Closed Manifolds with Small Excess

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Abstract

In this paper, we study closed Riemannian manifolds with small excess. We show that a closed connected Riemannian manifold with Ricci curvature and injectivity radius bounded from below is homeomorphic to a sphere if it has sufficiently small excess. We also show that a closed connected Riemannian manifold with weakly bounded geometry is a homotopy sphere if its excess is small enough.

1. Introduction and Main Theorems

A complete n-dimensional Riemannian manifold with Ricci curvature $\mathrm{Ric}_M \geq n-1$ has diameter $d(M) \leq \pi$ (see [CE]) and equality holds if and only if M is isometric to the unit n-sphere (see [Ch]). This lead to many investigations into the topology of manifolds with $\mathrm{Ric}_M \geq n-1$ and $d(M)$ close to π.

Without further conditions, Anderson and Otsu showed that one can not hope for such manifolds to be spheres.

When $n \geq 4$, there are n-dimensional complete Riemannian manifolds (see [A], [O]) with $H_2(M) \neq 0,\ \mathrm{Ric}_M \geq n-1,\ \mathrm{Vol}(M) \geq v$ and $d(M)$ close to π.

There have been several diameter pinching sphere theorems for n-manifolds with Ricci curvature bounded from below by $n-1$(see [I], [Sh], [W], [E], [P]), and with the exception of [P], most of them have since been generalized by Grove and Peterson in [GP2]. Moreover, Grove and Peterson initiated the idea of using excess to solve the diameter sphere problem. Recall that the

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excess $e(M)$ of a metric space is defined as

$$e(M) = \inf_{p,q\in M} \sup_{x\in M} (d(p,x) + d(q,x) - d(p,q)). \tag{1.1}$$

The importance of the diameter pinching condition " $\mathrm{Ric}_M \geq (n-1),\ d(M) \geq \pi - \epsilon$ " lies in the fact that such manifolds satisfy $e(M) \leq \varphi(n,\epsilon)$, where $\varphi(n,\epsilon) \to 0$ as $\epsilon \to 0$.

In [Pe], Petersen proved a sphere theorem for manifolds with Ricci curvature bounded from below by a negative constant.

Theorem ([Pe]). *Let $n \geq 2$ be an integer and $c,\ \kappa, V > 0$. There is an $\epsilon = \epsilon(n,c,\kappa,V) > 0$ such that any closed connected n-dimensional Riemannian manifold M with Ricci curvature $\mathrm{Ric}_M \geq -(n-1)\kappa^2$, injectivity radius $\mathrm{inj}_M \geq c$, volume $\mathrm{vol}(M) \leq V$, and excess $e(M) \leq \epsilon$ is homeomorphic to an n-sphere.*

In this paper, by using a different method, we strengthen Petersen's theorem as follows.

Theorem 1.1 *Given $c,\ \kappa > 0$ and an integer $n \geq 2$, there is an $\epsilon = \epsilon(n,c,\kappa) > 0$ such that any closed connected n-dimensional Riemannian manifold M with $\mathrm{Ric}_M \geq -(n-1)\kappa^2$, $\mathrm{inj}_M \geq c$, and $e(M) \leq \epsilon$ is homeomorphic to an n-sphere.*

It should be noticed that the counterexample obtained by Peterson in [Pe] shows that the Ricci curvature assumption in our Theorem 1.1 is necessary.

A complete manifold M is said to have weakly bounded geometry, if it satisfies the bounds: $K_M \geq K_0 > -\infty,\ v = \inf_{x\in M}\mathrm{vol}\ [B(x,1)] > 0$, where K_M is the sectional curvature of M and $B(x,r)$ denotes the open metric ball with center x and radius r. We have the following excess homotopy sphere theorem for closed manifolds with weakly bounded geometry.

Theorem 1.2 *Given $v, \kappa > 0$ and an integer $n \geq 2$, there is an $\epsilon = \epsilon(n,v,\kappa) > 0$ such that any closed connected n-dimensional Riemannian manifold M with sectional curvature $K_M \geq -\kappa^2$, $\inf_{x\in M}\mathrm{vol}[B(x,1)] \geq v$, and $e(M) \leq \epsilon$ is a homotopy sphere.*

2. A Proof of Theorem 1.1

Before proving Theorem 1.1, we list some facts we need. The following Toponogov type comparison-estimate obtained by Dai and Wei is an important tool in the proof of Theorem 1.1.

Lemma 2.1 ([DW]). *Let M be a complete n-dimensional manifold with* $\mathrm{Ric}_M \geq -(n-1)\kappa^2$ *and conjugate radius* $\mathrm{conj}_M \geq c > 0$. *There is a constant* $C = C(n, \kappa, c) > 0$ *such that if* $\gamma_i : [0, l_i] \to M$, $i = 1, 2$, *are minimizing normal geodesics from p, with* $l := \max(l_1, l_2) \leq \frac{1}{4}c$, *then*

$$d(\gamma_1(l_1), \gamma_2(l_2)) \leq e^{Cl^{\frac{1}{2}}} |l_1\gamma_1'(0) - l_2\gamma_2'(0)|.$$

Let M be a complete Riemannian manifold. For a point $p \in M$, we denote by S_pM and $d(p, x)$ the unit tangent sphere of M at p and the distance from p to x, respectively. Let $d_p(x) = d(p, x)$; then the distance function d_p is not a smooth function (on the cut locus of p). Hence the critical points of d_p are not defined in a usual sense. The notion of critical points of d_p was introduced by Grove-Shiohama in [GS].

A point $q (\neq p) \in M$ is called a critical point of d_p if there is, for any non-zero vector $v \in T_qM$, a minimal geodesic γ from q to p making an angle $\angle(v, \gamma'(0)) \leq \frac{\pi}{2}$ with v. We simply say that q is a critical point of p.

Lemma 2.2 (Isotopy Lemma, see [G], [Gr], [GS]). For any $r > 0$, let $B(p, r)$ be the open geodesic ball with center p and radius r. If $0 < r_1 < r_2 \leq \infty$ and if $\overline{B(p, r_2)} - B(p, r_1)$ contains no critical points of p, then $\overline{B(p, r_1)}$ is a deformation retraction of $\overline{B(p, r_2)}$. Moreover, $B(p, r_1)$ and $B(p, r_2)$ are homeomorphic.

Proof of Theorem 1.1. Let p, $q \in M$ be two points realizing the excess of M, i.e., $e(M)$ is the maximum of the excess function $e_{p,q}(x) = d(p, x) + d(q, x) - d(p, q)$. In this case the diameter $\mathrm{diam}(M)$ of M satisfies

$$\mathrm{diam}(M) \leq d(p, q) + e(M) \tag{2.1}$$

by triangle inequality. From $\mathrm{inj}_M \geq c$ we know that $\mathrm{conj}_M \geq c$. Take a positive number $l = l(n, \kappa, c) \leq \frac{1}{4}c$ such that

$$e^{Cl^{\frac{1}{2}}} \leq 1 + \frac{2}{25}, \tag{2.2}$$

where $C = C(n, \kappa, c)$ is as in Lemma 2.1. Now we choose the number ϵ in Theorem 1.1 to be $\epsilon = \frac{4l}{25}$. We claim that with this choice of ϵ any point $x \in M$ with $d(x,p) \geq \frac{c}{4}$ and $d(x,q) \geq \frac{c}{4}$ is not a critical point of p as well as of q. To see this, we fix an $x \in M - (B(p, \frac{c}{4}) \cup B(q, \frac{c}{4}))$ and let $\gamma_i : [0, l_i] \to M$, $i = 1, 2$ be minimizing geodesics from x to p and q, respectively. By definition, our claim is a consequence of

$$\theta := \angle\left(\gamma_1'(0), \gamma_2'(0)\right) > \frac{\pi}{2}. \tag{2.3}$$

To show (2.3), let $\tilde{p} = \gamma_1(l)$, $\tilde{q} = \gamma_2(l)$. From Lemma 2.1, we have

$$\begin{aligned} d(\tilde{p}, \tilde{q})^2 &\leq l^2 e^{2Cl^{\frac{1}{2}}} |\gamma_1'(0) - \gamma_2'(0)|^2 \\ &= 4l^2 e^{2Cl^{\frac{1}{2}}} \left(1 - \sin^2\left(\frac{\pi - \theta}{2}\right)\right). \end{aligned} \tag{2.4}$$

Since $d(\tilde{p}, \tilde{q}) \leq 2l$, one can easily deduce that

$$1 - \frac{d(\tilde{p}, \tilde{q})^2}{4l^2 e^{2Cl^{\frac{1}{2}}}} \leq 2\left(e^{Cl^{\frac{1}{2}}} - \frac{d(\tilde{p}, \tilde{q})}{2l}\right). \tag{2.5}$$

Substituting (2.2) and (2.5) into (2.4), we get

$$\sin^2\left(\frac{\pi - \theta}{2}\right) \leq 2\left(\frac{2}{25} + \frac{2l - d(\tilde{p}, \tilde{q})}{2l}\right). \tag{2.6}$$

On the other hand, from the triangle inequality and our assumption on the excess of M, we have

$$\begin{aligned} 2l - d(\tilde{p}, \tilde{q}) &= d(x, \tilde{p}) + d(x, \tilde{q}) - d(\tilde{p}, \tilde{q}) \\ &= d(x, p) - d(\tilde{p}, p) + d(x, q) - d(\tilde{q}, q) - d(\tilde{p}, \tilde{q}) \\ &\leq d(x, p) + d(x, q) - d(p, q) \\ &= e_{p,q}(x) \\ &\leq e(M) \\ &\leq \frac{4l}{25}. \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we obtain

$$\sin^2\left(\frac{\pi - \theta}{2}\right) \leq \frac{8}{25} < \sin^2\frac{\pi}{4}, \tag{2.8}$$

and so

$$\theta > \frac{\pi}{2}. \tag{2.9}$$

This proves our claim.

Let $M_1 = M - B(p, \frac{c}{4})$; we conclude from our claim and $\mathrm{inj}_M \geq c$ that M_1 contains no critical points of q. ¿From (2.1), $\mathrm{diam}(M) \geq \mathrm{inj}_M \geq c$ and our choice of ϵ, we have

$$\begin{aligned} d(p,q) &\geq \mathrm{diam}(M) - e(M) \\ &\geq c - \frac{4l}{25} \\ &\geq \frac{24}{25}c. \end{aligned} \tag{2.10}$$

Let $d = d(p,q)$; it is easy to show from triangle inequality that

$$\overline{B(q, d - \frac{c}{4})} \subset M - B(p, \frac{c}{4}). \tag{2.11}$$

Thus $\overline{B(q, d - \frac{c}{4})}$ contains no critical points of q. The isotopy Lemma then implies that $B(q, d - \frac{c}{4})$ is homeomorphic to $B(q, \frac{c}{2})$, the open n-disk. On the other hand, from

$$e(M) = \max_{x \in M}(d(p,x) + d(q,x) - d(p,q)) \leq \epsilon \leq \frac{c}{25}, \tag{2.12}$$

we have for any $x \in M - B(q, d - \frac{c}{4})$ that

$$\begin{aligned} d(x,p) &= (d(x,p) + d(x,q) - d(p,q)) - d(x,q) + d(p,q) \\ &\leq \frac{c}{25} - (d - \frac{c}{4}) + d \\ &< \left(\frac{2}{25} + \frac{1}{4}\right)c, \end{aligned} \tag{2.13}$$

and so

$$M = B(q, d - \frac{c}{4}) \cup B\left(p, (\frac{1}{4} + \frac{2}{25})c\right). \tag{2.14}$$

Thus M is covered by two open n-dimensional disks and therefore it is homeomorphic to an n-sphere by the generalized Schoenflies theorem, that states that any manifold M^n which is covered by two open n-dimensional disks, is homeomorphic to the sphere, S^n (see [Br]). This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

To prove Theorem 1.2, we state the following

Lemma 3.1 *Let M be a closed connected Riemannian n-manifold with $K_M \geq -\kappa^2$ and $\inf_{x\in M} \mathrm{vol}[B(x,1)] \geq v > 0$. Let $\mathrm{V}(n,-\kappa^2,1)$ denote the volume of a unit metric ball in a simply connected n-dimensional space form of curvature $-\kappa^2$. For any $\alpha > 0$ with $\alpha < v \cdot \frac{\pi}{2} V(n,-\kappa^2,1)^{-1}$, there is an $r = r(n,\kappa,v,\alpha) > 0$ such that if $p,q \in M$ are points where the directions of the set of minimal geodesics between p and q form $(\frac{\pi}{2}+\alpha)$-nets in S_pM and S_qM, then we have $d(p,q) \geq r$.*

Lemma 3.1 is slightly different from the *Main Lemma* in [GP1], where the conditions "vol$M \geq v$, diam$M \leq D$" are assumed. These assumptions are replaced in Lemma 3.1, by the condition "$\inf_{x\in M} \mathrm{vol}[B(x,1)] \geq v$", with no upper bound assumption on the diameter of M. However, the proof of Lemma 3.1 can be carried out by modifying the proof of the *Main Lemma* in [GP1] (cf. Lemma 1.5 in [GP3]) and thus it is omitted.

Proof of Theorem 1.2. We first notice that the Bishop-Gromov comparison theorem implies that the diameter of our manifold M satisfies $d(M) \geq d(n,\kappa,v)$ for some positive constant $d(n,\kappa,v)$ depending only on $n,\ \kappa, v$.

Let $\Delta \subset M \times M$ be the diagonal of $M \times M$ and for $r > 0$ let $\Delta(r) = \{(x,y) \in M\times M; d(x,y) < r\}$. By using Lemma 3.1 and the same arguments as in the proof of Theorem 1.6 in [GP1], we can find positive constants $r = r(n,\kappa,v),\ R = R(n,v,\kappa)$ with the property that there exists a differentiable strong deformation retract $H : \Delta(r) \times [0,1] \to \Delta(r)$ onto Δ in such a way that every retraction curve $t \to H(x,y,t)$ for $(x,y) \in \Delta(r)$ has length not greater than $R \cdot d(x,y)$ and the curve near the diagonal is a minimizing geodesic. Thus for each $p \in M$, $B(p,r_0)$ for a fixed $r_0 \in (0,r)$ is contractible in $B(p, R\cdot r_0)$. We can assume without loss of generality that $r \leq \frac{d}{3}$.

We choose the number ϵ in our Theorem 1.2 to be a sufficiently small positive solution $\epsilon = \epsilon(n,\kappa,v) < \frac{r}{16}$ of the following inequality

$$\cosh^2\left(\frac{\kappa r}{8}\right) - \cosh\left(\frac{\kappa r}{4} - \epsilon\kappa\right) < 0. \tag{3.1}$$

Now, let $p, q \in M$ be two points such that

$$\sup_{x\in M}(d(p,x)+d(q,x)-d(p,q)) = e(M) \leq \epsilon < \frac{r}{16}. \tag{3.2}$$

It then follows from (2.1) that

$$d(p,q) \geq d(M) - e(M) \geq d - \frac{r}{16} > r.$$

Thus $B(p,\frac{r}{2}) \cap B(q,\frac{r}{2}) = \emptyset$. Let $x \in M - (B(p,\frac{r}{4}) \cup B(q,\frac{r}{4}))$ be an arbitrary point and let $\gamma_i : [0,l_i] \longrightarrow M, i = 1,2$ be normal minimizing geodesics from x to p and q, respectively. Set $p' = \gamma_1(\frac{r}{8}),\ q' = \gamma_2(\frac{r}{8})$ and $\alpha = \angle(\gamma_1'(0), \gamma_2'(0))$. From the Toponogov comparison theorem we have

$$\cos\,\alpha\, \sinh^2\left(\frac{\kappa r}{8}\right) \leq \cosh^2\left(\frac{\kappa r}{8}\right) - \cosh\,(\kappa d(p',q')). \tag{3.3}$$

As in the proof of (2.7), one can deduce from the triangle inequality that

$$2\cdot\frac{r}{8} - d(p',q') \leq e_{p,q}(x) \leq e(M) \leq \epsilon, \tag{3.4}$$

or

$$d(p',q') \geq \frac{r}{4} - \epsilon. \tag{3.5}$$

Thus from (3.1) and (3.3) we obtain

$$\cos\,\alpha\, \sinh^2\left(\frac{\kappa r}{8}\right) \leq \cosh^2\left(\frac{\kappa r}{8}\right) - \cosh\left(\frac{\kappa r}{4} - \kappa\epsilon\right) < 0. \tag{3.6}$$

Hence $\alpha > \frac{\pi}{2}$, that is, x is not a critical point of p as well as of q and so $M - (B(p,\frac{r}{4}) \cup B(q,\frac{r}{4}))$ contains no critical points of p and q. The isotopy Lemma then implies that $M_1 := M - B(q,\frac{r}{4})$ contracts to $B(p,\frac{r}{4})$ which in turn contracts to a point in M. Similarly $M_2 := M - B(p,\frac{r}{4})$ contracts to a point in M. Now $M = M_1 \cup M_2$; then $H^*(M,M_i) \longrightarrow H^*(M)$ is surjective for $i = 1,2$ and therefore $H^*(M)$ has trivial cup product structure for any coefficient ring. Using Z_2 as coefficient field, it follows from Poincaré duality that $H_1(M,Z_2) = 0$. From the Mayer-Victoris sequence

$$0 \longrightarrow H_0(M_1 \cap M_2; Z_2) \longrightarrow H_0(M_1;Z_2) \oplus H_0(M_2;Z_2) \longrightarrow H_0(M;Z_2) \longrightarrow 0$$

we conclude that $M_1 \cap M_2$ is connected. This in turn implies that $\pi_1(M) = \{1\}$ by Van Kampen's theorem. Now any simply connected homology sphere is a homotopy sphere by theorems of Hurewicz and Whitehead. This completes the proof of Theorem 1.2.

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