Submanifolds associated to solutions of the Intrinsic Generalized Equation

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Abstract

The Intrinsic Generalized Equation is an n-dimensional system of nonlinear differential equations whose solutions correspond to Riemannian submanifolds $M^n(K)$ of constant sectional curvature K contained in pseudo-Riemannian manifolds $\overline{M}_s^{2n-1}(\overline{K})$ of index s, with $K \neq \overline{K}$, flat normal bundle and such that the normal principal curvatures are different from $K - \overline{K}$. The geometric properties of the submanifolds M associated to the solutions of the Intrinsic Generalized Equation, which are invariant under $(n-1)$-dimensional group of translations, are considered. It is shown that such submanifolds are foliated by $(n-1)$-dimensional flat submanifolds which have constant mean curvature in M. Moreover, each leaf of the foliation is itself foliated by curves of $\overline{M}$ which have constant curvatures.

1. Introduction

In this talk, we will consider a system of differential equations which characterizes submanifolds of constant sectional curvature K. We will see that although the point-Lie symmetry group of this system is small, the associated submanifolds have quite interesting geometric properties.

The system of differential equations mentioned above was obtained in the papers [2,4]. Its Lie point symmetry group was shown to consist of translations, when $K \neq 0$, and translations and dilations when $K = 0$ (see [5,7]).

In what follows, we will consider the solutions of the system which are invariant under an $(n-1)$-dimensional group of translations and provide the geometric properties of the submanifolds associated to these solutions. Similar results for the submanifolds associated to solutions invariant by dilations are not known.

2. System of differential equations associated to submanifolds of constant curvature

We consider Riemannian submanifolds $M^n(K)$ of constant sectional curvature K contained in pseudo-Riemannian manifolds $\overline{M}_s^{2n-1}(\overline{K})$ of index s, with $K \neq \overline{K}$. Such submanifolds correspond to solutions of an n-dimensional system of nonlinear differential equations, denominated in [1] by Intrinsic Generalized Equation, given in the following result.

Theorem 1. *A Riemannian manifold $M^n(K)$ isometrically immersed in a pseudo-Riemannian manifold $\overline{M}_s^{2n-1}(\overline{K})$ with $K \neq \overline{K}$, flat normal bundle and such that the principal normal curvatures are different from $K - \overline{K}$, is uniquely determined, up to rigid motions of $\overline{M}$, by a diagonal metric $g_{ij} = \delta_{ij} v_i^2$ where the functions v_i satisfy the following system of equations*

$$
\begin{aligned}
vJv^t &= 1 \\
\frac{\partial v_i}{\partial x_j} &= v_j h_{ji} \\
\frac{\partial h_{ij}}{\partial x_i} &+ \frac{\partial h_{ji}}{\partial x_j} + \sum_{s \neq i,j} h_{si} h_{sj} = -K v_i v_j, \quad i \neq j \qquad (2.1) \\
\frac{\partial h_{ij}}{\partial x_s} &= h_{is} h_{sj}, \quad i, j, s \quad \text{distinct}
\end{aligned}
$$

for $1 \leq i, j, s \leq n$ where

$$J = \text{diag}\,(\overbrace{1,\ldots,1}^{n-q\ times},\overbrace{-1,\ldots,-1}^{q\ times}), \quad 0 \leq q \leq n-1$$

with $q = s$ if $K < \overline{K}$ and $q = n - (s+1)$ if $K > \overline{K}$, and h is an off-diagonal $(n \times n)$-matrix function determined from v_i by the second equation of (2.1).

The system of equations (2.1), for $n = 2$ and $K \neq 0$ reduces to the sine-Gordon equation when $q = 0$ and to the elliptic sinh-Gordon equation when $q = 1$. Similarly, in the 2-dimensional case, when $K = 0$, (2.1) reduces to the homogeneous wave equation when $q = 0$ and to the Laplace equation when $q = 1$. Theorem 1 is a consequence of Theorems 2.1, 2.3 and Corollary 2.4 of [1].
The simplest example in the class of submanifolds considered in the above Theorem is the n-dimensional Clifford Torus contained in the unit sphere S^{2n-1}. In this case we have $q = 0$, $K = 0$, $\overline{K} = 1$, and the matrix function a is a constant orthogonal matrix. When M and $\overline{M}$ are Riemannian manifolds with $K \neq \overline{K}$, other examples are given by the so called toroidal submanifolds. These are generated by curves in such a way that each point of the curve describes an $(n-1)$-dimensional torus [6].
The symmetry group of (2.1) for the case $s = 0$ was first obtained by Tenenblat-Winternitz [7] and in the general case by Ferreira [5].

Theorem 2. *The symmetry group of local Lie-point transformations of the Intrinsic Generalized Equation for $n \geq 3$ is given by*

$$\begin{aligned} x_i' &= e^b x_i + a_i \\ v_i' &= v_i \\ h_{ij}' &= e^{-b} h_{ij} \end{aligned}$$

where $a_i, b \in \mathbf{R}$ and $b = 0$ when $K \neq 0$.

Several explicit solutions invariant under the symmetry group in terms of elliptic functions can be found in [3,5,7]. We will include a few examples.

Examples.

a) We consider the solutions of (2.1) , for $n = 3$, $q = 2$ and $K = -1, 0$ or 1 which are of the form $v(\xi)$, where $\xi = \sum_{i=1}^{3} \alpha_i x_i$, $\alpha_i \in \mathbf{R}\backslash\{0\}$ such that none of the coordinate functions v_j are constant. Then the solutions are given by

$$\begin{aligned}(v_{1,\xi})^2 &= b(a-b)(v_1^2 - \frac{c}{b})(v_1^2 - \frac{a-c}{a-b})\\ v_2^2 &= \frac{b}{a}(v_1^2 - \frac{c}{b})\\ v_3^2 &= \frac{a-b}{a}(v_1^2 - \frac{a-c}{a-b})\end{aligned}$$

where a, b are distinct, nonzero real numbers, $c \in \mathbf{R}$ and satisfy the relation

$$(a-b)(b\alpha_1^2 + a\alpha_2^2) + ab\alpha_3^2 + K = 0.$$

b) This example provides solutions of (2.1) which are invariant under an $(n-1)$-dimensional group of dilations (see [7] for details). We consider the basic invariant

$$\xi = \frac{\sum_{i=1}^{m_1} a_i x_i}{\sum_{i=1}^{m_2} b_i x_i}, \quad 1 \leq m_1 \leq m_2 \leq n$$

where a_i and b_i are real constants satisfying

$$\sum_{i=1}^{m_1} a_i^2 \neq 0, \quad \prod_{i=1}^{m_2} b_i \neq 0.$$

The invariant solutions will have the form

$$v_i = v_i(\xi), \quad 1 \leq i, j \leq n$$

Moreover, we are interested in solutions such that on any open subset of R^n, we have

$$v_i(\xi) \neq 0, \quad \forall\, i.$$

For $m_2 = 3$ and $q = 0$, the system of equations (2.1), with $K = 0$, reduces to the following system of ODEs

$$\begin{aligned}\dot{v}_1 &= c_1 \frac{v_2 v_3}{F_2 F_3}, \qquad \dot{v}_2 = c_2 \frac{v_3 v_1}{F_3 F_1}, \qquad \dot{v}_3 = c_3 \frac{v_1 v_2}{F_1 F_2}\\ v_j &= c_j, \quad j \geq 4,\end{aligned}$$

where

$$F_i = a_i - b_i\xi \qquad i = 1, 2, 3$$

$$c_2 = \epsilon_2 c_1, \;\; c_3 = \epsilon_3 c_1, \;\; \epsilon_1 = \pm 1, \;\; \epsilon_2 = \pm 1$$

$$a_1 + \epsilon_2 a_2 + \epsilon_3 a_3 = 0, \;\; b_1 + \epsilon_2 b_2 + \epsilon_3 b_3 = 0$$

and a_i, b_i and c_μ are constants.
The initial conditions $v_i(0)$ must be such that

$$\sum_{i=1}^{3} v_i^2(0) + \sum_{j=4}^{n} c_j^2 = 1.$$

3. Geometric properties of submanifolds

In what follows we will describe some geometric properties of the submanifolds associated to the solutions of the system (2.1) which are invariant under an $(n-1)$-dimensional group of translations. Such solutions are of the form

$$v(\xi) = (v_1(\xi), \ldots, v_n(\xi)) \tag{3.1}$$

where

$$\xi = \sum_{i=1}^{n} \alpha_i x_i \;, \quad \alpha_i \in \mathbf{R} \quad , \text{ and } \sum_{i=1}^{n} \alpha_i^2 \neq 0. \tag{3.2}$$

We will consider solutions defined in an open simply connected domain $\Omega \subset \mathbf{R}^n$. Since the set of $\xi \in \mathbf{R}$ such that $v_i(\xi) \neq 0 \;\; \forall i, \;\; 1 \leq i \leq n$ is an open subset of $\mathbf{R}$ we will assume in what follows that Ω is of the form

$$\Omega = \{ \, x \in \mathbf{R}^n \; : \xi_1 < \sum_{i=1}^{n} \alpha_i x_i < \xi_2 \, \}$$

where $\xi_1, \, \xi_2 \in [-\infty, \infty]$ and $\xi_1 < \xi_2$.
The following result provides the geometric properties of the Riemannian manifold defined by the diagonal metric v.

Theorem 3. The Riemannian manifold (Ω, g), where the metric g is defined by $g_{ij} = \delta_{ij} v_i^2(\xi)$, satisfies the following properties:

a) The sectional curvature of Ω is constant equal to K.

b) The hiperplanes of Ω defined by $\sum_{i=1}^{n} \alpha_i x_i = \xi_0$, $\xi_1 < \xi_0 < \xi_2$, with the induced metric are flat, have constant mean curvature (depending on ξ_0) and are geodesically parallel. Moreover, if $K < 0$ the hyperplanes cannot be minimally immersed in Ω.

c) For any set of vectors $U_1, \ldots, U_{n-1}$ generating the hyperplane $\sum_{i=1}^{n} \alpha_i x_i = 0$ the $(n-1)-$parameter group of translations $T_\varepsilon : \Omega \longrightarrow \Omega$ defined by

$$T_\varepsilon(x) = x + \varepsilon_1 U_1 + \cdots + \varepsilon_{n-1} U_{n-1} ,$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1}) \in \mathbf{R}^{n-1}$, are isometries of Ω.

For each solution of (2.1) of the form (3.1) (3.2), we consider the associated manifold of constant curvature, immersed in $\overline{M}(\overline{K})$ given by Theorem 1, $X : \Omega \to \overline{M}(\overline{K})$. Our next results provide the geometric properties of X, showing that such submanifolds are foliated by $(n-1)$-dimensional flat submanifolds which have constant mean curvature in M. Moreover, each leaf of the foliation is itself foliated by curves of $\overline{M}$ which have constant curvatures.

Theorem 4. Let q be any point of Ω and U a vector of the hyperplane $\sum_{i=1}^{n} \alpha_i x_i = 0$. Then

a)$\gamma(t) = X(q+tU)$, $t \in \mathbf{R}$, is a curve of $\overline{M}^{2n-1}(\overline{K})$ of constant curvatures.

b) $\gamma(t)$ is congruent to $\tilde{\gamma}(t) = X(\tilde{q} + tU)$ if q and $\tilde{q}$ are points of the same hyperplane $\sum_{i=1}^{n} \alpha_i x_i = \xi_0$.

Theorem 5. Assume that $\alpha_j = 0$ for $j \in L = \{j_1, ..., j_m\}$, for some m, $1 \leq m < n$. Then there exist m orthogonal vectors U_j, $j \in L$ of the hyperplane $\sum_{i=1}^{n} \alpha_i x_i = 0$ such that for each $q \in \Omega$ the curve

$$\gamma_j(t) = X(q + tU_j) \quad t \in \mathbf{R} \quad j \in L$$

is contained in a two dimensional totally geodesic submanifold of $\bar{M}$. Moreover, γ_j has nonzero constant curvature.

Theorem 6. Let $N_{\xi_0}^{n-1} = X(\mathcal{P}_{\xi_0})$ where $\mathcal{P}_{\xi_0}$ is the hyperplane of Ω given by $\sum_{i=1}^{n} \alpha_i x_i = \xi_0$. Then

a) The submanifold N_{ξ_0} is flat and has constant mean curvature in $M^n(K) = X(\Omega)$. Moreover, if $K < 0$ then N_{ξ_0} is not minimal.

b) The principal normal curvatures of M, are constant along N_{ξ_0}.

We conclude by observing that if $X(x_1, ..., x_n) \subset \overline{M}$ is the constant curvature submanifold associated to a solution of (2.1) of the form $v(\xi)$ where $\xi = \sum_{i=1}^{n} \alpha_i x_i$, then one can show that X is foliated by $(n-1)$-dimensional flat submanifold of constant mean curvature, and it can be reparametrized such that the coordinate curves of each leaf are curves of constant curvature of $\overline{M}$
More details and proofs of the geometric results presented in this talk can be found in [1].

References

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