Periodic averaging theorems for various types of equations

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Abstract

We prove a periodic averaging theorem for generalized ordinary differential equations and show that averaging theorems for ordinary differential equations with impulses and for dynamic equations on time scales follow easily from this general theorem. We also present a periodic averaging theorem for a large class of retarded equations.

Keywords: Periodic averaging, generalized ordinary differential equations, ordinary differential equations with impulses, dynamic equations on time scales, retarded functional differential equations, retarded difference equations

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1 Introduction

Classical averaging theorems for ordinary differential equations are concerned with the initial-value problem

$$x'(t) = \varepsilon f(t, x(t)) + \varepsilon^2 g(t, x(t), \varepsilon), \quad x(t_0) = x_0,$$

where $\varepsilon > 0$ is a small parameter. Assume that f is T-periodic in the first argument. Then, according to the periodic averaging theorem, we can obtain a good approximation of this initial-value problem by neglecting the ε^2-term and taking the average of f with respect to t. In other words, we consider the autonomous differential equation

$$y'(t) = \varepsilon f_0(y(t)), \quad y(t_0) = x_0,$$

where

$$f_0(y) = \frac{1}{T}\int_{t_0}^{t_0+T} f(t, y)\,\mathrm{d}t.$$

Different proofs of the periodic averaging theorem can be found e. g. in [5], [6], or [10]; these monographs also include many applications.

In this paper, we derive a periodic averaging theorem for generalized ordinary differential equations, which were introduced by Jaroslav Kurzweil in 1957 (see [4]). We then show that the classical averaging theorem (even with the possibility of including impulses) is a simple corollary of our theorem. As a second application, we obtain a periodic averaging theorem for dynamic equations on time scales. In the final section, we derive a periodic averaging theorem for a large class of retarded equations.

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2 Generalized ordinary differential equations

We start with a short summary of Kurzweil integration, which plays a crucial role in the theory of generalized ordinary differential equations.

A partition of a compact interval $[a,b]$ is a finite collection of point-interval pairs $(\tau_i,[s_{i-1},s_i])_{i=1}^m$, where $a=s_0\le s_1\le\ldots\le s_m=b$ and $\tau_i\in[s_{i-1},s_i]$, $i=1,2,\ldots,m$. Given a function $\delta:[a,b]\to\mathbb{R}^+$, we say that the partition $(\tau_i,[s_{i-1},s_i])_{i=1}^m$ is δ-fine whenever $[s_{i-1},s_i]\subset(\tau_i-\delta(\tau_i),\tau_i+\delta(\tau_i))$ for every $i=1,2,\ldots,m$.

A function $U:[a,b]\times[a,b]\to\mathbb{R}^n$ is called Kurzweil integrable on $[a,b]$, if there is an $I\in\mathbb{R}^n$ such that given an $\varepsilon>0$, there is a $\delta:[a,b]\to\mathbb{R}^+$ such that for every δ-fine partition $(\tau_i,[s_{i-1},s_i])_{i=1}^m$ of $[a,b]$, we have

$$\left\|\sum_{i=1}^m[U(\tau_i,s_i)-U(\tau_i,s_{i-1})]-I\right\|<\varepsilon.$$

In this case, we write $I=\int_a^b DU(\tau,t)$. This definition generalizes the well-known Henstock-Kurzweil integral of a function $f:[a,b]\to\mathbb{R}^n$, which is obtained by taking $U(\tau,t)=f(\tau)t$. Another important special case is the Kurzweil-Stieltjes integral of a function $f:[a,b]\to\mathbb{R}^n$ with respect to a function $g:[a,b]\to\mathbb{R}$, which corresponds to the choice $U(\tau,t)=f(\tau)g(t)$ and will be denoted by $\int_a^b f(s)\,\mathrm{d}g(s)$.

Consider a set $B\subset\mathbb{R}^n$, an interval $I\subset\mathbb{R}$ and a function $F:B\times I\to\mathbb{R}^n$. A function $x:I\to B$ is called a solution of the generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=DF(x,t),$$

whenever

$$x(b)-x(a)=\int_a^b DF(x(\tau),t)$$

for every a, $b\in I$.

A basic source in the theory of generalized ordinary differential equations is the book [7]. It is known that an ordinary differential equation $x'(t)=f(x(t),t)$ is equivalent to the generalized equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=DF(x,t),$$

where $F(x,t)=\int_{t_0}^t f(x,s)\,\mathrm{d}s$. However, generalized equations include many other types of equations such as impulsive equations, retarded functional differential equations, or dynamic equations on time scales.

Without loss of generality, we can always assume that the right-hand side of a generalized equation satisfies $F(x,0)=0$ for every $x\in B$. Otherwise, we let

$$\tilde{F}(x,t)=F(x,t)-F(x,0),\quad x\in B,\ t\in I,$$

and consider the equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=D\tilde{F}(x,t).$$

Then we have $\tilde{F}(x,0)=0$ for every $x\in B$, and it follows from the definition of the Kurzweil integral that the new equation has the same set of solutions as the original one.

Definition 1. Let $B\subset\mathbb{R}^n$, $I\subset\mathbb{R}$ an interval on the real line, $\Omega=B\times I$. Assume that $h:I\to\mathbb{R}$ is a nondecreasing function. We say that a function $F:\Omega\to\mathbb{R}^n$ belongs to the class $\mathcal{F}(\Omega,h)$, if it satisfies

$$\|F(x,s_2)-F(x,s_1)\|\le|h(s_2)-h(s_1)|$$

for every $x \in B$ and every s_1, $s_2 \in I$, and

$$\|F(x,s_2) - F(x,s_1) - F(y,s_2) + F(y,s_1)\| \le \|x-y\| \cdot |h(s_2) - h(s_1)|$$

for every x, $y \in B$ and every s_1, $s_2 \in I$.

The following existence theorem is proved in [7, Corollary 1.34]. The inequality follows easily from the definition of the Kurzweil-Stieltjes integral.

Theorem 2. *If $f : [a,b] \to \mathbb{R}^n$ is a regulated function and $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, then the integral $\int_a^b f(s)\,\mathrm{d}g(s)$ exists. Moreover,*

$$\left\| \int_a^b f(s)\,\mathrm{d}g(s) \right\| \le \int_a^b \|f(s)\|\,\mathrm{d}g(s).$$

The following lemma combines two statements from [7] (see Lemma 3.9 and Corollary 3.15).

Lemma 3. *Let $B \subset \mathbb{R}^n$, $\Omega = B \times [a,b]$. Assume that $F : \Omega \to \mathbb{R}^n$ belongs to the class $\mathcal{F}(\Omega, h)$. If $x : [a,b] \to B$ is a regulated function, then the integral $\int_a^b DF(x(\tau),t)$ exists and*

$$\left\| \int_a^b DF(x(\tau),t) \right\| \le h(b) - h(a).$$

We also need the following theorem, which can be found in [7, Lemma 3.12].

Lemma 4. *Let $B \subset \mathbb{R}^n$, $\Omega = B \times [a,b]$. Assume that $F : \Omega \to \mathbb{R}^n$ belongs to the class $\mathcal{F}(\Omega, h)$. Then every solution $x : [\alpha,\beta] \to B$ of the generalized ordinary differential equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t)$$

is a regulated function.

The following inequality will be useful in the proof of the averaging theorem.

Lemma 5. *Let $B \subset \mathbb{R}^n$, $\Omega = B \times [a,b]$. Assume that $F : \Omega \to \mathbb{R}^n$ belongs to the class $\mathcal{F}(\Omega, h)$. If x, $y : [a,b] \to B$ are regulated functions, then*

$$\left\| \int_a^b D[F(x(\tau),t) - F(y(\tau),t)] \right\| \le \int_a^b \|x(t) - y(t)\|\,\mathrm{d}h(t).$$

Proof. The Kurzweil-Stieltjes integral on the right-hand side exists, because h is nondecreasing and $\|x-y\|$ is regulated. For an arbitrary partition $(\tau_i, [s_{i-1}, s_i])_{i=1}^m$ of $[a,b]$, we have

$$\left\| \sum_{i=1}^m (F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1}) - F(y(\tau_i),s_i) + F(y(\tau_i),s_{i-1})) \right\| \le$$

$$\sum_{i=1}^m \|F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1}) - F(y(\tau_i),s_i) + F(y(\tau_i),s_{i-1})\| \le \sum_{i=1}^m \|x(\tau_i) - y(\tau_i)\|\,(h(s_i) - h(s_{i-1})).$$

Now, given an $\varepsilon > 0$, there is a partition $(\tau_i, [s_{i-1}, s_i])_{i=1}^m$ such that

$$\left\| \int_a^b D[F(x(\tau),t) - F(y(\tau),t)] - \sum_{i=1}^m (F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1}) - F(y(\tau_i),s_i) + F(y(\tau_i),s_{i-1})) \right\| < \varepsilon$$

and

$$\left|\int_a^b \|x(t)-y(t)\|\,\mathrm{d}h(t) - \sum_{i=1}^m \|x(\tau_i)-y(\tau_i)\|\,(h(s_i)-h(s_{i-1}))\right| < \varepsilon.$$

It follows that

$$\left\|\int_a^b D[F(x(\tau),t)-F(y(\tau),t)]\right\| \le \left\|\sum_{i=1}^m (F(x(\tau_i),s_i)-F(x(\tau_i),s_{i-1})-F(y(\tau_i),s_i)+F(y(\tau_i),s_{i-1}))\right\|$$

$$+\left\|\int_a^b D[F(x(\tau),t)-F(y(\tau),t)] - \sum_{i=1}^m (F(x(\tau_i),s_i)-F(x(\tau_i),s_{i-1})-F(y(\tau_i),s_i)+F(y(\tau_i),s_{i-1}))\right\|$$

$$< \sum_{i=1}^m \|x(\tau_i)-y(\tau_i)\|\,(h(s_i)-h(s_{i-1}))+\varepsilon$$

$$\le \left|\sum_{i=1}^m \|x(\tau_i)-y(\tau_i)\|\,(h(s_i)-h(s_{i-1})) - \int_a^b \|x(t)-y(t)\|\,\mathrm{d}h(t)\right| + \int_a^b \|x(t)-y(t)\|\,\mathrm{d}h(t)+\varepsilon$$

$$< 2\varepsilon + \int_a^b \|x(t)-y(t)\|\,\mathrm{d}h(t).$$

This proves the statement since ε can be arbitrarily small. □

The following theorem represents an analogue of Gronwall's inequality for the Kurzweil-Stieltjes integral; the proof can be found in [7, Corollary 1.43].

Theorem 6. *Let $h : [a,b] \to [0,\infty)$ be a nondecreasing left-continuous function, $k > 0$, $l \ge 0$. Assume that $\psi : [a,b] \to [0,\infty)$ is bounded and satisfies*

$$\psi(\xi) \le k + l\int_a^\xi \psi(\tau)\,\mathrm{d}h(\tau), \quad \xi \in [a,b].$$

Then $\psi(\xi) \le k e^{l(h(\xi)-h(a))}$ for every $\xi \in [a,b]$.

We proceed to our main result, which is a periodic averaging theorem for generalized ordinary differential equations. The proof is inspired by a proof of the classical averaging theorem for ordinary differential equations given in [6] (see Theorem 2.8.1 and Lemma 2.8.2).

Theorem 7. *Let $B \subset \mathbb{R}^n$, $\Omega = B \times [0,\infty)$, $\varepsilon_0 > 0$, $L > 0$. Consider functions $F : \Omega \to \mathbb{R}^n$ and $G : \Omega \times (0,\varepsilon_0] \to \mathbb{R}^n$ which satisfy the following conditions:*

1. *There exist nondecreasing left-continuous functions $h_1, h_2 : [0,\infty) \to [0,\infty)$ such that F belongs to the class $\mathcal{F}(\Omega, h_1)$, and for every fixed $\varepsilon \in (0,\varepsilon_0]$, the function $(x,t) \mapsto G(x,t,\varepsilon)$ belongs to the class $\mathcal{F}(\Omega, h_2)$.*
2. *$F(x,0) = 0$ and $G(x,0,\varepsilon) = 0$ for every $x \in B$, $\varepsilon \in (0,\varepsilon_0]$.*
3. *There exists a number $T > 0$ and a bounded Lipschitz-continuous function $M : B \to \mathbb{R}^n$ such that $F(x,t+T) - F(x,t) = M(x)$ for every $x \in B$ and $t \in [0,\infty)$.*
4. *There exists a constant $\alpha > 0$ such that $h_1(iT) - h_1((i-1)T) \le \alpha$ for every $i \in \mathbb{N}$.*
5. *There exists a constant $\beta > 0$ such that $|h_2(t)/t| \le \beta$ for every $t \ge L/\varepsilon_0$.*

Let

$$F_0(x) = \frac{F(x,T)}{T}, \quad x \in B.$$

Suppose that for every $\varepsilon \in (0, \varepsilon_0]$*, the initial-value problems*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = D\left[\varepsilon F(x,t) + \varepsilon^2 G(x,t,\varepsilon)\right], \quad x(0) = x_0(\varepsilon),$$

$$y'(t) = \varepsilon F_0(y(t)), \quad y(0) = y_0(\varepsilon)$$

have solutions x_ε*,* $y_\varepsilon : \left[0, \frac{L}{\varepsilon}\right] \to B$*. If there is a constant* $J > 0$ *such that* $\|x_0(\varepsilon) - y_0(\varepsilon)\| \leq J\varepsilon$ *for every* $\varepsilon \in (0, \varepsilon_0]$*, then there exists a constant* $K > 0$ *such that*

$$\|x_\varepsilon(t) - y_\varepsilon(t)\| \leq K\varepsilon$$

for every $\varepsilon \in (0, \varepsilon_0]$ *and* $t \in \left[0, \frac{L}{\varepsilon}\right]$*.*

Proof. If $x \in B$, then

$$\|F_0(x)\| = \left\|\frac{F(x,T)}{T}\right\| = \left\|\frac{F(x,T) - F(x,0)}{T}\right\| = \frac{\|M(x)\|}{T} \leq \frac{m}{T},$$

where m is a bound for M. Let l be a Lipschitz constant for M. The function $H : B \times [0, \infty) \to \mathbb{R}^n$ given by

$$H(x,t) = F_0(x)t = \frac{F(x,T)}{T}t$$

satisfies

$$\|H(x,s_2) - H(x,s_1)\| = \frac{1}{T}\|F(x,T)s_2 - F(x,T)s_1\| = \frac{1}{T}\|F(x,T)\|(s_2 - s_1) \leq \frac{m}{T}(s_2 - s_1)$$

and

$$\|H(x,s_2) - H(x,s_1) - H(y,s_2) + H(y,s_1)\| = \frac{1}{T}\|F(x,T)s_2 - F(x,T)s_1 - F(y,T)s_2 + F(y,T)s_1\|$$

$$= \frac{1}{T}\|F(x,T) - F(y,T)\|(s_2 - s_1) = \frac{1}{T}\|M(x) - M(y)\|(s_2 - s_1) \leq \frac{l}{T}\|x - y\|(s_2 - s_1)$$

for every x, $y \in B$ and every s_1, $s_2 \in [0, \infty)$, $s_1 \leq s_2$. It follows that H belongs to the class $\mathcal{F}(\Omega, h_3)$, where $h_3(t) = (m + l)t/T$. For every $t \in [0, L/\varepsilon]$, we have

$$x_\varepsilon(t) = x_0(\varepsilon) + \varepsilon \int_0^t DF(x_\varepsilon(\tau), s) + \varepsilon^2 \int_0^t DG(x_\varepsilon(\tau), s, \varepsilon),$$

$$y_\varepsilon(t) = y_0(\varepsilon) + \varepsilon \int_0^t F_0(y_\varepsilon(\tau))\,\mathrm{d}\tau = y_\varepsilon(0) + \varepsilon \int_0^t D[F_0(y_\varepsilon(\tau))s].$$

Consequently,

$$\|x_\varepsilon(t) - y_\varepsilon(t)\| = \left\|x_0(\varepsilon) - y_0(\varepsilon) + \varepsilon \int_0^t DF(x_\varepsilon(\tau), s) + \varepsilon^2 \int_0^t DG(x_\varepsilon(\tau), s, \varepsilon) - \varepsilon \int_0^t D[F_0(y_\varepsilon(\tau))s]\right\| \leq$$

$$\leq J\varepsilon + \varepsilon \left\|\int_0^t D[F(x_\varepsilon(\tau), s) - F(y_\varepsilon(\tau), s)]\right\| + \varepsilon \left\|\int_0^t D[F(y_\varepsilon(\tau), s) - F_0(y_\varepsilon(\tau))s]\right\| + \varepsilon^2 \left\|\int_0^t DG(x_\varepsilon(\tau), s, \varepsilon)\right\|.$$

According to Lemma 3, we have the estimate

$$\varepsilon^2 \left\| \int_0^t DG(x_\varepsilon(\tau), s, \varepsilon) \right\| \le \varepsilon^2 (h_2(t) - h_2(0)) \le \varepsilon^2 h_2(L/\varepsilon) = \varepsilon L \frac{h_2(L/\varepsilon)}{L/\varepsilon} \le \varepsilon L \beta.$$

Also, it follows from Lemma 5 that

$$\left\| \int_0^t D[F(x_\varepsilon(\tau), s) - F(y_\varepsilon(\tau), s)] \right\| \le \int_0^t \|x_\varepsilon(s) - y_\varepsilon(s)\| \, \mathrm{d}h_1(s).$$

Let p be the largest integer such that $pT \le t$. Then

$$\int_0^t D[F(y_\varepsilon(\tau), s) - F_0(y_\varepsilon(\tau))s] = \sum_{i=1}^{p} \int_{(i-1)T}^{iT} D[F(y_\varepsilon(\tau), s) - F_0(y_\varepsilon(\tau))s] + \int_{pT}^{t} D[F(y_\varepsilon(\tau), s) - F_0(y_\varepsilon(\tau))s]$$

For every $i \in \{1, \dots, p\}$, we obtain

$$\int_{(i-1)T}^{iT} D[F(y_\varepsilon(\tau), s) - F_0(y_\varepsilon(\tau))s] = \int_{(i-1)T}^{iT} D[F(y_\varepsilon(\tau), s) - F(y_\varepsilon(iT), s)]$$

$$- \int_{(i-1)T}^{iT} D[F_0(y_\varepsilon(\tau))s - F_0(y_\varepsilon(iT))s] + \int_{(i-1)T}^{iT} D[F(y_\varepsilon(iT), s) - F_0(y_\varepsilon(iT))s].$$

We estimate the first integral as follows:

$$\left\| \int_{(i-1)T}^{iT} D[F(y_\varepsilon(\tau), s) - F(y_\varepsilon(iT), s)] \right\| \le \int_{(i-1)T}^{iT} \|y_\varepsilon(s) - y_\varepsilon(iT)\| \, \mathrm{d}h_1(s)$$

Since y_ε satisfies $y'_\varepsilon(t) = \varepsilon F_0(y_\varepsilon(t))$, the mean value theorem gives

$$\|y_\varepsilon(s) - y_\varepsilon(iT)\| \le \varepsilon \frac{m}{T}(iT - s) \le \varepsilon m, \quad s \in [(i-1)T, iT],$$

and consequently

$$\int_{(i-1)T}^{iT} \|y_\varepsilon(s) - y_\varepsilon(iT)\| \, \mathrm{d}h_1(s) \le \varepsilon m (h_1(iT) - h_1((i-1)T)) \le \varepsilon m \alpha.$$

The same procedure applied to the second integral gives

$$\left\| \int_{(i-1)T}^{iT} D[F_0(y_\varepsilon(\tau))s - F_0(y_\varepsilon(iT))s] \right\| \le \varepsilon m (h_3(iT) - h_3((i-1)T)) \le \varepsilon m (m + l).$$

The third integral is zero, because for an arbitrary $y \in B$, we have

$$\int_{(i-1)T}^{iT} D[F(y, s) - F_0(y)s] = F(y, iT) - F(y, (i-1)T) - F_0(y)T = M(y) - F(y, T) = 0.$$

Since $pT \le L/\varepsilon$, we obtain

$$\left\| \sum_{i=1}^{p} \int_{(i-1)T}^{iT} D[F(y_\varepsilon(\tau), s) - F_0(y_\varepsilon(\tau))s] \right\| \le p\varepsilon m \alpha + p\varepsilon m (m + l) \le \frac{Lm\alpha}{T} + \frac{m(m+l)L}{T}.$$

Finally, the following estimate is a consequence of Lemma 3:

$$\left\|\int_{pT}^{t} D[F(y_\varepsilon(\tau),s)-F_0(y_\varepsilon(\tau))s]\right\| \leq \left\|\int_{pT}^{t} DF(y_\varepsilon(\tau),s)\right\| + \left\|\int_{pT}^{t} D[F_0(y_\varepsilon(\tau))s]\right\|$$

$$\leq h_1(t)-h_1(pT)+h_3(t)-h_3(pT) \leq h_1(pT+T)-h_1(pT)+h_3(pT+T)-h_3(pT) \leq \alpha+m+l$$

By combining the previous inequalities, we obtain

$$\left\|\int_{0}^{t} D[F(y_\varepsilon(\tau),s)-F_0(y_\varepsilon(\tau))s]\right\| \leq K,$$

where K is a certain constant. It follows that

$$\|x_\varepsilon(t)-y_\varepsilon(t)\| \leq \varepsilon\int_0^t \|x_\varepsilon(s)-y_\varepsilon(s)\|\,\mathrm{d}h_1(s)+\varepsilon(J+K+L\beta).$$

Since x_ε is a regulated function (we have used Lemma 4) and y_ε is a continuous functions, both of them must be bounded and we can apply Gronwall's inequality from Theorem 6 to obtain

$$\|x_\varepsilon(t)-y_\varepsilon(t)\| \leq e^{\varepsilon(h_1(t)-h_1(0))}\varepsilon(J+K+L\beta).$$

The proof is concluded by observing that

$$\varepsilon(h_1(t)-h_1(0)) \leq \varepsilon(h_1(L/\varepsilon)-h_1(0)) \leq \varepsilon(h_1(\lceil L/(\varepsilon T)\rceil\, T)-h_1(0))$$

$$\leq \varepsilon\left\lceil\frac{L}{\varepsilon T}\right\rceil\alpha \leq \varepsilon\left(\frac{L}{\varepsilon T}+1\right)\alpha \leq \left(\frac{L}{T}+\varepsilon_0\right)\alpha. \qquad \square$$

3 Ordinary differential equations with impulses

We now use the theorem from the previous section to obtain a periodic averaging theorem for ordinary differential equations with impulses. Given a set $B\subset\mathbb{R}^n$, a function $f: B\times[0,\infty)\to\mathbb{R}^n$, an increasing sequence of numbers $0\leq t_1<t_2<\cdots$, and a sequence of mappings $I_i: B\to\mathbb{R}^n$, $i\in\mathbb{N}$, consider the impulsive differential equation

$$x'(t)=\varepsilon f(x(t),t)+\varepsilon^2 g(x(t),t,\varepsilon),\quad t\in[0,\infty)\backslash\{t_1,t_2,\ldots\},$$

$$\Delta^+ x(t_i)=\varepsilon I_i(x(t_i)),\quad i\in\mathbb{N},$$

where $\Delta^+ x(t_i)=x(t_i+)-x(t_i)$.

Since we are interested in deriving a periodic averaging theorem, we will assume that f is T-periodic in the second argument and that the impulses are periodic in the following sense: There exists a $k\in\mathbb{N}$ such that $0\leq t_1<t_2<\cdots<t_k<T$ and for every integer $i>k$, we have $t_i=t_{i-k}+T$, $I_i=I_{i-k}$.

It is known (see Chapter 5 in [7]) that if f is a bounded function which is Lipschitz-continuous in the first argument and continuous in the second argument, and if the impulse operators I_i are bounded and Lipschitz-continuous, then the impulsive differential equation

$$x'(t)=f(x(t),t),\quad t\in[0,\infty)\backslash\{t_1,t_2,\ldots\},$$

$$\Delta^+ x(t_i)=I_i(x(t_i)),\quad i\in\mathbb{N},$$

is equivalent to a generalized ordinary differential equation with the right-hand side

$$F(x,t)=\int_0^t f(x,s)\,\mathrm{d}s+\sum_{i;\,0\leq t_i<t} I_i(x)=\int_0^t f(x,s)\,\mathrm{d}s+\sum_{i=1}^{\infty} I_i(x)H_{t_i}(t),$$

where H_v denotes the characteristic function of (v,∞), i.e. $H_v(t)=0$ for $t\leq v$ and $H_v(t)=1$ for $t>v$.

Theorem 8. *Assume that $B \subset \mathbb{R}^n$, $\Omega = B \times [0, \infty)$, $T > 0$, $\varepsilon_0 > 0$, $L > 0$. Consider functions $f : \Omega \to \mathbb{R}^n$ and $g : \Omega \times (0, \varepsilon_0] \to \mathbb{R}^n$ which are bounded, Lipschitz-continuous in the first argument and continuous in the second argument. Moreover, let f be T-periodic in the second argument. Assume that $k \in \mathbb{N}$, $0 \le t_1 < t_2 < \cdots < t_k < T$, and that $I_i : B \to \mathbb{R}^n$, $i = 1, 2, \ldots, k$ are bounded and Lipschitz-continuous functions. For every integer $i > k$, define t_i and I_i by the recursive formulas $t_i = t_{i-k} + T$ and $I_i = I_{i-k}$. Denote*

$$f_0(x) = \frac{1}{T} \int_0^T f(x, s) \, \mathrm{d}s \quad \text{and} \quad I_0(x) = \frac{1}{T} \sum_{i=1}^{k} I_i(x)$$

for every $x \in B$. Suppose that for every $\varepsilon \in (0, \varepsilon_0]$, the impulsive equation

$$x'(t) = \varepsilon f(x(t), t) + \varepsilon^2 g(x(t), t, \varepsilon) \quad t \in [0, \infty) \backslash \{t_1, t_2, \ldots\},$$

$$\Delta^+ x(t_i) = \varepsilon I_i(x(t_i)), \quad i \in \mathbb{N}, \quad x(0) = x_0(\varepsilon)$$

and the ordinary differential equation

$$y'(t) = \varepsilon(f_0(y(t)) + I_0(y(t))), \quad y(0) = y_0(\varepsilon)$$

have solutions x_ε, $y_\varepsilon : [0, \frac{L}{\varepsilon}] \to B$. If there is a constant $J > 0$ such that $\|x_0(\varepsilon) - y_0(\varepsilon)\| \le J\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$, then there exists a constant $K > 0$ such that

$$\|x_\varepsilon(t) - y_\varepsilon(t)\| \le K\varepsilon$$

for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, \frac{L}{\varepsilon}]$.

Proof. Let

$$F(x, t) = \int_0^t f(x, s) \, \mathrm{d}s + \sum_{i=1}^{\infty} I_i(x) H_{t_i}(t),$$

$$G(x, t, \varepsilon) = \int_0^t g(x, s, \varepsilon) \, \mathrm{d}s.$$

Given an $\varepsilon \in (0, \varepsilon_0]$, the function x_ε satisfies

$$\frac{\mathrm{d}x_\varepsilon}{\mathrm{d}\tau} = D[\varepsilon F(x_\varepsilon, t) + \varepsilon^2 G(x_\varepsilon, t, \varepsilon)].$$

According to the assumptions, there exists a constant $C > 0$ such that

$$\|f(x, t)\| \le C, \quad \|f(x, t) - f(y, t)\| \le C\|x - y\|$$

for every x, $y \in B$, $t \in [0, \infty)$, a constant $D > 0$ such that

$$\|I_i(x)\| \le D, \quad \|I_i(x) - I_i(y)\| \le D\|x - y\|$$

for every x, $y \in B$ and $i \in \mathbb{N}$, and a constant $N > 0$ such that

$$\|g(x, t, \varepsilon)\| \le N, \quad \|g(x, t, \varepsilon) - g(y, t, \varepsilon)\| \le N\|x - y\|$$

for every x, $y \in B$, $t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$. The function $h_1 : [0, \infty) \to \mathbb{R}$ given by

$$h_1(t) = Ct + D \sum_{i=1}^{\infty} H_{t_i}(t)$$

is left-continuous and nondecreasing. If $0 \le u \le t$, then

$$\|F(x,t) - F(x,u)\| = \left\| \int_u^t f(x,s)\,\mathrm{d}s + \sum_{i=1}^{\infty} I_i(x)(H_{t_i}(t) - H_{t_i}(u)) \right\| \le$$

$$\le \int_u^t \|f(x,s)\|\,\mathrm{d}s + \sum_{i=1}^{\infty} \|I_i(x)\|(H_{t_i}(t) - H_{t_i}(u)) \le C(t-u) + D\sum_{i=1}^{\infty}(H_{t_i}(t) - H_{t_i}(u)) = h_1(t) - h_1(u)$$

and

$$\|F(x,t) - F(x,u) - F(y,t) + F(y,u)\| = \left\| \int_u^t (f(x,s) - f(y,s))\,\mathrm{d}s + \sum_{i=1}^{\infty} (I_i(x) - I_i(y))\,(H_{t_i}(t) - H_{t_i}(u)) \right\|$$

$$\le \int_u^t \|f(x,s) - f(y,s)\|\,\mathrm{d}s + \sum_{i=1}^{\infty} \|I_i(x) - I_i(y)\|\,(H_{t_i}(t) - H_{t_i}(u)) \le$$

$$\le \|x-y\| \left(C(t-u) + D\sum_{i=1}^{\infty}(H_{t_i}(t) - H_{t_i}(u)) \right) = \|x-y\|(h_1(t) - h_1(u)).$$

It follows that F belongs to the class $\mathcal{F}(\Omega, h_1)$. Define $h_2 : [0,\infty) \to \mathbb{R}$ by $h_2(t) = Nt$. If $0 \le u \le t$, then

$$\|G(x,t,\varepsilon) - G(x,u,\varepsilon)\| = \left\| \int_u^t g(x,s,\varepsilon)\,\mathrm{d}s \right\| \le N(t-u) = h_2(t) - h_2(u).$$

Also, if $0 \le u \le t$ and $x,\, y \in B$, we have

$$\|G(x,t,\varepsilon) - G(x,u,\varepsilon) - G(y,t,\varepsilon) + G(y,u,\varepsilon)\| = \left\| \int_u^t (g(x,s,\varepsilon) - g(y,s,\varepsilon))\,\mathrm{d}s \right\| \le$$

$$\le N\|x-y\|(t-u) = \|x-y\|(h_2(t) - h_2(u)).$$

Therefore, for every fixed $\varepsilon \in (0, \varepsilon_0]$, the function $(x,t) \mapsto G(x,t,\varepsilon)$ belongs to the class $\mathcal{F}(\Omega, h_2)$. It is clear that $F(x,0) = 0$ and $G(x,0,\varepsilon) = 0$ for every $x \in B$. Moreover, for every $t \ge 0$, the difference

$$F(x,t+T) - F(x,t) = \int_t^{t+T} f(x,s)\,\mathrm{d}s + \sum_{i;\ t \le t_i < t+T} I_i(x) = \int_0^T f(x,s)\,\mathrm{d}s + \sum_{i;\ 0 \le t_i < T} I_i(x)$$

is independent of t, so we can define $M(x) = F(x,t+T) - F(x,t)$. The following calculations show that M is bounded and Lipschitz-continuous:

$$\|M(x)\| = \|F(x,T) - F(x,0)\| = \left\| \int_0^T f(x,s)\,\mathrm{d}s + \sum_{i=1}^{k} I_i(x) \right\| \le CT + kD$$

$$\|M(x) - M(y)\| = \|F(x,T) - F(y,T) - F(x,0) + F(y,0)\| =$$

$$= \left\| \int_0^T (f(x,s) - f(y,s))\,\mathrm{d}s + \sum_{i=1}^{k} (I_i(x) - I_i(y)) \right\| \le \int_0^T \|f(x,s) - f(y,s)\|\,\mathrm{d}s + \sum_{i=1}^{k} \|I_i(x) - I_i(y)\|$$

$$\le CT\|x-y\| + kD\|x-y\| = \|x-y\|(CT + kD)$$

For every $j \in \mathbb{N}$, we have

$$h_1(jT) - h_1((j-1)T) = CjT + D\sum_{i=1}^{\infty} H_{t_i}(jT) - C(j-1)T - D\sum_{i=1}^{\infty} H_{t_i}((j-1)T) =$$

$$= CT + D \sum_{i;\,(j-1)T \le t_i < jT} 1 = CT + Dk.$$

Finally, note that $|h_2(t)/t| = N$ for every $t > 0$. We see that the assumptions of Theorem 7 are satisfied. To conclude the proof, it is now sufficient to define

$$F_0(x) = \frac{F(x,T)}{T} = \frac{1}{T}\int_0^T f(x,s)\,\mathrm{d}s + \frac{1}{T}\sum_{i=1}^{k} I_i(x) = f_0(x) + I_0(x)$$

and apply Theorem 7. □

4 Dynamic equations on time scales

In this section, we use Theorem 7 to derive a periodic averaging theorem for dynamic equations on time scales. We assume that the reader is familiar with the basic notions of time scales calculus as described in [1], and with integration on time scales as presented in [2]. According to [8], dynamic equations on time scales can be converted to generalized ordinary differential equations. Before describing the corresponding procedure, we introduce the following notation, which is taken over from [8].

Let $\mathbb{T}$ be a time scale. If t is a real number such that $t \le \sup \mathbb{T}$, let

$$t^* = \inf\{s \in \mathbb{T}; s \ge t\}.$$

Since $\mathbb{T}$ is a closed set, we have $t^* \in \mathbb{T}$. Further, let

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup\mathbb{T}] & \text{if } \sup\mathbb{T} < \infty, \\ (-\infty,\infty) & \text{otherwise.} \end{cases}$$

Given a function $f : \mathbb{T} \to \mathbb{R}^n$, we define a function $f^* : \mathbb{T}^* \to \mathbb{R}^n$ by

$$f^*(t) = f(t^*), \quad t \in \mathbb{T}^*.$$

The following theorem, which is a special case of Theorem 12 from [8], describes a one-to-one correspondence between the solutions of a dynamic equation and the solutions of a certain generalized ordinary differential equation.

Theorem 9. *Consider a bounded set $B \subset \mathbb{R}^n$ and a bounded Lipschitz-continuous function $f : B \times \mathbb{T} \to \mathbb{R}^n$. Moreover, assume that f is rd-continuous, i.e. the function $t \mapsto f(x(t),t)$ is rd-continuous whenever $x : \mathbb{T} \to B$ is a continuous function. If $x : \mathbb{T} \to B$ is a solution of*

$$x^{\Delta}(t) = f(x(t),t), \;\; t \in \mathbb{T}, \tag{1}$$

then $x^* : \mathbb{T}^* \to B$ is a solution of generalized ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x,t), \;\; t \in \mathbb{T}^*, \tag{2}$$

where

$$F(x,t) = \int_{t_0}^{t} f(x,s^*)\,\mathrm{d}u(s), \;\; x \in B, \;\; t \in \mathbb{T}^*,$$

$t_0 \in \mathbb{T}$ is an arbitrary fixed number, and $u(s) = s^*$ for every $s \in \mathbb{T}^*$. Conversely, every solution $y : \mathbb{T}^* \to B$ of (2) has the form $y = x^*$, where $x : \mathbb{T} \to B$ is a solution of (1).

We now proceed to the periodic averaging theorem for dynamic equations on time scales.

Definition 10. Let $T > 0$ be a real number. A time scale $\mathbb{T}$ is called T-periodic if $t \in \mathbb{T}$ implies $t+T \in \mathbb{T}$ and $\mu(t) = \mu(t+T)$.

Theorem 11. *Let $\mathbb{T}$ be a T-periodic time scale, $t_0 \in \mathbb{T}$, $\varepsilon_0 > 0$, $L > 0$, $B \subset \mathbb{R}^n$ bounded. Consider a pair of bounded Lipschitz-continuous functions $f : B \times [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^n$ and $g : B \times [t_0, \infty)_{\mathbb{T}} \times (0, \varepsilon_0] \to \mathbb{R}^n$. Assume that f is T-periodic in the second variable, and that both f and g are rd-continuous. Define $f_0 : B \to \mathbb{R}^n$ by*

$$f_0(x) = \frac{1}{T} \int_{t_0}^{t_0+T} f(x,s)\Delta s, \quad x \in B.$$

Suppose that for every $\varepsilon \in (0, \varepsilon_0]$, the dynamic equation

$$x^{\Delta}(t) = \varepsilon f(x(t), t) + \varepsilon^2 g(x(t), t, \varepsilon), \quad x(t_0) = x_0(\varepsilon)$$

has a solution $x_\varepsilon : [t_0, t_0 + \frac{L}{\varepsilon}]_{\mathbb{T}} \to B$, and the ordinary differential equation

$$y'(t) = \varepsilon f_0(y(t)), \quad y(t_0) = y_0(\varepsilon)$$

has a solution $y_\varepsilon : [t_0, t_0 + \frac{L}{\varepsilon}] \to B$. If there is a constant $J > 0$ such that $\|x_0(\varepsilon) - y_0(\varepsilon)\| \le J\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$, then there exists a constant $K > 0$ such that

$$\|x_\varepsilon(t) - y_\varepsilon(t)\| \le K\varepsilon,$$

for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [t_0, t_0 + \frac{L}{\varepsilon}]_{\mathbb{T}}$.

Proof. Without loss of generality, we can assume that $t_0 = 0$; otherwise, consider a shifted problem with the time scale $\tilde{\mathbb{T}} = \{t - t_0; t \in \mathbb{T}\}$ and right-hand side $\tilde{f}(x,t) = f(x, t_0 + t)$. According to the assumptions, there exist constants $m, l > 0$ such that

$$\|f(x,t)\| \le m, \quad \|g(x,t,\varepsilon)\| \le m,$$

$$\|f(x,t) - f(y,t)\| \le l\|x-y\|, \quad \|g(x,t,\varepsilon) - g(y,t,\varepsilon)\| \le l\|x-y\|$$

for every x, $y \in B$, $t \in [0,\infty)_{\mathbb{T}}$, $\varepsilon \in (0, \varepsilon_0]$. Let $u(t) = t^*$, $h_1(t) = h_2(t) = (m+l)u(t)$,

$$F(x,t) = \int_0^t f(x, s^*)\,\mathrm{d}u(s), \ x \in B, \ t \in [0,\infty),$$

$$G(x,t,\varepsilon) = \int_0^t g(x, s^*, \varepsilon)\,\mathrm{d}u(s), \ x \in B, \ t \in [0,\infty).$$

If $0 \le t_1 \le t_2$ and x, $y \in B$, then

$$\|F(x,t_2) - F(x,t_1)\| = \left\| \int_{t_1}^{t_2} f(x,s^*)\,\mathrm{d}u(s) \right\| \le m(u(t_2) - u(t_1)) \le h_1(t_2) - h_1(t_1),$$

$$\|F(x,t_2) - F(x,t_1) - F(y,t_2) + F(y,t_1)\| = \left\| \int_{t_1}^{t_2} (f(x,s^*) - f(y,s^*))\,\mathrm{d}u(s) \right\| \le$$
$$\le l\|x-y\|(u(t_2) - u(t_1)) \le \|x-y\|(h_1(t_2) - h_1(t_1)).$$

It follows that F belongs to the class $\mathcal{F}(\Omega, h_1)$. Similarly, if $0 \le t_1 \le t_2$ and x, $y \in B$, then

$$\|G(x,t_2,\varepsilon) - G(x,t_1,\varepsilon)\| = \left\| \int_{t_1}^{t_2} g(x,s^*,\varepsilon)\,\mathrm{d}u(s) \right\| \le m(u(t_2) - u(t_1)) \le h_2(t_2) - h_2(t_1),$$

$$\|G(x,t_2,\varepsilon)-G(x,t_1,\varepsilon)-G(y,t_2,\varepsilon)+G(y,t_1,\varepsilon)\| = \left\|\int_{t_1}^{t_2}(g(x,s^*,\varepsilon)-g(y,s^*,\varepsilon))\,\mathrm{d}u(s)\right\| \leq$$

$$\leq l\|x-y\|(u(t_2)-u(t_1)) \leq \|x-y\|(h_2(t_2)-h_2(t_1)).$$

Therefore, for every fixed $\varepsilon \in (0,\varepsilon_0]$, the function $(x,t)\mapsto G(x,t,\varepsilon)$ belongs to the class $\mathcal{F}(\Omega,h_2)$. It is clear that $F(x,0)=0$ and $G(x,0,\varepsilon)=0$. Since $\mathbb{T}$ is a T-periodic time scale, the function u is also T-periodic. The function f is T-periodic in the second argument and it follows that the difference

$$F(x,t+T)-F(x,t) = \int_t^{t+T} f(x,s^*)\,\mathrm{d}u(s) = \int_0^T f(x,s^*)\,\mathrm{d}u(s)$$

does not depend on t. The function $M(x)=F(x,t+T)-F(x,t)$ satisfies

$$\|M(x)\| = \left\|\int_0^T f(x,s^*)\,\mathrm{d}u(s)\right\| \leq m(u(T)-u(0)) = mT,$$

$$\|M(x)-M(y)\| = \left\|\int_0^T (f(x,s^*)-f(y,s^*))\,\mathrm{d}u(s)\right\| \leq l\|x-y\|(u(T)-u(0)) = l\|x-y\|T,$$

i.e. M is a bounded Lipschitz-continuous function. For every $i\in\mathbb{N}$, we have

$$h_1(iT)-h_1((i-1)T) = (m+l)(u(iT)-u((i-1)T)) = (m+l)(iT-(i-1)T) = (m+l)T.$$

If $t \geq L/\varepsilon_0$, then

$$\left|\frac{h_2(t)}{t}\right| = (m+l)\frac{t^*}{t} \leq (m+l)\frac{t+T}{t} = (m+l)\left(1+\frac{T}{t}\right) \leq (m+l)\left(1+\frac{T\varepsilon_0}{L}\right).$$

Thus we have checked that all assumptions of Theorem 7 are satisfied. Moreover,

$$F_0(x) = \frac{F(x,T)}{T} = \frac{1}{T}\int_0^T f(x,s^*)\,\mathrm{d}u(s) = f_0(x),$$

where the last equality follows from Theorem 5 in [8]. By Theorem 9, for every $\varepsilon\in(0,\varepsilon_0]$, the function $x_\varepsilon^* : [t_0, t_0+\frac{L}{\varepsilon}] \to B$ satisfies

$$\frac{\mathrm{d}x_\varepsilon^*}{\mathrm{d}\tau} = D[\varepsilon F(x_\varepsilon^*,t)+\varepsilon^2 G(x_\varepsilon^*,t,\varepsilon)], \quad x_\varepsilon^*(0)=x_0(\varepsilon).$$

According to Theorem 7, there exists a constant $K>0$ such that

$$\|x_\varepsilon^*(t)-y_\varepsilon(t)\| \leq K\varepsilon$$

for every $t\in[0,\frac{L}{\varepsilon}]$, which proves the theorem. □

Note that in our Theorem 11, the averaged equation is an ordinary differential equation, while in a similar Theorem 9 obtained in [9], the averaged equation is a dynamic equation on the same time scale as the original equation.

5 Retarded equations

Let $r > 0$ be a given number. The theory of retarded functional differential equations is usually concerned with the initial-value problem

$$x'(t) = f(x_t, t), \quad x_{t_0} = \phi,$$

where x_t is given by the formula $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$. The equivalent integral form is

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\, \mathrm{d}s, \quad x_{t_0} = \phi.$$

We will focus on slightly more general problems of the form

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\, \mathrm{d}h(s), \quad x_{t_0} = \phi,$$

where the Kurzweil-Stieltjes integral on the right-hand side is taken with respect to a nondecreasing function h. More precisely, we are interested in deriving a periodic averaging theorem for the equation

$$x(t) = x(0) + \varepsilon \int_0^t f(x_s, s)\, \mathrm{d}h(s) + \varepsilon^2 \int_0^t g(x_s, s, \varepsilon)\, \mathrm{d}h(s), \quad x_0 = \phi.$$

Before proceeding to the averaging theorem, we need the following auxiliary lemma.

Lemma 12. *If $y : [a - r, b] \to \mathbb{R}^n$ is a regulated function, then $s \mapsto \|y_s\|_\infty$ is regulated on $[a, b]$.*

Proof. We will show that $\lim_{s \to s_0-} \|y_s\|_\infty$ exists for every $s_0 \in (a, b]$. The function y is regulated, and therefore satisfies the Cauchy condition at $s_0 - r$ and s_0: Given an arbitrary $\varepsilon > 0$, there exists a $\delta \in (0, s_0 - a)$ such that

$$\|y(u) - y(v)\| < \varepsilon, \quad u, v \in (s_0 - r - \delta, s_0 - r), \tag{3}$$

and

$$\|y(u) - y(v)\| < \varepsilon, \quad u, v \in (s_0 - \delta, s_0). \tag{4}$$

Now, consider a pair of numbers s_1, s_2 such that $s_0 - \delta < s_1 < s_2 < s_0$. For every $s \in [s_1 - r, s_2 - r]$, it follows from (3) that

$$\|y(s)\| \le \|y(s_2 - r)\| + \varepsilon \le \|y_{s_2}\|_\infty + \varepsilon.$$

It is also clear that $\|y(s)\| \le \|y_{s_2}\|_\infty$ for every $s \in [s_2 - r, s_1]$. Consequently, $\|y_{s_1}\|_\infty \le \|y_{s_2}\|_\infty + \varepsilon$. Using (4) in a similar way, we obtain $\|y_{s_2}\|_\infty \le \|y_{s_1}\|_\infty + \varepsilon$. It follows that

$$\big| \|y_{s_1}\|_\infty - \|y_{s_2}\|_\infty \big| \le \varepsilon, \quad s_1, s_2 \in (s_0 - \delta, s_0),$$

i.e. the Cauchy condition for the existence of $\lim_{s \to s_0-} \|y_s\|_\infty$ is satisfied. The existence of $\lim_{s \to s_0+} \|y_s\|_\infty$ for $s_0 \in [a, b)$ can be proved similarly. □

The proof of the periodic averaging theorem for retarded equations follows the same basic idea as the proof of Theorem 7. Certain details are inspired by the paper [3], which is devoted to nonperiodic averaging. Given a set $B \subset \mathbb{R}^n$, we use the symbol $G([a, b], B)$ to denote the set of all regulated functions $f : [a, b] \to B$.

Theorem 13. *Let $\varepsilon_0 > 0$, $L > 0$, $B \subset \mathbb{R}^n$, $X = G([-r, 0], B)$. Consider a pair of bounded functions $f : X \times [0, \infty) \to \mathbb{R}^n$, $g : X \times [0, \infty) \times (0, \varepsilon_0] \to \mathbb{R}^n$ and a nondecreasing left-continuous function $h : [0, \infty) \to \mathbb{R}$ such that the following conditions are satisfied:*

1. *The integral* $\int_0^b f(y_t,t)\,\mathrm{d}h(t)$ *exists for every* $b>0$ *and* $y\in G([-r,b],B)$.
2. f *is* T*-periodic in the second variable.*
3. *There is a constant* $\alpha>0$ *such that* $h(t+T)-h(t)=\alpha$ *for every* $t\geq 0$.
4. *There is a constant* $C>0$ *such that for* $x,y\in X$ *and* $t\in[0,\infty)$,

$$\|f(x,t)-f(y,t)\|\leq C\,\|x-y\|_\infty\,.$$

5. *The integral*

$$f_0(x)=\frac{1}{T}\int_0^T f(x,s)\,\mathrm{d}h(s)$$

exists for every $x\in X$.

Let $\phi\in X$. *Suppose that for every* $\varepsilon\in(0,\varepsilon_0]$, *the initial-value problems*

$$\begin{aligned}x(t)&=x(0)+\varepsilon\int_0^t f(x_s,s)\,\mathrm{d}h(s)+\varepsilon^2\int_0^t g(x_s,s,\varepsilon)\,\mathrm{d}h(s),\ \ x_0=\phi,\\ y(t)&=y(0)+\varepsilon\int_0^t f_0(y_s)\,\mathrm{d}s,\ \ y_0=\phi\end{aligned}$$

have solutions x^ε, $y^\varepsilon:\left[-r,\frac{L}{\varepsilon}\right]\to B$. *Then there exists a constant* $J>0$ *such that*

$$\|x^\varepsilon(t)-y^\varepsilon(t)\|\leq J\varepsilon$$

for every $\varepsilon\in(0,\varepsilon_0]$ *and* $t\in[0,\frac{L}{\varepsilon}]$.

Proof. There is a constant $M>0$ such that $\|f(x,t)\|\leq M$ and $\|g(x,t,\varepsilon)\|\leq M$ for every $x\in X$, $t\in[0,\infty)$ and $\varepsilon\in(0,\varepsilon_0]$. It follows that

$$\|f_0(x)\|=\left\|\frac{1}{T}\int_0^T f(x,s)\,\mathrm{d}h(s)\right\|\leq\frac{M}{T}(h(T)-h(0))=\frac{M\alpha}{T}$$

for every $x\in X$. Thus if $\varepsilon\in(0,\varepsilon_0]$, s, $t\in[0,\infty)$, $s\geq t$, the solution y^ε satisfies

$$\|y^\varepsilon(s+\theta)-y^\varepsilon(t+\theta)\|=\left\|\varepsilon\int_{t+\theta}^{s+\theta} f_0(y^\varepsilon_\sigma)\,\mathrm{d}\sigma\right\|\leq\frac{\varepsilon M(s-t)\alpha}{T},\ \ \theta\in[-r,0],$$

$$\|y^\varepsilon_s-y^\varepsilon_t\|_\infty=\sup_{\theta\in[-r,0]}\|y^\varepsilon(s+\theta)-y^\varepsilon(t+\theta)\|\leq\frac{\varepsilon M(s-t)\alpha}{T}. \tag{5}$$

For every $t\in[0,L/\varepsilon]$, we have

$$\|x^\varepsilon(t)-y^\varepsilon(t)\|=\left\|\varepsilon\int_0^t f(x^\varepsilon_s,s)\,\mathrm{d}h(s)+\varepsilon^2\int_0^t g(x^\varepsilon_s,s,\varepsilon)\,\mathrm{d}h(s)-\varepsilon\int_0^t f_0(y^\varepsilon_s)\,\mathrm{d}s\right\|\leq$$

$$\leq\varepsilon\left\|\int_0^t (f(x^\varepsilon_s,s)-f(y^\varepsilon_s,s))\,\mathrm{d}h(s)\right\|+\varepsilon\left\|\int_0^t f(y^\varepsilon_s,s)\,\mathrm{d}h(s)-\int_0^t f_0(y^\varepsilon_s)\,\mathrm{d}s\right\|+\varepsilon^2\left\|\int_0^t g(x^\varepsilon_s,s,\varepsilon)\,\mathrm{d}h(s)\right\|\leq$$

$$\leq\varepsilon\int_0^t C\|x^\varepsilon_s-y^\varepsilon_s\|_\infty\,\mathrm{d}h(s)+\varepsilon\left\|\int_0^t f\left(y^\varepsilon_s,s\right)\,\mathrm{d}h(s)-\int_0^t f_0\left(y^\varepsilon_s\right)\,\mathrm{d}s\right\|+\varepsilon^2 M(h(t)-h(0)). \tag{6}$$

(Note that the integral $\int_0^t C\|x_s^\varepsilon - y_s^\varepsilon\|_\infty \, \mathrm{d}h(s)$ is guaranteed to exist by Lemma 12, while the existence of $\int_0^t f\left(y_s^\varepsilon, s\right) \, \mathrm{d}h(s)$ follows from assumption 1.) First, we estimate the second term. Let p be the largest integer such that $pT \le t$. Then

$$\left\| \int_0^t f\left(y_s^\varepsilon, s\right) \, \mathrm{d}h(s) - \int_0^t f_0\left(y_s^\varepsilon\right) \, \mathrm{d}s \right\| \le$$

$$\le \sum_{i=1}^p \left\| \int_{(i-1)T}^{iT} (f(y_s^\varepsilon, s) - f(y_{(i-1)T}^\varepsilon, s)) \, \mathrm{d}h(s) \right\| + \sum_{i=1}^p \left\| \int_{(i-1)T}^{iT} f(y_{(i-1)T}^\varepsilon, s) \, \mathrm{d}h(s) - \int_{(i-1)T}^{iT} f_0(y_{(i-1)T}^\varepsilon) \, \mathrm{d}s \right\|$$

$$+ \sum_{i=1}^p \left\| \int_{(i-1)T}^{iT} (f_0(y_{(i-1)T}^\varepsilon) - f_0(y_s^\varepsilon)) \, \mathrm{d}s \right\| + \left\| \int_{pT}^t f(y_s^\varepsilon, s) \, \mathrm{d}h(s) - \int_{pT}^t f_0(y_s^\varepsilon) \, \mathrm{d}s \right\|.$$

For every $i \in \{1, 2, \ldots, p\}$ and every $s \in [(i-1)T, iT]$, inequality (5) gives

$$\|y_s^\varepsilon - y_{(i-1)T}^\varepsilon\|_\infty \le \frac{M\varepsilon\alpha(s - (i-1)T)}{T} \le M\varepsilon\alpha.$$

Using this estimate together with the fact that $pT \le \frac{L}{\varepsilon}$, we obtain

$$\sum_{i=1}^p \left\| \int_{(i-1)T}^{iT} (f(y_s^\varepsilon, s) - f(y_{(i-1)T}^\varepsilon, s)) \, \mathrm{d}h(s) \right\| \le \sum_{i=1}^p CM\varepsilon\alpha(h(iT) - h((i-1)T)) = CM\varepsilon\alpha^2 p \le \frac{CML\alpha^2}{T}.$$

When $s \ge t \ge 0$ and $y \in G([-r, s], B)$, then

$$\|f_0(y_s) - f_0(y_t)\| = \frac{1}{T} \left\| \int_0^T (f(y_s, \sigma) - f(y_t, \sigma)) \, \mathrm{d}h(\sigma) \right\| \le \frac{C}{T}\|y_s - y_t\|_\infty (h(T) - h(0)) = \frac{C}{T}\|y_s - y_t\|_\infty \alpha.$$

Thus

$$\sum_{i=1}^p \left\| \int_{(i-1)T}^{iT} (f_0(y_s^\varepsilon) - f_0(y_{(i-1)T}^\varepsilon)) \, \mathrm{d}s \right\| \le \sum_{i=1}^p \int_{(i-1)T}^{iT} \left\| f_0(y_s^\varepsilon) - f_0(y_{(i-1)T}^\varepsilon) \right\| \, \mathrm{d}s \le$$

$$\le \frac{C}{T}\alpha \sum_{i=1}^p \int_{(i-1)T}^{iT} \|y_s^\varepsilon - y_{(i-1)T}^\varepsilon\|_\infty \, \mathrm{d}s \le \frac{C}{T}\alpha \sum_{i=1}^p \varepsilon M\alpha T = \varepsilon MC\alpha^2 p \le \frac{MCL\alpha^2}{T}.$$

The fact that f is T-periodic in the second variable and the definition of f^0 imply

$$\sum_{i=1}^p \left\| \int_{(i-1)T}^{iT} f(y_{(i-1)T}^\varepsilon, s) \, \mathrm{d}h(s) - \int_{(i-1)T}^{iT} f_0(y_{(i-1)T}^\varepsilon) \, \mathrm{d}s \right\| =$$

$$= \sum_{i=1}^p \left\| \int_0^T f(y_{(i-1)T}^\varepsilon, s) \, \mathrm{d}h(s) - f_0(y_{(i-1)T}^\varepsilon) T \right\| = 0.$$

Finally,

$$\left\| \int_{pT}^t f(y_s^\varepsilon, s) \, \mathrm{d}h(s) - \int_{pT}^t f_0(y_s^\varepsilon) \, \mathrm{d}s \right\| \le \left\| \int_{pT}^t f(y_s^\varepsilon, s) \, \mathrm{d}h(s) \right\| + \int_{pT}^t \|f_0(y_s^\varepsilon)\| \, \mathrm{d}s \le$$

$$\le M(h(t) - h(pT)) + \frac{M\alpha}{T}(t - pT) \le M(h((p+1)T) - h(pT)) + \frac{M\alpha}{T}T = M\alpha + M\alpha = 2M\alpha.$$

By combination of the previous results, we obtain

$$\left\|\int_0^t f(y_s^\varepsilon, s)\,\mathrm{d}h(s) - \int_0^t f_0(y_s^\varepsilon)\,\mathrm{d}s\right\| \le \frac{2MCL\alpha^2}{T} + 2M\alpha.$$

Denote the constant on the right-hand side by K. Returning back to inequality (6), we see that

$$\|x^\varepsilon(t) - y^\varepsilon(t)\| \le \varepsilon \int_0^t C\|x_s^\varepsilon - y_s^\varepsilon\|_\infty\,\mathrm{d}h(s) + \varepsilon K + \varepsilon^2 M(h(t) - h(0)).$$

Let $\psi(s) = \sup_{\tau\in[0,s]} \|x^\varepsilon(\tau) - y^\varepsilon(\tau)\|$. Since x^ε and y^ε are regulated, it is not difficult to see that ψ is also regulated and therefore Kurzweil-Stieltjes integrable with respect to the function h. For every $u \in [0,t]$, we have

$$\|x^\varepsilon(u)-y^\varepsilon(u)\| \le \varepsilon\int_0^u C\psi(s)\,\mathrm{d}h(s)+\varepsilon K+\varepsilon^2 M(h(u)-h(0)) \le \varepsilon\int_0^t C\psi(s)\,\mathrm{d}h(s)+\varepsilon K+\varepsilon^2 M(h(t)-h(0)).$$

Consequently,

$$\psi(t) \le \varepsilon\int_0^t C\psi(s)\,\mathrm{d}h(s) + \varepsilon K + \varepsilon^2 M(h(t) - h(0)).$$

Next, note that

$$\varepsilon(h(t)-h(0)) \le \varepsilon(h(L/\varepsilon)-h(0)) \le \varepsilon(h(\lceil L/(\varepsilon T)\rceil\, T)-h(0)) \le \varepsilon\left\lceil\frac{L}{\varepsilon T}\right\rceil\alpha \le \varepsilon\left(\frac{L}{\varepsilon T}+1\right)\alpha \le \left(\frac{L}{T}+\varepsilon_0\right)\alpha.$$

Thus

$$\psi(t) \le \varepsilon\int_0^t C\psi(s)\,\mathrm{d}h(s) + \varepsilon K + \varepsilon M\left(\frac{L}{T}+\varepsilon_0\right)\alpha.$$

Gronwall's inequality from Theorem 6 gives

$$\psi(t) \le e^{\varepsilon C(h(t)-h(0))}\left(K + M\left(\frac{L}{T}+\varepsilon_0\right)\alpha\right)\varepsilon \le e^{C\left(\frac{L}{T}+\varepsilon_0\right)\alpha}\left(K + M\left(\frac{L}{T}+\varepsilon_0\right)\alpha\right)\varepsilon.$$

It follows that if we let $J = e^{C\left(\frac{L}{T}+\varepsilon_0\right)\alpha}\left(K + M\left(\frac{L}{T}+\varepsilon_0\right)\alpha\right)$, then

$$\|x^\varepsilon(t) - y^\varepsilon(t)\| \le \psi(t) \le J\varepsilon$$

for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, \frac{L}{\varepsilon}]$. □

In the special case $h(t) = t$, we obtain a periodic averaging theorem for the usual type of retarded functional differential equations. However, our theorem is much more general. The following example shows that it is applicable even to retarded difference equations.

Example 14. Consider the function $h(t) = \lceil t\rceil$ and assume that $x : [-r,\infty) \to \mathbb{R}^n$ satisfies

$$x(t) = x(0) + \int_0^t f(x_s, s)\,\mathrm{d}h(s), \quad t \in [0,\infty). \tag{7}$$

It follows from the properties of the Kurzweil-Stieltjes integral that for every integer $k \ge 0$, the function x is constant on $(k, k+1]$, and $x(k+) = x(k) + f(x_k, k)(h(k+) - h(k)) = x(k) + f(x_k, k)$. Using this observation, we see that a retarded difference equation of the form

$$a(k+1) - a(k) = F(k, a(k), a(k-1), \dots, a(k-r)), \quad k \in \mathbb{N}_0,$$

is equivalent to the integral equation (7), where $h(t) = \lceil t \rceil$ and $f(y,t) = F(\lceil t \rceil, y(0), y(-1), \ldots, y(-r))$ for every $t \geq 0$ and $y \in G([-r,0], \mathbb{R}^n)$. Indeed, every solution x of this integral equation must be constant on $(k, k+1]$ and satisfy

$$x(k+1) = x(k+) = x(k) + f(x_k, k) = x(k) + F(k, x(k), x(k-1), \ldots, x(k-r))$$

for every integer $k \geq 0$. Thus our averaging theorem is applicable to retarded difference equations of the form

$$a(k+1) - a(k) = \varepsilon F(k, a(k), a(k-1), \ldots, a(k-r)), \quad k \in \mathbb{N}_0,$$

where $\varepsilon \in (0, \varepsilon_0]$ is a small parameter. Assuming that F is T-periodic in the first argument (where T is a positive integer), the corresponding averaged equation has the form

$$y(t) = y(0) + \varepsilon \int_0^t f_0(y_s) \, \mathrm{d}s,$$

where the function f_0 is given by

$$f_0(y) = \frac{1}{T} \int_0^T f(y,s) \, \mathrm{d}h(s) = \frac{1}{T} \sum_{i=0}^{T-1} \int_i^{i+1} f(y,s) \, \mathrm{d}h(s) =$$

$$= \frac{1}{T} \sum_{i=0}^{T-1} f(y,i) = \frac{1}{T} \sum_{i=0}^{T-1} F(i, y(0), y(-1), \ldots, y(-r))$$

for every $y \in G([-r,0], \mathbb{R}^n)$.

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