On Ribaucour transformations

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Abstract

Minimal surfaces in R^3 are associated to solutions $\phi(u_1, u_2)$ of the nonlinear differential equation $\Delta\phi = e^{-2\phi}$. We consider Ribaucour transformations for minimal surfaces in R^3. The analytic interpretation of this transformation provides solutions for the differential equation from a given solution. By applying the theory to the catenoid and to Enneper's surface, we generate 3-parameter families of minimal surfaces and their corresponding solutions ϕ.

1.Introduction.

Ribaucour transformations for hypersurfaces, parametrized by lines of curvature, were classically studied by Bianchi [Bi]. Imposing certain conditions, they can be applied to obtain surfaces of constant Gaussian curvature from a given such surface. Analytically, it corresponds to composing appropriate Bäcklund transformations for the classical sine-Gordon equation and the elliptic sinh-Gordon equation.

Similarly, by using Ribaucour transfomations one may obtain minimal surfaces from a given such surface. In this case the analytic interpretation provides solutions for the differential equation $\Delta\phi = e^{-2\phi}$. Although the transfomation for minimal surfaces is well known, it has not been sufficiently explored.

In this paper, we exhibit 3-parameter families of minimal surfaces associated to Enneper's surface and to the catenoid, by applying Ribaucour transformations. Moreover, we provide the analytic interpretation of our geometric results in terms of explicit solutions for the equation $\Delta\phi = e^{-2\phi}$. Detailed proofs of our results will appear elsewhere.

2. Ribaucour transformation for minimal surfaces.

In this section, we first recall the theory of Ribaucour transformation for surfaces. For the proofs and more details see [CFT1] and [Bi]. We then restrict ourselves to Ribaucour transformations between minimal surfaces.

*Partially supported by CNPq and FAPDF

A *sphere congruence* in R^3 is a 2-parameter family of spheres, with a differentiable radius function, whose centers lie on an surface M_0 contained in R^3. An *involute* of a sphere congruence is a surface M of R^3 such that M, at each point, is tangent to a sphere of the sphere congruence. Two surfaces M and $\tilde{M}$ are said to be *associated by a sphere congruence* if there is a difeomorphism $\psi : M \to \tilde{M}$ such that at corresponding points p and $\psi(p)$ the sufaces are tangent to the same sphere of the sphere congruence. It follows from this definition that the normal lines at corresponding points intersect at an equidistant point on the center surface M_0. An important special case occurs when ψ preserves lines of curvature.

Let M and $\tilde{M}$ be orientable surfaces of R^3. We denote by N and $\tilde{N}$ their Gauss map. We say that M and $\tilde{M}$ are *associated by a Ribaucour transformation*, if and only if, there exists a differentiable function h defined on M and a diffeomorphism $\psi : M \to \tilde{M}$ such that

a) $p + h(p)N(p) = \psi(p) + h(p)\tilde{N}(\psi(p))$, for all $p \in M$.

b) The subset $p + h(p)N(p)$, $p \in M$ is an 2-dimensional surface.

c) ψ preserves lines of curvature.

One can also consider surfaces locally associated by Ribaucour transformations or parametrized surfaces associated by such transformations.

The following theorem gives a characterization of Ribaucour transformations. For the proof and more details for hypersurfaces M^n of R^{n+1} see [CFT1].

Theorem 2.1. *Let M be an orientable surface of R^3. Let e_i, $1 \leq i \leq 2$ be orthonormal principal directions, λ^i the corresponding principal curvatures. A surface $\tilde{M}$ is associated to M by a Ribaucour transformation, if and only if, M and $\tilde{M}$ are associated by a sphere congruence whose radius function $h : M \to R$ satisfies*

$$dZ^j(e_i) + Z^i\omega_{ij}(e_i) - Z^iZ^j\lambda^i = 0, \quad 1 \leq i \neq j \leq 2. \tag{1}$$

where

$$Z^i = \frac{dh(e_i)}{1 + h\lambda^i}$$

and ω_{ij} are the connection forms of the frame e_i.

Remark 2.2. Locally, whenever $X(u_1, u_2)$ is a parametrization of the surface M by lines of curvature, the function $h(u_1, u_2)$ is a differentiable function which satisfies the following equation, equivalent to (1),

$$h_{,12} - \frac{1 + h\lambda^1}{1 + h\lambda^2}\Gamma^2_{12}h_{,2} - \frac{1 + h\lambda^2}{1 + h\lambda^1}\Gamma^1_{12}h_{,1} - \left(\frac{\lambda^2}{1 + h\lambda^2} + \frac{\lambda^1}{1 + h\lambda^1}\right)h_{,1}h_{,2} = 0$$

where Γ^i_{12} are the Christoffel symbols and $h_{,1}$ and $h_{,12}$ denote the partial derivatives of h with respect to u_1 and to $u_1\, u_2$ respectively.

One can linearize the problem of obtaining the function h, as we will see in the following proposition. We will consider e_1, e_2 to be orthonormal principal directions and ω_1, ω_2 the dual frame.

Proposition 2.3. *Suppose that h is a nonvanishing function which satisfies equation (1) then*

$$\psi = \frac{1}{h}\sum_{i=1}^{2} Z^i \omega_i$$

is a closed 1-form and there exists a nonvanishing function Ω defined on a simply connected domain such that

$$d\Omega(e_i) = \frac{\Omega}{h} Z^i.$$

For each nonvanishing function h, which is a solution of (1), we consider Ω as above and we define

$$\Omega_i = d\Omega(e_i), \qquad W = \frac{\Omega}{h}.$$

With this notation, equation (1) is equivalent to a linear system given in the following result.

Proposition 2.4. *A nonvanishing function h is a solution of (1) defined on a simply connected domain, if and only if, $h = \Omega/W$ where Ω and W are nonvanishing functions which satisfy*

$$\begin{aligned} d\Omega_i(e_j) &= \Omega_j \omega_{ij}(e_j), \qquad \text{for } i \neq j, &(2)\\ d\Omega &= \sum_{i=1}^{n} \Omega_i \omega_i, &(3)\\ dW &= -\sum_{i=1}^{n} \Omega_i \lambda^i \omega_i. &(4) \end{aligned}$$

For each solution Ω_i, $1 \leq i \leq 2$, of (2), there exists a 2-parameter family of solutions of the system (3), (4). In fact, equation (2) is the integrability condition of the system of equations (3), (4) for Ω and W.

The Ribaucour transformation of a surface is given in terms of the solutions of the above system (see [CFT1] for details).

Theorem 2.5. *Let M be an orientable surface of R^3 parametrized by $X : U \subset R^2 \to M$. Assume e_i, $1 \le i \le 2$ are orthonormal principal directions, λ^i the corresponding principal curvatures and N is a unit vector field normal to M. A surface $\tilde{M}$ is locally associated to M, by a Ribaucour transformation, if and only if, there exist differentiable functions $W, \Omega, \Omega_i : V \subset U \to R$, which satisfy (2)-(4) and $\tilde{X} : V \subset R^2 \to \tilde{M}$, is a parametrization of $\tilde{M}$ given by*

$$\tilde{X} = X - \frac{2\Omega}{\sum_i (\Omega_i)^2 + W^2} \left(\sum_i \Omega_i e_i - WN \right). \tag{5}$$

We now recall a classical result which gives a sufficient condition for a Ribaucour transformation to transform a minimal surface into another minimal surface.

Theorem 2.6. *Let M and $\tilde{M}$ be surfaces of R^3, with no parabolic points, which are associated by a Ribaucour transformation, such that the normal lines intersect at a distance function h. If $h = \Omega/W$ is not constant along the lines of curvature and the functions Ω_i ,Ω and W satisfy (2)-(4) and the additional relation*

$$\Omega_1^2 + \Omega_2^2 + W^2 = 2c\Omega W,$$

where $c \neq 0$ is a real constant, then $\tilde{M}$ is a minimal surface, if and only if M is minimal.

One can also prove a permutability theorem for Ribaucour transformations between minimal surfaces. We will give its analytic interpretation as a superposition formula for equation (7).

We conclude this section, by recalling the well known correspondence between minimal surfaces and solutions of the differential equation $\Delta\phi = e^{-2\phi}$ (see [Bi], [E], [T1]).

Proposition 2.7. *Minimal surfaces in R^3 with no parabolic points are in correspondence with solutions of the equation*

$$\Delta\phi = e^{-2\phi}. \tag{6}$$

Simple examples are given by the catenoid, which is associated to the function

$$\phi = \log(\cosh u_1),$$

and the Enneper's surface that corresponds to the function

$$\phi = \log\left[1 + \frac{u_1^2}{4} + \frac{u_2^2}{4}\right].$$

The analytic interpretation of Ribaucour transformation for minimal surfaces is given by the following result.

Theorem 2.8. *Let $\phi(u_1, u_2)$ be a solution of the equation*

$$\Delta\phi = e^{-2\phi} \tag{7}$$

Then for each constant $c \neq 0$, the following system of equations for Ω_i, Ω and W

$$\begin{aligned} \Omega_{i,u_j} &= \Omega_j \phi_{,u_i} \qquad 1 \leq i \neq j \leq 2 \\ \Omega_{,u_i} &= \Omega_i e^{\phi} \\ W_{,u_i} &= (-1)^{i+1} \Omega_i e^{-\phi} \\ \Omega_1^2 + \Omega_2^2 + W^2 &= 2c\Omega W \end{aligned} \tag{8}$$

is integrable and a new solution of (7) is given by the function $\tilde{\phi}$ defined by

$$e^{\tilde{\phi}} = e^{-\phi} \frac{\Omega}{W}.$$

The solution $\tilde{\phi}$ is said to be *associated to ϕ by the Ribaucour transformation* and the constant c.

Theorem 2.9. (Superposition formula) *Let ϕ be a solution of (7). Let ϕ' and ϕ'' be solutions of (7) associated to ϕ by the Ribaucour transformation (8) for two distinct nonzero constants c' and c'' respectively. Let Ω' , W' and Ω'', W'' be the corresponding solutions of (8). Then ϕ^* defined algebraically by*

$$e^{\phi^*} = e^{\phi} \left(\frac{-B + c' e^{\phi''} + c'' e^{\phi'}}{-B + c' e^{\phi'} + c'' e^{\phi''}} \right),$$

where

$$B = e^{-\phi} + e^{\phi} < \nabla(\log W'), \nabla(\log W'') >,$$

is a new solution of (7).

3. Families of minimal surfaces associated to Enneper's surface and to the catenoid.

In this section, by using Ribaucour transformations, we will provide 3-parameter families of minimal surfaces from Enneper's surface and from the catenoid. Moreover, we will give families of solutions of equation (7) associated to these families of minimal surfaces.

Proposition 3.1. *Consider Enneper's surface parametrized by*

$$X=\left(u_1-\frac{u_1^3}{3}+u_1u_2^2,\ u_2-\frac{u_2^3}{3}+u_2u_1^2,\ u_1^2-u_2^2\right) \tag{9}$$

$\tilde{X}$ is a minimal surface associated to X as in Theorem 2.5., if and only if, up to a rigid motion of R^3,

$$\tilde{X}=X+\frac{1}{c}(-u_1,u_2,1)-\frac{1}{2c(f+g)}(f'X_{u_1}-g'X_{u_2}) \tag{10}$$

where

$$\begin{aligned} f(u_1) &= a_1e^{2\sqrt{c}u_1}+a_2e^{-2\sqrt{c}u_1} \\ g(u_2) &= b_1\cos(2\sqrt{c}u_2)+b_2\sin(2\sqrt{c}u_2) \end{aligned}$$

the constants are real numbers satisfying $c>0$, $a_1^2+a_2^2\neq 0$ and $b_1^2+b_2^2-4a_1a_2=0$.

Remark 3.2. The family of minimal surfaces given by (10) includes Enneper's surface. In fact, if we choose the constants $a_1=b_1=b_2=0$ and $a_2\neq 0$, by considering the variable $\overline{u}=u+1/\sqrt{c}$, we obtain the Enneper surface translated by $(3/c^{3/2},\ 0,\ 0)$. A similar result holds for the case $a_2=b_1=b_2=0$ and $a_1\neq 0$.

Similar results are obtained by considering the catenoid.

Proposition 3.3. *Consider the catenoid parametrized by*

$$X=(\cos u_2\cosh u_1,\ \sin u_2\cosh u_1,\ u_1)$$

$\tilde{X}$ is a minimal surface associated to X by a Ribaucour transformation as in Theorem 2.8, if and only if,

$$\tilde{X}=X-\frac{\cosh u_1}{c}(\cos u_2,\sin u_2,0)+\frac{1}{c(f+g)}(f'X_{u_1}-g'X_{u_2}) \tag{11}$$

where $c\neq 0$, $f(u_1)$ and $g(u_2)$ are solutions of the equations

$$f''+(2c-1)f=g''-(2c-1)g=c_1,$$

c_1 is a real constant, $f+g$ does not vanish and the initial conditions for f and g must satisfy

$$\left((f')^2+(g')^2+(2c-1)(f^2-g^2)-2c_1(f+g)\right)(u_1^0,u_2^0)=0.$$

Remark 3.4. The catenoid is included in the family of surfaces given by (11). In fact, we choose $c=1/2$, $c_1=0$, $f=a_1$, $g=a_2$, $a_1+a_2\neq 0$.

The analytic interpretation of the results given above are the following:
a) We consider the solution ϕ of equation (7) associated to the catenoid,

$$\phi = \log(\cosh u_1).$$

For any $c \neq 0$, the solutions of (8) are given by

$$\begin{aligned} \Omega_1 &= -\tanh u_1(f+g) + f' \\ \Omega_2 &= -g' \\ W &= (f+g)/\cosh u_1 \\ \Omega &= (\cosh^2 u_1 \Delta W/2c + W/c \end{aligned}$$

where $f(u_1)$ and $g(u_2)$ satisfy the differential equation

$$f'' + (2c-1)f = g'' - (2c-1)g$$

Hence, the function $\tilde{\phi}$

$$e^{\tilde{\phi}} = \frac{\Omega}{f+g}$$

is a new solution for equation (7). More solutions are explicitly obtained by using the superposition formula of Theorem 2.9.

b) We consider the function

$$\phi = \log\left[1 + \frac{u_1^2}{4} + \frac{u_2^2}{4}\right]$$

which is a solution of (7), associated to Enneper's surface. For any $c \neq 0$, the solutions of (8) are given by

$$\begin{aligned} \Omega_1 &= -u_1(f+g)/(2\varphi) + f' \\ \Omega_2 &= u_2(f+g)/(2\varphi) - g' \\ W &= (f+g)/\varphi \\ \Omega &= \varphi^2 \Delta W/2c + W/c \end{aligned}$$

where $\varphi = 1 + u_1^2/4 + u_2^1/4$, $\quad f(u_1)$ and $g(u_2)$ satisfy the differential equation

$$f'' - 2cf = g'' + 2cg.$$

Hence, the function

$$e^{\tilde{\phi}} = \frac{\Omega}{f+g}$$

is a new solution of (7). More explicit solutions are obtained by using the superposition formula.

We refer to [CFT2] for the proofs of Theorems 3.1, 3.2 and for geometric properties of the families of minimal surfaces given in these theorems. The reader is also referred to [T2] for a computer animated video which provides illustrative examples of these minimal surfaces.

References

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