BACKLUND TRANSFORMATIONS FOR A CLASS OF SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract

Backlund Transformations and Superposition formulae are given for a class of systems of nonlinear differential equations for matrix valued functions of n independent variables. When $n = 2$ the equations reduce to the wave equation, the sine-Gordon equation, the Laplace equation and the elliptic sinh-Gordon equation.

Introduction

Since Backlund obtained the so called Backlund Transformation for the sine-Gordon equation in 1875, several differential equations such as Korteweg-de Vries (KdV), Modified Korteweg-de Vries (MKdV), and others were shown to have such transformations. Very little is known for differential equations for functions depending on more than two variables. In 1980, a geometric generalization of the sine-Gordon equation was obtained together with its Backlund Transformation [13] and Superposition Formula [14]. Solutions for these generalized equation are orthogonal matrix functions which correspond to hyperbolic

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results were obtained [9] for an n-dimensional (nonlinear when $n \geq 3$) generalization of the homogeneous wave equation, whose solutions correspond to $n-$dimensional flat submanifolds of the unit sphere $\mathbf{S}^{2n-1} \subset \mathbf{R}^{2n}$.

Another geometric interpretation for the classical sine-Gordon equation and many other evolution equations such as KdV, MKdV, Burgers, sinh-Gordon was given in [4, 7, 8], where these equations were shown to be associated to hiperbolic metrics on open subsets of $\mathbf{R}^2$. Motivated by this interpretation, the intrinsic generalization of the sine-Gordon and the wave equations were introduced in [1], where it was also given a transformation which provides new solutions for the equations from a given one. These intrinsic generalized equations are differential equations for unit vector fileds in $\mathbf{R}^n$, whose generic solutions define metrics with constant sectional curvature, on open subsets of $\mathbf{R}^n$.

More recently [3], a similar theory was developped for the generalized and the intrinsic genralized Laplace and elliptic sinh-Gordon equations. These equations are also related to manifolds of constant curvature [6, 15].

In this paper, we show that the generalized equations mentioned above are particular cases of classes of equations which also have Backlund Transformations and Superposition Formulae. We first consider the following system of differential equations for a real or complex valued $(n \times n)-$ matrix function a, defined on an open subset of $\mathbf{R}^n$

$$\begin{aligned} aJa^t &= J \\ \frac{\partial a_{ki}}{\partial x_j} &= a_{kj}h_{ji}, \quad i \neq j, \quad h_{ii} = 0 \\ \frac{\partial h_{ij}}{\partial x_i} &+ \frac{\partial h_{ji}}{\partial x_j} + \sum_{s\neq i,j} h_{si}h_{sj} = -K a_{1i}a_{1j}, \quad i \neq j \\ \frac{\partial h_{ij}}{\partial x_s} &= h_{is}h_{sj}, \quad i, j, s \quad \text{distinct} \end{aligned} \tag{1}$$

where

$$1 \leq i, j, s \leq n \quad J = \text{diag}\,(\overbrace{1, \ldots, 1}^{p\ times}, \overbrace{-1, \ldots, -1}^{q\ times}), \quad p + q = n \quad p \geq 1. \tag{2}$$

function a, which is reduced to a first order system by the matrix h defined by the second equation. Moreover, when $n = 2$ and $p = 2$, the system (1) reduces to

$$u_{x_1x_1} - u_{x_2x_2} = -K \sin u$$

which is the sine-Gordon equation when $K \neq 0$ and the homogeneous wave equation when $K = 0$. Whenever $n = 2$ and $p = 1$ the system (1) reduces to

$$u_{x_1x_1} + u_{x_2x_2} = -K \sinh\, u$$

which is the elliptic sinh-Gordon equation when $K \neq 0$ and the Laplace equation when $K = 0$. We will call (1) the *Generalized Equation* following previous denominations. The system of equations (1) for the real valued matrix function $a_{ij}(x_1, \ldots, x_n)$ has the following geometrical interpretation.

Let M be an n-dimensional Riemannian manifold of constant sectional curvature K isometrically immersed in a $(2n-1)$-dimensional Riemannian or pseudo-Riemannian manifold $\overline{M}$ of constant sectional curvature $\overline{K}$, with $K < \overline{K}$. Assume there exist local coordinates $x_1, \ldots, x_n$ on M such that, the first and second fundamental forms are simultaneously diagonal,i.e.

$$I = \sum_{i=1}^{n} a_{1i}^2 dx_i^2 \qquad II = \sqrt{\overline{K} - K} \sum_{i=2,j=1}^{n} J_{ii} a_{ij} a_{1j} dx_j^2 e_{n+i-1}$$

where a_{ij} are differentiable functions of $x_1, \ldots, x_n$, J_{ii} is the diagonal term of the matrix J and $e_{n+1}, \ldots, e_{2n-1}$ is a frame normal to M. Then the structure equations of the immersion, i.e. the Gauss and Codazzi equations, provide a system of second order differentiable equations which must be satisfied by the matrix function a_{ij}. This system of equations is reduced to a first order system by introducing an off-diagonal matrix function h_{ij} defined in terms of the metric by $h_{ij} = \frac{1}{a_{1i}} \frac{\partial a_{1j}}{\partial x_i}$. With the above conditions, the Gauss and Codazzi equations are exactly the differential equations given in (1). The algebraic condition $aJa^t = J$ is partially imposed by the Gauss equation. More details will appear in a joint paper with J.L. Barbosa. One should observe that for the case $p = 1$, curvature $K > \overline{K}$ [3,15].

In order to state our main results for the generalized equation, we rewrite (1) in matrix notation. Let e_j be the diagonal $(n \times n)$-matrix given by $(e_j)_{rs} = \delta_{rj}\delta_{sj}$. We define the 1-form matrices

$$E = \sum_{j=1}^{n} e_j dx_j \qquad C = hE - Eh^t. \tag{3}$$

Then the system of equations (1) reduces to

$$\begin{aligned} aJa^t &= J & (4) \\ da \wedge E &= aE \wedge C & (5) \\ dC &= C \wedge C - \frac{K}{2} EJ \wedge QE, & (6) \end{aligned}$$

where

$$Q = 2Jv^t v - I, \quad v = (a_{11}, \dots, a_{1n}), \tag{7}$$

I is the identity matrix of order n and J is the diagonal matrix defined by (2).

Our first result provides a Backlund Transformation which shows that for a given solution of (1), one can obtain a one-parameter family of solutions satisfying an initial condition. We will denote by $\tilde{I}$ the diagonal $(n \times n)$-matrix defined by $\tilde{I} = diag(1, -1, \dots, -1)$.

Theorem 1. *Let $\Omega \subset \mathbf{R}^n$ be a simply connected domain and a a solution of* (1) *defined on Ω. Then, for each constant z, $z \neq 0$ if $K \neq 0$, the initial value problem*

$$\begin{cases} dX + XJ^{-1/2}CJ^{1/2} = JA_zJ - XA_z^tX \\ X(x_\circ) = X_\circ, \quad x_\circ \in \Omega, \end{cases} \qquad BT(z)$$

where

$$A_z = JM_z aEJ^{-1/2} \qquad M_z = \frac{1}{2}\left(zI - \frac{K}{z}\tilde{I}\right). \tag{8}$$

has a unique solution X. Moreover, if $X_0 \in \mathcal{O}(p,q)$ then $X \in \mathcal{O}(p,q)$ and it is a solution of (1).

ciated to a by $BT(z)$.

The Backlund Transformation given in Theorem 1, when $p = n$, is a consequence of a geometrical result, namely the generalization of Backlund Theorem [9]. One considers a geodesic congruence between two $n-$dimensional submanifolds M and M' of a $(2n-1)$-dimensional space form $\overline{M}$ of constant sectional curvature $\overline{K}$. Such a congruence is a diffeomorphism $l : M \to M'$ such that for each point $p \in M$ and $p' = l(p)$ there exists a unique geodesic in $\overline{M}$ joining p and p' whose tangent vectors at p and p' are tangent to M and M' respectively. Moreover, one requires the following conditions: the distance between M and M' on $\overline{M}$ is a constant independent of p; the $(n-1)$ angles between the normal spaces at p and p' are all equal to a constant independent of p; the normal bundles of M and M' are flat; the bundle map given by the orthogonal projection commutes with the normal connections. Under these conditions, one proves that M and M' have the same constant curvature K and $K < \overline{K}$. Moreover, one shows that given an $n-$dimensional submanifold M of $\overline{M}$ with constant sectional curvature $K < \overline{K}$, there exists an $n-$parameter family of submanifolds M', which are related to M by a geodesic congruence as described above. The analytic interpretation of this geometric result provides the Backlund Transformation given by $BT(z)$ when $p = n$, i.e. $J = I$. For $p \neq n$, the transformation of Theorem 1 may provide complex solutions and, up to now, it does not have a geometric interpretation.

The following theorem provides a Superposition Formula which shows that given a solution a of (1) and a_1, a_2 two solutions obtained by solving $\mathrm{BT}(z_i)$ for constants z_i, $i = 1, 2$, $z_1 \neq z_2$, then a fourth solution can be obtained algebraically. In what follows we denote by M_i the matrix M_{z_i} defined in (8) for the constant z_i.

Theorem 2. *Let* a *be a solution of* (1) *and let ,* a_i, $i = 1, 2$, *be solutions associated to* a *by* $\mathrm{BT}(z_i)$, $z_1 \neq z_2$. *Then there exists a solution* a^*, *associated to* a_1 *and* a_2 *by* $\mathrm{BT}(z_2)$ *and* $\mathrm{BT}(z_1)$ *respectively, given*

$$a^*Ja^t = J(a_1Ja_2^tM_1 - JM_2)^{-1}(JM_1 - a_1Ja_2^tM_2). \tag{9}$$

Although the solutions given by Theorem 1 may be complex valued, whenever $p \neq n$, by using conveniently the Superposition formula, one can obtain, from a given real solution of (1), a family of real solutions of the same equation. This is the content of our next result.

Corollary *Let* a *be a real solution of the system* (1) *and let* a_1 *be a solution associated to* a *by* BT(z_1). *Then the real valued matrix function* a^*, *given by*

$$a^*Ja^t = (a_1\bar{a}_1^tM_1 - J\bar{M}_1)^{-1}(a_1\bar{a}_1^t\bar{M}_1 - JM_1), \tag{10}$$

is another solution of (1), *where* $\bar{a}_1$ *and* $\bar{M}_1$ *are the complex conjugate of* a_1 *and* M_1, *respectively.*

We observe that for $n = p = 2$ and $K = -1$, Theorems 1 and 2 reduce to the classical Backlund Transformation and Superposition Formula for the sine-Gordon equation. Similarly, when $n = 2$, $p = 1$ and $K = 1$ the above Theorems reduce to the classical results for the elliptic sinh-Gordon equation [2, 5]. Moreover, for any n, when $p = n$ or $p = 1$ the results reduce to the particular cases obtained in previous results [3, 9, 13, 14]. Examples of applications of the above theorems were given in [10], where one and two soliton solutions for equation (1) with $p = n$ were explicitly given. Such solutions were obtained by applying Theorem 1 to the trivial solution $a = I$ and then applying Theorem 2. In section 2, we provide an example for the above Corollary.

Now we introduce the *Intrinsic Generalized Equation*, which is a system of equations for a pair $\{v, h\}$, where $v = (v_1, \ldots, v_n)$ and h is an off-diagonal $(n \times n)$-matrix function defined on $\Omega \subset \mathbf{R}^n$, which satisfy

$$vJv^t \;=\; 1$$

$$\begin{aligned}
\frac{\partial h_{ij}}{\partial x_i} &+ \frac{\partial h_{ji}}{\partial x_j} + \sum_{s\neq i,j} h_{si}h_{sj} = -Kv_iv_j, \quad i \neq j \qquad (11)\\
\frac{\partial h_{ij}}{\partial x_s} &= h_{is}h_{sj}, \quad i,j,s \quad \text{distinct}\\
J_{ii}\frac{\partial h_{ji}}{\partial x_i} &+ J_{jj}\frac{\partial h_{ij}}{\partial x_j} + \sum_{s\neq i,j} J_{ss}h_{is}h_{js} = 0, \quad i \neq j,
\end{aligned}$$

where $1 \leq i,j,s \leq n$.

We observe that whenever v_i are real functions which do not vanish on an open subset $U \subset \mathbf{R}^n$, they define a Riemannian metric on U with constant sectional curvature K. As in the case of the generalized equation, when $n = 2$ the above equation also reduces to the wave equation ($p = 2,\ K = 0$), the sine-Gordon equation ($p = 2,\ K \neq 0$), the Laplace equation ($p = 1,\ K = 0$) and the elliptic sinh-Gordon equation ($p = 1,\ K \neq 0$).

In order to state our main theorem for the Intrinsic Generalized Equation, we rewrite (11) in matrix notation as

$$\begin{aligned}
vJv^t &= 1 & (12)\\
dv &= -vBJ & (13)\\
dC &= C \wedge C - \frac{K}{2}EJ \wedge QE & (14)\\
dB &= BJ \wedge B. & (15)
\end{aligned}$$

where $J,\ C,\ E$ are given by (2), (3) and

$$B = Jh^tE - EhJ \qquad (16)$$

Our next result provides a Backlund Transformation for (11). We introduce the notation Z^o for the off-diagonal matrix obtained from Z .

Theorem 3. *Let* $\Omega \subset \mathbf{R}^n$ *be a simply connected domain of* $\mathbf{R}^n$ *and let* $(v,\ h)$ *be a solution of* (11) *defined in* Ω. *Then, for each* z, $z \neq 0$ *if* $K \neq 0$, *the Initial Value Problem*

$$\left\{\begin{array}{l}
dY + YJ^{-1/2}CJ^{1/2} = BJY + JF_zJ - YF_z^tY\\
Y(x_\circ) = Y_\circ, \quad x_\circ \in \Omega
\end{array}\right. \qquad \tilde{B}T(z)$$

$$F_z = JN_zEJ^{-1/2} \qquad\qquad N_z = \frac{1}{2}\left(zI - \frac{K}{z}Q\right), \tag{17}$$

has a unique solution Y. *Moreover, if* $Y_\circ \in \mathcal{O}(p,q)$, *then* $Y \in \mathcal{O}(p,q)$ *and the pair* $(\tilde{v}, \tilde{h})$, *defined by*

$$\tilde{v} = vY \qquad\qquad \tilde{h} = J^{-1/2}h^tJ^{1/2} - J^{1/2}(N_zY)^\circ, \tag{18}$$

is a new solution of (11).

This result, when $p = n$ or $p = 1$, reduces to the cases obtained previously in [1] and [3] respectively. Examples of one-soliton solutions for (11) with $p = n = 3$ were given explicitly in [11], by applying Theorem 3 to the trivial solution $v = (1, 0, 0)$ $h \equiv 0$.

This paper is organized as follows: in section 1, we introduce some preliminary results that will be used in the rest of the paper and we prove Theorem 1; in section 2, we prove Theorem 2 and the Corollary and finally in section 3 we prove Theorem 3. The results of this paper were partially announced in [12].

1. Proof of Theorem 1.

In order to prove Theorem 1, we need some tecnical results stated in the following lemmas.

Lemma 1.1: *The matrices* a Q B *and* C *defined by (3), (7) and (16) satisfy the relations*

$$aQ = \tilde{I}a \qquad \text{where} \qquad \tilde{I} = diag(1, -1, \ldots, -1) \tag{19}$$

$$Q^t = JQJ, \qquad\qquad Q^2 = I. \tag{20}$$

$$E \wedge C + BJ \wedge E = 0, \qquad\qquad C \wedge E + E \wedge JB = 0. \tag{21}$$

Moreover, if $dv = -vBJ$, *where* v *is defined by (7), then*

$$dQ = BJQ - QBJ. \tag{22}$$

$$\begin{aligned}(aQ)_{ij} &= 2\sum_k (aJ)_{ik}a_{1k}a_{1j} - a_{ij} = \\ &= 2(aJa^t)_{i1}a_{1j} - a_{ij} = \\ &= 2J_{i1}a_{1j} - a_{ij} = (\tilde{I}a)_{ij}\end{aligned}$$

This proves (19). Equations (20) and (21) are identities trivially verified and relation (22) follows from differentiating $Q = 2Jv^tv - I$. In fact,

$$\begin{aligned}dQ &= 2J(dv^tv + v^tdv) \\ &= 2J(JBv^tv - v^tvBJ) \\ &= BJ(2Jv^tv - I) - (2Jv^tv - I)BJ.\end{aligned}$$

■

Lemma 1.2: *Let* $a \in \mathcal{O}(p,q)$ *and let* h *be an off-diagonal matrix function satisfying the condition*

$$\frac{\partial a_{ij}}{\partial x_r} = a_{ir}h_{rj}, \quad j \neq r, \quad i,j,r \geq 1.$$

Then $da = aEh - aJh^tEJ$.

Proof: Since the matrix a^tJda is skew-symmetric, it is sufficient to show that

$$(a^tJda)_{ij} = J_{ii}h_{ij}dx_i - J_{jj}h_{ji}dx_j \ , \ i \neq j.$$

In fact,

$$\begin{aligned}(a^tJda)_{ij} &= \textstyle\sum_r \sum_{s\neq j} a^t_{ir}J_{rr}\frac{\partial a_{rj}}{\partial x_s}dx_s + \sum_r a^t_{ir}J_{rr}\frac{\partial a_{rj}}{\partial x_j}dx_j \\ &= \textstyle\sum_{s\neq j}\sum_r a^t_{ir}J_{rr}a_{rs}h_{sj}dx_s - \sum_r a_{rj}J_{rr}a_{rj}h_{ji}dx_j \\ &= J_{ii}h_{ij}dx_i - J_{jj}h_{ji}dx_j.\end{aligned}$$

■

Lemma 1.3: *Let* $A \equiv A_z$. *If* a *satisfies* (1), *then the following relations hold*

(b) $dA = A \wedge J^{1/2} C J^{-1/2}$

(c) $dA^t = J^{-1/2} C J^{1/2} \wedge A^t$

(d) $A^t \wedge JA = -\frac{K}{2} E J^{1/2} \wedge Q E J^{-1/2}$

Proof: The relation (a) follows from elementary properties of the matrices E and J. Item (b) follows from the fact that a satisfies (5). The relation (c) is obtained by transposing (b) and using the skew-symmetry of C. In order to prove (d) we observe that M is a diagonal matrix and

$$M^2 \; = \; \lambda I \; - \; \frac{K}{2} \tilde{I} \; , \qquad \lambda = \frac{1}{4} \left(z^2 \; + \; \frac{K^2}{z^2} \right) \; .$$

Then, it follows from (4) and (19) that

$$\begin{aligned} a^t M^t J M a \;\; &= \;\; \lambda J \; - \; \tfrac{K}{2} a^t J \tilde{I} a, \\ &= \;\; \lambda J \; - \; \tfrac{K}{2} J Q. \;\; ; \end{aligned}$$

Therefore,

$$E a^t M^t \wedge J M a E \; = \; -\frac{K}{2} E J \wedge Q E \; ,$$

which proves (d).

■

Lemma 1.4: *Suppose that $X(x) \in \mathcal{O}(p,q)$ satisfies* BT(z). *Then for each i, $1 \leq i \leq n$ there exists l, such that $X_{li}(x) \neq 0$. Define the matrix $\tilde{h} = (\tilde{h}_{ij})$ by*

$$\tilde{h}_{ij}(x) \; = \; \frac{1}{X_{li}} \frac{\partial X_{lj}}{\partial x_i}(x), \quad i \neq j, \quad \tilde{h}_{ii} \; = \; 0.$$

Then $\tilde{h}$ is well defined and

$$\tilde{h} \; = \; J^{-1/2} h^t J^{1/2} \; - \; J^{-1/2} (a^t M_z J X)^{\circ}. \tag{23}$$

$$dX_{lj} = J_{ll}A_{lj}J_{jj} - \sum_{r,s} X_{lr}A^t_{rs}X_{sj} - \sum_{r} X_{lr}J_{rr}^{-1/2}C_{rj}J_{jj}^{1/2}.$$

Since $A = JMaEJ^{-1/2}$ and $C = hE - Eh^t$, we get, for $i \neq j$,

$$\frac{1}{X_{li}}\frac{\partial X_{lj}}{\partial x_i} = J_{ii}^{-1/2}h^t_{ij}J_{jj}^{1/2} - \sum_{s}(J^{-1/2}a^tMJ)_{is}X_{sj}.$$

where the right hand sidedoes not depend on l. Hence, $\tilde{h}_{ij} = (J^{-1/2}h^tJ^{1/2})_{ij} - (J^{-1/2}a^tMJX)_{ij}$ for $i \neq j$. Since $\tilde{h}$ is off-diagonal, we conclude that it satisfies (23).

■

Let $\tilde{h}$ be the matrix defined in Lemma 1.4. We consider the matrix $\tilde{C}$ associated to $\tilde{h}$, $\tilde{C} = \tilde{h}E - E\tilde{h}^t$. Introducing the notation

$$\alpha = X^tJM_zaJ^{-1/2} \qquad\qquad \beta = J^{1/2}hJ^{-1/2}, \tag{24}$$

we can write

$$\tilde{C} = E(\alpha - \beta) - (\alpha^t - \beta^t)E. \tag{25}$$

Lemma 1.5: *Let a be a solution of* (1) *and suppose that* $X \in O(p,q)$ *satisfies* BT(z) *and* $dX \wedge E = XE \wedge \tilde{C}$. *Then α and β satisfy the relations*

(a) $d(E\alpha) + E\beta \wedge E\alpha = E\alpha \wedge (E\alpha - E\beta + \beta^tE)$

(b) $d(\alpha^tE) - \alpha^tE \wedge \beta^tE = (\beta^tE - E\beta - \alpha^tE) \wedge \alpha^tE$

(c) $d(E\beta) + E\beta \wedge \left(E\beta - \frac{1}{2}\beta^tE\right) = d(\beta^tE) + \left(\frac{1}{2}E\beta - \beta^tE\right) \wedge \beta^tE$

(d) $E\alpha \wedge \alpha^tE = -\frac{K}{2}EJ \wedge \tilde{Q}E$,

where $\tilde{Q} = 2J\tilde{v}^t\tilde{v} - I$, *and* $\tilde{v} = (X_{11}, \ldots, X_{1n})$.

$$E \wedge dX^t = E(\beta - \alpha) \wedge EX^t.$$

Therefore, using (24) we get

$$E \wedge d\alpha = -E\alpha \wedge E\alpha + E\beta \wedge E\alpha + EX^tJ \wedge M_z daJ^{-1/2}.$$

Hence, in order to obtain the relation (a) it is sufficient to show that

$$EX^tJ \wedge M_z daJ^{-1/2} = E\alpha \wedge E\beta - E\alpha \wedge \beta^t E.$$

This follows easily from Lemma 1.2. Relation (b) is obtained by transposing (a).

From Lemma 1.2, taking exterior derivative of da, we get

$$da \wedge (Eh - Jh^tEJ) - aE \wedge dh - aJdh^t \wedge EJ = 0.$$

Since matriz a satisfies (5), we obtain relation (c). Relation (d) follows from the fact that $a(x)$ and $X(x)$ are in $\mathcal{O}(p,q)$ and also because X and $\tilde{Q}$ satisfy the equality $X\tilde{Q} = \tilde{I}X$, given in (19).

∎

Proof of Theorem 1: We first show that the system BT(z) is integrable. Let $\mathcal{I}$ be the ideal generated by the 1-form matrices

$$P = dX + XJ^{-1/2}CJ^{1/2} - JA_zJ + XA_z^tX. \qquad (26)$$

Then $\mathcal{I}$ is closed under exterior differentiation. In fact,

$$\begin{aligned} dP &= dX \wedge (A_z^tX + J^{-1/2}CJ^{1/2}) - XA_z^t \wedge dX \\ &+ XJ^{-1/2}dCJ^{1/2} + XdA_z^tX - JdA_zJ. \end{aligned}$$

It follows from (26) and the relations of Lemma 1.3 that

$$\begin{aligned} dP &= P \wedge (A_z^tX + J^{-1/2}CJ^{1/2}) - XA_z^t \wedge P + \\ &+ XJ^{-1/2}(dC - C \wedge C + \frac{K}{2}EJ \wedge QE)J^{1/2}. \end{aligned} \qquad (27)$$

If the initial value $X_\circ$ satisfies $X_\circ J X_\circ^t = J$, the solution X of BT(z) satisfies $XJX^t = J$ identically. In fact, if X satisfies BT(z), then

$$d(XJX^t - J) = (XJX^t - J)H + H^t(XJX^t - J),$$

where $H = -A_z X^t$. This is a linear problem for $XJX^t - J$, whose value at $x_\circ$ is zero. From the uniqueness of the solution we get that $XJX^t = J$ in Ω.

We now show that X is a solution of (1) or equivalently (4)-(6). Since X satisfies BT(z) and $A_z \wedge E = 0.$, we get

$$dX \wedge E = -XA_z^t \wedge XE - XJ^{-1/2}CJ^{1/2} \wedge E. \tag{28}$$

Using Lemma 1.4, we introduce the matrix $\tilde{h}$ which satisfies (23) and $\tilde{C} = \tilde{h}E - E\tilde{h}^t$. Then,

$$\begin{aligned} XE \wedge \tilde{C} &= XE \wedge \tilde{h}E \\ &= XE \wedge J^{-1/2}h^t J^{1/2}E - XE \wedge J^{-1/2}(a^t M_z JX)^\circ E, \end{aligned}$$

Using the identity

$$Y^\circ = Y - \sum_{j=1}^{n} e_j Y e_j$$

and (8), we obtain

$$XE \wedge \tilde{C} = -XJ^{-1/2}C \wedge J^{1/2}E - XA_z^t \wedge XE, \tag{29}$$

since $E \wedge E = 0$ and, for each j, the matrix $e_j a^t M_z JXe_j$ is diagonal. From (28) and (29), we conclude that X satisfies equation (5). Finally, with the notation introduced in (24), we consider the following equations obtained from (25),

$$d\tilde{C} = E \wedge d(\beta - \alpha) + d(\beta^t - \alpha^t) \wedge E \tag{30}$$

and

$$\begin{aligned} \tilde{C} \wedge \tilde{C} &= E(\alpha - \beta) \wedge E(\alpha - \beta) - \\ &- E(\alpha - \beta) \wedge (\alpha^t - \beta^t)E + (\alpha^t - \beta^t)E \wedge (\alpha^t - \beta^t)E. \end{aligned} \tag{31}$$

$$d\tilde{C} = \tilde{C} \wedge \tilde{C} - \frac{K}{2} EJ \wedge \tilde{Q} E,$$

i.e., equation (6) is also satisfied by X.

2. Proof of Theorem 2 and Corollary.

Before proving Theorem 2 we observe that if a_1, a_2 are solutions of (1) associated to a by BT(z$_1$) and BT(z$_2$) respectively, then the matrix function $a_1 J a_2^t M_1 - J M_2$, which appears on the right hand side of (9) is generically invertible in open subsets of $\mathbf{R}^n$.

Proof of Theorem 2: By hipothesis, a_1 and a_2 satisfy equation BT(z), i. e.,

$$da_i + a_i J^{-1/2} C J^{1/2} = J A_i J - a_i A_i^t a_i, \quad i = 1, 2,$$

where $A_i \equiv A_{z_i}$.

We introduce the auxiliary notation

$$F = a_1 J a_2^t M_1 - J M_2 \qquad G = J M_1 - a_1 J a_2^t M_2$$

$$A_{ij} = J M_j a_i E J^{-1/2}, \quad i, j = 1, 2.$$

We need to show that the matrix a^* given by

$$a^* J a^t = J F^{-1} G \tag{32}$$

satisfies the equations

$$a^* J (a^*)^t = J \tag{33}$$

and

$$da^* + a^* J^{-1/2} C_i J^{1/2} = J A_{ij} J - a^* A_{ij}^t a^*, \quad i \neq j, \quad i, j = 1, 2, \tag{34}$$

where $C_i = h_i E - E h_i^t$ and h_i is the matrix associated to the solution a_i.

From (8), we have

$$M_1^2 - M_2^2 = (\lambda_1 - \lambda_2) I, \quad \text{where} \quad \lambda_i = \frac{1}{4} \left(z_i^2 + \frac{K^2}{z_i^2} \right). \tag{35}$$

G verify the relation $GJG^t = FJF^t$, therefore equation (33) holds.

Differentiating (32), and substituting da_1, da_2^t we get the expression

$$\begin{aligned} da^* &= JF^{-1}GJ(da - aA_{22}^t a) - a^* EJ^{-1/2} a_2^t M_1 J a^* + \\ &+ JF^{-1} a_1 (A_1^t M_1 - A_2^t M_2) a. \end{aligned} \tag{36}$$

But $C_2 = h_2 E - Eh_2^t$, and from (23),

$$h_2 = J^{-1/2} h^t J^{1/2} - J^{-1/2} (a^t M_2 J a_2)^o.$$

Moreover, from Lemma 1.2, we have $da = aEh - aJh^t EJ$. Therefore,

$$\begin{aligned} da^* &= JF^{-1} a_1 (A_1^t M_1 - A_2^t M_2) a - JF^{-1} G M_2 J a_2 E J^{1/2} - \\ &- a^* EJ^{-1/2} a_2^t M_1 J a^* - a^* J^{-1/2} C_2 J^{1/2}. \end{aligned} \tag{37}$$

Substituting A_1^t, A_2^t and G respectively by their expressions, the right hand side of (37), which contain those terms, reduce to

$$JF^{-1}(a_1 J a_2^t M_1 - JM_2) J M_1 a_2 E J^{1/2},$$

which is exactly $JA_{21}J$. Therefore, from (37), we finally obtain

$$da^* + a^* J^{-1/2} C_2 J^{1/2} = JA_{21}J - a^* A_{21}^t a^*.$$

Hence, a^* is a solution of (1) associated to a_2 by [BT(z_1)].

Similarly, one shows that (34) holds for $i = 1$, $j = 2$, since (9) is equivalent to

$$a^* J a^t = J(a_2 J a_1^t M_2 - JM_1)^{-1} (JM_2 - a_2 J a_1^t M_1).$$

■

$$da_1 + a_1 J^{-1/2} C J^{1/2} = J A_1 J - a_1 A_1^t a_1,$$

where $A_1 = J M_1 a E J^{-1/2}$ and $M_1 = \frac{1}{2}\left(z_1 I - \frac{K}{z_1} J\right)$. We coonsider $M_2 = -J\bar{M}_1$ and $a_2 = -J\bar{a}_1 J$. Then

$$\begin{aligned} da_2 + a_2 J^{-1/2} C J^{1/2} &= -J d\bar{a}_1 J - J\bar{a}_1 J^{1/2} C J^{1/2} \\ &= J A_2 J - a_2 A_2^t a_2, \end{aligned}$$

since a and C are real matrices and $\overline{J^{-1/2}} = J^{1/2}$. Moreover $a_2 J a_2^t = J$. Therefore, a_2 is a solution of (1), associated to a by [BT(z_2)]. Applying Theorem 2 to a_1 and a_2 , we obtain a solution a^* defined by

$$a^* J a^t = J(a_1 J a_2^t M_1 - J M_2)^{-1}(J M_1 - a_1 J a_2^t M_2)$$

which reduces to (10) after substituting a_2 and M_2. Now,

$$\begin{aligned} \bar{a}^* J a^t &= (\bar{a}_1 a_1^t \bar{M}_1 - J M_1)^{-1}(\bar{a}_1 a_1^t M_1 - J\bar{M}_1) \\ &= (a_1 \bar{a}_1^t M_1 - J\bar{M}_1)^{-1}(a_1 \bar{a}_1^t \bar{M}_1 - J M_1) \\ &= a^* J a^t, \end{aligned}$$

since a_1 satisfies (4). Therefore, a^* is a real solution of (1).

■

We conclude this section by giving an example of an application of the Corollary. We consider equation (1) with $n = 2,\ p = 1$ and $K = 1$. We start with the trivial solution $a = I$. Applying Theorem 1 we obtain a one-parameter family of complex valued solutions of (1), associated to a by BT(z), given by

$$a_1 = \frac{1}{1 - f^2}\begin{pmatrix} 1 + f^2 & 2f \\ 2f & 1 + f^2 \end{pmatrix},$$

where

$$f(x_1, x_2) = exp(bx_1 + icx_2) \quad b = -\frac{1}{2}(z - z^{-1}),\ c = -\frac{1}{2}(z + z^{-1}), \quad \text{and } z \in \mathbf{R}\backslash\{0\}.$$

The Corollary provides real solutions given by

$$a^* = \frac{1}{1 - g^2}\begin{pmatrix} 1 + g^2 & 2g \\ 2g & 1 + g^2 \end{pmatrix}, \quad \text{where} \quad g(x_1, x_2) = \frac{b \cos cx_2)}{c \cosh bx_1}.$$

In this section we prove Theorem 3 which gives a Backlund Transformation for the Intrinsic Generalized Equation. In order to prove this theorem, we will establish some relations in the following Lemmas.

Lemma 3.1: *The matrices* F_z *ande* N_z *introduced in* (17) *verify the following relations*:

(a) $N_z^t = JN_zJ$

(b) $N_z^tJN_z = \lambda J - \frac{K}{2}JQ, \quad \lambda = \frac{1}{4}\left(z^2 + \frac{K^2}{z^2}\right)$

(c) $dF_z = JB \wedge F_z + F_z \wedge J^{1/2}CJ^{-1/2}$

(d) $dF_z^t = F_z^t \wedge BJ + J^{-1/2}CJ^{1/2} \wedge F_z^t$

(e) $F_z^t \wedge JF_z = -\frac{K}{2}J^{1/2}E \wedge QEJ^{-1/2}$.

Proof: Relation (a) is an immediate consequence of the equality $Q^t = JQJ$, in (20). From (a), $N_z^tJN_z = JN_z^2$. Therefore using (17) and (20), we obtain relation (b). Now, it follows from (22), (17) and (21) that

$$\begin{aligned} dF_z &= -\tfrac{K}{2z}J(BJQ - QBJ) \wedge EJ^{-1/2}, \\ &= JB \wedge F_z - JN_z(BJ \wedge E)J^{-1/2}, \\ &= JB \wedge F_z + JN_z(E \wedge C)J^{-1/2}. \end{aligned}$$

Hence, relation (c) holds and by transposition we get (d). Finally, (e) follows directly from (b). ■

Using the identity

$$S^\circ = S - \sum_{j=1}^{n} e_jSe_j$$

and the matrices B, C and Q defined, respectively, by (16), (3) and (7), we verify that the matrices $\tilde{B}$, $\tilde{C}$ and $\tilde{Q}$, corresponding to the pair $(\tilde{v}, \tilde{h})$, are given by

$$\begin{aligned} B &= J \quad CJ \quad + J \quad (EN_zYJ \quad - J \quad Y \quad N_zE)J \quad , \\ \tilde{C} &= J^{1/2}BJ^{1/2} - J^{1/2}(N_zYEJ^{-1/2} - J^{-1/2}EY^tN_z^t)J^{1/2}, \\ \tilde{Q} &= JY^tJQY + JY^tJY - I. \end{aligned}$$

It follows from Lemma 3.1 (a), that the matrix $\tilde{B}$ is simply written as

$$\tilde{B} = J^{-1/2}CJ^{-1/2} + F_z^tYJ - JY^tF_z, \tag{38}$$

where F_z is the matrix defined by (17). Moreover, substituting

$$G = N_zYEJ^{-1/2}, \tag{39}$$

we can write

$$J^{-1/2}\tilde{C}J^{-1/2} = B - G + G^t. \tag{40}$$

Lemma 3.2: *Let $G = N_zYEJ^{-1/2}$. If Y satisfies $\tilde{BT}(z)$ and $YJY^t = J$, then the following relations hold*

(a) $G \wedge JG^t = 0$

(b) $dG + G \wedge JG = G \wedge JB + BJ \wedge G$

(c) $dG^t - G^t \wedge JG^t = G^t \wedge JB + BJ \wedge G^t$

(d) $G^t \wedge JG = -\frac{K}{2}J^{1/2}E \wedge \tilde{Q}EJ^{-1/2}$.

Proof: Item (a) is immediate. It follows from (17), $\tilde{BT}(z)$ and the use of (21) and Lemma 3.1 (a) that relation (b) consists in annihilating the expression

$$(dN_z + N_zBJ - BJN_z) \wedge YEJ^{-1/2},$$

Since, by (17) and (22)

$$\begin{aligned} 2zdN_z &= -KdQ, \\ &= K(QBJ - BJQ), \end{aligned}$$

it follows from (17) that $dN_z = BJN_z - N_zBJ$. Relation (c) can be obtained by transposing (b) and using the fact that B and C are skewsymmetric. Finally, since $\tilde{Q} = JY^tJQY$, we conclude the proof, by using Lemma 3.1 (b).

■

The system $\tilde{BT}(z)$ is integrable. In order to prove that, we consider $\mathcal{I}$ the ideal generated by the 1-form matrix

$$P = dY + YJ^{-1/2}CJ^{1/2} - BJY - JFJ + YF^tY. \tag{41}$$

Then $\mathcal{I}$ is closed under exterior differentiation. In fact,

$$\begin{aligned} dP &= dY \wedge (J^{-1/2}CJ^{1/2} + F^tY) + (BJ - YF^t) \wedge dY + \\ &+ YJ^{-1/2}dCJ^{1/2} - dBJY - JdFJ + YdF^tY. \end{aligned}$$

Therefore, from (41) and $F \wedge JF^t = 0$ we get

$$\begin{aligned} dP &= P \wedge (J^{-1/2}CJ^{1/2} + F^tY) + (BJ - YF^t) \wedge P + \\ &+ YJ^{-1/2}(dC - C \wedge C + \frac{K}{2}EJ \wedge QE)J^{1/2} - (dB - BJ \wedge B)JY, \end{aligned} \tag{42}$$

by applying Lemma 3.1 (c)(d)(e). Since (v, h) satisfies (14) and (15), it foollows from (42) that $d\mathcal{I} \subset \mathcal{I}$.

Suppose that $Y_0 \in \mathcal{O}(p, q)$ and let Y be a solution of $\tilde{BT}(z)$. Then,

$$d(YJY^t - J) = H(YJY^t - J) + (YJY^t - J)H^t,$$

where $H = BJ - YF^t$. This is a linear problem for $YJY^t - J$, whose value at x_0 is zero. From the uniqueness of the solution, we get $YJY^t = J$ in Ω.

The pair $(\tilde{v}, \tilde{h})$, given by (18), is also a solution of (11). In fact, since v satisfies (12) and $YJY^t = J$, it follows that $\tilde{v}J\tilde{v}^t = 1$. From (13), $\tilde{BT}(z)$ and (18) we get

$$\begin{aligned} d\tilde{v} &= dvY + vdY \\ &= \tilde{v}(JY^tF - F^tYJ - J^{-1/2}CJ^{-1/2})J, \end{aligned}$$

i. e., using (38), $d\tilde{v} = -\tilde{v}\tilde{B}J$. Hence, equation (13) is satisfied by $(\tilde{v}, \tilde{h})$.

$$J^{-1/2} d\tilde{C} J^{-1/2} = dB - dG + dG^t \tag{43}$$

and

$$J^{-1/2}\tilde{C} \wedge \tilde{C} J^{-1/2} = (B - G + G^t) \wedge J(B - G + G^t). \tag{44}$$

Subtracting (44) from (43) and using Lemma 3.2 we obtain

$$d\tilde{C} = \tilde{C} \wedge \tilde{C} - \frac{K}{2} EJ \wedge \tilde{Q}E + J^{1/2}(dB - BJ \wedge B)J^{1/2}.$$

Since (v, h) satisfies (15), we conclude that equation (14) is also satisfied by $(\tilde{v}, \tilde{h})$.

Finally, from (38) we get

$$d\tilde{B} - \tilde{B}J \wedge \tilde{B} = J^{-1/2}(dC - C \wedge C + \frac{K}{2} EJ \wedge QE)J^{-1/2},$$

by applying Lemma 3.1 and using the fact that $F \wedge JF^t = 0$. Therefore, equation (15) is also satisfied by $(\tilde{v}, \tilde{h})$, since (v, h) satisfies (14).

■

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