On Differential Systems Describing Surfaces of Constant Curvature

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Abstract

The geometric notion of a differential system describing surfaces of constant nonzero Gaussian curvature is introduced. The nonlinear Schrödinger equation (NLS) with $\kappa = 1$ and $\kappa = -1$ are shown to describe a family of spherical surfaces (s.s.) and pseudospherical surfaces (p.s.s.) respectively. The Schrödinger flow of maps into S^2 (the HF model) and its generalized version, the Landau-Lifschitz equation, are shown to describe spherical surfaces. The Schrödinger flow of maps into H^2 (the M-HF model) provides another example of a system describing pseudospherical surfaces. New differential systems describing surfaces of nonzero constant Gaussian curvature are obtained. Furthermore, we give a characterization of evolution systems which describe surfaces of nonzero constant Gaussian curvature. In particular, we determine all differential systems of type

$$\begin{cases} u_t = -v_{xx} + H_{11}(u,v)u_x + H_{12}(u,v)v_x + H_{13}(u,v) \\ v_t = u_{xx} + H_{21}(u,v)u_x + H_{22}(u,v)v_x + H_{23}(u,v), \end{cases}$$

which describe η-pseudospherical or η-spherical surfaces. As an application, we obtain 4-parameter family of such systems for a complex valued function $q = u+iv$ given by

$$iq_t + q_{xx} \pm i\gamma(|q|^2 q)_x - i\alpha q_x \pm \sigma|q|^2 q - \beta q = 0,$$

where $\sigma \geq 0$ if $\gamma = 0$. Particular cases of this family, obtained by the vanishing of the parameters, are the linear equations, the NLS equation, the derivative nonlinear Schrödinger equation (DNLS) and the mixed NLS-DNLS equation.

Key words: differential systems, pseudospherical surfaces, spherical surfaces, evolution systems.

1. Introduction

In 1979, Sasaki [20] observed that a class of nonlinear differential equations, such as KdV, MKdV and SG which can be solved by the AKNS 2×2 inverse scattering method [1], was related to pseudospherical surfaces. The geometric notion of a differential equation, for a real function, which describes a pseudoshperical surface (p.s.s.) was actually introduced in the literature by Chern and Tenenblat in [4], where equations of type

$$u_t = F(u, u_x, \cdots, \partial^k u/\partial x^k) \tag{1}$$

were studied systematically. Later, in [11,16], this concept was applied to other type of differential equations. A generic solution of these equations provides a metric defined on an open subset in R^2, for which the Gaussian curvature is -1.

Such an equation is characterized as being the integrability condition of a linear problem of the form

$$\phi_x = \left(\begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix} + A\right)\phi, \quad \phi_t = B\phi,$$

where η is a parameter, B is a 2×2 matrix of trace zero, and A is a 2×2 off-diagonal matrix depending on η, u and its derivatives. Examples of this class of equations are (real) equations of AKNS type. Other examples, which are not AKNS, can be found in [11,16,17]. Geometric interpretation of special properties such as infinite number of conservation laws and Bäcklund transformations for solutions of a differential equation which describes p.s.s. have been systematically exploited in [2-4,19]. In 1995, Kamran and Tenenblat [12], extending the results of [4], gave a complete classification of the evolution equations of type (1) which describe p.s.s. by considering equations which are the integrability condition of a linear problem of the form $\phi_x = \Omega_1\phi$, $\phi_t = \Omega_2\phi$, where Ω_1 and Ω_2 are 2×2 traceless matrix functions of η, u and its derivatives. Moreover, they proved that there exists, under a technical assumption, a smooth mapping transforming any generic solution of one such equation into a solution of the other. This geometric notion of scalar differential equations was also generalized to differential equations of type $u_t = F(x, u, u_x, \cdots, \partial^k u/\partial x^k)$ by Reyes recently in [19].

Although a deep understanding of differential equations describing p.s.s. has been displayed in [2-5, 11-12,16-20], very little is known for complex differential equations or, in other words, for differential systems for two real functions, such as the nonlinear Schrödinger equation (NLS),

$$iq_t + q_{xx} + 2\kappa|q|^2 q = 0,$$

where the subscripts denote partial derivatives and κ is a real constant, which models a wide range of physical phenomens (see, for examples, [7-9]). This equation can be written in the real form as follows ($q = u + iv$),

$$\begin{cases} u_t + v_{xx} + 2\kappa(u^2+v^2)v = 0, \\ -v_t + u_{xx} + 2\kappa(u^2+v^2)u = 0. \end{cases}$$

Without loss of gerenality, we will denote, as usual, by NLS^+ and NLS^- the NLS with $\kappa = 1$ and $\kappa = -1$, respectively. The AKNS system motivated the notion of a differential equation which describes p.s.s. (see [4,20]), and the nonlinear Schrödinger equation is a main example in the AKNS hierarchies [1]. Therefore, it is very interesting to investigate whether the NLS describes p.s.s.. Moreover, the same problem can be considered for the Heisenberg ferromagnet model (HF model) given by $\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}$, where $\mathbf{S} = (s_1, s_2, s_3)$ is a point on the unit sphere in R^3. This is an important equation in condensed matter physics [7,9,15], which is a differential system for two independent functions, say s_1, s_2. Another such system is the Landau-Lifschitz equation [7,13-15], which is a generalization of the HF model. Motivated by these important evolution systems, in this paper, we are interested in determining whether differential systems of the following type:

$$\begin{cases} u_t = F(u, \partial u/\partial x, \cdots, \partial^k u/\partial x^k, v, \partial v/\partial x, \cdots, \partial^r v/\partial x^r) \\ v_t = G(u, \partial u/\partial x, \cdots, \partial^k u/\partial x^k, v, \partial v/\partial x, \cdots, \partial^r v/\partial x^r) \end{cases} \tag{2}$$

can also describe pseudospherical surfaces, where k and r are some positive integers. If this is the case how does the geometric properties of the surfaces may provide analytic information for such a differential system?

We find that the notion of Chern-Tenenblat's geometric approach to differential equations may extend to differential systems and it provides a new tool for studying the integrability of partial differential systems.

This paper is organized as follows: In Section 2, we generalize the notion of a differential equation describing p.s.s. to a differential system describing surfaces of nonzero constant curvature. Then we show that the NLS^+ describes spherical surfaces of constant curvature 1 (s.s.) and the NLS^- describes pseudospherical surfaces of constant curvature -1 (p.s.s.), respectively. This shows that the notions of a differential equation describing p.s.s. and s.s. are necessary for studying differential systems. Moreover, we give a general characterization of the differential systems which describe spherical surfaces or pseudospherical surfaces. Some other differential systems, such as the HF model and the Landau-Lifschitz equation, are presented as examples of differential systems describing surfaces of nonzero constant curvature. In Section 3, motivated by [4], we introduce special classes of differential systems which describe η-p.s.s. and η-s.s.. Then we give a complete classification of such differential systems of the forms $iq_t + q_{xx} + H_1(q, \bar{q})q_x + H_2(q, \bar{q})\bar{q}_x + H_3(q, \bar{q}) = 0$, where $q = u + iv$ is a complex valued function. Moreover, we characterize the systems of type (2) with $k = r = 1$, which describe η-surfaces of constant curvature -1 and 1. Finally, in section 4, we include the conclusion and some remarks.

2. Differential systems describing surfaces of constant curvature

A differential equation for a complex-valued function $q(x, t)$, or equivalently a differential system for two real-valued functions $u(x, t)$ and $v(x, t)$, is said to describe pseudospherical surfaces (p.s.s.) (resp. spherical surfaces (s.s.)) if it is the necessary and sufficient condition for the existence of smooth (real) functions $f_{ij}, 1 \leq i \leq 3, 1 \leq j \leq 2$, depending only on u, v and their derivatives, such that the 1-forms

$$\omega_i = f_{i1}dx + f_{i2}dt, \quad 1 \leq i \leq 3,$$

satisfy the structure equations of a surface of constant Gaussian curvature -1 (resp. 1), that is,

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_3, \quad d\omega_3 = \delta\omega_1 \wedge \omega_2, \tag{3}$$

where $\delta = 1$ (resp. $\delta = -1$). One can easily see that (3) is equivalent to saying that

$$d\phi = \Omega\phi,$$

where $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and the coefficient matrix $\Omega = \frac{1}{2}\begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix}$ (resp. $\Omega = \frac{1}{2}\begin{pmatrix} i\omega_2 & \omega_1 + i\omega_3 \\ -\omega_1 + i\omega_3 & -i\omega_2 \end{pmatrix}$. is a completely integrable equation, i.e. $d\Omega - \Omega \wedge \Omega = 0$.

Suppose that a differential system for u and v is given by (2), where k and r are some positive integers. It will be convenient to introduce the following notation for the derivatives of u and v,

$$z_i = \frac{\partial^i u}{\partial x^i}, \quad 0 \leq i \leq k; \quad y_j = \frac{\partial^j v}{\partial x^j}, \quad 0 \leq j \leq r \tag{4}$$

and to view $(x, t, z_0, \cdots, z_k, y_0, \cdots, y_r)$ as local coordinates in an open subset U of a manifold M. System (2) can be rewritten as

$$\begin{cases} z_{0t} = F(z_0, \cdots, z_k, y_0, \cdots, y_r), \\ y_{0t} = G(z_0, \cdots, z_k, y_0, \cdots, y_r). \end{cases}$$

Definition An *evolution system (2) is said to describe p.s.s. (resp. s.s.)*, if there exist 1-forms ω_1, ω_2 and ω_3 on U, given by

$$\omega_i = f_{i1}(z_0, \cdots, z_k, y_0, \cdots, y_r)dx + f_{i2}(z_0, \cdots, z_k, y_0, \cdots, y_r)dt \tag{5}$$

such that the differential ideal I_1 of $\wedge^2 T^*U$ generated by the 2-forms

$$\Pi_1 = d\omega_1 - \omega_3 \wedge \omega_2, \quad \Pi_2 = d\omega_2 - \omega_1 \wedge \omega_3, \quad \Pi_3 = d\omega_3 - \delta\omega_1 \wedge \omega_2, \tag{6}$$

where $\delta = 1$ (resp. $\delta = -1$) and the differential ideal I_2 of $\wedge^2 T^*U$ generated by the forms

$$\begin{array}{lll} dz_0 \wedge dx + F dx \wedge dt, & dz_l \wedge dt - z_{l+1} dx \wedge dt, & 0 \leq l \leq k-1, \\ dy_0 \wedge dx + G dx \wedge dt, & dy_l \wedge dt - y_{l+1} dx \wedge dt, & 0 \leq l \leq r-1 \end{array} \tag{7}$$

coincide, i.e.

$$I_1 = I_2.$$

Given an evolution system (2) describing p.s.s. (resp. s.s.), we consider ($u(x,t)$, $v(x,t)$) a local solution defined on $V \subset R^2$, where $f_{11}f_{22} - f_{12}f_{21}$ does not vanish, then

$$ds^2 = \omega_1^2 + \omega_2^2$$

defines a Riemannian metric of constant Gaussian curvature equal to -1 (resp. 1) on V, whose connection 1-form is given by ω_3. In this paper, we shall restrict ourselves to solutions satisfying this requirement, i.e. $\omega_1 \wedge \omega_2 \neq 0$.

Example 1. The NLS$^+$ equation,

$$iq_t + q_{xx} + 2|q|^2 q = 0$$

or equivalently in real form ($q(x,t) = u(x,t) + iv(x,t)$),

$$\begin{cases} u_t + v_{xx} + 2(u^2+v^2)v = 0, \\ -v_t + u_{xx} + 2(u^2+v^2)u = 0, \end{cases} \tag{8}$$

is a differential system describing s.s., where the 1-forms ω_i, $1 \le i \le 3$, are given by (5) with

$$\begin{gathered} f_{11} = 2v, \quad f_{21} = 2\eta, \quad f_{31} = -2u, \quad f_{12} = -4\eta v + 2u_x, \\ f_{22} = -4\eta^2 + 2(u^2+v^2), \quad f_{32} = 4\eta u + 2v_x \end{gathered}$$

and η is a spectral parameter. In fact, one can easily verify that these forms satisfy (3) with $\delta = -1$.

Example 2. The NLS$^-$ equation,

$$iq_t + q_{xx} - 2|q|^2 q = 0$$

or equivalently in real form ($q(x,t) = u(x,t) + iv(x,t)$),

$$\begin{cases} u_t + v_{xx} - 2(u^2+v^2)v = 0, \\ -v_t + u_{xx} - 2(u^2+v^2)u = 0, \end{cases} \tag{9}$$

is a differential system describing p.s.s., where ω_i, $1 \le i \le 3$ are given by (5) with

$$\begin{gathered} f_{11} = 2u, \quad f_{21} = -2v, \quad f_{31} = 2\eta, \quad f_{12} = -4\eta u - 2v_x, \\ f_{22} = 4\eta v - 2u_x, \quad f_{32} = -4\eta^2 - 2(u^2+v^2) \end{gathered}$$

and η is a spectral parameter. In fact, one can show that (9) is equivalent to (3) with $\delta = 1$.

Example 3. The HF model—the Schrödinger flow of maps into $S^2 \hookrightarrow R^3$ [6] (or the Landau-Lifschitz equation for an isotropic chain [10]),

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \tag{10}$$

where $\mathbf{S} = (s_1(x,t), s_2(x,t), s_3(x,t)) \in R^3$ with $s_1^2 + s_2^2 + s_3^2 = 1$, and $\times$ denotes the cross product, is a differential system for two independent functions describing s.s., where ω_i, $1 \le i \le 3$ are given by (5) with

$$f_{11} = 2\eta s_1, \quad f_{21} = -2\eta s_3, \quad f_{31} = 2\eta s_2, \quad f_{12} = 4\eta^2 s_1 + 2\eta s_2 s_{3x} - 2\eta s_3 s_{2x},$$
$$f_{22} = -4\eta^2 s_3 + 2\eta s_2 s_{1x} - 2\eta s_1 s_{2x}, \quad f_{32} = 4\eta^2 s_2 - 2\eta s_1 s_{3x} + 2\eta s_3 s_{1x}$$

and η is a spectral parameter. It is a straightforward computation to show that (10) is equivalent to (3) with $\delta = -1$.

Example 4. The M-HF model—the Schrödinger flow of maps into $H^2 \hookrightarrow R^{2+1}$ (see [6]),

$$\mathbf{S}_t = \mathbf{S} \dot{\times} \mathbf{S}_{xx}, \tag{11}$$

where $\mathbf{S} = (s_1(x,t), s_2(x,t), s_3(x,t)) \in R^{2+1}$ satisfies $s_1^2 + s_2^2 - s_3^2 = -1$ with $s_3 > 0$, and $\dot{\times}$ denotes the pseudo cross product, i.e. for arbitrary two vectors $\mathbf{a}, \mathbf{b} \in R^{2+1}$,

$$\mathbf{a} \dot{\times} \mathbf{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, -(a_1 b_2 - a_2 b_1)),$$

is a differential system for two independent functions describing p.s.s.. In fact, we consider ω_i, $1 \le i \le 3$, given by (5) with

$$f_{11} = 2\eta s_2, \quad f_{21} = 2\eta s_1, \quad f_{31} = -2\eta s_3, \quad f_{12} = 4\eta^2 s_2 - 2\eta s_1 s_{3x} + 2\eta s_3 s_{1x},$$
$$f_{22} = 4\eta^2 s_1 + 2\eta s_2 s_{3x} - 2\eta s_3 s_{2x}, \quad f_{32} = -4\eta^2 s_3 - 2\eta s_2 s_{1x} + 2\eta s_1 s_{2x}$$

and η is a spectral parameter. Then (11) is equivalent to (3) with $\delta = 1$.

Example 5.

$$u_t = (2up)_x + 4vp, \quad v_t = (2vp)_x - 2up, \tag{12}$$

where $p = p(u,v) \neq 0$ is an arbitrary function. This is a new family of differential systems describing s.s. with 1-forms ω_i, $1 \le i \le 3$, given by (5), where

$$f_{11} = u, \quad f_{21} = 1 + v, \quad f_{31} = -1 + v,$$
$$f_{12} = 2up, \quad f_{22} = 2vp, \quad f_{32} = 2vp.$$

Example 6.

$$u_t = 2h, \quad v_t = h_x - uh, \tag{13}$$

where $h = h(u,v) \neq 0$ is an arbitrary function. This new family of differential systems describes p.s.s. by considering $\omega_i = f_{i1} dx + f_{i2} dt$ with

$$f_{11} = u, \quad f_{21} = 1 + v, \quad f_{31} = -1 + v,$$
$$f_{12} = 0, \quad f_{22} = h, \quad f_{32} = h.$$

The examples above show that it is important to study the general theory of evolution systems (2) which describe pseudospherical surfaces or spherical surfaces. For this purpose, we begin with the following useful lemma.

Lemma 1 *The necessary and sufficient conditions for an evolution system*

$$\left\{\begin{array}{l} z_{0t} = F(z_0, \cdots, z_k, y_0, \cdots, y_r), \\ y_{0t} = G(z_0, \cdots, z_k, y_0, \cdots, y_r). \end{array}\right. \tag{14}$$

to describe a pseudospherical (resp. spherical) surface, with ω_i given by (5), are

$$\begin{aligned} & f_{i1,z_j} = 0, \quad f_{i1,y_l} = 0, \quad f_{i2,z_k} = 0, \quad f_{i2,y_r} = 0, \\ & \qquad 1 \le i \le 3, \quad 1 \le j \le k, \quad 1 \le l \le r, \end{aligned} \tag{15}$$

$$\left| \begin{array}{cc} f_{11,z_0} & f_{11,y_0} \\ f_{21,z_0} & f_{21,y_0} \end{array} \right|^2 + \left| \begin{array}{cc} f_{21,z_0} & f_{21,y_0} \\ f_{31,z_0} & f_{31,y_0} \end{array} \right|^2 + \left| \begin{array}{cc} f_{31,z_0} & f_{31,y_0} \\ f_{11,z_0} & f_{11,y_0} \end{array} \right|^2 \neq 0, \tag{16}$$

$$-f_{11,z_0}F - f_{11,y_0}G + B_{12} + D_{12} + f_{21}f_{32} - f_{22}f_{31} = 0, \tag{17}$$

$$-f_{21,z_0}F - f_{21,y_0}G + B_{22} + D_{22} - f_{11}f_{32} + f_{12}f_{31} = 0, \tag{18}$$

$$-f_{31,z_0}F - f_{31,y_0}G + B_{32} + D_{32} - \delta(f_{11}f_{22} - f_{12}f_{21}) = 0, \tag{19}$$

where

$$B_{j2} = \sum_{l=0}^{k-1} f_{j2,z_l} z_{l+1}, \quad D_{j2} = \sum_{l=0}^{r-1} f_{j2,y_l} y_{l+1}, \quad 1 \le j \le 3 \tag{20}$$

and $\delta = 1$ (resp. $\delta = -1$).

Proof. From the structure equations (3), we know that

$$\begin{aligned} & \sum_{i=0}^{k} f_{11,z_i} dz_i \wedge dx + \sum_{i=0}^{r} f_{11,y_i} dy_i \wedge dx + \sum_{i=0}^{k} f_{12,z_i} dz_i \wedge dt + \sum_{i=0}^{r} f_{12,y_i} dy_i \wedge dt \\ & \qquad + (f_{21}f_{32} - f_{22}f_{31}) dx \wedge dt = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} & \sum_{i=0}^{k} f_{21,z_i} dz_i \wedge dx + \sum_{i=0}^{r} f_{21,y_i} dy_i \wedge dx + \sum_{i=0}^{k} f_{22,z_i} dz_i \wedge dt + \sum_{i=0}^{r} f_{22,y_i} dy_i \wedge dt \\ & \qquad + (-f_{11}f_{32} + f_{12}f_{31}) dx \wedge dt = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} & \sum_{i=0}^{k} f_{31,z_i} dz_i \wedge dx + \sum_{i=0}^{r} f_{31,y_i} dy_i \wedge dx + \sum_{i=0}^{k} f_{32,z_i} dz_i \wedge dt + \sum_{i=0}^{r} f_{32,y_i} dy_i \wedge dt \\ & \qquad - \delta(f_{11}f_{22} - f_{12}f_{21}) dx \wedge dt = 0. \end{aligned} \tag{23}$$

Equations (15) and (17)-(19) are obtained just by substituting equation (7) into (21-23). Finally, the constraint (16) has to be satisfied, otherwise, system (14) could not be necessary and sufficient for (21-23) to hold. □

We see from (16) that all three determinants

$$Z_1 = \left| \begin{array}{cc} f_{21,z_0} & f_{21,y_0} \\ f_{31,z_0} & f_{31,y_0} \end{array} \right|, \quad Z_2 = \left| \begin{array}{cc} f_{31,z_0} & f_{31,y_0} \\ f_{11,z_0} & f_{11,y_0} \end{array} \right|, \quad Z_3 = \left| \begin{array}{cc} f_{11,z_0} & f_{11,y_0} \\ f_{21,z_0} & f_{21,y_0} \end{array} \right| \tag{24}$$

cannot be identically zero. Therefore there will be three cases in our classification of evolution systems describing p.s.s. and s.s., respectively. We shall state the results for all three cases for p.s.s. (resp. s.s.) and give the proofs only for the first case, since the proofs of the remaining cases are identical modulo sign changes and permutation of indices in the calculations.

Theorem 1 *Let $f_{ij}, 1 \leq i \leq 3, 1 \leq j \leq 2$ be smooth functions of $z_0, \cdots, z_k$ and $y_0, \cdots, y_r$ satisfying (15) and let Z_i be the functions defined by (24). If $Z_3 \neq 0$, then an evolution system (14) describes p.s.s. (resp. s.s.) with associated 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ $(1 \leq i \leq 3)$, if and only if,*

$$\begin{aligned} z_{0t} = & \frac{1}{Z_3}[f_{21,y_0}(B_{12}+D_{12}) - f_{11,y_0}(B_{22}+D_{22}) - f_{31}(f_{22}f_{21,y_0} + f_{12}f_{11,y_0}) \\ & + f_{32}(f_{21}f_{21,y_0} + f_{11}f_{11,y_0})], \end{aligned} \tag{25}$$

$$\begin{aligned} y_{0t} = & \frac{1}{Z_3}[-f_{21,z_0}(B_{12}+D_{12}) + f_{11,z_0}(B_{22}+D_{22}) + f_{31}(f_{22}f_{21,z_0} + f_{12}f_{11,z_0}) \\ & - f_{32}(f_{21}f_{21,z_0} + f_{11}f_{11,z_0})], \end{aligned} \tag{26}$$

where B_{i2}, D_{i2} are given by (20), $\delta = 1$ (resp. $\delta = -1$) and f_{ij} satisfy the equation

$$\begin{aligned} & Z_1(B_{12}+D_{12}) + Z_2(B_{22}+D_{22}) + Z_3(B_{32}+D_{32}) + Z_1(f_{21}f_{32} - f_{22}f_{31}) \\ & + Z_2(-f_{11}f_{32} + f_{12}f_{31}) - \delta Z_3(f_{11}f_{22} - f_{12}f_{21}) = 0. \end{aligned} \tag{27}$$

Proof. Since $Z_3 \neq 0$, the system of equations (17-19) is rewritten in the following equivalent form,

$$\begin{aligned} & Z_3F - [f_{21,y_0}(B_{12}+D_{12}) - f_{11,y_0}(B_{22}+D_{22}) - f_{31}(f_{22}f_{21,y_0} \\ & + f_{12}f_{11,y_0}) + f_{32}(f_{21}f_{21,y_0} + f_{11}f_{11,y_0})] = 0, \end{aligned} \tag{28}$$

$$\begin{aligned} & Z_3G - [-f_{21,z_0}(B_{12}+D_{12}) + f_{11,z_0}(B_{22}+D_{22}) + f_{31}(f_{22}f_{21,z_0} \\ & + f_{12}f_{11,z_0)}) - f_{32}(f_{21}f_{21,y_0} + f_{11}f_{11,y_0})] = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} & Z_1(B_{12}+D_{12}) + Z_2(B_{22}+D_{22}) + Z_3(B_{32}+D_{32}) + Z_1(f_{21}f_{32} - f_{22}f_{31}) \\ & + Z_2(-f_{11}f_{32} + f_{12}f_{31}) - \delta Z_3(f_{11}f_{22} - f_{12}f_{21}) = 0. \end{aligned} \tag{30}$$

Then the conclusion follows from (28-30) easily.

The converse is a straightforward computation. □

Theorem 2 *Let $f_{ij}, 1 \leq i \leq 3, 1 \leq j \leq 2$ be smooth functions of $z_0, \cdots, z_k$ and $y_0, \cdots, y_r$ satisfying (15) and let Z_i be the functions defined by (24). If $Z_2 \neq 0$, then an evolution system (14) describes p.s.s. (resp. s.s.), with associated 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ $(1 \leq i \leq 3)$, if and only if*

$$\begin{aligned} z_{0t} = & \frac{1}{Z_2}[f_{11,y_0}(B_{32}+D_{32}) - f_{31,y_0}(B_{12}+D_{12}) + f_{22}(-\delta f_{11}f_{11,y_0} + f_{31}f_{31,y_0}) \\ & + f_{21}(\delta f_{12}f_{11,y_0} - f_{32}f_{31,y_0})], \end{aligned} \tag{31}$$

$$\begin{aligned} y_{0t} = & \frac{1}{Z_2}[-f_{11,z_0}(B_{32}+D_{32}) + f_{31,z_0}(B_{12}+D_{12}) + f_{22}(\delta f_{11}f_{11,z_0} - f_{31}f_{31,z_0}) \\ & + f_{21}(-\delta f_{12}f_{11,z_0} + f_{32}f_{31,z_0})], \end{aligned} \tag{32}$$

where B_{i2}, D_{i2} are given by (20), $\delta = 1$ (resp. $\delta = -1$) and f_{ij} satisfy the equation

$$\begin{aligned} & Z_1(B_{12}+D_{12}) + Z_2(B_{22}+D_{22}) + Z_3(B_{32}+D_{32}) + Z_1(f_{21}f_{32} - f_{22}f_{31}) \\ & + Z_2(-f_{11}f_{32} + f_{12}f_{31}) - \delta Z_3(f_{11}f_{22} - f_{12}f_{21}) = 0. \end{aligned}$$

Theorem 3 *Let $f_{ij}, 1 \leq i \leq 3, 1 \leq j \leq 2$ be smooth functions of $z_0, \cdots, z_k$ and $y_0, \cdots, y_r$ satisfying (15) and let Z_i be the functions defined by (24). If $Z_1 \neq 0$, then an evolution system (14) describes p.s.s. (resp. s.s.), with associated 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ $(1 \leq i \leq 3)$, if and only if,*

$$\begin{aligned} z_{0t} = \frac{1}{Z_1}[&f_{31,y_0}(B_{22}+D_{22}) - f_{21,y_0}(B_{32}+D_{32}) - f_{11}(f_{32}f_{31,y_0} - \delta f_{22}f_{21,y_0}) \\ &+f_{12}(f_{31}f_{31,y_0} - \delta f_{21}f_{21,y_0})], \end{aligned} \tag{33}$$

$$\begin{aligned} y_{0t} = \frac{1}{Z_1}[&-f_{31,z_0}(B_{22}+D_{22}) + f_{21,z_0}(B_{32}+D_{32}) + f_{11}(f_{32}f_{31,z_0} - \delta f_{22}f_{21,z_0}) \\ &-f_{12}(f_{31}f_{31,z_0} + \delta f_{21}f_{21,z_0})], \end{aligned} \tag{34}$$

where B_{i2}, D_{i2} are given by (20), $\delta = 1$ (resp. $\delta = -1$) and f_{ij} satisfy the equation

$$\begin{aligned} &Z_1(B_{12}+D_{12}) + Z_2(B_{22}+D_{22}) + Z_3(B_{32}+D_{32}) + Z_1(f_{21}f_{32} - f_{22}f_{31}) \\ &+Z_2(-f_{11}f_{32} + f_{12}f_{31}) - \delta Z_3(f_{11}f_{22} - f_{12}f_{21}) = 0. \end{aligned}$$

We conclude this section by considering the Landau-Lifschitz equation, which has attracted many authors' attention in recent decades (see [7,9,13-15]) as a generalized form of the HF model. We will show that it has the same geometric feature as the HF model described in Example 3.

Example 7. The Landau-Lifschitz equation, for a spin chain with an easy plane, is given by

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times J\mathbf{S} \tag{35}$$

where $\mathbf{S} = (s_1(x,t), s_2(x,t), s_3(x,t)) \in R^3$ satisfies $s_1^2 + s_2^2 + s_3^2 = 1$, and J is a diagonal matrix,

$$J = \mathrm{diag}(0, 0, -\rho^2)$$

which characterizes the easy plane, ρ is a positive constant. The Lax pair for this equation is given by (see [10])

$$\Omega_1 = \begin{pmatrix} -i\mu s_3 & \eta s \\ -\eta\bar{s} & i\mu s_3 \end{pmatrix}, \quad s = s_1 - is_2,$$

$$\Omega_2 = \begin{pmatrix} i2\eta^2 s_3 - i\mu(s_1 s_{2x} - s_2 s_{1x}) & -2\eta\mu s + i\eta(s s_{3x} - s_3 s_x) \\ 2\eta\mu\bar{s} + i\eta(\bar{s}s_{3x} - s_3\bar{s}_x) & -i2\eta^2 s_3 + i\mu(s_1 s_{2x} - s_2 s_{1x}) \end{pmatrix},$$

where the parameters μ and η satisfy $\mu^2 = \eta^2 + \rho^2$. Note that (35) is a differential system for two independent functions, say s_1 and s_2. This differential system describes spherical surfaces, where ω_i, $1 \leq i \leq 3$ are given by (5) with

$$\begin{aligned} &f_{11} = 2\eta s_1, f_{21} = -2\mu s_3, f_{31} = -2\eta s_2, f_{12} = -4\eta\mu s_1 + 2\eta s_2 s_{3x} - 2\eta s_3 s_{2x}, \\ &f_{22} = 4\eta^2 s_3 - 2\mu s_1 s_{2x} + 2\mu s_2 s_{1x}, \quad f_{32} = 4\eta\mu s_2 + 2\eta s_1 s_{3x} - 2\eta s_3 s_{1x}. \end{aligned}$$

It is easy to see that, when $\rho = 0$, these $\omega_i (1 \leq i \leq 3)$ reduce just to those of the HF model mentioned in Example 3 by fixing $\mu \to -\eta$ and $s_1 \to s_1, s_2 \to -s_2, s_3 \to -s_3$.

Example 8. Consider the Landau-Lifschitz equation (35) for the spin chain with complete anisotropic, where $J = \mathrm{diag}(J_1, J_2, J_3)$ is a diagonal matrix, (say, $J_3 > J_2 > J_1$). The Lax pair is given by (see [15])

$$\begin{array}{c}\Omega_1 = -i\sum_\alpha w_\alpha(\eta)s_\alpha\sigma_\alpha, \\ \Omega_2 = -i\sum_{\alpha,\beta,\gamma} w_\alpha(\eta)s_\beta s_{\gamma_x}\sigma_\alpha\varepsilon^{\alpha\beta\gamma} + i\sum_{\alpha,\beta,\gamma} w_\beta(\eta)w_\gamma(\eta)s_\alpha\sigma_\alpha|\varepsilon^{\alpha\beta\gamma}|,\end{array} \tag{36}$$

where σ_α $(\alpha = 1, 2, 3)$ are Pauli matrices, $\varepsilon^{\alpha\beta\gamma}$ is the completely antisymmetric tensor and $w_\alpha(\eta)$ are elliptic functions in the rectangle $R = \{\eta : |\mathrm{Re}\eta| \leq 2k, |\mathrm{Im}\eta| \leq 2k'\}$,

$$w_1(\eta) = \frac{\rho}{\mathrm{sn}(\eta, k)}, \quad w_2(\eta) = \frac{\rho\,\mathrm{dn}(\eta, k)}{\mathrm{sn}(\eta, k)}, \quad w_3(\eta) = \frac{\rho\,\mathrm{cn}(\eta, k)}{\mathrm{sn}(\eta, k)},$$

$$k = \sqrt{\frac{J_2 - J_1}{J_3 - J_1}}, \quad \rho = \frac{1}{2}\sqrt{J_3 - J_1}.$$

One can see that this differential system describes spherical surfaces, where ω_i, $1 \leq i \leq 3$, are given by (5) with

$$\begin{array}{c}f_{11} = 2w_1(\eta)s_1, \quad f_{21} = -2w_3(\eta)s_3, \quad f_{31} = -2w_2(\eta)s_2, \\ f_{12} = -4w_2(\eta)w_3(\eta)s_1 + 2w_1(\eta)(s_2s_{3x} - s_3s_{2x}), \\ f_{22} = 4w_1(\eta)w_2(\eta)s_3 - 2w_3(\eta)(s_1s_{2x} - s_2s_{1x}), \\ f_{32} = 4w_3(\eta)w_1(\eta)s_2 - 2w_2(\eta)(s_3s_{1x} - s_1s_{3x}).\end{array}$$

Remark 1 Similarly, we can define the Minkowski-Landau-Lifschitz equation from the Schrödinger flow of maps into H^2 (see [6]) as follows,

$$\mathbf{S}_t = \mathbf{S}\dot{\times}\mathbf{S}_{xx} + \mathbf{S}\dot{\times}J\mathbf{S} \tag{37}$$

where $\mathbf{S} = (s_1(x,t), s_2(x,t), s_3(x,t)) \in R^{2+1}$ satisfying $s_1^2 + s_2^2 - s_3^2 = -1$ with $s_3 > 0$ and J is a diagonal matrix with $J_3 \geq J_2 \geq J_1$.

The geometric explanations for some dynamical properties such as the existence of infinite number of conservation laws and Bäcklund transformations of the systems might also be discussed in a similar way to [4,20]. However, we would like to pay special attention to classifying certain type of differential systems describing η-p.s.s. and η-s.s. in the following section.

3. Theorems of Classification

Motivated by the AKNS system, the notion of a differential equation describing η-p.s.s. was introduced by Chern and Tenenblat in [4]. We say that a differential equation *describes* an η-p.s.s., if it describes a p.s.s. with $f_{21} = \eta$. For such a differential equation, there is a one-parameter (η) family of linear problems for which the equation is the integrability condition and one may hope to solve it by inverse scattering techniques (see [1,2,4]). Similarly, we should have the same notion for a differential system describing η-p.s.s. or η-s.s.. That is, a differential system *describes* an η-p.s.s. (resp. η-s.s.) if it describes a p.s.s. (resp. s.s.) with either $f_{11} = \eta$, or $f_{21} = \eta$, or $f_{31} = \eta$. We see from Example 1 and 2 that the NLS$^+$ describes η-s.s. and the NLS$^-$ describes η-p.s.s. respectively. By the transformation $\widetilde{\omega}_1 = \omega_2, \widetilde{\omega}_2 = \omega_1$ and $\widetilde{\omega}_3 = -\omega_3$, we can transfer the case of $f_{11} = \eta$ into the case of $f_{21} = \eta$. So there are essentially two classes of η-p.s.s. (resp. η-s.s.). One class, which we call class one, with $f_{31} = \eta$. Another class, which we call class two, with $f_{21} = \eta$. We also observe that the property of describing an η-p.s.s. (resp. η-s.s.) is not preserved by a change of independent variables in general, although the property of describing a p.s.s. (resp. s.s.) is preserved under such change of variables. Moreover, it does not exclude the possibility of the differential systems and their corresponding f_{ij} to depend on η. Hence we are interested in differential systems describing η-p.s.s. (resp. η-s.s.) which are independent of the parameter η, although their corresponding f_{ij} may depend on η. One of the important problems in this respect is, of course, to determine all such differential systems of certain type describing η-p.s.s (resp. η-s.s.), and this is what will be done in this section.

A series of classification theorems of scalar differential equations of certain type describing η-p.s.s. was given in [4,17,18]. In this section, motivated by the nonlinear Schrödinger equation in Examples 1 and 2, we shall first classify all such differential systems of the form,

$$\left\{ \begin{array}{l} u_t = -v_{xx} + H_{11}(u,v)u_x + H_{12}(u,v)v_x + H_{13}(u,v), \\ v_t = u_{xx} + H_{21}(u,v)u_x + H_{22}(u,v)v_x + H_{23}(u,v), \end{array} \right. \tag{38}$$

where H_{ij} $(1 \le i \le 2, 1 \le j \le 3)$ are some functions of u and v.

Theorem 4 *Up to a transformation of type: $u + \gamma_3 \to u$, $v + \gamma_4 \to v$ for some real constants γ_3 and γ_4, all the differential systems of the form (38), which describe η-p.s.s. (resp. η-s.s.) of class one (i.e., with $f_{31} = \eta$), are*

$$\left\{ \begin{array}{l} u_t = -v_{xx} + \delta\gamma[(u^2+v^2)u]_x + \alpha u_x + \delta\sigma(u^2+v^2)v + \beta v, \\ v_t = u_{xx} + \delta\gamma[(u^2+v^2)v]_x + \alpha v_x - \delta\sigma(u^2+v^2)u - \beta u, \end{array} \right. \tag{39}$$

where $\delta = 1$ (resp. $\delta = -1$), α, β, γ and σ are constants such that $\sigma \ge 0$ if $\gamma = 0$. Moreover, (39) is the integrability condition of the linear problem

$$\phi_x = \Omega_1 \phi, \quad \phi_t = \Omega_2 \phi,$$

where

$$\Omega_1 = \frac{1}{2} \begin{pmatrix} Qv & Qu - \eta \\ Qu + \eta & -Qv \end{pmatrix},$$

$$\Omega_2 = \frac{1}{2} \begin{pmatrix} Q(Sv + u_x) & Q(Su - v_x) - \eta S - \beta - \sigma(u^2+v^2) \\ Q(Su - v_x) + \eta S + \beta + \sigma(u^2+v^2) & -Q(Sv + u_x) \end{pmatrix}$$

if $\delta = 1$, and

$$\Omega_1 = \frac{1}{2} \begin{pmatrix} iQv & Qu + i\eta \\ -Qu + i\eta & -iQv \end{pmatrix},$$

$$\Omega_2 = \frac{1}{2} \begin{pmatrix} iQ(u_x + Sv) & A + iB \\ -A + iB & -iQ(u_x + Sv) \end{pmatrix}$$

if $\delta = -1$, in which $Q = \sqrt{2(\gamma u + \sigma)}$, $S = \eta + \alpha + \delta\gamma(u^2+v^2)$, $A = Q(-v_x + Su)$ and $B = \eta S + \beta - \sigma(u^2+v^2)$.

Proof. We shall use the notation introduced in Section **2**. For any differential system of type (2) with $k = r = 2$ describing η-p.s.s. (resp. η-s.s.) of class one, it follows from Lemma 1 that $f_{i1} = f_{i1}(\eta, z_0, y_0)$ $(i = 1, 2)$, $f_{31} = \eta$ and $f_{i2} = f_{i2}(\eta, z_0, z_1, y_0, y_1)$ $(i = 1, 2, 3)$, which implies that $Z_1 = Z_2 = 0$ and $Z_3 \neq 0$. Hence, we have

$$B_{i2} = f_{i2_{z_0}} z_1 + f_{i2_{z_1}} z_2, \quad D_{i2} = f_{i2_{y_0}} y_1 + f_{i2_{y_1}} y_2. \tag{40}$$

Now equations (25-27) in Theorem 1 can be expressed, with aid of (40), as follows,

$$\begin{aligned}
z_{0t} &= \frac{1}{Z_3}[(f_{21_{y_0}} f_{12_{z_1}} - f_{11_{y_0}} f_{22_{z_1}}) z_2 + (f_{21_{y_0}} f_{12_{y_1}} - f_{11_{y_0}} f_{22_{y_1}}) y_2 \\
&\quad + (f_{21_{y_0}} f_{12_{z_0}} - f_{11_{y_0}} f_{22_{z_0}}) z_1 + (f_{21_{y_0}} f_{12_{y_0}} - f_{11_{y_0}} f_{22_{y_0}}) y_1 \\
&\quad - (f_{22} f_{21_{y_0}} + f_{12} f_{11_{y_0}}) f_{31} + (f_{21} f_{21_{y_0}} + f_{11} f_{11_{y_0}}) f_{32}], \qquad (41) \\
y_{0t} &= \frac{1}{Z_3}[(-f_{21_{z_0}} f_{12_{z_1}} + f_{11_{z_0}} f_{22_{z_1}}) z_2 + (-f_{21_{z_0}} f_{12_{y_1}} + f_{11_{z_0}} f_{22_{y_1}}) y_2 \\
&\quad + (-f_{21_{z_0}} f_{12_{z_0}} + f_{11_{z_0}} f_{22_{z_0}}) z_1 + (-f_{21_{z_0}} f_{12_{y_0}} + f_{11_{z_0}} f_{22_{y_0}}) y_1 \\
&\quad + (f_{22} f_{21_{z_0}} + f_{12} f_{11_{z_0}}) f_{31} - (f_{21} f_{21_{z_0}} + f_{11} f_{11_{z_0}}) f_{32}], \qquad (42) \\
0 &= f_{32_{z_1}} z_2 + f_{32_{y_1}} y_2 + f_{32_{z_0}} z_1 + f_{32_{y_0}} y_1 + \delta(-f_{11} f_{22} + f_{21} f_{12}). \qquad (43)
\end{aligned}$$

The requirement of the differential systems to be of the form (38), implies that

$$\begin{cases} f_{21_{y_0}} f_{12_{z_1}} - f_{11_{y_0}} f_{22_{z_1}} = 0 \\ -f_{21_{z_0}} f_{12_{z_1}} + f_{11_{z_0}} f_{22_{z_1}} = Z_3, \end{cases} \qquad \begin{cases} f_{21_{y_0}} f_{12_{y_1}} - f_{11_{y_0}} f_{22_{y_1}} = -Z_3 \\ -f_{21_{z_0}} f_{12_{y_1}} + f_{11_{z_0}} f_{22_{y_1}} = 0. \end{cases}$$

Since $Z_3 \neq 0$, the unique solutions of the above equations are,

$$f_{12_{z_1}} = f_{11_{y_0}}, \quad f_{22_{z_1}} = f_{21_{y_0}}, \tag{44}$$

$$f_{12_{y_1}} = -f_{11_{z_0}}, \quad f_{22_{y_1}} = -f_{21_{z_0}}. \tag{45}$$

From (43), we obtain

$$f_{32_{z_1}} = 0, \quad f_{32_{y_1}} = 0. \tag{46}$$

Thus, it follows from (44-46), that

$$\begin{cases} f_{12} = f_{11_{y_0}} z_1 - f_{11_{z_0}} y_1 + l_{12} \\ f_{22} = f_{21_{y_0}} z_1 - f_{21_{z_0}} y_1 + l_{22} \\ f_{32} = l_{32} \end{cases} \tag{47}$$

for some functions $l_{i2} = l_{i2}(\eta, z_0, y_0)$ $(1 \leq i \leq 3)$ which depend only on η, z_0 and y_0. From (47) and (43), we see that

$$\begin{cases} l_{32_{z_0}} = \delta(f_{11} f_{21_{y_0}} - f_{21} f_{11_{y_0}}) \\ l_{32_{y_0}} = -\delta(f_{11} f_{21_{z_0}} - f_{21} f_{11_{z_0}}) \end{cases} \tag{48}$$

and

$$f_{11} l_{22} = f_{21} l_{12}. \tag{49}$$

The integrability condition of (48) yields

$$f_{11}(f_{21 z_0 z_0} + f_{21 y_0 y_0}) = f_{21}(f_{11 z_0 z_0} + f_{11 y_0 y_0}). \tag{50}$$

It follows from (47) and (41,42), since all the coefficients of z_1^2, $z_1 y_1$ and y_1^2 in (41) and (42) must vanish, that

$$\begin{cases} f_{21 y_0} f_{11 z_0 y_0} - f_{11 y_0} f_{21 z_0 y_0} = 0 \\ -f_{21 z_0} f_{11 z_0 y_0} + f_{11 z_0} f_{21 z_0 y_0} = 0, \end{cases}$$

$$\begin{cases} -f_{21 y_0}(f_{11 z_0 z_0} - f_{11 y_0 y_0}) + f_{11 y_0}(f_{21 z_0 z_0} - f_{21 y_0 y_0}) = 0 \\ f_{21 z_0}(f_{11 z_0 z_0} - f_{11 y_0 y_0}) - f_{11 z_0}(f_{21 z_0 z_0} - f_{21 y_0 y_0}) = 0. \end{cases}$$

The above equations have the following unique solutions:

$$\begin{aligned} &f_{11 z_0 y_0} = 0, \quad f_{21 z_0 y_0} = 0, \\ &f_{11 z_0 z_0} - f_{11 y_0 y_0} = 0, \quad f_{21 z_0 z_0} - f_{21 y_0 y_0} = 0. \end{aligned}$$

Therefore, we obtain

$$\begin{cases} f_{11} = a z_0^2 + a y_0^2 + a_1 z_0 + b_1 y_0 + c_1, \\ f_{21} = b z_0^2 + b y_0^2 + a_2 z_0 + b_2 y_0 + c_2, \end{cases}$$

where a_1, a_2, b_1, b_2, c_1 and c_2 are some constants depending on η. It is not difficult to see from (50), that $a = b = 0$. Therefore

$$f_{11} = a_1 z_0 + b_1 y_0 + c_1, \quad f_{21} = a_2 z_0 + b_2 y_0 + c_2, \tag{51}$$

and from (47) and (48) we have

$$f_{12} = b_1 z_1 - a_1 y_1 + l_{12}, \quad f_{22} = b_2 z_1 - a_2 y_1 + l_{22} \tag{52}$$

and

$$l_{32} = \delta[\frac{1}{2}(a_1 b_2 - a_2 b_1)(z_0^2 + y_0^2) + (c_1 b_2 - c_2 b_1) z_0 + (a_1 c_2 - a_2 c_1) y_0] + C(\eta) \tag{53}$$

for some constant $C(\eta)$ depending on η.

We must require that all the coefficients in z_1, y_1 and the constant terms in the right handside of (41) and (42) are independent of η, i.e., the following expressions

$$\frac{1}{Z_3}\left[f_{21 y_0} l_{12 z_0} - f_{11 y_0} l_{22 z_0} - \eta(f_{21 y_0} f_{21 y_0} + f_{11 y_0} f_{11 y_0})\right], \tag{54}$$

$$\frac{1}{Z_3}\left[f_{21 y_0} l_{12 y_0} - f_{11 y_0} l_{22 y_0} + \eta(f_{21 z_0} f_{21 y_0} + f_{11 z_0} f_{11 y_0})\right], \tag{55}$$

$$\frac{1}{Z_3}\left[-f_{21 z_0} l_{12 z_0} + f_{11 z_0} l_{22 z_0} + \eta(f_{21 y_0} f_{21 z_0} + f_{11 y_0} f_{11 z_0})\right], \tag{56}$$

$$\frac{1}{Z_3}\left[-f_{21 z_0} l_{12 y_0} + f_{11 z_0} l_{22 y_0} - \eta(f_{21 z_0} f_{21 z_0} + f_{11 z_0} f_{11 z_0})\right], \tag{57}$$

$$\frac{1}{Z_3}\left[-(l_{22} f_{21 y_0} + l_{12} f_{11 y_0})\eta + (f_{21} f_{21 y_0} + f_{11} f_{11 y_0}) l_{32}\right], \tag{58}$$

$$\frac{1}{Z_3}\left[(l_{22} f_{21 z_0} + l_{12} f_{11 z_0})\eta - (f_{21} f_{21 z_0} + f_{11} f_{11 z_0}) l_{32}\right] \tag{59}$$

do not depend on η. In what follows, we shall use these facts to determine f_{2j} ($j = 1,2,3$) explicitly.

First of all, note that (49) is equivalent to the existence of a function $p = p(\eta, z_0, y_0)$ (since $f_{11}f_{21} \neq 0$) such that

$$l_{12} = pf_{11}, \quad l_{22} = pf_{21}. \tag{60}$$

It follows from (60), that (58) and (59) are equivalent to saying that

$$(-\eta p + l_{32})\frac{(a_1b_1 + a_2b_2)z_0 + (b_2^2 + b_1^2)y_0 + c_2b_2 + c_1b_1}{a_1b_2 - a_2b_1} \tag{61}$$

and

$$(\eta p - l_{32})\frac{(a_1^2 + a_2^2)z_0 + (a_1b_1 + a_2b_2)y_0 + c_1a_1 + c_2a_2}{a_1b_2 - a_2b_1}$$

are independent of η respectively, which implies that

$$\frac{(a_1b_1 + a_2b_2)z_0 + (b_2^2 + b_1^2)y_0 + c_2b_2 + c_1b_1}{(a_1^2 + a_2^2)z_0 + (a_1b_1 + a_2b_2)y_0 + c_1a_1 + c_2a_2}$$

is independent of η. Let the first derivative of the above expression with respect to η be 0. Then we have

$$a_1b_1 + a_2b_2 = \gamma_1(a_1^2 + a_2^2), \tag{62}$$
$$b_1^2 + b_2^2 = \gamma_2(a_1^2 + a_2^2), \tag{63}$$
$$c_1a_1 + c_2a_2 = \gamma_3(a_1^2 + a_2^2), \tag{64}$$
$$c_1b_1 + c_2b_2 = \gamma_4(a_1^2 + a_2^2), \tag{65}$$

where γ_i $(1 \le i \le 4)$ are constants which do not depend on η.

Next, by using the expressions (51) for f_{11} and f_{21}, into Eqs. (54-57) we conclude that the following expressions

$$\begin{aligned}
&\frac{1}{a_1b_2 - a_2b_1}\left[b_2l_{12z_0} - b_1l_{22z_0} - \eta(b_2^2 + b_1^2)\right],\\
&\frac{1}{a_1b_2 - a_2b_1}\left[b_2l_{12y_0} - b_1l_{22y_0} + \eta(a_1b_1 + a_2b_2)\right],\\
&\frac{1}{a_1b_2 - a_2b_1}\left[-a_2l_{12z_0} + a_1l_{22z_0} + \eta(a_1b_1 + a_2b_2)\right],\\
&\frac{1}{a_1b_2 - a_2b_1}\left[-a_2l_{12y_0} + a_1l_{22y_0} - \eta(a_1^2 + a_2^2)\right],
\end{aligned}$$

are functions of z_0 and y_0 which are independent of η. Therefore, we have

$$l_{12} = (b_2z_0 - a_2y_0)\eta + b_1h + a_1g + \widetilde{g}(\eta), \tag{66}$$
$$l_{22} = (-b_1z_0 + a_1y_0)\eta + b_2h + a_2g + \widetilde{h}(\eta), \tag{67}$$

where $\widetilde{h}(\eta)$ and $\widetilde{g}(\eta)$ depend on η. Substituting (66) and (67) into (49), we have

$$\begin{aligned}
&[(a_1b_1 + a_2b_2)(z_0^2 - y_0^2) + (b_1^2 + b_2^2 - a_1^2 - a_2^2)z_0y_0\\
&+(c_1b_1 + c_2b_2)z_0 - (c_1a_1 + c_2a_2)y_0]\eta\\
=\;&(a_1b_2 - a_2b_1)(z_0h - y_0g) + (c_1b_2 - c_2b_1)h + (c_1a_2 - c_2a_1)g\\
&+(a_1\widetilde{h} - a_2\widetilde{g})z_0 + (b_1\widetilde{h} - b_2\widetilde{g})y_0 - c_2\widetilde{g} + c_1\widetilde{h}.
\end{aligned} \tag{68}$$

Taking the second derivatives with respect to z_0 and y_0 on both sides of (68), we have

$$2(a_1b_1+a_2b_2)\eta = Z_3(z_0h-y_0g)_{z_0z_0}+(c_1b_2-c_2b_1)h_{z_0z_0}+(c_1a_2-c_2a_1)g_{z_0z_0}, \tag{69}$$

$$-2(a_1b_1+a_2b_2)\eta = Z_3(z_0h-y_0g)_{y_0y_0}+(c_1b_2-c_2b_1)h_{y_0y_0}+(c_1a_2-c_2a_1)g_{y_0y_0}, \tag{70}$$

$$(b_1^2+b_2^2-a_1^2-a_2^2)\eta = Z_3(z_0h-y_0g)_{z_0y_0}+(c_1b_2-c_2b_1)h_{z_0y_0}+(c_1a_2-c_2a_1)g_{z_0y_0}. \tag{71}$$

From (62-65) we have

$$\begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} = \begin{pmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{pmatrix}\begin{pmatrix} \gamma_1 & \gamma_3 \\ \gamma_2 & \gamma_4 \end{pmatrix}(a_1^2+a_2^2)/Z_3. \tag{72}$$

Multiplying (72) on the left by the matrix $\begin{pmatrix} -a_2 & a_1 \\ -c_2 & c_1 \end{pmatrix}$, we conclude that

$$\begin{aligned} Z_3 = a_1b_2-a_2b_1 &= (-\gamma_1^2+\gamma_2)(a_1^2+a_2^2)^2/Z_3, && (73)\\ c_1a_2-c_2a_1 &= (\gamma_3\gamma_1-\gamma_4)Z_3/(\gamma_2-\gamma_1^2), && (74)\\ c_1b_2-c_2b_1 &= (\gamma_3\gamma_2-\gamma_4\gamma_1)Z_3/(\gamma_2-\gamma_1^2). && (75)\end{aligned}$$

It follows that $Z_3^2 = (-\gamma_1^2+\gamma_2)(a_1^2+a_2^2)^2$, which implies $\gamma_2-\gamma_1^2>0$. A useful consequence of (72) and (73) is that

$$\eta p - l_{32} \quad \text{is independent of } \eta. \tag{76}$$

This follows from (62-65) and the fact that expression (61) is independent of η.

From (74), (75), (69) and (71), we obtain

$$\begin{aligned} 2\frac{a_1b_1+a_2b_2}{Z_3}\eta &= (z_0h-y_0g)_{z_0z_0}+\frac{\gamma_3\gamma_2-\gamma_4\gamma_1}{\gamma_2-\gamma_1^2}h_{z_0z_0}+\frac{\gamma_3\gamma_1-\gamma_4}{\gamma_2-\gamma_1^2}g_{z_0z_0}, && (77)\\ \frac{b_1^2+b_2^2-a_1^2-a_2^2}{Z_3}\eta &= (z_0h-y_0g)_{z_0y_0}+\frac{\gamma_3\gamma_2-\gamma_4\gamma_1}{\gamma_2-\gamma_1^2}h_{z_0y_0}+\frac{\gamma_3\gamma_1-\gamma_4}{\gamma_2-\gamma_1^2}g_{z_0y_0}. && (78)\end{aligned}$$

Since the right handside of (77) and (78) are independent of η, it follows that

$$\gamma_1 = 0, \quad \gamma_2 = 1. \tag{79}$$

Now we should give a summary of the constant coefficients a_i, b_i and c_i $(i=1,2)$ appearing in (51) before going on with further discussion. Without loss of generality (one may refer to the transformation $\widetilde{\omega}_1=\omega_2$, $\widetilde{\omega}_2=\omega_1$ and $\widetilde{\omega}_3=-\omega_3$ mentioned before), we may assume $a_1\neq 0$ and set $a_2=ca_1$ for some $c=c(\eta)$ which may depend only on η. The first and second equation of (79) yield $b_1=-cb_2$ and $b_2=\delta_0a_1$ respectively, where $\delta_0^2=1$. From (72), we have $c_1=(\gamma_3-\gamma_4c\delta_0)a_1$ and $c_2=(\gamma_3c+\gamma_4\delta_0)a_1$.

Substituting the above relations into (68), after taking the first derivative with respect to η, we have

$$\begin{aligned} \widetilde{h} &= c_2\delta_0\eta+\sigma_1b_2+\sigma_2a_2, && (80)\\ \widetilde{g} &= c_1\delta_0\eta+\sigma_1b_1+\sigma_2a_1 && (81)\end{aligned}$$

for some constants σ_i $(i = 1, 2)$ not depending on η. From (80,81) and (60), with aid of (51) and (66,67), a straightforward computation shows that

$$p = \delta_0\eta + \frac{-\delta_0 c(h+\sigma_1) + g + \sigma_2}{z_0 - \delta_0 c y_0 + \gamma_3 - \gamma_4 c\delta_0} = \delta_0\eta + \frac{\delta_0(h+\sigma_1) + c(g+\sigma_2)}{cz_0 + \delta_0 y_0 + \gamma_3 c + \gamma_4\delta_0}, \tag{82}$$

which yields

$$(z_0 + \gamma_3)(f + \sigma_1) = (y_0 + \gamma_4)(g + \sigma_2). \tag{83}$$

Therefore, we have

$$p = \delta_0\eta + \frac{g + \sigma_2}{z_0 + \gamma_3}. \tag{84}$$

We would like to point out that the second term in the (last equality) right hand side of (84) is independent of η. This is an important fact for us to deduce the explicit expression for f_{ij}. Now, replacing $z_0 + \gamma_3$ by z_0 and $y_0 + \gamma_4$ by y_0 respectively, it follows from (51,52), (53), (66,67) and (84) that

$$f_{11} = (z_0 - \delta_0 c y_0)a_1, \quad f_{21} = (cz_0 + \delta_0 y_0)a_1, \tag{85}$$
$$f_{12} = \left[(\delta_0 z_0 - c y_0)\eta - \delta_0 c(h+\sigma_1) + (g+\sigma_2) - c\delta_0 z_1 - y_1\right] a_1, \tag{86}$$
$$f_{22} = \left[(\delta_0 c z_0 + y_0)\eta + \delta_0(h+\sigma_1) + c(g+\sigma_2) + \delta_0 z_1 - c y_1\right] a_1, \tag{87}$$
$$f_{32} = l_{32} = \frac{\delta}{2}(z_0^2 + y_0^2)\delta_0 a_1^2(1+c^2) + C(\eta), \tag{88}$$
$$l_{12}/f_{11} = l_{21}/f_{22} = p = \delta_0\eta + \Phi, \tag{89}$$

where $\Phi = (g+\sigma_2)/z_0 = (h+\sigma_1)/y_0$ is a function which is independent of η. It is very easy to verify that a differential system of the form (38) is invariant under the transformation $z_0 + \gamma_3 \to z_0$, $y_0 + \gamma_4 \to y_0$ for arbitrary constants γ_3 and γ_4. So there is no loss of generality in considering these transformations. From (74), (88,89), we have

$$\eta p - l_{32} = \delta_0\eta^2 + \Phi\eta - \frac{\delta\delta_0}{2}(z_0^2 + y_0^2)a_1^2(1+c^2) - C(\eta) = \Psi \tag{90}$$

for some function Ψ which is independent of η. The second derivative of (90) with respect to η implies that

$$C(\eta) = \delta_0\eta^2 + \alpha\eta + \beta, \quad \text{and} \quad a_1^2(1+c^2) = \gamma\eta + \sigma,$$

where α, β, γ and σ are constants that are independent of η and $\sigma \geq 0$ if $\gamma = 0$. Hence

$$\Phi = \frac{\delta\delta_0}{2}(z_0^2 + y_0^2)\gamma + \alpha$$
$$\Psi = -\frac{\delta\delta_0}{2}(z_0^2 + y_0^2)\sigma - \beta.$$

The parameter c does not appear in the differential system, therefore for simplicity, we consider $c = 0$ from now on. From the above expressions for $C(\eta)$, a_1, Φ and Ψ and

(84-89), we have

$$f_{11} = (\gamma\eta + \sigma)^{1/2} z_0, \quad f_{21} = \delta_0(\gamma\eta + \sigma)^{1/2} y_0, \quad f_{31} = \eta,$$
$$f_{12} = -(\gamma\eta + \sigma)^{1/2} y_1 + [\delta_0\eta + \delta\frac{\delta_0}{2}(z_0^2 + y_0^2)\gamma + \alpha](\gamma\eta + \sigma)^{1/2} z_0,$$
$$f_{22} = \delta_0(\gamma\eta + \sigma)^{1/2} z_1 + [\delta_0\eta + \delta\frac{\delta_0}{2}(z_0^2 + y_0^2)\gamma + \alpha]\delta_0(\gamma\eta + \sigma)^{1/2} y_0,$$
$$f_{32} = \delta\frac{\delta_0}{2}(z_0^2 + y_0^2)(\gamma\eta + \sigma) + \delta_0\eta^2 + \alpha\eta + \beta.$$

The corresponding differential systems are as follows,

$$\begin{cases} z_{0t} = -y_2 + \delta\frac{\gamma\delta_0}{2}[(z_0^2 + y_0^2)z_0]_x + \alpha z_1 + \delta\frac{\sigma}{2}(z_0^2 + y_0^2)y_0 + \beta\delta_0 y_0 \\ y_{0t} = z_2 + \delta\frac{\gamma\delta_0}{2}[(z_0^2 + y_0^2)y_0]_x + \alpha y_1 - \delta\frac{\sigma}{2}(z_0^2 + y_0^2)z_0 - \beta\delta_0 z_0. \end{cases} \tag{91}$$

Renaming $\frac{\sigma}{2}$ by σ and $\frac{\gamma}{2}$ by γ, since γ and β are arbitrary constants we may assume $\delta_0 = 1$, (91) is reduced to (39). This completes the proof of Theorem 4. □

Theorem 5 *(i). There are no differential systems of the form (38) describing η-p.s.s. of class two (i.e., with $f_{21} = \eta$).*

(ii). Up to a transformation of type: $u + \gamma_3 \to u$, $v + \gamma_4 \to v$ for some real constants γ_3 and γ_4, all the differential systems of the form (38), which describe η-s.s. of class two (i.e., with $f_{21} = \eta$), are

$$\begin{cases} u_t = -v_{xx} + \gamma[(u^2 + v^2)u]_x - \alpha u_x - \sigma(u^2 + v^2)v - \beta v, \\ v_t = u_{xx} + \gamma[(u^2 + v^2)v]_x + \alpha v_x + \sigma(u^2 + v^2)u + \beta u, \end{cases} \tag{92}$$

where α, β, γ and σ are real constants such that $\sigma \geq 0$ if $\gamma = 0$.

Proof. We shall use the notation introduced in Theorem 2 and Theorem 4. Consider a differential system of type (2) with $k = r = 2$ which describes η-p.s.s. (resp. η-s.s.) of class two. It follows from Lemma 1 that $f_{i1} = f_{i1}(\eta, z_0, y_0)$ $(i = 1, 3)$, $f_{21} = \eta$, $f_{i2} = f_{i2}(\eta, z_0, z_1, y_0, y_1)$ $(i = 1, 2, 3)$. Hence the determinants defined in (24) $Z_1 = Z_3 = 0$, $Z_2 \neq 0$ and

$$B_{i2} = f_{i2_{z_0}} z_1 + f_{i2_{z_1}} z_2, \quad D_{i2} = f_{i2_{y_0}} y_1 + f_{i2_{y_1}} y_2.$$

With the same arguments as in the proof of Theorem 4, under the restriction that the differential systems are of the from (38), by using (31), (32) and (27), we similarly have:

$$\begin{aligned} f_{11} &= a_1(\eta)z_0 + b_1(\eta)y_0 + c_1(\eta), \\ f_{21} &= \eta, \\ f_{31} &= a_3(\eta)z_0 + b_3(\eta)y_0 + c_3(\eta), \end{aligned} \tag{93}$$

where $a_i(\eta)$, $b_i(\eta)$ and $c_i(\eta)$ $(i = 1, 3)$ are constants depending only on η,

$$\begin{cases} f_{12} = b_1 z_1 - a_1 y_1 + l_{12} \\ f_{22} = l_{22} \\ f_{32} = b_3 z_1 - a_3 y_1 + l_{32} \end{cases} \tag{94}$$

and

$$f_{11}l_{32} = f_{31}l_{12}, \quad \text{i.e., there exists} \quad p \quad \text{such that} \quad l_{32} = pf_{31}, \quad l_{12} = pf_{11}, \tag{95}$$

in which $l_{i2}(1 \leq i \leq 3)$ are functions of z_0, y_0 which may depend on η. From (93), (94) and (31), (32), we obtain

$$\begin{aligned}
z_{0t} &= -y_2 + \frac{1}{Z_2}\{[b_1 l_{32z_0} - b_3 l_{12z_0} + \eta(\delta b_1^2 - b_3^2)]z_1 \\
&\quad +[b_1 l_{32y_0} - b_3 l_{12y_0} + \eta(-\delta a_1 b_1 + a_3 b_3)]y_1 \\
&\quad +(p\eta - l_{22})[(\delta a_1 b_1 - a_3 b_3)z_0 + (\delta b_1^2 - b_3^2)y_0 + \delta b_1 c_1 - b_3 c_3]\}, \qquad (96) \\
y_{0t} &= z_2 + \frac{1}{Z_2}\{[-a_1 l_{32z_0} + a_3 l_{12z_0} + \eta(-\delta a_1 b_1 + a_3 b_3)]z_1 \\
&\quad +[-a_1 l_{32y_0} + a_3 l_{12y_0} + \eta(\delta a_1^2 - a_3^2)]y_1 \\
&\quad +(-p\eta + l_{22})[(\delta a_1^2 - a_3^2)z_0 + (\delta a_1 b_1 - a_3 b_3)y_0 + \delta a_1 c_1 - a_3 c_3]\}. \qquad (97)
\end{aligned}$$

Similar to the proof of Theorem 4, the requirement of the constant terms in (96) and (97) are independent of η leads to

$$\left\{\begin{array}{c}
\delta a_1 b_1 - a_3 b_3 = \gamma_1(\delta a_1^2 - a_3^2) \\
\delta b_1^2 - b_3^2 = \gamma_2(\delta a_1^2 - a_3^2) \\
\delta b_1 c_1 - b_3 c_3 = \gamma_4(\delta a_1^2 - a_3^2) \\
\delta a_1 c_1 - a_3 c_3 = \gamma_3(\delta a_1^2 - a_3^2).
\end{array}\right.$$

Here we have assumed that $\delta a_1^2 - a_3^2 \neq 0$ (otherwise we may replace it by $\delta b_1^2 - b_3^2$ in the above relations), which implies that

$$Z_2^2 = \delta(\gamma_1^2 - \gamma_2)(\delta a_1^2 - a_3^2)^2 \tag{98}$$

and

$$b_3 c_1 - c_3 b_1 = \frac{\gamma_2\gamma_3 - \gamma_1\gamma_4}{\gamma_1^2 - \gamma_2} Z_2, \quad a_3 c_1 - c_3 a_1 = \frac{\gamma_1\gamma_3 - \gamma_4}{\gamma_1^2 - \gamma_2} Z_2. \tag{99}$$

On the other hand, the requirement of the coefficients of z_1 and y_1 in (96-97) to be independent of η yields

$$\left\{\begin{array}{l}
l_{12} = (-b_3 z_0 + a_3 y_0)\eta + b_1 h + a_1 g + \widetilde{g}(\eta), \\
l_{32} = (-b_1 z_0 + a_1 y_0)\eta\delta + b_3 h + a_3 g + \widetilde{h}(\eta)
\end{array}\right. \tag{100}$$

for some functions h and g not depending on η and some constants $\widetilde{h}(\eta)$ and $\widetilde{g}(\eta)$ depending on η. From equations (100) and (95), using (99), with the same argument as in the proof of Theorem 4, we obtain

$$\delta a_1 b_1 - a_3 b_3 = 0, \quad \delta b_1^2 - b_3^2 = \delta a_1^2 - a_3^2, \quad \text{i.e.} \quad \gamma_1 = 0, \quad \gamma_2 = 1. \tag{101}$$

Now, it follows from (101) and (98), that

$$Z_2^2 = -\delta(\delta a_1^2 - a_3^2)^2,$$

which holds only when $\delta=-1$. This proves that there are no differential systems of the form (38) describing η-p.s.s. of class two. When $\delta=-1$, a straightforward calculation shows that all the differential systems of the form (38) describing η-s.s. of class two are

$$\begin{cases} z_{0t} = -y_2 + \frac{\delta_0\gamma}{2}[(z_0^2+y_0^2)z_0]_x - \alpha z_1 - \frac{\sigma}{2}(z_0^2+y_0^2)y_0 - \beta\delta_0 y_0, \\ y_{0t} = z_2 + \frac{\delta_0\gamma}{2}[(z_0^2+y_0^2)y_0]_x + \alpha y_1 + \frac{\sigma}{2}(z_0^2+y_0^2)z_0 + \beta\delta_0 z_0, \end{cases}$$

where $\delta_0=\pm1$, α, β, γ and σ are constants (one can also refer to the transformation: $\widetilde{\omega}_1=-\omega_1$, $\widetilde{\omega}_2=\omega_3$, $\widetilde{\omega}_3=\omega_2$ in this case). Since the constants α, β, γ and σ are arbitrary, by renaming the constants, we get (92). This completes the proof of Theorem 5. $\square$

Remark 2 Since the differential system (92) in Theorem 5 is in fact system (39) with $\delta=-1$ in Theorem 4, we only need to discuss (39). The differential system (39) is a four parameter family of systems which describe η-p.s.s. (resp. η-s.s.) when $\delta=1$ (resp. $\delta=-1$). Particular cases of this family are:

(i) Linear systems ($\gamma=\sigma=0$),

$$\begin{cases} u_t = -v_{xx} + \alpha u_x + \beta v, \\ v_t = u_{xx} + \alpha v_x - \beta u, \end{cases} \quad \text{or} \quad i(q_t - \alpha q_x) + q_{xx} - \beta q = 0.$$

(ii) The nonlinear Schrödinger equation ($\alpha=\beta=\gamma=0$, $\delta=\pm1, \sigma\geq 0$) (NLS),

$$\begin{cases} u_t = -v_{xx} - \delta\sigma(u^2+v^2)v, \\ v_t = u_{xx} + \delta\sigma(u^2+v^2)u, \end{cases}, \quad \text{or} \quad iq_t + q_{xx} + \delta\sigma|q|^2 q = 0.$$

(iii) The derivative nonlinear Schrödinger equation ($\alpha=\beta=\sigma=0$) (DNLS) [8],

$$\begin{cases} u_t = -v_{xx} + \delta\gamma[(u^2+v^2)u]_x, \\ v_t = u_{xx} + \delta\gamma[(u^2+v^2)v]_x, \end{cases}, \quad \text{or} \quad iq_t + q_{xx} - i\delta\gamma(|q|^2 q)_x = 0.$$

(iv) The mixed NLS-DNLS equation ($\alpha=\beta=0$, $\gamma\neq0$, $\sigma\neq0$) [21],

$$\begin{cases} u_t = -v_{xx} + \delta\gamma[(u^2+v^2)u]_x + \delta\sigma(u^2+v^2)v, \\ v_t = u_{xx} + \delta\gamma[(u^2+v^2)v]_x - \delta\sigma(u^2+v^2)u, \end{cases}$$
$$\text{or} \quad iq_t + q_{xx} - i\delta\gamma(|q|^2 q)_x - \delta\sigma|q|^2 q = 0.$$

The general case (i.e., $\alpha\neq0$, $\beta\neq0$, $\gamma\neq0$ and $\sigma\neq0$) is a new family of differential systems describing η-p.s.s. (resp. η-s.s.), namely

$$iq_t + q_{xx} - i\delta\gamma(|q|^2 q)_x - i\alpha q_x - \delta\sigma|q|^2 q - \beta q = 0.$$

Remark 3 It follows from Theorem 4 and 5 that the NLS$^+$ describes only η-s.s. both classes one and two, while the NLS$^-$ describes only η-p.s.s. of class one. In contrast, however, it is interesting to see that the derivative nonlinear Schrödinger equation (DNLS) has a different geometric character. That is, the DNLS$^+$ or the DNLS$^-$ not only describes η-p.s.s., but also describes η-s.s.. For example, from Theorem 4, the DNLS$^-$

$$\left\{\begin{array}{r} u_t = -v_{xx} + 2[(u^2+v^2)u]_x, \\ v_t = u_{xx} + 2[(u^2+v^2)v]_x \end{array}\right.$$

describes η-p.s.s. (resp. η-s.s.), where the 1-forms $\omega_i, 1 \leq i \leq 3$ are given by (5) with

$$\begin{array}{l} f_{11} = 2\eta^{1/2}u, \quad f_{21} = 2\delta\eta^{1/2}v, \quad f_{31} = \eta, \\ f_{12} = 2\eta^{1/2}\left(-v_x + [\delta\eta + 2(u^2+v^2)]u\right), \\ f_{22} = 2\eta^{1/2}\left(\delta u_x + \delta[\delta\eta + 2(z_0^2+y_0^2)]v\right), \\ f_{32} = 2(u^2+v^2)\eta + \delta\eta^2 \end{array}$$

and $\delta = 1$ (resp. $\delta = -1$).

Next, we classify all the nontrivial differential systems of type (2) with $k = r = 1$ describing η-surfaces of nonzero constant curvature. A trivial differential system of type (2) with $k = r = 1$ means that, under a linear transformation of the variables t and x, all the coefficients of z_1 and y_1 vanish simutaneously.

Theorem 6 *(i). There are no differential systems of type (2) with $k = r = 1$ describing η-p.s.s. (resp. η-s.s.) of class one.*

(ii). All the nontrivial differrential systems of type (2) with $k = r = 1$ describing η-p.s.s. (resp. η-s.s.) of class two are as follows:

$$\left\{\begin{array}{l} u_t = \frac{1}{D}\left[(Dp - p_u P\psi_v)u_x - p_v P\psi_v v_x + \delta(p-\alpha)P_v\right], \\ v_t = \frac{1}{D}\left[p_u P\psi_u u_x + (Dp + p_v P\psi_u)v_x - \delta(p-\alpha)P_u\right], \end{array}\right. \tag{102}$$

where $\delta = 1$ (resp. -1), α is a constant, p, P and ψ are arbitrary functions of u and v such that p is not a constant, $p \neq \alpha$ pointwise and $D = P_v\psi_u - P_u\psi_v$ does not vanish. Moreover, (102) is the integrability condition of the linear problem

$$\phi_x = \Omega_1\phi, \quad \phi_t = \Omega_2\phi,$$

where

$$\Omega_1 = \frac{1}{2}\begin{pmatrix} \eta & Pe^{-\eta\psi} \\ Pe^{\eta\psi} & -\eta \end{pmatrix}, \quad \Omega_2 = \frac{1}{2}\begin{pmatrix} \eta\alpha & pPe^{-\eta\psi} \\ pPe^{\eta\psi} & -\eta\alpha \end{pmatrix}$$

if $\delta = 1$, and

$$\Omega_1 = \frac{1}{2}\begin{pmatrix} i\eta & Pe^{i\eta\psi} \\ -Pe^{-i\eta\psi} & -i\eta \end{pmatrix}, \quad \Omega_2 = \frac{1}{2}\begin{pmatrix} i\eta\alpha & pPe^{i\eta\psi} \\ -pPe^{-i\eta\psi} & -i\eta\alpha \end{pmatrix}$$

if $\delta = -1$.

Proof. i) For a system of type (2) with $k = r = 1$ describing η-p.s.s. (resp. η-s.s.) of class one, we have $f_{ij} = f_{ij}(\eta, z_0; w_0)$ $(1 \leq i \leq 3, 1 \leq j \leq 2)$, $f_{31} = \eta$ and the corresponding determinants (24) are $Z_1 = Z_2 = 0$, $Z_3 \neq 0$ from Lemma 1. It follows from (27) that

$$f_{11}f_{22} - f_{12}f_{21} = 0,$$

which is a contradiction to the requirement that the metric is nondegenerate. Therefore, there exist no such systems.

ii) In this case, we have that $f_{ij} = f_{ij}(\eta, z_0, y_0)$ with $f_{21} = \eta$, and the corresponding determinants (24) are $Z_1 = Z_3 = 0$, $Z_2 \neq 0$. Substituting these relations into (31), (32) and (27), we see that

$$f_{22} = c = c(\eta), \quad \text{and} \quad f_{11}f_{32} = f_{31}f_{12} \tag{103}$$

and

$$\begin{aligned} z_{0t} &= \frac{1}{Z_2}[(f_{11y_0}f_{32z_0} - f_{31y_0}f_{12z_0})z_1 + (f_{11y_0}f_{32y_0} - f_{31y_0}f_{12y_0})y_1 \\ &\quad +(\eta p - c(\eta))(\delta f_{11}f_{11y_0} - f_{31}f_{31y_0})], \end{aligned} \tag{104}$$

$$\begin{aligned} y_{0t} &= \frac{1}{Z_2}[(-f_{11z_0}f_{32z_0} + f_{31z_0}f_{12z_0})z_1 + (-f_{11z_0}f_{32y_0} + f_{31z_0}f_{12y_0})y_1 \\ &\quad +(-\eta p + c(\eta))(\delta f_{11}f_{11z_0} - f_{31}f_{31z_0})], \end{aligned} \tag{105}$$

here we have set $p = f_{12}/f_{11} = f_{32}/f_{31}$ because of the second relation of (103). From the requirement of the constant terms and the coefficients of z_1 and y_1 in (104) and (105) being independent of η, we have

$$(\eta p - c)\frac{\delta f_{11}f_{11y_0} - f_{31}f_{31y_0}}{Z_2}, (-\eta p + c)\frac{\delta f_{11}f_{11z_0} - f_{31}f_{31z_0}}{Z_2} \quad \text{are indep. of } \eta \tag{106}$$

and

$$\begin{cases} p_{z_0}(f_{11y_0}f_{31} - f_{31y_0}f_{11}) = (A-p)Z_2 \\ p_{z_0}(-f_{11z_0}f_{31} + f_{31z_0}f_{11}) = BZ_2 \\ p_{y_0}(f_{11y_0}f_{31} - f_{31y_0}f_{11}) = CZ_2 \\ p_{y_0}(-f_{11z_0}f_{31} + f_{31z_0}f_{11}) = (D-p)Z_2 \end{cases} \tag{107}$$

where A, B, C and D are η-independent functions of z_0 and y_0. From (107), we obtain

$$p_{y_0}(A-p) = p_{z_0}C, \quad p_{y_0}B = p_{z_0}(D-p), \quad (A-p)(D-p) = BC. \tag{108}$$

The third equation of (108) implies that p is an η-independent function. Furthermore, p is not a constant. Notice that the general solution to the first two equations of (108) is,

$$A = p + p_{z_0}Q_1, \quad C = p_{y_0}Q_1, \quad D = p + p_{y_0}Q_2, \quad B = p_{z_0}Q_2,$$

in which Q_1 and Q_2 are η-independent functions, defined by

$$Q_1 = \frac{f_{11y_0}f_{31} - f_{31y_0}f_{11}}{Z_2}, \quad Q_2 = \frac{-f_{11z_0}f_{31} + f_{31z_0}f_{11}}{Z_2}. \tag{109}$$

It follows from (109) that

$$f_{11} = Q_1 f_{11\,z_0} + Q_2 f_{11\,y_0}, \quad f_{31} = Q_1 f_{31\,z_0} + Q_2 f_{31\,y_0}. \tag{110}$$

As a consequence of (110) we have

$$Q_2 \left(f_{11} f_{11\,y_0} - f_{31} f_{31\,y_0} \right) = -Q_1 \left(\delta f_{11} f_{11\,z_0} - f_{31} f_{31\,z_0} \right) + \delta f_{11}^2 - f_{31}^2$$

which implies, combining it with the implication of (106): $\frac{\delta f_{11} f_{11\,y_0} - f_{31} f_{31\,y_0}}{\delta f_{11} f_{11\,z_0} - f_{31} f_{31\,z_0}}$ is independent of η, that

$$\delta f_{11}^2 - f_{31}^2 \quad \text{is independent of } \eta. \tag{111}$$

From (110) and (106), we have that $Z_2 = (\eta p - c)h$ for some η-independent function h . On the other hand, from (109) again, we have

$$\begin{aligned} f_{11\,y_0 z_0} f_{31} - f_{31\,y_0 z_0} f_{11} &= (Q_{1\,z_0} - 1) Z_2 + Q_1 Z_{2\,z_0}, \\ -f_{11\,z_0 y_0} f_{31} + f_{31\,z_0 y_0} f_{11} &= (Q_{2\,y_0} - 1) Z_2 + Q_2 Z_{2\,y_0} \end{aligned}$$

which imply that $Q_1 Z_{2\,z_0} + Q_2 Z_{2\,y_0} + (Q_{1\,z_0} + Q_{2\,y_0} - 2) Z_2 = 0$. Repalcing $Z_2 = (\eta p - c)h$ into this equality, we see that $c(\eta) = \alpha\eta$ for a constant α and hence $Z_2 = \eta H$ for $H = (p - \alpha)h$ which is independent of η.

Without loss of generality, we consider $f_{11}^2 - f_{31}^2 = P^2$ (we will get the same conclusion when $f_{11}^2 - f_{31}^2 = -P^2$) (resp. $-f_{11}^2 - f_{31}^2 = -P^2$), where $P \neq 0$ is independent of η from (111). Since f_{11} and f_{31} satisfy (110), it is easy to verify that P satisfies $P = Q_1 P_{z_0} + Q_2 P_{y_0}$, which implies that $P \neq$const.. Furthermore, we see that $f_{11} = P \cosh\theta$ (resp. $f_{11} = P\cos\theta$) and $f_{31} = P\sinh\theta$ (resp. $f_{31} = P\sin\theta$) for some η-dependent function θ. Substituting this expression of f_{11} (or f_{31}) into Eq. (110), by using $P = Q_1 P_{z_0} + Q_2 P_{y_0}$, then the restriction on θ reads,

$$Q_1 \theta_{z_0} + Q_2 \theta_{y_0} = 0.$$

From $Z_2 = P(P_{y_0}\theta_{z_0} - P_{z_0}\theta_{y_0}) = \frac{\theta_{z_0}}{Q_2} P^2 = -\frac{\theta_{y_0}}{Q_1} P^2$, we obtain that $\theta_{z_0} = \eta Q_2 H / P^2$ and $\theta_{y_0} = -\eta Q_1 H / P^2$. Hence $\theta = \eta\psi$, for a nonconstant η-independent function ψ. Finally, we have P, ψ and p are arbitrary functions of z_0, y_0 such that $P_{y_0}\psi_{z_0} - P_{z_0}\psi_{y_0} \neq 0$. Moreover,

$$\begin{gathered} f_{11} = P\cosh\eta\psi \quad (\text{resp. } P\cos\eta\psi), \quad f_{31} = P\sinh\eta\psi \quad (\text{resp. } P\sin\eta\psi), \\ f_{21} = \eta, \quad f_{12} = p f_{11}, \quad f_{22} = \eta\alpha, \quad f_{32} = p f_{31}, \end{gathered}$$

in which, the requirement of $\omega_1 \wedge \omega_2 \neq 0$ leads to $p \neq \alpha$ in the domain of z_0 and y_0. Since

$$Q_1 = -\frac{P\psi_{y_0}}{P_{y_0}\psi_{z_0} - P_{z_0}\psi_{y_0}}, \quad Q_2 = \frac{P\psi_{z_0}}{P_{y_0}\psi_{z_0} - P_{z_0}\psi_{y_0}}, \quad Z_2 = P\eta(P_{y_0}\psi_{z_0} - P_{z_0}\psi_{y_0}),$$

we conclude that the corresponding differential system is just (102). This completes the proof of Theorem 6. □

We conclude this section with an application of this theorem.

Example 9. Consider $P = v$, $\psi = u$, $p = u^2 + v^2$, $\alpha = -1$, hence by considering

$$f_{11} = v\cosh\eta u, \quad f_{31} = v\sinh\eta u,$$
$$f_{21} = \eta, \quad f_{12} = (u^2+v^2)f_{11}, \quad f_{22} = -\eta, \quad f_{32} = (u^2+v^2)f_{31},$$

we get $D = 1$ and the first-order differential system describing η-p.s.s. is given by

$$\begin{cases} u_t = (u^2+v^2)u_x + u^2 + v^2 + 1, \\ v_t = 2uvu_x + (u^2+3v^2)v_x. \end{cases}$$

It follows from the expressions of f_{ij} that this system is the integrability condition of the linear system

$$\phi_x = \frac{1}{2}\begin{pmatrix} \eta & ve^{-\eta u} \\ ve^{\eta u} & -\eta \end{pmatrix}\phi, \quad \phi_t = \frac{1}{2}\begin{pmatrix} -\eta & (u^2+v^2)ve^{-\eta u} \\ (u^2+v^2)ve^{\eta u} & \eta \end{pmatrix}\phi.$$

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