TOROIDAL SUBMANIFOLDS OF CONSTANT NON POSITIVE CURVATURE

M. L. Rabelo **K. Tenenblat** *

Introduction

The $n-$dimensional hyperbolic submanifolds M^n of the euclidean space $\mathbf{R}^{2n-1}$ are in correspondence with solutions of a certain system of differential equations called the *intrinsic generalized sine-Gordon equation.* Similarly, flat submanifolds M^n of the unit sphere S^{2n-1}, correspond to solutions of the *intrinsic generalized wave equation.* [A], [BT].

In this paper, we consider particular solutions of these equations which depend only on one variable and we obtain the associated submanifolds as being generated by curves, in such a way that each point of the curve describes an (n-1)-dimensional torus. These are called *toroidal submanifolds.*

In the case of constant negative curvature the associated submanifolds are generated by curves which are given explicitly in terms of a family of elliptic functions when $n \leq 3$. For $n \geq 4$, the submanifold is generated by a tractrix in $\mathbf{R}^n$ (see Theorem 2.1 Proposition 2.1). Moreover, we show that these are the only hyperbolic toroidal submanifolds. As an immediate consequence, we prove that there are no complete toroidal submanifolds M^n in $\mathbf{R}^{2n-1}$, with constant sectional curvature -1. This is also contained in the result of Aminov [A2].

Similarly, the flat submanifolds M^n of S^{2n-1} which correspond to the special solutions of the intrinsic generalized wave equation, are the Clifford torus and the toroidal submanifolds generated by a family of curves contained in a two-dimensional sphere (see Theorem 3.1). Moreover, we show that these characterize the flat toroidal submanifolds of S^{2n-1}. In particular, we conclude that the Clifford torus T^n is the only complete flat toroidal submanifold of the sphere S^{2n-1}.

*The authors were partially supported by CNPq.

realized isometrically in $\mathbf{R}^3$. It has been conjectured for a long time that hyperbolic n-space cannot be isometrically immersed in $\mathbf{R}^{2n-1}$, $n \geq 2$. Partial results in this direction were obtained by Xavier [X] and Aminov [A2]. The difficulty of the problem lies in the study of global solutions of the system of differential equations which is highly non linear. One should observe that Okayasu [O] has obtained a $O(2) \times O(2)-$invariant complete hypersurface M^3 of constant negative scalar curvature in $\mathbf{R}^4$.

1. Preliminaries

In this section, we present the results and definitions that will be used in the following sections. We denote by $M^n(K)$ an $n-$dimensional manifold of constant sectional curvature K.

It is well known that for submanifolds $M^n(K) \subset \bar{M}^{2n-1}(\bar{K})$, $K < \bar{K}$, there exists a parametrization by lines of curvature, such that the metric (g_{ij}) $1 \leq i,j \leq n$ is diagonal with constant trace 1 and the normal bundle is flat [A] [M] [T]. More precisely

Theorem 1.1 *Let $M^n(K)$ be a riemannian manifold isometrically immersed in $\bar{M}^{2n-1}(\bar{K})$, such that $K < \bar{K}$. Then locally there exist coordinates $(x_1, \ldots, x_n)$ such that the first and second fundamental forms are given by*

$$I = \sum_{i=1}^{n} a_{1i}^2 \, \mathrm{d}x_i^2 \qquad II = \sum_{i=1,j=2}^{n} a_{ji}a_{1i} \, \mathrm{d}x_i^2 e_{n+j-1} \tag{1}$$

where $\sum_{i=1}^n a_{1i}^2 = 1$ and e_{n+j-1} is an orthonormal frame normal to M.

From now on we normalize the curvatures by considering $\bar{K} - K = 1$. Under the conditions of Theorem 1.1, one can show that the $n \times n$ matrix function $a = (a_{ij})$ satisfies the following system of equations

$$aa^t = I, \tag{2}$$

$$\frac{\partial a_{1i}}{\partial x_j} = a_{1j}h_{ji} \qquad i \neq j, \tag{3}$$

$$\frac{\partial h_{ij}}{\partial x_s} = h_{is}h_{sj} \qquad i, j, s \text{ distinct}, \tag{5}$$

$$\frac{\partial a_{jl}}{\partial x_i} = a_{ji}h_{il} \qquad i \neq l, \quad j \geq 2, \tag{6}$$

where the off diagonal matrix function $h = (h_{ij})$ is defined by (3). When $K = 0$, this is the *generalized wave equation* (GWE) and when $K = -1$, this is the *generalized sine-Gordon equation* (GSGE) (see [T] [TC] [TT]).

The above equations are equivalent to the Gauss and Codazzi equations. As a consequence of the Fundamental Theorem for Submanifolds, given a matrix function a, which satisfies the equations (2)-(6) defined on a simply connected open subset $\Omega \subset \mathbf{R}^n$, there exists an isometric immersion $X : \Omega \subset \mathbf{R}^n \longrightarrow \bar{M}^{2n-1}(\bar{K})$ such that the first and second fundamental forms are given by (1).

In the two-dimensional case, the Codazzi equation (6) is a consequence of the Gauss equation (4) and (5). Motivated by this result, intrinsic generalizations for the sine-Gordon and the wave equations were introduced in [BT] as stated in the following result.

Theorem 1.2 *Let Ω be a simply connected open subset of $\mathbf{R}^n$, with coordinates $x_1, \ldots, x_n$ and $g = g_{ij}$ a riemannian metric which is diagonal, with tr $g \equiv 1$ and constant sectional curvature K. Then the smooth function $v : \Omega \longrightarrow \mathbf{R}^n$, $v = (v_1, \ldots, v_n)$ defined by $v_i^2 = g_{ii}$ satisfies*

$$vv^t = 1 \tag{7}$$

$$\frac{\partial v_i}{\partial x_j} = v_j h_{ji} \qquad i \neq j, \tag{8}$$

$$\frac{\partial h_{ij}}{\partial x_i} + \frac{\partial h_{ji}}{\partial x_j} + \sum_{s\neq i, s\neq j} h_{si}h_{sj} = -K v_i v_j, \qquad i \neq j, \tag{9}$$

$$\frac{\partial h_{ij}}{\partial x_s} = h_{is}h_{sj}, \qquad i, s, j \text{ distinct}, \tag{10}$$

where $1 \leq i, j, s \leq n$ and h_{ij} is an off diagonal matrix function given by (7). Conversely, given a solution of (7)-(10) such that $v_i(x) \neq 0$, for all $x \in \Omega$, then $g_{ij} = \delta_{ij}v_i^2$ defines a metric on Ω which satisfies the above conditions.

As it was shown in [BT], whenever $v_i(x) \neq 0$, for all $x \in \Omega$ and $1 \leq i \leq n$, then

$$\frac{\partial v_i}{\partial x_i} = -\sum_{j\neq i} v_j h_{ij} \tag{11}$$

$$\frac{\partial h_{ji}}{\partial x_i} + \frac{\partial h_{ij}}{\partial x_j} + \sum_{s\neq i, s\neq j} h_{js} h_{is} = 0. \tag{12}$$

The system (7)-(12) was called the *Intrinsic Generalized Wave Equation* (IGWE) when the constant $K = 0$ and the *Intrinsic Generalized Sine-Gordon Equation* (IGSGE) when $K = -1$.

In what follows, we consider solutions v_i which define metrics as in Theorem 1.2, hence $v_i(x) \neq 0, \forall x \in \Omega$. Therefore, the intrinsic equations can be reduced to (7)-(10). Under these conditions , the relation between the generalized equations is stated in the following theorem (see Theorem 2 [BT] and [A]).

Theorem 1.3

(i) *If a is a solution of the GWE, then each row of a, whose elements do not vanish, is a solution of the IGWE.*

(ii) *Suppose a is a solution of the GSGE. Then the first row of a is a solution of the IGSGE, whenever its elements do not vanish.*

(iii) *Conversely, if v is a solution of the IGWE (resp. IGSGE) whose coordinate functions do not vanish on a simply connected domain $\Omega \subset I\!\!R^n$, then there exists a solution a on Ω for the GWE (resp. GSGE) whose first row is v.*

It follows from the above results that the solutions v of the IGWE (resp. IGSGE), which do not vanish on a simply connected domain $\Omega \subset \mathbf{R}^n$ are in correspondence with the isometric immersions $X : \Omega \longrightarrow S^{2n-1}$ (resp. $X : \Omega \longrightarrow \mathbf{R}^{2n-1}$) of constant curvature $K = 0$ (resp. $K = -1$). Such an immersion is determined up to rigid motions and is called *the immersion associate to v*

As an example we consider the solution of the IGSGE

$$v_1 = \text{tgh } x_1 \quad v_j = c_j \text{ sech } x_1 \quad j \geq 2$$

where $c_j$and $\sum_{j=2}^{n} c_j^2 = 1, x_1 > 0$. The associated immersion is

$$X = (\text{tgh } x_1 - x_1, c_2\text{sech } x_1 \cos x_2, c_2\text{sech } x_1 \sin x_2, \ldots, c_n\text{sech } x_1 \cos x_n, c_n\text{sech } x_1 sinx_n)$$

dimension $n-1$ in $\mathbf{R}^{2n-2}$.

Similarly, the Clifford torus

$$X(x_1 \dots x_n) = (c_1 \cos x_1, c_1 \sin x_1, \dots c_n \cos x_n, c_n \sin x_n)$$

is the immersion associated to the constant solution $v_i = c_i \neq 0, \sum_{i=1}^n c_i^2 = 1$, of the IGWE.

For later use, we introduce the following definitions.

Definition 1.1

a) Let

$$\alpha(x_1) = (f_1(x_1, \dots, f_n(x_1)) \quad x_1 \in I \subset \mathbf{R}$$

be a parametrization of a regular curve in $\mathbf{R}^n, n \geq 3$ such that $f_i(x_1) \neq 0, i \geq 2, \forall x_1 \in I$. The submanifold, which up to a rigid motion of $\mathbf{R}^{2n-1}$, is given by

$$(f_1(x_1), f_2(x_1) \cos x_2, f_2(x_1) \sin x_2, \dots, f_n(x_1 \cos x_n, f_n(x_1 \sin x_n),$$

is called a *toroidal submanifold* M^n of $\mathbf{R}^{2n-1}$ *generated by the curve* α.

b) Let

$$\beta(x_1) = (f_0(x_1), f_1(x_1), \dots, f_n(x_1)) \quad x_1 \in I \subset \mathbf{R}$$

be a parametrization of a regular curve in $\mathbf{R}^{n+1}, n \geq 3$, such that $f_i(x_1) \neq 0, i \geq 2, \forall x_1 \in I$. The submanifold , which up to a rigid motion of $\mathbf{R}^{2n}$ is given by

$$(f_0(x_1), f_1(x_1), f_2(x_1) \cos x_2, f_2(x_1) \sin x_2, \dots, f_n \cos x_n, f_n(x_1) \sin x_n)$$

is called a *toroidal submanifold* M^n of $\mathbf{R}^{2n}$ *generated by the curve* β .

A toroidal submanifold is generated by a curve in such a way that each point of the curve describes a flat (n-1)-dimensional torus T^{n-1} contained in R^{2n-2}. The immersion generated by the tractrix and the Clifford torus, given above, are examples of toroidal submanifolds.

2. Hiperbolic submanifolds of Euclidean space

In this section, we first obtain the solutions of the IGSGE which depend only on one independent variable. These solutions are invariant by the local symmetry group of terms of elementary functions or elliptic functions. Then we show that the isometric immersions $M^n \hookrightarrow \mathbf{R}^{2n-1}$ of constant curvature $K \equiv -1$ associated to these solutions are , up to rigid motions, toroidal submanifolds.

Moreover, we prove that these are the only toroidal submanifolds of R^{2n-1} with constant curvature . In particular, we conclude that these are no complete toroidal submanifolds $M^n \subset \mathbf{R}^{2n-1}$ with $K \equiv -1$.

Proposition 2.1 *Let $v = (v_1, \ldots, v_n), n \geq 2$, be a solution of the IGSGE which depends only on x_1, such that $v_i(x_1) \neq 0, 1 \leq i \leq n$ for x_1 in an open interval $I \subset \mathbf{R}$.*

(i) *If* n=2 , *then*

$$\begin{aligned} (v_1^{'})^2 &= (1 - v_1^2)(1 + c - v_1^2), \\ v_2^2 &= 1 - v_1^2, \end{aligned} \tag{13}$$

where $c \in \mathbf{R}$ and $1 + c > 0$.

(ii) *If $n = 3$, then*

$$\begin{aligned} (v_1^{'})^2 &= (\frac{b^2}{c^2} - v_1^2)(1 + c^2 - b^2 - v_1^2), \\ v_2^2 &= \frac{c^2}{1 + c^2}(\frac{b^2}{c^2} - v_1^2), \\ v_3^2 &= \frac{1}{1 + c^2}(1 + c^2 - b^2 - v_1^2), \end{aligned} \tag{14}$$

where $b \neq 0, c \neq 0$ and $1 + c^2 - b^2 > 0$.

(iii) *If $n \geq 4$ then*

$$\begin{aligned} v_1 &= \pm \mathrm{tgh}(x_1 - a), \\ v_j &= c_j \mathrm{sech}(x_1 - a) \quad , j \geq 2 \end{aligned} \tag{15}$$

where $a, c_j \neq 0 \in \mathbf{R}$, and $\sum_{j=2}^{n} c_j^2 = 1$.

Proof: It follows from the hypothesis that equations (7) - (10) reduce to

$$vv^t = 1 \tag{16}$$

$$h_{1i} = \frac{v_i^{'}}{v_1} \quad , i \geq 2 \tag{17}$$

$$h_{1i}h_{1j} = v_i vj, \quad i \neq j, \quad i,j \geq 2 \tag{19}$$

$$h'_{1j} = v_1 vj \quad , j \geq 2 \tag{20}$$

If $n = 2$, it is easy to see that this system is equivalent to (13).

If $n \geq 3$, using (17), the equations (19) and (20) reduce to

$$\frac{v'_i}{v_1}\frac{v'_j}{v_1} = v_i v_j \quad \text{for } i \neq j, \quad i,j \geq 2 \tag{21}$$

$$\left(\frac{v'_j}{v_1}\right)' = v_1 v_j, \quad j \geq 2 \tag{22}$$

It follows from (21) that $v'_i \neq 0, \forall i \geq 2$ and

$$v_1 vj = \frac{v'_i v'_j}{v_i v_1}, \quad i \neq j, \quad i,j \geq 2.$$

Substituting into (22) we obtain

$$\left(\frac{v'_j}{v_1}\right)' = \frac{v'_i}{v_i}\frac{v'_j}{v_1}, \quad i \neq j, \quad i,j \geq 2$$

hence

$$v'_j = c_{ji} v_1 v_i \quad , i \neq j, i,j \geq 2 \tag{23}$$

where c_{ji} is a nonzero real constant.

If $n \geq 4$, then it follows from (23) that

$$v_j = b_j w \quad j \geq 2 \tag{24}$$

where w is a nowhere zero differentiable function of x_1. Therefore, (21) reduces to

$$\left(\frac{w'}{v_1}\right)^2 = w^2 \tag{25}$$

Hence we have $v_1 = \varepsilon \frac{w'}{w}$, where $\varepsilon^2 = 1$ and using (24), the equation (16) gives

$$(w')^2 = w^2(1 - \sum_{j=2}^{n} b_j^2 w^2)$$

Therefore

$$w = \pm \frac{1}{\sqrt{Z}\cosh(x_1 - a)}, \quad where \quad Z = \sum_{j=2}^{n} b_j^2,$$

$$c_j = \frac{b_j}{\sqrt{\sum_{j=2}^n b_j^2}}, \quad j \geq 2.$$

If $n = 3$, then it follows from (23) that

$$v_2^{'} = cv_1v_3 \tag{26}$$

where $c = c_{23}$ is a nonzero constant. Substituting this expression into (22) we get

$$v_3^{'} = \frac{-k}{c}v_1v_2. \tag{27}$$

Since $\sum_{i=1}^3 v_iv_i^{'} = 0$, we obtain from (26) and (27) that

$$v_1^{'} = -(c + \frac{1}{c})v_2v_3 \tag{28}$$

From the quotient of (28) and (26) we have

$$\frac{v_1^{'}}{v_2^{'}} = -\frac{c^2+1}{c^2}\frac{v_2}{v_1}$$

hence

$$v_2^2 = \frac{1}{1+c^2}(b^2 - c^2v_1^2) \tag{29}$$

It follows from (16) that

$$v_3^2 = \frac{1+c^2-b^2-v_1^2}{1+c^2} \tag{30}$$

Therefore, using (28) we conclude that v satisfies (14).

Remark 2.1 We observe that when $b^2 = c^2$, the equations (14) give v in terms of elementary functions exactly as in (15). Moreover, for each solution given in Proposition 2.1 there exists a value $x_0 \in \mathbf{R}$ such that v_1 tends to zero when x approaches x_0.

For the proof of Theorem 2.1, we will need the following technical result

Lemma 2.1 Suppose v_j are the functions given by Proposition 2.1, then

$$v_1^{''} - v_1^3 + \sum_{j=2}^n \frac{(v_j^{'})^2}{v_1} + v_1 = 0 \tag{31}$$

$$\left(\frac{v_j}{v_1}\right) - v_j^2 + 1 = r_j^2 \quad j \geq 2 \tag{32}$$

$$\left(\frac{v_j'}{v_1^2}\right)' = r_j^2 \frac{v_j}{v_1^2} \quad j \geq 2, \tag{33}$$

where

$$\begin{aligned} r_2^2 &= 1 + c \quad \text{if} \quad n = 2, \\ r_2^2 &= 1 + c^2 - b^2, \quad r_3^2 = \frac{b^2}{c^2}, \quad \text{if} \quad n = 3, \\ r_j^2 &= 1, \quad j \geq 2, \quad \text{if} \quad n \geq 4. \end{aligned} \tag{34}$$

Proof: Equation (32) is an immediate consequence of (16) (17) and (20). It is a straightforward computation to verify that (31) and (34) are satisfied in each case of Proposition 2.1. We now prove (33). From (16),(17) and (20) we have

$$\begin{aligned} \left(\frac{v_j'}{v_1^2}\right)' &= h_{1j}' \frac{1}{v_1} - h_{1j} \frac{v_1'}{v_1^2} = v_j + \frac{h_{1j}}{v_1^2} \sum_{k \geq 2} \frac{v_k v_k'}{v_1} \\ &= v_j + \frac{h_{1j}}{v_1^2} \sum_{k \geq 2} v_k h_{1k} = v_j + \frac{h_{1j}^2 v_j}{v_1^2} + \frac{h_{1j}}{v_1^2} \sum_{k \geq 2, k \neq j} v_k h_{1k} \end{aligned}$$

Now, using (19) we have

$$\begin{aligned} \left(\frac{v_j'}{v_1^2}\right)' &= v_j + \frac{h_{1j}^2 v_j}{v_1^2} + \frac{v_j}{v_1^2} \sum_{k \geq 2, k \neq j} v_k^2 \\ &= v_j + \frac{h_{1j}^2 v_j}{v_1^2} + \frac{v_j}{v_1^2}(1 - v_1^2 - v_j^2) \\ &= \frac{v_j}{v_1^2}(h_{1j}^2 - v_j^2 + 1) = \frac{v_j}{v_1^2} r_j^2 \end{aligned}$$

where the last equality follows from (32).

We now prove the main theorems of this section.

Theorem 2.1 *The submanifolds $M^n \subset \mathbf{R}^{2n-1}$ with constant sectional curvature $K \equiv -1$, associated to the solutions of the IGSGE given in Proposition 2.1 are, up to a rigid motion.*

$$\left(-\int v_1^2 dx_1, \frac{v_2}{\sqrt{1+c}}\right), \quad \text{if } n=2,$$

where v_1, v_2 *are defined by (13).*

(ii) *The toroidal submanifolds generated by the curves*

$$\left(-\int v_1^2 dx_1, \frac{v_2}{\sqrt{1+c^2-b^2}}, \frac{cv_3}{b}\right), \quad if n=3,$$

where $v_i, 1 \leq i \leq 3$, *are given by (14) and*

(iii) *The toroidal submanifold generated by the curve*

$$(\text{ tgh } x_1 - x_1, c_2 \text{ sech } x_1, \ldots, c_n \text{ sech } x_1) \quad , if n \geq 4,$$

where $\sum_{j=2}^{n} c_j^2 = 1$.

Proof: Let $v = (v_1, \ldots, v_n)$ be a solution of the IGSGE as in Proposition 2.1. By theorem B.T.... there exists a solution $(a)_{ij}$ for the GSGE such that $a_{1j} = v_j$, for $1 \leq j \leq n$. Moreover, $a_{ij}, 2 \leq i \leq n$, also depend only on x_1.

The submanifold associated to v is determined by the first and second fundamental forms given by

$$I = \sum_{i=1}^{n} a_{1i}^2 dx_i^2 \quad \text{and} \quad II = \sum_{j=2,i=1}^{n} (a_{ji} a_{1i} dx_i^2) e_{n+j-1}$$

where $e_{n+j-1}, 2 \leq j \leq n$, is an orthonormal basis for the normal bundle. It follows from the fundamental theorem for submanifolds of the euclidean space, that the immersion $X : \Omega \subset \mathbf{R}^n \longrightarrow \mathbf{R}^{2n-1}$, where Ω is simply connected, is determined, up to a rigid motion, by solving the following system of differential equations for the vector fields X_{x_i} and $e_{n+s-1}, \quad 1 \leq i \leq n, \quad 2 \leq s \leq n$:

$$\begin{aligned} X_{x_i x_j} &= \sum_{k=1}^{n} \Gamma_{ij}^k X_k + \sum_{r=2}^{n} a_{1i} a_{ri} \delta_{ij} e_{n+r-1} \quad , 1 \leq i, j \leq n \\ e_{n+s-1, x_i} &= -\frac{a_{si}}{a_{1i}} X_{x_i} \quad , 2 \leq s \leq n. \end{aligned}$$

Since the riemannian metric g is a diagonal matrix and the functions $g_{ii} = a_{1i}^2$ depend only on x_1, the Christoffel symbols Γ_{ij}^k are given by

$$\Gamma_{ij}^k = 0 \quad \text{for i,j,k distinct;}$$

$$\Gamma^i_{i1} = \Gamma^i_{1i} = \frac{a'_{1i}}{a_{1i}};$$

$$\Gamma'_{ii} = -\frac{a_{1i}a'_{1i}}{a_{11}^2}, \quad \text{for} \quad i \neq 1,$$

where the prime means derivative with respect to x_1. Therefore the above system reduces to

$$X_{x_1x_1} = \frac{a'_{11}}{a_{11}} X_{x_1} + a_{11}\sum_{s=2}^{n} a_{s1}e_{n+s-1} \tag{35}$$

$$X_{x_ix_i} = -\frac{a_{1i}a'_{1i}}{a_{11}^2} X_{x_1} + a_{1i}\sum_{s=2}^{n} a_{si}e_{n+s-1} \quad , i \neq 1, \tag{36}$$

$$X_{x_ix_j} = 0 \quad i \neq j,\ i \neq 1,\ j \neq 1, \tag{37}$$

$$X_{x_1x_j} = \frac{a'_{1j}}{a_{1j}} X_{x_j} \quad , j \neq 1, \tag{38}$$

$$e_{n+s-1,x_i} = -\frac{a_{si}}{a_{1i}} X_{x_i} \quad , 2 \leq s \leq n. \tag{39}$$

It follows from (37) and (38) that

$$X_{x_j} = \frac{\partial F_j(x_j)}{\partial x_j} a_{1j} \quad , j \geq 2, \tag{40}$$

where $F_j : \mathbf{R} \longrightarrow \mathbf{R}^{2n-1}$ is a differentiable function which depends only on x_j , with $\frac{\partial F_j}{\partial x_j} \neq 0$. Equations (39) with $i \geq 2$ and (40) give

$$e_{n+s-1} = -\sum_{i=2}^{n} a_{si}F_i + G_s(x_1) \quad , s \geq 2, \tag{41}$$

where $G_s : \mathbf{R} \longrightarrow \mathbf{R}^{2n-1}$ is a differentiable function of x_1. From equation (39) with $i = 1$, we get, using (41) and GSGE, that

$$G'_s = a_{s1}\psi(x_1) \quad , s \geq 2, \tag{42}$$

where $\psi : \mathbf{R} \longrightarrow \mathbf{R}^{2n-1}$ is a differentiable function of x_1, and

$$X_{x_1} = \sum_{j=2}^{n} a'_{1j}F_j - a_{11}\psi. \tag{43}$$

We observe that in order to obtain (43) we have used the fact that there exists $s \geq 2$ such that $a_{si} \neq 0$ (the matrix (a_{ij}) is orthogonal and $a_{1i}^2 = g_{ii} \neq 0, 1 \leq i \leq n$).

(22) and (32), where $v_i = a_{1i}$, we obtain

$$\psi' + \sum_{s=2}^{n} a_{s1} G_s = 0 \tag{44}$$

and

$$\frac{\partial^2 F_j}{\partial x_j^2} + r_j^2 F_j - \frac{a'_{1j}}{a_{11}}\psi - \sum_{s=2}^{n} a_{sj} G_s = 0 \quad , j \geq 2, \tag{45}$$

where $r_j \neq 0$ is given by (34).

Since ψ, G_s and a_{ij} depend only on x_1 and F_j depends on x_j, $\left(\frac{\partial F_j}{\partial x_j} \neq 0\right)$ we necessarily have

$$\frac{a'_{1j}}{a_{11}}\psi + \sum_{s=2}^{n} a_{sj} G_s = C_j \quad , j \geq 2, \tag{46}$$

where $C_j \in \mathbf{R}^{2n-1}, j \geq 2$. Hence, F_j is determined by equation (45), which reduces to

$$\frac{\partial^2 F_j}{\partial x_j^2} + r_j^2 F_j = C_j \quad , j \geq 2.$$

Therefore,

$$F_j = \frac{C_j}{r_j^2} + A_j \cos r_j x_j + B_j \sin r_j x_j \quad , j \geq 2 \tag{47}$$

where A_j and $B_j \in \mathbf{R}^{2n-1}$.

Multiplying equation (46) by a_{1j} and adding up for $j \geq 2$, it follows from (16) and (44) that ψ satisfies

$$\psi' - \frac{a'_{11}}{a_{11}}\psi - \sum_{j=2}^{n} \frac{a_{1j}}{a_{11}} C_j = 0.$$

Therefore

$$\psi = a_{11} \sum_{j=2}^{n} C_j \int \frac{a_{1j}}{a_{11}^2} dx_1 = \sum_{j=2}^{n} \frac{C_j}{r_j^2} \frac{a'_{1j}}{a_{11}} + a_{11} A_1, \tag{48}$$

where the integral is determined by (33) and $A_1 \in \mathbf{R}^{2n-1}$ is a constant vector.

Since (a_{ij}) is an orthogonal matrix function, it follows from (44) and (46) that

$$G_s = -a_{s1}\psi' + \sum_{j\geq 2} a_{sj}(C_j - \frac{a_{1j}}{a_{11}}\psi),$$

hence, using (48), we obtain

$$G_s = \sum_{j=2}^{n} \frac{C_j}{r_j^2} a_{sj} - \sum_{A=1}^{n} a_{sA} a'_{1A} A_1 \tag{49}$$

equation (42) is an identity.

In order to obtain the local parametrization X we need to integrate (40) and (43), where F_j and ψ are given by (47) and (48) respectively, i.e.,

$$X_{x_j} = r_j a_{1j}(-A_j \sin r_j x_j + B_j \cos r_j x_j) \quad , j \geq 2 \tag{50}$$

$$X_{x_1} = \sum_{j=2}^{n} a'_{1j}(A_j \cos r_j x_j + B_j \sin r_j x_j) - a_{11}^2 A_1. \tag{51}$$

Therefore, we have

$$X(x_1, x_2, \ldots, x_n) = -\int_{x_1^0}^{x1} a_{11}^2 dx_1 A_1 + \sum_{j=2}^{n} a_{1j}(A_j \cos r_j x_j + B_j \sin r_j x_j), \tag{52}$$

where we choose the initial condition at a point

$$p_0 = (x_1^0, 0, \ldots, 0), \quad X(p_0) = \sum_{j=2}^{n} a_{1j}(x_1^0) A_j.$$

Observe that using (49) the normal vector fields given by (41) reduce to

$$e_{n+s-1} = -\sum_{j=2}^{n} a_{sj}(A_j \cos r_j x_j + B_j \sin r_j x_j) - \sum_{c=1}^{n} a_{sc} a'_{1c} A_1, \quad s \geq 2,$$

where the constant vectors A_j, B_j are determined by the initial conditions

$$X_{x_i}(p_0), 1 \leq i \leq n, \quad \text{and} \quad e_{n+s-1}(p_0), \quad 2 \leq s \leq n.$$

We conclude the proof of the theorem by choosing these conditions appropriately.

Theorem 2.2
a) The submanifolds $M^n \subset \mathbf{R}^{2n-1}, n \geq 3$, given by Theorem 2.1 are, up to a rigid motion, the only toroidal submanifolds of $\mathbf{R}^{2n-1}$ with constant sectional curvature $K \equiv -1$.
b) There are no complete toroidal submanifolds $M^n \subset \mathbf{R}^{2n-1}$, with $K \equiv -1$.

Proof: A toroidal submanifold is locally given by

$$X = (f_1(x_1), f_2(x_1) \cos x_2, f_2(x_1) \sin x_2, \ldots, f_n(x_1) \cos x_n, f_n(x_1) \sin x_n)$$

where $x_1 \in I \subset \mathbf{R}, -\pi < x_j < \pi$ and $f_i(x_1) \neq 0, \forall x_1 \in I, 1 \leq i \leq n, n \geq 3$. We will show that there exists a change of coordinates such that the metric (g_{ij}) is a diagonal matrix whose trace is equal to 1.

for $x_1 \in [a,b]$. Consider a change of variables $x_j = b_j \tilde{x}_j, 2 \leq j \leq n,$ where $b_j^2 < \frac{1}{B^2}$.

Now we can change the variable x_1 so that

$$\sum_{i=1}^{n}(f_i')^2 + \sum_{j=2}^{n} b_j^2 f_j^2 = 1 \tag{53}$$

where f_i' is the derivative with respect to the new variable. Hence, without loss of generality we may assume M^n locally given by

$$X = (f_1(x_1), f_2(x_1)\cos b_2x_2, f_2(x_1)\sin b_2x_2, \ldots, f_n(x_1)\cos b_nx_n, f_n(x_1)\sin b_nx_n)$$

where $x_1 \in (a,b)$, b_j are constants and equation (53) is satisfied.

The first fundamental form of this parametrization is given by

$$I = \sum_{i=1}^{n} v_i^2(x_1)dx_i^2,$$

where

$$v_1^2(x_1) = \sum_{i=1}^{n}(f_i')^2(x_1), \quad v_j^2(x_1) = b_j^2 f_j^2(x_1), \quad 2 \leq j \leq n.$$

Moreover, $\sum_{i=1}^{n} v_i^2 = 1$. Since the manifold has curvature $K \equiv -1$, it follows from Theorem B.T. that $v : (a,b) \subset \mathbf{R} \longrightarrow S^{n-1} \subset \mathbf{R}^n$ is a solution of IGSGE which depends only on x_1. Therefore, we are in the cases (ii) or (iii) of Proposition 2.1.

It is easy to see that in the case (ii) we can consider

$$b_2^2 = 1 + c^2 - b^2 \quad \text{and} \quad b_3^2 = \frac{b^2}{c^2}.$$

In the other case we can choose $b_j = 1$, for each j. The proof of b) is an immediate consequence of the first part of the theorem and Remark 2.1.

3. Flat submanifolds of the unit sphere

In this section we obtain the results analogous to those in section 2, by considering the correspondence between flat submanifolds of the sphere and solutions of the IGWE. We first obtain the solutions of the IGWE which depend only on one independent variable and then we show that the associated flat isometric immersions in the unit

S^2 and the Clifford torus. Moreover we prove that these are the only flat toroidal submanifolds of S^{2n-1}. In particular, we conclude that the Clifford torus is the only complete one.

Let $v = (v_1, \ldots, v_n), n \geq 2$, *be a solution of the IGWE which depends only on* x_1, *with* $v_i(x_1) \neq 0, \forall 1 \leq i, \leq n$, *for* x_1 *in an open interval* $I \subset \mathbf{R}$. *Then*

$$\begin{aligned} v_1 &= \sqrt{1-c^2}\sin(\lambda x_1 - a) \\ v_{j_0} &= \pm\sqrt{1-c^2}\cos(\lambda x_1 - a) \\ v_j &= c_j, \quad j \geq 2,\ j \neq j_0 \end{aligned} \tag{54}$$

where $\sum_{j=3}^{n} c_j^2 = c^2,\ c_j \neq 0,\ \lambda, a, c_j \in \mathbf{R}$ and $\mid c \mid < 1$. *When* $\lambda = 0$, *then* $I = \mathbf{R}$ *and* $a \neq l\pi/2$ *for any integer* l*; when* $\lambda \neq 0$, *then* $x_1 \in I$ *such that* $l\pi/2 < \lambda x_1 - a < (l+1)\pi/2$.

Proof: It follows from the hypothesis that equations (7)-(10) reduce to

$$vv^t = 1 \tag{55}$$

$$h_{1i} = \frac{v_i'}{v_1}, \quad i \geq 2 \tag{56}$$

$$h_{ji} = 0, \quad j \geq 2, \quad 1 \leq i \leq n \tag{57}$$

$$h_{1i}h_{1j} = 0, \quad i \neq j \quad i, j \geq 2 \tag{58}$$

$$h_{1j}' = 0, \quad j \geq 2 \tag{59}$$

From (56) and (59) we have

$$\frac{v_j'}{v_1} = b_j, \quad j \geq 2. \tag{60}$$

If $n = 2$, it follows that

$$\begin{aligned} v_1 &= \sin(\lambda x_1 - a) \\ v_2 &= -\cos(\lambda x_1 - a) \end{aligned}$$

where $a \in \mathbf{R}$, is not a multiple of $\frac{\pi}{2}$ when $\lambda = 0$.
If $n \geq 3$, it follows from (56), (58) and (60) that

$$b_i b_j = 0, \quad \text{for} \quad i, j \geq 2, \quad i \neq j.$$

Therefore, there exists at most one $b_{j_0} \neq 0$, with $j_0 \geq 2$. Hence

$$v_{j_0}' = \lambda v_1, \quad v_j = c_j, \quad j \geq 2,\ j \neq j_0. \tag{61}$$

We observe that when $\lambda = 0$, the solution (54) is constant. Let v_1, v_{j_0} be as in (54), with $\lambda \neq 0$, and let C_2 be a real constant. Then the solutions Θ for the ordinary differential equation

$$\Theta'' - 2\frac{v_1'}{v_1}\Theta' + \Theta + v_{j_0}C = 0 \tag{62}$$

are given by

$$\Theta = f\cos(\lambda x_1 - a) - \frac{f'}{\lambda}\sin(\lambda x_1 - a) \tag{63}$$

where

$$f = A\cos rx_1 + B\sin rx_1 \mp \frac{\sqrt{1-c^2}}{r^2}C, \quad A, B \in \mathbf{R},$$

and $r^2 = \lambda^2 + 1$. This remark will be used in the following theorem.

Theorem 3.1 *The flat submanifolds $M^n \subset S^{2n-1}$, associated to the solutions of the IGWE given in Proposition 3.1 are up to a rigid motion,*
(i) The Clifford torus

$$(c_1\cos x_1, c_1\sin x_1, \ldots, c_n\cos x_n, c_n\sin x_n)$$

where $v_i = c_i \neq 0,\ 1 \leq i \leq n$, whenever $\lambda = 0$.
(ii) The toroidal submanifolds generated by the curves

$$(\delta f_0, \delta f_1, \delta f_2, c_3, \ldots, c_n), \quad \delta = \sqrt{1-c^2}$$

where $r^2 = \lambda^2 + 1,\ c^2 = \sum_{j=3}^n c_j^2$ and

$$\begin{aligned} f_0 &= \frac{\lambda}{r}\sin rx_1\cos(\lambda x_1 - a) - \cos rx_1\sin(\lambda x_1 - a) \\ f_1 &= \frac{\lambda}{r}\cos rx_1\cos(\lambda x_1 - a) + \sin rx_1\sin(\lambda x_1 - a) \\ f_2 &= -\frac{1}{r}\cos(\lambda x_1 - a) \end{aligned} \tag{64}$$

whenever $\lambda \neq 0$.

Proof: Let $v = (v_1, \ldots, v_n)$ be a solution of the IGWE given by Proposition 3.1, where without loss of generality we will assume $j_0 = 2$. It follows from Theorem 1.3, that there exists a solution (a_{ij}) for the GWE such that $a_{1j} = v_j$, for $1 \leq j \leq n$.

determines the first and second fundamental form of the flat submanifold associated to v.

As in the proof of Theorem 2.1 such submanifolds are given, up to a rigid motion, locally by immersions $X : \Omega \subset \mathbf{R}^n \longrightarrow S^{2n-1} \subset \mathbf{R}^{2n}$, where Ω is simply connected. In order to obtain X, we consider the adapted orthonormal frame in $\mathbf{R}^{2n}$, defined by

$$e_i = \frac{X_{x_i}}{a_{1i}}, \quad 1 \leq i \leq n, \quad e_{n+j-1}, \quad 2 \leq j \leq n \quad \text{and} \quad X.$$

It is not difficult to see that X has to satisfy the system

$$\begin{aligned} X_{x_i x_j} &= \sum_{k=1}^{n} \Gamma_{ij}^k X_k + \sum_{k=2}^{n} a_{1i} a_{ki} \delta_{ij} e_{n+k-1} - \delta_{ij} a_{1i}^2 X, \quad 1 \leq i,j \leq n, \\ e_{n+s-1,x_i} &= -\frac{a_{si}}{a_{1i}} X_{x_i}. \qquad 2 \leq s \leq n \end{aligned}$$

Substituting the expressions of the Cristoffel symbols this system reduces to

$$X_{x_1 x_1} = \frac{a_{11}^{'}}{a_{11}} X_{x_1} + a_{11} \sum_{s=2}^{n} a_{s1} e_{n+s-1} - a_{11}^2 X \tag{65}$$

$$X_{x_i x_i} = \frac{-a_{1i} a_{1i}^{'}}{a_{11}^2} X_{x_1} + a_{1i} \sum_{s=2}^{n} a_{si} e_{n+s-1} - a_{1i}^2 X \quad i \geq 2 \tag{66}$$

$$X_{x_i x_j} = 0, \quad i \neq j,, \quad i \neq 1, \quad j \neq 1 \tag{67}$$

$$X_{x_1 x_j} = \frac{a_{1j}^{'}}{a_{1j}} X_{x_j} \quad j \neq 1 \tag{68}$$

$$e_{n+s-1,x_i} = \frac{-a_{si}}{a_{1i}} X_{x_i} \quad s \neq 1 \tag{69}$$

It follows from (67) and (68) that

$$X_{x_j} = \frac{\partial F_j}{\partial x_j}(x_j) a_{1j}, \quad j \geq 2 \tag{70}$$

where $F_j : \mathbf{R} \longrightarrow \mathbf{R}^{2n}$ is a differentiable function which depends only on x_j, with $\frac{\partial F_j}{\partial x_j} \neq 0$. Equations (69) with $i \geq 2$ and (70) give

$$e_{n+s-1} = -\sum_{i=1}^{n} a_{si} F_i + G_s(x_1), \quad s \geq 2 \tag{71}$$

where $G_s : \mathbf{R} \longrightarrow \mathbf{R}^{2n}$ is a differentiable function of x_1. From equation (69) with $i = 1$, we get using (71) and the GWE that

$$G_s^{'} = a_{s1} \psi(x_1), \quad s \geq 2, \tag{72}$$

$$X_{x_1} = \sum_{j=2}^{n} a'_{1j} F_j - a_{11}\psi \tag{73}$$

Now integrating (70) and (73) we obtain

$$X = \sum_{j=2}^{n} F_j a_{1j} + \varphi(x_1), \tag{74}$$

where

$$\psi = -\frac{\varphi'}{a_{11}}. \tag{75}$$

Now we substitute (70), (71), (73)-(75) into (65) and (66). Using the expressions of (54) we obtain

$$\left(\frac{\varphi'}{a_{11}}\right)' + a_{11}\varphi - \sum_{s=2}^{n} a_{s1} G_s = 0, \tag{76}$$

$$\frac{a'_{1k}}{a_{11}^2}\varphi' - \sum_{s=2}^{n} a_{sk} G_s + a_{1k}\varphi = -C_k, \quad k \geq 2 \tag{77}$$

$$\frac{\partial^2 F_k}{\partial x_k^2} + r_k^2 F_k = C_k, \quad k \geq 2, \tag{78}$$

where $r_k = 1$ for $k \geq 3$ and $r_2^2 = \lambda^2 + 1$. Therefore

$$F_k = \frac{C_k}{r_k^2} + A_k \cos r_k x_k + B_k \sin r_k x_k, \quad k \geq 2, \tag{79}$$

where $A_k, B_k, C_k \in \mathbf{R}^{2n}$.

Multiplying equation (67) by a_{1k} and adding up for $k \geq 2$, it follows from (55) and (76) that φ satisfies

$$\varphi'' - 2\frac{a'_{11}}{a_{11}}\varphi' + \varphi + \sum_{k=2}^{n} a_{1k} C_k = 0. \tag{80}$$

Since (a_{ij}) is an orthogonal matrix function, it follows from (76) and (77) that

$$G_s = a_{s1}\left[\left(\frac{\varphi'}{a_{11}}\right)' + a_{11}\varphi\right] + \sum_{k=2}^{n} a_{sk}\left(\frac{a'_{1k}}{a_{11}^2}\varphi' + a_{1k}\varphi + C_k\right), \quad s \geq 2 \tag{81}$$

i) Suppose $\lambda = 0$, then $a_{1j} = c_j$, $1 \leq j \leq n$ and $r_k = 1$, $2 \leq k \leq n$. Therefore, the solution $\varphi : \mathbf{R} \longrightarrow \mathbf{R}^{2n}$ of (80) is given by

$$\varphi = A_1 \cos x_1 + B_1 \sin x_1 - \sum_{k=2}^{n} a_{1k} C_k$$

$$G_s = \frac{a_{s1}}{a_{11}}\varphi'' + \sum_{k=2}^{n} a_{sk}C_k.$$

Hence, it follows from (71) and (74) that

$$e_{n+s-1} = -\sum_{k=2}^{n} a_{sk}(A_k \cos x_k + B_k \sin x_k) - \frac{a_{s1}}{a_{11}}(A_1 \cos x_1 + B_1 \sin x_1)$$

and

$$X = \sum_{k=2}^{n} a_{1k}(A_k \cos x_k + B_k \sin x_k) + A_1 \cos x_1 + B_1 \sin x_1.$$

By considering adequate initial conditions at the origin, we conclude that, up to a rigid motion, the immersion describes the Clifford torus.

ii) Suppose $\lambda \neq 0$, then it follows from (62) and (63) that the solution φ of (80) is given by

$$\varphi = f \cos(\lambda x_1 - a) - \frac{f'}{\lambda}\sin(\lambda x_1 - a) - \sum_{k=3}^{n} a_{1k}C_k,$$

where

$$f = B_1 \cos r x_1 + A_1 \sin r x_1 \mp \frac{\sqrt{1-c^2}}{r^2}C_2, \quad A_1, B_1 \in \mathbf{R}^{2n}$$

and $r^2 = r_2^2 = \lambda^2 + 1$. Hence, we get from (81) that

$$G_s = \frac{1}{\sqrt{1-c^2}}\left(a_{s1}\frac{f'}{\lambda} - a_{s2}(\lambda^2 f + f'')\right) + \sum_{k=2}^{n} a_{sk}C_k.$$

Therefore, it follows from (71) and (74) that

$$e_{n+s-1} = -\sum_{j=2}^{n} a_{sj}(A_j \cos r_j x_j + B_j \sin r_j x_j) + \frac{1}{\sqrt{1-c^2}}\left(\frac{a_{s1}}{\lambda}f' - \frac{1}{r_2^2}a_{s2}f''\right)$$

where r_j is given by (78) and

$$X = \sum_{j=2}^{n} a_{1j}(A_j \cos r_j x_j + B_j \sin r_j x_j) - \frac{f''}{r^2}\cos(\lambda x_1 - a) - \frac{f'}{\lambda}\sin(\lambda x_1 - a).$$

By choosing the initial conditions appropriately we conclude that, up to a rigid motion, the immersion is given by

$$X = (\delta f_0, \delta f_1, \delta f_2 \cos r x_2, \delta f_2 \sin r x_2, c_3 \cos x_3, c_3 \sin x_3, \ldots, c_n \cos x_n, c_n \sin x_n),$$

where $\delta = \sqrt{1-c^2}$ and f_0, f_1, f_2 are defined by (64).

orem 3.1 (ii) is contained on a two-dimensional sphere (see Fig.1)

The following theorem can be proved by using the same arguments as in Theorem 2.2

Theorem 3.2

a) The submanifolds $M^n \subset S^{2n-1}$ *given by Theorem 3.1 are, up to a rigid motion, the only toroidal flat submanifolds of* $\mathbf{R}^{2n}$ *contained in* S^{2n-1}.

b) The only complete toroidal flat submanifolds $M^n \subset S^{2n-1}$ *is the Clifford torus.*

References

[A1] Aminov, Ju. A., *On the immersion of domains of* $n-$*dimensional Lobacevskii space in* $(2n-1)-$*dimensional Euclidean space*, Soviet Math. Dokl. 18, (1977), 1210-1213.

[A2] Aminov, Ju. A., *Isometric immersions with flat connection, of domains of* $n-$*dimensional Lobachevsky space into Euclidean spaces. A model of a gauge field*, Math. USSR Sbornik 65, (1990), 279-303.

[BT] Beals, R.; Tenenblat, K., *An intrinsic generalization for the wave and sine-Gordon equations*, Differential Geometry, Lawson, B., et all Editors, Pitman Monographs and Surveys in Pure and Applied Math. # 52, 1991, 25-46.

[M] Moore, J.D., *Isometric immersions of space forms in a space form*, Pac. J. of Math. 40, (1972), 157-166.

[O] Okayasu, O., $O(2)\times O(2)-$*invariant hypersurfaces with constant negative scalar curvature in* E^4, Proc. of AMS 107, (1989), 1045-1050.

[T] Tenenblat, K., *Backlund's theorem for submanifolds of space forms and a generalized wave equation*, Bol.Soc. Bras. Mat. 16, (1985), 69-94.

[TT] Tenenblat, K.; Terng, C.L., *Backlund's theorem for n-dimensional submanifolds of* $\Re^{2n-1}$, Ann.of Math.,111, (1980), 477–490.

and its soliton theory, Ann.of Math. 111, (1980), 491–510.

[TW] Tenenblat, K.; Winternitz, P., *On the Symmetry Group of the Intrinsic Generalized Wave and sine-Gordon Equations*, to appear.

[X] Xavier, F., *A non-immersion theorem for hyperbolic manifolds*, Comment. Math. Helv. 60, (1985), 280-283.

Universidade de Brasília
Departamento de Matemática
70910 Brasília, D.F. Brazil