

**ON METRICS SATISFYING EQUATION $R_{ij} - Kg_{ij}/2 = T_{ij}$
FOR CONSTANT TENSORS T**

Romildo Pina

Keti Tenenblat *

Abstract

Necessary and sufficient conditions are given on a constant symmetric tensor T_{ij} on R^n , $n \geq 3$, for which there exist metrics \bar{g} , conformal to a pseudo-euclidean metric g , such that $\bar{R}_{ij} - \bar{K}\bar{g}_{ij}/2 = T_{ij}$, where \bar{R}_{ij} and \bar{K} are the Ricci tensor and the scalar curvature of \bar{g} . All solutions \bar{g} are given explicitly and it is shown that there are no complete metrics \bar{g} conformal and nonhomothetic to g .

In the problem section of the Seminar on Differential Geometry edited by S.T. Yau [Y], the first problem on Ricci curvature is the following:

Find necessary and sufficient conditions on a symmetric tensor T_{ij} on a compact manifold so that one can find a metric g_{ij} to satisfy $R_{ij} - Kg_{ij}/2 = T_{ij}$ whence R_{ij} is the Ricci tensor and K is the scalar curvature of g_{ij} . (P)

If g_{ij} is the Lorentz metric on a four-dimensional manifold, this is simply the Einstein field equation. Whenever the tensor T represents a physical field such as electromagnetic field, perfect fluid type, pure radiation field and vacuum ($T = 0$), the above equation has been studied in several papers most of them dealing with solutions which are invariant under some symmetry group of the equation (see [KSMH] for details). When the metric g_{ij} is conformal to the Minkowski space time, then the solutions in the vacuum case, are necessarily flat and apparently not known explicitly (see [KSMH]). In the remaining cases, all the solutions conformal to the Minkowski metric are known. We refer to [MTT] for the pure radiation or the null electromagnetic field, to [CL] and [S] for the non-null electromagnetic field, and finally to [S] and [B] if T is a perfect fluid. Besides these special results, as far as we know, very little is known about problem (P) with respect to other manifolds, its dimension or the tensor T .

Our purpose in this paper is to solve problem (P) in R^n , $n \geq 3$, for constant symmetric tensors of the form

$$T = \sum_{i,j=1}^n \varepsilon_j c_{ij} dx_i \otimes dx_j \quad \text{with} \quad c_{ij} \in R, \quad \text{and} \quad \varepsilon_j c_{ij} = \varepsilon_i c_{ji} \quad (1)$$

requiring the metric to be conformal to the pseudo-Euclidean metric (R^n, g) , $g_{ij} = \delta_{ij}\varepsilon_i$, $\varepsilon_i = \pm 1$, where at least one eigenvalue ε_i is positive. We want to find metrics \bar{g} such that

$$\begin{cases} \bar{g} = \frac{1}{\varphi^2}g \\ \text{Ric } \bar{g} - \frac{\bar{K}}{2}\bar{g} = T. \end{cases} \quad (2)$$

*Partially supported by CNPq and PRONEX. MSC 2000: 53C21, 53C50, 53C80. Key words: Ricci tensor, conformal metric, scalar curvature.

Before stating our results, we observe that since dimensions higher than 4 are considered in some theories in Physics, it is important to treat this problem in any dimension. Moreover, T being a constant tensor in the standard coordinates of R^n is not a property preserved under a change of coordinates. The requirement of being constant in our paper implies that T is covariantly constant in the standard flat metric g . However, T in general will not be covariantly constant in the metric \bar{g} conformal to g .

We consider the linear functions β_i , $1 \leq i \leq n$ defined for each $x = (x_1, \dots, x_n) \in R^n$ by $\beta_i(x) = (n-1)(n-2)x_i - (n-3) \sum_{k=1}^n x_k$. For a fixed pseudo-Euclidean metric $g_{ij} = \delta_{ij}\varepsilon_i$, we consider the following subsets of R^n

$$\begin{aligned} D &= \{x \in R^n; \varepsilon_j \beta_j(x) \geq 0 \forall j, 1 \leq j \leq n\}, \\ L &= \{x \in R^n; \varepsilon_j \beta_j(x) \leq 0 \forall j, 1 \leq j \leq n\} \\ \pi_i &= \{x \in R^n; \beta_i(x) = 0\}, \quad 1 \leq i \leq n. \end{aligned}$$

D and L are nonempty subsets of R^n . With this notation we can now state our results.

Theorem 1. *Let (R^n, g) be a pseudo-Euclidean space and let T be a non diagonal symmetric tensor as in (1) such that $\sum_i c_{ii} \neq 0$. Then there exists a metric \bar{g} solving (2), if and only if, $c = (c_{11}, \dots, c_{nn}) \in D \setminus \{\pi_\ell \cup \pi_k\}$ for some $\ell \neq k$ and*

$$c_{ij} = \frac{\varepsilon_j \gamma_i \gamma_j}{(n-1)(n-2)} \sqrt{\varepsilon_i \varepsilon_j \beta_i \beta_j(c)} \quad \forall i \neq j \quad (3)$$

where $\gamma_j = \pm 1$ for $1 \leq j \leq n$. For any such fixed tensor T , the solutions are $\bar{g} = g/\varphi^2$ where

$$\varphi(x) = k \exp \left(\frac{\delta}{(n-2)\sqrt{(n-1)}} \sum_j \gamma_j \sqrt{\varepsilon_j \beta_j(c)} x_j \right) \quad (4)$$

where k is a nonzero constant and $\delta = \pm 1$.

In Theorem 1, for each $c \in D \setminus \{\pi_\ell \cup \pi_k\}$, the expressions (3) define at least two and generically 2^{n-1} tensors T .

Theorem 2 *If T be a non diagonal tensor as in (1) such that $\sum_i c_{ii} = 0$, then there exists \bar{g} solving (2), if and only if $c = (c_{11}, \dots, c_{nn}) \in (D \cup L) \setminus \{\pi_\ell \cup \pi_k\}$ for some $\ell \neq k$ and*

$$c_{ij} = \begin{cases} \varepsilon_j \gamma_i \gamma_j \sqrt{\varepsilon_i \varepsilon_j c_{ii} c_{jj}} & \forall i \neq j \quad \text{if } c \in D \setminus \{\pi_\ell \cup \pi_k\} \\ -\varepsilon_j \gamma_i \gamma_j \sqrt{\varepsilon_i \varepsilon_j c_{ii} c_{jj}} & \forall i \neq j \quad \text{if } c \in L \setminus \{\pi_\ell \cup \pi_k\} \end{cases} \quad (5)$$

where $\gamma_j = \pm 1$ for $1 \leq j \leq n$. In this case, for any such fixed tensor T , $\bar{K} = 0$. Moreover, the function φ is constant if g is the Euclidean metric and otherwise it is given by

$$\varphi(x) = \begin{cases} k_1 \exp \left(\sum_j h_j(x_j) \right) + k_2 \exp \left(-\sum_j h_j(x_j) \right), & \text{if } c \in D \setminus \{\pi_\ell \cup \pi_k\} \\ k_1 \cos \left(\sum_j h_j(x_j) \right) + k_2 \sin \left(\sum_j h_j(x_j) \right), & \text{if } c \in L \setminus \{\pi_\ell \cup \pi_k\} \end{cases} \quad (6)$$

and

$$h_j(x_j) = \begin{cases} \sqrt{\varepsilon_j c_{jj}} \gamma_j x_j / \sqrt{n-2} & \text{if } c \in D \setminus \{\pi_\ell \cup \pi_k\}, \\ \sqrt{-\varepsilon_j c_{jj}} \gamma_j x_j / \sqrt{n-2} & \text{if } c \in L \setminus \{\pi_\ell \cup \pi_k\}. \end{cases} \quad (7)$$

The functions φ given in Theorem 2. satisfy $\Delta_g \varphi = \|\nabla_g \varphi\|^2 = 0$.

Theorem 3. *If $T = \sum_{i=1}^n \varepsilon_i c_{ii} dx_i^2$ is a non zero diagonal tensor, then there exists a solution \bar{g} of (2), if and only if,*

$$T = \begin{cases} b\varepsilon_k dx_k^2 & \text{if } n = 3, \\ b \sum_{i \neq k, i=1}^n \varepsilon_i dx_i^2 + \frac{n-1}{n-3} b\varepsilon_k dx_k^2 & \text{if } n \geq 4, \end{cases} \quad (8)$$

for some fixed k , $1 \leq k \leq n$ where b is a real constant such that $b\varepsilon_k > 0$. In this case,

$$\bar{g}_{ij} = \begin{cases} \delta_{ij} \varepsilon_i \exp(a - 2\delta \sqrt{b\varepsilon_k} x_k) & \text{if } n = 3, \\ \delta_{ij} \varepsilon_i \exp\left(a - 2\delta \sqrt{\frac{2b\varepsilon_k}{(n-2)(n-3)}} x_k\right) & \text{if } n \geq 4, \end{cases} \quad (9)$$

where $\delta = \pm 1$ and $a \in R$.

Theorem 4. *If $T = 0$, then there exists a solution \bar{g} of (2), if and only if,*

$$\varphi = \sum_{j=1}^n (A\varepsilon_j x_j^2 + B_j x_j + C_j), \quad \text{where} \quad 4A \sum_j C_j - \sum_j \varepsilon_j B_j^2 = 0 \quad (10)$$

and A, C_j, B_j are real constants. In this case, $\bar{K} \equiv 0$, i.e. $\text{Ric } \bar{g} \equiv 0$.

As a consequence of the above theorems we obtain:

Corollary 5. *Let (R^n, g) be a pseudo-Euclidean space. For any constant symmetric tensor T , there are no complete metrics \bar{g} , conformal and nonhomothetic to g , such that $\text{Ric } \bar{g} - \bar{K} \bar{g}/2 = T$.*

The techniques used to prove our results are similar to those introduced in [PT]. We first recall a well known result (see for example [KR]), that if $\bar{g} = g/\varphi^2$, then

$$\text{Ric } \bar{g} - \text{Ric } g = \frac{1}{\varphi^2} \left\{ (n-2)\varphi \text{Hess}_g(\varphi) + \left(\varphi \Delta_g \varphi - (n-1)\|\nabla_g \varphi\|^2 \right) g \right\}.$$

Hence the scalar curvature $\bar{K} = \sum \bar{g}^{ij} \bar{R}_{ij}$ is given by $\bar{K} = (n-1)(2\varphi \Delta_g \varphi - n\|\nabla_g \varphi\|^2)$. Therefore, one proves that solving problem (2) is equivalent to studying the system

$$\begin{cases} \varphi_{x_i x_i} = \varepsilon_i \left(\lambda_i \varphi + \frac{\|\nabla_g \varphi\|^2}{2\varphi} \right) \\ \varphi_{x_i x_j} = \frac{\varepsilon_j c_{ij}}{n-2} \varphi \end{cases} \quad 1 \leq i \neq j \leq n, \quad (11)$$

where $\lambda_i = \frac{c_{ii}}{n-2} - \frac{\sum_{\ell} c_{\ell\ell}}{(n-1)(n-2)}$. If φ is a solution of (11) then one can show that

$$c_{ji} \varphi_{x_i} = \frac{\beta_i(c)}{(n-1)(n-2)} \varphi_{x_j} \quad \forall i \neq j. \quad (12)$$

Using (11) and (12), we can prove that if T is non-diagonal and φ is a solution of (11) then $\|\nabla_g \varphi\|^2/(2\varphi) = -\sum_k \lambda_k \varphi/(n-2)$. It follows from this equation that the diagonal elements of

T are such that $c = (c_{11}, \dots, c_{nn}) \in (D \cup L) \setminus \{\pi_r \cup \pi_\ell\}$ for some pair $(r, \ell), 1 \leq r \neq \ell \leq n$, and the non-diagonal elements are determined by c as in (3).

If $\sum_i c_{ii} \neq 0$, and $c \in L \setminus \{\pi_r \cup \pi_\ell\}$ then $\varphi = 0$, hence we conclude that in this case, $c \in D \setminus \{\pi_r \cup \pi_\ell\}$ and φ is given by (4). If $\sum_i c_{ii} = 0$, then (11) reduces to $\varphi_{x_i x_i} = \varepsilon_i c_{ii} \varphi / (n - 2)$ and $\varphi_{x_i x_j} = \varepsilon_j c_{ij} \varphi / (n - 2)$ for $i \neq j$. Therefore, one can show that φ is given by (6) and the elements of T satisfy (5).

In order to prove Theorem 3. we observe that if T is a nonzero diagonal tensor, it follows from (12), that φ is not constant and $0 = \beta_i(c) \varphi_{x_j} \forall i \neq j$. Let k be such that $\varphi_{x_k} \neq 0$. If $n \geq 4$, then for all $i \neq k$, $c_{ii} = b$, where $b \neq 0$ is a real constant and $\beta_i = 0$. We conclude that $c_{kk} = (n - 1)b / (n - 3)$. If $n = 3$, then $c_{ii} = 0$ for all $i \neq k$ and $c_{kk} = b \neq 0$. In both cases φ depends only on x_k . Therefore T is given by (8) and the system (11) reduces to ordinary differential equations whose solution provides \bar{g} as in (9).

The proof of Theorem 4 follows immediately from the fact that φ satisfies the system of equations (11), where $\lambda_i = 0$ for all i .

The converse of Theorems 1 - 4 follows from a straightforward computation.

For each fixed tensor T as in Theorems 1 or 3, there exist two semi-Riemannian metrics (given by $\delta = \pm 1$) in the same conformal class which have pointwise the same Ricci tensor. Since they are not homothetic to each other, it follows from the results of [F] and [KR] that they are not complete. A similar argument applies to the metrics obtained in Theorem 2 when $c \in D \setminus \{\pi_\ell \cup \pi_k\}$. In the remaining cases, the metric $\bar{g} = g / \varphi^2$ has singularity points. This completes the proof of Corollary 5.

References

- [B] Barnes, A., *Space-times of embedding class one in general relativity*, General Relativity and Gravitation, 5, (1974), 147-161.
- [CL] Cahen, M. and Leroy, J. *Exact solutions of the Einstein-Maxwell equations*, J. Math. Mech. 16, (1966), 501-508.
- [F] Ferrand, J., *Sur une class de morphismes conformes*, C. R. . Sci. Paris 296 (1983), 681-684.
- [KR] Kühnel, W.; Rademacher, H.B., *Conformal diffeomorphisms preserving the Ricci tensor*, Proc. of the American Mathematical Society, Vol 123, (1995), 2841-2848.
- [KSMH] Kramer, D., Stephani, H., MacCallun M.A.H. and Herlt, E. , *Exact solutions of Einstein's field equations*, Cambridge University Press, 1980.
- [MTT] McLenaghan, R.G., Tarig, N. and Tupper, B.O.J. *Conformally flat solutions of the Einstein-Maxwell equations for the null electromagnetic fields*, J. Math. Phys. 16, (1975), 829-831.
- [PT] Pina, R., Tenenblat, K. , *Conformal metrics and Ricci tensors in the pseudo-euclidean space*, Proc. Amer. Math. Soc. 129, (2001), 1149-1160.
- [S] Stephani, H., *Konform flache Gravitationsfelder*, Comm. in Math. Phys. 5, (1967) 337-342.
- [Y] Yau, S.T. Ed. *Seminar on Differential Geometry*, Princeton University Press, 1982.

R. Pina
IME, Universidade Federal de Goiás
74001-970 Goiânia, GO, Brazil
romildo@mat.ufg.br

K. Tenenblat
Depto de Matemática, Universidade de Brasília
70910-900, Brasília, DF, Brazil
keti@mat.unb.br