LAPLACE TRANSFORMATION IN HIGHER DIMENSIONS

Niky Kamran **Keti Tenenblat**

1. Introduction.

The study of the deep relationship between differential geometry and partial differential equations has a long and distinguished history, going back to the works of Darboux, Lie, Bäcklund, Goursat and E. Cartan. This relationship stems from the fact that most of the local properties of surfaces are expressed naturally in terms of partial differential equations. For example, the condition that a graph $z = z(x, y)$ have constant Gaussian curvature gives rise to a Monge - Ampère equation for $z(x, y)$. Equivalently, the same condition is expressed by the sine-Gordon equation for negative curvature or elliptic sinh-Gordon equation for positive curvature.

It is therefore very important to study the transformations of surfaces that preserve geometric properties expressible as partial differential equations, since the analytic formulation of these transformations will give rise to mappings preserving the class of partial differential equations under consideration. Ultimately, one expects the geometry of the transformed surface to have considerable implications at the analytic level, notably in terms of the explicit integrability of the underlying partial differential equations.

Perhaps the best known example of such a transformation of surfaces is Bäcklund's transformation, which takes a surface of constant negative Gaussian curvature (a pseudospherical surface) into another such surface. Since pseudospherical surfaces correspond to solutions of the sine - Gordon equation, the analytic formulation of this transformation will define a mapping taking solutions of the sine - Gordon equation into other solutions. Bäcklund transformations have become a very important ingredient in the theory of solitons solutions of completely integrable equations.

An equally interesting but perhaps lesser known example of a transformation of surfaces is the Laplace transformation. One starts from a surface S admitting a net of curves which is conjugate for the second fundamental form. Taking the net to be the parametric net with parameters u and v, one has

$$X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v, \tag{1}$$

which is a linear hyperbolic partial differential equation for $X(u, v)$. The two Laplace transforms S_1 and S_{-1} of S , denoted respectively by $L_1(S)$ and $L_{-1}(S)$, are defined geometrically as follows. One takes a coordinate curve $X(v, u_0)$ and considers the developable surface obtained by taking the tangent lines to the coordinate curves $X(u_0, v)$ at the points of $X(v, u_0)$. On such a line, there is a point X_1 where the line is tangent to the edge of regression. As u and v vary, one obtains in general a surface $S_1 = L_1(S)$ parametrized by $X_1(u, v)$. Likewise, if we reverse the role of u and v in the above construction, one obtains in general a surface $S_{-1} = L_{-1}(S)$ parametrized by $X_{-1}(u, v)$. The remarkable fact here is that the coordinate net is also conjugate for the transformed surfaces S_1 and S_{-1} , so that $X_1(u, v)$ and $X_{-1}(u, v)$ also satisfy a differential equation of the the form (1).

It is not difficult to prove that the two Laplace transformations are inverses of each other, in the sense that

$$L_{-1}(L_1(S)) = L_1(L_{-1}(S)) = S$$

for any surface S with a conjugate net.

Of particular interest is the case in which one of the Laplace transforms degenerates into a curve. Indeed, the pde can in this case be integrated explicitly by quadratures, as we shall see below in Section 2. This geometric construction thus leads to a beautiful method of integration for the linear hyperbolic equations arising from surfaces S which degenerate into curves after a finite sequence of Laplace transformations. One integrates the equation obtained at the end of the sequence by quadratures and works one's way back to the original equation by using the inversion formula displayed above. The classical treatises of Darboux [Da] and Goursat [Gou] contain many interesting results obtained by the application of this method of integration.

It is a tantalizing and difficult problem to try to generalize these constructions from the case of surfaces to that of submanifolds of higher dimension. The higher-dimensional generalization of Bäcklund's construction has been obtained by Tenenblat and Terng [TT], who also derived the corresponding Bäcklund transformation of a system of partial differential equations which generalizes the sine-Gordon equation. This work has led to important developments in the area of multi-dimensional integrable partial differential equations, such as the geometrical generalization of the wave, elliptic sinh-Gordon and Laplace equations [ABT], [BT], [T], [To], [CT].

Chern [Ch1, Ch2] has given a beautiful geometric description of a generalization of the Laplace transformation to a class of n-dimensional submanifolds in projective space, previously studied by Elie Cartan [Ca] in a different context. These submanifolds, which Chern calls Cartan manifolds, admit a parametrization by a conjugate net. For each n-dimensional Cartan manifold, Chern shows how to construct $n(n-1)$ generalized Laplace transforms which, generically, will also be n-dimensional Cartan manifolds. This result also appears in a subsequent paper by Geidel'man [G].

In this paper, we consider Cartan manifolds in Euclidean space rather than in projective space. Analytically, this will imply that the functions giving the parametrization of the submanifold satisfy an overdetermined system of second order partial differential equations of the form

$$X_{ij} = \Gamma^i_{ij} X_i + \Gamma^j_{ij} X_j, \qquad 1 \leq i \neq j \leq n. \tag{1'}$$

We first carry out the Euclidean version of Chern's generalized Laplace transformation. This will define in general $n(n-1)$ Laplace transforms for a given Cartan manifold. To each of the transforms of a Cartan manifold there corresponds therefore an overdetermined system of the same form (1').

We then obtain a transformation for overdetermined systems of the form

$$y_{,lk} + a_{lk}^{l} y_{,l} + a_{lk}^{k} y_{,k} + c_{lk} y + h_{lk} = 0 \qquad 1 \leq k \neq l \leq n \tag{2}$$

which generalizes the classical Laplace transformation for linear second-order hyperbolic equations in the plane. This is done by working out the analytical form of the geometric construction of Chern. We then apply this transformation to the problem of solving (2) for smooth Cauchy data given by prescribing the values of y along n curves passing through a given point x^0. In particular, we will show that the cases where the transformed submanifold degenerates to a curve are exactly those for which the integration of the original system of partial differential equations reduces to that of a system of the same type involving $n-1$ independent variables. This gives rise to a method of integration for these systems.

We want to emphasize the fact that the analytic expression of this generalization of the Laplace transformation could not have been inferred on a strictly formal basis, just from the known formulas in the case of surfaces . The guiding principle throughout this paper is the geometry of the problem, motivated by the geometric construction of Chern.

We should mention that the systems (2), to which the higher-dimensional Laplace transformation is applicable, play a decisive role in the analysis of an important class of partial differential equations in several independent variables. Indeed, the conserved quantities for semi-Hamiltonian, strongly hyperbolic systems of hydrodynamic type ([Tsa]) are precisely governed by overdetermined systems (2) to which the higher-dimensional Laplace transformation is applicable. For other applications of such systems see also [Ti].

In Section 2, we recall the method of Laplace for the integration of the linear second order hyperbolic equations in the plane and give its geometric interpretation in terms of the classical Laplace transformation of surfaces. In Section 3, we work out the Euclidean version of Chern's Laplace transformation for n-dimensional Cartan manifolds. Moreover, we characterize the Cartan manifolds for which a Laplace transform reduces to a curve. In Section 4, we derive the analytic expression of the generalized Laplace transformation as it applies to overdetermined systems of the form (2). Using the characterization obtained in Section 3, we define higher dimensional Laplace invariants. Finally, we prove our fundamental reduction theorem for the integration of systems for which the higher dimensional Laplace invariants vanish.

2. The classical Laplace Transformation for linear second-order hyperbolic equations and its geometrical interpretation

Consider a second-order partial differential equation for a real valued function $z(u,v)$ given by

$$z_{uv} + az_u + bz_v + cz + l = 0 \tag{3}$$

where a, b, c and l are differentiable functions of u and v. The classical Laplace method for solving this equation is given as follows. Consider the Laplace invariant

$$h = a_u + ab - c.$$

If $h = 0$, then (3) reduces to

$$\frac{\partial}{\partial u}(z_v + az) + b(z_v + az) + l = 0.$$

Letting

$$S = e^s(z_v + az), \quad \text{where} \quad s = \int b\ du$$

and differentiating with respect to u, we obtain

$$S = -\int e^s l\ du + F(v).$$

Hence,

$$z_v + az = e^{-s} S.$$

Similarly, letting

$$\tilde{S} = e^{\tilde{s}} z \quad \text{where} \quad \tilde{s} = \int a\ dv,$$

we conclude that

$$z = e^{-\tilde{s}} \left[- \int e^{\tilde{s}-s} (\int e^s l \; du - F(v)) dv + G(u) \right],$$

where F and G are arbitrary differentiable functions. The functions F and G are determined by the initial conditions $z(u_0, v)$ and $z(u, v_0)$. In fact,

$$\begin{aligned} F(v) &= \left[e^s (\tilde{s}_v z + z_v) + \int e^s l \; du \right] (u_0, v) \\ G(u) &= \left[e^{\tilde{s}} z + \int e^{\tilde{s}-s} [\int e^s l \; du - F(v)] \; dv \right] (u, v_0). \end{aligned}$$

Similarly, if the Laplace invariant

$$k = b_v + ab - c$$

vanishes, we can also explicitly solve (3).

The classical transformation of Laplace is a transformation theory for differential equations of the form (3). Assume $h \neq 0$ and define

$$z_1 = z_v + az.$$

Then one can easily see that (3) is transformed into a differential equation of the same form for z_1, namely

$$z_{1,uv} + (a - \frac{h_v}{h}) z_{1,u} + b z_{1,v} + [ab + h((\frac{b}{h})_v - 1)] z_1 + l(a - \frac{h_v}{h}) + l_v = 0.$$

Similarly, if $k \neq 0$ we consider

$$z_{-1} = z_u + bz$$

which also transforms (3) into an equation of the same type for z_{-1}. Those two functions z_1 and z_{-1} are said to be the $\mathcal{L}_1$ and $\mathcal{L}_{-1}$ Laplace transforms of z. Moreover, one can invert those transformations. In fact, it is not difficult to see that when $h \neq 0$ and $k \neq 0$,

$$z = [\mathcal{L}_{-1}(z_1) + l]/h \quad \text{and} \quad z = [\mathcal{L}_1(z_{-1}) + l]/k.$$

The basic idea for this method is to apply a sequence of Laplace transformations to a given equation (3) until eventually it is transformed into one which has a vanishing Laplace invariant. This equation is integrated and then using the inverse transformation one obtains a solution for the given initial differential equation.

There is a geometrical construction for surfaces [Ch3], which corresponds to the Laplace method described above. Before going into the geometry, we would like to make the following observations:

1. Given a differential equation of the form (3), we have seen that if $h = 0$, we obtain linear first-order equations first for $z_v + az$ and then for z, which we can easily solve. It is easy to see that when $a \neq 0$ one can also solve first for $z_v/a + z$ and then for z. A similar observation also holds for the case $k = 0$ and $b \neq 0$.

2. The Laplace invariants h and k are independent of the nonhomogeneous term l of equation (3). In fact, one can apply the Laplace method to solve the nonhomogeneous equation if and only if one can apply it to the homogeneous equation ($l = 0$).

3. There is a close relationship between solutions of

$$z_{uv} + az_u + bz_v + cz = 0 \tag{4}$$

and solutions of a corresponding equation of the form

$$Z_{uv} + AZ_u + BZ_v = 0. \tag{5}$$

In fact, given a nonzero solution λ of (4), the solutions z of (4) and the solutions Z of (5), where

$$A = a + \lambda_v/\lambda \quad B = b + \lambda_u/\lambda,$$

are related by $z = \lambda Z$ [Da].

In view of the observations 2 and 3 above, we will restrict ourselves to equations of the form (5) for the geometrical interpretation of the Laplace transformation. Observation 1 will be useful to relate the geometrical construction with the Laplace method.

Consider a parametrized surface $X(u,v)$ in $\mathbf{R}^3$ or $\mathbf{R}^4$ such that the coordinate curves form a conjugate net. The mixed second order derivative of X is then given by

$$X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v \tag{6}$$

where $\Gamma^i_{12}, i = 1,2$ are the Christoffel symbols of the surface X. Hence, the vector valued function X satisfies an equation of the form (5). The surfaces we consider will be *generic*, in the sense that we shall assume X_u, X_v, X_{uu}, and X_{vv} to be linearly independent in the ambient space.

Suppose that $\Gamma^1_{12} \neq 0$ and consider the ruled surface defined by

$$Y(t,u,v_0) = X(u,v_0) + tX_v(u,v_0).$$

The tangent plane to this surface at $t=0$ is generated by the vectors $X_u(u,v_0)$ and $X_v(u,v_0)$. Letting u vary, we obtain a one-parameter family of tangent planes given by

$$T_i(p,u) =< p - X(u,v_0), N_i(u,v_0) >= 0 \quad p \in \mathbf{R}^3 \text{ or } \mathbf{R}^4,$$

where the N_i, $i=1$ or $i=1,2$ (according to whether the surface is immersed in $\mathbf{R}^3$ or $\mathbf{R}^4$) are linearly independent normal vector fields to the surface X spanning the normal space of X. The characteristic line of the surface Y is the intersection of the planes

$$T_i(p,u) = 0, \tag{7}$$

$$T_{i,u}(p,u) = 0. \tag{8}$$

It follows that the direction of the characteristic line is given by $X_v(u,v_0)$. In fact, for each u, the line

$$p(t) = X(u,v_0) + tX_v(u,v_0)$$

is contained in the tangent plane (7) and also in the neighboring planes (8). This follows from the identity $< X_v, N_{i,u} >= 0$, which is a consequence of the fact that the surface X is parametrized by conjugate coordinates.

The surface Y consists of the tangent lines to a curve which is classically called the edge of regression of the one-parameter family of planes $T_i(p,u)$. This terminology originates from the fact that the tangent developable of a curve in $\mathbf{R}^3$ consists of two sheets which meet along the curve in a sharp edge (the edge of regression) [S]. The edge of regression of the surface Y is determined by the intersection of the planes

$$T_i(p,u) = 0,$$
$$T_{i,u}(p,u) = 0,$$
$$T_{i,uu}(p,u) = 0.$$

It follows that for each u, there exists a unique point $p(t_0)$ on the line $p(t)$ such that $Y(t_0,u,v_0)$ is on the edge of regression. In fact,

$$T_{i,uu}(p(t),u) = -(1+t\Gamma_{12}^1) < X_u, N_{i,u} > .$$

Since we are assuming the surface is generic, we conclude that $t = -1/\Gamma_{12}^1$. As u and v vary, we obtain in general, a map $X_1(u,v)$ given by

$$X_1 = X - X_v/\Gamma_{12}^1, \tag{9}$$

whose differential satisfies

$$\begin{pmatrix} X_{1,u} \\ X_{1,v} \end{pmatrix} = \begin{pmatrix} h/\Gamma_{12}^1 & 0 \\ \star & -1 \end{pmatrix} \begin{pmatrix} X_v \\ X_{vv} \end{pmatrix} /\Gamma_{12}^1 \tag{10}$$

where

$$h = -\Gamma_{12,u}^1 + \Gamma_{12}^1\Gamma_{12}^2$$

Since X_v and X_{vv} are assumed to be linearly independent, it follows from (10) that $X_1(u,v)$ will be a parametrized surface if and only if h is not zero. Moreover, we have

$$X_{1,uv} = \left[\left(\log\frac{h}{(\Gamma_{12}^1)^2}\right)_v + \Gamma_{12}^1 + (\log\Gamma_{12}^1)_v\right] X_{1,u} - \frac{h}{\Gamma_{12}^1} X_{1,v}, \tag{11}$$

so that the coordinate curves will also form a conjugate net on X_1. We will say that X_1 is the $\mathcal{L}_1$-*Laplace Transform* of the surface X and we will denote it by

$X_1 = \mathcal{L}_1(X)$. From (10) we see that X_1 reduces to a curve if and only if $h = 0$. In that case, it is easy to determine the original surface X from its Christoffel symbols $\Gamma^1_{12}, \Gamma^2_{12}$. In fact, from (6) we have

$$\left(\frac{X_v}{\Gamma^1_{12}}\right)_u = X_u.$$

Hence, we can integrate in u and we finally get

$$X = e^{-J}\left(\int e^J \Gamma^1_{12} V(v)\, dv + U(u)\right),$$

where $J = -\int \Gamma^1_{12}\, dv$, V and U are arbitrary vector valued functions of v and u respectively. The functions V and U are determined by the initial conditions $X(u_0, v)$ and $X(u, v_0)$. In fact,

$$\begin{aligned} V(v) &= \left[(-X + \frac{1}{\Gamma^1_{12}} X_v)\right](u_0, v), \\ U(u) &= \left[e^J X - \int e^J \Gamma^1_{12} V\, dv\right](u, v_0). \end{aligned}$$

Similarly, one can do the same construction by interchanging u and v obtaining what is called the $\mathcal{L}_{-1}$-*Laplace Transform* of the surface X given by

$$X_{-1} = X - X_u/\Gamma^2_{12}. \tag{12}$$

The two Laplace transformations $\mathcal{L}_1$ and $\mathcal{L}_{-1}$ are inverses of each other. In fact, it follows from (9), (11) and (12) that when $h \neq 0$,

$$\mathcal{L}_{-1}(\mathcal{L}_1(X)) = X.$$

Similarly, when $k \neq 0$ we have

$$\mathcal{L}_1(\mathcal{L}_{-1}(X)) = X.$$

3. The geometry of Cartan manifolds

In [Ch1,Ch2], Chern considered a class of manifolds in projective space, which he called Cartan manifolds, since they had been previously considered by Cartan in a different context [Ca]. Chern showed that there exists a transformation that generalizes for these manifolds the classical Laplace transformation for surfaces admitting a conjugate net. In this section, we give the Euclidean version of this transformation.

Definition A Riemannian n-dimensional manifold M^n isometrically immersed in $\mathbf{R}^{2n}$ is said to be a *Cartan manifold* if there exist local coordinates $(x_1 \ldots x_n)$ such that the net of coordinate curves is conjugate and the osculating space is $2n$-dimensional.

We recall that given an isometric immersion $X(x_1, \ldots x_n)$, the net of coordinate curves is conjugate whenever the second fundamental forms are simultaneously diagonalized. In this case, we say that the manifold is parametrized by conjugate coordinates. The osculating space (or more precisely the second order osculating space) of X at a point $x = (x_1, \ldots, x_n)$ is the subspace of $\mathbf{R}^{2n}$ generated by the first and second order derivatives of X, at the point x. We now provide some explicit examples to which we will refer later in the paper.

Examples of Cartan manifolds:
a) The Clifford torus parametrized by

$$X(x_1, \ldots, x_n) = (\cos(x_1), \sin(x_1), \ldots, \cos(x_n), \sin(x_n)).$$

b) The flat toroidal $n-$dimensional submanifold of the unit sphere contained in $\mathbf{R}^{2n}$ (see [RT]), given by

$$X = (\delta f_0, \delta f_1, \delta f_2 \cos(x_2), \delta f_2 \sin(x_2), c_3 \cos(x_3), c_3 \sin(x_3), \ldots, c_n \cos(x_n), c_n \sin(x_n)),$$

where $\delta = \sqrt{1-c^2}$, $c^2 = \sum_{j=3}^{n} c_j^2$, $c_j \neq 0$ and $0 < x_1 < \pi/2\lambda$.

$$\begin{aligned} f_0 &= \frac{\lambda}{r} \sin r x_1 \cos \lambda x_1 - \cos r x_1 \sin \lambda x_1 \\ f_1 &= \frac{\lambda}{r} \cos r x_1 \cos \lambda x_1 + \sin r x_1 \sin \lambda x_1 \\ f_2 &= -\frac{1}{r} \cos \lambda x_1 \end{aligned}$$

where λ and r are nonzero real numbers which satisfy $r^2 = \lambda^2 + 1$.

c) The manifold described by

$$X(x_1, ...x_n) = (g_0(x_1, x_2), g_1(x_1, x_2), \cos(x_2), \sin(x_2), ..., \cos(x_n), \sin(x_n)),$$

where

$$g_0 = \frac{x_1x_2 - 1}{x_2^2} e^{x_1x_2} \qquad g_1 = \frac{x_1^2x_2^2 - 2x_1x_2 + 2}{x_2^3} e^{x_1x_2}$$

and $x_1 > 0$ $x_2 > 0$.

We recall that if a submanifold can be parametrized by conjugate coordinates then its normal bundle is necessarily flat. Therefore, any submanifold M^n of $\mathbf{R}^{2n}$, whose normal bundle is not flat, is not a Cartan manifold.

We will use the following range of indices

$$1 \leq i, j, k, l \leq n,$$

and we will denote as usual by Γ_{ij}^k the Christoffel symbols for a given parametrization $X : U \subset \mathbf{R}^n \to \mathbf{R}^{2n}$. Moreover, X_i and X_{jk} will denote the derivative of X with respect to x_i and the second derivative of X with respect to x_j and x_k respectively.

Lemma *If $X : U \subset \mathbf{R}^n \to \mathbf{R}^{2n}$ is a Cartan manifold parametrized by conjugate coordinates, then for each i, j with $i \neq j$ the vector field X_{ij} lies in the space spanned by X_i and X_j, i. e.*

$$X_{ij} = \Gamma_{ij}^i X_i + \Gamma_{ij}^j X_j \qquad i \neq j.$$

Moreover, the Christoffel symbols satisfy

$$\frac{\partial \Gamma_{ik}^k}{\partial x_l} + \Gamma_{ik}^k \Gamma_{kl}^k - \Gamma_{il}^i \Gamma_{ik}^k - \Gamma_{il}^l \Gamma_{lk}^k = 0 \qquad l, i, k \text{ distinct.} \tag{13}$$

Proof: Suppose $X(x_1, \ldots, x_n)$ is a Cartan manifold in $\mathbf{R}^{2n}$ parametrized by conjugate coordinates. Then the second fundamental forms are diagonalized.

Moreover, the $2n$ vector fields X_i, X_{ii}, $1 \leq i \leq n$, are linearly independent. Therefore, we have

$$X_{ij} = \sum_{k=1}^{n} \Gamma_{ij}^k X_k, \quad i \neq j \quad \text{and} \quad X_{ii} = \sum_{k=1}^{n} \Gamma_{ii}^k X_k + N_i,$$

where N_i is the normal component of X_{ii}. For each i we consider

$$\begin{aligned} dX_i &= X_{ii}dx_i + \sum_{j\neq i} X_{ij}dx_j \\ &= N_i dx_i + \sum_k \left(\Gamma_{ii}^k dx_i + \sum_{j\neq i} \Gamma_{ij}^k dx_j \right) X_k. \end{aligned}$$

Taking exterior derivative of this equation we get

$$\begin{aligned} 0 &= dN_i \wedge dx_i + \\ &+ \sum_k \left[d\Gamma_{ii}^k \wedge dx_i X_k + \Gamma_{ii}^k dX_k \wedge dx_i + \sum_{j\neq i} \left(d\Gamma_{ij}^k \wedge dx_j X_k + \Gamma_{ij}^k dX_k \wedge dx_j \right) \right]. \end{aligned}$$

Introducing the notation

$$\frac{\partial N_i}{\partial x_l} = \sum_j (b_{il}^j X_j + r_{il}^j N_j), \tag{14}$$

it follows that for each i, k fixed

$$\begin{aligned} &\sum_l \left(b_{il}^k + \frac{\partial \Gamma_{ii}^k}{\partial x_l} + \Gamma_{ii}^l \Gamma_{ll}^k + \sum_{s\neq l} \Gamma_{ii}^s \Gamma_{sl}^k \right) dx_l \wedge dx_i + \\ &+ \sum_l \sum_{j\neq i} \left(\frac{\partial \Gamma_{ij}^k}{\partial x_l} + \Gamma_{ij}^l \Gamma_{ll}^k + \sum_{s\neq l} \Gamma_{ij}^s \Gamma_{sl}^k \right) dx_l \wedge dx_j = 0, \end{aligned} \tag{15}$$

and

$$\sum_l r_{il}^k dx_l \wedge dx_i + \Gamma_{ii}^k dx_k \wedge dx_i + \sum_{j\neq i} \Gamma_{ij}^k dx_k \wedge dx_j = 0.$$

From the last equation we obtain that

$$\Gamma_{ij}^k = 0 \qquad r_{ij}^k = 0 \qquad \text{for all } i, j, k \text{ distinct.}$$

$$r_{ik}^k + \Gamma_{ii}^k = 0 \qquad\qquad r_{ik}^i - \Gamma_{ik}^i = 0 \qquad \text{for all } i \neq k.$$

Therefore, for each $i \neq j$, X_{ij} is in the space generated by X_i and X_j. Finally, the relation (13) follows from the coefficient of $dx_l \wedge dx_k$, in (15) where k, i and l are distinct. $\square$

It should be pointed out that we have not extracted the full set of integrability conditions for the immersion X to define a Cartan manifold. The remaining conditions, which will not be used in our analysis, can be obtained, without difficulty, by considering all the coefficients of the 2-form given by (15). In fact, one has to add the following condition

$$\frac{\partial N_i}{\partial x_l} = \sum_{k=1}^{n} b_{il}^k X_k - \Gamma_{ii}^l N_l + \Gamma_{il}^i N_i \quad i \neq l,$$

where the functions b_{il}^k are given by

$$\begin{aligned}
b_{il}^k &= -\frac{\partial \Gamma_{ii}^k}{\partial x_l} + \Gamma_{ii}^l \Gamma_{ll}^k - \Gamma_{ii}^k \Gamma_{kl}^k + \Gamma_{il}^i \Gamma_{ii}^k \quad i, l, k \text{ distinct}, \\
b_{ik}^k &= -\frac{\partial \Gamma_{ii}^k}{\partial x_k} + \frac{\partial \Gamma_{ik}^k}{\partial x_i} - \sum_s \Gamma_{ii}^s \Gamma_{sk}^k + \Gamma_{ik}^i \Gamma_{ii}^k + (\Gamma_{ik}^k)^2 \quad i \neq k, \\
b_{ik}^i &= -\frac{\partial \Gamma_{ii}^i}{\partial x_k} + \frac{\partial \Gamma_{ik}^i}{\partial x_i} - \Gamma_{ii}^k \Gamma_{kk}^i + \Gamma_{ik}^k \Gamma_{ki}^i \quad i \neq k.
\end{aligned}$$

The full set of integrability conditions was obtained by Cartan [Ca], where he also showed that the degree of generality of such immersions is given by $n(n-1)$ functions of two variables.

In what follows, we will associate to each n-dimensional Cartan manifold X, in general a family of $n(n-1)$ manifolds which will also be Cartan manifolds. This will be achieved by considering the edge of regression of ruled manifolds constructed from X.

Consider a Cartan manifold X parametrized by conjugate coordinates. Then, it follows from the above Lemma that the mixed second order derivatives of X are given by

$$X_{lk} = \Gamma_{lk}^l X_l + \Gamma_{lk}^k X_k, \quad l \neq k.$$

For each $(n-1)$-dimensional submanifold of X, given by fixing $x_j = x_j^0$, consider the ruled manifold defined by

$$Y(t, x_1, .., x_j^0, .., x_n) = X(x_1, .., x_j^0, .., x_n) + tX_j(x_1, .., x_j^0, .., x_n).$$

The tangent space to this manifold at $t = 0$ is generated by the vectors $X_k(x_1, .., x_j^0, ..x_n)$, $1 \leq k \leq n$. This gives rise to a $n-1$-parameter family of tangent spaces described by the system of equations

$$T_l(p, x_1, .., x_j^0, .., x_n) =< p - X, N_l >= 0 \quad p \in \mathbf{R}^{2n}, \quad 1 \leq l \leq n,$$

where N_l are normal vector fields to the manifold X and the right hand side is evaluated at $(x_1, .., x_j^0, .., x_n)$. The characteristic line of the manifold Y is the intersection of the spaces

$$T_l(p, x_1, .., x_j^0, .., x_n) = 0, \quad 1 \leq l \leq n, \tag{16}$$

$$T_{l,k}(p, x_1, .., x_j^0, .., x_n) = 0, \quad \forall k, k \neq j. \tag{17}$$

It follows that the direction of the characteristic line is given by X_j. In fact, at each point $(x_1, .., x_j^0, .., x_n)$, the line

$$p(t) = X + tX_j$$

is contained in the tangent space (16) and also in the neighboring spaces (17). This results from the identity

$$< X_j, N_{l,k} >= 0 \qquad \forall k \neq j, \forall l$$

implied by the fact that X is a Cartan manifold.

Now for each $i \neq j$, motivated by the 2-dimensional case described in section 2, we define the *edge of regression* of the manifold Y in the direction i to be the intersection of the spaces

$$T_l(p, x_1, .., x_j^0, .., x_n) = 0, \qquad \forall l,, \ 1 \leq l \leq n,$$

$$T_{l,k}(p, x_1, .., x_j^0, .., x_n) = 0, \qquad \forall k, \ k \neq j,$$

$$T_{l,ki}(p, x_1, .., x_j^0, .., x_n) = 0, \qquad \text{for fixed } i \neq j.$$

This intersection is a unique point on the characteristic line $p(t)$. In fact, since

$$T_{l,ki}(p(t), x_1, .., x_j^0, .., x_n) = -(1 + t\Gamma_{ij}^i) < X_i, N_{l,k} > \quad \forall l \neq j, k \neq j,$$

we conclude that $t = -1/\Gamma_{ij}^i$. As x_j^0 varies, we obtain

$$Y(x_1, \ldots, x_n) = X - X_j/\Gamma_{ij}^i, \qquad i \neq j.$$

The map Y will be called the (i,j)-*Laplace Transform* of X, where (i,j) is an ordered pair.

Our next result shows that a Laplace Transform of a Cartan manifold is generically also a Cartan manifold. In order to state the theorem, we need to introduce some notation. We consider an n-dimensional Cartan manifold in $\mathbf{R}^{2n}$ parametrized by conjugate coordinates. For each ordered pair $(i,j), i \neq j$ for which Γ_{ij}^i is nonzero, we define the functions:

$$\begin{aligned} M_{lj} &= \frac{\Gamma_{ij,l}^i}{\Gamma_{ij}^i} - \Gamma_{jl}^j, \\ M_{ll} &= \Gamma_{ij}^i - \Gamma_{jl}^l, \quad \forall l,\ l \neq j, \\ M_{jj} &= \frac{\Gamma_{ij,j}^i}{\Gamma_{ij}^i} + \Gamma_{ij}^i, \end{aligned} \tag{18}$$

where $\Gamma_{ij,l}^i$ denotes the derivative of Γ_{ij}^i with respect to x_l.

Theorem1: *Let $X(x_1, \ldots, x_n)$ be a Cartan manifold in $\mathbf{R}^{2n}$ parametrized by conjugate coordinates. Consider an ordered pair $(i,j), i \neq j$ such that $\Gamma_{ij}^i \neq 0$. Then the map*

$$Y = X - \frac{1}{\Gamma_{ij}^i} X_j \tag{19}$$

defines generically a Cartan manifold.

Proof: ¿From equation (19) it follows that

$$\begin{aligned} Y_j &= (M_{jj}X_j - X_{jj})/\Gamma_{ij}^i, \\ Y_i &= M_{ij}X_j/\Gamma_{ij}^i, \\ Y_k &= (M_{kj}X_j + M_{kk}X_k)/\Gamma_{ij}^i \quad \forall k, k \neq i, k \neq j. \end{aligned} \tag{20}$$

Hence, we have

$$\begin{pmatrix} Y_i \\ Y_k \\ \vdots \\ Y_j \end{pmatrix} = \frac{1}{\Gamma^i_{ij}} \begin{pmatrix} M_{ij} & 0 & \cdots & 0 \\ M_{kj} & M_{kk} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ M_{jj} & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} X_j \\ X_k \\ \vdots \\ X_{jj} \end{pmatrix} \quad k \neq i, k \neq j. \tag{21}$$

Therefore, the tangent space of Y is n-dimensional if and only if,

$$M_{ij} \prod_{k\neq i, k\neq j} M_{kk} \neq 0.$$

This implies that generically Y defines an n-dimensional parametrized submanifold of $\mathbf{R}^{2n}$.

In order to show that Y is generically a Cartan manifold, we need to show that the set of vector fields $Y_l, Y_{ll}, 1 \leq l \leq n$, are generically linearly independent and moreover that Y is parametrized by conjugate coordinates.

A straightforward computation shows that

$$\begin{aligned} Y_{ii} &= \left[(\frac{M_{ij}}{\Gamma^i_{ij}})_{,i} + \frac{M_{ij}}{\Gamma^i_{ij}} \Gamma^j_{ij} \right] X_j + M_{ij} X_i, \\ Y_{kk} &= \left[(\frac{M_{kj}}{\Gamma^i_{ij}})_{,k} + \frac{M_{kj}}{\Gamma^i_{ij}} \Gamma^j_{kj} \right] X_j + \left[(\frac{M_{kk}}{\Gamma^i_{ij}})_{,k} + \frac{M_{kj}}{\Gamma^i_{ij}} \Gamma^k_{kj} \right] X_k + \frac{M_{kk}}{\Gamma^i_{ij}} X_{kk}, \\ Y_{jj} &= (\frac{M_{jj}}{\Gamma^i_{ij}})_{,j} X_j + \left[1 - 2(\frac{1}{\Gamma^i_{ij}})_{,j} \right] X_{jj} - \frac{1}{\Gamma^i_{ij}} X_{jjj}, \end{aligned}$$

for all $k, k \neq i, k \neq j$ where, using the notation introduced in (14) we have

$$\begin{aligned} X_{jjj} &= \left(\Gamma^j_{jj,j} + \Gamma^i_{jj}\Gamma^j_{ij} + \sum_{k\neq i, k\neq j} \Gamma^k_{jj}\Gamma^j_{kj} + b^j_{jj} - \sum_l r^l_{jj}\Gamma^j_{ll} \right) X_j \\ &+ \sum_{k\neq i, k\neq j} \left(\Gamma^k_{jj,j} + \Gamma^k_{jj}\Gamma^k_{kj} + b^k_{jj} - \sum_l r^l_{jj}\Gamma^k_{ll} \right) X_k \\ &+ \left(\Gamma^i_{jj,j} + \Gamma^i_{jj}\Gamma^i_{ij} + b^i_{jj} - \sum_l r^l_{jj}\Gamma^i_{ll} \right) X_i \\ &+ (\Gamma^j_{jj} + r^j_{jj}) X_{jj} + \sum_{l\neq j} r^l_{jj} X_{ll}. \end{aligned}$$

Hence,

$$\begin{pmatrix} Y_i \\ Y_k \\ \vdots \\ Y_{ii} \\ Y_j \\ Y_{kk} \\ \vdots \\ Y_{jj} \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} X_j \\ X_k \\ \vdots \\ X_i \\ X_{jj} \\ X_{kk} \\ \vdots \\ X_{ii} \end{pmatrix} \qquad k \neq i, k \neq j,$$

where A, B and C are $n \times n$ lower triangular matrices

$$A = \frac{1}{\Gamma^i_{ij}} \begin{pmatrix} M_{ij} & 0 & \cdots & 0 \\ M_{kj} & M_{kk} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ * & 0 & 0 & M_{ij}\Gamma^i_{ij} \end{pmatrix},$$

and

$$C = \frac{1}{\Gamma^i_{ij}} \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & M_{kk} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ * & * & * & -r^i_{jj} \end{pmatrix}.$$

The determinant of the above matrix is equal to

$$r^i_{jj} M^2_{ij} \prod_{k\neq i, k\neq j} M^2_{kk} / (\Gamma^i_{ij})^{2n-1}.$$

Since $X_1, \ldots, X_n, X_{11}, \ldots, X_{nn}$ are linearly independent, it follows that generically $Y_1, \ldots, Y_n, Y_{11}, \ldots, Y_{nn}$ are linearly independent. Hence the osculating space of Y is $2n$-dimensional.

Finally, we need to show that Y is parametrized by conjugate coordinates. ¿From (20), we obtain upon differentiation the following expressions

$$\begin{aligned} Y_{ij} &= (\frac{M_{ij}}{\Gamma^i_{ij}})_{,j} X_j + \frac{M_{ij}}{\Gamma^i_{ij}} X_{jj} \\ Y_{ik} &= \left[(\frac{M_{ij}}{\Gamma^i_{ij}})_{,k} + \frac{M_{ij}}{\Gamma^i_{ij}} \Gamma^j_{jk} \right] X_j + \frac{M_{ij}}{\Gamma^i_{ij}} \Gamma^k_{jk} X_k \\ Y_{kj} &= \left[(\frac{M_{kj}}{\Gamma^i_{ij}})_{,j} + \frac{M_{kk}}{\Gamma^i_{ij}} \Gamma^j_{jk} \right] X_j + \left[(\frac{M_{kk}}{\Gamma^i_{ij}})_{,j} + \frac{M_{kk}}{\Gamma^i_{ij}} \Gamma^k_{jk} \right] X_k + \frac{M_{kj}}{\Gamma^i_{ij}} X_{jj} \\ Y_{kl} &= \left[(\frac{M_{kj}}{\Gamma^i_{ij}})_{,l} + \frac{M_{kj}}{\Gamma^i_{ij}} \Gamma^j_{jl} \right] X_j + \left[(\frac{M_{kk}}{\Gamma^i_{ij}})_{,l} + \frac{M_{kk}}{\Gamma^i_{ij}} \Gamma^k_{kl} \right] X_k + \left(\frac{M_{kj}}{\Gamma^i_{ij}} \Gamma^l_{jl} + \frac{M_{kk}}{\Gamma^i_{ij}} \Gamma^l_{kl} \right) X_l. \end{aligned} \tag{22}$$

where k, l are such that $k \neq l$ and are distinct from i, j. We know from (21) that the tangent space to Y is generated by the set $X_l, l \neq i, X_{jj}$. Eqs. (22) therefore show that each of the mixed partials $Y_{kl}, k \neq l$ lies in the subspace spanned by Y_k and Y_l. This concludes the proof of the theorem. □

We will illustrate the geometrical Laplace transformation by considering the examples of Cartan manifolds given in this section. The Clifford torus, given in Example a), is a special Cartan manifold, since its Christoffel symbols Γ^i_{ij} vanish for all pairs (i, j), $i \neq j$. Therefore, the Laplace transformation does not apply.

Consider the $n-$dimensional toroidal submanifold given in Example b). It is easy to see that $\Gamma^2_{12} = -\lambda \tan \lambda x_1$ and $\Gamma^i_{ji} = 0$ for all pairs $(i, j) \neq (2, 1)$. By applying the $(2, 1)-$Laplace transformation to the immersion X, we obtain an $(n-1)-$dimensional Cartan manifold contained in $\mathbf{R}^{2n-2}$, given by

$$Y(x_1, x_3, ..., x_n) = (\delta \tilde{f}_0, \delta \tilde{f}_1, 0, 0, c_3 \cos x_3, \sin x_3, ..., c_3 \cos x_n, \sin x_n),$$

where

$$\tilde{f}_0 = \frac{r}{\lambda} \sin r x_1 \cos \lambda x_1 - \cos r x_1 \sin \lambda x_1 \quad \tilde{f}_1 = \frac{r}{\lambda} \cos r x_1 \cos \lambda x_1 - \sin r x_1 \sin \lambda x_1.$$

We observe that $M_{21} = 0$ and $M_{kk} = \Gamma^2_{12}$ for all k distinct from 1 and 2. This explains the dimension of Y.

Finally, we consider the Cartan manifold given in Example c). In this example $\Gamma^1_{12} = x_1$ and $\Gamma^i_{ij} = 0$ for all pairs $(i, j) \neq (1, 2)$. By applying the $(1, 2)-$Laplace transformation to the immersion X, we obtain $Y = X - X_2/x_1$, which is also an $n-$dimensional Cartan manifold. This is a consequence of the fact that M_{12} and M_{kk} do not vanish for all k distinct from 1 and 2. In fact, $M_{12} = 1$ and $M_{kk} = x_1$.

The above examples illustrate the geometrical Laplace transformation and the role of the functions M_{ij} and M_{kk} in the dimension of the transformed manifold.

Remark If Y is the (i, j)-Laplace Transform of a Cartan manifold X, then generically Y will also be a Cartan manifold. Therefore, the mixed partial derivatives Y_{kl} will be a linear combination of Y_k and Y_l. In what follows we will compute those derivatives. From (21) we have that

$$\begin{aligned} X_j &= \frac{\Gamma^i_{ij}}{M_{ij}} Y_i \\ X_k &= \frac{\Gamma^i_{ij}}{M_{kk}} \left(Y_k - \frac{M_{kj}}{M_{ij}} Y_i \right) \\ X_{jj} &= \Gamma^i_{ij} \left(\frac{M_{jj}}{M_{ij}} Y_i - Y_j \right) \end{aligned} \tag{23}$$

where k, l are such that $k \neq l$ and are distinct from i, j. Therefore, using (22) we obtain

$$Y_{ij} = \left[(\frac{M_{ij}}{\Gamma^i_{ij}})_{,j} \frac{\Gamma^i_{ij}}{M_{ij}} + M_{jj} \right] Y_i - M_{ij} Y_j, \tag{24}$$

$$\begin{aligned} Y_{ik} &= \left[(\frac{M_{ij}}{\Gamma^i_{ij}})_{,k} \frac{\Gamma^i_{ij}}{M_{ij}} + \Gamma^j_{jk} - M_{kj} \frac{\Gamma^k_{jk}}{M_{kk}} \right] Y_i + M_{ij} \frac{\Gamma^k_{jk}}{M_{kk}} Y_k, \\ Y_{kj} &= \left[(\frac{M_{kk}}{\Gamma^i_{ij}})_{,j} \frac{\Gamma^i_{ij}}{M_{kk}} + \Gamma^k_{kj} \right] Y_k - M_{kj} Y_j, \\ Y_{kl} &= \left[(\frac{M_{kk}}{\Gamma^i_{ij}})_{,l} \frac{\Gamma^i_{ij}}{M_{kk}} + \Gamma^k_{kl} \right] Y_k + \frac{1}{M_{ll}} \left(M_{kj} \Gamma^l_{jl} + M_{kk} \Gamma^l_{kl} \right) Y_l, \end{aligned}$$

where k and l are distinct from i and j.

We observe that the coefficients of Y_i in the expressions of Y_{kj} and Y_{kl} vanish as a consequence of the property (13).

It follows from the definition there are at most $n(n-1)$ Laplace transforms for a given Cartan manifold. Moreover, it is particularly simple to characterize the Cartan manifolds X for which the (i,j)-Laplace Transform degenerates to a curve.

Corollary: *The (i,j)-Laplace Transform of a Cartan manifold X reduces to a curve if and only if*

$$M_{ij} = M_{kk} = 0 \quad \forall k, k \neq i, k \neq j.$$

Proof: Let Y be the (i, j)-Laplace Transform of X. Then, it follows from (21) that the tangent space of Y is generated by X_l, $l \neq i$ and X_{jj}. Therefore, Y reduces to a curve if and only if the $n \times n$ matrix in (21) has rank one, i.e. $M_{ij} = M_{kk} = 0$ for all $k \neq i, k \neq j$, since as a consequence of (13), we have

$$M_{kj} + \frac{\Gamma^i_{ik}}{\Gamma^i_{ij}} M_{kk} = 0.$$

□

It is easy to see that generically the Laplace transformation is invertible.

Proposition: *If $M_{ij} \neq 0$, the inverse of the (i, j)-Laplace transform exists and it is given by the (j,i)-Laplace transform.*

Proof: If Y is the (i, j)-Laplace Transform of X, then it follows from (24) that the (j, i)-Laplace Transform of Y is given by

$$Z = Y + \frac{1}{M_{ij}} Y_i.$$

Substituting the expressions of Y and Y_i in terms of X and its derivatives given by (19) and (20) we conclude that $Z = X$.

4. The generalized method of Laplace for systems of second order PDEs

In this section we consider a system of partial differential equations for a scalar or vector valued function y of $x_1, \ldots, x_n$ given by

$$y_{,lk} + a^l_{lk} y_{,l} + a^k_{lk} y_{,k} + c_{lk} y + h_{lk} = 0 \qquad 1 \leq k \neq l \leq n, \tag{25}$$

where the functions a, c and h are symmetric in the lower indices and depend differentiably on $x_1, \ldots, x_n$. In this equation, $y_{,k}$ and $y_{,lk}$ denote the derivatives of y with respect to x_k and the second order derivative of y with respect to x_l and x_k respectively. We will be interested in solving (25) for smooth Cauchy data given by $y(x_1^0, ..., x_l, ..., x_n^0) = f_l(x_l), 1 \leq l \leq n$.

Motivated by the Corollary in the previous section and the analysis of the two dimensional case, we will provide a generalized method of Laplace for solving systems of equations of the form (25). We start with preliminary results which will essentially provide the higher dimensional Laplace invariants.

Proposition: *Consider a system of equations of the form*

$$y_{,lk} + a^l_{lk} y_{,l} + a^k_{lk} y_{,k} + c_{lk} y = 0 \qquad k \neq l \tag{26}$$

for $y(x_1, \ldots, x_n)$ and let λ be a nonvanishing solution of (26). Then the function $y = \lambda Y$ satisfies (26) if and only if Y satisfies the system of equations

$$Y_{,lk} + A^l_{lk} Y_{,l} + A^k_{lk} Y_{,k} = 0 \quad k \neq l, \tag{27}$$

where

$$A^l_{lk} = a^l_{lk} + \lambda_{,k}/\lambda, \qquad A^k_{lk} = a^k_{lk} + \lambda_{,l}/\lambda. \tag{28}$$

Proof: From $y = \lambda Y$, we get

$$\begin{aligned} y_{,l} &= \lambda_{,l} Y + \lambda Y_{,l}, \\ y_{,k} &= \lambda_{,k} Y + \lambda Y_{,k}, \\ y_{,lk} &= \lambda_{,lk} Y + \lambda_{,l} Y_{,k} + \lambda_{,k} Y_{,l} + \lambda Y_{,lk}. \end{aligned}$$

From the fact that λ satisfies (26) it follows that

$$y_{,lk} + a^l_{lk} y_{,l} + a^k_{lk} y_{,k} + c_{lk} y = \lambda(Y_{,lk} + A^l_{lk} Y_{,l} + A^k_{lk} Y_{,k})$$

This concludes the proof. □

The proposition above shows that one obtains the set of solutions of (26) by considering the set of solutions of (27) and a particular solution of (26).

Proposition: *Consider the system of equations (26) and (27) as above. For any ordered pair (i,j) such that $A^i_{ij} \neq 0$, consider the functions defined by*

$$\begin{aligned} M_{ij} &= A^j_{ij} + A^i_{ij,i}/A^i_{ij}, \\ M_{ll} &= A^l_{lj} - A^i_{ij} \qquad \forall l,\ l \neq j. \end{aligned} \tag{29}$$

Then

$$M_{ij} = M_{ll} = 0 \qquad \forall l,\ l \neq j$$

if and only if

$$m_{ij} = m_{ll} = 0,$$

where

$$\begin{aligned} m_{ij} &= a^i_{ij,i} + a^i_{ij}a^j_{ij} - c_{ij}, \\ m_{ll} &= a^l_{lj} - a^i_{ij}, \qquad \forall l,\ l \neq j. \end{aligned}$$

Proof: The proof follows immediately from the relations (28) and (29) . In fact, one gets

$$\begin{aligned} M_{ij} &= m_{ij}/(a^i_{ij} + \frac{\lambda_{,j}}{\lambda}), \\ M_{ll} &= m_{ll}. \end{aligned}$$

□

We observe that if $y(x_1, \ldots, x_n)$ is a $2n$-vector valued function and λ is a scalar valued function which solves (26) then $Y = y/\lambda$ is a $2n$-valued function whch satisfies an equation of the form (27). Whenever Y describes a Cartan manifold, the condition $M_{ij} = M_{ll} = 0,\ l \neq j$, characterizes the fact that the $(i,j)-$Laplace Transform of Y reduces to a curve, as we have seen in the Corollary of Section 3.

Any solution y of (25) which is sufficiently differentiable must satisfy the integrability conditions

$$y_{,ljk} = y_{,lkj} \quad \text{for } l, j, k \text{ distinct.}$$

Moreover, recall that we want to solve (25) for given initial conditions

$$y(x^0_1, ..., x^0_{l-1}, x_l, x^0_{l+1}, ..., x^0_n) = f_l(x_l)$$

for all $l, 1 \leq l \leq n$ and for any fixed point $(x_1^0, ..., x_n^0)$. Therefore, the following relations must be satisfied, for all l, k, j distinct.

$$\begin{aligned} & a^l_{lk,j} - a^l_{lj,k} = 0, \\ & c_{lj} = a^k_{lk,j} - a^k_{lk} a^k_{kj} + a^l_{lj} a^k_{lk} + a^j_{lj} a^k_{jk}, \\ & c_{lk,j} - c_{lj,k} + a^l_{lj} c_{lk} + (a^j_{lj} - a^k_{lk}) c_{kj} - a^l_{lk} c_{lj} = 0, \\ & h_{lk,j} - h_{lj,k} + a^l_{lj} h_{lk} + (a^j_{lj} - a^k_{lk}) h_{kj} - a^l_{lk} h_{lj} = 0. \end{aligned} \tag{30}$$

¿From now on, we will assume that (30) are valid. We observe that the first equation in (30) follows from the second one since $c_{lj} = c_{jl}$. Moreover, whenever c_{lk} and h_{lk} are zero then (30) reduces just to the second equation and this one coincides with (13) in the case y is a vector valued function parametrizing a Cartan manifold.

Let (i, j), $i \neq j$, be an ordered pair. Define the functions

$$\begin{aligned} m_{ij} &= a^i_{ij,i} + a^i_{ij} a^j_{ij} - c_{ij}, \\ m_{ll} &= a^l_{lj} - a^i_{ij}, \qquad \forall l, l \neq j, l \neq i. \end{aligned} \tag{31}$$

These functions will be called the (i, j)-*higher dimensional Laplace invariants* of (25). They are invariant under the scaling of y by a nonvanishing function of $x_1, ..., x_n$. We observe that whenever $n = 2$ the above expressions reduce to the classical Laplace invariants h and k described in Section 2. More precisely, h is the $(1, 2)$-Laplace invariant and k is the $(2, 1)$-Laplace invariant.

We can now prove the following reduction theorem.

Theorem2: *Consider a system of differential equations of the form (25) for y, with given initial conditions $y(x_1^0, ..., x_l, ..., x_n^0) = f_l(x_l), 1 \leq l \leq n$. Let (i, j) be such that the (i, j)- higher dimensional Laplace invariants vanish. Then the general solution of (25) is given by*

$$y = Q + e^{-J} G(\hat{x}_j), \tag{32}$$

where

$$Q = -e^{-J} \int e^{J-I} \left[\int e^I h_{ij} dx_i - F(x_j) \right] dx_j, \tag{33}$$

$$I=\int a^j_{ij}dx_i, \qquad J=\int a^i_{ij}dx_j, \tag{34}$$

F is an arbitrary function of x_j and $G(x_1,\ldots,\hat{x}_j,\ldots,x_n)$ does not depend on x_j and the antiderivative I can be chosen so that G satisfies a system of partial differential equations in $n-1$ variables $x_1,\ldots,\hat{x}_j,\ldots,x_n$ given by

$$G_{,lk}+g^l_{lk}G_{,l}+g^k_{lk}G_{,k}+b_{lk}G+r_{lk}=0 \quad l\neq k \text{ distinct from } j. \tag{35}$$

Moreover,

$$g^l_{lk}=a^l_{lk}-J_{,k}, \qquad g^k_{lk}=a^k_{lk}-J_{,l}$$

$$b_{lk}=c_{lk}+J_{,k}J_{,l}-J_{,lk}-a^l_{lk}J_{,l}-a^k_{lk}J_{,k}$$

and

$$r_{lk}=e^J(h_{lk}+Q_{,lk}+a^l_{lk}Q_{,l}+a^k_{lk}Q_{,k}+c_{lk}Q).$$

The function $F(x_j)$ and the initial conditions for G are determined by

$$F(x_j)=\left[e^I(a^i_{ij}f_j(x_j)+f'_j(x_j))+\int e^I h_{ij}\;dx_i\right](x^0_1,...,x_j,...,x^0_n),$$

$$G(x^0_1,...,x_l,...,x^0_n)=e^J(f_l(x_l)-Q)(x^0_1,...,x_l,...,x^0_n), \qquad \forall l, 1\leq l\leq n, l\neq j.$$

Proof: If $m_{ij}=m_{kk}=0$, then it follows from (30) that

$$a^i_{ij,k}+a^i_{ij}a^j_{kj}-c_{kj}=0$$

and the equations

$$y_{,ij}+a^i_{ij}y_{,i}+a^j_{ij}y_{,j}+c_{ij}y+h_{ij}=0,$$
$$y_{,jk}+a^j_{jk}y_{,j}+a^k_{jk}y_{,k}+c_{jk}y+h_{jk}=0,$$

reduce to the following

$$\frac{\partial}{\partial x_i}(y_{,j}+a^i_{ij}y)+a^j_{ij}(y_{,j}+a^i_{ij}y)+h_{ij}=0,$$
$$\frac{\partial}{\partial x_k}(y_{,j}+a^i_{ij}y)+a^j_{jk}(y_{,j}+a^i_{ij}y)+h_{jk}=0.$$

With the notation introduced in (34) we consider

$$Z=e^I(y_{,j}+a^i_{ij}y). \tag{36}$$

Then, we have

$$Z_{,i} = -e^I h_{ij} \quad \text{and} \quad Z_{,k} = -e^I h_{jk} \qquad \forall k, k \neq i, k \neq j. \tag{37}$$

¿From (30), using the fact that $m_{ll} = 0$, we have

$$h_{ij,k} - h_{jk,i} + a^j_{jk} h_{ij} - a^j_{ij} h_{jk} = 0 \quad \forall k, k \neq i, k \neq j$$

Therefore, we can integrate equations (37). In fact,

$$Z = -\int e^I h_{ij} dx_i + F(x_j) \tag{38}$$

where F depends only on x_j. From (36) we get

$$y_{,j} + a^i_{ij} y = e^{-I} Z.$$

Therefore, letting

$$T = e^J y, \tag{39}$$

we obtain that

$$T_{,j} = -e^{J-I} (\int e^I h_{ij} dx_i - F).$$

Hence

$$T = -\int e^{J-I} \left[\int e^I h_{ij} dx_i - F(x_j) \right] dx_j + G(x_1, \ldots, \hat{x}_j, \ldots, x_n).$$

where G does not depend on x_j. Using (39), we conclude that y is given by (32) where G satisfies (35). We observe that the system (35) is obtained by substituting the expression of y into the differential equations (25) for $l, k, l \neq k$ distinct from j. It follows from (30) that I can be chosen so that $I_{,i} = a^j_{ij}$ and $I_{,k} = a^j_{jk}$ for all $k, k \neq i, k \neq j$. The vanishing of the invariants combined with (30), implies that $g^l_{lk}, g^k_{lk}, b_{lk}$ and r_{lk} do not depend on x_j. Moreover, as a consequence of (30) it is easy to show that the coefficients of equation (35) also satisfy the properties (30). This completes the proof of the theorem. □

As we have seen in Section 2, given a differential equation of the form (3) such that the Laplace invariants do not vanish, one defines the Laplace transforms

$\mathcal{L}_1$ or $\mathcal{L}_{-1}$. These are again equations of the same type and one looks for the invariants of the new equation. In what follows we will consider the higher dimensional version of the Laplace transformation.

Consider a system of equations (25) for y. For an ordered pair (i, j) we define

$$\tilde{y} = y_{,j} + a^i_{ij} y \tag{40}$$

to be the (i, j)-*Laplace transform of* y which we denote by $\tilde{y} = \mathcal{L}_{(i,j)}(y)$. In the next result we show that just as in the two dimensional case $\tilde{y}$ will satisfy a system of differential equations of the same type as (25).

Theorem 3: *Consider a system of differential equations (25) for y. Let $\tilde{y}$ be the (i, j)-Laplace Transform of y. If the invariants m_{ij} and m_{kk} do not vanish for all k, $k \neq i, k \neq j$, then $\tilde{y}$ satisfies a system of differential equations of the same type.*

Proof: This result follows from a lengthy but straightforward computation. In fact, consider the (i, j)-Laplace transform of y given by (40) and define

$$m_{kj} = a^i_{ij,k} + a^i_{ij} a^j_{kj} - c_{kj}.$$

Then, we have

$$\begin{aligned} \tilde{y}_{,i} &= -a^j_{ij} y_{,j} + (m_{ij} - a^i_{ij} a^j_{ij}) y - h_{ij} & (41) \\ \tilde{y}_{,j} &= a^i_{ij,j} y + a^i_{ij} y_{,j} + y_{,jj} & (42) \\ \tilde{y}_{,k} &= (m_{kj} - a^j_{kj} a^i_{ij}) y - m_{kk} y_{,k} - a^j_{jk} y_{,j} - h_{jk} & (43) \end{aligned}$$

where $k \neq i$ $k \neq j$. From equations(40) and (41) we get y and $y_{,j}$ in terms of $\tilde{y}$ and $\tilde{y}_{,i}$.

$$\begin{aligned} y &= (a^j_{ij} \tilde{y} + \tilde{y}_{,i} + h_{ij}) / m_{ij} & (44) \\ y_{,j} &= \left[(m_{ij} - a^i_{ij} a^j_{ij}) \tilde{y} - a^i_{ij} (\tilde{y}_{,i} + h_{ij}) \right] / m_{ij}. & (45) \end{aligned}$$

Then $y_{,jj}$ and $y_{,k}$ are obtained from (42)-(45). In fact,

$$y_{,k} = \left[-\tilde{y}_{,k} + \frac{m_{kj}}{m_{ij}}\tilde{y}_{,i} + (\frac{m_{kj}}{m_{ij}}a^j_{ij} - a^j_{jk})\tilde{y} + \frac{m_{kj}}{m_{ij}}h_{ij} - h_{jk}\right]/m_{kk} \quad (46)$$

$$y_{,jj} = \tilde{y}_{,j} - \left\{(a^i_{ij,j} - a^i_{ij}a^i_{ij})\tilde{y}_{,i} + \left[a^i_{ij,j}a^j_{ij} + a^i_{ij}(m_{ij} - a^i_{ij}a^j_{ij})\right]\tilde{y}\right\}/m_{ij}$$
$$-h_{ij}(a^i_{ij,j} - a^i_{ij}a^i_{ij})/m_{ij}. \quad (47)$$

Computing the second order derivatives of $\tilde{y}$, it follows from the equations (43)-(45) that $\tilde{y}$ satisfies a system of differential equations

$$\tilde{y}_{,lk} + \tilde{a}^l_{lk}\tilde{y}_{,l} + \tilde{a}^k_{lk}\tilde{y}_{,k} + \tilde{c}_{lk}\tilde{y} + \tilde{h}_{lk} = 0 \quad k \neq l. \quad (48)$$

We observe that the coefficients of $\tilde{y}_{,i}$ in the expressions of $\tilde{y}_{,jk}$ and $\tilde{y}_{,lk}$ where l and k are distinct from i and j vanish as a consequence of (30). Moreover, we can explicitly give the coefficients of (48) in terms of the coefficients of the equation for y. In fact, using (30) we have

$$\tilde{a}^i_{ij} = a^i_{ij} - \frac{m_{ij,j}}{m_{ij}}, \qquad \tilde{a}^j_{ij} = a^j_{ij}, \quad (49)$$

$$\tilde{c}_{ij} = a^j_{ij}a^i_{ij} + m_{ij}[(\frac{a^j_{ij}}{m_{ij}})_{,j} - 1], \qquad \tilde{h}_{ij} = h_{ij}\tilde{a}^i_{ij} + h_{ij,j}. \quad (50)$$

For each $k, k \neq i, k \neq j$ we have

$$\tilde{a}^i_{ik} = -(\log m_{ij})_k + a^i_{ik} \qquad \tilde{a}^k_{ik} = a^j_{ij} + \frac{m_{ij}}{m_{kk}} \quad (51)$$

$$\tilde{c}_{ik} = m_{ij}[(\frac{a^j_{ij}}{m_{ij}})_{,k} + \frac{a^j_{jk}}{m_{kk}}] + a^j_{ij}a^i_{ik} \qquad \tilde{h}_{ik} = h_{ij}\tilde{a}^i_{ik} + h_{jk}\frac{m_{ij}}{m_{kk}} + h_{ij,k} \quad (52)$$

$$\tilde{a}^j_{jk} = a^j_{jk} \qquad \tilde{a}^k_{jk} = a^k_{jk} - (\log m_{kk})_j \quad (53)$$

$$\tilde{c}_{jk} = 2a^j_{ij}a^i_{ij,k}m_{kk}/m_{ij} + a^j_{jk}(\tilde{a}^k_{jk} - m_{kk}) - m_{kj} + a^j_{jk,j} \quad (54)$$

$$\tilde{h}_{jk} = h_{jk}(\tilde{a}^k_{jk} - m_{kk}) - h_{jk,j} \quad (55)$$

Moreover, when $n \geq 4$ we have for each k and l distinct from i and j

$$\tilde{a}^k_{kl} = a^k_{kl} - \frac{m_{kk,l}}{m_{kk}}, \qquad \tilde{a}^l_{kl} = a^j_{kj} + P_{kl} \quad (56)$$

$$\tilde{c}_{kl} = a^j_{jk,l} + a^j_{jl} P_{kl} + a^j_{jk} \tilde{a}^k_{kl}, \qquad \tilde{h}_{kl} = h_{jl} P_{kl} + h_{jk} \tilde{a}^k_{kl} - m_{kk} h_{kl} + h_{jk,l} \tag{57}$$

where

$$P_{kl} = (-a^i_{ik} + a^l_{kl}) m_{kk}/m_{ll}.$$

□

As in the two dimensional case, we can generically invert the Laplace transformation.

Theorem 4: *Consider a system of differential equations (25) for y and let $\tilde{y} = \mathcal{L}_{(i,j)}(y)$. If $m_{ij} \neq 0$, then the inverse of the (i,j)-Laplace transform exists and it is given by*

$$y = [\mathcal{L}_{(j,i)}(\tilde{y}) + h_{ij}]/m_{ij}.$$

Proof: Let $\tilde{y} = \mathcal{L}_{(i,j)}(y)$ and $y* = \mathcal{L}_{(j,i)}(\tilde{y})$. Hence,

$$y* = \tilde{a}^j_{ij} \tilde{y} + \tilde{y}_{,i}.$$

It follows from (40), (43) and (51) that

$$y* = m_{ij} y - h_{ij}.$$

Therefore, since $m_{ij} \neq 0$ we obtain

$$y = (y * + h_{ij})/m_{ij}$$

and the theorem is proved. □

We conclude this section with some illustrative examples.

Example 1: The simplest example of a system of differential equations treated in this section is given by

$$y_{,lk} = 0 \qquad l \neq k, 1 \leq l, k \leq n.$$

The relations (30) are trivially satisfied and Theorem 2 can be aplied at each level to reduce the problem to a system with one less independent variable.

This gives the general solution of this system

$$y = \sum_{i=1}^{n} f_i(x_i) + c$$

with the given initial conditions

$$y(0, ..., 0, x_i, 0, ..., 0) = f_i(x_i) \quad \text{and } f_i(0) = -\frac{c}{n-1}.$$

Example 2: Consider the system of equations given by

$$y_{,lk} + y_l + y_k + y + 1 = 0 \qquad l \neq k, 1 \leq l, k \leq n.$$

The relations (30) are satisfied and we can apply Theorem 2. For the sake of simplicity we will take $n = 3$ and we consider the initial conditions to be

$$y(x_1, 0, 0) = f(x_1), \quad y(0, x_2, 0) = g(x_2), \quad y(0, 0, x_3) = l(x_3).$$

We first take the pair $(i, j) = (2, 3)$. It is easy to see that the $(2, 3)$-higher dimensional invariants vanish. Then, it follows from Theorem 2, after relabelling of the arbitrary functions, that

$$y = -1 + e^{-(x_1+x_2)} F(x_3) + e^{-x_3} G(x_1, x_2),$$

where $F(x_3)$ is determined by the Cauchy data $l(x_3)$ and G satisfies the equation

$$G_{,12} + G_{,1} + G_{,2} + G = 0.$$

This reduced equation has vanishing Laplace invariants. Its general solution is given by

$$G = e^{-x_2} H(x_1) + e^{-x_1} K(x_2)$$

so that

$$y = -1 + e^{-(x_1+x_2)} F(x_3) + e^{-(x_1+x_3)} K(x_2) + e^{-(x_2+x_3)} H(x_1),$$

where H and K are determined by the remaining Cauchy data.

Example 3 Consider the system of differential equations for $y(x_1, x_2, x_3)$ given by

$$y_{,12} + \frac{x_1 + x_3}{x_2} y_{,1} + \log x_2 \; y_{,2} + \frac{1}{x_2}[1 + (x_1 + x_3)\log x_2]y = 0$$
$$y_{,13} + (e^{x_1} + \log x_2)y_{,1} + (-1 + \log x_2)y_{,3} + \log x_2(e^{x_1} + \log x_2 - 1)y = 0$$
$$y_{,23} + \frac{x_1 + x_3}{x_2} y_{,3} + \frac{1}{x_2} y = 0$$

We consider the ordered pair $(1, 2)$. It is easy to see that the $(1, 2)$-higher dimensional invariants defined by (31) vanish. It then follows from Theorem 3 that

$$y = Q_{12} + e^{-J} G(x_1, x_3) \tag{58}$$

where

$$Q_{12} = e^{-J} \int e^{J-I} F(x_2) dx_2, \qquad I = \int \log x_2 dx_1, \qquad J = \int (\frac{x_1 + x_3}{x_2}) dx_2,$$

where $F(x_2)$ is an arbitrary function and G satisfies the differential equation

$$G_{,13} + g^1_{13} G_{,1} + g^3_{13} G_{,3} + b_{13} G = 0 \tag{59}$$

where

$$g^1_{13} = e^{x_1} + p_{,3}, \qquad g^3_{13} = -1 + p_{,1}$$

and

$$b_{13} = p_{,13} + p_{,3} p_{,1} + p_{,3} + e^{x_1} p_{,1}.$$

We observe that the integral J is defined up to a function which depends on x_1 and x_3 only. So let

$$J = (x_1 + x_3)\log x_2 - p(x_1, x_3).$$

For the sake of simplicity we choose $F = 0$ (although the integration could be carried out in principle with $F \neq 0$). The differential equation for G can be solved by the classical Laplace method since one of its two invariants vanishes. For example, one can choose the function p (e.g. $p = 0$) to satisfy

$$p_{,3} = 0.$$

The general solution of this equation is given by

$$p = f(x_1)$$

where f is arbitrary. Then it follows that

$$G(x_1, x_3) = e^{-\tilde{s}} \left(\int e^{\tilde{s}-s} R(x_3) dx_3 + H(x_1) \right),$$

where

$$\tilde{s} = \int e^{x_1} dx_3, \quad \text{and} \quad s = \int (-1 + p_{,1}) dx_1.$$

Hence, the solution for our initial system of equations is given by

$$y = e^{-J-\tilde{s}} \left(\int e^{\tilde{s}-s} R(x_3) dx_3 + H(x_1) \right).$$

As a concluding remark, we observe that the higher dimensional Laplace transformation and the method of integration presented in this paper can be implemented without difficulty in any simbolic computation language.

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N. Kamran, *McGill University, Montreal, Canada*
e-mail: nkamran@gauss.math.mcgill.ca

K. Tenenblat, *Universidade de Brasilia, Brasilia, Brazil*
e-mail: keti@mat.unb.br