx = o(\varepsilon_R) = o_R(1),$$

which proves item (b), concluding the proof of the lemma. □

**Lemma 1.4.9.** *Assume that $(V_1)$–$(V_4)$ and $(f_1)$–$(f_3)$ hold true. Then, there exists $R_0 \geq 1$ such that for any $R \geq R_0$, there exists a unique positive constant $s := S^R$ such that*

$$U^R\left(\frac{\cdot}{s}\right) \in \mathcal{P}_V^G,$$

*where $U^R$ is given as in (1.4.8). Moreover, there exist $\sigma_0 \in (0, 1/2)$ and $S_0 > 1$ such that $S^R \in (\sigma_0, S_0)$ for any $R \geq R_0$. In addition, $S^R$ is a continuous function of the variable $R$.*

*Proof.* Denote, $w_- := w_-^R = w(\cdot - Ry)$, $w_+ := w_+^R = w(\cdot + Ry)$ and $U := U^R = w_-^R + w_+^R$. Let $\xi_V : (0, +\infty) \to \mathbb{R}$ be defined by

$$\xi_V(s) := I_V(U(\cdot/s)) = \frac{s^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla U|^2 dx + \frac{s^N}{2}\int_{\mathbb{R}^N} V(sx)U^2 dx - s^N\int_{\mathbb{R}^N} F(U)dx.$$

Then, $U(\cdot/s) \in \mathcal{P}_V^G$ if and only if $\xi_V'(s) = 0$, where

$$\begin{aligned}\xi_V'(s) &= \frac{N-2}{2}s^{N-3}\int_{\mathbb{R}^N} |\nabla U|^2 dx \\ &\quad + N s^{N-3}\left[s^2\left(\frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{\nabla V(sx)\cdot(sx)}{N} + V(sx)\right)U^2 dx - \int_{\mathbb{R}^N} F(U)dx\right)\right].\end{aligned}$$

Since $s > 0$, we have $\xi_V'(s) = 0$ if and only if

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla U|^2 dx = Ns^2\left[\int_{\mathbb{R}^N} F(U)dx - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{\nabla V(sx)\cdot(sx)}{N} + V(sx)\right)U^2 dx\right].$$

Observe that

$$\int_{\mathbb{R}^N} U^2 dx = \int_{\mathbb{R}^N} (w_+ + w_-)^2\, dx \le 2\int_{\mathbb{R}^N}\left[(w_+)^2 + (w_-)^2\right]dx = 4\int_{\mathbb{R}^N} w^2 dx,$$

which gives that $\|U\|_2$ is bounded uniformly for any $R \ge 1$. Since $\int_{\mathbb{R}^N} |\nabla w|^2 dx > 0$, using $(V_2)$ and Lemma 1.4.8, there exists $R_1 > 1$, sufficiently large, and $\sigma_0 \in (0, 1/2)$ sufficiently small such that

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla U|^2 dx - Ns^2\left[\int_{\mathbb{R}^N} F(U)dx - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{\nabla V(sx)\cdot(sx)}{N} + V(sx)\right)U^2 dx\right] > 0,$$

and so it holds $\xi_V'(s) > 0$, for every $s \in (0, \sigma_0]$ and $R \ge R_1$.

Now let us define a function $\psi_V : (\sigma_0, +\infty) \to \mathbb{R}$ by

$$\psi_V(s) = s^2\left[\int_{\mathbb{R}^N} F(U)dx - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{\nabla V(sx)\cdot(sx)}{N} + V(sx)\right)U^2 dx\right].$$

Note that

$$\begin{aligned}\psi_V'(s) = {} & 2s\left[\int_{\mathbb{R}^N} F(U)dx - \frac{1}{2}\int_{\mathbb{R}^N} V(sx)U^2 dx\right] \\ & - \frac{s}{2}\left[(N+3)\int_{\mathbb{R}^N} \frac{\nabla V(sx)\cdot(sx)}{N}U^2 dx + \int_{\mathbb{R}^N} \frac{(sx)H(sx)(sx)}{N}U^2 dx\right].\end{aligned}$$

Observe that

$$(1+|sx|)^{-k} \le \begin{cases} \sigma_0^{-k}(1+|x|)^{-k}, & \text{if } \sigma_0 < s \le 1 \\ (1+|x|)^{-k}, & \text{if } 1 \le s. \end{cases}$$

Therefore, using the hypothesis $(V_2)$, we obtain constants $C_1, C_2 > 0$ such that

$$\int_{\mathbb{R}^N} |V(sx)|U^2 dx \le C_1\int_{\mathbb{R}^N} (1+|x|)^{-k}\left[w_- + w_+\right]^2 dx,$$

$$\int_{\mathbb{R}^N} |\nabla V(sx)\cdot(sx)|U^2 dx \le C_2\int_{\mathbb{R}^N} (1+|x|)^{-k}\left[w_- + w_+\right]^2 dx,$$

for every $s > \sigma_0$. Thus, using the inequalities in (1.1.2) and applying Lemma 1.4.1(b), we obtain

$$\int_{\mathbb{R}^N} |V(sx)|U^2 dx = o_R(1), \qquad \int_{\mathbb{R}^N} |\nabla V(sx)\cdot(sx)|U^2 dx = o_R(1), \tag{1.4.10}$$

where $o_R(1) \to 0$ as $R \to +\infty$. Furthermore, note that

$$\int_{\mathbb{R}^N} |(sx)H(sx)(sx)|\, U^2 dx \leq 2\int_{\mathbb{R}^N} |(sx)H(sx)(sx)| \left[(w_-)^2 + (w_+)^2\right] dx.$$

Let us prove that $\int_{\mathbb{R}^N} |(sx)H(sx)(sx)|(w_-)^2\, dx = o_R(1)$. Indeed, let $\varepsilon > 0$ be given arbitrarily. Since $\|w\|_2 > 0$, using the hypothesis $(V_4)$, we can take $\bar\rho > 0$ sufficiently large such that if $s > \sigma_0$ and $|x| \geq \bar\rho/\sigma_0$, then

$$|(sx)H(sx)(sx)| < \frac{\varepsilon}{4\|w\|_2^2}.$$

So, for all $s > \sigma_0$, we have

$$\int_{|x| \geq \bar\rho/\sigma_0} |(sx)H(sx)(sx)|(w_-)^2\, dx \leq \frac{\varepsilon}{4\|w\|_2^2}\int_{\mathbb{R}^N} (w_-)^2\, dx \leq \frac{\varepsilon}{4\|w\|_2^2}\int_{\mathbb{R}^N} w^2 dx \leq \frac{\varepsilon}{4}. \tag{1.4.11}$$

On the other hand, as $\lim\limits_{|x| \to \infty} |(x)H(x)(x)| = 0$, there exists a constant $C_3 > 0$ such that

$$|(sx)H(sx)(sx)| \leq C_3, \quad \text{for every} \quad s > \sigma_0 \quad \text{and} \quad |x| \leq \bar\rho/\sigma_0.$$

Thus, using (1.1.2), for every $s > \sigma_0$ and $R > 2\bar\rho/\sigma_0$, we obtain

$$\begin{aligned}\int_{|x| \leq \bar\rho/\sigma_0} |(sx)H(sx)(sx)|(w_-)^2\, dx &\leq C_3\int_{|x| \leq \bar\rho/\sigma_0} (w(x - Ry))^2\, dx \\ &\leq C\int_{|x| \leq \bar\rho/\sigma_0} (1 + |x - Ry|)^{-(N-2)}\, dx \leq C\int_{|x| \leq \bar\rho/\sigma_0} (|Ry| - |x|)^{-(N-2)} dx \\ &\leq C\left(R - \frac{R}{2}\right)^{-(N-2)} \leq CR^{-(N-2)}.\end{aligned} \tag{1.4.12}$$

Therefore, inequalities (1.4.11) and (1.4.12) give that

$$\int_{\mathbb{R}^N} |(sx)H(sx)(sx)|(w_-)^2\, dx \leq \frac{\varepsilon}{4} + CR^{-(N-2)}, \tag{1.4.13}$$

for every $s > \sigma_0$. By an analogous procedure, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |(sx)H(sx)(sx)|(w_+)^2\, dx \leq \frac{\varepsilon}{4} + CR^{-(N-2)}, \tag{1.4.14}$$

for every $s > \sigma_0$. From (1.4.13) and (1.4.14), we obtain

$$\int_{\mathbb{R}^N} |(sx)H(sx)(sx)| U^2 dx \le 2\int_{\mathbb{R}^N} |(sx)H(sx)(sx)| \left[(w_-)^2 + (w_+)^2\right] dx \le \varepsilon + CR^{-(N-2)}, \tag{1.4.15}$$

for every $s > \sigma_0$. Since $\varepsilon > 0$ was taken arbitrarily, it follows from (1.4.15) that

$$\int_{\mathbb{R}^N} |(sx)H(sx)(sx)| U^2 dx = o_R(1). \tag{1.4.16}$$

Thus, knowing that $\int_{\mathbb{R}^N} F(w)dx > 0$, using the hypotheses $(V_2)$, $(V_4)$, Lemma 1.4.8(b), (1.4.10) and (1.4.16), there exists $R_1 \ge 1$ sufficiently large such that

$$\begin{aligned}\psi_V'(s) = 2s&\left[\int_{\mathbb{R}^N} F(U)dx - \frac{1}{2}\int_{\mathbb{R}^N} V(sx)U^2dx\right]\\ &-\frac{s}{2}\left[(N+3)\int_{\mathbb{R}^N}\frac{\nabla V(sx)\cdot(sx)}{N}U^2dx + \int_{\mathbb{R}^N}\frac{(sx)H(sx)(sx)}{N}U^2dx\right] > 0,\end{aligned}$$

for every $s > \sigma_0$ and $R \ge R_1$ sufficiently large. This means that $\psi_V(s)$ is increasing for $s > \sigma_0$ and $R$ taken sufficiently large. This implies that the term in the brackets for $\xi_V'(s)$ is decreasing for $s > \sigma_0$, and goes to $-\infty$ as $s \to +\infty$. Therefore, there is a unique $s = S^R > \sigma_0$ such that $\xi_V'(s) = 0$, i.e. $U^R(\cdot/s) \in \mathcal{P}_V^G$. Furthermore, again by Lemma 1.4.8(b) and (1.1.4) there exist $R_2 \ge 1$, sufficiently large, and $S_0 > 1$ such that $\xi_V'(s) < 0$, for all $s > S_0$ and $R \ge R_2$. Taking $R_0 = \max\{R_1, R_2\}$ the result follows. Finally, from the uniform estimates for $U$, $\nabla U$ and $F(U)$ with respect to $R \ge R_0$, the continuity of $S^R$ in this variable is clear, and the proof is complete. □

From here on, consider $S^R$ as obtained in Lemma 1.4.9.

**Lemma 1.4.10.** *Assume that $(V_1)$–$(V_4)$ and $(f_1)$–$(f_3)$ hold true. Then, it holds that*

$$\lim_{R\to+\infty} S^R = 1.$$

*Proof.* By Lemma 1.4.9, there exist constants $R_0 \ge 1$, $S_0 > 1$ and $\sigma_0 \in (0, 1/2)$ such that $S^R \in (\sigma_0, S_0)$ for any $R \ge R_0$. Denoting $w_- := w_-^R = w(\cdot - Ry)$ and $w_+ := w_+^R = w(\cdot + Ry)$, we have

$$\begin{aligned}J_0(w_- + w_+) &= \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla w_- + \nabla w_+|^2 - N\int_{\mathbb{R}^N} F(w_- + w_+)\\ &= \left[\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla w|^2 - N\int_{\mathbb{R}^N}F(w)\right] + \left[\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla w|^2 - N\int_{\mathbb{R}^N}F(w)\right]\\ &\quad + (N-2)\int_{\mathbb{R}^N}\nabla w_- \nabla w_+ - N\int_{\mathbb{R}^N}[F(w_- + w_+) - F(w_-) - F(w_+)].\end{aligned}$$

Since $J_0(w) = 0$, it follows that

$$J_0(w_- + w_+) = (N-2)\int_{\mathbb{R}^N}\nabla w_- \nabla w_+ - N\int_{\mathbb{R}^N}[F(w_- + w_+) - F(w_-) - F(w_+)]. \tag{1.4.17}$$

Lemma 1.4.5 yields

$$\int_{\mathbb{R}^N}|\nabla w_- \nabla w_+| \le CR^{-(N-2)}. \tag{1.4.18}$$

On the other hand, using definition (1.4.4) and its estimates and Corollary 1.4.7, we get

$$\begin{aligned}&\int_{\mathbb{R}^N}|F(w_- + w_+) - F(w_-) - F(w_+)|\\ &\le \int_{\mathbb{R}^N}|F(w_- + w_+) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-|\\ &\quad + \int_{\mathbb{R}^N}|f(w_-)w_+ + f(w_+)w_-|\\ &= o(\varepsilon_R) + 2\varepsilon_R \le CR^{-(N-2)}.\end{aligned} \tag{1.4.19}$$

Therefore, by inequalities (1.4.17), (1.4.18) and (1.4.19), there exists $C > 0$ such that

$$|J_0(w_- + w_+)| \le CR^{-(N-2)}, \tag{1.4.20}$$

and so, $J_0(w_- + w_+) \to 0$ as $R \to \infty$. Then, using hypothesis $(V_2)$, we obtain

$$\begin{aligned}J_V(U^R) &= J_0(w_- + w_+) + \frac{N}{2}\int_{\mathbb{R}^N}\left(\frac{\nabla V(x)\cdot x}{N} + V(x)\right)(w_- + w_+)^2dx\\ &\le J_0(w_- + w_+) + C\int_{\mathbb{R}^N}(1+|x|)^{-k}(w_- + w_+)^2dx,\end{aligned} \tag{1.4.21}$$

and again using (1.1.2) and Lemma 1.4.1(b) the last integral above is bounded by $CR^{-(N-2)}$. From (1.4.20) and (1.4.21), we get

$$\left|J_V(U^R)\right| \le CR^{-(N-2)}.$$

Therefore, $J_V\left(U^R\right) = o_R(1)$, where $o_R(1) \to 0$ as $R \to \infty$, which implies that

$$\lim_{R\to+\infty} S^R \to 1,$$

by uniqueness of $S^R$ and continuity with respect to $R$. This proves the lemma. □

**Lemma 1.4.11.** *Assume that* $(V_1)$–$(V_2)$ *hold true and take* $S_0 > 1$ *and* $0 < \sigma_0 < 1/2$. *Then, there exists a constant* $\tau > N-2$ *such that*

$$s^N\int_{\mathbb{R}^N} |V(sx)|\left[\left(w^R_-\right)^2 + \left(w^R_+\right)^2\right]dx \le CR^{-\tau},$$

*for every* $s \in (\sigma_0, S_0)$ *and* $R \ge 1$.

*Proof.* Denote, as before, $w_- := w^R_{-y} = w(\cdot - Ry)$ and $w_+ := w^R_{+y} = w(\cdot + Ry)$. Thus, by hypothesis $(V_2)$ and decay estimates (1.1.2), we have

$$\begin{aligned} s^N\int_{\mathbb{R}^N} |V(sx)|\left[(w_-)^2 + (w_+)^2\right]dx &= s^N\int_{\mathbb{R}^N} |V(sx)|(w_-)^2dx + s^N\int_{\mathbb{R}^N} |V(sx)|(w_+)^2dx \\ &\le CS_0^N\left\{\int_{\mathbb{R}^N}\frac{dx}{(1+|sx|)^k(1+|x-Ry|)^{2(N-2)}} + \int_{\mathbb{R}^N}\frac{dx}{(1+|sx|)^k(1+|x+Ry|)^{2(N-2)}}\right\}, \end{aligned}$$

for every $s \in (\sigma_0, S_0)$ and $R \ge 1$. Since $0 < \sigma_0 < 1/2$ and $|sx| \ge \sigma_0|x|$, then by Lemma 1.4.1(b), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N}\frac{dx}{(1+|sx|)^k(1+|x-Ry|)^{2(N-2)}} &\le \int_{\mathbb{R}^N}\frac{dx}{(1+\sigma_0|x|)^k(1+|x-Ry|)^{2(N-2)}} \\ &\le \sigma_0^{-k}\int_{\mathbb{R}^N}\frac{dx}{(1+|x|)^k(1+|x-Ry|)^{2(N-2)}} \le CR^{-\tau}, \end{aligned}$$

where $\tau = \min\{k, 2(N-2), k+2(N-2)-N\} > N-2$. Similarly,

$$\int_{\mathbb{R}^N}\frac{dx}{(1+|sx|)^k(1+|x+Ry|)^{2(N-2)}} \le CR^{-\tau},$$

and so, the lemma is proved. □

**Proposition 1.4.12.** *Assume that* $(V_1)$–$(V_4)$, $(f_1)$–$(f_3)$ *hold true. Then, there exist* $L > 2$ *large enough and* $R_4 \ge 1$ *such that*

$$I_V\left(U^R\left(\frac{\cdot}{s}\right)\right) < 2I_0(w) = 2p_0, \quad \textit{for all } s \in (0, L] \textit{ and all } R \ge R_4 \tag{1.4.22}$$

*and*

$$I_V\left(U^R\left(\frac{\cdot}{L}\right)\right) < 0, \quad \textit{for all } R \ge R_4. \tag{1.4.23}$$

***Proof.*** By Lemma 1.4.9, there exist constants $R_0 \geq 1$, $\sigma_0 \in (0,1/2)$ and $S_0 > 1$ such that $S^R \in (\sigma_0, S_0)$ for every $R \geq R_0$. Thus, changing the variables $sz = x$ and, for simplicity, denoting $w_- := w_-^R$ and $w_+ := w_+^R$, we have

$$
\begin{aligned}
I_V\Big(U^R\Big(\frac{\cdot}{s}\Big)\Big) &= \frac{s^{N-2}}{2}\Big[\int_{\mathbb{R}^N} |\nabla w_-|^2 dz - 2s^2\int_{\mathbb{R}^N} F(w_-)dz\Big] \\
&\quad + \frac{s^{N-2}}{2}\Big[\int_{\mathbb{R}^N} |\nabla w_+|^2 dz - 2s^2\int_{\mathbb{R}^N} F(w_+)dz\Big] \\
&\quad + \frac{s^N}{2}\int_{\mathbb{R}^N} V(sz)[w_- + w_+]^2 dz + s^{N-2}\int_{\mathbb{R}^N} \nabla w_- \nabla w_+ \, dz \\
&\quad - s^N\int_{\mathbb{R}^N} [F(w_- + w_+) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-]\,dz \\
&\quad - s^N\int_{\mathbb{R}^N} [f(w_-)w_+ + f(w_+)w_-]\,dz \\
&\leq I_0\Big(w_-\Big(\frac{\cdot}{s}\Big)\Big) + I_0\Big(w_+\Big(\frac{\cdot}{s}\Big)\Big) + s^N\int_{\mathbb{R}^N} |V(sz)|\big[(w_-)^2 + (w_+)^2\big]dz \\
&\quad + s^N\int_{\mathbb{R}^N} |F(w_- + w_+) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-|\,dz \\
&\quad + s^{N-2}\int_{\mathbb{R}^N} \big[\nabla w_- \nabla w_+ - s^2 f(w_-)w_+ - s^2 f(w_+)w_-\big]dz \\
&= 2I_0\Big(w\Big(\frac{\cdot}{s}\Big)\Big) + s^N\int_{\mathbb{R}^N} |V(sz)|\big[(w_-)^2 + (w_+)^2\big]dz \\
&\quad + s^N\int_{\mathbb{R}^N} |F(w_- + w_+) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-|\,dz \\
&\quad + s^{N-2}\int_{\mathbb{R}^N} \big[\nabla w_- \nabla w_+ - s^2 f(w_-)w_+ - s^2 f(w_+)w_-\big]dz.
\end{aligned}
$$

Since $p_0 = I_0(w) = \max_{t>0} I_0\big(w\big(\frac{\cdot}{t}\big)\big) > 0$, it follows that

$$
I_0\Big(w\Big(\frac{\cdot}{s}\Big)\Big) \leq p_0, \qquad \text{for all } s \in (0, \infty).
$$

Let us set

$$
\begin{aligned}
(I_1) &:= s^N\int_{\mathbb{R}^N} |V(sz)|\big[(w_-)^2 + (w_+)^2\big]dz, \\
(I_2) &:= s^N\int_{\mathbb{R}^N} |F(w_- + w_+) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-|\,dz, \\
(I_3) &:= s^{N-2}\int_{\mathbb{R}^N} \big[\nabla w_- \nabla w_+ - s^2 f(w_-)w_+ - s^2 f(w_+)w_-\big]dz.
\end{aligned}
$$

To show (1.4.22) and (1.4.23), we will estimate $(I_1)$, $(I_2)$ and $(I_3)$. Take $L > 2$ large enough. By Lemma 1.4.11, we obtain

$$(I_1) \leq CR^{-\tau},$$

where $\tau > N-2$ for all $N \geq 3$, and hence, $(I_1) = o(\varepsilon_R)$, for every $s \in (0, L]$ and $R \geq 1$. Moreover, Corollary 1.4.7 yields

$$(I_2) = o(\varepsilon_R),$$

for all $N \geq 3$, for every $s \in (0, L]$ and $R \geq 1$.

Using the fact that $w$ is a solution of $(P_0)$, we also have

$$\int_{\mathbb{R}^N} \nabla w_- \nabla w_+ \, dz = \int_{\mathbb{R}^N} f(w_-)w_+ \, dz = \int_{\mathbb{R}^N} f(w_+)w_- \, dz,$$

and so

$$\int_{\mathbb{R}^N} \nabla w_+ \nabla w_- \, dz = \frac{1}{2}\int_{\mathbb{R}^N} [f(w_-)w_+ + f(w_+)w_-]\, dz.$$

Thus,

$$\begin{aligned}(I_3) &= s^{N-2}\int_{\mathbb{R}^N} \left[\nabla w_- \nabla w_+ - s^2 f(w_-)w_+ - s^2 f(w_+)w_-\right] dz\\ &= \left(\frac{1}{2} - s^2\right) s^{N-2}\int_{\mathbb{R}^N} [f(w_-)w_+ + f(w_+)w_-]\, dz\\ &= (1-2s^2)s^{N-2}\varepsilon_R,\end{aligned}$$

where $\varepsilon_R = \int_{\mathbb{R}^N} f(w_-)w_+ \, dz = \int_{\mathbb{R}^N} f(w_+)w_- \, dz$. So, there exist $0 < \delta < 1/4$ and $C_0 > 0$ such that

$$(I_3) = (1-2s^2)s^{N-2}\varepsilon_R \leq -C_0\,\varepsilon_R, \tag{1.4.24}$$

for every $s \in [1-\delta, 1+\delta]$. Therefore, by previous estimates, there exists $R_1 \geq 1$ sufficiently large such that

$$I_V\left(U^R\left(\frac{\cdot}{s}\right)\right) \leq 2I_0\left(w\left(\frac{\cdot}{s}\right)\right) + (I_1) + (I_2) + (I_3) \leq 2p_0 - C_0\,\varepsilon_R + o(\varepsilon_R) < 2p_0, \tag{1.4.25}$$

for every $s \in [1-\delta, 1+\delta]$ and $R \geq R_1$.

Next, let us show that there exists $R_2 \geq 1$ such that

$$I_V\left(U^R\left(\frac{\cdot}{s}\right)\right) < 2p_0 \quad \text{for all } s \in (0, 1-\delta) \cup (1+\delta, L] \text{ and all } R \geq R_2.$$

Note that hypothesis $(V_2)$, the pointwise limit $\lim_{R\to\infty} U^R(x) = 0$ and Lebesgue dominated convergence theorem imply that

$$s^N\int_{\mathbb{R}^N}|V(sz)|\big[(w_+)^2+(w_-)^2\big]dz \to 0, \qquad \text{as } R\to+\infty, \tag{1.4.26}$$

uniformly in $s\in(0,L]$. Also, by Corollary 1.4.7,

$$s^N\int_{\mathbb{R}^N}|F(w_++w_-)-F(w_+)-F(w_-)-f(w_+)w_- - f(w_-)w_+|\,dz \to 0 \tag{1.4.27}$$

as $R\to+\infty$, uniformly in $s\in(0,L]$. Furthermore, applying Lemmas 1.4.3, 1.4.4 and 1.4.5, we may conclude that

$$s^{N-2}\int_{\mathbb{R}^N}\big[\nabla w_+\nabla w_- - s^2 f(w_+)w_- - s^2 f(w_-)w_+\big]dz \to 0 \tag{1.4.28}$$

as $R\to+\infty$, uniformly in $s\in(0,L]$. Hence, it follows from (1.4.26), (1.4.27) and (1.4.28) that

$$\Big|I_V\Big(U^R\Big(\frac{\cdot}{s}\Big)\Big)-2I_0\Big(w\Big(\frac{\cdot}{s}\Big)\Big)\Big| \to 0 \qquad \text{as } R\to+\infty, \tag{1.4.29}$$

uniformly in $s\in(0,L]$. From (1.4.29) and recalling that the map $t\mapsto I_0\big(w\big(\frac{\cdot}{t}\big)\big)$ is strictly increasing in $(0,1]$ and strictly decreasing in $[1,\infty)$ and $I_0(w)=p_0$, it follows that $I_0\big(w\big(\frac{\cdot}{t}\big)\big)<I_0(w)$ for all $t\neq 1$, and so there exists $R_2\geq R_1$ such that

$$I_V\Big(U^R\Big(\frac{\cdot}{s}\Big)\Big)<2p_0, \qquad \text{for all } s\in(0,1-\delta)\cup(1+\delta,L] \text{ and all } R\geq R_2. \tag{1.4.30}$$

Thus, from (1.4.25) and (1.4.30), we conclude that

$$I_V\Big(U^R\Big(\frac{\cdot}{s}\Big)\Big)<2p_0, \qquad \text{for all } s\in(0,L] \text{ and all } R\geq R_2. \tag{1.4.31}$$

Finally, we will prove that (1.4.23) occurs. We claim that $I_0\big(w\big(\frac{\cdot}{L}\big)\big)<0$. Indeed, as $w$ is a solution of problem ($P_0$), it follows that

$$\int_{\mathbb{R}^N}F(w)dx=\frac{N-2}{2N}\int_{\mathbb{R}^N}|\nabla w|^2dx>0,$$

and so, for $L>2$ large enough, we obtain

$$I_0\Big(w\Big(\frac{\cdot}{L}\Big)\Big)=\frac{L^{N-2}}{2}\Big[\int_{\mathbb{R}^N}|\nabla w|^2dx-2L^2\int_{\mathbb{R}^N}F(w)dx\Big]<0. \tag{1.4.32}$$

Thus, using the fact that $I_0\big(w\big(\frac{\cdot}{L}\big)\big) < 0$ and (1.4.29), there exists $R_3 \geq 1$ such that

$$I_V\Big(U^R\Big(\frac{\cdot}{L}\Big)\Big) < I_0\Big(w\Big(\frac{\cdot}{L}\Big)\Big) < 0, \qquad \text{for all } R \geq R_3. \tag{1.4.33}$$

Therefore, taking $R_4 := \max\{R_2, R_3\}$, we get from (1.4.31) and (1.4.33) that

$$I_V\Big(U^R\Big(\frac{\cdot}{s}\Big)\Big) < 2p_0, \qquad \text{for all } s \in (0, L] \text{ and all } R \geq R_4$$

and

$$I_V\Big(U^R\Big(\frac{\cdot}{L}\Big)\Big) < 0, \qquad \text{for all } R \geq R_4,$$

concluding the proof of the proposition. □

**Lemma 1.4.13.** *Assume that* $(f_1)$–$(f_3)$ *hold true and let* $w$ *be a ground state solution to* $(P_0)$*, which is positive, radially symmetric and decreasing in the radial direction. Then, there exists a path* $\gamma_0 \in C\big([0,1], D_G^{1,2}(\mathbb{R}^N)\big)$*, with* $\gamma_0(0) = 0$ *and* $I_0(\gamma_0(1)) < 0$*, such that*

$$w \in \gamma_0([0,1]), \qquad \max_{t\in[0,1]} I_0(\gamma_0(t)) = I_0(w) = m_0.$$

*Proof.* By hypothesis, for any $g \in G$ and $x \in \mathbb{R}^N$, we have $w(gx) = w(|gx|) = w(|x|) = w(x)$, and so $w \in D_G^{1,2}(\mathbb{R}^N)$. Moreover, $w$ is a ground state solution to $(P_0)$, which is positive, radially symmetric and decreasing in the radial direction. Then, we can define a continuous path $\alpha : [0, \infty) \to D_G^{1,2}(\mathbb{R}^N)$, putting $\alpha(t) := w(\cdot/t)$ for $t > 0$ and $\alpha(0) := 0$. Thus, by construction, it follows that $I_0(\alpha(0)) = 0$ and, for every $t > 0$, we have

$$I_0(\alpha(t)) = I_0\,(w\,(\cdot/t)) = \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla w|^2 dx - t^N\int_{\mathbb{R}^N} F(w)dx.$$

Therefore, deriving the above expression, we obtain

$$\begin{aligned}\frac{d}{dt}I_0(\alpha(t)) &= \frac{N-2}{2}t^{N-3}\int_{\mathbb{R}^N} |\nabla w|^2 dx - Nt^{N-1}\int_{\mathbb{R}^N} F(w)dx \\ &= t^{N-3}\left[\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla w|^2 dx - Nt^2\int_{\mathbb{R}^N} F(w)\,dx\right].\end{aligned}$$

Since $w$ is a solution to $(P_0)$, then $w$ satisfies the Pohozaev identity

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla w|^2 dx = N\int_{\mathbb{R}^N} F(w)dx,$$

and thus,

$$\frac{d}{dt} I_0(\alpha(t)) = N t^{N-3}(1-t^2)\int_{\mathbb{R}^N} F(w)dx.$$

Since $Nt^{N-3}\int_{\mathbb{R}^N} F(w)dx > 0$, for every $t > 0$, it follows that the map $t \mapsto I_0(\alpha(t))$ reaches the maximum value at $t = 1$. Choosing $T > 0$ sufficiently large, we have

$$\max_{0\leq t\leq T} I_0(\alpha(t)) = I_0(\alpha(1)) = I_0(w) = m_0 \quad \text{and} \quad I_0(\alpha(T)) < 0.$$

Considering the path $\gamma_0 : [0,1] \to D_G^{1,2}(\mathbb{R}^N)$, defined by $\gamma_0(t) := \alpha(tT)$, the result follows. □

**Lemma 1.4.14.** *Assume that* $(V_1)$–$(V_2)$ *and* $(f_1)$–$(f_3)$ *hold true. Then, the functional* $I_V$ *satisfies the geometrical properties of the mountain pass theorem.*

*Proof.* Observe that $I_V(0) = 0$. Moreover, using the hypothesis $(f_1)$ and the continuity of the embedding $D_G^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$, we get

$$\begin{aligned} I_V(u) &= \frac{1}{2}\|u\|_V^2 - \int_{\mathbb{R}^N} F(u)dx \geq \frac{1}{2}\|u\|_V^2 - A_2\|u\|_{2^*}^{2^*} \\ &\geq \frac{1}{2}\|u\|_V^2 - C_1A_2\|u\|_V^{2^*} = \left[\frac{1}{2} - C_1A_2\|u\|_V^{2^*-2}\right]\|u\|_V^2. \end{aligned}$$

Since $2^*-2 > 0$, taking $\rho := \min\left\{1, \left(\frac{1}{4C_1A_2}\right)^{1/(2^*-2)}\right\} > 0$, we have: if $u \in D_G^{1,2}(\mathbb{R}^N) \setminus \{0\}$, with $\|u\|_V = \rho$, then

$$I_V(u) \geq \left[\frac{1}{2} - C_1A_2\|u\|_V^{2^*-2}\right]\|u\|_V^2 \geq \frac{\|u\|_V^2}{4} = \frac{\rho^2}{4} > 0.$$

On the other hand, if $w$ is a ground state solution to ($P_0$), positive, radially symmetric and decreasing in the radial direction, then for any $g \in G$ and $x \in \mathbb{R}^N$, we have $w(gx) = w(|gx|) = w(|x|) = w(x)$, and so $w \in D_G^{1,2}(\mathbb{R}^N)$. Moreover, from Lemma 1.4.13, for $L > 1$ sufficiently large, there exists a path $\gamma : [0,L] \to D_G^{1,2}(\mathbb{R}^N)$ defined by $\gamma(0) = 0$ and $\gamma(t) = w(\cdot/t)$, for $t \in (0,L]$. We may observe that $\gamma$ satisfies

$$\gamma(0) = 0, \quad \gamma(1) = w, \quad I_0(\gamma(L)) < 0, \tag{1.4.34}$$

$$I_0(\gamma(t)) < I_0(w), \quad \text{for all } t \neq 1. \tag{1.4.35}$$

Fix $L > 2$ sufficiently large such that (1.4.34) holds. Arguing as in Proposition 1.4.12, see expression (1.4.29), it follows that

$$\left| I_V\left(U^R\left(\frac{\cdot}{t}\right)\right) - 2I_0\left(w\left(\frac{\cdot}{t}\right)\right)\right| \to 0 \qquad \text{as } R \to +\infty,$$

uniformly in $t \in (0, L]$. Using the fact that $I_0\left(w\left(\frac{\cdot}{L}\right)\right) = I_0(\gamma(L)) < 0$, we conclude that

$$I_V\left(U^R\left(\frac{\cdot}{L}\right)\right) < 0,$$

for $R \gg 1$ sufficiently large. Therefore, the functional $I_V$ satisfies the geometrical properties of the mountain pass theorem, concluding the proof. □

**Proof of Theorem 1.1.1.** Let us apply the mountain pass theorem of Ambrosetti-Rabinowitz [3]. We define a mountain pass level for $I_V$ on $D_G^{1,2}(\mathbb{R}^N)$ by

$$c_V := \inf_{\gamma \in \Gamma_V} \max_{0 \le t \le 1} I_V(\gamma(t)), \qquad \Gamma_V := \left\{\gamma \in C([0,1], D_G^{1,2}(\mathbb{R}^N)) : \gamma(0) = 0, I_V(\gamma(1)) < 0\right\}.$$

Since $I_V$ satisfies the geometrical properties of the mountain pass theorem, then $c_V > 0$ and there exists a Cerami sequence $(u_n) \subset D_G^{1,2}(\mathbb{R}^N)$ for $I_V$ at level $c_V$. By Lemma 1.3.2, $(u_n)$ has a bounded subsequence that we will denote by $(u_n)$. From (1.4.32), we may choose $L > 2$ such that $I_0\left(w\left(\frac{\cdot}{L}\right)\right) < 0$. Next, consider the following path:

$$\gamma(t) = \begin{cases} U^R\left(\frac{\cdot}{Lt}\right), & \text{if } t \in (0,1], \\ 0, & \text{if } t = 0. \end{cases}$$

Note that $\gamma \in \Gamma_V$ and, also by Proposition 1.4.12, we may choose $R \gg 1$ sufficiently large such that

$$I_V(\gamma(t)) < 2p_0, \quad \text{for all } t \in [0,1],$$

and so $c_V < 2p_0$. On the other hand, recalling that $c_V > 0$ and $\ell(G)p_0 \ge 2p_0$, from Corollary 1.3.8, there exists $u \in D_G^{1,2}(\mathbb{R}^N) \setminus \{0\}$ such that $u_n \to u$ strongly in $D_G^{1,2}(\mathbb{R}^N)$, i.e. $u$ is a nontrivial critical point of $I_V$ such that $I_V(u) = c_V$. Therefore, it follows that $u$ is a nontrivial solution of problem ($P_G$). Using the maximum principle we conclude that $u$ is positive, proving the theorem.

**Remark 1.4.15.** Assuming that the potential $V$ is invariant under a group action $G \subset O(N)$, with $\ell(G) \in (2, \infty]$ and $d_G \in (0, 2]$, under assumptions ($V_1$)–($V_4$) and ($f_1$)–($f_3$), we may prove that Theorem 1.1.1 also holds.

**Remark 1.4.16.** Assuming that the potential $V$ is invariant under a group action $G \subset O(N)$, with $\ell(G) \in [2, \infty)$ and $d_G \in (0, 2]$, under assumptions $(V_1)$–$(V_4)$ and $(f_1)$–$(f_3)$, we may prove that Theorem 1.1.1 also holds.

To prove this, we took as basis two important papers by Hirata [22, p. 182–190] and [23, p. 3180–3188]. We define

$$U^R := \sum_{j=1}^{\ell(G)} w(\cdot - Re_j), \tag{1.4.36}$$

where $e_1, \dots, e_{\ell(G)} \in \mathbb{S}^{N-1}$ and $d_G \in (0, 2]$, as in (0.0.1) and (0.0.2). Moreover, for $i, j = 1, \dots, \ell(G)$, we denote

$$\varepsilon_R := \int_{\mathbb{R}^N} f(w(x - Re_i))w(x - Re_j)dx = \int_{\mathbb{R}^N} f(w(x - Re_j))w(x - Re_i)dx. \tag{1.4.37}$$

Following the same ideas applied when we assume that $\ell(G) = 2$, we can prove that there exist $L > 2$ large enough and $R_4 \geq 1$ such that

$$I_V\Big(U^R\Big(\frac{\cdot}{s}\Big)\Big) < \ell(G)I_0(w) = \ell(G)p_0, \quad \text{for all } s \in (0, L] \text{ and all } R \geq R_4$$

and

$$I_V\Big(U^R\Big(\frac{\cdot}{L}\Big)\Big) < 0, \quad \text{for all } R \geq R_4.$$

From the above inequalities and as $I_V$ satisfies the geometrical properties of the mountain pass theorem, the result follows by Lemma 1.3.2 and Corollary 1.3.8, using that

$$0 < c_V < \ell(G)p_0.$$

## 1.5 Appendix

**Lemma 1.5.1.** *Under the assumptions of Lemma 1.3.5, for any integer $j \in \{1, \dots, k\}$, there exist $R$ and $\alpha$ positive constants such that, for $n$ sufficiently large,*

$$\int_{B_R(y_n^j)} \left| \sum_{i=1}^{k} \nabla w^i(x - y_n^i) \right|^2 dx \geq \alpha > 0, \tag{1.5.1}$$

*where $w^i$ is a nontrivial solution of $(P_0)$ and, as $n \to \infty$, $|y_n^i| \to \infty$ and $|y_n^i - y_n^j| \to \infty$ if $i \neq j$.*

*Proof.* For $1 \le i \le k$, $w^i \in H^1(\mathbb{R}^N)$ is a nontrivial function, so there is $\alpha_i > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla w^i(x)|^2 dx > 2\alpha_i > 0.$$

We may choose $R_i > 0$ sufficiently large such that

$$\int_{B_{R_i}(0)} |\nabla w^i(x)|^2 dx \ge 2\alpha_i > 0.$$

Take $R = \max\{R_1, \dots, R_k\}$ and $\alpha = \min\{\alpha_1, \dots, \alpha_k\}$, and a fixed $j \in \{1, \dots, k\}$. Then,

$$\begin{aligned}
\int_{B_R(y_n^j)} \left| \sum_{i=1}^k \nabla w^i(x - y_n^i) \right|^2 dx &\ge \int_{B_{R_j}(y_n^j)} \left[ |\nabla w^j(x - y_n^j)|^2 - \left| \sum_{i \neq j}^k \nabla w^i(x - y_n^i) \right|^2 \right] dx \\
&= \int_{B_{R_j}(0)} |\nabla w^j(z)|^2 dz - \int_{B_{R_j}(y_n^j)} \left| \sum_{i \neq j}^k \nabla w^i(x - y_n^i) \right|^2 dx \\
&\ge 2\alpha_j - \int_{B_{R_j}(0)} \left| \sum_{i \neq j}^k \nabla w^i(x - (y_n^i - y_n^j)) \right|^2 dx \\
&\ge 2\alpha_j - C \int_{B_{R_j}(0)} \sum_{i \neq j}^k |\nabla w^i(x - (y_n^i - y_n^j))|^2 dx \\
&= 2\alpha_j - C \sum_{i \neq j}^k \int_{B_{R_j}(0)} |\nabla w^i(x - (y_n^i - y_n^j))|^2 dx. \qquad (1.5.2)
\end{aligned}$$

Since $|y_n^i - y_n^j| \to \infty$ as $n \to \infty$, if $i \neq j$, it follows that

$$\int_{B_R(0)} |\nabla w^i(x - (y_n^i - y_n^j))|^2 dx = o_n(1),$$

for $1 \le i \le k$, $i \neq j$. Thus, (1.5.2) $\ge \alpha_j \ge \alpha > 0$ for $n$ sufficiently large, that is,

$$2\alpha_j - C \sum_{i \neq j}^k \int_{B_{R_j}(0)} |\nabla w^i(x - (y_n^i - y_n^j))|^2 dx \ge \alpha_j \ge \alpha > 0,$$

and this proves (1.5.1). □

Chapter 2

# Nonlinear Schrödinger equations with general nonlinearities

## 2.1 Introduction

Our goal in this chapter is to show the existence of a positive bound state solution for the problem

$$-\Delta u + V(x)u = f(u), \qquad u \in H^1(\mathbb{R}^N), \; N \ge 3, \qquad (P)$$

where the potential $V$ is a positive function and the nonlinearity $f$, under very mild assumptions, is asymptotically linear or superlinear and subcritical at infinity, not satisfying any monotonicity condition. The existence of a solution to this problem is established in situations where a ground state solution is not attained.
We will assume that the potential $V$ is invariant under a group action $G \subset O(N)$ and we try to find a positive solution in the space of $G$-symmetric functions

$$H^1_G(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u(gx) = u(x), \forall g \in G, \forall x \in \mathbb{R}^N\}.$$

As in the first chapter, we will consider the case that $G \subset O(N)$ is closed subgroup with the following property: for any $x \in \mathbb{S}^{N-1}$, there exists $g \in G$ such that $gx \neq x$. This means that G acts effectively on $\mathbb{S}^{N-1}$, that is, $G$ satisfies

$$\#\{gy : g \in G\} \in [2, \infty], \qquad \text{for all } y \in \mathbb{S}^{N-1},$$

where $\#\{\ \}$ denotes the cardinal number of sets and $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$. We will define

$$\ell(G) := \min\{\#Gx : x \in \mathbb{S}^{N-1}\}.$$

We also observe that in this work we are going to consider only the case $\ell(G)$ finite and

$$\ell(G) \in [2, \infty).$$

In fact, for simplicity, our study is focused in the case $\ell(G) = 2$, but could clearly be extended to finite $\ell(G) > 2$.
Let $S$ be the best constant that satisfies Gagliardo-Nirenberg-Sobolev inequality (0.2.1).

Throughout Chapter 2, we will consider the potential $V$ under assumptions $(\widetilde{V}_1)$–$(\widetilde{V}_4)$ and the nonlinearity $f$ under assumptions $(\widetilde{f}_1)$–$(\widetilde{f}_4)$.

Observe that $F(0) = 0$ and by $(\widetilde{f}_1)$, $F(s) \geq 0$ for $s > 0$.
Under assumptions $(\widetilde{f}_1)$–$(\widetilde{f}_3)$, the classical result of Berestycki and Lions [10, Theorem 1] establishes the existence of a ground state solution $w \in C^2(\mathbb{R}^N)$ to the limit problem at infinity

$$-\Delta u + V_\infty u = f(u), \qquad u \in H^1(\mathbb{R}^N), \tag{$P_\infty$}$$

where $w$ is positive, radially symmetric and decreasing in the radial direction, see also [4] and [32]. It is well known, see [21], which there exist constants $A_5, A_6 > 0$ such that

$$A_5(1+|x|)^{-\frac{N-1}{2}} e^{-\sqrt{V_\infty}|x|} \leq \left|D^i w(x)\right| \leq A_6(1+|x|)^{-\frac{N-1}{2}} e^{-\sqrt{V_\infty}|x|}, \qquad i = 0, 1. \tag{2.1.1}$$

As in first chapter, by virtue of $G$-invariant property, we do not need the uniqueness of positive solution for the limit problem $(P_\infty)$. Since $H^1(\mathbb{R}^N)$ is not compactly embedded into $L^{p_i+1}(\mathbb{R}^N)$, for $i = 1, 2$, then the mountain pass minimax value for corresponding functional may not be attained. However, as we are assuming that the potential $V$ and the function $f$ are invariant under finite effective group action $G$, we will show that the mountain pass minimax value for functional restricted to the subspace $H^1_G(\mathbb{R}^N)$ is attained.
Now we can restate our main result of existence of a solution in Chapter 2.

**Theorem 2.1.1.** *Assume that $(\widetilde{V}_1)$–$(\widetilde{V}_4)$ and $(\widetilde{f}_1)$–$(\widetilde{f}_4)$ hold true. Then, problem $(P)$ has a positive solution $u \in H^1_G(\mathbb{R}^N)$.*

Assumptions $(\widetilde{f}_1)$–$(\widetilde{f}_2)$ imply that, for all $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|F(s)| \leq \frac{\varepsilon}{2} s^2 + C_\varepsilon |s|^{2^*}. \tag{2.1.2}$$

Hypotheses $(\widetilde{V}_1)$, $(\widetilde{V}_3)$ and $(\widetilde{V}_4)$ imply that, for all $x \in \mathbb{R}^N$, there exist constants $A_2, A_3, A_4 \in \mathbb{R}$ such that

$$|V(x) - V_\infty| \leq A_2, \quad |\nabla V(x) \cdot x| \leq A_3, \quad |xH(x)x| \leq A_4. \tag{2.1.3}$$

## 2.2 Pohozaev manifold structure and preliminary results

Associated with problem $(P)$, we define the functional $I_V : H^1_G(\mathbb{R}^N) \to \mathbb{R}$ by

$$I_V(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)dx - \int_{\mathbb{R}^N} F(u)dx$$

Let us define the functional $J_V : H^1_G(\mathbb{R}^N) \to \mathbb{R}$ by

$$J_V(u) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2}\int_{\mathbb{R}^N} \left(\frac{\nabla V(x) \cdot x}{N} + V(x)\right)u^2 dx - N\int_{\mathbb{R}^N} F(u)dx,$$

and define the Pohozaev manifold associated to the problem $(P)$ by

$$\mathcal{P}_V^G := \{u \in H^1_G(\mathbb{R}^N) \setminus \{0\} : J_V(u) = 0\}.$$

Likewise the Pohozaev manifold $\mathcal{P}_\infty$ associated to the limit problem $(P_\infty)$. Set

$$\mathcal{P}_\infty := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : J_\infty(u) = 0\},$$

where

$$J_\infty(u) := \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - N\int_{\mathbb{R}^N} \left(F(u) - V_\infty \frac{u^2}{2}\right)dx.$$

We recall that solutions of $(P_\infty)$ are critical points of the functional

$$I_\infty(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_\infty u^2\right)dx - \int_{\mathbb{R}^N} F(u)dx, \qquad u \in H^1(\mathbb{R}^N).$$

We also recall that $w$ is a ground state solution of the limit problem $(P_\infty)$ if

$$I_\infty(w) = m := \inf\{I_\infty(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\} \text{ is a solution of } (P_\infty)\}. \tag{2.2.1}$$

We will denote

$$p_\infty = \inf_{u \in \mathcal{P}_\infty} I_\infty(u). \tag{2.2.2}$$

Next lemma was inspired by [24] and [28]. The arguments used to prove its can be found there.

**Lemma 2.2.1.** *Assume that* $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ *hold true. Then,* $m = p_1$.

*Proof.* To prove this lemma, we follow the same ideas found in [28, Lemma 2.4]. Consider

$$S_1 := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} G_1(u)dx = 1 \right\},$$

where $G_1(u) := F(u) - \frac{V_1}{2}u^2$, and let $\;: S_1 \to P_1$ be defined by

$$(u)(x) := u\left(\frac{x}{t_u}\right), \quad t_u := \left(\frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{1/2} = \left(\frac{N-2}{2N}\right)^{1/2} \|\nabla u\|_2.$$

Observe that establishes a bijective correspondence between $S_1$ and $P_1$. Moreover, for every $u \in S_1$, we have

$$\begin{aligned} I_1((u)) &= \frac{t_u^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - t_u^N \int_{\mathbb{R}^N} G_1(u)dx \\ &= t_u^{N-2}\left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - t_u^2\right] \\ &= \left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} \|\nabla u\|_2^{N-2}\left[\frac{1}{2}\|\nabla u\|_2^2 - \frac{N-2}{2N}\|\nabla u\|_2^2\right] \\ &= \frac{1}{N}\left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} \|\nabla u\|_2^N, \end{aligned}$$

and so

$$p_1 = \inf_{u \in P_1} I_1(u) = \inf_{u \in S_1} I_1((u)) = \inf_{u \in S_1} \frac{1}{N}\left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} \|\nabla u\|_2^N = m,$$

since the infimum is achieved and the corresponding value equals the least energy level $m$. This can be proved by performing calculations similar to those of [13, Lemma 1(i)]. □

We define $f(s) := -f(-s)$ for $s < 0$. So, it follows from hypotheses $(\widetilde{f}_1)$ and $(\widetilde{f}_2)$ that $f \in C^1(\mathbb{R})$ and it is an odd function. Note that, if $u$ is a positive solution of problem $(P)$ for this new function, it is also a solution of $(P)$ for the original function $f$. Hereafter, we shall consider this extension, and establish the existence of a positive solution for $(P)$. Since $f \in C^1(\mathbb{R})$ and $f$ satisfies $(\widetilde{f}_1)$–$(\widetilde{f}_3)$, a classical result of Berestycki and Lions establishes the existence of a ground state solution $w \in C^2(\mathbb{R}^N)$ to problem $(P_1)$, which is positive, radially symmetric and decreasing in the radial direction, see [10, Theorem 4]. Next we will consider the space of $G$-symmetric functions in $H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, for $2 \leq p \leq 2^*$, with its scalar product and norm

$$\langle u, v\rangle_V := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)dx, \qquad \|u\|_V^2 := \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)dx. \quad (2.2.3)$$

Let us denote $\|\cdot\|_q$ the $L^q(\mathbb{R}^N)$-norm, for all $q \in [1, \infty)$ and $C$, $C_i$ are positive constants which may vary from line to line. By assumptions $(\widetilde{V}_1)$ and $(\widetilde{V}_2)$, we can see that the expressions in (2.2.3) are well defined and that $\|\cdot\|_V$ is a norm in $H^1_G(\mathbb{R}^N)$, which is equivalent to the standard one. We will write

$$\langle u, v\rangle := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V_\infty uv)dx, \qquad \|u\|^2 := \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_\infty u^2\right)dx.$$

**Remark 2.2.2.** Throughout this chapter, to denote an inner product or norm in the space $H^1(\mathbb{R}^N)$, we will use the same notations adopted for the subspace of functions $G$-symmetric $H^1_G(\mathbb{R}^N)$.

Consider the following problem in the space of $G$-symmetric functions $H^1_G(\mathbb{R}^N)$, for $N \geq 3$,

$$-\Delta u + V(x)u = f(u), \qquad u \in H^1_G(\mathbb{R}^N). \quad (P_G)$$

We claim that solutions of $(P_G)$ are also solutions of $(P)$. Indeed, note that the action of $G$ on $H^1(\mathbb{R}^N)$ is isometric and, furthermore, we can easily see that the functional $I_V$ defined in the whole space $H^1(\mathbb{R}^N)$ satisfies $I_V(gu) = I_V(u)$, for all $g \in G$ and all $u \in H^1(\mathbb{R}^N)$. So, by the principle of symmetric criticality (see [36, Theorem 1.28]), it follows that if $u_0$ is a weak solution of problem $(P_G)$, that is, if $u_0$ is a critical point of the restricted functional $I_V$, restricted to $H^1_G(\mathbb{R}^N)$, then $u_0$ is a critical point of $I_V$ in the whole space $H^1(\mathbb{R}^N)$. In fact, to show that $u_0$ is a critical point of the functional $I_V$ in $H^1(\mathbb{R}^N)$, it suffices to show that $I_V(u_0)\mathfrak{v} = 0$, for all $\mathfrak{v} \in \left(H^1_G(\mathbb{R}^N)\right)^\perp$, and this is a consequence of the following lemma, which holds for all $u \in H^1_G(\mathbb{R}^N)$, not only critical points of $I_V$.

**Lemma 2.2.3.** *Assume that $(\widetilde{V}_1)$–$(\widetilde{V}_2)$ and $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ hold true. Then,*

$$I_V'(u)\mathfrak{v} = 0, \quad \textit{for any } u \in H^1_G(\mathbb{R}^N) \textit{ and } \mathfrak{v} \in \left(H^1_G(\mathbb{R}^N)\right)^\perp.$$

*Proof.* To prove this lemma, just follow the same ideas used to prove Lemma 1.2.3, substituting $D^{1,2}(\mathbb{R}^N)$ by $H^1(\mathbb{R}^N)$. □

## 2.3 Bounded Palais-Smale sequences

Recall that a sequence $(u_n)$ in $H^1_G(\mathbb{R}^N)$ is said to be a $(PS)_d$*-sequence for* $I_V$*, with* $d \in \mathbb{R}$, if $I_V(u_n) \to d$ and $I'_V(u_n) \to 0$ in $H_G^{-1}(\mathbb{R}^N)$. A sequence $(u_n)$ in $H^1_G(\mathbb{R}^N)$ is said to be a *Cerami sequence for* $I_V$ *at level* $d \in \mathbb{R}$, denoted by $(Ce)_d$, if $I_V(u_n) \to d$ and $\|I'_V(u_n)\|_{H_G^{-1}(\mathbb{R}^N)}(1+\|u_n\|_V) \to 0$.

**Lemma 2.3.1.** *Assume that* $(\widetilde{f}_1)$–$(\widetilde{f}_4)$ *hold true and let* $(u_n)$ *in* $H^1_G(\mathbb{R}^N)$ *be a Cerami sequence for* $I_V$ *at level* $d \in \mathbb{R}$*. Then,* $(u_n)$ *has a bounded subsequence.*

*Proof.* Suppose, by contradiction, that $(u_n)$ has no bounded subsequence. Then, we can assume that $u_n \neq 0$ for all $n \in \mathbb{N}$ and $\|u_n\|_V \to +\infty$. Let us define $\tilde{u}_n := u_n/\|u_n\|_V$ for all $n \in \mathbb{N}$. Thus, $(\tilde{u}_n)$ is a bounded sequence and $\|\tilde{u}_n\|_V = 1$. Hence, up to a subsequence, it holds $\tilde{u}_n \rightharpoonup \tilde{u}$ in $H^1_G(\mathbb{R}^N)$. Therefore, one of the two cases occurs:

$$\text{Case 1:} \quad \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\tilde{u}_n|^2 dx > 0;$$

$$\text{Case 2:} \quad \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\tilde{u}_n|^2 dx = 0.$$

First, let us suppose that Case 2 occurs, and let $L > 1$ be an arbitrary constant. Then, we have,

$$I_V\left(\frac{L}{\|u_n\|_V}u_n\right) = \frac{L^2}{2} - \int_{\mathbb{R}^N} F\left(\frac{L}{\|u_n\|_V}u_n\right)dx.$$

So, using hypothesis $(\widetilde{f}_2)$, we obtain

$$\int_{\mathbb{R}^N} F\left(\frac{L}{\|u_n\|_V}u_n\right)dx \leq A_1 L^{p_1+1}\int_{\mathbb{R}^N} |\tilde{u}_n|^{p_1+1}dx + A_1 L^{p_2+1}\int_{\mathbb{R}^N} |\tilde{u}_n|^{p_2+1}dx.$$

Since $1 < p_1 \leq p_2 < 2^* - 1$, it follows from Lions' lemma [29] that

$$\int_{\mathbb{R}^N} |\tilde{u}_n|^{p_1+1}dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |\tilde{u}_n|^{p_2+1}dx \to 0,$$

and so

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(L\tilde{u}_n) = \lim_{n\to\infty} \int_{\mathbb{R}^N} F\left(\frac{L}{\|u_n\|_V}u_n\right)dx = 0.$$

By hypothesis $(\widetilde{f}_1)$ and using that $f(s) = -f(-s)$ for $s < 0$, we have $F(s) \geq 0$ for all $s \in \mathbb{R}$. Hence,

$$I_V\left(\frac{L}{\|u_n\|_V}u_n\right) = \frac{L^2}{2} - \int_{\mathbb{R}^N} F\left(\frac{L}{\|u_n\|_V}u_n\right)dx \geq \frac{L^2}{4}$$

for $n$ sufficiently large. Since $\|u_n\|_V \to +\infty$, then $\frac{L}{\|u_n\|_V} \in (0,1)$, for $n$ sufficiently large. So, there exists $n_1 \in \mathbb{N}$ such that

$$\max_{t\in[0,1]} I_V(tu_n) \geq I_V\left(\frac{L}{\|u_n\|_V}u_n\right) \geq \frac{L^2}{4},$$

for all $n \geq n_1$. Let $t_n \in [0,1]$ be such that $I_V(t_nu_n) := \max_{t\in[0,1]} I_V(tu_n)$. Thus,

$$I_V(t_nu_n) \geq \frac{L^2}{4}, \tag{2.3.1}$$

for all $n \geq n_1$. Since $t_n \leq 1$, using $(\widetilde{f}_4)$ and the fact that $f(s) = -f(-s)$ for $s < 0$, we obtain

$$\begin{aligned}
I_V(t_nu_n) &= I_V(t_nu_n) - \frac{1}{2}I_V'(t_nu_n)(t_nu_n) + o_n(1)\\
&= \int_{\mathbb{R}^N}\left(\frac{1}{2}f(t_nu_n)(t_nu_n) - F(t_nu_n)\right)dx + o_n(1)\\
&\leq D\int_{\mathbb{R}^N}\left(\frac{1}{2}f(u_n)u_n - F(u_n)\right)dx + o_n(1)\\
&= D\left(I_V(u_n) - \frac{1}{2}I_V'(u_n)u_n\right) + o_n(1)\\
&= Dd + o_n(1).
\end{aligned}$$

So, there exists $n_2 \in \mathbb{N}$ such that

$$I_V(t_nu_n) \leq 2Dd, \tag{2.3.2}$$

for all $n \geq n_2$. Taking $n_0 := \max\{n_1, n_2\}$, it follows from (2.3.1) and (2.3.2) that

$$\frac{L^2}{4} \leq I_V(t_nu_n) \leq 2Dd,$$

for all $n \geq n_0$. Taking $L > 3\sqrt{Dd}$, we come to a contradiction.

Now suppose that Case 1 occurs, that is, there exists $\delta > 0$ such that

$$\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|\tilde{u}_n|^2dx = \delta.$$

If $(y_n) \subset \mathbb{R}^N$ is a sequence such that $|y_n| \to \infty$ and $\int_{B_1(y_n)}|\tilde{u}_n|^2dx > \delta/2$, whereas that

$\tilde{u}_n(\cdot + y_n) \rightharpoonup \tilde{u}$, we obtain

$$\int_{B_1(0)} |\tilde{u}_n(x + y_n)|^2 > \frac{\delta}{2},$$

and so

$$\int_{B_1(0)} |\tilde{u}(x)|^2 dx \geq \frac{\delta}{2},$$

showing that $\tilde{u} \not\equiv 0$. Thus, there exists a subset of positive Lebesgue measure $\Omega \subset B_1(0)$ such that

$$0 < |\tilde{u}(x)| = \lim_{n \to \infty} |\tilde{u}_n(x + y_n)| = \lim_{n \to \infty} \frac{|u_n(x + y_n)|}{\|u_n\|_V}, \quad \forall x \in \Omega.$$

Since $\|u_n\|_V \to +\infty$, it follows that

$$|u_n(x + y_n)| \to +\infty, \quad \forall x \in \Omega.$$

Then, using the hypothesis $(\widetilde{f}_4)$ and Fatou lemma, we obtain

$$\begin{aligned}
\liminf_{n \to \infty} \int_{\mathbb{R}^N} \left[\frac{1}{2} f(u_n(x + y_n))u_n(x + y_n) - F(u_n(x + y_n))\right] dx \\
\geq \liminf_{n \to \infty} \int_{\Omega} \left[\frac{1}{2} f(u_n(x + y_n))u_n(x + y_n) - F(u_n(x + y_n))\right] dx \\
\geq \int_{\Omega} \liminf_{n \to \infty} \left[\frac{1}{2} f(u_n(x + y_n))u_n(x + y_n) - F(u_n(x + y_n))\right] dx \\
= +\infty.
\end{aligned}$$

On the other hand, we have

$$|I_V'(u_n)u_n| \leq \|I_V'(u_n)\|_{H_G^{-1}(\mathbb{R}^N)} \|u_n\|_V \leq \|I_V'(u_n)\|_{H_G^{-1}(\mathbb{R}^N)} (1 + \|u_n\|_V) \to 0,$$

and so, $I_V'(u_n)u_n = o_n(1)$. Therefore, for $n$ sufficiently large, we have

$$\int_{\mathbb{R}^N} \left[\frac{1}{2} f(u_n(x + y_n))u_n(x + y_n) - F(u_n(x + y_n))\right] dx = I_V(u_n) - \frac{1}{2} I_V'(u_n)u_n \leq d + 1,$$

which gives a contradiction.
If $(y_n)$ is bounded, then there exists $R > 1$ such that $|y_n| \leq R$ for all $n \in \mathbb{N}$ and

$$\int_{B_{2R}(0)} |\tilde{u}_n(x + y_n)|^2 dx \geq \int_{B_1(0)} |\tilde{u}_n(x + y_n)|^2 dx > \frac{\delta}{2}.$$

Since $\tilde{u}_n(\cdot + y_n) \to \tilde{u}$ in $B_{2R}(0)$, it follows that

$$\int_{B_1(0)} |\tilde{u}(x)|^2 dx \geq \frac{\delta}{2}.$$

Similarly to the previous case, there exists $\Omega_1 \subset B_1(0)$, with $|\Omega_1| > 0$ such that

$$\lim_{n\to\infty} \frac{|u_n(x+y_n)|}{\|u_n\|_V} = \lim_{n\to\infty} |\tilde{u}_n(x+y_n)| = |\tilde{u}(x)| \neq 0, \quad \forall x \in \Omega_1.$$

The argument follows as in the previous case where $|y_n| \to +\infty$ and we arrive at a contradiction. Therefore, neither Case 1 nor Case 2 can occur and lemma is proved. □

Next, let us present the standard result about the splitting of bounded $(PS)$ sequences. This lemma is a version of the concentration compactness of P.L. Lions [29] and found in [34]. Before proving the result, we will need the following versions of Brezis-Lieb lemma. The proof of this lemma is similar to the proof of Lemma 1.3.5, but unlike Chapter 1, here we will only use the assumptions and the fact that $H^1_G(\mathbb{R}^N)$ is continuously embedded into $L^{p_i+1}(\mathbb{R}^N)$, $i = 1, 2$.

**Lemma 2.3.2.** *Assume that $(\widetilde{V}_1)$–$(\widetilde{V}_3)$ and $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ hold true. Let $(u_n)$ be a bounded sequence in $H^1_G(\mathbb{R}^N)$ such that $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$. Then, the following statements hold true:*

(a) $\|u_n\|_V^2 = \|u_n - u\|^2 + \|u\|_V^2 + o_n(1)$;

(b) $\displaystyle\int_{\mathbb{R}^N} |f(u_n) - f(u)||\varphi| dx = o_n(1)$, *for every* $\varphi \in C_0^\infty(\mathbb{R}^N)$;

(c) $\displaystyle\int_{\mathbb{R}^N} F(u_n) dx - \int_{\mathbb{R}^N} F(u_n - u) dx = \int_{\mathbb{R}^N} F(u) dx + o_n(1)$;

(d) $f(u_n) - f(u_n - u) \to f(u)$ *in* $H_G^{-1}(\mathbb{R}^N)$.

*Proof.* Since $(u_n) \subset H^1_G(\mathbb{R}^N)$, it follows that $u_n(gx) = u_n(x)$ for any $g \in G$ and $x \in \mathbb{R}^N$. Thus, as $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$, we have

$$u(gx) = \lim_{n\to\infty} u_n(gx) = \lim_{n\to\infty} u_n(x) = u(x) \quad \text{for a.e. } x \in \mathbb{R}^N,$$

which shows that $u \in H^1_G(\mathbb{R}^N)$.

Next, for each $n \in \mathbb{N}$, define $v_n := u_n - u$. Thus, as $u_n$ is bounded and $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$, then $(v_n)$ is bounded and, up to a subsequence, $v_n \rightharpoonup 0$ in $H^1_G(\mathbb{R}^N)$.

(a) As $u_n \rightharpoonup u$ in $H^1_G(\mathbb{R}^N)$, it follows that $\langle u_n, u\rangle_V \to \langle u, u\rangle_V = \|u\|_V^2$. Hence, we have

$$
\begin{aligned}
\|v_n\|_V^2 &= \|u_n - u\|_V^2 = \langle u_n - u, u_n - u\rangle_V \\
&= \langle u_n, u_n\rangle_V - \langle u_n, u\rangle_V - \langle u, u_n\rangle_V + \langle u, u\rangle_V \\
&= \|u_n\|_V^2 - 2\langle u_n, u\rangle_V + \|u\|_V^2 \\
&= \|u_n\|_V^2 - \|u\|_V^2 + o_n(1).
\end{aligned} \tag{2.3.3}
$$

On the other hand, we have

$$
\begin{aligned}
\|v_n\|_V^2 &= \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)v_n^2\right)dx \\
&= \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V_\infty v_n^2\right)dx + \int_{\mathbb{R}^N} [V(x) - V_\infty]\, v_n^2 dx \\
&= \|v_n\|^2 + \int_{\mathbb{R}^N} [V(x) - V_\infty]\, v_n^2 dx.
\end{aligned}
$$

As $(v_n)$ is bounded in $H^1_G(\mathbb{R}^N)$ and $v_n(x) \to 0$ for a.e. $x \in \mathbb{R}^N$, there exists $M > 0$ such that $\|v_n\|_2 \leq M$ for all $n \in \mathbb{N}$ and, up to a subsequence, $v_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, by $(\widetilde{V}_1)$, we have $V(x) \to V_\infty$ as $|x| \to +\infty$. Thus, given $\varepsilon > 0$ there exists $R \geq 1$ such that if $|x| \geq R$ then $|V(x) - V_\infty| < \varepsilon/M^2$. Hence,

$$
\int_{\mathbb{R}^N \setminus B_R(0)} |V(x) - V_\infty| v_n^2 dx \leq \frac{\varepsilon}{M^2}\int_{\mathbb{R}^N \setminus B_R(0)} v_n^2 dx \leq \varepsilon.
$$

Thus, by (2.1.3), it follows that

$$
\int_{B_R(0)} |V(x) - V_\infty| v_n^2 dx \leq A_2 \int_{B_R(0)} v_n^2 dx = o_n(1).
$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$
\|v_n\|_V^2 = \|v_n\|^2 + o_n(1). \tag{2.3.4}
$$

Substituting (2.3.4) in (2.3.3), it follows that

$$
\|u_n\|_V^2 = \|v_n\|^2 + \|u\|_V^2 + o_n(1),
$$

proving item (a).

(b) By hypothesis $(\widetilde{f}_2)$ and the fact that $f(s) = -f(-s)$, for $s < 0$, we have

$$|f'(s)| \leq A_1\left(|s|^{p_1-1} + |s|^{p_2-1}\right), \quad \forall\, s \in \mathbb{R}.$$

By the mean value theorem, there exists $\xi \in (0,1)$ such that

$$\begin{aligned}
|f(u_n) - f(u)| &= |f'(u + \xi(u_n - u))||u_n - u| \\
&\leq A_1\left(|u + \xi(u_n - u)|^{p_1-1} + |u + \xi(u_n - u)|^{p_2-1}\right)|u_n - u| \\
&\leq A_1\left[(|u| + |u_n - u|)^{p_1-1} + (|u| + |u_n - u|)^{p_2-1}\right]|u_n - u|.
\end{aligned}$$

Observe that for $i = 1, 2$, we have

$$(|u| + |u_n - u|)^{p_i-1} \leq (2\max\{|u|, |u_n - u|\})^{p_i-1} \leq 2^{p_i-1}\left(|u|^{p_i-1} + |u_n - u|^{p_i-1}\right),$$

and so

$$\begin{aligned}
|f(u_n) - f(u)| &\leq A_1\left[(|u| + |u_n - u|)^{p_1-1} + (|u| + |u_n - u|)^{p_2-1}\right]|u_n - u| \qquad (2.3.5)\\
&\leq C_1\left[\left(|u|^{p_1-1} + |u_n - u|^{p_1-1}\right) + \left(|u|^{p_2-1} + |u_n - u|^{p_2-1}\right)\right]|u_n - u| \\
&= C_1\left[\left(|u|^{p_1-1}|u_n - u| + |u_n - u|^{p_1}\right) + \left(|u|^{p_2-1}|u_n - u| + |u_n - u|^{p_2}\right)\right].
\end{aligned}$$

Since $(u_n)$ is bounded in $H^1_G(\mathbb{R}^N)$ and, passing to a subsequence, $u_n \rightharpoonup u$ and $u_n \to u$ strongly in $L^{p_i+1}_{\text{loc}}(\mathbb{R}^N)$, $i = 1, 2$, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $i = 1, 2$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^{p_i-1}|u_n - u||\varphi| dx &\leq \left(\int_{\mathbb{R}^N} \left(|u|^{p_i-1}\right)^{\frac{p_i+1}{p_i-1}} dx\right)^{\frac{p_i-1}{p_i+1}} \left(\int_{\mathbb{R}^N} (|u_n - u||\varphi|)^{\frac{p_i+1}{2}} dx\right)^{\frac{2}{p_i+1}} \\
&= \left(\int_{\mathbb{R}^N} |u|^{p_i+1} dx\right)^{\frac{p_i-1}{p_i+1}} \left(\int_{\text{supp}(\varphi)} (|u_n - u||\varphi|)^{\frac{p_i+1}{2}} dx\right)^{\frac{2}{p_i+1}} \\
&\leq \|u\|^{p_i-1}_{p_i+1}\|\varphi\|_{p_i+1} \left(\int_{\text{supp}(\varphi)} |u_n - u|^{p_i+1} dx\right)^{\frac{1}{p_i+1}} \\
&\leq C\|\varphi\|_V \left(\int_{\text{supp}(\varphi)} |u_n - u|^{p_i+1} dx\right)^{\frac{1}{p_i+1}} = o_n(1).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n - u|^{p_i}|\varphi| dx &= \int_{\text{supp}(\varphi)} |u_n - u|^{p_i}|\varphi| dx \\ &\leq \left(\int_{\text{supp}(\varphi)} (|u_n - u|^{p_i})^{\frac{p_i+1}{p_i}} dx\right)^{\frac{p_i}{p_i+1}} \left(\int_{\text{supp}(\varphi)} |\varphi|^{p_i+1} dx\right)^{\frac{1}{p_i+1}} \\ &\leq C\|\varphi\|_V \left(\int_{\text{supp}(\varphi)} |u_n - u|^{p_i+1} dx\right)^{\frac{p_i}{p_i+1}} = o_n(1).\end{aligned}$$

Therefore, we conclude that

$$\int_{\mathbb{R}^N} |f(u_n) - f(u)||\varphi| dx = o_n(1), \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^N),$$

which proves item (b).
(c) By hypothesis $(\widetilde{f}_2)$, we have $|F(u)| \leq A_1(|u|^{p_1+1} + |u|^{p_2+1})$. Thus, arguing as in (2.3.5) and using $(\widetilde{f}_2)$, we obtain

$$\begin{aligned}|F(u_n) - F(v_n)| &= |F(u + v_n) - F(v_n)| \\ &\leq A_1[(|v_n| + |u|)^{p_1} + (|v_n| + |u|)^{p_2}]|u| \leq C_1[(|v_n|^{p_1} + |u|^{p_1}) + (|v_n|^{p_2} + |u|^{p_2})]|u| \\ &= C_1\left[\left(|v_n|^{p_1}|u| + |u|^{p_1+1}\right) + \left(|v_n|^{p_2}|u| + |u|^{p_2+1}\right)\right],\end{aligned}$$

and so

$$\begin{aligned}|F(u_n) - F(v_n) - F(u)| &\leq |F(u_n) - F(v_n)| + |F(u)| \\ &\leq C_1\left[\left(|v_n|^{p_1}|u| + |u|^{p_1+1}\right) + \left(|v_n|^{p_2}|u| + |u|^{p_2+1}\right)\right] + A_1\left(|u|^{p_1+1} + |u|^{p_2+1}\right) \\ &= C_1(|v_n|^{p_1}|u| + |v_n|^{p_2}|u|) + (C_1 + A_1)\left(|u|^{p_1+1} + |u|^{p_2+1}\right).\end{aligned}$$

Since $(v_n)$ is bounded in $H^1_G(\mathbb{R}^N)$ and $H^1_G(\mathbb{R}^N)$ is continuously embedded into $L^{p_i+1}(\mathbb{R}^N)$, $i = 1, 2$, there exists a constant $M_i > 0$ such that

$$\left(\int_{|x|>R} |v_n|^{p_i+1} dx\right)^{\frac{p_i}{p_i+1}} \leq M_i.$$

So, given $\varepsilon > 0$, we may choose $R > 1$ sufficiently large such that

$$
\begin{aligned}
\int_{|x|>R} |F(u_n) - F(v_n) - F(u)| dx &\leq \int_{|x|>R} |F(u_n) - F(v_n)| dx + \int_{|x|>R} |F(u)| dx \\
&\leq C_1 \left[ \int_{|x|>R} |v_n|^{p_1} |u| dx + \int_{|x|>R} |v_n|^{p_2} |u| dx \right] \\
&\quad + (C_1 + A_1) \left[ \int_{|x|>R} |u|^{p_1+1} dx + \int_{|x|>R} |u|^{p_2+1} dx \right] \\
&\leq C_1 \left( \int_{|x|>R} |v_n|^{p_1+1} dx \right)^{\frac{p_1}{p_1+1}} \left( \int_{|x|>R} |u|^{p_1+1} dx \right)^{\frac{1}{p_1+1}} \\
&\quad + C_1 \left( \int_{|x|>R} |v_n|^{p_2+1} dx \right)^{\frac{p_2}{p_2+1}} \left( \int_{|x|>R} |u|^{p_2+1} dx \right)^{\frac{1}{p_2+1}} \\
&\quad + (C_1 + A_1) \left[ \int_{|x|>R} |u|^{p_1+1} dx + \int_{|x|>R} |u|^{p_2+1} dx \right] \\
&\leq C_1 \left[ M_1 \left( \int_{|x|>R} |u|^{p_1+1} dx \right)^{\frac{1}{p_1+1}} + M_2 \left( \int_{|x|>R} |u|^{p_2+1} dx \right)^{\frac{1}{p_2+1}} \right] \\
&\quad + (C_1 + A_1) \left[ \int_{|x|>R} |u|^{p_1+1} dx + \int_{|x|>R} |u|^{p_2+1} dx \right] \\
&< \varepsilon.
\end{aligned}
$$

On the other hand, using assumption that $(v_n)$ is bounded and $v_n(x) \to 0$ for a.e. $x \in \mathbb{R}^N$

again, passing to a subsequence, $v_n \to 0$ strongly in $L^{p_i+1}_{\mathrm{loc}}(\mathbb{R}^N)$, and so

$$
\begin{aligned}
\int_{|x|\geq R} |F(u_n) - F(v_n) - F(u)|dx &\leq \int_{|x|\geq R} |F(u_n) - F(u)|dx + \int_{|x|\geq R} |F(v_n)|dx \\
&\leq C_2\left[\int_{|x|\geq R} |u|^{p_1}|v_n|dx + \int_{|x|\geq R} |u|^{p_2}|v_n|dx\right] \\
&\quad + (C_2 + A_1)\left[\int_{|x|\geq R} |v_n|^{p_1+1}dx + \int_{|x|\geq R} |v_n|^{p_2+1}dx\right] \\
&\leq C_2\left(\int_{|x|\geq R} |u|^{p_1+1}dx\right)^{\frac{p_1}{p_1+1}} \left(\int_{|x|\geq R} |v_n|^{p_1+1}dx\right)^{\frac{1}{p_1+1}} \\
&\quad + C_2\left(\int_{|x|\geq R} |u|^{p_2+1}dx\right)^{\frac{p_2}{p_2+1}} \left(\int_{|x|\geq R} |v_n|^{p_2+1}dx\right)^{\frac{1}{p_2+1}} \\
&\quad + (C_2 + A_1)\left[\int_{|x|\geq R} |v_n|^{p_1+1}dx + \int_{|x|\geq R} |v_n|^{p_2+1}dx\right] \\
&\leq C_2\left[\|u\|^{p_1}_{p_1+1}\left(\int_{|x|\geq R} |v_n|^{p_1+1}dx\right)^{\frac{1}{p_1+1}} + \|u\|^{p_2}_{p_2+1}\left(\int_{|x|\geq R} |v_n|^{p_2+1}dx\right)^{\frac{1}{p_2+1}}\right] \\
&\quad + (C_2 + A_1)\left[\int_{|x|\geq R} |v_n|^{p_1+1}dx + \int_{|x|\geq R} |v_n|^{p_2+1}dx\right] \\
&< \varepsilon,
\end{aligned}
$$

if $n \in \mathbb{N}$ is large enough, which proves item (c).

(d) Again, by hypothesis $(\widetilde{f}_2)$ and the fact that $f(s) = -f(-s)$, for $s < 0$, arguing as in (b), see (2.3.5), we obtain

$$
|f(u_n) - f(u_n - u)| \leq C_1\left[\left(|u_n - u|^{p_1-1}|u| + |u|^{p_1}\right) + \left(|u_n - u|^{p_2-1}|u| + |u|^{p_2}\right)\right],
$$

and so,

$$
\begin{aligned}
|f(u_n) - f(u_n - u) - f(u)| &\leq |f(u_n) - f(u_n - u)| + |f(u)| \\
&\leq C_1\left[\left(|u_n - u|^{p_1-1}|u| + |u|^{p_1}\right) + \left(|u_n - u|^{p_2-1}|u| + |u|^{p_2}\right)\right] \\
&\quad + A_1(|u|^{p_1} + |u|^{p_2}) \\
&= C_1\left(|u_n - u|^{p_1-1}|u| + |u_n - u|^{p_2-1}|u|\right) \\
&\quad + (C_1 + A_1)(|u|^{p_1} + |u|^{p_2}).
\end{aligned}
$$

Let $\varphi \in H^1_G(\mathbb{R}^N)$ and $R > 0$ be. Then,

$$\begin{aligned}\int_{|x|>R} |f(u_n) - f(u_n - u) - f(u)||\varphi| dx &\leq C_1\left(\int_{|x|>R} |u_n - u|^{p_1 - 1}|u||\varphi| dx + \int_{|x|>R} |u_n - u|^{p_2 - 1}|u||\varphi| dx\right)\\ &\quad + (C_1 + A_1)\left(\int_{|x|>R} |u|^{p_1}|\varphi| dx + \int_{|x|>R} |u|^{p_2}|\varphi| dx\right).\end{aligned}$$

Since $(v_n)$ is bounded in $H^1_G(\mathbb{R}^N)$, where $v_n := u_n - u$, and $H^1_G(\mathbb{R}^N)$ is continuously embedded into $L^{p_i+1}(\mathbb{R}^N)$, $i = 1, 2$, we have

$$\begin{aligned}\int_{|x|>R} |u_n - u|^{p_i - 1}|u||\varphi| dx &\leq \left(\int_{|x|>R} \left(|u_n - u|^{p_i - 1}|u|\right)^{\frac{p_1+1}{p_i}} dx\right)^{\frac{p_i}{p_i+1}} \left(\int_{|x|>R} |\varphi|^{p_i+1} dx\right)^{\frac{1}{p_i+1}}\\ &\leq \left[\left(\int_{|x|>R} |u_n - u|^{p_i+1} dx\right)^{\frac{p_i - 1}{p_i}} \left(\int_{|x|>R} |u|^{p_i+1} dx\right)^{\frac{1}{p_i}}\right]^{\frac{p_i}{p_i+1}} \left(\int_{|x|>R} |\varphi|^{p_i+1} dx\right)^{\frac{1}{p_i+1}}\\ &= \left(\int_{|x|>R} |u_n - u|^{p_i+1} dx\right)^{\frac{p_i-1}{p_i+1}} \left(\int_{|x|>R} |u|^{p_i+1} dx\right)^{\frac{1}{p_i+1}} \left(\int_{|x|>R} |\varphi|^{p_i+1} dx\right)^{\frac{1}{p_i+1}}\\ &\leq \|u_n - u\|^{p_i - 1}_{p_i+1}\|\varphi\|_{p_i+1}\left(\int_{|x|>R} |u|^{p_i+1} dx\right)^{\frac{1}{p_i+1}}\\ &\leq C\|\varphi\|_V\left(\int_{|x|>R} |u|^{p_i+1} dx\right)^{\frac{1}{p_i+1}}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\int_{|x|>R} |u|^{p_i}|\varphi| dx &\leq \left(\int_{|x|>R} |u|^{p_i+1}| dx\right)^{\frac{p_i}{p_i+1}} \left(\int_{|x|>R} |\varphi|^{p_i+1}| dx\right)^{\frac{1}{p_i+1}}\\ &\leq C\|\varphi\|_V\left(\int_{|x|>R} |u|^{p_i+1}| dx\right)^{\frac{p_i}{p_i+1}}.\end{aligned}$$

Thus, given $\varepsilon > 0$, we may choose $R > 1$ sufficiently large such that

$$\int_{|x|>R} |f(u_n) - f(u_n - u) - f(u)||\varphi| dx \leq \frac{\varepsilon}{2}\|\varphi\|_V. \tag{2.3.6}$$

On the other hand, from (2.3.5) and hypothesis $(\widetilde{f}_2)$, we get

$$
\begin{aligned}
|f(u_n) - f(u_n - u) - f(u)| &\le |f(u_n) - f(u)| + |f(u_n - u)| \\
&\le C_1\big[\big(|u|^{p_1-1}|u_n - u| + |u_n - u|^{p_1}\big) + \big(|u|^{p_2-1}|u_n - u| + |u_n - u|^{p_2}\big)\big] \\
&\quad + A_1(|u_n - u|^{p_1} + |u_n - u|^{p_2}) \\
&= C_1\big(|u|^{p_1-1}|u_n - u| + |u|^{p_2-1}|u_n - u|\big) \\
&\quad + (C_1 + A_1)(|u_n - u|^{p_1} + |u_n - u|^{p_2}),
\end{aligned}
$$

and so, we have

$$
\begin{aligned}
\int_{|x|\le R} |u|^{p_i-1}|u_n - u||\varphi|dx &\le \left(\int_{|x|\le R} \big(|u|^{p_i-1}|u_n - u|\big)^{\frac{p_i+1}{p_i}} dx\right)^{\frac{p_i}{p_i+1}} \left(\int_{|x|\le R} |\varphi|^{p_i+1}dx\right)^{\frac{1}{p_i+1}} \\
&\le \left[\left(\int_{|x|\le R} |u|^{p_i+1}dx\right)^{\frac{p_i-1}{p_i}} \left(\int_{|x|\le R} |u_n - u|^{p_i+1}dx\right)^{\frac{1}{p_i}}\right]^{\frac{p_i}{p_i+1}} \left(\int_{|x|\le R} |\varphi|^{p_i+1}dx\right)^{\frac{1}{p_i+1}} \\
&= \left(\int_{|x|\le R} |u|^{p_i+1}dx\right)^{\frac{p_i-1}{p_i+1}} \left(\int_{|x|\le R} |u_n - u|^{p_i+1}dx\right)^{\frac{1}{p_i+1}} \left(\int_{|x|\le R} |\varphi|^{p_i+1}dx\right)^{\frac{1}{p_i+1}} \\
&\le \|u\|_{p_i+1}^{p_i-1}\|\varphi\|_{p_i+1}\left(\int_{|x|\le R} |u_n - u|^{p_i+1}dx\right)^{\frac{1}{p_i+1}} \\
&\le C\|\varphi\|_V\left(\int_{|x|\le R} |u|^{p_i+1}dx\right)^{\frac{1}{p_i+1}}
\end{aligned}
$$

and we also have

$$
\begin{aligned}
\int_{|x|\le R} |u_n - u|^{p_i}|\varphi|dx &\le \left(\int_{|x|\le R} |u_n - u|^{p_i+1}dx\right)^{\frac{p_i}{p_i+1}} \left(\int_{|x|\le R} |\varphi|^{p_i+1}dx\right)^{\frac{1}{p_i+1}} \\
&\le C\|\varphi\|_V\left(\int_{|x|\le R} |u_n - u|^{p_i+1}dx\right)^{\frac{p_i}{p_i+1}}.
\end{aligned}
$$

Hence, as $u_n \to u$ strongly in $L^{p_i+1}_{\mathrm{loc}}(\mathbb{R}^N)$, $i = 1, 2$, we obtain

$$
\int_{|x|\le R} |f(u_n) - f(u_n - u) - f(u)||\varphi|dx \le \frac{\varepsilon}{2}\|\varphi\|_V, \tag{2.3.7}
$$

for $n \in \mathbb{N}$ sufficiently large. Therefore, from (2.3.6) and (2.3.7), given $\varepsilon > 0$ and $\varphi \in H^1_G(\mathbb{R}^N)$, it follows that

$$
\left|\int_{\mathbb{R}^N} [f(u_n) - f(u_n - u) - f(u)]\,\varphi dx\right| \le \varepsilon\|\varphi\|_V,
$$

for $n \in \mathbb{N}$ sufficiently large, which proves item (d). □

**Lemma 2.3.3** (Splitting). *Assume that* $(\widetilde{V}_1)$–$(\widetilde{V}_3)$ *and* $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ *hold true. Let* $c \in \mathbb{R}$ *and* $(u_n)$ *be a bounded sequence in* $H^1_G(\mathbb{R}^N)$ *such that*

$$I_V(u_n) \to c \quad \text{and} \quad I'_V(u_n) \to 0 \quad \text{in } {H_G}^{-1}(\mathbb{R}^N).$$

*Then, passing* $(u_n)$ *to a subsequence, if necessary, there exist a solution* $u \in H^1_G(\mathbb{R}^N)$ *of problem* $(P_G)$, *a number* $k \in \mathbb{N} \cup \{0\}$, $k$ *sequences* $(y_n^j) \subset \mathbb{R}^N$, $1 \leq j \leq k$ *and* $k$ *nontrivial solutions* $w^1, \ldots, w^k$ *of the limit problem* $(P_\infty)$, *satisfying:*

$(i)$ $u_n \rightharpoonup u$ *weakly in* $H^1_G(\mathbb{R}^N)$;

$(ii)$ *for any* $i, j = 1, \ldots, k$, $|y_n^j| \to \infty$ *and* $|y_n^j - y_n^i| \to \infty$, *if* $i \neq j$;

$(iii)$ $u_n - u - \sum_{j=1}^{k} w^j(\cdot - y_n^j) \to 0$ *in* $H^1(\mathbb{R}^N)$;

$(iv)$ $c = I_V(u) + \sum_{j=1}^{k} I_\infty(w^j)$,

*for* $k \in \mathbb{N}$. *In the case* $k = 0$, *the above holds without* $w^j$, $(y_n^j)$.

The proof of this lemma is entirely analogous to the proof of Lemma 1.3.6, but unlike Chapter 1, where we used Lemma 1.3.1 if strong convergence does not occur, here we will use Lions' Lemma and follow the same ideas, and so on we get the result.

*Proof.* Since $(u_n) \subset H^1_G(\mathbb{R}^N)$ is a $(PS)_c$-sequence for $I_V$ restricted to $H^1_G(\mathbb{R}^N)$, it follows from Lemma 2.2.3 that $I'_V(u_n)v = 0$ for any $v \in \left(H^1_G(\mathbb{R}^N)\right)^*$, and so $(u_n)$ is also $(PS)_c$-sequence for $I_V$ defined in the whole space $H^1(\mathbb{R}^N)$. As $(u_n)$ is bounded, passing to a subsequence, we get $u \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$. Let us show that $u \in H^1_G(\mathbb{R}^N)$. In fact, as $(u_n) \subset H^1_G(\mathbb{R}^N)$, we have $u_n(gx) = u_n(x)$ for any $g \in G$ and $x \in \mathbb{R}^N$, and so

$$u(gx) = \lim_{n \to \infty} u_n(gx) = \lim_{n \to \infty} u_n(x) = u(x) \quad \text{a.e. } x \in \mathbb{R}^N,$$

which shows that $u \in H^1_G(\mathbb{R}^N)$. It follows from weak convergence and Lemma 2.3.2(b) that, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} o_n(1) = I'_V(u_n)\varphi &= \int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + V(x)u_n\varphi)dx - \int_{\mathbb{R}^N} f(u_n)\varphi\, dx \\ &= \int_{\mathbb{R}^N} (\nabla u \nabla \varphi + V(x)u\varphi)dx - \int_{\mathbb{R}^N} f(u)\varphi\, dx + o_n(1) \\ &= I'_V(u)\varphi + o_n(1), \end{aligned}$$

which shows that $I_V'(u)\varphi = 0$, and so, as $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, it follows that $I_V'(u)v = 0$ for any $v \in H^1(\mathbb{R}^N)$. Since $u \in H_G^1(\mathbb{R}^N)$ and $I_V'(u)\tilde{v} = 0$ for any $\tilde{v} \in \left(H_G^1(\mathbb{R}^N)\right)^\perp$, we conclude that $u$ is a critical point of functional $I_V$ restricted to $H_G^1(\mathbb{R}^N)$, and so $u$ is a solution of problem $(P_G)$. Now, for each $n \in \mathbb{N}$, we define $u_{n,1} := u_n - u$. So, up to a subsequence, we have $u_{n,1} \rightharpoonup 0$ in $H_G^1(\mathbb{R}^N)$. We state that if

$$\lim_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_{n,1}|^2dx\right) = 0, \tag{2.3.8}$$

then $u_n \to u$ in $H_G^1(\mathbb{R}^N)$, and so the lemma occurs for $k = 0$. In fact, we have

$$\begin{aligned} I_V'(u_n)u_{n,1} &= \int_{\mathbb{R}^N}(\nabla u_n\nabla u_{n,1} + V(x)u_nu_{n,1})dx - \int_{\mathbb{R}^N}f(u_n)u_{n,1}dx\\ &= \int_{\mathbb{R}^N}\left(|\nabla u_{n,1}|^2 + \nabla u\nabla u_{n,1} + V(x)u_{n,1}^2 + V(x)uu_{n,1}\right)dx - \int_{\mathbb{R}^N}f(u_n)u_{n,1}dx\\ &= \|u_{n,1}\|_V^2 + \langle u, u_{n,1}\rangle_V - \int_{\mathbb{R}^N}f(u_n)u_{n,1}dx, \end{aligned}$$

and thus, using that $I_V'(u)u_{n,1} = 0$, we obtain

$$\begin{aligned} \|u_{n,1}\|_V^2 &= I_V'(u_n)u_{n,1} - \langle u, u_{n,1}\rangle_V + \int_{\mathbb{R}^N}f(u_n)u_{n,1}dx\\ &= I_V'(u_n)u_{n,1} - \int_{\mathbb{R}^N}f(u)u_{n,1}dx + \int_{\mathbb{R}^N}f(u_n)u_{n,1}dx. \end{aligned} \tag{2.3.9}$$

Since $(u_n)$ is bounded in $H_G^1(\mathbb{R}^N)$, it follows from definition of $u_{n,1}$ that $(u_{n,1})$ is a bounded sequence. Thus, as $I_V'(u_n) \to 0$ in ${H_G}^{-1}(\mathbb{R}^N)$, by hypothesis, it follows that $I_V'(u_n)u_{n,1} \to 0$. By assumption $(\widetilde{f_2})$, Hölder inequality and by the continuity of the embedding of $H_G^1(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$, $q \in (2, 2^*)$, we have

$$\begin{aligned} \left|\int_{\mathbb{R}^N}f(u_n)u_{n,1}dx\right| &\le \int_{\mathbb{R}^N}|f(u_n)||u_{n,1}|dx \le A_1\int_{\mathbb{R}^N}(|u_n|^{p_1} + |u_n|^{p_2})|u_{n,1}|dx\\ &\le A_1\left[\|u_n\|_{p_1+1}^{p_1}\|u_{n,1}\|_{p_1+1} + \|u_n\|_{p_2+1}^{p_2}\|u_{n,1}\|_{p_2+1}\right]\\ &\le C[\|u_n\|_V^{p_1}\|u_{n,1}\|_{p_1+1} + \|u_n\|_V^{p_2}\|u_{n,1}\|_{p_2+1}]. \end{aligned} \tag{2.3.10}$$

So if (2.3.8) holds, as $(u_{n,1})$ is bounded, it follows from Lions' lemma [29] that, as $n \to \infty$, $u_{n,1} \to 0$ in $L^q(\mathbb{R}^N)$, for all $q \in (2, 2^*)$. Since $2 < p_1 + 1 \le p_2 + 1 < 2^*$, we conclude that

$$\|u_{n,1}\|_{p_1+1} \to 0 \quad \text{and} \quad \|u_{n,1}\|_{p_2+1} \to 0. \tag{2.3.11}$$

As $(u_n)$ is bounded in $H^1_G(\mathbb{R}^N)$, it follows from (2.3.10) and (2.3.11) that

$$\int_{\mathbb{R}^N} f(u_n)u_{n,1}dx \to 0.$$

Similarly, we have

$$\int_{\mathbb{R}^N} f(u)u_{n,1}dx \to 0.$$

Therefore, doing $n \to \infty$ in (2.3.9), we conclude that

$$u_{n,1} \to 0, \quad \text{i.e.} \quad u_n \to u \quad \text{strongly in } H^1_G(\mathbb{R}^N),$$

which shows that the lemma occurs for $k = 0$.

Suppose now that there exists $\delta > 0$ such that

$$\lim_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_{n,1}|^2 dx \right) = \delta. \tag{2.3.12}$$

We showed in Lemma 2.3.2 that the following statements hold:

(a) $\|u_n\|^2_V = \|u_{n,1}\|^2 + \|u\|^2_V + o_n(1)$;

(b) $\displaystyle\int_{\mathbb{R}^N} |f(u_n) - f(u)||\varphi|dx = o_n(1)$, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$;

(c) $\displaystyle\int_{\mathbb{R}^N} [F(u_n) - F(u_{n,1}) - F(u)]dx = o_n(1)$;

(d) $f(u_n) - f(u_{n,1}) \to f(u)$ in $H_G^{-1}(\mathbb{R}^N)$.

Therefore, it follows from (a) and (c) that

$$\begin{aligned} I_V(u_n) - I_\infty(u_{n,1}) - I_V(u) &= \frac{1}{2}\|u_n\|^2_V - \int_{\mathbb{R}^N} F(u_n)dx - \frac{1}{2}\|u_{n,1}\|^2 + \int_{\mathbb{R}^N} F(u_{n,1})dx \\ &\quad - \frac{1}{2}\|u\|^2_V + \int_{\mathbb{R}^N} F(u)dx \\ &= \frac{1}{2}\left[\|u_n\|^2_V - \|u_{n,1}\|^2 - \|u\|^2_V\right] \\ &\quad - \int_{\mathbb{R}^N} [F(u_n) - F(u_{n,1}) - F(u)]dx \\ &= o_n(1), \end{aligned}$$

and thus,

$$I_V(u_n) = I_V(u) + I_\infty(u_{n,1}) + o_n(1). \tag{2.3.13}$$

Next, we will show that $I'_V(u_{n,1}) \to 0$ in $H_G^{-1}(\mathbb{R}^N)$. Indeed, by hypothesis, $I'_V(u_n) \to 0$ in $H_G^{-1}(\mathbb{R}^N)$ and so it follows that $I'_V(u_n)v \to 0$, for any $v \in H^1_G(\mathbb{R}^N)$. So, we have

$$\begin{aligned} o_n(1) &= I'_V(u_n)v = I'_V(u_{n,1}+u)v \\ &= \int_{\mathbb{R}^N} (\nabla u_{n,1}\nabla v + V(x)u_{n,1}v)dx + \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx \\ &\quad - \int_{\mathbb{R}^N} f(u_{n,1}+u)vdx \\ &= I'_V(u_{n,1})v + \int_{\mathbb{R}^N} f(u_{n,1})vdx + I'_V(u)v + \int_{\mathbb{R}^N} f(u)vdx \\ &\quad - \int_{\mathbb{R}^N} f(u_n)vdx \\ &= I'_V(u_{n,1})v + I'_V(u)v - \int_{\mathbb{R}^N} [f(u_n) - f(u_{n,1}) - f(u)]vdx. \end{aligned}$$

The fact that $I'_V(u) = 0$ and item (d) imply that

$$I'_V(u_{n,1})v = o_n(1), \quad \text{for all } v \in H^1_G(\mathbb{R}^N),$$

which shows that, as $n \to \infty$, $I'_V(u_{n,1}) \to 0$ in $H_G^{-1}(\mathbb{R}^N)$. Now observe that, by (2.3.12), we obtain a sequence $(y_n^1) \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_n^1)} |u_{n,1}(x)|^2 dx > \frac{\delta}{2}. \tag{2.3.14}$$

Consider a sequence $(v_n^1)$ defined by

$$v_n^1 := u_{n,1}(\cdot + y_n^1).$$

Since $(u_{n,1})$ is bounded in $H^1_G(\mathbb{R}^N)$, then $(v_n^1)$ is bounded in $H^1(\mathbb{R}^N)$, and so we have, up to a subsequence,

$$\begin{cases} v_n^1 \rightharpoonup w^1, & \text{weakly in } H^1(\mathbb{R}^N), \\ v_n^1 \to w^1, & \text{strongly in } L^2_{loc}(\mathbb{R}^N), \\ v_n^1(x) \to w^1(x), & \text{a.e. } x \in \mathbb{R}^N. \end{cases}$$

Since $v_n^1 \to w^1$ in $L^2_{loc}(\mathbb{R}^N)$ and

$$\int_{B_1(0)} \left|v_n^1(x)\right|^2 dx = \int_{B_1(0)} \left|u_{n,1}(x+y_n^1)\right|^2 dx > \delta/2,$$

it follows that

$$\int_{B_1(0)} |w^1(x)|^2 dx \geq \delta/2,$$

and so $w^1 \not\equiv 0$. The fact that $u_{n,1} \rightharpoonup 0$ in $H^1_G(\mathbb{R}^N)$ implies that $(y^1_n)$ is unbounded and, passing to a subsequence, we may assume that $|y^1_n| \to \infty$.

So, about the sequence $(u_{n,1})$ the following statements hold:

(a1) $\|u_n\|^2_V = \|u_{n,1}\|^2 + \|u\|^2_V + o_n(1)$;

(b1) $I_V(u_n) = I_V(u) + I_\infty(u_{n,1}) + o_n(1)$;

(c1) $I'_V(u_{n,1}) \to 0$ in $H_G^{-1}(\mathbb{R}^N)$.

Next, we shall show that $w^1$ is a nontrivial solution of the limit problem ($P_\infty$). So, as $(u_{n,1}) \subset H^1_G(\mathbb{R}^N)$, by Lemma 2.2.3, we have $I'_V(u_{n,1})v = 0$ for any $v \in \left(H^1_G(\mathbb{R}^N)\right)^*$, and so $I'_V(u_{n,1}) \to 0$ in $H^{-1}(\mathbb{R}^N)$. Moreover, given $\varphi \in C_0^\infty(\mathbb{R}^N)$, as $u_{n,1} \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, using (2.1.3) and Hölder inequality, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} |V(x) - V_\infty||u_{n,1}||\varphi|\, dx &= \int_{\text{supp}(\varphi)} |V(x) - V_\infty||u_{n,1}||\varphi|\, dx \\
&\leq A_2\left(\int_{\text{supp}(\varphi)} |u_{n,1}|^2 dx\right)^{1/2}\left(\int_{\text{supp}(\varphi)} |\varphi|^2 dx\right)^{1/2} \\
&\leq C\|\varphi\|_V\left(\int_{\text{supp}(\varphi)} |u_{n,1}|^2 dx\right)^{1/2} = o_n(1), \qquad (2.3.15)
\end{aligned}$$

and so,

$$\begin{aligned}
o_n(1) = I'_V(u_{n,1})\varphi &= \int_{\mathbb{R}^N} (\nabla u_{n,1} \nabla \varphi + V(x)u_{n,1}\varphi)dx - \int_{\mathbb{R}^N} f(u_{n,1})\varphi dx \\
&= \int_{\mathbb{R}^N} (\nabla u_{n,1} \nabla \varphi + V_\infty u_{n,1}\varphi)dx - \int_{\mathbb{R}^N} f(u_{n,1})\varphi dx + \int_{\mathbb{R}^N} [V(x) - V_\infty]u_{n,1}\varphi dx \\
&= I'_\infty(u_{n,1})\varphi + \int_{\mathbb{R}^N} [V(x) - V_\infty]u_{n,1}\varphi dx \\
&= I'_\infty(u_{n,1})\varphi + o_n(1).
\end{aligned}$$

Therefore,

$$I'_\infty(u_{n,1})\varphi = o_n(1), \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N),$$

and it implies that, as $n \to \infty$, $I'_\infty(u_{n,1}) \to 0$ in $H^{-1}(\mathbb{R}^N)$. So, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that

$$\|I'_\infty(u_{n,1})\|_{H^{-1}(\mathbb{R}^N)} = \sup_{\|\varphi\| \leq 1} |I'_\infty(u_{n,1})\varphi| < \varepsilon, \quad \varphi \in C_0^\infty(\mathbb{R}^N).$$

Given $\varphi \in C_0^\infty(\mathbb{R}^N)$, we define $\varphi_n^1 := \varphi(\cdot - y_n^1)$. Thus,

$$\begin{aligned}
\sup_{\|\varphi\|\le 1} |I_\infty'(v_n^1)\varphi| &= \sup_{\|\varphi\|\le 1} |I_\infty'(u_{n,1}(\cdot + y_n^1))\varphi| = \sup_{\|\varphi(\cdot - y_n^1)\|\le 1} |I_\infty'(u_{n,1})\varphi(\cdot - y_n^1)| \\
&= \sup_{\|\varphi_n^1\|\le 1} |I_\infty'(u_{n,1})\varphi_n^1| \le \sup_{\|\phi\|\le 1} |I_\infty'(u_{n,1})\phi| < \varepsilon, \quad \phi \in C_0^\infty(\mathbb{R}^N),
\end{aligned}$$

for $n \in \mathbb{N}$ large enough. So, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, of weak convergence $v_n^1 \rightharpoonup w^1$ in $H^1(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N} \left[\nabla v_n^1 \nabla \varphi + V(x) v_n^1 \varphi\right] dx = \int_{\mathbb{R}^N} \left[\nabla w^1 \nabla \varphi + V(x) w^1 \varphi\right] dx + o_n(1)$$

and arguing as in (2.3.15), as $v_n^1 \to w^1$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} [V(x) - V_\infty] v_n^1 \varphi \, dx = \int_{\mathbb{R}^N} [V(x) - V_\infty] w^1 \varphi \, dx + o_n(1).$$

Furthermore, using the same ideas applied in Lemma 2.3.2(b), it follows that

$$\int_{\mathbb{R}^N} f(v_n^1)\varphi \, dx = \int_{\mathbb{R}^N} f(w^1)\varphi \, dx + o_n(1).$$

Therefore, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
o_n(1) &= I_\infty'(v_n^1)\varphi = \int_{\mathbb{R}^N} \left[\nabla v_n^1 \nabla \varphi + V_\infty v_n^1 \varphi\right] dx - \int_{\mathbb{R}^N} f(v_n^1)\varphi \, dx \\
&= \int_{\mathbb{R}^N} \left[\nabla v_n^1 \nabla \varphi + V(x) v_n^1 \varphi\right] dx - \int_{\mathbb{R}^N} f(v_n^1)\varphi \, dx - \int_{\mathbb{R}^N} [V(x) - V_\infty] v_n^1 \varphi \, dx \\
&= \int_{\mathbb{R}^N} \left[\nabla w^1 \nabla \varphi + V(x) w^1 \varphi\right] dx - \int_{\mathbb{R}^N} f(w^1)\varphi \, dx \\
&\quad - \int_{\mathbb{R}^N} [V(x) - V_\infty] w^1 \varphi \, dx + o_n(1) \\
&= \int_{\mathbb{R}^N} \left[\nabla w^1 \nabla \varphi + V_\infty w^1 \varphi\right] dx - \int_{\mathbb{R}^N} f(w^1)\varphi \, dx + o_n(1) \\
&= I_\infty'(w^1)\varphi + o_n(1),
\end{aligned}$$

which shows that $I_\infty'(w^1)\varphi = 0$, and so, $w^1$ is a nontrivial solution of the limit problem $(P_\infty)$.

Let us define now

$$u_{n,2} := u_{n,1} - w^1(\cdot - y_n^1).$$

So, as before, we have

(a2) $\|u_n\|_V^2 = \|u_{n,2}\|^2 + \|u\|_V^2 + \|w^1\|^2 + o_n(1)$;

(b2) $I_V(u_n) = I_V(u) + I_\infty(u_{n,2}) + I_\infty(w^1) + o_n(1)$;

(c2) $I'_\infty(u_{n,2}) \to 0$ in $H^{-1}(\mathbb{R}^N)$.

The verification of these items follows the same argument used previously in the analogous items for the sequence $(u_{n,1})$, with the necessary adaptations. Indeed, if follows from (a1) that

$$\begin{aligned}\|u_{n,2}\|^2 &= \langle u_{n,1} - w^1(\cdot - y_n^1), u_{n,1} - w^1(\cdot - y_n^1)\rangle \\ &= \|u_{n,1}\|^2 + \|w^1(\cdot - y_n^1)\|^2 - 2\langle u_{n,1}, w^1(\cdot - y_n^1)\rangle \\ &= o_n(1) + \|u_n\|_V^2 - \|u\|_V^2 + \|w^1(\cdot - y_n^1)\|^2 - 2\langle u_{n,1}, w^1(\cdot - y_n^1)\rangle. \end{aligned} \tag{2.3.16}$$

Making a change of variables, we obtain

$$\begin{aligned}\|w^1(\cdot - y_n^1)\|^2 &= \int_{\mathbb{R}^N} \left[|\nabla w^1(x - y_n^1)|^2 + V_\infty (w^1(x - y_n^1))^2\right] dx \\ &= \int_{\mathbb{R}^N} \left[|\nabla w^1(x)|^2 + V_\infty (w^1(x))^2\right] dx = \|w^1\|^2. \end{aligned} \tag{2.3.17}$$

Moreover, we have

$$\begin{cases} v_n^1 \rightharpoonup w^1, & \text{weakly in } H^1(\mathbb{R}^N), \\ v_n^1 \to w^1, & \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^N), \\ v_n^1(x) \to w^1(x), & \text{a.e. } x \in \mathbb{R}^N. \end{cases}$$

Thus, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, using (2.1.3) and Hölder inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \left[\nabla v_n^1 \nabla \varphi + V_\infty v_n^1 \varphi\right] dx &= \int_{\mathbb{R}^N} \left[\nabla v_n^1 \nabla \varphi + V(x) v_n^1 \varphi\right] dx - \int_{\mathbb{R}^N} [V(x) - V_\infty] v_n^1 \varphi dx \\ &= \int_{\mathbb{R}^N} \left[\nabla w^1 \nabla \varphi + V(x) w^1 \varphi\right] dx \\ &\quad - \int_{\mathbb{R}^N} [V(x) - V_\infty] w^1 \varphi dx + o_n(1) \\ &= \int_{\mathbb{R}^N} \left[\nabla w^1 \nabla \varphi + V_\infty w^1 \varphi\right] dx + o_n(1)\end{aligned}$$

and so, as $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, it follows that

$$\int_{\mathbb{R}^N} \left[\nabla v_n^1 \nabla u + V_\infty v_n^1 u\right] dx = \int_{\mathbb{R}^N} \left[\nabla w^1 \nabla u + V_\infty w^1 u\right] dx + o_n(1),$$

for all $u \in H^1(\mathbb{R}^N)$. In particular, for $u = w^1$, we get

$$\begin{aligned}\int_{\mathbb{R}^N} \left[\nabla v_n^1 \nabla w^1 + V_\infty v_n^1 w^1\right] dx &= \int_{\mathbb{R}^N} \left[|\nabla w^1|^2 + V_\infty (w^1)^2\right] dx + o_n(1) \\ &= \|w^1\|^2 + o_n(1).\end{aligned}$$

So, we have

$$\begin{aligned}\langle u_{n,1}, w^1(\cdot - y_n^1)\rangle &= \int_{\mathbb{R}^N} \left[\nabla u_{n,1}(x) \nabla w^1(x - y_n^1) + V_\infty u_{n,1}(x) w^1(x - y_n^1)\right] dx \\ &= \int_{\mathbb{R}^N} \left[\nabla u_{n,1}(x + y_n^1) \nabla w^1(x) + V_\infty u_{n,1}(x + y_n^1) w^1(x)\right] dx \\ &= \int_{\mathbb{R}^N} \left[\nabla v_n^1(x) \nabla w^1(x) + V_\infty v_n^1(x) w^1(x)\right] dx \\ &= \|w^1\|^2 + o_n(1). \end{aligned} \tag{2.3.18}$$

Substituting (2.3.17) and (2.3.18) in (2.3.16), it follows that

$$\|u_n\|_V^2 = \|u_{n,2}\|^2 + \|u\|_V^2 + \|w^1\|^2 + o_n(1),$$

proving (a2).
Using the previous results obtained in (a2) and (c), we have

$$\begin{aligned}&I_V(u_n) - I_V(u) - I_\infty(u_{n,2}) - I_\infty(w^1) \\ &= \frac{1}{2}\|u_n\|_V^2 - \int_{\mathbb{R}^N} F(u_n) dx - \frac{1}{2}\|u\|_V^2 + \int_{\mathbb{R}^N} F(u) dx \\ &\quad - \frac{1}{2}\|u_{n,2}\|^2 + \int_{\mathbb{R}^N} F(u_{n,2}) dx - \frac{1}{2}\|w^1\|^2 + \int_{\mathbb{R}^N} F(w^1) dx \\ &= \frac{1}{2}\left[\|u_n\|_V^2 - \|u\|_V^2 - \|u_{n,2}\|^2 - \|w^1\|^2\right] - \int_{\mathbb{R}^N} [F(u_n) - F(u_{n,1}) - F(u)] dx \\ &\quad - \int_{\mathbb{R}^N} [F(u_{n,1}) - F(u_{n,2})] dx + \int_{\mathbb{R}^N} F(w^1) dx \\ &= o_n(1) - \int_{\mathbb{R}^N} \left[F(u_{n,1}(x + y_n^1)) - F(u_{n,2}(x + y_n^1))\right] dx + \int_{\mathbb{R}^N} F(w^1) dx \\ &= o_n(1) - \int_{\mathbb{R}^N} \left[F(u_{n,1}(x + y_n^1)) - F(u_{n,2}(x + y_n^1)) - F(w^1(x))\right] dx \\ &= o_n(1) - \int_{\mathbb{R}^N} \left[F(v_n^1) - F(v_n^1 - w^1)) - F(w^1)\right] dx.\end{aligned}$$

Following the same ideas as Lemma 2.3.2(c), changing the space $H_G^1(\mathbb{R}^N)$ by $H^1(\mathbb{R}^N)$, $u_n$

by $v_n^1$ and $u$ by $w^1$, we conclude that

$$\int_{\mathbb{R}^N} \left[F(v_n^1) - F(v_n^1 - w^1)) - F(w^1)\right] dx = o_n(1),$$

and so

$$I_V(u_n) = I_V(u) + I_\infty(u_{n,2}) + I_\infty(w^1) + o_n(1),$$

which proves (b2).

Next, we will show that $I'_\infty(u_{n,2}) \to 0$ in $H^{-1}(\mathbb{R}^N)$. The fact that $I'_V(u_{n,1}) \to 0$ in $H_G^{-1}(\mathbb{R}^N)$ implies that, by Lemma 2.2.3, $I'_V(u_{n,1}) \to 0$ in $H^{-1}(\mathbb{R}^N)$, and so $I'_V(u_{n,1})\varphi \to 0$, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. On the other hand, as $I'_\infty(w^1) = 0$, we have

$$\begin{aligned}
I'_V(u_{n,1})\varphi &= I'_V(u_{n,2} + w^1(\cdot - y_n^1))\varphi \\
&= \int_{\mathbb{R}^N} (\nabla u_{n,2}(x)\nabla\varphi(x) + V(x)u_{n,2}(x)\varphi(x))dx \\
&\quad + \int_{\mathbb{R}^N} \left(\nabla w^1(x - y_n^1)\nabla\varphi(x) + V(x)w^1(x - y_n^1)\varphi(x)\right)dx \\
&\quad - \int_{\mathbb{R}^N} f(u_{n,2}(x) + w^1(x - y_n^1))\varphi(x)dx \\
&= I'_V(u_{n,2})\varphi + \int_{\mathbb{R}^N} f(u_{n,2}(x))\varphi(x)dx \\
&\quad + \int_{\mathbb{R}^N} \left(\nabla w^1(x)\nabla\varphi(x + y_n^1) + (x + y_n^1)w^1(x)\varphi(x + y_n^1)\right)dx \\
&\quad - \int_{\mathbb{R}^N} f(u_{n,1}(x))\varphi(x)dx \\
&= I'_V(u_{n,2})\varphi + \int_{\mathbb{R}^N} f(u_{n,2}(x))\varphi(x)dx \\
&\quad + \int_{\mathbb{R}^N} \left(\nabla w^1(x)\nabla\varphi(x + y_n^1) + V_\infty w^1(x)\varphi(x + y_n^1)\right)dx \\
&\quad + \int_{\mathbb{R}^N} \left[V(x + y_n^1) - V_\infty\right] w^1(x)\varphi(x + y_n^1)dx - \int_{\mathbb{R}^N} f(u_{n,1}(x))\varphi(x)dx \\
&= I'_V(u_{n,2})\varphi + \int_{\mathbb{R}^N} f(u_{n,2}(x))\varphi(x)dx \\
&\quad + I'_\infty(w^1)\varphi(\cdot + y_n^1) + \int_{\mathbb{R}^N} f(w^1(x))\varphi(x + y_n^1)dx \\
&\quad + \int_{\mathbb{R}^N} \left[V(x + y_n^1) - V_\infty\right] w^1(x)\varphi(x + y_n^1)dx - \int_{\mathbb{R}^N} f(u_{n,1}(x))\varphi(x)dx \\
&= I'_V(u_{n,2})\varphi + \int_{\mathbb{R}^N} \left[V(x + y_n^1) - V_\infty\right] w^1(x)\varphi(x + y_n^1)dx \\
&\quad - \int_{\mathbb{R}^N} \left[f(u_{n,1}(x + y_n^1)) - f(u_{n,2}(x + y_n^1)) - f(w^1(x))\right]\varphi(x + y_n^1)dx.
\end{aligned}$$

Using $(\widetilde{V}_1)$ and applying Lebesgue dominated convergence theorem, it follows that

$$\int_{\mathbb{R}^N}\left[V(x+y_n^1) - V_\infty\right]w^1(x)\varphi(x+y_n^1)dx = o_n(1)$$

and, following the same ideas as in Lemma 2.3.2(d), we have

$$\begin{aligned}\int_{\mathbb{R}^N}&\left[f(u_{n,1}(x+y_n^1)) - f(u_{n,2}(x+y_n^1)) - f(w^1(x))\right]\varphi(x+y_n^1)dx\\ &= \int_{\mathbb{R}^N}\left[f(v_n^1) - f(v_n^1 - w^1) - f(w^1)\right]\varphi(x+y_n^1)dx = o_n(1).\end{aligned}$$

Hence,

$$I_V'(u_{n,1})\varphi = I_V'(u_{n,2})\varphi + o_n(1), \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N),$$

which shows that, as $n \to \infty$, $I_V'(u_{n,2}) \to 0$ in $H^{-1}(\mathbb{R}^N)$. Furthermore, arguing as in (2.3.15), we get

$$\int_{\mathbb{R}^N}|V(x) - V_\infty||u_{n,2}||\varphi|\,dx = o_n(1),$$

and thus,

$$\begin{aligned}o_n(1) = I_V'(u_{n,2})\varphi &= \int_{\mathbb{R}^N}(\nabla u_{n,2}\nabla\varphi + V(x)u_{n,2}\varphi)dx - \int_{\mathbb{R}^N}f(u_{n,2})\varphi dx\\ &= \int_{\mathbb{R}^N}(\nabla u_{n,2}\nabla\varphi + V_\infty u_{n,2}\varphi)dx - \int_{\mathbb{R}^N}f(u_{n,2})\varphi dx + \int_{\mathbb{R}^N}[V(x) - V_\infty]u_{n,2}\varphi dx\\ &= I_\infty'(u_{n,2})\varphi + \int_{\mathbb{R}^N}[V(x) - V_\infty]u_{n,2}\varphi dx\\ &= I_\infty'(u_{n,2})\varphi + o_n(1).\end{aligned}$$

Therefore,

$$I_\infty'(u_{n,2})\varphi = o_n(1), \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N),$$

and so, as $n \to \infty$, $I_\infty'(u_{n,2}) \to 0$ in $H^{-1}(\mathbb{R}^N)$, proving (c2).

Thus, if $u_{n,2} \to 0$ strongly in $H^1(\mathbb{R}^N)$, we have completed the proof. Otherwise, if $u_{n,2} \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ and does not converge strongly to zero, we take $u_{n,3} := u_{n,2} - w^2(\cdot - y_n^2)$ and repeat the argument. Hence, we obtain

$$I_V(u_n) = I_V(u) + I_\infty(w^1) + I_\infty(w^2) + o_n(1).$$

Continuing this way, we get a sequence of points $(y_n^j) \subset \mathbb{R}^N$ such that $|y_n^j| \to \infty$, $|y_n^j - y_n^i| \to \infty$ if $i \neq j$ and sequences of functions $u_{n,j} := u_{n,j-1} - w^{j-1}(\cdot - y_n^{j-1})$, $j \geq 2$,

such that
$$u_{n,j}(\cdot + y_n^j) \rightharpoonup w^j \quad \text{in } H^1(\mathbb{R}^N),$$
where $w^j$ is a nontrivial solution of the limit problem $(P_\infty)$. Since $I_\infty(w^j) \geq m = p_\infty$ and $I_V(u_n) \to c$, there exists a positive integer $k$ such that
$$I_V(u_n) = I_V(u) + \sum_{j=1}^{k} I_\infty(w^j) + o_n(1),$$
and the proof of lemma is complete. □

Note that as in Remark 1.3.7 in Chapter 1, if $u \not\equiv 0$ is a solution of $(P_G)$ then $u \in \mathcal{P}_V^G$ and it holds $I_V(u) > 0$.

**Corollary 2.3.4.** *Assume that $(\widetilde{V}_1)$–$(\widetilde{V}_3)$ and $(\widetilde{f}_1)$–$(\widetilde{f}_4)$ hold true. Let $(u_n) \subset H_G^1(\mathbb{R}^N)$ be a bounded $(PS)_c$-sequence for $I_V$ restricted to $H_G^1(\mathbb{R}^N)$. If $0 < c < \ell(G)p_\infty$, where $p_\infty$ is given in (2.2.2), then the functional $I_V$ has a nontrivial critical point $u \in H_G^1(\mathbb{R}^N)$ such that $I_V(u) = c$.*

*Proof.* To prove this corollary, just follow the same ideas applied in Corollary 1.3.8, substituting $D^{1,2}(\mathbb{R}^N)$ by $H^1(\mathbb{R}^N)$. □

## 2.4 Existence of a critical point

In this section we will prove the main result of this chapter. Its proof requires some important estimates and the previous lemmas.

In what follows, for simplicity, we will consider $G = O(N-1) \times \mathbb{Z}_2 \subset O(N)$, where $\mathbb{Z}_2 := \{id, -id\}$, $\ell(G) = 2$ and $d_G = 2$. That is, for all $g \in G$, we have
$$g(x_1, \ldots, x_{N-1}, x_N) = (g_1(x_1, \ldots, x_{N-1}), -x_N),$$
where $g_1 \in O(N-1)$. Moreover, we will denote $y = (0, \ldots, 0, 1) \in \mathbb{R}^N$ and $w$ a ground state solution of the limit problem $(P_\infty)$, which is positive, radially symmetric and decreasing in the radial direction, such that $I_\infty(w) = m$. Observe that, for any $g \in G$ and $x \in \mathbb{R}^N$, we have $w(gx) = w(|gx|) = w(|x|) = w(x)$ which shows that $w \in H_G^1(\mathbb{R}^N)$.

As in the first chapter, we will construct a positive solution of $(P_G)$ exploiting the interaction of two translated bumps. Let us denote $B_r(x_0) := \{x \in \mathbb{R}^N : |x - x_0| \leq r\}$. For any $R > 0$ and $y = (0, \ldots, 0, 1) \in \mathbb{R}^N$, we define
$$w_-^R := w(\cdot - Ry), \quad w_+^R := w(\cdot + Ry). \tag{2.4.1}$$

In the next lemmas we study the interaction of powers of these two translated solitons.

**Lemma 2.4.1.** *If $\mu_2 > \mu_1 \geq 0$, then there exists $C_1 > 0$ such that, for all $x_1, x_2 \in \mathbb{R}^N$,*

$$\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} dx \leq C_1 e^{-\mu_1|x_1-x_2|}.$$

*If $\mu_2 > \mu_3 \geq \mu_1 \geq 0$, then there exists $C_2 > 0$ such that, for all $x_1, x_2, x_3 \in \mathbb{R}^N$,*

$$\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} e^{-\mu_3|x-x_3|} dx \leq C_2 e^{-\frac{\mu_1}{2}(|x_1-x_2|+|x_1-x_3|+|x_2-x_3|)}.$$

*Proof.* Note that

$$\begin{aligned} \mu_1|x_1-x_2| + (\mu_2-\mu_1)|x-x_2| &\leq \mu_1(|x-x_1| + |x-x_2|) + (\mu_2-\mu_1)|x-x_2| \\ &= \mu_1|x-x_1| + \mu_2|x-x_2|. \end{aligned}$$

Similarly, we also obtain the following inequalities

$$\mu_1|x_1-x_3| + (\mu_3-\mu_1)|x-x_3| \leq \mu_1|x-x_1| + \mu_3|x-x_3|$$

and

$$\mu_3|x_3-x_2| + (\mu_2-\mu_3)|x-x_2| \leq \mu_3|x-x_3| + \mu_2|x-x_2|.$$

Therefore, by first inequality, there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} dx \leq \int_{\mathbb{R}^N} e^{-\mu_1|x_1-x_2|} e^{-(\mu_2-\mu_1)|x-x_2|} dx \leq C_1 e^{-\mu_1|x_1-x_2|}.$$

On the other hand, as $\mu_2 > \mu_1$ and $\mu_3 \geq \mu_1$, it follows that

$$\begin{aligned} \mu_1(|x_1-x_2| + |x_1-x_3| + |x_2-x_3|) + (\mu_2-\mu_1)|x-x_2| \\ \leq 2(\mu_1|x-x_1| + \mu_2|x-x_2| + \mu_3|x-x_3|), \end{aligned}$$

and so, there exists $C_2 > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} e^{-\mu_3|x-x_3|} dx &\leq \int_{\mathbb{R}^N} e^{-\frac{\mu_1}{2}(|x_1-x_2|+|x_1-x_3|+|x_2-x_3|)} e^{-\frac{(\mu_2-\mu_1)}{2}|x-x_2|} dx \\ &\leq C_2 e^{-\frac{\mu_1}{2}(|x_1-x_2|+|x_1-x_3|+|x_2-x_3|)}. \end{aligned}$$

□

**Lemma 2.4.2.** *Let $0 \leq q_1 < q_2 < \infty$. Then, for any $R \geq 1$, there exist constants*

$C_1, C_2 > 0$ *such that the following inequalities hold:*

$$\int_{\mathbb{R}^N} \left(w^R\right)^{q_2} \left(w_+^R\right)^{q_1} \leq C_1 R^{-q_1\frac{N-1}{2}} e^{-2q_1\sqrt{V_1}R} \tag{2.4.2}$$

*and*

$$\int_{\mathbb{R}^N} \left(w_+^R\right)^{q_2} \left(w^R\right)^{q_1} \leq C_2 R^{-q_1\frac{N-1}{2}} e^{-2q_1\sqrt{V_1}R}. \tag{2.4.3}$$

*Proof.* Note that, by making a change of variables and using (2.1.1), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(w^R\right)^{q_2} \left(w_+^R\right)^{q_1} dx &= \int_{\mathbb{R}^N} (w(x))^{q_2} (w(x+2Ry))^{q_1} dx \\
&\leq C\int_{\mathbb{R}^N} (1+|x|)^{-q_2\frac{N-1}{2}} e^{-q_2\sqrt{V_1}|x|}(1+|x+2Ry|)^{-q_1\frac{N-1}{2}} e^{-q_1\sqrt{V_1}|x+2Ry|} dx \\
&\leq C\int_{B_R(0)} e^{-q_2\sqrt{V_1}|x|}(1+|x+2Ry|)^{-q_1\frac{N-1}{2}} e^{-q_1\sqrt{V_1}|x+2Ry|} dx \\
&\quad + C\int_{\mathbb{R}^N\setminus B_R(0)} (1+|x|)^{-q_2\frac{N-1}{2}} e^{-q_2\sqrt{V_1}|x|} e^{-q_1\sqrt{V_1}|x+2Ry|} dx \\
&\leq CR^{-q_1\frac{N-1}{2}}\int_{B_R(0)} e^{-q_2\sqrt{V_1}|x|} e^{-q_1\sqrt{V_1}|x+2Ry|} dx \\
&\quad + CR^{-q_2\frac{N-1}{2}}\int_{\mathbb{R}^N\setminus B_R(0)} e^{-q_2\sqrt{V_1}|x|} e^{-q_1\sqrt{V_1}|x+2Ry|} dx \\
&\leq CR^{-q_1\frac{N-1}{2}}\int_{\mathbb{R}^N} e^{-q_2\sqrt{V_1}|x|} e^{-q_1\sqrt{V_1}|x+2Ry|} dx.
\end{aligned}$$

Therefore, by Lemma 2.4.1, there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} \left(w^R\right)^{q_2} \left(w_+^R\right)^{q_1} \leq C_1 R^{-q_1\frac{N-1}{2}} e^{-2q_1\sqrt{V_1}R}.$$

Similarly, we get a constant $C_2 > 0$ such that

$$\int_{\mathbb{R}^N} \left(w_+^R\right)^{q_2} \left(w^R\right)^{q_1} \leq C_2 R^{-q_1\frac{N-1}{2}} e^{-2q_1\sqrt{V_1}R}.$$

□

Next, let us define

$$\varepsilon_R := \int_{\mathbb{R}^N} f\left(w^R\right) w_+^R \, dx = \int_{\mathbb{R}^N} f\left(w_+^R\right) w^R \, dx \tag{2.4.4}$$

and we will obtain some estimates for $\varepsilon_R$.

**Lemma 2.4.3.** *Assume that* $(\widetilde{f}_1)$–$(\widetilde{f}_2)$ *hold true. Then, for any* $R \geq 1$, *there exists a constant* $C_3 > 0$ *such that*

$$\varepsilon_R \leq C_3 R^{-\frac{N-1}{2}} e^{-2\sqrt{V_1} R}. \tag{2.4.5}$$

*Proof.* Using hypothesis $(\widetilde{f}_2)$, we obtain

$$\begin{aligned}\varepsilon_R &= \int_{\mathbb{R}^N} f(w^R_{+y})\, w^R_{-y}\, dx \\ &\leq A_1 \int_{\mathbb{R}^N} (w^R_{+y})^{p_1} w^R_{-y}\, dx + A_1 \int_{\mathbb{R}^N} (w^R_{+y})^{p_2} w^R_{-y}\, dx.\end{aligned}$$

Since $1 < p_1 \leq p_2 < 2^* - 1$, applying Lemma 2.4.2 with $q_1 = 1$ and $q_2 = p_1$ or $p_2$, we find $C_3 > 0$ such that (2.4.5) holds true. □

Note that $-\Delta w(0) + V_1 w(0) = f(w(0))$, where $w(0)$ is maximum point of the positive radial ground state solution $w$ of the limit problem $(P_1)$. Hence, $-\Delta w(0) \geq 0$ and so $f(w(0)) - V_1 w(0) \geq 0$, or equivalently $f(w(0))/w(0) - V_1 > 0$. Since the function $f(s)/s$ is continuous and $f(w(0))/w(0) - V_1 > 0$, there exists $r_0 = r_0(f, V_1, w) > 0$ (which depends only on $f$, $V_1$ and $w$) such that $f(w(x))/w(x) - V_1/2 > 0$ in the ball $B_{r_0}(0)$.

**Lemma 2.4.4.** *Assume that* $(\widetilde{f}_1)$–$(\widetilde{f}_2)$ *hold true. Then, for any* $R \geq 1$, *there exists a constant* $C_4 > 0$ *such that*

$$\varepsilon_R \geq C_4 R^{-\frac{N-1}{2}} e^{-2\sqrt{V_1} R}. \tag{2.4.6}$$

*Proof.* In the above considerations, since $r_0$ is a constant independent of $R$ and $y$, we can assume without loss of generality that $r_0 = 1$. So it follows that $f(w(x))/w(x) - V_1/2 > 0$ in the ball $B_1(0)$. Then, by making a change of variables and using (2.1.1), for any $R \geq 1$, we obtain

$$\begin{aligned}\varepsilon_R &= \int_{\mathbb{R}^N} f(w(x - Ry))w(x + Ry)dx = \int_{\mathbb{R}^N} f(w(z))w(z + 2Ry)dz \\ &\geq \int_{B_1(0)} f(w(z))w(z + 2Ry)dz \geq \int_{B_1(0)} \frac{V_1}{2} w(z)w(z + 2Ry)dz \\ &\geq C\int_{B_1(0)} (1 + |z|)^{-\frac{N-1}{2}} e^{-\sqrt{V_1}|z|}(1 + |z + 2Ry|)^{-\frac{N-1}{2}} e^{-\sqrt{V_1}|z+2Ry|} dz \\ &\geq C\int_{B_1(0)} (1 + |z|)^{-\frac{N-1}{2}} e^{-\sqrt{V_1}|z|}(1 + |z + 2Ry|)^{-\frac{N-1}{2}} e^{-\sqrt{V_1}|z|} e^{-2\sqrt{V_1} R} dz \\ &\geq C|B_1(0)| e^{-2\sqrt{V_1}} R^{-\frac{N-1}{2}} e^{-2\sqrt{V_1} R} \geq CR^{-\frac{N-1}{2}} e^{-2\sqrt{V_1} R}.\end{aligned}$$

Therefore, for any $R \geq 1$, there exists a constant $C_4 > 0$ such that

$$\varepsilon_R \leq C_4 R^{-\frac{N-1}{2}} e^{-2\sqrt{V_1} R}.$$

□

We will also need the estimates from [1, Lemma 2.2]. Let us define the sum of the two translated solitons

$$U^R := w_+^R + w_-^R, \tag{2.4.7}$$

and present some of its properties and estimates. Following the same ideas applied in the first chapter, we can conclude that $U^R \in H^1_G(\mathbb{R}^N)$.

**Corollary 2.4.5.** *Assume that $(\widetilde{f}_1)$–$(\widetilde{f}_2)$ hold true. Then, it holds*

$$\int_{\mathbb{R}^N} \left| F(U^R) - F(w_-^R) - F(w_+^R) - f(w_-^R) w_+^R - f(w_+^R) w_-^R \right| dx = o(\varepsilon_R). \tag{2.4.8}$$

*Proof.* Set $w_- := w_-^R$, $w_+ := w_+^R$ and $U := U^R$. Using [1, Lemma 2.2], since $w_-$, $w_+$ and $U$ are bounded uniformly $R$, there exist constants $C > 0$ and $\sigma \in (1/2, 1]$ such that

$$\int_{\mathbb{R}^N} |F(U) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-|\, dx \leq C \int_{\mathbb{R}^N} (w_- w_+)^{2\sigma}\, dx.$$

Note that, by (2.1.1), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} (w_- w_+)^{2\sigma} dx &= \int_{\mathbb{R}^N} (w(x - Ry))^{2\sigma} (w(x + Ry))^{2\sigma} dx = \int_{\mathbb{R}^N} (w(x))^{2\sigma} (w(x + 2Ry))^{2\sigma} dx \\
&\leq C \int_{\mathbb{R}^N} (1 + |x|)^{-\sigma(N-1)} e^{-2\sigma\sqrt{V_1}|x|} (1 + |x + 2Ry|)^{-\sigma(N-1)} e^{-2\sigma\sqrt{V_1}|x+2Ry|} dx \\
&\leq C \int_{B_R(0)} e^{-2\sigma\sqrt{V_1}|x|} (1 + |x + 2Ry|)^{-\sigma(N-1)} e^{-2\sigma\sqrt{V_1}|x+2Ry|} dx \\
&\quad + C \int_{\mathbb{R}^N \setminus B_R(0)} (1 + |x|)^{-\sigma(N-1)} e^{-2\sigma\sqrt{V_1}|x|} e^{-2\sigma\sqrt{V_1}|x+2Ry|} dx \\
&\leq C R^{-\sigma(N-1)} \int_{B_R(0)} e^{-2\sigma\sqrt{V_1}|x|} e^{-2\sigma\sqrt{V_1}|x+2Ry|} dx \\
&\quad + C R^{-\sigma(N-1)} \int_{\mathbb{R}^N \setminus B_R(0)} e^{-2\sigma\sqrt{V_1}|x|} e^{-2\sigma\sqrt{V_1}|x+2Ry|} dx \\
&\leq C R^{-\sigma(N-1)} \int_{\mathbb{R}^N} e^{-2\sigma\sqrt{V_1}|x|} e^{-2\sigma\sqrt{V_1}|x+2Ry|} dx \\
&\leq C R^{-\sigma(N-1)} \int_{\mathbb{R}^N} e^{-\sqrt{V_1}|x|} e^{-2\sigma\sqrt{V_1}|x+2Ry|} dx.
\end{aligned}$$

Hence, it follows from Lemma 2.4.2, with $q_1 = 1$ and $q_2 = 2\sigma > 1$, that there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} (w - w_+)^{2\sigma} dx \leq C_1 R^{-\sigma(N-1)} e^{-2\sqrt{V_1} R} < C_1 R^{-\frac{N-1}{2}} e^{-2\sqrt{V_1} R},$$

which yields (2.4.8), proving the corollary. □

**Lemma 2.4.6.** *Assume that $(\widetilde{V}_1)$–$(\widetilde{V}_2)$ and $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ hold true and let $\mu \in (0,1)$ be. Then, for any $R \geq 1$ and $y \in \partial B_1(0)$, the following statements hold:*

$$\int_{\mathbb{R}^N} \left| \nabla w^R_{+y} \nabla w^R_{-y} \right| dx \leq C_1 R^{-\frac{N-1}{2}} e^{-2\mu\sqrt{V_1} R} = o_R(1) \tag{2.4.9}$$

*and*

$$\int_{\mathbb{R}^N} w^R_{+y} w^R_{-y}\, dx \leq C_2 R^{-\frac{N-1}{2}} e^{-2\mu\sqrt{V_1} R} = o_R(1), \tag{2.4.10}$$

*where $o_R(1) \to 0$ as $R \to +\infty$.*

*Proof.* Note that, by making a change of variables and using (2.1.1), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} \left| \nabla w^R_{+y} \nabla w^R_{-y} \right| dx &= \int_{\mathbb{R}^N} |\nabla w(x - Ry) \nabla w(x + Ry)|\, dx \\
&\leq C \int_{\mathbb{R}^N} (1+|x|)^{-\frac{N-1}{2}} e^{-\sqrt{V_1}|x|} (1 + |x + 2Ry|)^{-\frac{N-1}{2}} e^{-\sqrt{V_1}|x+2Ry|} dx \\
&\leq C \int_{B_R(0)} e^{-\sqrt{V_1}|x|} (1 + |x + 2Ry|)^{-\frac{N-1}{2}} e^{-\sqrt{V_1}|x+2Ry|} dx \\
&\quad + C \int_{\mathbb{R}^N \setminus B_R(0)} (1+|x|)^{-\frac{N-1}{2}} e^{-\sqrt{V_1}|x|} e^{-\sqrt{V_1}|x+2Ry|} dx \\
&\leq C R^{-\frac{N-1}{2}} \int_{B_R(0)} e^{-\sqrt{V_1}|x|} e^{-\sqrt{V_1}|x+2Ry|} dx \\
&\quad + C R^{-\frac{N-1}{2}} \int_{\mathbb{R}^N \setminus B_R(0)} e^{-\sqrt{V_1}|x|} e^{-\sqrt{V_1}|x+2Ry|} dx \\
&\leq C R^{-\frac{N-1}{2}} \int_{\mathbb{R}^N} e^{-\sqrt{V_1}|x|} e^{-\sqrt{V_1}|x+2Ry|} dx.
\end{aligned}$$

Since $\mu \in (0,1)$, it follows that

$$\int_{\mathbb{R}^N} \left| \nabla w^R_{+y} \nabla w^R_{-y} \right| dx \leq C R^{-\frac{N-1}{2}} \int_{\mathbb{R}^N} e^{-\mu\sqrt{V_1}|x|} e^{-\sqrt{V_1}|x+2Ry|} dx,$$

and so, by Lemma 2.4.1, there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} \left| \nabla w^R_{+y} \nabla w^R_{-y} \right| dx \leq C_1 R^{-\frac{N-1}{2}} e^{-2\mu\sqrt{V_1} R},$$

which proves (2.4.9). Similarly, we show that (2.4.10) also holds true, and the proof of the lemma is complete. □

**Lemma 2.4.7.** *Assume that* $(\widetilde{V}_1)$–$(\widetilde{V}_2)$ *and* $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ *hold true. Then, the following statements hold:*

(a) $\displaystyle\int_{\mathbb{R}^N} |\nabla U^R|^2\,dx = 2\int_{\mathbb{R}^N} |\nabla w_-|^2 dx + o_R(1)$;

(b) $\displaystyle\int_{\mathbb{R}^N} \left(U^R\right)^2 dx = 2\int_{\mathbb{R}^N} w_-^2 dx + o_R(1)$;

(c) $\displaystyle\int_{\mathbb{R}^N} F\left(U^R\right) dx = 2\int_{\mathbb{R}^N} F(w_-)dx + o_R(1)$;

(d) $\displaystyle\int_{\mathbb{R}^N} \left(F\left(U^R\right) - \frac{V_\infty}{2}\left(U^R\right)^2\right) dx = \frac{2}{2^*}\int_{\mathbb{R}^N} |\nabla w_-|^2 dx + o_R(1)$,

*where* $o_R(1) \to 0$ *as* $R \to +\infty$.

*Proof.* Set $w_- := w_-^R$, $w_+ := w_+^R$ and $U := U^R$. Then,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla U|^2 dx &= \int_{\mathbb{R}^N} |\nabla w_-|^2 dx + 2\int_{\mathbb{R}^N} \nabla w_- \cdot \nabla w_+ dx + \int_{\mathbb{R}^N} |\nabla w_-|^2 dx \\ &= 2\int_{\mathbb{R}^N} |\nabla w_-|^2 dx + 2\int_{\mathbb{R}^N} \nabla w_- \cdot \nabla w_+\, dx.\end{aligned}$$

By (2.4.9), we have

$$\int_{\mathbb{R}^N} |\nabla w_- \cdot \nabla w_+|\, dx = o_R(1),$$

proving item (a), and by (2.4.10), we have

$$\int_{\mathbb{R}^N} w_-\, w_+\, dx = o_R(1),$$

so this implies that

$$\int_{\mathbb{R}^N} U^2 dx = \int_{\mathbb{R}^N} w_-^2 dx + 2\int_{\mathbb{R}^N} w_-\, w_+\, dx + \int_{\mathbb{R}^N} w_-^2 dx = 2\int_{\mathbb{R}^N} w_-^2 dx + o_R(1),$$

proving item (b). We also have

$$\begin{aligned}\int_{\mathbb{R}^N} F(U)dx - 2\int_{\mathbb{R}^N} F(w_-)dx &= \int_{\mathbb{R}^N} F(U)dx - \int_{\mathbb{R}^N} F(w_-)dx - \int_{\mathbb{R}^N} F(w_+)dx \\ &= \int_{\mathbb{R}^N} [F(U) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-]dx + \\ &\quad + \int_{\mathbb{R}^N} [f(w_-)w_+ + f(w_+)w_-]dx.\end{aligned}$$

By Corollary 2.4.5, it follows that

$$\int_{\mathbb{R}^N} |F(U) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-|\, dx = o_R(1). \tag{2.4.11}$$

On the other hand, by definition (2.4.4) and Lemma 2.4.3, we also have

$$\int_{\mathbb{R}^N} [f(w_-)w_+ + f(w_+)w_-]\, dx = 2\varepsilon_R = o_R(1), \tag{2.4.12}$$

and so (c) follows. Now, we denote

$$G_1(u) := F(u) - \frac{V_1}{2}u^2. \tag{2.4.13}$$

Thus, using (2.4.10), (2.4.11) and (2.4.12), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} G_1(U)dx &= \int_{\mathbb{R}^N} \left(F(w_- + w_+) - \frac{V_1}{2}(w_- + w_+)^2\right) dx \\
&= \int_{\mathbb{R}^N} \left(F(w_-) - \frac{V_1}{2}(w_-)^2\right) dx + \int_{\mathbb{R}^N} \left(F(w_+) - \frac{V_1}{2}(w_+)^2\right) dx \\
&\quad + \int_{\mathbb{R}^N} [F(w_- + w_+) - F(w_-) - F(w_+)]\, dx - \int_{\mathbb{R}^N} V_1 w_- w_+\, dx \\
&= 2\int_{\mathbb{R}^N} G_1(w)dx - \int_{\mathbb{R}^N} V_1 w_- w_+\, dx + \int_{\mathbb{R}^N} [f(w_-)w_+ + f(w_+)w_-]\, dx \\
&\quad + \int_{\mathbb{R}^N} [F(w_- + w_+) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-]\, dx \\
&= 2\int_{\mathbb{R}^N} G_1(w)dx + o_R(1).
\end{aligned}$$

Since $w$ is a solution of problem ($P_1$), it follows that

$$\int_{\mathbb{R}^N} G_1(w)dx = \int_{\mathbb{R}^N} \left(F(w) - \frac{V_1}{2}w^2\right) dx = \frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla w|^2 dx,$$

which proves (d), concluding the proof of lemma. □

**Lemma 2.4.8.** *Assume that $(\widetilde{V}_1)$, $(\widetilde{V}_3)$ and $(\widetilde{V}_4)$ hold true and let $a \le s \le b$, for positive numbers $a$ and $b$. Then, the following statements hold:*

(a) $\displaystyle\int_{\mathbb{R}^N} |V(sx) - V_1|(U^R)^2\, dx = o_R(1)$;

(b) $\displaystyle\int_{\mathbb{R}^N} |\nabla V(sx)\cdot(sx)|(U^R)^2\, dx = o_R(1)$;

(c) $\displaystyle\int_{\mathbb{R}^N} |(sx)H(sx)(sx)|\left(U^R\right)^2 dx = o_R(1)$,

*where* $o_R(1) \to 0$ *as* $R \to +\infty$.

*Proof.* Let us prove only the item (a). The other items can be proved analogously. To simplify the notation, let us consider $w_- := w_-^R$, $w_+ := w_+^R$ and $U := U^R$.

Let $\varepsilon > 0$ be given arbitrarily. Since $\|w\|_2^2 = \displaystyle\int_{\mathbb{R}^N} w^2 dx > 0$, using the hypothesis $(\widetilde{V}_1)$, we get $\tau > 0$ large enough and fixed such that

$$|V(sx) - V_\infty| < \frac{\varepsilon}{4\|w\|_2^2}$$

for any $a \le s \le b$ and $|x| \ge \tau$. Hence,

$$\int_{|x|\ge\tau} |V(sx) - V_\infty|(w_-)^2 dx \le \frac{\varepsilon}{4\|w\|_2^2}\int_{|x|\ge\tau}(w_-)^2 dx \le \frac{\varepsilon}{4\|w\|_2^2}\int_{\mathbb{R}^N} w^2 dx = \frac{\varepsilon}{4}. \qquad (2.4.14)$$

On the other hand, for any $a \le s \le b$ and $R > \max\{1, \tau\}$, using (2.1.3) and (2.1.1), we obtain

$$\begin{aligned}\int_{|x|\le\tau} |V(sx) - V_\infty|(w_-)^2 dx &\le A_2\int_{|x|\le\tau}(w_-)^2 dx\\ &\le C\int_{|x|\le\tau}(1+|x-Ry|)^{1-N} e^{-2\sqrt{V_\infty}|x-Ry|}dx \le C\int_{|x|\le\tau} e^{-2\sqrt{V_\infty}|x-Ry|}dx\\ &\le C\int_{|x|\le\tau} e^{-2\sqrt{V_\infty}(|Ry|-|x|)}dx \le Ce^{-2\sqrt{V_\infty}(R-\tau)}|B_\tau(0)| \le Ce^{-\sqrt{V_\infty}R}. \qquad (2.4.15)\end{aligned}$$

So by (2.4.14) and (2.4.15), it follows that

$$\int_{\mathbb{R}^N} |V(sx) - V_\infty|(w_-)^2 dx \le \frac{\varepsilon}{4} + Ce^{-\sqrt{V_\infty}R},$$

Similarly, for any $a \le s \le b$ and $R > \max\{1, \tau\}$, we get

$$\int_{\mathbb{R}^N} |V(sx) - V_\infty|(w_+)^2 dx \le \frac{\varepsilon}{4} + Ce^{-\sqrt{V_\infty}R},$$

Therefore, for any $a \le s \le b$ and $R > \max\{1, \tau\}$, as

$$U^2 = (w_- + w_+)^2 \le 2\left(w_-\right)^2 + 2\left(w_+\right)^2,$$

it follows that

$$\int_{\mathbb{R}^N} |V(sx) - V_\infty| U^2 dx \le \varepsilon + Ce^{-\sqrt{V_\infty}R}.$$

Since $\varepsilon > 0$ was taken arbitrarily, we conclude that

$$\int_{\mathbb{R}^N} |V(sx) - V_\infty| U^2 dx = o_R(1)$$

which proves item (a). Using $(\widetilde{V}_3)$ and $(\widetilde{V}_4)$, proceeding as before, we can prove (b) and (c), respectively. □

**Lemma 2.4.9.** *Assume that $(\widetilde{V}_1)$–$(\widetilde{V}_4)$ and $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ hold true. Then, there exists $R_0 \geq 1$ such that for any $R \geq R_0$, there exists a unique positive constant $s := S^R$ such that*

$$U^R\Big(\frac{\cdot}{s}\Big) \in \mathcal{P}_V^G,$$

*where $U^R$ is given as in (2.4.7). Moreover, there exist $\sigma_0 \in (0, 1/2)$ and $S_0 > 1$ such that $S^R \in (\sigma_0, S_0)$ for any $R \geq R_0$. In addition, $S^R$ is a continuous function of the variable $R$.*

*Proof.* Denote $w_- := w_-^R = w(\cdot - Ry)$, $w_+ := w_+^R = w(\cdot + Ry)$ and $U := U^R = w_-^R + w_+^R$. Let $\xi : (0, +\infty) \to \mathbb{R}$ be defined by

$$\xi(s) := I_V(U(\cdot/s)) = \frac{s^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla U|^2 dx + \frac{s^N}{2}\int_{\mathbb{R}^N} V(sx)U^2 dx - s^N \int_{\mathbb{R}^N} F(U) dx.$$

Thus, $U(\cdot/s) \in \mathcal{P}_V^G$ if and only if $\xi'(s) = 0$, where

$$\begin{aligned} \xi'(s) &= \frac{N-2}{2} s^{N-3} \int_{\mathbb{R}^N} |\nabla U|^2 dx \\ &\quad + N s^{N-3}\left[s^2\left(\frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{\nabla V(sx)\cdot(sx)}{N} + V(sx)\right)U^2 dx - \int_{\mathbb{R}^N} F(U) dx\right)\right]. \end{aligned}$$

Since $s > 0$, we have $\xi'(s) = 0$ if and only if

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla U|^2 dx = N s^2 \left[\int_{\mathbb{R}^N} F(U) dx - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{\nabla V(sx)\cdot(sx)}{N} + V(sx)\right)U^2 dx\right].$$

Note that

$$\int_{\mathbb{R}^N} U^2 dx = \int_{\mathbb{R}^N} (w_- + w_+)^2\, dx \leq 2\int_{\mathbb{R}^N} \left[(w_-)^2 + (w_+)^2\right] dx = 4\int_{\mathbb{R}^N} w^2 dx,$$

which shows that $\|U\|_2$ is bounded uniformly for any $R \geq 1$. Since $\int_{\mathbb{R}^N} |\nabla w|^2 dx > 0$, using assumptions $(\widetilde{V}_1)$ and $(\widetilde{V}_3)$ and Lemma 2.4.7, there exist $R_1 \geq 1$ sufficiently large and $\sigma_0 \in (0, 1/2)$ sufficiently small such that

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla U|^2 dx > Ns^2\left[\int_{\mathbb{R}^N} F(U)dx - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{\nabla V(sx)\,(sx)}{N} + V(sx)\right)U^2 dx\right],$$

and so it holds $\xi'(s) > 0$, for every $s \in (0, \sigma_0]$ and $R \geq R_1$.

Now let us define a function $\varphi : (\sigma_0, +\infty) \to \mathbb{R}$ by

$$\varphi(s) = s^2\left[\int_{\mathbb{R}^N} F(U)dx - \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{\nabla V(sx)\,(sx)}{N} + V(sx)\right)U^2 dx\right].$$

Note that, denoting

$$G_\infty(U) := F(U) - \frac{V_\infty}{2}U^2,$$

as in (2.4.13), we obtain

$$\begin{aligned}\varphi'(s) = {} & 2s\left[\int_{\mathbb{R}^N} G_\infty(U)dx - \frac{1}{2}\int_{\mathbb{R}^N}[V(sx) - V_\infty]U^2 dx\right]\\ & - \frac{s}{2}\left[(N+3)\int_{\mathbb{R}^N}\left(\frac{\nabla V(sx)\,(sx)}{N}\right)U^2 dx + \int_{\mathbb{R}^N}\left(\frac{(sx)H(sx)(sx)}{N}\right)U^2 dx\right]\end{aligned}$$

and so

$$\begin{aligned}\varphi'(s) \geq {} & 2s\left[\int_{\mathbb{R}^N} G_\infty(U)dx - \frac{1}{2}\int_{\mathbb{R}^N}|V(sx) - V_\infty|U^2 dx\right]\\ & - \frac{s}{2}\left[(N+3)\int_{\mathbb{R}^N}\left|\frac{\nabla V(sx)\,(sx)}{N}\right|U^2 dx + \int_{\mathbb{R}^N}\left|\frac{(sx)H(sx)(sx)}{N}\right|U^2 dx\right].\end{aligned} \tag{2.4.16}$$

By Lemma 2.4.7(d), there exists $R_2 \geq 1$ sufficiently large such that

$$2\int_{\mathbb{R}^N} G_\infty(U)dx \geq \frac{1}{2}\int_{\mathbb{R}^N}|\nabla w|^2 dx, \tag{2.4.17}$$

for every $R \geq R_2$. The bounds given by (2.1.3), the pointwise limit $\lim_{R\to\infty} U^R(x) = 0$ and Lebesgue dominated convergence theorem or applying Lemma 2.4.8 imply that

$$\begin{aligned}&\int_{\mathbb{R}^N}|V(sx) - V_\infty|U^2 dx + \frac{N+3}{2}\int_{\mathbb{R}^N}\left|\frac{\nabla V(sx)\,(sx)}{N}\right|U^2 dx\\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N}\left|\frac{(sx)H(sx)(sx)}{N}\right|U^2 dx = o_R(1).\end{aligned}$$

Thus, since $\int_{\mathbb{R}^N} |\nabla w|^2 dx > 0$, there exists $R_3 \geq 1$ sufficiently large such that

$$\int_{\mathbb{R}^N} |V(sx) - V_\infty| U^2 dx + \frac{N+3}{2}\int_{\mathbb{R}^N} \left|\frac{\nabla V(sx)\cdot(sx)}{N}\right| U^2 dx + \frac{1}{2}\int_{\mathbb{R}^N}\left|\frac{(sx)H(sx)(sx)}{N}\right| U^2 dx \leq \frac{1}{2\cdot 2}\int_{\mathbb{R}^N} |\nabla w|^2 dx, \tag{2.4.18}$$

for every $s > \sigma_0$ and $R \geq R_3$. Therefore, taking $R_4 := \max\{R_1, R_2, R_3\}$ and substituting (2.4.17) and (2.4.18) in (2.4.16), we obtain

$$\varphi'(s) \geq s\left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 dx - \frac{1}{2\cdot 2}\int_{\mathbb{R}^N} |\nabla w|^2 dx\right] > \frac{\sigma_0}{2\cdot 2}\int_{\mathbb{R}^N} |\nabla w|^2 dx > 0,$$

for every $s > \sigma_0$ and $R \geq R_4$. This means that $\varphi(s)$ is increasing for $s > \sigma_0$ and $R$ taken sufficiently large. This implies that the term in the brackets for $\xi'(s)$ is decreasing for $s > \sigma_0$, and goes to $-\infty$ as $s \to +\infty$. Therefore, there is a unique $s = S^R > \sigma_0$ such that $\xi'(s) = 0$, i.e. $U^R(\cdot/s) \in \mathcal{P}_V^G$. Furthermore, again by Lemma 2.4.7(c) and (2.1.3), there exist $R_5 \geq 1$, sufficiently large, and $S_0 > 1$ such that $\xi'(s) < 0$, for all $s > S_0$ and $R \geq R_5$. Taking $R_0 = \max\{R_4, R_5\}$ the result follows. Finally, from the uniform estimates for $U$, $\nabla U$, $F(U)$ and $G_\infty(U)$ with respect to $R \geq R_0$, the continuity of $S^R$ in this variable is clear, and the proof is complete. □

From here on, let us consider $S^R$ as obtained in Lemma 2.4.9, $0 < \sigma_0 < S^R < S_0$.

**Lemma 2.4.10.** *Assume that* $(\widetilde{V}_1)$–$(\widetilde{V}_4)$ *and* $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ *hold true. Then, it holds that*

$$\lim_{R\to+\infty} S^R = 1. \tag{2.4.19}$$

*Proof.* The proof follows the same ideas as Lemma 1.4.10, changing $J_0$ by $J_\infty$. By Lemma 2.4.9, there exist constants $R_0 \geq 1$, $S_0 > 1$ and $\sigma_0 \in (0, 1/2)$ such that $S^R \in (\sigma_0, S_0)$ for every $R \geq R_0$. Denoting $w_- := w_-^R = w(\cdot - Ry)$ and $w_+ := w_+^R = w(\cdot + Ry)$, we have

$$\begin{aligned}
J_1(w_- + w_+) &= \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla w_- + \nabla w_+|^2\, dx + \frac{N}{2}\int_{\mathbb{R}^N} V_1 (w_- + w_+)^2\, dx \\
&\quad - N\int_{\mathbb{R}^N} F(w_- + w_+)\, dx \\
&= \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla w_-|^2 dx - N\int_{\mathbb{R}^N}\left(F(w_-) - \frac{V_1}{2} w_-^2\right) dx \\
&\quad + \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla w_+|^2 dx - N\int_{\mathbb{R}^N}\left(F(w_+) - \frac{V_1}{2} w_+^2\right) dx \\
&\quad + (N-2)\int_{\mathbb{R}^N} \nabla w_- \cdot \nabla w_+\, dx + N\int_{\mathbb{R}^N} V_1 w_- w_+\, dx \\
&\quad - N\int_{\mathbb{R}^N} [F(w_- + w_+) - F(w_-) - F(w_+)]\, dx.
\end{aligned}$$

Since $J_1(w) = 0$, it follows that

$$\begin{aligned}
J_1(w_- + w_+) = (N-2)&\int_{\mathbb{R}^N} \nabla w_- \cdot \nabla w_+\, dx + N\int_{\mathbb{R}^N} V_1 w_- w_+\, dx \\
&- N\int_{\mathbb{R}^N} [F(w_- + w_+) - F(w_-) - F(w_+)]\, dx.
\end{aligned}$$

By (2.4.9) and (2.4.10), we obtain

$$\int_{\mathbb{R}^N} |\nabla w_- \cdot \nabla w_+|\, dx = \int_{\mathbb{R}^N} |\nabla w(x - Ry) \cdot \nabla w(x + Ry)| dx = o_R(1)$$

and

$$\int_{\mathbb{R}^N} w_- w_+\, dx = \int_{\mathbb{R}^N} w(x - Ry) w(x + Ry) dx = o_R(1),$$

where $o_R(1) \to 0$ as $R \to +\infty$. On the other hand, since $w$ is solution of ($P_1$), applying Corollary 2.4.5, and Lemmas 2.4.3 and 2.4.4, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |F(w_- + w_+) - F(w_-) - F(w_+)|\, dx \\
&\qquad \leq \int_{\mathbb{R}^N} |F(w_- + w_+) - F(w_-) - F(w_+) - f(w_-)w_+ - f(w_+)w_-|\, dx \\
&\qquad\quad + \int_{\mathbb{R}^N} |f(w_-)w_+ + f(w_+)w_-|\, dx = o_R(1).
\end{aligned}$$

Hence,

$$|J_1(w_- + w_+)| = o_R(1), \tag{2.4.20}$$

so this implies that $J_\infty(w_- + w_+) \to 0$ as $R \to \infty$. The bounds given by (2.1.3), the pointwise limit $\lim_{R\to\infty} U^R(x) = 0$ and Lebesgue dominated convergence theorem imply that

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla V(x)\cdot x|(w_- + w_+)^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N} |V(x) - V_\infty|(w_- + w_+)^2\,dx = o_R(1). \quad (2.4.21)$$

Since

$$\begin{aligned} J_V(U^R) &= J_V(w_- + w_+) \\ &= J_\infty(w_- + w_+) + \frac{1}{2}\int_{\mathbb{R}^N} \nabla V(x)\cdot x\,(w_- + w_+)^2 dx + \frac{N}{2}\int_{\mathbb{R}^N} [V(x) - V_\infty](w_- + w_+)^2 dx, \end{aligned}$$

it follows from (2.4.20) and (2.4.21) that

$$\left|J_V(U^R)\right| = o_R(1).$$

Therefore, $J_V(U^R) \to 0$ as $R \to +\infty$, which implies that

$$\lim_{R\to+\infty} S^R \to 1,$$

by uniqueness of $S^R$ and continuity with respect to $R$. This proves the lemma. □

The previous lemma states that we can choose $\epsilon > 0$ sufficiently small and find $R_6 \geq 1$ such that $k\,S^R > 2\sqrt{V_\infty}$, for any $R \geq R_6$, for $k$ presented in hypothesis $(\widetilde{V}_2)$.

The next lemma gives a precise estimate of the interaction between the potential term $V - V_\infty$ and a translated copy of a ground state solution.

**Lemma 2.4.11.** *Assume that* $(\widetilde{V}_1)$–$(\widetilde{V}_2)$ *and* $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ *hold true and let* $s > 0$ *be such that* $ks > 2\sqrt{V_\infty}$. *Then, for any* $R \geq 1$,

$$\int_{\mathbb{R}^N} [V(sx) - V_\infty]\left(w_-^R + w_+^R\right)^2 dx = o(\varepsilon_R),$$

*where* $o(\varepsilon_R) \to 0$ *as* $R \to \infty$.

*Proof.* First let us prove that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} [V(sx) - V_\infty]\left(w_-^R\right)^2 dx \leq C R^{-(N-1)} e^{-2\sqrt{V_\infty}R}. \quad (2.4.22)$$

Observe that, by hypothesis $(\widetilde{V}_2)$ and (2.1.1), there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} [V(sx) - V_1](w^R)^2\, dx \le C\int_{\mathbb{R}^N} e^{-ks|x|}(1 + |x - Ry|)^{-(N-1)}e^{-2\sqrt{V_1}|x-Ry|}\, dx,$$

for any $R \ge 1$. Thus, from the fact that $ks > 2\sqrt{V_1}$, we may fix $\rho \in (0,1)$ such that $ks > ks(1-\rho) > 2\sqrt{V_1}$. So by Lemma 2.4.1, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_{\rho R}(Ry)} e^{-ks|x|}(1 + |x - Ry|)^{-(N-1)}e^{-2\sqrt{V_1}|x-Ry|}\, dx \le CR^{-(N-1)}e^{-2\sqrt{V_1}R}. \tag{2.4.23}$$

On the other hand, for all $x \in B_{\rho R}(0)$, it holds that

$$ks|x + Ry| \ge ks(R|y| - |x|) \ge ksR(1-\rho) > 2\sqrt{V_1}R.$$

Making a change of variables, we obtain

$$\begin{aligned}
&\int_{B_{\rho R}(Ry)} e^{-ks|x|}(1 + |x - Ry|)^{-(N-1)}e^{-2\sqrt{V_1}|x-Ry|}\, dx \\
&\qquad = \int_{B_{\rho R}(0)} e^{-ks|x+Ry|}(1 + |x|)^{-(N-1)}e^{-2\sqrt{V_1}|x|}\, dx \\
&\qquad \le e^{-ksR(1-\rho)}\int_{B_{\rho R}(0)} (1 + |x|)^{-(N-1)}dx \le CRe^{-ksR(1-\rho)} \\
&\qquad \le CR^{-(N-1)}e^{-2\sqrt{V_1}R}.
\end{aligned} \tag{2.4.24}$$

Hence, it follows from (2.4.23) and (2.4.24) that (2.4.22) occurs. Similarly, we get a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} [V(sx) - V_1](w^R_+)^2\, dx \le CR^{-(N-1)}e^{-2\sqrt{V_1}R}. \tag{2.4.25}$$

Now let us prove that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} [V(sx) - V_1]\, w^R w^R_+\, dx \le CR^{-(N-1)}e^{-2\sqrt{V_1}R}. \tag{2.4.26}$$

Set $\Omega := \mathbb{R}^N \setminus [B_{\rho R}(Ry) \cup B_{\rho R}(-Ry)]$. Using $(V_2)$ and (2.1.1), we get

$$\begin{aligned}
\int_\Omega [V(sx) - V_1]\, w^R w^R_+ dx &\le A_0\int_\Omega e^{-ks|x|}w^R w^R_+ dx \\
&\le C\int_\Omega e^{-ks|x|}(1 + |x - Ry|)^{-\frac{N-1}{2}}e^{-\sqrt{V_1}|x-Ry|}(1 + |x + Ry|)^{-\frac{N-1}{2}}e^{-\sqrt{V_1}|x+Ry|}\, dx,
\end{aligned}$$

for any $R \geq 1$. From the second inequality in Lemma 2.4.1, we obtain

$$\int e^{-k s|x|}(1+|x-Ry|)^{-\frac{N-1}{2}} e^{-\sqrt{V_1}|x-Ry|}(1+|x+Ry|)^{-\frac{N-1}{2}} e^{-\sqrt{V_1}|x+Ry|}\, dx$$
$$\leq CR^{-(N-1)} e^{-\frac{1}{2}\sqrt{V_1}(R+R+2R)} = CR^{-(N-1)} e^{-2\sqrt{V_1}R}.$$

The integrals on $B_{\rho R}(Ry)$ and $B_{\rho R}(-Ry)$ are estimated by the same argument of (2.4.24). Note that these balls are disjoint. Thus, we conclude that (2.4.26) holds true. Therefore, by (2.4.22), (2.4.25) and (2.4.26), the lemma is proved. □

**Proposition 2.4.12.** *Assume that $(\widetilde{V}_1)$–$(\widetilde{V}_4)$ and $(\widetilde{f}_1)$–$(\widetilde{f}_4)$ hold true. Then, there exist $L > 2$ large enough and $R_4 \geq 1$ such that*

$$I_V\left(U^R\left(\frac{\cdot}{s}\right)\right) < 2I_1(w) = 2p_1, \quad \textit{for all } s \in (0, L] \textit{ and all } R \geq R_4 \tag{2.4.27}$$

*and*

$$I_V\left(U^R\left(\frac{\cdot}{L}\right)\right) < 0, \quad \textit{for all } R \geq R_4. \tag{2.4.28}$$

*Proof.* By Lemma 2.4.9, there exist constants $R_0 \geq 1$, $\sigma_0 \in (0, 1/2)$ and $S_0 > 1$ such that $S^R \in (\sigma_0, S_0)$ for every $R \geq R_0$. So, changing the variables $sz = x$ and denoting $w_- := w_-^R = w(\cdot - Ry)$ and $w_+ := w_+^R = w(\cdot + Ry)$, where $y = (0, \dots, 0, 1) \in \mathbb{R}^N$, we have

$$
\begin{aligned}
I_V\left(U^R\left(\frac{\cdot}{\varepsilon}\right)\right) &= \varepsilon^{N-2}\left[\frac{1}{2}\int_{\mathbb{R}^N}|\nabla w_+|^2dz - \varepsilon^2\int_{\mathbb{R}^N}\left(F(w_+) - \frac{V_\infty}{2}(w_+)^2\right)dz\right] \\
&\quad + \varepsilon^{N-2}\left[\frac{1}{2}\int_{\mathbb{R}^N}|\nabla w_-|^2dz - \varepsilon^2\int_{\mathbb{R}^N}\left(F(w_-) - \frac{V_\infty}{2}(w_-)^2\right)dz\right] \\
&\quad + \frac{\varepsilon^N}{2}\left[\int_{\mathbb{R}^N}[V(\varepsilon z) - V_\infty]\left[(w_+)^2 + (w_-)^2\right]dz + 2\int_{\mathbb{R}^N}V(\varepsilon z)w_+w_-\,dz\right] \\
&\quad - \varepsilon^N\int_{\mathbb{R}^N}[F(w_+ + w_-) - F(w_+) - F(w_-) - f(w_+)w_- - f(w_-)w_+]dz \\
&\quad + \varepsilon^{N-2}\int_{\mathbb{R}^N}\left[\nabla w_+\cdot\nabla w_- - \varepsilon^2 f(w_+)w_- - \varepsilon^2 f(w_-)w_+\right]dz \\
&\le I_\infty\left(w_+\left(\frac{\cdot}{\varepsilon}\right)\right) + I_\infty\left(w_-\left(\frac{\cdot}{\varepsilon}\right)\right) + \frac{\varepsilon^N}{2}\int_{\mathbb{R}^N}[V(\varepsilon z) - V_\infty](w_+ + w_-)^2\,dz \\
&\quad + \varepsilon^{N-2}\int_{\mathbb{R}^N}\left[\nabla w_+\cdot\nabla w_- + \varepsilon^2 V_\infty w_+w_- - \varepsilon^2 f(w_+)w_- - \varepsilon^2 f(w_-)w_+\right]dz \\
&\quad + \varepsilon^N\int_{\mathbb{R}^N}|F(w_+ + w_-) - F(w_+) - F(w_-) - f(w_+)w_- - f(w_-)w_+|dz \\
&\le 2I_\infty\left(w\left(\frac{\cdot}{\varepsilon}\right)\right) + \frac{\varepsilon^N}{2}\int_{\mathbb{R}^N}[V(\varepsilon z) - V_\infty](w_+ + w_-)^2\,dz \\
&\quad + \varepsilon^{N-2}\int_{\mathbb{R}^N}\left[\nabla w_+\cdot\nabla w_- + \varepsilon^2 V_\infty w_+w_- - \varepsilon^2 f(w_+)w_- - \varepsilon^2 f(w_-)w_+\right]dz \\
&\quad + \varepsilon^N\int_{\mathbb{R}^N}|F(w_+ + w_-) - F(w_+) - F(w_-) - f(w_+)w_- - f(w_-)w_+|dz.
\end{aligned}
$$

Since $p_\infty = I_\infty(w) = \max_{t>0} I_\infty\left(w\left(\frac{\cdot}{t}\right)\right) > 0$, then

$$
I_\infty\left(w\left(\frac{\cdot}{\varepsilon}\right)\right) \le p_\infty, \qquad \text{for all } \varepsilon \in (0, \infty). \tag{2.4.29}
$$

Let us set

$$
\begin{aligned}
I_1 &:= \frac{\varepsilon^N}{2}\int_{\mathbb{R}^N}[V(\varepsilon z) - V_\infty](w_+ + w_-)^2\,dz, \\
I_2 &:= \varepsilon^{N-2}\int_{\mathbb{R}^N}\left[\nabla w_+\cdot\nabla w_- + \varepsilon^2 V_\infty w_+w_- - \varepsilon^2 f(w_+)w_- - \varepsilon^2 f(w_-)w_+\right]dz, \\
I_3 &:= \varepsilon^N\int_{\mathbb{R}^N}|F(w_+ + w_-) - F(w_+) - F(w_-) - f(w_+)w_- - f(w_-)w_+|dz.
\end{aligned}
$$

To show (2.4.27) and (2.4.28), we will estimate $I_1$, $I_2$ and $I_3$. Take $L > 2$ large enough. By hypothesis $(\widetilde{V}_2)$, we have $k > 2\sqrt{V_\infty}$ and so, there exists $0 < \delta_1 < 1/4$ sufficiently small such that $ks > 2\sqrt{V_\infty}$, for all $s \geq 1 - \delta_1$. So, by Lemma 2.4.11, we obtain

$$I_1 = \frac{s^N}{2}\int_{\mathbb{R}^N} [V(sz) - V_\infty](w_+ + w_-)^2\,dz = o(\varepsilon_R), \tag{2.4.30}$$

for every $s \in [1-\delta_1, L]$ and $R \geq 1$, where $o(\varepsilon_R) \to 0$ as $R \to +\infty$.

Using the fact that $w$ is a solution of $(P_\infty)$, we get

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla w_+ \nabla w_-\,dz &= \int_{\mathbb{R}^N} f(w_+)w_-\,dz - \int_{\mathbb{R}^N} V_\infty w_+ w_-\,dz\\ &= \int_{\mathbb{R}^N} f(w_-)w_+\,dz - \int_{\mathbb{R}^N} V_\infty w_- w_+\,dz,\end{aligned}$$

and so

$$\begin{aligned}&\lim_{s\to 1}\int_{\mathbb{R}^N}\left[\nabla w_+ \nabla w_- + s^2 V_\infty w_+ w_- - s^2\left(\frac{f(w_+)w_- + f(w_-)w_+}{2}\right)\right]dz\\ &= \int_{\mathbb{R}^N}\left[\nabla w_+ \nabla w_- + V_\infty w_+ w_- - \left(\frac{f(w_+)w_- + f(w_-)w_+}{2}\right)\right]dz = 0,\end{aligned}$$

for any $R \geq 1$. Since $\int_{\mathbb{R}^N}[f(w_+)w_- + f(w_-)w_+]\,dz > 0$, there exists $0 < \delta_2 < 1/4$ sufficiently small such that

$$\frac{3s^2}{2}\int_{\mathbb{R}^N}\left(\frac{f(w_+)w_- + f(w_-)w_+}{2}\right)dz \geq \int_{\mathbb{R}^N}\left[\nabla w_+ \nabla w_- + s^2 V_\infty w_+ w_-\right]dz, \tag{2.4.31}$$

for every $s \in [1-\delta_2, 1+\delta_2]$ and $R \geq 1$.

From inequality (2.4.31), we obtain a constant $C_0 > 0$ such that

$$\begin{aligned}I_2 &= s^{N-2}\int_{\mathbb{R}^N}\left[\nabla w_+ \nabla w_- + s^2 V_\infty w_+ w_- - s^2 f(w_+)w_- - s^2 f(w_-)w_+\right]dz\\ &\leq -\frac{s^N}{4}\int_{\mathbb{R}^N}[f(w_+)w_- + f(w_-)w_+]\,dz = -\frac{s^N \varepsilon_R}{2} \leq -C_0\,\varepsilon_R,\end{aligned} \tag{2.4.32}$$

for every $s \in [1-\delta_2, 1+\delta_2]$ and $R \geq 1$.

By Corollary 2.4.5, it follows that

$$\begin{aligned}I_3 &\leq s^N\int_{\mathbb{R}^N}|F(w_+ + w_-) - F(w_+) - F(w_-) - f(w_+)w_- - f(w_-)w_+|\,dz\\ &= o(\varepsilon_R),\end{aligned} \tag{2.4.33}$$

for every $s \in (0, L]$ and $R \geq 1$. Hence, taking $\delta := \min\{\delta_1, \delta_2\}$, by previous estimates

(2.4.29), (2.4.30), (2.4.32) and (2.4.33), there exists $R_1 \geq 1$ sufficiently large such that

$$I_V\left(U^R\left(\frac{\cdot}{s}\right)\right) \leq 2I_\infty\left(w\left(\frac{\cdot}{s}\right)\right) - C_0\,\varepsilon_R + o(\varepsilon_R) < 2p_\infty, \tag{2.4.34}$$

for every $s \in [1-\delta, 1+\delta]$ and $R \geq R_1$.

Next, note that the first bound given by (2.1.3), the pointwise limit $\lim_{R\to\infty} U^R(x) = 0$ and Lebesgue dominated convergence theorem imply that

$$\frac{s^N}{2}\int_{\mathbb{R}^N} |V(sz) - V_\infty|(w_+ + w_-)^2\,dz \to 0, \qquad \text{as } R \to +\infty, \tag{2.4.35}$$

uniformly in $s \in (0, L]$. Also, by Lemmas 2.4.3, 2.4.4 and 2.4.6, we have

$$s^{N-2}\int_{\mathbb{R}^N}\left[\nabla w_+ \nabla w_- + s^2 V_\infty w_+ w_- - s^2 f(w_+)w_- - s^2 f(w_-)w_+\right]dz \to 0 \tag{2.4.36}$$

and, by Corollary 2.4.5,

$$s^N\int_{\mathbb{R}^N}|F(w_+ + w_-) - F(w_+) - F(w_-) - f(w_+)w_- - f(w_-)w_+|\,dz \to 0 \tag{2.4.37}$$

as $R \to +\infty$, uniformly in $s \in (0, L]$. Hence, by (2.4.30), (2.4.32) and (2.4.33), applying (2.4.35), (2.4.36) and (2.4.37), it holds

$$\left|I_V\left(U^R\left(\frac{\cdot}{s}\right)\right) - 2I_\infty\left(w\left(\frac{\cdot}{s}\right)\right)\right| \to 0 \qquad \text{as } R \to +\infty, \tag{2.4.38}$$

uniformly in $s \in (0, L]$. From (2.4.38), and recalling that the map $t \mapsto I_\infty\left(w\left(\frac{\cdot}{t}\right)\right)$ is strictly increasing in $(0, 1]$ and strictly decreasing in $[1, \infty)$ and $I_\infty(w) = p_\infty$, it follows that $I_\infty\left(w\left(\frac{\cdot}{t}\right)\right) < I_\infty(w)$ for all $t \neq 1$, and so there exists $R_2 \geq R_1$ such that

$$I_V\left(U^R\left(\frac{\cdot}{s}\right)\right) < 2p_\infty, \qquad \text{for all } s \in (0, 1-\delta) \cup (1+\delta, L] \text{ and all } R \geq R_2. \tag{2.4.39}$$

Thus, from (2.4.34) and (2.4.39), we conclude that

$$I_V\left(U^R\left(\frac{\cdot}{s}\right)\right) < 2p_\infty, \qquad \text{for all } s \in (0, L] \text{ and all } R \geq R_2. \tag{2.4.40}$$

Finally, we will prove that (2.4.28) occurs. We claim that $I_\infty\left(w\left(\frac{\cdot}{L}\right)\right) < 0$. Indeed, as $w$ is a solution of problem $(P_\infty)$, it follows that

$$\int_{\mathbb{R}^N}\left(F(w) - \frac{V_\infty}{2}w^2\right)dx = \frac{N-2}{2N}\int_{\mathbb{R}^N}|\nabla w|^2\,dx > 0,$$

and so, for $L > 2$ large enough, we obtain

$$I_1\left(w\left(\frac{\cdot}{L}\right)\right) = \frac{L^{N-2}}{2}\left[\int_{\mathbb{R}^N} |\nabla w|^2 dx - 2L^2\int_{\mathbb{R}^N}\left(F(w) - \frac{V_1}{2}w^2\right)dx\right]$$
$$= \frac{L^{N-2}}{2}\left[\int_{\mathbb{R}^N} |\nabla w|^2 dx - \frac{L^2(N-2)}{N}\int_{\mathbb{R}^N} |\nabla w|^2 dx\right] < 0. \qquad (2.4.41)$$

Thus, using that $I_1\left(w\left(\frac{\cdot}{L}\right)\right) < 0$ and (2.4.38), there exists $R_3 \geq 1$ such that

$$I_V\left(U^R\left(\frac{\cdot}{L}\right)\right) < I_1\left(w\left(\frac{\cdot}{L}\right)\right) < 0, \qquad \text{for all } R \geq R_3. \qquad (2.4.42)$$

Therefore, taking $R_4 := \max\{R_2, R_3\}$, we obtain from (2.4.40) and (2.4.42) that

$$I_V\left(U^R\left(\frac{\cdot}{s}\right)\right) < 2p_1, \qquad \text{for all } s \in (0, L] \text{ and all } R \geq R_4$$

and

$$I_V\left(U^R\left(\frac{\cdot}{L}\right)\right) < 0, \qquad \text{for all } R \geq R_4,$$

concluding the proof of the proposition. □

**Lemma 2.4.13.** *Assume that $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ hold true and let $w$ be a ground state solution of $(P_1)$, which is positive, radially symmetric and decreasing in the radial direction. Then, there exists a path $\gamma \in C\left([0,1], H^1_G(\mathbb{R}^N)\right)$, with $\gamma(0) = 0$ and $I_1(\gamma(1)) < 0$, such that*

$$w \in \gamma([0,1]), \quad \max_{t\in[0,1]} I_1(\gamma(t)) = I_1(w) = m.$$

*Proof.* By hypothesis, for any $g \in G$ and $x \in \mathbb{R}^N$, we have $w(gx) = w(|gx|) = w(|x|) = w(x)$, and so $w \in H^1_G(\mathbb{R}^N)$. Moreover, $w$ is a ground state solution to $(P_1)$, which is positive, radially symmetric and decreasing in the radial direction. Then, we can define a continuous path $\alpha : [0, \infty) \to H^1_G(\mathbb{R}^N)$, putting $\alpha(t) := w(\cdot/t)$ for $t > 0$ and $\alpha(0) := 0$. Thus, by construction, it follows that $I_1(\alpha(0)) = 0$ and, for every $t > 0$, we have

$$I_1(\alpha(t)) = I_1(w(\cdot/t)) = \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla w|^2 dx - t^N\int_{\mathbb{R}^N} G_1(w)dx,$$

where $G_1(w) := F(w) - V_1 \dfrac{w^2}{2}$. Therefore, deriving the above expression, we obtain

$$\begin{aligned}\frac{d}{dt} I_1(\alpha(t)) &= \frac{N-2}{2} t^{N-3} \int_{\mathbb{R}^N} |\nabla w|^2 dx - N t^{N-1} \int_{\mathbb{R}^N} G_1(w) dx \\ &= t^{N-3} \left[ \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla w|^2 dx - N t^2 \int_{\mathbb{R}^N} G_1(w)\, dx \right].\end{aligned}$$

Since $w$ is a solution of $(P_1)$, then $w$ satisfies the Pohozaev identity

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla w|^2 dx = N \int_{\mathbb{R}^N} G_1(w) dx,$$

and thus,

$$\frac{d}{dt} I_1(\alpha(t)) = N t^{N-3} (1 - t^2) \int_{\mathbb{R}^N} G_1(w) dx.$$

As $N t^{N-3} \int_{\mathbb{R}^N} G_1(w) dx > 0$, for every $t > 0$, it follows that the map $t \mapsto I_1(\alpha(t))$ reaches the maximum value at $t = 1$. Choosing $T > 0$ sufficiently large, we have

$$\max_{0 \le t \le T} I_1(\alpha(t)) = I_1(\alpha(1)) = I_1(w) = m \quad \text{and} \quad I_1(\alpha(T)) < 0.$$

Considering the path $\gamma : [0,1] \to H^1_G(\mathbb{R}^N)$, defined by $\gamma(t) := \alpha(tT)$, the result follows. □

**Lemma 2.4.14.** *Assume that $(\widetilde{V}_1)$–$(\widetilde{V}_3)$ and $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ hold true. Then, the functional $I_V$ satisfies the geometrical properties of the mountain pass theorem.*

*Proof.* Note that $I_G(0) = 0$. Moreover, for every $u \in H^1_G(\mathbb{R}^N)$, by $(\widetilde{V}_1)$ and (2.1.2), taking $\varepsilon = \dfrac{\inf_{x \in \mathbb{R}^N} V(x)}{2}$, we get $C_\varepsilon > 0$ such that

$$\begin{aligned} I_V(u) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x) u^2\right) dx - \int_{\mathbb{R}^N} F(u) dx \\ &\ge \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x) u^2\right) dx - \int_{\mathbb{R}^N} \left[\frac{\varepsilon}{2} u^2 + C_\varepsilon |u|^{2^*}\right] dx \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + (V(x) - \varepsilon) u^2\right) dx - \int_{\mathbb{R}^N} C_\varepsilon |u|^{2^*} dx \\ &\ge \frac{1}{4} \|u\|_V^2 - C_\varepsilon \|u\|_{2^*}^{2^*}. \end{aligned}$$

By the continuity of the embedding $H^1_G(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$, there exists a constant $C_1 > 0$ such that

$$I_V(u) \ge \frac{1}{4} \|u\|_V^2 - C_1 \|u\|_V^{2^*} = \left(\frac{1}{4} - C_1 \|u\|_V^{2^*-2}\right) \|u\|_V^2.$$

Since $2 \quad 2 > 0$, taking $\varrho := \min\left\{1, \left(\frac{1}{8C_1}\right)^{1/(2 \quad 2)}\right\} > 0$, we have: if $u \in H^1_G(\mathbb{R}^N) \setminus \{0\}$, with $\|u\|_V = \varrho$, then

$$I_V(u) \quad \left(\frac{1}{4} \quad C_1\|u\|_V^{\quad 2}\right)\|u\|_V^2 \quad \frac{\|u\|_V^2}{8} = \frac{\varrho^2}{8} > 0.$$

On the other hand, if $w$ is a ground state solution to $(P_1)$, positive, radially symmetric and decreasing in the radial direction, then for any $g \in G$ and $x \in \mathbb{R}^N$, we have $w(gx) = w(|gx|) = w(|x|) = w(x)$, and so $w \in H^1_G(\mathbb{R}^N)$. Furthermore, using the same idea applied by Jeanjean-Tanaka in [24], see also Lemma 2.4.13, take $L > 2$ large enough and define $\gamma : [0, L] \to H^1_G(\mathbb{R}^N)$ by $\gamma(0) = 0$ and $\gamma(t) = w(\cdot/t)$, for $t \in (0, L]$. We may observe that $\gamma$ is a path that satisfies

$$\gamma(0) = 0, \quad \gamma(1) = w, \quad I_1(\gamma(L)) < 0, \tag{2.4.43}$$

$$I_1(\gamma(t)) < I_1(w), \quad \text{for all} \quad t \neq 1. \tag{2.4.44}$$

Fix $L > 2$ large enough such that (2.4.43) holds. Arguing as in Proposition 2.4.12, see expression (2.4.38), it follows that

$$\left|I_V\left(U^R\left(\frac{\cdot}{t}\right)\right) \quad 2I_1\left(w\left(\frac{\cdot}{t}\right)\right)\right| \to 0 \qquad \text{as} \quad R \to +\infty,$$

uniformly in $t \in (0, L]$. Using that $I_1\left(w\left(\frac{\cdot}{L}\right)\right) = I_1(\gamma(L)) < 0$, we conclude that

$$I_V\left(U^R\left(\frac{\cdot}{L}\right)\right) < 0,$$

for $R \quad 1$ sufficiently large. Therefore, the functional $I_V$ satisfies the geometrical properties of the mountain pass theorem, concluding the proof. □

**Proof of Theorem 2.1.1.** Let us apply the mountain pass theorem of Ambrosetti-Rabinowitz [3]. We define a mountain pass level for $I_V$ on $H^1_G(\mathbb{R}^N)$ by

$$c_V := \inf_{\gamma \in \ _V} \max_{0 \quad t \quad 1} I_V(\gamma(t)), \qquad _V := \left\{\gamma \in C([0,1], H^1_G(\mathbb{R}^N)) : \gamma(0) = 0, I_V(\gamma(1)) < 0\right\}.$$

Since $I_V$ satisfies the geometrical properties of the mountain pass theorem, then $c_V > 0$ and there exists a Cerami sequence $(u_n) \quad H^1_G(\mathbb{R}^N)$ for $I_V$ at level $c_V$. By Lemma 2.3.1, $(u_n)$ contains a bounded subsequence, still denoted by $(u_n)$. As in the proof of Proposition 2.4.12, more precisely, from (2.4.41), we may choose $L > 2$ large enough such that $I_\infty\left(w\left(\frac{\cdot}{L}\right)\right) < 0$. Next, consider the following path:

$$\gamma(t) = \begin{cases} U^R\left(\frac{\cdot}{Lt}\right), & \text{if } t \in (0,1], \\ 0, & \text{if } t = 0. \end{cases}$$

Note that $\gamma \in \Gamma_V$ and, also by Proposition 2.4.12, we may choose $R \geq 1$ sufficiently large such that

$$I_V(\gamma(t)) < 2p_\infty, \quad \text{for all } t \in [0,1],$$

and so $c_V < 2p_\infty$. Hence, recalling that $c_V > 0$ and $\ell(G)p_\infty \geq 2p_\infty$, we have

$$0 < c_V < 2p_\infty \leq \ell(G)p_\infty.$$

From Corollary 2.3.4, there exists $u \in H^1_G(\mathbb{R}^N)\setminus\{0\}$ such that $u_n \to u$ strongly in $H^1_G(\mathbb{R}^N)$, i.e. $u$ is a nontrivial critical point of $I_V$ restricted to $H^1_G(\mathbb{R}^N)$ such that $I_V(u) = c_V$. Therefore, it follows that $u$ is a nontrivial solution of problem $(P_G)$. Using the maximum principle we conclude that $u$ is positive, proving the theorem.

Note that as in Remark 1.4.16 in Chapter 1, assuming that the potential $V$ is invariant under a group action $G \subset O(N)$ and under assumptions $(\widetilde{V}_1)$–$(\widetilde{V}_4)$ and $(\widetilde{f}_1)$–$(\widetilde{f}_4)$, we can prove that Theorem 2.1.1 also holds, for $\ell(G) \in (2, \infty)$ and $d_G \in (0,2]$.

As before, to prove this, we took as basis the following papers by Hirata [22, p. 182–190] and [23, p. 3180–3188]. Unlike Hirata's work, we are not assuming that $f(s)/s$ is increasing and so, to prove the necessary estimates, we will use [1, Lemma 2.2]. We define

$$U^R := \sum_{j=1}^{\ell(G)} w(\cdot - Re_j), \tag{2.4.45}$$

where $e_1, \ldots, e_{\ell(G)} \in \mathbb{S}^{N-1}$ and $d_G \in (0,2]$, as in (0.0.1) and (0.0.2). Moreover, for $i, j = 1, \ldots, \ell(G)$, we denote

$$\varepsilon_R := \sum_{i \neq j}^{\ell(G)} \int_{\mathbb{R}^N} f(w(x - Re_i))w(x - Re_j)dx. \tag{2.4.46}$$

Following the same ideas applied when we assume that $\ell(G) = 2$ and $d_G = 2$, we get $C_1, C_2 > 0$ such that

$$C_1 R^{-\frac{N-1}{2}} e^{-d_G\sqrt{V_\infty}R} \leq \varepsilon_R \leq C_2 R^{-\frac{N-1}{2}} e^{-d_G\sqrt{V_\infty}R}.$$

Take $L > 2$ large enough and note that

$$I_V\left(U^R\left(\frac{}{s}\right)\right) = I_V\left(\sum_{j=1}^{\ell(G)} w\left(\frac{}{s} \quad Re_j\right)\right) \quad I_1\left(\sum_{j=1}^{\ell(G)} w\left(\frac{}{s} \quad Re_j\right)\right)$$
$$+ I_1\left(\sum_{j=1}^{\ell(G)} w\left(\frac{}{s} \quad Re_j\right)\right) \quad \sum_{j=1}^{\ell(G)} I_1\left(w\left(\frac{}{s} \quad Re_j\right)\right)$$
$$+ \ell(G) I_1\left(w\left(\frac{}{s}\right)\right).$$

Set

$$(I) := I_V\left(\sum_{j=1}^{\ell(G)} w\left(\frac{}{s} \quad Re_j\right)\right) \quad I_1\left(\sum_{j=1}^{\ell(G)} w\left(\frac{}{s} \quad Re_j\right)\right),$$
$$(II) := I_1\left(\sum_{j=1}^{\ell(G)} w\left(\frac{}{s} \quad Re_j\right)\right) \quad \sum_{j=1}^{\ell(G)} I_1\left(w\left(\frac{}{s} \quad Re_j\right)\right).$$

Observe that

$$(I) = I_V\left(\sum_{j=1}^{\ell(G)} w\left(\frac{}{s} \quad Re_j\right)\right) \quad I_1\left(\sum_{j=1}^{\ell(G)} w\left(\frac{}{s} \quad Re_j\right)\right)$$
$$= \frac{1}{2}\int_{\mathbb{R}^N} [V(x) \quad V_1]\left(\sum_{j=1}^{\ell(G)} w\left(\frac{x}{s} \quad Re_j\right)\right)^2 dx$$
$$= \frac{s^N}{2}\int_{\mathbb{R}^N} [V(sz) \quad V_1]\left(\sum_{j=1}^{\ell(G)} w(z \quad Re_j)\right)^2 dz$$
$$\frac{A_0 s^N}{2}\int_{\mathbb{R}^N} e^{\quad ksjzj}\left(\sum_{j=1}^{\ell(G)} w(z \quad Re_j)\right)^2 dz$$
$$\frac{A_0 s^N}{2}\int_{\mathbb{R}^N} e^{\quad ksjzj}\sum_{j=1}^{\ell(G)} C\,(w(z \quad Re_j))^2\, dz$$
$$C s^N \sum_{j=1}^{\ell(G)}\int_{\mathbb{R}^N} e^{\quad ksjzj}(w(z \quad Re_j))^2\, dz$$
$$C s^N \sum_{j=1}^{\ell(G)}\int_{\mathbb{R}^N} e^{\quad ksjzj}(1 + jz \quad Re_j j)^{\quad N+1} e^{\quad 2^p\overline{V_1} jz \quad Re_j j} dz.$$

As $k > d_G\sqrt{V_1}$, there exists $0 < \delta_1 < 1/4$ such that $ks > d_G\sqrt{V_1}$ for all $s \geq 1 - \delta_1$. So, following the same ideas applied to prove (2.4.22), we arrive that

$$\begin{aligned}
&\sum_{j=1}^{\ell(G)} \int_{\mathbb{R}^N} e^{-ks|z|}(1+|z - Re_j|)^{-N+1} e^{-2\sqrt{V_1}|z-Re_j|}dz \\
&\quad\leq \sum_{j=1}^{\ell(G)} \int_{\mathbb{R}^N} e^{-d_G\sqrt{V_1}|z|}(1+|z - Re_j|)^{-N+1} e^{-2\sqrt{V_1}|z-Re_j|}dz \\
&\quad\leq CR^{-(N+1)}e^{-d_G\sqrt{V_1}R},
\end{aligned} \tag{2.4.47}$$

for all $s \geq 1 - \delta_1$. It follows from (2.4.47) that, for any $s \geq 1 - \delta_1$ and $R \geq 1$,

$$(I) \leq CR^{-(N+1)}e^{-d_G\sqrt{V_1}R} = o(\varepsilon_R). \tag{2.4.48}$$

Next, we will estimate $(II)$. Denoting $w_j := w(\cdot - Re_j)$ for $j = 1, \ldots, \ell(G)$, we have

$$\begin{aligned}
(II) &= I_1\left(\sum_{j=1}^{\ell(G)} w\left(\frac{\cdot}{s} - Re_j\right)\right) - \sum_{j=1}^{\ell(G)} I_1\left(w\left(\frac{\cdot}{s} - Re_j\right)\right) \\
&= \frac{s^{N-2}}{2}\int_{\mathbb{R}^N}\left|\sum_{j=1}^{\ell(G)} \nabla w_j\right|^2 dx + \frac{s^N}{2}\int_{\mathbb{R}^N} V_1\left(\sum_{j=1}^{\ell(G)} w_j\right)^2 dx \\
&\quad - s^N\int_{\mathbb{R}^N} F\left(\sum_{j=1}^{\ell(G)} w_j\right)dx \\
&\quad - \frac{s^{N-2}}{2}\sum_{j=1}^{\ell(G)}\int_{\mathbb{R}^N}|\nabla w_j|^2dx - \frac{s^N}{2}\sum_{j=1}^{\ell(G)}\int_{\mathbb{R}^N} V_1 w_j^2dx \\
&\quad + s^N\sum_{j=1}^{\ell(G)}\int_{\mathbb{R}^N} F(w_j)dx \\
&= \frac{s^{N-2}}{2}\sum_{i\neq j}^{\ell(G)}\int_{\mathbb{R}^N}\left[\nabla w_i \nabla w_j + s^2 V_1 w_i w_j\right]dx \\
&\quad - s^N\int_{\mathbb{R}^N}\left[F\left(\sum_{j=1}^{\ell(G)} w_j\right) - \sum_{j=1}^{\ell(G)} F(w_j) - \sum_{i\neq j}^{\ell(G)} f(w_i)w_j\right]dx \\
&\quad - s^N\int_{\mathbb{R}^N}\sum_{i\neq j}^{\ell(G)} f(w_i)w_j\,dx.
\end{aligned}$$

Note that $(II) \le (II.1) + (II.2)$, where

$$(II.1) := \frac{s^{N-2}}{2}\sum_{i\neq j}^{\ell(G)}\int_{\mathbb{R}^N}\left[\nabla w_i \nabla w_j + s^2 V_1 w_i w_j - 2s^2 f(w_i)w_j\right]dx,$$

$$(II.2) := s^N\int_{\mathbb{R}^N}\left|F\left(\sum_{j=1}^{\ell(G)} w_j\right) - \sum_{j=1}^{\ell(G)} F(w_j) - \sum_{i\neq j}^{\ell(G)} f(w_i)w_j\right|dx.$$

Using that $w$ is a solution of $(P_1)$, arguing as in the proof of Proposition 2.4.12, we obtain constants $0 < \delta_2 < 1/4$ and $C_0 > 0$ such that

$$(II.1) = \frac{s^{N-2}}{2}\sum_{i\neq j}^{\ell(G)}\int_{\mathbb{R}^N}\left[\nabla w_i \nabla w_j + s^2 V_1 w_i w_j - 2s^2 f(w_i)w_j\right]dx \le -C_0\,\varepsilon_R, \qquad (2.4.49)$$

for every $s \in [1-\delta_2, 1+\delta_2]$ and $R \ge 1$. On the other hand, using [1, Lemma 2.2], we obtain $\alpha \in (1/2, 1]$ such that

$$\int_{\mathbb{R}^N}\left|F\left(\sum_{j=1}^{\ell(G)} w_j\right) - \sum_{j=1}^{\ell(G)} F(w_j) - \sum_{i\neq j}^{\ell(G)} f(w_i)w_j\right| \le C\int_{\mathbb{R}^N}\left(\sum_{i<j}^{\ell(G)}|w_iw_j|^{2\alpha} + \sum_{i<j<l}^{\ell(G)}|w_iw_jw_l|^{2/3}\right)$$

Again, following the same ideas applied when we assume that $\ell(G) = 2$ and $d_G = 2$, for $i, j \in \{1, \dots, \ell(G)\}$ with $i \neq j$, since $\alpha > 1/2$ we get

$$\int_{\mathbb{R}^N}(w_iw_j)^{2\alpha}\,dx \le CR^{-\alpha(N-1)}e^{-d_G\sqrt{V_1}} = o(\varepsilon_R). \qquad (2.4.50)$$

Next, we fix $\rho \in (0, d_G/3)$ and consider $\epsilon \in (0, \sqrt{V_1})$ sufficiently small. Note that, for all $z \in B_{\rho R}(0)$, for $i, j \in \{1, \dots, \ell(G)\}$ with $i \neq j$, it holds

$$1 + |z + R(e_i - e_j)| \ge 1 + d_G R - \rho R > \frac{2}{3}d_G R. \qquad (2.4.51)$$

So, using (2.4.51), (2.1.1) and second inequality in Lemma 2.4.1, for $i, j, l \in \{1, \dots, \ell(G)\}$

with $i < j < l$, making a change of variables, we obtain

$$
\begin{aligned}
\int_{B_{\rho R}(Re_i)} |w_i w_j w_l|^{2/3} dx
&\le CR^{-\frac{2}{3}(N-1)} \int_{B_{\rho R}(0)} (1+|z|)^{-\frac{N-1}{3}} e^{-\frac{2}{3}\sqrt{V_1}|z|} e^{-\frac{2}{3}\sqrt{V_1}|z+R(e_i-e_j)|} e^{-\frac{2}{3}\sqrt{V_1}|z+R(e_i-e_l)|} dz \\
&\le CR^{-\frac{2}{3}(N-1)} \int_{B_{\rho R}(0)} e^{-\frac{2}{3}(\sqrt{V_1}-\epsilon)|z|} e^{-\frac{2}{3}(\sqrt{V_1}-\epsilon)|z+R(e_i-e_j)|} e^{-\frac{2}{3}\sqrt{V_1}|z+R(e_i-e_l)|} dz \\
&\le CR^{-\frac{2}{3}(N-1)} e^{-\frac{1}{3}(\sqrt{V_1}-\epsilon)(|e_i-e_j|+|e_i-e_l|+|e_j-e_l|)R} \\
&\le CR^{-\frac{2}{3}(N-1)} e^{-d_G(\sqrt{V_1}-\epsilon)R}. \qquad (2.4.52)
\end{aligned}
$$

Similarly, we obtain

$$
\int_{B_{\rho R}(Re_j)} |w_i w_j w_l|^{2/3} dx \le CR^{-\frac{2}{3}(N-1)} e^{-d_G(\sqrt{V_1}-\epsilon)R} \qquad (2.4.53)
$$

and

$$
\int_{B_{\rho R}(Re_l)} |w_i w_j w_l|^{2/3} dx \le CR^{-\frac{2}{3}(N-1)} e^{-d_G(\sqrt{V_1}-\epsilon)R}. \qquad (2.4.54)
$$

Note that the balls $B_{\rho R}(Re_i)$, $B_{\rho R}(Re_j)$ and $B_{\rho R}(Re_l)$ are two by two disjoint. So, taking $\Omega := B_{\rho R}(Re_i) \cup B_{\rho R}(Re_j) \cup B_{\rho R}(Re_l)$, it follows from (2.4.52), (2.4.53) and (2.4.54) that

$$
\int_{\Omega} |w_i w_j w_l|^{2/3} dx \le CR^{-\frac{2}{3}(N-1)} e^{-d_G(\sqrt{V_1}-\epsilon)R}. \qquad (2.4.55)
$$

On the other hand, using (2.1.1) and second inequality in Lemma 2.4.1 again, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N \setminus \Omega} |w_i w_j w_l|^{2/3} dx
&\le CR^{-(N-1)} \int_{\mathbb{R}^N \setminus \Omega} e^{-\frac{2}{3}\sqrt{V_1}|x-Re_i|} e^{-\frac{2}{3}\sqrt{V_1}|x-Re_j|} e^{-\frac{2}{3}\sqrt{V_1}|x-Re_l|} dx \\
&\le CR^{-(N-1)} \int_{\mathbb{R}^N \setminus \Omega} e^{-\frac{2}{3}(\sqrt{V_1}-\epsilon)|x-Re_i|} e^{-\frac{2}{3}(\sqrt{V_1}-\epsilon)|x-Re_j|} e^{-\frac{2}{3}\sqrt{V_1}|x-Re_l|} dx \\
&\le CR^{-(N-1)} e^{-\frac{1}{3}(\sqrt{V_1}-\epsilon)(|e_i-e_j|+|e_i-e_l|+|e_j-e_l|)R} \\
&\le CR^{-(N-1)} e^{-d_G(\sqrt{V_1}-\epsilon)R}. \qquad (2.4.56)
\end{aligned}
$$

It follows from (2.4.55) and (2.4.56) that

$$
\int_{\mathbb{R}^N} |w_i w_j w_l|^{2/3} dx \le CR^{-\frac{2}{3}(N-1)} e^{-d_G(\sqrt{V_1}-\epsilon)R}
$$

and so, making $\epsilon \to 0$, we conclude that

$$\int_{\mathbb{R}^N} |w_i w_j w_l|^{2/3} dx \leq C R^{\frac{2}{3}(N-1)} e^{-d_G \sqrt{V_\infty} R} = o(\varepsilon_R). \tag{2.4.57}$$

Since the map $t \mapsto I_\infty\left(w\left(\frac{\cdot}{t}\right)\right)$ is strictly increasing in $(0, 1]$ and strictly decreasing in $[1, \infty)$ and $I_\infty(w) = p_\infty$, it follows that $I_\infty\left(w\left(\frac{\cdot}{t}\right)\right) < p_\infty$ for all $t \neq 1$. So, taking $\delta := \min\{\delta_1, \delta_2\}$, from (2.4.48), (2.4.50) and (2.4.57), we get $R_1 \geq 1$ such that

$$I_V\left(U^R\left(\frac{\cdot}{s}\right)\right) \leq \ell(G) I_\infty\left(w\left(\frac{\cdot}{s}\right)\right) + o(\varepsilon_R) - C_0\, \varepsilon_R < \ell(G) p_\infty, \tag{2.4.58}$$

for every $s \in [1-\delta, 1+\delta]$ and $R \geq R_1$. Again, arguing as in the proof of Proposition 2.4.12, we obtain $R_2, R_3 \geq 1$ such that

$$I_V\left(U^R\left(\frac{\cdot}{s}\right)\right) < \ell(G) p_\infty, \qquad \text{for all } s \in (0, 1-\delta) \cup (1+\delta, L] \text{ and all } R \geq R_2 \tag{2.4.59}$$

and

$$I_V\left(U^R\left(\frac{\cdot}{L}\right)\right) < I_\infty\left(w\left(\frac{\cdot}{L}\right)\right) < 0, \qquad \text{for all } \ R \geq R_3. \tag{2.4.60}$$

Taking $R_4 := \max\{R_1, R_2, R_3\}$, it follows from (2.4.58), (2.4.59) and (2.4.60) that

$$I_V\left(U^R\left(\frac{\cdot}{s}\right)\right) < \ell(G) p_\infty, \qquad \text{for all } \ s \in (0, L] \ \text{ and all } \ R \geq R_4$$

and

$$I_V\left(U^R\left(\frac{\cdot}{L}\right)\right) < 0, \qquad \text{for all } \ R \geq R_4.$$

From the above inequalities and as $I_V$ satisfies the geometrical properties of the mountain pass theorem, the proof of the statement follows from Lemma 2.3.1 and Corollary 2.3.4.

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