HYDRODYNAMIC SYSTEMS AND THE HIGHER-DIMENSIONAL LAPLACE TRANSFORMATION OF CARTAN SUBMANIFOLDS

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Dedicated to Jiri Patera and Pavel Winternitz
on the occasion of their 60th birthday

1. Introduction.

Our purpose in this paper is to show how the recently discovered higher-dimensional Laplace transformation [KT] for overdetermined systems of partial differential equations, can be applied to the study of the class of systems of hydrodynamic type which are rich in conservation laws [Se], also known as semi-Hamiltonian hydrodynamic systems [Tsa]. This class includes such important examples as the electrophoresis system, the equations of ideal chromatography, the Benney system and the system describing non-linear electromagnetic plane waves [Tsa],[Se]. We will demonstrate that the higher-dimensional Laplace transformation, induced by the geometric Laplace transformation of Cartan manifolds [Ch], is an effective tool for computing in closed form the infinitely many conserved densities admitted by the hydrodynamic systems belonging to this class. We will also show that there is a transformation of hydrodynamic systems which is induced by the higher-dimensional Laplace transformation. (A similar result has been announced in a preprint of Ferapontov [Fe1].) As such, this transformation is a powerful tool for generating new systems which are rich in conservation laws from known ones.

Our paper is organized as follows. In Section 2, we recall briefly the basic definitions and fundamental properties of the class of hydrodynamic systems under consideration. In Section 3, we show how the higher-dimensional Laplace transformation of Cartan manifolds gives rise to a transformation mapping hydrodynamic systems which are rich in conservation laws to other such sytems. We also characterize those hydrodynamic systems whose conserved densities are governed by an overdetermined system with

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vanishing Laplace invariants, so that their conserved densities may be determined by quadratures from those of a lower-dimensional system of the same type. In Section 4, we give some examples which illustrate how our transformation and reduction theorems can be applied to compute conserved densities and transformations of hydrodynamic systems of interest in applications.

2. Hydrodynamic Systems Rich in Conservation Laws.

Our goal in this section is to recall some fundamental results of D. Serre [Se] and Tsarev [Tsa] concerning hydrodynamic systems which are rich in conservation laws. We shall see in Section 3 that there is a well-defined action of the higher-dimensional Laplace transformation on these systems which is induced by the transformation of their conserved densities.

By a *hydrodynamic system*, we mean as usual a determined system of n first-order quasilinear partial differential equations of the form

$$u^i_{,t} = \sum_{i=1}^{n} v^i_{\ j}(u^1, \cdots, u^n)\, u^j_{,x}\ , \tag{2.1}$$

for n functions $u^i(x,t)$, $1 \leq i \leq n$. The coefficients $v^i_{\ j}, 1 \leq i, j \leq n$, thus transform like the components of a $(1,1)$-tensor under smooth and invertible changes of variables $\bar{u}^i = f^i(u^1, \cdots, u^n), 1 \leq i \leq n$.

We shall consider only those hydrodynamic systems which are *diagonalizable* and *strongly hyperbolic*, meaning that there exist coordinates $\bar{u}^1, \cdots, \bar{u}^n$, usually called Riemann invariants, in which the $(1,1)$-tensor $(v^i_{\ j})$ is diagonal, with distinct eigenvalues $\bar{v}^i(u^1, \cdots, u^n), 1 \leq i \leq n$. Since we will always work in a coordinate system composed of Riemann invariants, we will drop the overbars and write our hydrodynamic system in diagonal form as

$$u^i_{,t} = v^i(u^1, \cdots, u^n)\, u^i_{,x}, \qquad 1 \leq i \leq n, \tag{2.2}$$

A *conservation law*, or an *entropy-flux pair*, for a hydrodynamic system (2.1) consists in a pair of functions $(P(u^1, \cdots, u^n), Q(u^1, \cdots, u^n))$, such that

$$\frac{DP}{Dt} + \frac{DQ}{Dx} = 0, \tag{2.3}$$

whenever $(u^1, \cdots, u^n)$ is a solution of (2.1). If the hydrodynamic system is diagonalizable and strongly hyperbolic, then it is straightforward to show that P must satisfy the overdetermined system of $\frac{n(n-1)}{2}$ second-order equations given by

$$P_{,ij} - \frac{v^i_{,j}}{v^j - v^i} P_{,i} - \frac{v^j_{,i}}{v^i - v^j} P_{,j} = 0, \qquad 1 \leq i \neq j \leq n, \tag{2.4}$$

where $P_{,i} = \frac{\partial}{\partial u^i}$, etc... We will refer to the solutions P of (2.4) as *conserved densities* for the hydrodynamic system (2.2). If (2.2) is to have any non-trivial conserved densities, then the coefficients $v^i, 1 \leq i \leq n$, of (2.2) must satisfy the constraints

$$\left(\frac{v^i_{,j}}{v^j - v^i}\right)_{,k} = \left(\frac{v^i_{,k}}{v^k - v^i}\right)_{,j}, \tag{2.5}$$

for all $1 \leq i \neq j \neq k \leq n$. It is a non-trivial result, proved independently by D.Serre [Se] and Tsarev [Tsa] that the conditions (2.5) are also sufficient for the existence of infinitely many conserved densities, which form a complete set in the periodic and rapidly decreasing cases. We shall limit ourselves to stating the existence theorem, which appears already in the lectures of Darboux [Da] on orthogonal coordinate systems.

Theorem 2.1: *A strongly hyperbolic hydrodynamic system (2.2) whose coefficients satisfy the conditions (2.5) has infinitely many linearly independent conserved densities depending on n arbitrary functions of one variable.*

In view of Theorem 2.1, the hydrodynamic systems whose coefficients satisfy (2.5) are called *rich in conservation laws* by D. Serre [Se]. They are also called semi-Hamiltonian by Tsarev [Tsa]. It has been conjectured by Ferapontov [Fe2] that all such systems can be written in Hamiltonian form,

$$u^i_{,t} = \sum_{j=1}^{n} A^{ij} \frac{\delta H}{\delta u^j},$$

for some Hamiltonian $(0,2)$-tensor-valued operator A^{ij}, which will generally be *non-local* and some Hamiltonian density H which will always be *local.*

An important technical tool in the proof of Theorem 2.1 and in the application of the higher-dimensional Laplace transformation to systems of hydrodynamic type is provided by the following existence theorem, due also to Darboux [Da].

Theorem 2.2: *Let $\Gamma^i_{ij}(u^1, \cdots, u^n), 1 \leq i \neq j \leq n$, be a collection of $n(n-1)$ smooth functions, which are commutative in the lower indices, satisfying,*

$$\Gamma^i_{il,k} = \Gamma^i_{ik,l}, \qquad 1 \leq i \neq k \neq l \leq n \tag{2.6}$$

$$\Gamma^k_{ik,l} + \Gamma^k_{ik}\Gamma^k_{kl} - \Gamma^i_{il}\Gamma^k_{ik} - \Gamma^l_{il}\Gamma^k_{lk} = 0, \qquad 1 \le i \ne k \ne l \le n. \tag{2.7}$$

Then the linear first-order system

$$\frac{w^i{}_{,j}}{w^j - w^i} = \Gamma^i_{ij}, \qquad 1 \le i \ne j \le n, \tag{2.8}$$

(which is overdetermined for $n > 2$) admits infinitely many linearly independent solutions depending on n arbitrary functions of one variable.

Corollary 2.3: *If $w^i(u^1, \cdots, w^n)$ is a solution of (2.8), then the hydrodynamic system*

$$u^i_{,t} = w^i(u^1, \cdots, u^n)\, u^i_{,x}, \qquad 1 \le i \le n, \tag{2.9}$$

will automatically be rich in conservation laws.

In fact Tsarev [Tsa] proved that the flows which are solutions of the systems (2.9) corresponding to different solutions of (2.8) are commuting flows.

We conclude this section by listing some well-known examples of hydrodynamic systems which are rich in conservation laws:

i) The electrophoresis system [Se]:

$$\begin{aligned} u^1_{,t} &= 0, \\ u^i_{,t} &= u^1 \left(\textstyle\prod_{j=2}^n u^j\right) u^i_{,x}, \qquad 2 \le i \le n. \end{aligned} \tag{2.10}$$

ii) The ideal chromatography system [Tsa]:

$$u^i_{,t} = u^i \left(\prod_{j=1}^n u^j\right) u^i_{,x}, \qquad 1 \le i \le n. \tag{2.11}$$

3. Applications of the Higher-Dimensional Laplace Transformation to Hydrodynamic Systems which are Rich in Conservation Laws.

In this section, we will show that the higher-dimensional Laplace transformation of n-dimensional Cartan submanifolds of $\mathbf{R}^{2n}$, which was developed in [KT], induces naturally a new kind of transformation mapping hydrodynamic systems which are rich in conservation laws into other such systems. Moreover, we will show that the Laplace transformation can be used in order to obtain conserved densities for such systems.

Laplace transformations for Cartan submanifolds were first considerd by Chern [Ch] in projective space. In this section, we state the Euclidean version of this transformation so as to set up the basic notation. We refer the reader to [KT] for all the proofs and details.

Definition: *A Riemannian n-dimensional manifold M^n isometrically immersed in $\mathbf{R}^{2n}$ is said to be a Cartan submanifold if there exist local coordinates $(u^1 \cdots u^n)$ such that the net of coordinate curves is conjugate and the osculating space is $2n$-dimensional.*

It follows from this definition that given a parametrization of a Cartan manifold by conjugate curves, $X : U \subset \mathbf{R}^n \to \mathbf{R}^{2n}$, then the second fundamental forms are simultaneously diagonalized. If we denote as usual by Γ^k_{ij} the Christoffel symbols for this parametrization one shows (see Lemma in [KT]) that for each i,j with $i \neq j$ the vector field X_{ij} lies in the space spanned by X_i and X_j, i. e.

$$X_{,ij} = \Gamma^i_{ij} X_{,i} + \Gamma^j_{ij} X_{,j} \qquad i \neq j.$$

Moreover, the Christoffel symbols are rather strongly constrained since they satisfy (2.6) and (2.7).

It has been shown that to each n-dimensional Cartan submanifold X, one can associate in general a family of $n(n-1)$ submanifolds which will also be Cartan submanifolds. The construction can be summarized as follows (we refer the reader to [KT] for the details). For a fixed ordered pair $(i,j), 1 \leq i \neq j \leq n$, such that $\Gamma^i_{ij} \neq 0$, consider the edge of regression of the ruled submanifold constructed from X by taking the envelope of the $n-1$-parameter family of n-planes tangent to X along the coordinate $n-1$-dimensional coordinate hypersurface $u^j = u^j_0$. By letting u^j_0 vary, we obtain the map

$$Y(u^1, \cdots, u^n)) = X - X_{,j}/\Gamma^i_{ij}, \qquad i \neq j,$$

which will be called the (i,j)-*Laplace Transform* of X. A Laplace Transform of a Cartan submanifold is generically also a Cartan submanifold. In order to state the theorem, we need to introduce some notation. For each ordered pair $(i,j), i \neq j$ for which Γ^i_{ij} is nonzero, we define the functions:

$$M_{ij} \;=\; \frac{\Gamma^i_{ij,i}}{\Gamma^i_{ij}} - \Gamma^j_{ij}, \tag{3.2}$$

$$M_{ijl} \;=\; \Gamma^i_{ij} - \Gamma^l_{jl}, \quad \forall l,\ l \neq j.$$

The functions M_{ij} and M_{ijl} are called the *(i,j)-Laplace invariants* of the Cartan submanifold X. We observe that in order to decrease the number of indices used to denote the invariants associated to distinct pairs (i,j), the notation introduced in (3.2) is slightly different from the one used in [KT].

Theorem 3.1: *Let $X(u^1, \cdots, u^n)$ be a Cartan submanifold of $\mathbf{R}^{2n}$ parametrized by conjugate coordinates. Consider an ordered pair $(i,j), i \neq j$ such that $\Gamma^i_{ij} \neq 0$. If the (i,j)-Laplace invariants M_{ij} and M_{ijl} are non-zero, then the map*

$$Y = X - \frac{1}{\Gamma^i_{ij}} X_{,j}$$

defines a Cartan manifold, that is, Y satisfies

$$Y_{k\ell} = \tilde{\Gamma}^k_{k\ell}\, Y_{,k} + \tilde{\Gamma}^\ell_{k\ell}\, Y_{,\ell}, \qquad 1 \leq k \neq \ell \leq n,$$

where the coefficients $\tilde{\Gamma}^k_{k\ell}, 1 \leq k \neq \ell \leq n$ are given by the equations

$$\begin{aligned}
Y_{,ij} &= \left(\frac{M_{ij,j}}{M_{ij}} + \Gamma^i_{ij}\right) Y_{,i} - M_{ij} Y_{,j} \\
Y_{,ik} &= \left(M_{ij,k} + \Gamma^i_{ik}\right) Y_{,i} + \frac{M_{ij}}{M_{ijk}} \Gamma^k_{jk} Y_{,k} \qquad (3.3)\\
Y_{,kj} &= \left[\left(\frac{M_{ijk}}{\Gamma^i_{ij}}\right)_{,j} \frac{\Gamma^i_{ij}}{M_{ijk}} + \Gamma^k_{kj}\right] Y_{,k} + \frac{\Gamma^i_{ik}}{\Gamma^i_{ij}} M_{ijk} Y_{,j} \\
Y_{,k\ell} &= \left[\left(\frac{M_{ijk}}{\Gamma^i_{ij}}\right)_{,\ell} \frac{\Gamma^i_{ij}}{M_{ijk}} + \Gamma^k_{k\ell}\right] Y_{,k} + \frac{M_{ijk}}{M_{ij\ell}} \left(\Gamma^\ell_{k\ell} - \frac{\Gamma^i_{ik}\Gamma^\ell_{j\ell}}{\Gamma^i_{ij}}\right) Y_{,\ell}
\end{aligned}$$

It follows from the definition there are at most $n(n-1)$ Laplace transforms for a given Cartan manifold. It is easy to see that generically the Laplace transformation is invertible.

Proposition 3.2: *If $M_{ij} \neq 0$, the inverse of the (i,j)-Laplace transform exists and it is given by the (j,i)-Laplace transform, i. e.*

$$X = Y + Y_{,i}/M_{ij}. \qquad (3.4)$$

The vanishing of the higher-dimensional Laplace invariants has a simple geometric interpretation which is given in the following proposition:

Proposition 3.3: *The* (i,j)*-Laplace Transform of a Cartan submanifold* X *reduces to a curve if and only if*

$$M_{ij} = M_{ijk} = 0 \qquad \forall k, k \neq i, k \neq j.$$

At the analytic level, the vanishing of the higher-dimensional Laplace invariants leads to an interesting reduction theorem which will play a significant role in the applications of the higher-dimensional Laplace transformation to hydrodynamic systems.

Theorem 3.4: *Suppose that the immersion* $X : U \subset \mathbf{R}^n \to \mathbf{R}^{2n}$ *defines a Cartan submanifold parametrized by conjugate coordinates, so that we have*

$$X_{,ij} = \Gamma^i_{ij} X_{,i} + \Gamma^j_{ij} X_{,j} \qquad i \neq j.$$

If for some ordered pair (i,j)*, the higher-dimensional Laplace invariants* M_{ij} *and* $M_{ijk}, 1 \leq k \leq n, k \neq i, k \neq j$*, are identically zero, then* X *is given by*

$$X = Q + e^J G(\hat{u^j}),$$

where

$$Q = e^J \int e^{I-J} F(u^j) du^j,$$

$$I = \int \Gamma^j_{ij} du^i, \qquad J = \int \Gamma^i_{ij} du^j,$$

where F *is an arbitrary function of* x_j*,* $G(u^1, \cdots, \hat{u^j}, \cdots, u^n)$ *does not depend on* x_j *and where the antiderivative* I *is such that* $I_{,k} = -\Gamma^j_{jk}$ *for* $k \neq i, k \neq j$*. Then* G *satisfies a linear system in* $n-1$ *independent variables* $u^1, \cdots, \hat{u^j}, \cdots, u^n$ *of the form*

$$G_{,kl} + g^k_{kl} G_{,k} + g^\ell_{kl} G_{,\ell} + b_{kl} G = 0, \qquad k \neq l \ \text{ distinct from } j,$$

where

$$g^l_{lk} = \Gamma^l_{lk} - J_{,k}, \qquad g^k_{lk} = \Gamma^k_{lk} - J_{,l},$$

$$b_{lk} = J_{,k} J_{,l} - J_{,lk} - \Gamma^l_{lk} J_{,l} - \Gamma^k_{lk} J_{,k}.$$

We are now ready to apply the above results concerning Cartan submanifolds to the class of hydrodynamic systems which are rich in conservation laws and to their conserved densities. We begin with some general theoretical results which follow naturally from the transformation theorems. These theoretical results are similar to those recently announced by Ferapontov [Fe1]. Our approach is based on the Laplace transformations of Cartan submanifolds given in this section. We shall also make use of the reduction theorem (Theorem 3.4) and Theorem 3.1 in the computation of the conserved densities for concrete examples.

From Theorem 2.1 and Corollary 2.2, we see that if $X : U \subset \mathbf{R}^n \to \mathbf{R}^{2n}$ is a Cartan submanifold parametrized by conjugate coordinates, then every component of the $\mathbf{R}^{2n}$-valued function X is a conserved density for an infinite number of hydrodynamic systems wich are rich in conservation laws and whose flows are commuting. The higher-dimensional Laplace transformation will therefore induce a transformation at the level of these hydrodynamic systems. We have the following result:

Theorem 3.5: *Let*

$$u^i_{,t} = v^i(u^1, \cdots, u^n)\, u^i_{,x}, \qquad 1 \le i \le n,$$

be a strongly hyperbolic hydrodynamic system which is rich in conservation laws, meaning that

$$v^i \ne v^j, \qquad 1 \le i \ne j \le n,$$
$$\left(\frac{v^i_{,j}}{v^j - v^i}\right)_{,k} = \left(\frac{v^i_{,k}}{v^k - v^i}\right)_{,j},$$

in some open set U of $\mathbf{R}^n$ with coordinates $(u^1, \cdots, u^n)$. Suppose that for some ordered pair $(i,j), 1 \le i \ne j \le n$, the higher-dimensional Laplace invariants of the corresponding system (2.4) are non-zero in U,

$$\begin{aligned} M_{ij} &= \frac{v^i_{,ij}}{v^i_{,j}} + \frac{v^i_{,i}}{v^j - v^i} \ne 0, \\ M_{ijk} &= -\frac{v^k_{,j}}{v^j - v^k} + \frac{v^i_{,j}}{v^j - v^i} \ne 0, \end{aligned}$$

for all $1 \le k \le n, k \ne i, k \ne j$. Consider now the coefficients $\tilde{\Gamma}^k_{kl}$ of the (i,j)-transformed system, given by (3.3). Then we have:

i) The system

$$\frac{\tilde{v}^k_{,l}}{\tilde{v}^l - \tilde{v}^k} = \tilde{\Gamma}^k_{kl}(u^1, \cdots, u^n) \qquad 1 \le k \ne l \le n, \tag{3.5}$$

admits infinitely many solutions $\tilde{v}^k(u^1, \cdots, u^n)$ depending on n functions of one variable.

ii) The corresponding hydrodynamic systems

$$u^i_{,t} = \tilde{v}^i(u^1, \cdots, u^n)\, u^i_{,x}, \qquad 1 \le i \le n,$$

are rich in conservation laws. The flows corresponding to different solutions of (3.5) are commuting.

Ferapontov [Fe1] gives an explicit formula of $\tilde{v}$ in terms of v which provides a particular solution of (3.5) (see our last example in this paper).

The following result shows that the higher-dimensional Laplace transformation induces a one-to-one correspondence between the conserved densities for any pair of hydrodynamic systems related as in Theorem 3.5.

Theorem 3.6: *Suppose that we have two strongly hyperbolic hydrodynamic systems which are rich in conservation laws and related as in Theorem 3.5,*

$$u^i_{,t} = v^i(u^1, \cdots, u^n)\, u^i_{,x}, \qquad 1 \le i \le n, \tag{3.6}$$

$$u^i_{,t} = \tilde{v}^i(u^1, \cdots, u^n)\, u^i_{,x}, \qquad 1 \le i \le n, \tag{3.7}$$

Then there is a one-to-one correspondence between the conserved densities $P(u^1, \cdots, u^n)$ for (3.6) and $\tilde{P}(u^1, \cdots, u^n)$ for (3.7), given by

$$\begin{aligned} \tilde{P} &= P - P_{,j}/\Gamma^i_{ij} \\ P &= \tilde{P} + \tilde{P}_{,i}/M_{ij}. \end{aligned} \tag{3.8}$$

where M_{ij} is given by (3.2).

In what follows, we will show how the Laplace transformation can be used in order to obtain conserved densities for hydrodynamic systems. We will denote the (i,j)-Laplace transform of X by $\mathcal{L}_{(i,j)}(X)$. We will first consider the ideal chromatography system.

Proposition 3.7: *The conserved densities $P(u^1 \ldots, u^N)$, $N \ge 2$ for the chromatography system*

$$u^i_{,t} = u^i \prod_{\ell=1}^{N} u^\ell u^i_{,x}, \qquad 1 \le i \le N,$$

are given inductively as follows:

$$P^{(N)} = \frac{1}{u^1} \sum_{k=2, i=0}^{N} (-1)^i \frac{\partial^i}{\partial u^k} \left((u^1 - u^k)^N \right) \frac{\partial^{N-i} U_k}{\partial u^k} + U_1. \tag{3.9}$$

$$P^{(n)} = P^{(n+1)} - \frac{u^1(u^1 - u^{n+1})}{(n+1)u^{n+1}} P^{(n+1)}_{,1} \qquad 2 \le n \le N-1, \text{ if } N \ge 3.$$

$$P = P^{(2)} - \frac{u^1(u^1 - u^2)}{2u^2} P^{(2)}_{,1}. \tag{3.10}$$

where U_ℓ are arbitrary differentiable function of u^ℓ.

Proof. We will prove this result in three steps: We will first consider the system satisfied by the conserved densities P for the chromatography system. Then we will apply to P the composition of Laplace transformations

$$P^{(N)} = \mathcal{L}_{(1,N)} \circ \ldots \circ \mathcal{L}_{(1,2)}(P)$$

and we will solve the system satisfied by $P^{(N)}$. Finally, we will obtain P by inverting the Laplace transformations.

We start observing that it follows from (2.4) that the conserved densities for the chromatography system must satisfy the overdetermined system given by

$$P_{,ij} = \Gamma^i_{ij} P_{,i} + \Gamma^j_{ij} P_{,j} \quad \text{where} \quad \Gamma^i_{ij} = \frac{u^i}{u^j(u^j - u^i)} \qquad 1 \le i \ne j \le N. \tag{3.11}$$

We will show that the general solution P for this system is given by (3.10).

We first compute the (i,j)-Laplace invariants of the system (3.11). For a fixed pair (i,j), $i \ne j$, it follows from (3.5) that

$$M_{ij} = -2\Gamma^j_{ij}, \qquad M_{ijk} = \Gamma^i_{ij} - \Gamma^k_{kj}, \quad k \text{ distinct from } i \text{ and } j \tag{3.12}$$

<u>**Claim:**</u> If $N \ge 3$, then for any integer n, $2 \le n < N$, the composition of Laplace transformations given by

$$P^{(n)} = \mathcal{L}_{(1,n)} \circ \ldots \circ \mathcal{L}_{(1,3)} \circ \mathcal{L}_{(1,2)}(P)$$

satisfies the system of equations

$$P^{(n)}_{,1k} = \begin{cases} n\Gamma^k_{1k} P^{(n)}_{,k} & \text{if } 2 \le k \le n; \\ \Gamma^1_{1k} P^{(n)}_{,1} + n\Gamma^k_{1k} P^{(n)}_{,k} & \text{if } n+1 \le k \le N \end{cases} \tag{3.13}$$

$$P^{(n)}_{,\ell k} = \begin{cases} 0 & \text{if } 2 \le \ell \ne k \le n; \\ \Gamma^\ell_{\ell k} P^{(n)}_{,\ell} & \text{if } 2 \le \ell \le n, \ \ n+1 \le k \le N; \\ \Gamma^\ell_{\ell k} P^{(n)}_{,\ell} + \Gamma^k_{\ell k} P^{(n)}_{,k} & \text{if } n+1 \le \ell \ne k \le N. \end{cases} \tag{3.14}$$

The claim will be proved by induction on n. First we show that our claim holds for $n = 2$. In fact, we consider $P^{(2)}$ defined by

$$P^{(2)} = \mathcal{L}_{(1,2)}(P).$$

We will use (3.3) where $i = 1$ and $j = 2$. From (3.12) and (3.11) we observe that

$$\frac{\partial}{\partial u^2}(log M_{12}) + \Gamma^1_{12} = 0.$$

Hence, it follows from the first equation of (3.3) that

$$P^{(2)}_{,12} = 2\Gamma^2_{12} P^{(2)}_{,2}.$$

From the second equation of (3.3), using (3.12) and (3.11) we get

$$P^{(2)}_{,1k} = \frac{\Gamma^1_{12}\Gamma^2_{2k}}{\Gamma^1_{12} - \Gamma^k_{k2}} P^{(2)}_{,1} - \frac{2\Gamma^2_{12}\Gamma^k_{2k}}{\Gamma^1_{12} - \Gamma^k_{k2}} P^{(2)}_{,k}$$

which reduces to

$$P^{(2)}_{,1k} = \Gamma^1_{1k} P^{(2)}_{,1} + 2\Gamma^k_{1k} P^{(2)}_{,k}.$$

Similarly, from the third equation of (3.3) we get that

$$P^{(2)}_{,2k} = \Gamma^2_{2k} P^{(2)}_{,2} \qquad 3 \le k \le N.$$

Finally, from the last equation of (3.3) we obtain

$$P^{(2)}_{,\ell k} = \Gamma^k_{k\ell} P^{(2)}_{,k} + \frac{1}{M_{12\ell}}\left(-\Gamma^\ell_{2\ell}\Gamma^2_{2k} + M_{12k}\Gamma^\ell_{k\ell}\right) P^{(2)}_{,\ell}, \qquad \text{for } \; 3 \le k \ne \ell \le N,$$

which reduces to

$$P^{(2)}_{,\ell k} = \Gamma^k_{k\ell} P^{(2)}_{,k} + \Gamma^\ell_{k\ell} P^{(2)}_{,\ell}$$

when we use equations (3.12) and (3.11). This concludes the first step of our claim. Now assuming the induction hypothesis (3.13) and (3.14) we will prove it will also hold for $n + 1$. We consider

$$P^{(n+1)} = \mathcal{L}_{(1,n+1)}(P^{(n)}).$$

We observe that the $(1, n + 1)$-Laplace invariants for the system (3.13) and (3.14) are

$$\begin{aligned} \overset{(n)}{M}_{1n+1} &= -(n+1)\Gamma^{n+1}_{1n+1} \\ \overset{(n)}{M}_{1n+1k} &= \Gamma^1_{1n+1} - \Gamma^k_{kn+1} \end{aligned} \tag{3.15}$$

where k is distinct from 1 and $n + 1$. In fact, this follows from the fact that

$$\overset{(n)}{M}_{1n+1} = \frac{\partial}{\partial u^1}(log \Gamma^1_{1n+1}) - n\Gamma^{n+1}_{1n+1}.$$

Now, we will obtain the system of equations which are satisfied by $P^{(n+1)}$ by applying (3.3), with $i = 1$ and $j = n+1$, to the system (3.13) and (3.14). From the first equation of (3.3), we obtain

$$P^{(n+1)}_{,1n+1} = (n+1)\Gamma^{n+1}_{1n+1} P^{(n+1)}_{,n+1}.$$

For k distinct from 1 and $n+1$ it follows from the second equation of (3.3) and from (3.11) that

$$P^{(n+1)}_{,1k} = \begin{cases} (n+1)\Gamma^{k}_{1k} P^{(n+1)}_{,k} & \text{if } 2 \leq k \leq n, \\ \Gamma^{1}_{1k} P^{(n+1)}_{,1} + (n+1)\Gamma^{k}_{1k} P^{(n+1)}_{,k} & \text{if } n+1 < k \leq N, \end{cases}$$

From the third equation of (3.11), we get

$$P^{(n+1)}_{,n+1k} = \begin{cases} 0 & \text{if } 2 \leq k < n+2 \\ \Gamma^{n+1}_{u+1k} P^{(n+1)}_{,n+1} & \text{if } n+2 \leq k \leq N. \end{cases}$$

Finally, it follows from the last equation of (3.3), that for k and ℓ distinct from 1 and $n+1$

$$P^{(n+1)}_{,k\ell} = \begin{cases} 0 & \text{if } 2 \leq k \neq \ell < n+1, \\ \Gamma^{\ell}_{k\ell} P^{(n+1)}_{,\ell} & \text{if } 2 \leq \ell < n+1, \quad n+2 \leq k \leq N, \\ \Gamma^{k}_{k\ell} P^{(n+1)}_{,k} + \Gamma^{\ell}_{k\ell} P^{(n+1)}_{,\ell} & \text{if } n+2 \leq \ell \neq k \leq N. \end{cases}$$

This concludes the proof of our claim.

As a consequence we get that

$$P^{N-1} = \mathcal{L}_{(1,N-1)} \circ \ldots \circ \mathcal{L}_{(1,2)}(P)$$

satisfies the system of equations

$$P^{(N-1)}_{,1k} = \begin{cases} (N-1)\Gamma^{k}_{1k} P^{(N-1)}_{,k} & 2 \leq k \leq N-1 \\ \Gamma^{1}_{1N} P^{(N-1)}_{,1} + (N-1)\Gamma^{N}_{1N} P^{(N-1)}_{,k} & \text{if } k = N \end{cases}$$

$$P^{(N-1)}_{,\ell k} = \begin{cases} 0 & \text{if } 2 \leq \ell \neq k \leq N-1 \\ \Gamma^{\ell}_{\ell N} P^{(N-1)}_{,\ell} & \text{if } 2 \leq \ell \leq N-1, \quad k = N. \end{cases}$$

Hence

$$P^{(N)} = \mathcal{L}_{(1,N)} \circ \ldots \circ \mathcal{L}_{(1,2)}(P)$$

satisfies the system of equations

$$P^{(N)}_{,1k} = N\Gamma^k_{1k} P^{(N)}_{,k} \qquad 2 \le k \le N,$$
$$P^{(N)}_{,\ell k} = 0 \qquad \text{if } 2 \le \ell \ne \le N.$$

It follows that

$$P^{(N)} = \sum_{k=2}^{N} \int \left(1 - \frac{u^k}{u^1}\right)^N U_k(u^k) du^k + U_1(u^1)$$

where U_i are arbitrary functions of u^i. This last expression reduces to (3.9).

Now using the inversion of the Laplace transformation (3.4), we obtain from (3.15) that

$$P^{(N-1)} = P^N - \frac{1}{N\Gamma^N_{1N}} P^N_{,1}.$$

Inductively, we obtain

$$P^{(n)} = P^{(n+1)} - \frac{1}{(n+1)\Gamma^{n+1}_{1n+1}} P^{(n+1)}_{,1} \qquad 2 \le n \le N-1$$

and

$$P = P^{(2)} - \frac{1}{2\Gamma^2_{12}} P^{(2)}_{1},$$

which gives the general solution of the system (3.11).

□

As a consequence of Theorem 3.4 and the previous Proposition we will obtain the conserved densities for another system of hydrodynamic type.

Proposition 3.8: *The conserved densities $P(u^1 \dots, u^N)$, $N \ge 2$ for the eletrophoresis system*

$$\begin{aligned} u^1_{,t} &= 0 \\ u^i_{,t} &= -u^i \prod_{\ell=1}^{N} u^\ell u^i_{,x}, \qquad 2 \le i \le N. \end{aligned}$$

are given by

$$P = U_1(u^1) + \frac{1}{u^1} G(u^2 \dots u_N) \tag{3.16}$$

where G is defined inductively as follows: Let

$$\begin{aligned}
G^{(N-1)} &= \frac{1}{u^2} \sum_{k=3,i=0}^{N} (-1)^i \frac{\partial^i}{\partial u^k} \left((u^2 - u^k)^{N-1}\right) \frac{\partial^{N-1-i} U_k}{\partial u^k} + U_2. \\
G^{(n-1)} &= G^{(n)} - \frac{u^2(u^2 - u^{n+1})}{n\, u^{n+1}} G^{(n)}_{,2} \qquad 2 \leq n-1 \leq N-2 \quad \textit{if } N \geq 4 \\
G &= G^{(2)} - \frac{u^2(u^2 - u^3)}{2u^3} G^{(2)}_{,2}
\end{aligned}$$

where $U_i(u^i)$ are arbitrary differentiable functions of u^i.

Proof. It follows from (2.4) that the conserved densities $P(u^1, \dots, u_N)$ must satisfy the system of equations

$$P_{,ij} = \Gamma^i_{ij} P_{,i} + \Gamma^j_{ij} P_{,j} \qquad i \neq j \quad 1 \leq i \neq j \leq N \tag{3.17}$$

where

$$\Gamma^1_{1k} = 0, \quad \Gamma^k_{k1} = -\frac{1}{u^1}, \quad \Gamma^k_{kj} = \frac{u^k}{u^j(u^j - u^k)} \qquad 2 \leq k \neq j \leq N. \tag{3.18}$$

We want to obtain the general solution for this system.

We consider the (2,1)-Laplace invariants for the system (3.17) (3.18). It follows from (3.18) that

$$M_{21} = M_{21k} = 0 \qquad \forall k,\ 3 \leq k \leq N.$$

Applying Theorem 3.4, we get that its general solution P is given by (3.16) and $G(u_2 \dots u_N)$ satisfies the differential system

$$G_{,ij} = \Gamma^i_{ij} G_{,i} + \Gamma^j_{ij} G_{,j}, \qquad 2 \leq i \neq j \leq N,$$

where

$$\Gamma^i_{ij} = \frac{u^i}{u^j(u^j - u^i)}.$$

In order to apply Proposition 3.7 we change our notation by considering

$$G(v^1, \dots v^{N-1})$$

where $v^i = u^{i+1}$ for $1 \leq i \leq N-1$, and

$$\tilde{\Gamma}^i_{ij} = \frac{v^i}{v^j(v^j - v^i)} = \Gamma^{i+1}_{i+1\, j+1}, \qquad i \neq j, \quad 1 \leq i \neq j \leq N-1.$$

It follows from Proposition 3.7 that G is defined inductively as follows:

$$\begin{aligned} G^{(N-1)} &= \sum_{k=2}^{N-1} \left(1 - \frac{v^k}{v^1}\right)^{N-1} V_\ell(v^\ell)dv^\ell + V_1(v^1) \\ G^{(n-1)} &= G^{(n)} - \frac{1}{n\tilde{\Gamma}^n_{1n}} \qquad 2 \leq n-1 \leq N-2 \ \text{ if } \ N \geq 4 \\ G &= G^{(2)} - \frac{1}{2\tilde{\Gamma}^2_{12}} G^{(2)}_{,v^1} \end{aligned}$$

Replacing v^ℓ and $\tilde{\Gamma}$ we obtain

$$\begin{aligned} G^{(N-1)} &= \sum_{k=2}^{N-1} \int \left(1 - \frac{u^{k+1}}{u^2}\right)^{N-1} U_{\ell+1}(u^{\ell+1})du^{\ell+1} + U_2(u^2) \\ G^{(n-1)} &= G^{(n)} - \frac{1}{n\Gamma^{n+1}_{2n+1}} G^{(n)}_{,2} \qquad 2 \leq n-1 \leq N-2 \ \text{ if } \ N \geq 4 \\ G &= G^{(2)} - \frac{1}{2\Gamma^3_{23}} G^{(2)}_{,2} \end{aligned}$$

which concludes the proof of Proposition 3.8.

□

Similar arguments will show

Proposition 3.10: *The conserved quatities* P *for the system*

$$u^i_{,x} = -u^i \prod_{\ell=1}^{N} u^\ell u^i_{,t}, \qquad 1 \leq i \leq N,$$

are given by the system of equations

$$P_{,ij} = \frac{1}{u^j - u^i}(P_{,i} - P_{,j}),$$

whose general solution is given inductively by

$$\begin{aligned} P^{(N)} &= \sum_{k=2,i=0}^{N} (-1)^i \frac{\partial^i}{\partial u^k} \left((u^1 - u^k)^N\right) \ \frac{\partial^{N-i} U_k}{\partial u^k} + U_1. \\ P^{(n)} &= P^{(n+1)} - \frac{u^1 - u^{n+1}}{(n+1)} \ P_1^{(n+1)} \qquad 2 \leq n \leq N-1, \ \textit{if}\ N \geq 3. \\ P &= P^{(2)} - \frac{u^1 - u^2}{2} \ P_1^{(2)}. \end{aligned}$$

where U_ℓ are arbitrary differentiable function of u^ℓ.

As an application of Theorem 3.6. we provide the following example of hydrodynamic systems related by a Laplace transformation.

Example: Consider the systems

$$\begin{aligned} u^1_{,t} &= 0, \\ u^i_{,t} &= u^i \left(\prod_{j=1}^{3} u^j \right) u^i_{,x}, \qquad 2 \leq i \leq 3. \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} u^1_{,t} &= -(u^1)^2 U_{1,1} u^1_{,x}, \\ u^2{}_{,}t &= u^1(U_1 + U_2) u^2_{,x}, \\ u^3{}_{,}t &= u^1(U_1 + f(u^2, u^3)) u^3_{,x} \end{aligned} \tag{3.20}$$

where

$$f = \left(\frac{u^2}{u^2 - u^3} \right)^2 \left[2 \int \frac{u^3(u^2 - u^3)}{(u^2)^3} U_2 du^2 + U_3 \right]$$

and U_i are arbitrary differentiable functions of u^i. Then the electrophoresis system (3.19) and system (3.20) are related as in Theorem 3.5 by the $(2,3)$ -Laplace transformation. Therefore, the conserved densities $\tilde{P}$ of (3.20) are obtained from the conserved densities P of (3.19) by the relation

$$\tilde{P} = P - \frac{u^3(u^3 - u^2)}{u^2} P_{,3}.$$

In fact, the conserved densities $P(u^1, u^2, u^3)$ for (3.19) must satisfy (3.17), (3.18), where $N = 3$. Since the $(2,3)$-Laplace invariants $M_{23} = -2\Gamma^3_{23}$ and $M_{231} = \Gamma^2_{23}$ are non zero, it follows from Theorem 3.1 that the $(2,3)$-Laplace transform of the system (3.17) is given by

$$\tilde{P}_{k\ell} = \tilde{\Gamma}^k_{k\ell}\, \tilde{P}_{,k} + \tilde{\Gamma}^\ell_{k\ell}\, \tilde{P}_{,\ell}, \qquad 1 \leq k \neq \ell \leq 3, \tag{3.21}$$

where the coefficients are given by $\tilde{\Gamma}^1_{1\ell} = 0$, $\tilde{\Gamma}^\ell_{1\ell} = \Gamma^1_{1\ell}$ for $\ell \neq 1$, $\tilde{\Gamma}^2_{23} = 0$, and $\tilde{\Gamma}^3_{23} = 2\Gamma^3_{23}$. Applying Theorem 3.5, by solving the system

$$\frac{\tilde{v}^k_{,l}}{\tilde{v}^l - \tilde{v}^k} = \tilde{\Gamma}^k_{kl} \qquad 1 \leq k \neq l \leq 3,$$

we obtain (3.20). The relation between the conserved densities is given by Theorem 3.6 as in (3.21). We conclude observing that by choosing the particular functions $U_1 = 0$, $U_2 = (u^2)^3$ and $U_3 = (u^3)^3$ in (3.20), we obtain the transformed system given in [Fe1].

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