GAUGE EQUIVALENCE OF DIFFERENTIAL EQUATIONS DESCRIBING SURFACES OF CONSTANT GAUSSIAN CURVATURE

Qing Ding, Keti Tenenblat
Institute of Mathematics, Fudan University, Shanghai 200433, CHINA
email: ding@mat.unb.br
Departamento de Matemática, Universidade de Brasilia,
Brasilia, DF 70910-900, BRAZIL
e-mail: keti@mat.unb.br

Abstract: In this paper, we show that, given two differential equations or systems describing pseudospherical surfaces (resp. spherical surfaces), there exists a local gauge transformation that transforms a generic solution of one into any generic solution of the other. This reveals an interesting phenomenon of differential systems describing surfaces of constant Gaussian curvature.
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1. Introduction

It is well-known (see Chern and Tenenblat [2], Ding and Tenenblat [5], Kamran and Tenenblat [9], Reyes [11] and Sasaki[12]) that a differential equation for a real-valued function $u(x,t)$, or a differential system for a 2-vector-valued function $u(x,t)$, is said to describe pseudospherical surface (p.s.s) (resp. spherical surface (s.s.)) if it is the necessary and sufficient condition for the existence of smooth (real) functions f_{ij}, $1 \leq i \leq 3, 1 \leq j \leq 2$, depending only on u and a finite number of derivatives, such that the one-forms

$$\omega_i = f_{i1}dx + f_{i2}dt, \quad 1 \leq i \leq 3, \tag{1}$$

satisfy the structure equations of a surface of constant Gaussian curvature -1 (resp. $+1$),

$$\omega_1 = \omega_3 \wedge \omega_2, \quad \omega_2 = \omega_1 \wedge \omega_3, \quad \omega_3 = \delta\omega_1 \wedge \omega_2, \tag{2}$$

where $\delta = 1$ (reps. $\delta = -1$).

It is a straightforward computation to verify that (2) is equivalent to saying that

$$d\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \Omega \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3}$$

where $\Omega = \frac{1}{2}\begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix}$ is $sl(2,R)$ valued 1-form matrix for the case of equations describing p.s.s. (resp. $\Omega = \frac{1}{2}\begin{pmatrix} i\omega_3 & \omega_2 + i\omega_1 \\ -\omega_2 + i\omega_1 & -i\omega_3 \end{pmatrix}$ is $su(2)$ valued for the case of equations describing s.s.), is a completely integrable system, i.e. $d\Omega - \Omega \wedge \Omega = 0$. The one-form Ω may be regarded as defining a connection on a principle $SL(2,R)$ (resp. $SU(2)$) bundle over R^2 and the equation expresses the fact that the curvature $F = d\Omega - \Omega \wedge \Omega$ of this connection vanishes. It should be pointed out that the integrability of (3) or equivalently the structure equations (2) are invariant under the gauge transformations: $\Omega \to \Omega^A = A\Omega A^{-1} + dAA^{-1}$, where A is an $SL(2,R)$ (resp. $SU(2)$) valued smooth matrix function.

It is known now that the SG, KdV, MKdV, Burgers' equation, NLS$^-$, M-HF model, etc. describe p.s.s. (see Chern and Tenenblat [2], Ding and Tenenblat [5], Reyes [11], Sasaki [12]) and the NLS$^+$, HF model, Laudan-Lifschitz equation, DNLS, etc. describe s.s.(see Ding and Tenenblat [5]). For example,

1. The NLS$^+$ equation,

$$iq_t + q_{xx} + 2|q|^2 q = 0 \tag{4}$$

or equivalently in real form ($q(x,t) = u(x,t) + iv(x,t)$),

$$u_t + v_{xx} + 2(u^2 + v^2)u = 0, \quad -v_t + u_{xx} + 2(u^2 + v^2)v = 0$$

is a differential equation describing s.s. with

$$\begin{aligned} \omega_1 &= 2v dx + (-4\eta v + 2u_x)dt, \\ \omega_2 &= 2\eta dx + [-4\eta^2 + 2(u^2 + v^2)]dt, \\ \omega_3 &= -2u dx + (4\eta u + 2v_x)dt, \end{aligned} \tag{5}$$

where η is a spectral parameter.

2. The NLS$^-$ equation,

$$iq_t + q_{xx} - 2|q|^2 q = 0 \tag{6}$$

or equivalently in real form ($q(x,t) = u(x,t) + iv(x,t)$),

$$u_t + v_{xx} - 2(u^2 + v^2)u = 0, \quad -v_t + u_{xx} - 2(u^2 + v^2)v = 0$$

is a differential equation describing s.s. with

$$\begin{aligned} \omega_1 &= 2udx + (4\eta u - 2v_x)dt, \\ \omega_2 &= -2vdx - (4\eta v + u_x)dt, \\ \omega_3 &= -2\eta dx - [(4\eta^2 + 2(u^2 + v^2)]dt, \end{aligned} \tag{7}$$

where η is a spectral parameter.

3. The HF model (Faddeev and Takhtajan [6], Zakharov and Takhtajan [13])—-the Schrödinger flow of maps into $S^2 \hookrightarrow R^3$ (Ding [3]) (or the Laudan-Lifschitz equation for an isotropic chain),

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \tag{8}$$

where $\mathbf{S} = (s_1(x,t), s_2(x,t), s_3(x,t)) \in R^3$ with $s_1^2 + s_2^2 + s_3^2 = 1$ and × denotes the cross product in R^3. One can directly verify that this model describes s.s. with

$$\begin{aligned} \omega_1 &= -2\eta s_2 dx + (4\eta^2 s_2 + 2\eta s_1 s_{3x} - 2\eta s_1 s_{3x})dt, \\ \omega_2 &= -2\eta s_1 dx + (4\eta^2 s_1 - 2\eta s_2 s_{3x} + 2\eta s_3 s_{2x})dt, \\ \omega_3 &= 2\eta s_3 dx + (-4\eta^2 s_3 - 2\eta s_2 s_{1x} + 2\eta s_1 s_{2x})dt, \end{aligned} \tag{9}$$

where η is a spectral parameter.

4. The M-HF model—-the Schrödinger flow of maps into $H^2 \hookrightarrow R^{3+1}$ (Ding [3]),

$$\mathbf{S}_t = \mathbf{S} \dot{\times} \mathbf{S}_{xx}, \tag{10}$$

where $\mathbf{S} = (s_1(x,t), s_2(x,t), s_3(x,t)) \in R^{2+1}$ with $s_1^2 + s_2^2 - s_3^2 = -1$ ($s_3 > 0$) and $\dot{\times}$ denotes the pseudo cross product in R^{2+1}. It is a straightforward computation to verify that this model describes p.s.s. with

$$\begin{aligned} \omega_1 &= 2\eta s_2 dx + (4\eta^2 s_2 - 2\eta s_1 s_{3x} + 2\eta s_3 s_{1x})dt, \\ \omega_2 &= 2\eta s_1 dx + (4\eta^2 s_1 + 2\eta s_2 s_{3x} - 2\eta s_3 s_{2x})dt, \\ \omega_3 &= -2\eta s_3 dx + (4eta^2 s_3 + 2\eta s_1 s_{2x} - 2\eta s_2 s_{1x})dt, \end{aligned} \tag{11}$$

where η is a spectral parameter.

The dynamical properties of these soliton equations such as the existence of infinite number of conservation laws and symmetries, the Bäcklund transformations for such equations can be geometrically interpreted by relations (2) (see Chern and Tenenblat [2], Rabelo [10], Reyes [11] and Sasaki [12]). In [9], Kamran and Tenenblat proved by a geometric approach the following interesting result: given two differential equations describing p.s.s., there exists a transformation between a generic solution of one equation into any generic solution of the other. It seems likely that the above geometric approach provides a powerful tool in understanding the integrability of these equations. On the other hand, gauge transformations of the linear systems of soliton equations are very useful in studying such equations. Zakharov and Takhtajan [13] proved in 1979 that the NLS$^+$ is $SU(2)$-gauge equivalent to the HF model. Later this result was generalized to the matrix nonlinear Schrödinger equation by Honerkamp in [7]. Recently, as a dual geometric interpretation of the mentioned fact due to Zakharov and Takhtajan, Ding showed that there are $SU(1,1)$-gauge transformation (see Ding [3]) and, also, $SL(2,R)$-gauge transformation (see Ding and Tenenblat [4]) between the NLS$^-$ equation and the M-HF model. These results show that some integrable models are classically equivalent and differ only by the choice of coordinates.

In this paper, we shall generalize the result of Kamran and Tenenblat to differential equations or systems describing surfaces of constant Gaussian curvature and interprete them geometrically in terms of local gauge transformations. This reveals a quite universal phenomenon that local gauge transformations always exist between differential equations or systems decribing p.s.s. (resp. s.s.).

2. Local gauge transformations between generic solutions

We denote a (1+1) differential equation describing p.s.s. (resp. s.s.) by $E[u(x,t)] = 0$, where u might be a vector-valued function and the square brackets indicate that E is a function of u and a finite number of its partial derivatives

To each solution of a differential equation $E[u(x,t)] = 0$ describing surfaces of constant Gaussian curvature, which satisfies the generity condition: $\omega_1 \wedge \omega_2 \neq 0$, i.e., $\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \neq 0$, on $(x,t) \in W \subset R^2$, one can associate a two dimensional metric $ds^2 = \omega_1^2 + \omega_2^2$ of Gaussian curvature -1 (resp. $+1$) with connection 1-forms given by ω_3. Therefore, solutions for which $\omega_1 \wedge \omega_2 \neq 0$ will be called generic solutions.

Now we prove the existence of local gauge transformations between generic solutions of equations describing p.s.s. (resp. s.s.). Let us begin with the following useful lemma.

Lemma 1 *Let*

$$E[u(x,t)] = 0 \tag{12}$$

and

$$\widetilde{E}[\widetilde{u}(\widetilde{x},\widetilde{t})] = 0 \tag{13}$$

be differential equations describing p.s.s. (resp. s.s.), with associated 1-forms given by $\omega_i = f_{i1}dx + f_{i2}dt$ *and* $\widetilde{\omega}_k = \widetilde{f}_{k1}d\widetilde{x} + \widetilde{f}_{k2}d\widetilde{t}$*,* $1 \leq i,k \leq 3$*, where* $u(x,t)$ *and* $\widetilde{u}(\widetilde{x},\widetilde{t})$ *are real-valued or vector-valued functions. Then, for any generic solution* $u(x,t)$ *to Eq.(12) and any generic solution* $\widetilde{u}(\widetilde{x},\widetilde{t})$ *to Eq.(13), there exists a local diffeomorphism* $\Phi : (x,t) \in V \subset W \to (\widetilde{x},\widetilde{t}) \in \widetilde{V} \subset \widetilde{W}$*, where* V *and* $\widetilde{V}$ *are open subsets of the domains* W *and* $\widetilde{W}$ *of* u *and* $\widetilde{u}$ *respectively, and a smooth function* $\alpha(x,t) : W \to R$ *such that*

$$\widetilde{\omega}_1(\widetilde{x}(x,t),\widetilde{t}(x,t)) = \cos\alpha(x,t)\omega_1(x,t) + \sin\alpha(x,t)\omega_2(x,t), \tag{14}$$

$$\widetilde{\omega}_2(\widetilde{x}(x,t),\widetilde{t}(x,t)) = -\sin\alpha(x,t)\omega_1(x,t) + \cos\alpha(x,t)\omega_2(x,t), \tag{15}$$

$$\widetilde{\omega}_3(\widetilde{x}(x,t),\widetilde{t}(x,t)) = \omega_3(x,t) + d\alpha(x,t), \tag{16}$$

where we have substituted the expression for Φ *as* $\widetilde{x} = \widetilde{x}(x,t)$*,* $\widetilde{t} = \widetilde{t}(x,t)$ *and used the invariant of first-order differential of a function under the change of independent variables in (14-16).*

Proof. We shall only show the case for differential equations describing s.s., the proof of the case for differential equations describing p.s.s. was given in [9] by Kamran and Tenenblat.

First, we show that there exists a (local) diffeomorphism $\Psi : V \subset W \to U$, where V and U are some open subsets of W and $R^2 = S^2 - \{\infty\}$ respectively, and a smooth function σ such that

$$\theta_1 = \cos\sigma\omega_1 + \sin\sigma\omega_2, \tag{17}$$

$$\theta_2 = -\sin\sigma\omega_1 + \cos\sigma\omega_2, \tag{18}$$

$$\theta_3 = \omega_3 + d\sigma, \tag{19}$$

where

$$\theta_1 = \frac{2d\bar{x}}{1+\bar{x}^2+\bar{t}^2}, \quad \theta_2 = \frac{2d\bar{t}}{1+\bar{x}^2+\bar{t}^2}, \quad \theta_3 = \frac{2\bar{t}d\bar{x} - 2\bar{x}d\bar{t}}{1+\bar{x}^2+\bar{t}^2} \tag{20}$$

are the 1-forms which give the standard spherical metric:

$$ds^2 = \theta_1^2 + \theta_2^2 = \frac{4(d\bar{x}^2 + d\bar{t}^2)}{(1+\bar{x}^2+\bar{t}^2)^2}$$

and its connection 1-form θ_3.

Now let Ψ be denoted by $\bar{x} = \bar{x}(x,t), \bar{t} = \bar{t}(x,t)$. The explicit form of Eqs.(17-19) in terms of $\bar{x}, \bar{t}, \sigma$ and their derivatives is given by

$$\frac{\partial \bar{x}}{\partial x} = \phi_1, \quad \frac{\partial \bar{t}}{\partial x} = \psi_1, \quad \frac{\partial \sigma}{\partial x} = \sigma_1, \tag{21}$$

$$\frac{\partial \bar{x}}{\partial t} = \phi_2, \quad \frac{\partial \bar{t}}{\partial t} = \psi_2, \quad \frac{\partial \sigma}{\partial t} = \sigma_2, \tag{22}$$

where

$$\phi_1 = \frac{1+\bar{x}^2+\bar{t}^2}{2}(\cos\sigma f_{11} + \sin\sigma f_{21}), \tag{23}$$

$$\phi_2 = \frac{1+\bar{x}^2+\bar{t}^2}{2}(-\sin\sigma f_{11} + \cos\sigma f_{21}), \tag{24}$$

$$\psi_1 = \frac{1+\bar{x}^2+\bar{t}^2}{2}(\cos\sigma f_{12} + \sin\sigma f_{22}), \tag{25}$$

$$\psi_2 = \frac{1+\bar{x}^2+\bar{t}^2}{2}(-\sin\sigma f_{12} + \cos\sigma f_{22}), \tag{26}$$

$$\sigma_1 = -f_{31} + [-\bar{x}(-\sin\sigma f_{11} + \cos\sigma f_{21}) + \bar{t}(\cos\sigma f_{11} + \sin\sigma f_{21})], \tag{27}$$

$$\sigma_2 = -f_{32} + [-\bar{x}(-\sin\sigma f_{12} + \cos\sigma f_{22}) + \bar{t}(\cos\sigma f_{12} + \sin\sigma f_{22})]. \tag{28}$$

Since $u(x,t)$ is a solution to Eq.(12), which describes spherical surfaces with associated 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$, $1 \le i \le 3$, we have

$$-\frac{\partial f_{11}}{\partial t} + \frac{\partial f_{12}}{\partial x} = f_{31}f_{22} - f_{32}f_{21}, \tag{29}$$

$$-\frac{\partial f_{21}}{\partial t} + \frac{\partial f_{22}}{\partial x} = f_{11}f_{32} - f_{12}f_{31}, \tag{30}$$

$$-\frac{\partial f_{31}}{\partial t} + \frac{\partial f_{32}}{\partial x} = -f_{11}f_{22} + f_{12}f_{21}. \tag{31}$$

It is a straightforward calculation that

$$\frac{\partial \phi_1}{\partial t} = \frac{\partial \phi_2}{\partial x}, \quad \frac{\partial \psi_1}{\partial t} = \frac{\partial \psi_2}{\partial x}, \quad \frac{\partial \sigma_1}{\partial t} = \frac{\partial \sigma_2}{\partial x} \tag{32}$$

by using (29-31) and (23-28). However, (32) is just the Frobenius conditions for Eqs.(17-19) for the existence of a local solution. Therefore, we have proved that such a local solution exists. To verify that the smooth mapping $\bar{x} = \bar{x}(x,t), \bar{t} = \bar{t}(x,t)$ obtained by solving the system (17-19) is a local diffeomorphism, all we have to do is to use the identity

$$\frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{t}}{\partial t} - \frac{\partial \bar{t}}{\partial x}\frac{\partial \bar{x}}{\partial t} = \frac{(1+\bar{x}^2+\bar{t}^2)^2}{4}(f_{11}f_{22} - f_{12}f_{21}). \tag{33}$$

The right-hand side of Eq.(33) is nonzero over W since we assumed that u is to be a generic solution. It follows that Ψ will be a diffeomorphism between $V \subset W$ and $U \subset S^2$.

The proof is concluded by observing that since σ and $\Psi : V \longrightarrow U$ solve Eqs.(17-19) and $\widetilde{\sigma}$ and $\Pi : \widetilde{V} \longrightarrow U$ solve the system corresponding to Eqs.(17-19) with σ replaced by $\widetilde{\sigma}$ and ω_i replaced by $\widetilde{\omega}_i$, it follows that $\Phi = \Pi^{-1} \odot \Psi : V \longrightarrow \widetilde{V}$ is a local diffeomorphism, as a composition of local diffeomorphisms, and it satisfies Eqs.(14-16) with $\alpha = \sigma - \widetilde{\sigma}$. □

Now, we interprete the above Lemma just as a local gauge transformation between the two corresponding connection 1-forms Ω and $\widetilde{\Omega}$ as follows:

Theorem 1 *For any generic solution to Eq.(12) and any generic solution to Eq.(13), where Eqs.(12) and (13) describe surfaces of the same nonzero constant curvature, i.e., p.s.s. or s.s., we let Ω and $\widetilde{\Omega}$ be the corresponding connection 1-form in (3) to these two equations respectively. Then, for any generic solution of Eq.(12) and a generic solution of Eq.(13), there exists a local diffeomorphism $\Phi : W \longrightarrow \widetilde{W}$ and a smooth function $\alpha(x,t) : W \longrightarrow R$ such that*

$$\widetilde{\Omega} = A\Omega A^{-1} + dAA^{-1} \quad \textit{for} \quad A = \begin{pmatrix} \cos\alpha/2 & -\sin\alpha/2 \\ \sin\alpha/2 & \cos\alpha/2 \end{pmatrix} \in SL(2,R) \tag{34}$$

if the two equations describe p.s.s., or

$$\widetilde{\Omega} = A\Omega A^{-1} + dAA^{-1} \quad \textit{for} \quad A = \begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{pmatrix} \in SU(2) \tag{35}$$

if the two equations describe s.s.. This implies that the two generic solutions are (local) gauge equivalent.

Proof. A direct computation shows that (14-16) is equivalent to (34)) when the two equations describe p.s.s. and to (35) when the two equations describe s.s.. □

Remark 1 It should be pointed out that, in order to have (35) satisfied, the connection 1-form Ω as described in (3) plays an essential role. The gauge transformations displayed previously (Ding [3, 4], Honerkamp [7], Zakharov and Takhtajan 13]) have no change of the independent variables (i.e. $\Phi = Id.$ in terms of the notation used in Theorem 1) and therefore are special versions of the present general cases. It is interesting to see that the special ones do not involve a change of independent variables but are complicated in the choice of gauge matrices, however, the present cases need generally a change of independent variables (by a local diffeomorphism), but are relatively simpler in the choice of gauge matrices.

Finally, we illustrate the present general gauge transformation by means of an example and give a geometric interpretation of the well-known (special) gauge equivalence between the NLS$^+$ and HF model, which are differential systems desribing s.s.. The geometric explanation of the gauge equivalence between tha NLS$^-$ and the M-HF model can be found in Ding [4].

Example 1. *Geometric interpretation of the $SU(2)$-gauge equivalence between the NLS$^+$ and HF model (Zakharov and Takhtajan [13]).* From (5), the corresponding connection 1-form of the NLS$^+$ is

$$\widetilde{\Omega} = (i\eta\sigma_3 + U)dx + \left[-2i\eta^2\sigma_3 - 2\eta U - i(U^2 + U_x)\sigma_3\right] dt,$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $U = \begin{pmatrix} 0 & v - iu \\ -v - iu & 0 \end{pmatrix}$. Similarly, from (8) we obtain the corresponding 1-form of the HF model as follows,

$$\Omega = -\eta S dx + \left[2\eta^2 S - \eta S_x S\right] dt,$$

where $S = \begin{pmatrix} -is_3 & s_1 + is_2 \\ -s_1 + is_2 & is_3 \end{pmatrix}$. Therefore, for any solution (u, v) to the NLS$^+$, if we let $A(x, t)$ be a fundament solution to

$$A_x = UA, \quad A_t = -i(U^2 + U_x)\sigma_3 A,$$

then the gauge transformation between the NLS$^+$ and the HF model can be now expressed as follows

$$\Omega \to \widetilde{\Omega}\Big|_{S=-iA^{-1}\sigma_3 A} = dAA^{-1} + A\Omega A^{-1}.$$

This gauge transformation maps a family of s.s. determining the NLS$^+$ into another family of s.s. determining the HF model.

Example 2. *Local gauge transformation.* The equation $E[u(x, t] = 0$ is given by (Kamran and Tenenblat [9])

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}, \tag{36}$$

which describes p.s.s. with 1-forms ω_i $(1 \le i \le 3)$ given by (1), where

$$f_{11} = u, \quad f_{12} = u_x, \quad f_{21} = 1, \quad f_{22} = 0, \quad f_{31} = u, \quad f_{32} = u_x.$$

The equation $\widetilde{E}[\widetilde{u}(\widetilde{x}, \widetilde{t})] = 0$ is given by Burgers' equation as follows,

$$\frac{\partial \widetilde{u}}{\partial \widetilde{t}} = \frac{\partial^2 \widetilde{u}}{\partial \widetilde{x}^2} - 2\widetilde{u}\frac{\partial \widetilde{u}}{\partial \widetilde{x}}, \tag{37}$$

which describes p.s.s. with 1-forms $\widetilde{\omega}_i$ $(1 \leq i \leq 3)$ given by (1), where

$$\widetilde{f}_{11} = \widetilde{u}, \quad \widetilde{f}_{12} = \widetilde{u}_x - \widetilde{u}^2, \quad \widetilde{f}_{21} = 1, \quad \widetilde{f}_{22} = -\widetilde{u}, \quad \widetilde{f}_{31} = 1, \quad \widetilde{f}_{32} = \widetilde{u}.$$

If we take the generic solution to (36) given by $u(x,t) = x + t$ and the generic solution to (37) given by $\widetilde{u}(\widetilde{x}, \widetilde{t}) = \frac{e^{\widetilde{x}+\widetilde{t}}}{1-e^{\widetilde{x}+\widetilde{t}}}$. Then the local diffepmorphism $\widetilde{x} = \widetilde{x}(x,t), \widetilde{t} = \widetilde{t}(x,t)$, and smooth function $\alpha(x,t)$ given by

$$\begin{aligned}
\widetilde{x}(x,t) &= \frac{-(x+t)}{e^x(1+(x+t-1)^2)}, \\
\widetilde{t}(x,t) &= -\widetilde{x}(x,t) + \log\left|1 - \frac{-(x+t)}{e^x(1+(x+t-1)^2)}\right|, \\
\cos\alpha &= \frac{2(x+t-1)}{1+(x+t-1)^2}, \quad \sin\alpha = \frac{1-(x+t-1)^2}{1+(x+t-1)^2}
\end{aligned}$$

solve the corresponding system given by (14-16). Thus the above two generic solutions to (36) and (37) respectively are related and can be transformed by the following (local) gauge transformation:

$$\widetilde{\Omega}\Big|_{\widetilde{u}(\widetilde{x},\widetilde{t})=e^{\widetilde{x}+\widetilde{t}}/(1-e^{\widetilde{x}+\widetilde{t}})} = A\,\Omega\big|_{u=x+t}\,A^{-1} + dAA^{-1},$$

where $A = \begin{pmatrix} \frac{2(x+t-1)}{1+(x+t-1)^2} & -\frac{1-(x+t-1)^2}{1+(x+t-1)^2} \\ \frac{1-(x+t-1)^2}{1+(x+t-1)^2} & \frac{2(x+t-1)}{1+(x+t-1)^2} \end{pmatrix}$.

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References

[1] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, The inverse scattering transform Furier analysis for nonlinear problems, *Stud. Appl. Math.* **53**, 249-315 (1974).

[2] S.S. Chern and K. Tenenblat, Pseudo-spherical surfaces and evolution equations, *Stud. Appl. Math.* **74**, 55-83 (1986).

[3] Q. Ding, A note on the NLS and the Schrödinger flow of maps, *Phys. Lett.* A **248**, 49-56 (1998).

[4] Q. Ding, The NLS$^-$ and the its $SL(2,R)$ structure, *J. Phys. A: Math. Gen.* **33** L325-L329 (2000).

[5] Q. Ding and K. Tenenblat, On differential systems describing surfaces of constant curvature, Preprint (2000).

[6] L.D. Faddeev and L.A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer-Verlag Berlin Heideberg 1987.

7 J. Honerkamp, Gauge equivalent of exactly integrable field theoretic models, *J. Math. Phys.* **22** 277-281 (1981).

[8] L.P. Jorge and K.Tenenblat, Linear problems associated to evolution equations of type $u_t = F(u, u_x, u_{xx}, u_t)$, *Stud. Appl. Math.* **77**, 103-117 (1987).

[9] N. Kamran and K. Tenenblat, On differential equations describing pseudo-spherical surfaces, *J. Diff. Equa.* **115**, 75-98 (1995).

[10] M.L. Rabelo, On evolution equations which describe pseudo-spherical surfaces, *Stud. Appl. Math.* **81** 222-248 (1989).

[11] E. Reyes, Pseudo-spherical surfaces and integrability of evolution equations, *J. Diff. Equa.* **147** 195-230 (1998).

[12] R. Sasaki, Soliton equations and pseudo-spherical surfaces, *Nucl. Phys. B* **154**, 343-357 (1979).

[13] V.E. Zakharov and L.A. Takhtajan, Equivalence of a nonlinear Schrödinger equation and a Heisenberg ferromagnet equation, *Theor. Math. Phys.* **38**, 17-26 (1979).