On Saddle Submanifolds of Riemannian Manifolds

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Abstract

Saddle submanifolds are considered. A characterization of such submanifolds of Euclidean space is given in terms of sectional curvature. Extending results of T. Frankel, K. Kenmotsu and C. Xia, we determine under what conditions two complete saddle submanifolds of a complete connected Riemannian manifold $\overline{M}$, with nonnegative k-Ricci curvature, must intersect. Moreover, if $\overline{M}$ has positive k-Ricci curvature and the dimension of a compact saddle submanifold satisfies a certain inequality then we show that the homomorphism of the fundamental groups $\pi_1(M)$ and $\pi_1(\overline{M})$ is surjective.

Hadamard [7] proved that on a complete surface with positive Gaussian curvature, every geodesic must meet every closed geodesic. In 1961, T. Frankel [4] generalized this theorem to Riemannian spaces of positive sectional curvature. He showed that two compact totally geodesic submanifolds M_1 and M_2 of an n-dimensional Riemannian manifold $\overline{M}^n$ of positive sectional curvature must necessarily intersect if the sum of their dimensions is at least equal to n. As a consequence of this theorem, in 1966, Frankel [5] showed that if $\overline{M}^n$ is complete and connected with strictly positive sectional curvature and M^ℓ is an ℓ-dimensional compact totally geodesic manifold immersed in $\overline{M}$, with $2\ell \geq n$, then the homomorphism of the fundamental groups $\Pi_1(M) \to \Pi_1(\overline{M})$ is surjective.

In 1981, Frankel's first theorem was generalized to k-saddle submanifolds [2] of a positively curved manifold. Recently K. Kenmotsu and C. Xia [8,9] extended Frankel's theorem to the case where the ambient Riemannian manifold has partially positive curvature. In this paper we generalize these last results to saddle submanifolds.

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We first recall the definition of manifolds with partially positive curvature [8,9]. Let M^n be an n-dimensional Riemannian manifold. We say that M has *positive* (resp. *nonnegative*) *$k-$Ricci curvature at a point p of M* if, for any $k+1$ orthonormal tangent vectors $e, e_1, ..., e_k$ at p we have $\sum_{i=1}^{k} K(e, e_i) > 0 (resp \geq 0)$, where $K(e, e_i)$ denotes the sectional curvature of the plane spanned by e and e_i. If M has positive (resp. nonnegative) $k-$ Ricci curvature at every point we say that M has *positive* (resp. *nonnegative*) *$k-$Ricci curvature.*

Thus, 1-Ricci curvature is equivalent to sectional curvature and $(n-1)$-Ricci curvature is equivalent to Ricci curvature. For later use, we introduce the notation

$$K(e, \Pi) = \sum_{i=1}^{k} K(e, e_i),$$

where Π is a k-dimensional subspace of the tangent space T_pM orthogonal to the vector e and $\{e_i\}$ is an orthonormal basis of Π.

The class of k-saddle submanifolds in higher dimensions was introduced by S. Sefel, who studied the structure of 1-saddle submanifolds in Euclidean space [10]. A submanifold M^ℓ of a Riemannian space $\overline{M}^n$ is a *k-saddle submanifold* if, at an arbitrary point $q \in M$, the second quadratic form with respect to each normal vector ξ satisfies $\max\{(r_\xi + s_\xi)/2, (r_\xi - s_\xi)/2\} \leq k$, where r_ξ and s_ξ denote the rank and signature of the second quadratic form respectively, i.e., after being reduced to a diagonal form, the second quadratic form has no more than k coefficients of the same sign.

We observe that totally geodesic submanifolds are 0-saddle submanifolds and ℓ-dimensional minimal submanifolds are $(\ell-1)$-saddle submanifolds. A Kähler submanifold M^ℓ of a Kähler manifold is an $\ell/2$-saddle submanifold.

Submanifolds M^ℓ of the Euclidean space $E^{\ell+p}$ whose q-dimensional sectional curvature is nonpositive are $p(q-1)$-saddle submanifolds [2]. In particular, submanifolds M^ℓ of nonpositive sectional curvature in $E^{\ell+p}$ are p-saddle submanifolds.

The definition of k-saddle submanifold is equivalent to the existence of asymptotic subspaces in the tangent spaces. A submanifold M^ℓ of $\overline{M}^n$ is said to be *k-asymptotic* if for each normal vector ξ at an arbitrary point $q \in M$, there exists a k-dimensional subspace L^k_ξ such that the second fundamental form with respect to ξ, restricted to L, vanishes. It is easy to see that M^ℓ is a k-saddle submanifold if and only if it is (at least) an $(\ell - k)$-asymptotic submanifold. If the plane L^k_ξ is independent of the normal direction ξ we say that M is *k-strongly asymptotic* [3]. The notion of asymptotic submanifold is essential in continous and infinitesimal bendings [11,12].

Our first result provides a characterization of k-saddle submanifolds of Euclidean space in terms of its sectional curvature. In order to state the result, we consider the class $\mathcal{C}_k$ of Riemannian manifolds M^ℓ, $1 \leq k \leq \ell - 1$, which satisfy the following property: for each $q \in M$, and for any $(k+1)$-dimensional subspace L^{k+1} of the tangent space T_qM, there exists a 2-dimensional plane $\sigma \subset L^{k+1}$ such that the sectional curvature $K(\sigma) \leq 0$. Thus, $\mathcal{C}_1$ is the class of manifolds with nonpositive curvature and for each k, we have $\mathcal{C}_k \subset \mathcal{C}_{k+1}, \forall k$. In section 1 we prove the following result:

Theorem 1: *A submanifold M^ℓ of the Euclidean space E^n is a k-saddle submanifold if and only if it satisfies the following conditions:*

a) *M with the induced metric belongs to the class $\mathcal{C}_k$.*

b) *Property a) is invariant under any nondegenerate affine transformation A of E^n, i.e. $A(M)$ with the induced metric belongs to $\mathcal{C}_k$.*

Theorem 1 for $k = 1$ was proved by S. Sefel [10]. The following results will be proved in section 2.

Theorem 2: *Let $\overline{M}^n$ be a complete Riemannian manifold of dimension $n > 2$. Let M^ℓ be a compact immersed s-saddle submanifold of dimension $\ell \geq 1$. Assume that along each geodesic ray $\gamma : [0, \infty) \to \overline{M}$ issuing orthogonally from M,*

$$\int_0^\infty K(\gamma'(t), \Pi_t)\ \mathrm{dt} > 0, \tag{1}$$

where Π_0 is any k-dimensional subspace of $T_{\gamma(0)}M$, Π_t is the parallel translation of Π_0 along γ and $\ell \geq s + k$. Then $\overline{M}$ is necessarily compact.

For minimal submanifolds, this theorem was proved by G. Galloway [6], with the weaker condition $\ell \geq k$. As an immediate consequence of Theorem 2 we have

Corollary 1: *Let $\overline{M}$ be a complete noncompact Riemannian manifold. Suppose along each ray $\gamma : [0, \infty) \to \overline{M}$ condition (1) holds for every family of k-planes Π_t, orthogonal to γ and parallel along γ. Then $\overline{M}$ contains no compact immersed s-saddle submanifolds of dimension greater or equal to $s + k$.*

Let M^ℓ be a submanifold of a Riemannian manifold $\overline{M}^{\ell+r}$. The extrinsic sectional curvature of M at a plane $\sigma \subset T_pM$ is the difference $K(\sigma) - \overline{K}(\sigma)$, where $\overline{K}$ is the sectional curvature of $\overline{M}$. In [1], it was shown that submanifolds M^ℓ of $\overline{M}^{\ell+s}$ with nonpositive extrinsic curvature are s-saddle submanifolds. Hence, as an immediate consequence we obtain the following:

Corollary 2: *Under the same conditions of Corollary 1, there are no compact immersed submanifolds M^ℓ of $\overline{M}^{\ell+s}$, with nonpositive extrinsic curvature, such that the dimension $\ell \geq s + k$.*

Theorem 3: *Let $\overline{M}^n$ be an n-dimensional complete connected Riemannian manifold with nonnegative k-Ricci curvature and let $M_1^{\ell_1}$ and $M_2^{\ell_2}$ be two complete immersed s_1-saddle and s_2-saddle submanifolds of dimensions ℓ_1 and ℓ_2 respectively. Assume M_1 is closed and M_2 is compact. If $\overline{M}$ has positive k-Ricci curvature either at all points of M_1 or at all points of M_2 and $\ell_1 + \ell_2 \geq n + k - 1 + s_1 + s_2$, then M_1 and M_2 must intersect.*

Theorem 4: *Let $\overline{M}^n$ be an n-dimensional complete connected Riemannian manifold with positive k-Ricci curvature. Suppose M^ℓ is a compact ℓ-dimensional s-saddle submanifold such that $2\ell \geq n + k - 1 + 2s$. Then the homomorphism of the fundamental groups $\Pi_1(M) \to \Pi_1(\overline{M})$ is surjective.*

1. A characterization of k-saddle submanifolds of Euclidean space

In this section we prove Theorem 1. We will need the following results.

Proposition 1: *The class of k-saddle submanifolds of E^n is invariant under* *nondegenerate affine transformations of* E^n*.*

Proof: Let M^ℓ be a k-saddle submanifold of E^n and let A be a nondegenerate affine transformation of E^n. Suppose M is locally described by a parametrization $X : U \subset E^\ell \to E^n$, $X(u_1, ..., u_\ell)$. Then $\tilde{X} = AX$ is a local parametrization of $\tilde{M} = A(M)$. Hence,

$$\tilde{X}_i = AX_i, \quad \tilde{X}_{ij} = AX_{ij}, \quad 1 \leq i, j \leq l,$$

where the lower indices indicate partial derivatives with respect to u_i, u_j.

We consider an arbitrary point $\tilde{q} \in \tilde{X}$ and $\tilde{\xi}$ a unit vector normal to $\tilde{X}$ at $\tilde{q}$. Then $\xi = A^t\tilde{\xi}$ is normal to ξ at $A^{-1}(\tilde{q})$. In fact,

$$< X_i, \xi >=< X_i, A^t\tilde{\xi} >=< AX_i, \tilde{\xi} >=< \tilde{X}_i, \tilde{\xi} >= 0.$$

The second fundamental form of $\tilde{X}$ with respect to $\tilde{\xi}$ at $\tilde{q}$ is given by

$$< \tilde{X}_{ij}, \tilde{\xi} >=< AX_{ij}, (A^{-1})^t\xi >=< X_{ij}, \xi >= |\xi| < X_{ij}, \frac{\xi}{|\xi|} > . \qquad (2)$$

Therefore, the second fundamental quadratic forms at corresponding points with respect to corresponding normals are multiple of each other. Hence, if X is a k-saddle submanifold then $\tilde{X}$ is also a k-saddle submanifold. □

Lemma 1: *Let* M^ℓ *be a submanifold of the Euclidean space* E^n*. Assume that the sectional curvature at a point* $q \in M$ *is positive for all 2-dimensional planes* σ *of a subspace* $L \subset T_qM$*. Then there exists a unit vector* ξ *normal to* M *at* q *such that the second fundamental form of* M *with respect to* ξ *restricted to* L *is positive definite.*

Proof: Let B be the second fundamental form of M at q. We define a vector $\tilde{H}$ normal to M at q given by the trace of B restricted to L. Since the sectional curvature of M restricted to L is positive $\tilde{H}$ is a nonzero vector.

We consider an orthonormal basis $\xi_1, ..., \xi_{n-\ell}$ normal to M at q such that ξ_1 is in the direction of $\tilde{H}$. Let $e_1, ..., e_k$, $k \leq \ell$ be an orthonormal basis of L which diagonalizes the second fundamental form in the direction of ξ_1 when restricted to L. We use the notation

$$b_{ij}^{\alpha} =< B(e_i, e_j), \xi_\alpha >, \qquad 1 \leq i,j \leq k, \quad 1 \leq \alpha \leq n-\ell.$$

It follows from the Gauss equation that the sectional curvature of the plane generated by e_i, e_j is given by

$$K(e_i, e_j) = b_{ii}^1 b_{jj}^1 + \sum_{\alpha=2}^{n-\ell} \left[b_{ii}^{\alpha} b_{jj}^{\alpha} - (b_{ij}^{\alpha})^2 \right].$$

For each i fixed, we sum over j, $j \neq i$, $1 \leq j \leq k$. Since ξ_1 is in the direction of $\tilde{H}$ we have $\tilde{H} = \sum_{j=1}^{k} b_{jj}^1 \xi_1$ and hence $\sum_{j=1}^{k} b_{ii}^{\alpha} = 0$ for $\alpha \geq 2$. Therefore we get

$$\sum_{j \neq i} K(e_i, e_j) = b_{ii}^1 (h - b_{ii}^1) - \sum_{\alpha=2}^{n-\ell} (b_{ii}^{\alpha})^2 - \sum_{\alpha=2}^{n-\ell} (b_{ij}^{\alpha})^2,$$

where we have used the notation $h = \sum_{j=1}^{k} b_{jj}^1$. Since the left hand side of the above equation is positive we conclude that

$$b_{ii}^1 (h - b_{ii}^1) > 0 \qquad \forall i, \quad 1 \leq i \leq \ell.$$

Without loss of generality we assume $h > 0$. Therefore, $b_{ii}^1 > 0$ and the second fundamental form of M at q with respect to ξ_1 restricted to L is positive definite.

□

The lemma above will be used in the proof of Theorem 1, however the lemma also holds with a weaker hypothesis on the curvature. In fact it is enough to assume that $\sum_{i=1}^{k} K(e, e_i)$, for all $k+1$ orthonormal vectors $e, e_i \in L$, is positive.

Proof of Theorem 1: Let M^ℓ be a k-saddle submanifold of E^n. Suppose that M endowed with the induced metric does not belong to the class $\mathcal{C}_k$. Then there exist a point $q \in M$ and a $(k+1)$-dimensional subspace L^{k+1} of T_qM, such that $K(\sigma) > 0$, for all 2-dimensional plane σ contained in L. It follows from Lemma 1 that there exists a normal vector ξ at q such that the restriction of the second fundamental form to L is positive definite and therefore it has $k+1$ positive eigenvalues. This contradiction concludes the proof of a). Property b) follows immediately from a) and Proposition 1.

Conversely, suppose that M^ℓ is a submanifold of E^n which satisfies a) and b). Suppose that M is not a k-saddle submanifold of E^n, then there exist a point $q \in M$ and a unit normal vector ξ at q such that the second fundamental form of M with respect to ξ has at least $k+1$ positive eigenvalues $\lambda_1, ..., \lambda_{k+1}$. Let L^{k+1} be the corresponding eigenspace. We consider q to be the origin of E^n and M to be locally given as a graph over the tangent space T_qM, i.e.,

$$X(x_1, ..., x_\ell, z^1, ..., z^{n-\ell}),$$

where z^α, $1 \le \alpha \le n-\ell$ are functions of $x_1, ..., x_\ell$, such that the coordinate axis $x_1, ..., x_{k+1}$ correspond to the subspace L^{k+1} and the vector ξ is the direction of the axes z^1. Then

$$\begin{aligned} z^\alpha(0) &= 0, \qquad \frac{\partial z^\alpha}{\partial x_i}(0) = 0, \\ b^\alpha_{ij}(0) &= \frac{\partial^2 z^\alpha}{\partial x_i \partial x_j}(0) \quad \text{for } 1 \le i,j \le \ell, \quad 1 \le \alpha \le n-\ell. \end{aligned} \tag{3}$$

Let A be the affine transformation of E^n given by

$$A = \begin{pmatrix} I_\ell & & 0 \\ & h & \\ 0 & & I_{n-\ell-1} \end{pmatrix} \qquad h > 0,$$

where I_r is the $r \times r$ identity matrix. We consider the submanifold $\tilde{X} = AX$ and the normal vector $\tilde{\xi} = A^{-1}\xi$ at $\tilde{q} = Aq = 0$. It follows from (3) and the definition of A that at $\tilde{q}$

$$\begin{aligned} \tilde{b}^1_{ij} &= h b^1_{ij}, \\ \tilde{b}^\alpha_{ij} &= b^\alpha_{ij}, \quad 2 \le \alpha \le n-\ell. \end{aligned}$$

Moreover, $\tilde{L}^{k+1} = AL^{k+1} = L^{k+1}$. We consider the sectional curvature $\tilde{K}$ of $\tilde{X}$ at $\tilde{q}$ along $\tilde{L}$. For each orthonormal vectors v and w of $\tilde{L}$ we have

$$\begin{aligned} \tilde{K}(v,w) &= h^2 \left[< B(v,v), \xi >< B(w,w), \xi > - < B(v,w), \xi >^2 \right] \\ &+ \sum_{\alpha=2}^{n-\ell} \left[< B(v,v), \xi^\alpha >< B(w,w), \xi^\alpha > - < B(v,w), \xi^\alpha >^2 \right]. \end{aligned}$$

If we denote $v = \sum_i v_i e_i$ and $w = \sum_i w_i e_i$ we have

$$< B(v,v),\xi >< B(w,w),\xi > - < B(v,w),\xi >^2 = \sum_{i<j}(v_i w_j - v_j w_i)^2 \lambda_i \lambda_j > c^2 > 0$$

It follows that for h large enough, the sectional curvature $\tilde{K}$ at $\tilde{q}$ is positive for any plane in $\tilde{L}$. This contradicts condition b) and concludes the proof. □

2. Intersection of saddle submanifolds

In this section we prove Theorems 2, 3 and 4 by modifying conveniently the arguments used in [2,5,8]. Let M be a submanifold of $\overline{M}$, $p \in M$ and ξ a normal vector at p. A subspace L of T_pM is called *asymptotic* with respect to the normal ξ, if the second fundamental form at p in the direction ξ vanishes on this subspace. We will need the following lemmas:

Lemma 2. *Let M^ℓ be an ℓ-dimensional immersed submanifold of $\overline{M}^n$. Let $\gamma : [0,\infty) \to \overline{M}$ be a geodesic ray parametrized by arclength, orthogonal to M at $p = \gamma(0)$. Let Π_0 be a k-dimensional subspace of T_pM which is asymptotic with respect to $\gamma'(0)$ and Π_t its parallel translation along γ. Suppose the initial value problem*

$$\begin{aligned} x'' + \frac{1}{k}K(\gamma'(t),\Pi_t)x = 0 \\ x(0) = 1, \quad x'(0) = 0 \end{aligned} \qquad (4)$$

has a solution defined on $[0,\infty)$ which vanishes at some point of $(0,\infty)$, then there exists a focal point to M along γ.

Proof: The proof uses standard techniques of Morse Index Theory. By hypothesis there exists a solution $\phi : [0,\infty) \to \mathbf{R}$ of (4) such that $\phi(t_0) = 0$ for some $t_0 > 0$.

Let $\mathcal{V}$ be the set of smooth vector fields $V(t)$ along γ, $t \in [0,t_0]$, orthogonal to γ which satisfy $V(0) \in T_pM$, $V(t_0) = 0$ and $V'(0) = 0$. We consider the index form

$$I(V,W) = -\int_0^{t_0} < V'' + R(X,\gamma')\gamma', W > dt + < S_{\gamma'(0)}V(0), W(0) > .$$

for $V, W \in \mathcal{V}$, where $S_{\gamma'(0)}$ is the Weingarten map of the immersion at p.

Let $e_1, \ldots, e_k$ be an orthonormal basis of Π_0. By parallel translation we extend this basis to orthonormal vector fields along $\gamma|_{[0,t_0]}$. For each i, $1 \leq i \leq k$, we define

$$V_i(t) = \phi(t) e_i(t).$$

Since ϕ satisfies (4), it follows that $V_i \in \mathcal{V}$ and

$$I(V_i, V_i) = -\int_0^{t_0} (\phi'' + K(e_i, \gamma')\phi)\phi dt + < S_{\gamma'(0)} e_i(0), e_i(0) > .$$

The vector $e_i(0)$ is asymptotic with respect to $\gamma'(0)$, therefore the last term in the above equation vanishes. Hence

$$\sum_{i=1}^{k} I(V_i, V_i) = -k \int_0^{t_0} \left(\phi'' + \frac{1}{k} K(\gamma'(t), \Pi_t)\right) \phi dt = 0,$$

since ϕ is a solution of (4).

It follows that $I(V_i, V_i) \leq 0$ for some i. However, by standard index form results we must have $I(V_i, V_i) > 0$ unless there is some point on $\gamma|_{[0,t_0]}$ which is a focal point to M^ℓ along $\gamma|_{[0,t_0]}$. $\square$

Lemma 3. [13] *If* $\int_0^\infty f(t)dt > 0$ *then the initial value problem*

$$x'' + f(t)x = 0$$
$$x(0) = 1, \quad x'(0) = 0$$

has a solution defined on $[0, \infty)$ *which vanishes at some point of* $(0, \infty)$.

Before proving Theorem 2, we recall that, as a consequence of the definition of an s-saddle submanifold M^ℓ of $\overline{M}^n$, at each point of M, there exists an asymptotic plane with respect to each normal whose dimension is at least $(\ell - s)$.

Proof of theorem 2: Suppose $\overline{M}$ is not compact. Then there exists a sequence of points $\{q_i\}$ such that $d(q_i, M) \to \infty$, where d is the metric distance function. Let $\gamma_i : [0, t_i] \to \overline{M}$ be a geodesic joining $p_i = \gamma_i(0) \in M$ to q_i, whose length realizes the distance from q_i to M^ℓ. Then $\xi_i = \gamma_i'(0)$ is necessarily orthogonal do M^ℓ.

Since the unit normal bundle of M is compact there is a subsequence $\{(p_j, \xi_j)\}$ which converges to a point (p, ξ) in the unit normal bundle. Let $\gamma : [0, \infty) \to \overline{M}$ be the geodesic emanating from p with initial velocity ξ.

For each t the length of the segment $\gamma|_{[0,t]}$ realizes the distance between $\gamma(t)$ and M^ℓ. In particular γ is free of focal points. M is an s-saddle submanifold, therefore there exists an asymptotic plane Π_0 with respect to the normal $\gamma'(0)$ at the point $p = \gamma(0)$, whose dimension is greater or equal to $\ell - s \geq k$. It follows from Lemmas 2 and 3 that there is a focal point along γ. This contradiction concludes the proof. □

Proof of theorem 3: We consider $\overline{M}^n$ a complete Riemannian manifold with nonnegative $k-$ Ricci curvature and let $M_1^{\ell_1}$ and $M_2^{\ell_2}$ be two complete immersed s_1-saddle and s_2-saddle submanifolds. Assume M_1 is closed, M_2 is compact and $\overline{M}^n$ has positive $k-$ Ricci curvature either at all points of M_1 or at all points of M_2. Also we assume $\ell_1 + \ell_2 \geq n + k - 1 + s_1 + s_2$ and we want to prove that M_1 and M_2 must intersect. Suppose the contrary and let $\gamma : [0, a] \to \overline{M}$ be a normal geodesic joining $p \in M_1$ to $q \in M_2$, which realizes the minimum distance between these submanifolds.

Now consider an orthonormal basis $e_1(0), \ldots, e_{\ell_1 - s_1}(0)$ of a plane $\Pi_0 \subset T_pM$, which is asymptotic with respect to the normal $\gamma'(0)$ and take their parallel translations $e_i(t)$ along γ.

The condition $(\ell_1 - s_1) + (\ell_2 - s_2) \geq n + k - 1$ implies that at least k of the vectors $e_1(a), \ldots, e_{\ell_1 - s_1}(a)$ are asymptotic with respect to the normal $\gamma'(a)$ at the point q. We assume that $e_1(a), \ldots, e_k(a)$ are these vectors. Each vector field $e_i(t)$, $1 \leq i \leq k$ gives rise to a variation of γ whose variational curves have their endpoints on M_1 and M_2 and have length $L_{e_i}(u)$. Since γ minimizes the distance between M_1 and M_2 we have $L_{e_i}'(0) = 0$. By the second variation formula

$$\begin{aligned} L''_{e_i}(0) &= < S_{\gamma'(a)}(e_i(a)), e_i(a) > - < S_{\gamma'(0)}(e_i(0)), e_i(0) > \\ &\quad - \textstyle\int_0^a K(\gamma'(t), e_i(t))dt, \quad i \leq i \leq k \end{aligned}$$

where $S_{\gamma'(0)}$ and $S_{\gamma'(a)}$ are the Weingarten maps of M_1 and M_2 with respect to the normal $\gamma'(0)$ and $\gamma'(a)$, respectively. Since $e_i(a), e_i(0), \ 1 \leq i \leq k$, are asymptotic vectors it follows that

$$L''_{e_i}(0) = -\int_0^a K(\gamma'(t), e_i(t))dt. \tag{5}$$

Moreover, the k-Ricci curvature of $\overline{M}$ is nonnegative, hence

$$\sum_{i=1}^{k} K(\gamma'(t), e_i(t)) \geq 0.$$

Since $\overline{M}$ has positive k-Ricci curvature either at all points of M_1 or at all points of M_2, we know that

$$\sum_{i=1}^{k} K(\gamma'(0), e_i(0)) > 0$$

or

$$\sum_{i=1}^{k} K(\gamma'(a), e_i(a)) > 0.$$

Substituting these formulas into the sum of (5) over i we get $\sum_{i=1}^{k} L''_{e_i}(0) < 0$. This implies that $L''_{e_i}(0) < 0$ for some i, which contradicts the assumption on the minimality of γ. Hence M_1 and M_2 must intersect. $\square$

Proof of theorem 4: Let $\overline{M}^n$ be a complete, connected n-dimensional Riemannian manifold with positive k-Ricci curvature. Suppose M^ℓ is a compact ℓ-dimensional s-saddle submanifold of $\overline{M}$, such that $2\ell \geq n+k-1+2s$. We want to prove that the homomorphism of the fundamental groups $\pi_1(M) \to \pi_1(\overline{M})$ is surjective.

First we note that $k \leq n-1$, hence the hyphothesis on the dimensions implies that $\ell \geq s+k$. It then follows from Theorem 2 that $\overline{M}$ is compact and using the condition on the k-Ricci curvature of $\overline{M}$, we get that the Ricci curvature of

$\overline{M}$ is positive. Therefore Bonnet-Myers Theorem implies that $\pi_1(\overline{M})$ is finite. It also follows from Theorem 3 that M^ℓ is connected.

Now we take the universal covering $\tilde{M}^k$ of $\overline{M}$ with the covering metric. Then the k-Ricci curvature of $\tilde{M}$ is also positive. We denote by $N^\ell = \pi^{-1}(M^\ell)$ the inverse image of M^ℓ under the projection map $\pi : \tilde{M}^n \to \overline{M}^n$.

We naturally have that N^ℓ is a compact s-saddle submanifold. By Theorem 2 all components of N^ℓ must intersect each other and hence N^ℓ is indeed connected. Thus N^ℓ is a covering space of M^ℓ. Then we conclude the proof by using the same arguments as in the proof of the main theorem of [5]. Namely, let α be a loop in $\overline{M}$ with base point $p \in M$. Lift α to a curve $\tilde{\alpha}_1$ in $\tilde{M}$ joining two points $\tilde{p}_1$ and $\tilde{p}_2$ whose projection by π is the point p. Since $\pi^{-1}(M)$ is connected there is a curve $\tilde{\alpha}_2$ in $\pi^{-1}(M)$ joining $\tilde{p}_1$ to $\tilde{p}_2$, which is homotopic to $\tilde{\alpha}_1$ in $\tilde{M}$ since $\tilde{M}$ is simply connected. Then $\pi(\tilde{\alpha}_2)$ is a loop in M^ℓ which is homotopic in $\overline{M}^n$ to $\alpha = \pi(\tilde{\alpha}_1)$ $\square$

It is well known that every compact minimal submanifold M in a complete manifold with nonpositive sectional curvature has an infinite fundamental group. An analogous result holds for k-saddle submanifolds. It is easy to prove that there is no compact k-saddle submanifold M^ℓ of euclidean space with $k < \ell$. The same result holds when the ambient space is replaced by any simply connected Riemannian manifold with nonpositive curvature. The simplest proof of this follows from the second variation formula. An immediate consequence is the following: If $\overline{M}^n$ is complete with nonpositive sectional curvature then every compact immersed k-saddle submanifold M^ℓ, $k < \ell$, has an infinite fundamental group. In fact, this is true, otherwise the complete inverse image of M^ℓ in the universal covering of $\overline{M}$ would have compact components.

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