Minimal surfaces obtained by Ribaucour transformations

A. V. Corro W. Ferreira K. Tenenblat *

Abstract

We consider Ribaucour transformations between minimal surfaces and we relate such transformations to generating planar embedded ends. Applying Ribaucour transformations to Enneper's surface and to the catenoid, we obtain new families of complete, minimal surfaces, of genus zero, immersed in R^3, with infinitely many embedded planar ends or with any finite number of such ends. Moreover, each surface has one or two nonplanar ends. A particular family is obtained from the catenoid, for each pair (n, m), $n \neq m$, such that $n/m > 0$ is an irreducible rational number. For any such pair, we get a 1-parameter family of finite total curvature, complete minimal surfaces with $n+2$ ends, n embedded planar ends and 2 nonplanar ends of geometric index m, whose total curvature is $-4\pi(n+m)$. The analytic interpretation of a Ribaucour transformation as a Bäcklund transformation and a superposition formula for the nonlinear differential equation $\Delta\phi = e^{-2\phi}$ is included.

Introduction.

In the last two decades, a great activity in the field of minimal surfaces has been the construction of new complete minimal surfaces in $\mathbf{R^3}$ (see for example [CHM], [Co], [HM], [JM]). The main tool in such constructions has been the Weierstrass representation of minimal surfaces. In this paper, we exhibit the first families of complete minimal surfaces based upon the use of the Ribaucour transformation between minimal surfaces.

Ribaucour transformations for hypersurfaces, parametrized by lines of curvature, were classically studied by Bianchi [Bi]. They can be applied to obtain surfaces of constant Gaussian curvature from a given such surface. Similarly, by using Ribaucour transfomations one may obtain minimal surfaces from a given such surface. Recently, they have been used to associate a Dupin hypersurface to another such submanifold [CFT1]. Although the transformation for minimal surfaces is a classical result, as far as we know, it has never been used in constructing minimal surfaces.

In this paper, applying Ribaucour transformation to Enneper's surface and to the catenoid, we are able to obtain new families of complete, minimal surfaces, of genus zero, immersed in R^3, with infinitely many embedded planar ends or with any finite number of

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such ends. Moreover, each surface has one or two nonplanar ends. (See [JM] for minimal surfaces with any finite number of catenoid ends and [CHM] for minimal surfaces with an infinite number of anular ends).

The families we obtain depend on two or three parameters. One of them is the parameter $c \neq 0$ of the Ribaucour transformation. According to the value of c, the minimal surfaces can be expressed conformally as a sphere punctured at a finite or infinite number of points of a circle. The other parameters appear from integrating the Ribaucour transformation and they affect the position of the planar ends of the minimal surfaces.

The family of minimal surfaces obtained from Enneper's surface depends on three parameters and it is defined, without loss of generality, for $c > 0$. Excluding Enneper's surface, each minimal surface has infinite total curvature, since it is conformally a sphere punctured at an infinite discrete set of points p_k contained on a circle and at the pole p_0. The set p_k converges to p_0 and the ends corresponding to p_k are planar.

The two parameter family of minimal surfaces obtained from the catenoid has more interesting aspects, since it includes three distinct classes of families of minimal surfaces $\tilde{X}_c$, depending on the value of the parameter c.

For $c = 1/2$, $\tilde{X}_{1/2}$ is a 2-parameter family which includes the catenoid. Excluding the catenoid, any minimal surface of $\tilde{X}_{1/2}$ is a 2-ended, genus zero, immersed minimal surface with infinite total curvature. It has one embedded planar end and the other one is not embedded.

If $c > 1/2$, or $0 \neq c < 1/2$ and $\sqrt{1-2c}$ is not a rational number, then $\tilde{X}_c$ is a 1-parameter family. Excluding the catenoid, any minimal surface of $\tilde{X}_c$ has infinite total curvature which is modelled on a sphere punctured at an infinite discrete set of points p_k and at the pole p_0. As in the case of the surfaces associated to Enneper's surface, the set p_k converges to p_0 and the ends corresponding to p_k are embedded planar ends.

When $0 \neq c < 1/2$ and $\sqrt{1-2c} = n/m$ is an irreducible rational number, with $n \neq m$, then we obtain a family of minimal surfaces that we denote by $\tilde{X}_{(n,m)}$. For each such pair (n, m), $\tilde{X}_{(n,m)}$ is a 1-parameter family of finite total curvature, complete minimal surfaces which can be expressed conformally as an $(n+2)$-punctured sphere. Its total curvature is $-4\pi(n+m)$, since it has n embedded planar ends and 2 non planar ends of geometric index m. The position of the planar ends depends on n and on the other parameter. As a corollary, we have that for any integer $j \geq 3$, there exists a finite number of 1-parameter families of immersed complete minimal surfaces $\tilde{X}_{(n,m)}$, whose total curvature is $-4\pi j$. The finite number follows from the distinct positive integers n, m as above such that $n + m = j$. Any such surface has at least 1 and at most $j-1$ embedded planar ends and two non-planar ends. In particular, any surface of the family $\tilde{X}_{(j,1)}$ has $j-1$ planar ends and two embedded catenoid ends.

It should be pointed out that any minimal surface of the family $\tilde{X}_c$ with $0 \neq c < 1/2$ which has infinite total curvature, is the limit of a sequence of minimal surfaces of type $\tilde{X}_{(n,m)}$ whose total curvature is $-4\pi(n+m)$. Moreover, the catenoid is the limiting surface of $\tilde{X}_c$ as c tends to $1/2$, for generic values of the other parameter.

In section 1, we recall the theory of Ribaucour transformations for surfaces in R^3 and we restrict ourselves to such transformations between minimal surfaces. We show that in this case these transformations generically introduce planar embedded ends. In section 2, we obtain families of parametrized minimal surfaces by applying Ribaucour transformations to Enneper's surface and to the catenoid. We show that any minimal surface of these families is complete and we describe its properties.

In section 3, we recall the classically known (see [Bi] [E]) correspondence between minimal surfaces and solutions of the differential equation $\Delta\phi = e^{-2\phi}$. We also provide the analytic interpretation of the Ribaucour transformation as a Bäcklund type transformation for this equation and its superposition formula. We conclude this section by exhibiting the solutions of this equation which correspond to the minimal surfaces obtained in section 2.

We observe that the superposition formula given in the last section, should be useful in producing new families of minimal surfaces.

1. Ribaucour transformation for minimal surfaces.

In this section, we first recall the theory of Ribaucour transformation for surfaces. For the proofs and more details see [Bi] and [CFT1]. We then restrict ourselves to Ribaucour transformations between minimal surfaces.

A *sphere congruence* in R^3 is a 2-parameter family of spheres, with a differentiable radius function, whose centers lie on an surface M_0 contained in R^3. An *involute* of a sphere congruence is a surface M of R^3 such that, at each point, M is tangent to a sphere of the sphere congruence. Two surfaces M and $\tilde{M}$ are said to be *associated by a sphere congruence* if there is a difeomorphism $\psi : M \to \tilde{M}$ such that at corresponding points p and $\psi(p)$ the sufaces are tangent to the same sphere of the sphere congruence. It follows from this definition that the normal lines at corresponding points intersect at an equidistant point on the center surface M_0. An important special case occurs when ψ preserves lines of curvature.

Let M be an orientable surface of R^3, with no umbilic points, whose Gauss map we denote by N. We say that $\tilde{M}$ is *associated to M by a Ribaucour transformation*, if and only if, there exists a differentiable function h defined on M and a diffeomorphism $\psi : M \to \tilde{M}$ such that

a) for all $p \in M$, $p + h(p)N(p) = \psi(p) + h(p)\tilde{N}(\psi(p))$, where $\tilde{N}$ is the Gauss map of $\tilde{M}$.

b) The subset $p + h(p)N(p),\ p \in M$, is a 2-dimensional submanifold.

c) ψ preserves lines of curvature.

We say that $\tilde{M}$ is *locally associated to M by a Ribaucour transformation* if for all $\tilde{p} \in \tilde{M}$ there exists a neighborhood of $\tilde{p}$ in $\tilde{M}$ which is associated by a Ribaucour transformation to an open subset of M. Similarly, one may consider the corresponding definitions for parametrized surfaces.

The following result gives a characterization of Ribaucour transformations. For the proof and more details see [CFT1].

Theorem 1.1. *Let M be an orientable surface of R^3, with no umbilic points, whose Gauss is N. Let e_i, $1 \leq i \leq 2$ be orthonormal principal directions, λ^i the corresponding principal curvatures, i.e. $dN(e_i) = \lambda_i e_i$. A surface $\tilde{M}$ is associated to M by a Ribaucour transformation, if and only if, M and $\tilde{M}$ are associated by a sphere congruence whose radius function $h : M \to R$ satisfies*

$$dZ^j(e_i) + Z^i \omega_{ij}(e_i) - Z^i Z^j \lambda^i = 0, \quad 1 \leq i \neq j \leq 2. \tag{1}$$

where

$$Z^i = \frac{dh(e_i)}{1 + h\lambda^i}$$

and ω_{ij} are the connection forms of the frame e_i.

Remark 1.2. Locally, whenever $X(u_1, u_2)$ is a parametrization of the surface M by lines of curvature, the function $h(u_1, u_2)$ is a differentiable function which satisfies a second order nonlinear partial differential equation corresponding to equation (1). However, one can linearize the problem of obtaining the function h. This is a consequence of the following result, where we will consider e_1, e_2 to be orthonormal principal directions and ω_1, ω_2 the dual frame.

Proposition 1.3. *Suppose that h is a nonvanishing function which satisfies equation (1) then*

$$\frac{1}{h} \sum_{i=1}^{2} Z^i \omega_i$$

is a closed 1-form and there exists a nonvanishing function Ω, defined on a simply connected domain, such that

$$d\Omega(e_i) = \frac{\Omega}{h} Z^i.$$

For each nonvanishing function h, which is a solution of (1), we consider Ω as above and we define

$$\Omega_i = d\Omega(e_i), \qquad W = \frac{\Omega}{h}.$$

With this notation,

$$dh(e_i) = \frac{\Omega_i}{W}(1 + \Omega\lambda^i/W) \qquad 1 + h\lambda^i = 1 + \Omega\lambda^i/W \qquad Z^i = \frac{\Omega_i}{W}. \tag{2}$$

Moreover, equation (1) is equivalent to a linear system given in the following result.

Proposition 1.4. *A function h is a solution of (1) defined on a simply connected domain, if and only if, $h = \Omega/W$ where Ω and W are functions which satisfy*

$$\begin{aligned} d\Omega_i(e_j) &= \Omega_j \omega_{ij}(e_j), \quad \text{for } i \neq j, && (3) \\ d\Omega &= \sum_{i=1}^{n} \Omega_i \omega_i, && (4) \\ dW &= -\sum_{i=1}^{n} \Omega_i \lambda^i \omega_i. && (5) \end{aligned}$$

For each solution Ω_i, $1 \leq i \leq 2$, of (3), there exists a 2-parameter family of solutions of the system (4), (5). In fact, equation (3) is the integrability condition of the system of equations (4), (5) for Ω and W.

One can show that the Ribaucour transformation of a surface is given in terms of the solutions of the above system (see [CFT1]).

Theorem 1.5. *Let M be an orientable surface of R^3 parametrized by $X : U \subset R^2 \to M$, with no umbilic points. Assume e_i, $1 \leq i \leq 2$ are orthogonal principal directions, λ^i the corresponding principal curvatures and N is a unit vector field normal to M. A surface $\tilde{M}$ is locally associated to M, by a Ribaucour transformation, if and only if, there exist differentiable functions $W, \Omega, \Omega_i : V \subset U \to R$, which satisfy (3)-(5) and $\tilde{X} : V \subset R^n \to \tilde{M}$, is a parametrization of $\tilde{M}$ given by*

$$\tilde{X} = X - \frac{2\Omega}{S} \left(\sum_i \Omega_i e_i - WN \right). \quad (6)$$

where

$$S = \sum_i (\Omega_i)^2 + W^2 \quad (7)$$

Moreover, the normal map of $\tilde{X}$ is given by

$$\tilde{N} = N + \frac{2W}{S} \left(\sum_i \Omega_i e_i - WN \right). \quad (8)$$

From now on, whenever we say that $\tilde{M}$ is locally associated to M by a Ribaucour transformation we are assuming that there are functions where Ω_i, Ω and W locally defined, satisfying the system (3)-(5). Moreover, we observe that since the normal lines at corresponding points intersect at a distance $h = \Omega/W$, it follows from (2) that $dh(e_i) \neq 0$ if and only if $\Omega_i \neq 0$.

Theorem 1.6. *Assume $\tilde{M}$ is locally associated to M by a Ribaucour transformation. Let e_i, $1 \leq i \leq 2$ be the principal directions and λ^i the corresponding principal curvatures of M. Then the principal curvatures of $\tilde{M}$ for each $1 \leq i \leq 2$ are given by*

$$\tilde{\lambda}^i = \frac{WT^i + \lambda^i S}{S - \Omega T^i} \tag{9}$$

where Ω_i, Ω and W satisfy the system (3)-(5), S is given by (7) and

$$T^i = 2(d\Omega_i(e_i) + \sum_k \Omega_k \omega_{ki}(e_i) - W\lambda^i). \tag{10}$$

We now recall a classical result which gives a sufficient condition for a Ribaucour transformation to transform a minimal surface into another minimal surface.

Theorem 1.7. *Let M and $\tilde{M}$ be orientable surfaces of R^3, with no parabolic points. Assume that $\tilde{M}$ is associated to M by a Ribaucour transformation, such that the functions Ω_i , Ω and W satisfy the additional relation*

$$d\Omega_i(e_i) = -\Omega_j \omega_{ji}(e_i) + W\lambda^i + c(W - \Omega\lambda^i), \tag{11}$$

where $c \neq 0$ is a real constant and the initial condition for (3)-(5) and (11) satisfies $S(p_0) = 2c\Omega W(p_0)$. Then

$$\Omega_1^2 + \Omega_2^2 + W^2 \equiv 2c\Omega W.$$

Moreover, $\tilde{M}$ is a minimal surface, if and only if M is minimal.

Proof. It follows from (7) and (10) that $dS(e_i) = \Omega^i T^i$. Moreover, the hypothesis (11) is equivalent to saying that

$$T^i = 2c(W - \Omega\lambda^i).$$

Therefore, one concludes, using (4) and (5), that

$$d(S - 2c\Omega W)(e_i) \equiv 0.$$

Hence the initial condition implies that $S \equiv 2c\Omega W$. From (9), we get

$$\tilde{\lambda}^i = \left(\frac{W}{\Omega}\right)^2 \frac{1}{\lambda^i},$$

which concludes the proof.

□

As a consequence of Theorems 1.5 and 1.7 minimal surfaces associated to a minimal surface X by a Ribaucour transformation are given by

$$\tilde{X} = X - \frac{1}{cW}\left(\sum_i \Omega_i e_i - WN\right) \tag{12}$$

and its normal map $\tilde{N}$ by

$$\tilde{N} = N + \frac{1}{c\Omega}\left(\sum_i \Omega_i e_i - WN\right). \qquad (13)$$

It follows from (6) that the points p_0 which annihilate $S = \sum_i(\Omega_i)^2 + W^2$ are ends of the minimal surface $\tilde{X}$. Our next result relates Ribaucour transformations to generating embedded planar ends p_0 and it describes the behaviour of $\tilde{X}$, in the neighborhood of p_0, whenever $\Omega(p_0) \neq 0$.

Proposition 1.8. *Let $\tilde{X} : D \setminus \{p_0\} \to R^3$ be a minimal surface locally associated by a Ribaucour transformation to minimal surface $X : D \to R^3$, such that the functions Ω, Ω_i, W are defined in D. Let $\tilde{N}$ and N be the normal maps of $\tilde{X}$ and X respectively.*

If $S(p_0) = 0$, $\Omega(p_0) \neq 0$ and $S(p) \neq 0$ for all $p \in D \setminus \{p_0\}$ then:

a) *For any divergent curve $\gamma : [0,1) \to D \setminus \{p_0\}$ such that $\lim_{t\to 1}\gamma(t) = p_0$ the length of $\tilde{X}(\gamma)$ is infinite.*

b) *The minimal surface $\tilde{X}$ has an embedded planar end at p_0, and $\lim_{p\to p_0}\tilde{N}(p) = N(p_0)$.*

Proof. **a)** Since the surface X has no planar points, without loss of generality, we may assume that the first and second fundamental forms of X are given by $I = \varphi^2(du_1^2 + du_2^2)$ and $II = C(du_1^2 - du_2^2)$ respectively, where C is a nonzero constant. Moreover, after a translation, we may also assume that $p_0 = 0$. It follows from Theorem 1.7. that the first fundamental form of $\tilde{X}$ is given by $\tilde{I} = \psi^2(du_1^2 + du_2^2)$, where $\psi = |C\Omega/(\varphi W)|$.

At p_0 we have $S(0) = 0$, therefore $\Omega_i(0) = W(0) = 0$. Moreover, it follows from equations (3)-(5) that $W_{u_i}(0) = W_{u_iu_j}(0) = 0$ for $i \neq j$ and $W_{u_iu_i}(0) = cC^2\Omega(0)/\varphi^2(0)$. Hence, by considering the Taylor expantion of W on a neighborhood of 0, we get

$$W(p) = \frac{cC^2\Omega(0)}{2\varphi^2(0)}|p|^2 + R, \qquad (14)$$

where $\lim_{|p|\to 0} R/|p|^2 = 0$. Therefore, since $\Omega(p_0) \neq 0$, we have

$$\lim_{|p|\to 0} |p|^2\psi = \frac{2\varphi(0)}{|cC|}.$$

Hence, there exists $\delta > 0$ such that $\forall p$, $0 < |p| < \delta$ we have $|p|^2\psi(p) > \varphi(0)/|cC|$.

Let $\gamma : [0,1) \to D \setminus \{0\}$ be a curve in the complex plane, given by $\gamma(t) = u_1(t) + iu_2(t)$ such that $\lim_{t\to 1}\gamma(t) = 0$. Then its length is

$$\ell(\gamma) = \int_0^1 \psi(\gamma(t))|\gamma'(t)|\,dt > \frac{\varphi(0)}{|cC|}\int_{t_0}^1 \frac{|\gamma'(t)|}{|\gamma(t)|^2}dt \geq \frac{\varphi(0)}{|cC|}\left|\int_{t_0}^1 \frac{\gamma'(t)}{(\gamma(t))^2}dt\right| = \infty.$$

b) The minimal surface $\tilde{X}$ and its normal map $\tilde{N}$ are given by (12) and (13) respectively. Therefore, $\tilde{N}$ extends continuously to p_0. We denote by $N_0 = \lim_{p\to 0} \tilde{N}(p) = N(0)$. We consider a rotation $\mathcal{R}$ in R^3 such that N_0 coincides with the south pole of the unit sphere S^2. Let F and G be the Weierstrass representation of $\mathcal{R} \circ \tilde{X}$. Since $G = \pi \circ \mathcal{R} \circ \tilde{N}$, where π is the stereographic projection of S^2 from the north pole, it follows that $G(0) = 0$. We will show that $p_0 = 0$ is a pole of order 2 of the function F. In fact

$$\lim_{p\to 0} \left| \frac{p^2 F}{2} \right| = \lim_{p\to 0} |p|^2 \left(\frac{|F|(1+|G|^2)}{2} \right) = \lim_{p\to 0} |p|^2 \psi = \frac{2\varphi(0)}{|cC|} > 0.$$

We conclude that p_0 is a pole of order 2 of F and hence $\tilde{X}(\tilde{D} \setminus \{p_0\})$ is an end of index 1, therefore it is embedded ([JM]).

We will now show that it is a planar end. We consider the oriented distance of $\tilde{X}$ to the plane orthogonal to N_0,

$$d(u_1, u_2) = < \tilde{X}(u_1, u_2), N_0 > .$$

It follows from (12) that

$$d = < X, N_0 > + \frac{< N, N_0 >}{c} - \frac{\Delta}{cW\varphi}, \qquad \text{where} \quad \Delta = \sum_i \Omega_i < X_{u_i}, N_0 > .$$

Since $\Delta(0) = \Delta_{u_i}(0) = \Delta_{u_i u_j}(0) = 0$ for $i \neq j$ and $\Delta_{u_i u_i}(0) = 2cC^2\Omega(0)/\varphi(0)$, we conclude that the Taylor expansion of Δ on a neighborhood of 0 is given by

$$\Delta(p) = cC^2 \frac{\Omega(0)}{\varphi(0)} |p|^2 + \tilde{R},$$

where $\lim_{p\to 0} \tilde{R}/|p|^2 = 0$. It follows from (14) that $\lim_{p\to 0} \Delta/(cW\varphi) = 2/c$. Hence, $\lim_{p\to 0} d(p) = < X(0), N_0 > -1/c$. This proves that $\tilde{X}(\tilde{D})$ is a planar end where $\tilde{D}$ is a punctured disk at p_0.

□

2. Families of minimal surfaces associated to Enneper's surface and to the catenoid.

In this section, by using Ribaucour transformations, we will obtain families of minimal surfaces starting from Enneper's surface and from the catenoid. We will describe its geometric properties, the number of ends and its total curvature.

Proposition 2.1. *Consider Enneper's surface parametrized by*

$$X(u_1, u_2) = \left(u_1 - \frac{u_1^3}{3} + u_1 u_2^2,\ u_2 - \frac{u_2^3}{3} + u_2 u_1^2,\ u_1^2 - u_2^2 \right) \tag{15}$$