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Let M^2 be a connected oriented two-dimensional Riemannian surface and let $X : M^2 \to R^N$ be a conformal immersion. We denote by

$$g : M^2 \to G_{2,N}$$

the generalized Gauss map, where $G_{2,n}$ is the Grassmannian of oriented 2-planes in R^N, and for each point p of M, $g(p)$ is the oriented tangent plane of the immersed surface $X(M)$ at $X(p)$.

The properties of the map g, related to the geometry of $X(M) \subset I\!R^n$ and the conformal structure, were studied by Hoffman and Osserman [1,2]. In particular, they obtained necessary and sufficient conditions, in the generic case, for a given map g to arrise as a Gauss map as above. Moreover, they showed [2] that if X is not a minimal immersion, then X is determined uniquely by g, up to translation and homothety in $I\!R^N$.

In [3], E. Vargasta considered the following problem: Under what conditions does there exist another conformal immersion $\bar{X} : M^2 \to I\!R^N$, such that the generalized Gauss ma $\bar{g}$ of $\bar{X}$ differs from g by an orientation-reversing congruence in $G_{2,N}$. In this case X and $\bar{X}$ are said to have the *same Gauss map* and *opposite orientation*. For $N = 3$, necessary and sufficient conditions on the principal curvatures of X, were obtained in [3], for the existence (at least locally) of such an $\bar{X}$, which is unique up to homothety and translation in $I\!R^3$.

These conditions are trivially satisfied by a rotation surface, a cyclic of Dupin or a surface of constant mean curvature, with has no umbilic points.

In this paper, we obtain a six parameter family of complete surfaces immersed in $I\!R^3$ which admit a conformal immersion with the same Gauss map and opposite orientation (Theorem 1). This family is characterized by the solutions of a third order nonlinear ordinary differential equation.

In order to state and prove our result we need to recall the following theorem which was proved in [3].

Theorem A: *Let $X : M^2 \to R^3$ be a conformal immersion without umbilic points. Let (x, y) be local coordinates by lines of curvature and let λ, μ be the principal curvatures.*

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and $\bar{X}$ is unique up to homothety and translation in $I\!R^3$.

(ii) *If M^2 is simply connected and equation (1) is satisfied then there exists a conformal immersion $\bar{X}$ of M^2 in $I\!R^3$ with the same Gauss map as X and opposite orientation.*

Equation (1) is trivially satisfied by a rotation surface, since the principal curvatures depend only on the variable which parametrizes the generating plane curve. For a Dupin sruface, one has $\lambda_x = \mu_y = 0$. Finally, another trivial solution is given by surfaces of constant mean curvature $(\lambda + \mu)/2$.

In order to provide a non trivial family of immersions which admit another conformal immersion with the same Gauss map and opposite orientation, we observe that basides equation (1) one needs to consider also the Gauss and Codazzi equations. The following lemma studies the solutions of a nonlinear third order ordinary differential equation which will be equivalent to the Gauss equation.

Lemma: *Consider the ordinary differential equation for $f(\xi)$*

$$\left(\frac{f''}{f'}D + aN + \frac{f}{2}\right)' = \frac{(af+b)^2}{2f'} \tag{2}$$

where

$$N = e^{-a\xi-\ell} - e^{a\xi-c}, \ \ D = e^{-a\xi-\ell} + e^{a\xi-c} \ \text{and} \ a \neq 0, \ b, c, \ell \in I\!R. \tag{3}$$

Given initial conditions $f(\xi_0)$, $f'(\xi_0) > 0$ and $f''(\xi_0) \in I\!R$, there exists a solution f defined uniquely in some neighborhood $I \subset I\!R$ of ξ_0, which satisfies the initial conditions and it is strictly increasing. Moreover, if $f(\xi_0) = -b/a$, then $I = I\!R$.

Proof: The existence of f follows from basic results on ordinary differential systems. We only need to prove that if $f(\xi_0) = -b/a$ then $I = I\!R$.

Suppose $f(\xi)$ is not defined for all $\xi \in I\!R$, then there exists $\xi_1 \in I\!R$ such that $\lim_{\xi \to \xi_1} f'(\xi) = 0$. Suppose ξ_1 is the first ξ, for which such a limit holds.

We may suppose without loss of generality that $\xi_0 < \xi_1$. Then it follows from equation (2) that

We claim that

$$\lim_{\xi \to \xi_1^-} (af(\xi) + b)^2 > 0. \tag{5}$$

In fact, otherwise this limit would be zero, i.e.

$$\lim_{\xi \to \xi_1^-} f(\xi) = -b/a.$$

Now since the initial conditions are $f(\xi_0) = -b/a$ and $f'(\xi_0) > 0$, then it would exist $\xi_2 \in (\xi_0, \xi_2)$ for which $f'(\xi_2) = 0$. This contradicts the fact that ξ_1 is the first ξ for which the derivative of f vanishes.

Therefore, from (4) and (5) we get

$$\lim_{\xi \to \xi_1^-} f'''(\xi) < 0.$$

Hence, there exists $\epsilon > 0$, such that $f'''(\xi) < 0$ for $\xi \in J = (\xi_1 - \epsilon, \xi_1)$, i.e. f'' is strictly decreasing in this interval and therefore

$$f''(\xi) > \lim_{\xi \to \xi_1^-} f''(\xi) = 0 \quad \text{for} \quad \xi \in J.$$

Hence $f'(\xi)$ is strictly increasing for $\xi \in J$. Since $\lim_{\xi \to \xi_1^-} f'(\xi) = 0$, it follows that $f'(\xi_3) < 0$ for some $\xi_3 \in J$. Now, $f'(\xi_0) > 0$ and $f'(\xi_3) < 0$ imply that there exists $\xi_4 \in (\xi_0, \xi_3)$ such that $f'(\xi_4) = 0$. This is a contradiction, since ξ_1 was chosen so that for any $\xi \in (\xi_0, \xi_1)$, f' does not vanish. ■

Using the above lemma we can prove our main result.

Theorem 1: *For each strictly increasing solution $f(\xi)$ of (2) defined on $I \subset R$, there exists an immersion, determined up to rigid motions of $I\!R^3$, $X : I \times I\!R \to R^3$, which is complete if $f(\xi_0) = -b/a$, has no umbilic points and it is foliated by circular helix curves. Moreover, there exists a conformal immersion $\bar{X}$ with the same Gauss map as X and opposite orientation. The immersion $\bar{X}$ is unique up to homothety and translation in $I\!R^3$.*

Proof: Let $f(\xi)$ be a strictly increasing solution of (2) defined for $\xi \in I \subset I\!R$. We consider the where

$$E = \frac{e^{-a\xi+c}}{f'(\xi)}, \quad G = \frac{e^{a\xi+\ell}}{f'(\xi)}, \tag{6}$$

$$e = \frac{1}{2}(f' + af + b)E, \quad g = \frac{1}{2}(-f' + af + b)G, \tag{7}$$

$$\xi = x - y. \tag{8}$$

f' denotes the derivative with respect to ξ and the functions are evaluated at $\xi = x - y$.

Since f is strictly increasing, the functions E and G are positive. We will show that the above quadratic forms satisfy the Gauss and Codazzi equations for a surface in $I\!R^3$.

We introduce the notation

$$\lambda(\xi) = \frac{1}{2}(f' + af + b) \qquad \mu = \frac{1}{2}(-f' + af + b). \tag{9}$$

Then

$$e = \lambda E, \qquad g = \mu G, \tag{10}$$

and the Codazzi equations reduce to

$$\frac{E'}{E} = -\frac{2\lambda'}{\lambda - \mu} \quad \text{and} \quad \frac{G'}{G} = \frac{2\mu'}{\lambda - \mu}.$$

Now, these equalities follow easily from (6) and (9).

The Gauss equation reduces to verifying

$$\lambda\mu = -\frac{1}{\sqrt{EG}} \left\{ \left(\frac{(\sqrt{E})_y}{\sqrt{G}} \right)_y + \left(\frac{(\sqrt{G}}{\sqrt{E}} \right)_x \right\}.$$

Substituting E, G, λ, and μ given by (6) and (9) and using the fact that $D' = -aN$ and $N' = -aD$, for D and N defined by (3), we obtain equation (2), which is satisfied since by hypothesis $f(\xi)$ is a solution of (2).

It follows from the fundamental theorem for surfaces that there exists an immersion $X : I \times I\!R \to I\!R^3$, defined for (ξ, η), $\xi \in I$, $\eta = x + y \in I\!R$, whose first and second fundamental forms are given by (6) and (7). This immersion is determined up to rigid motions of R^3.

If $f(\xi_0) = -b/a$, then the lemma asserts that f is defined for all $\xi \in I\!R$ and therefore the immersion $X : I\!R^2 \to I\!R^3$ is complete.

We now show that the immersion is foliated by circular helix curves. Passing through $\rho_0 = X(\xi_0, \eta_0)$, we consider the curve $\alpha(\eta) = X(\xi_0, \eta), \quad \eta \in I\!R$. Then $\alpha' = \frac{1}{2}(X_x + X_y)(\xi_0, \eta)$, and similarly α'' and α''' are sums of second and third order derivatives of X with respect to x and y, evaluated at (ξ_0, η). Therefore, α', α'' and α''' are linear combinations of X_x, X_y and N, where the coefficients depend on the first and second fundamental forms of X and its derivatives evaluated at (ξ_0, η). From (6) and (7) these fundamental forms depend only on ξ, and hence they are constant along α. We conclude that α is a circular helix curve since the curvature and torsion of α are constants which are determined by ξ_0.

Finally, Theorem A concludes the last part of the proof of our theorem. ■

The reader should reffer to the paper of Dajczer and Vergasta, in this volume, for the problem of conformal immersions of hypersurfaces M^n in $I\!R^{n+1}$, $n \geq 3$, which have the same Gauss map.

References

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