Minimal Surfaces of Rotation in Finsler Space with a Randers Metric

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Abstract

We consider Finsler spaces with a Randers metric $F = \alpha + \beta$, on the three dimensional real vector space, where α is the Euclidean metric and $\beta = bdx_3$ is a 1-form with norm b, $0 \leq b < 1$. By using the notion of mean curvature for immersions in Finsler spaces introduced by Z. Shen, we get the ordinary differential equation that characterizes the minimal surfaces of rotation around the x_3 axis. We prove that for every b, $0 \leq b < 1$, there exists, up to homothety, a unique forward complete minimal surface of rotation. The surface is embedded, symmetric with respect to a plane perpendicular to the rotation axis and it is generated by a concave plane curve. Moreover, for every b, $\frac{\sqrt{3}}{3} < b < 1$ there are non complete minimal surfaces of rotation, which include explicit minimal cones.

1. Introduction

The differential geometry of minimal surfaces in Riemannian manifolds has been extensively developped. However, minimal surfaces in Finsler spaces have not been studied and developped at the same pace. Actually, there are no examples of such surfaces, except the trivial ones. The fundamental contribution on this subject was given by Shen in [Sh1]. He introduced the notion of mean curvature for immersions into Finsler manifolds and he established some of its properties. As in the Riemannian case, if the mean curvature is identically zero, then the immersion is said to be minimal. The purpose of this paper is to present our investigation, which includes some of the results obtained in [So], on minimal surfaces of rotation in a three dimensional vector space equipped with a Randers metric.

Although Finsler metrics seem to be more complicated than Riemannian metrics, there exist good reasons to study Finsler spaces, since Randers spaces occur naturally, for example, in Physics applications and in some problems in Biology (see [AIM], [YS]).

In this paper, we consider V^{n+1} the standard $n+1$-dimensional real vector space equipped with a Randers metric $F_b(x,y) = \alpha(x,y) + \beta(x,y)$ where (x,y) is in the tangent bundle TV, α is the Euclidean metric and $\beta = bdx_{n+1}$ is a 1-form whose norm b satisfies $0 \leq b < 1$. We present the differential equation for a minimal immersion into an $n+1$-dimensional Randers space (Theorem 2). A particular case of the above equation enables us to establish the ordinary differential equation that characterizes the minimal surfaces of rotation around the x_3 axis (Theorem 4) in the Randers space V^3. By analyzing the solutions of this equation we prove our main result

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Theorem 1 *Up to homothety, there exists a unique forward complete minimal surface of rotation around the x_3 axis on a Randers space (V^3, F_b), for each b, $0 \leq b < 1$. The surface is embedded, symmetric with respect to a plane perpendicular to the rotation axis and it is generated by a concave plane curve. Moreover, when $\frac{\sqrt{3}}{3} < b < 1$, the slope of the tangent lines to the curve is bounded by $\pm\sqrt{1-b^2}/\sqrt{3b^2-1}$.*

This theorem reduces to the classical result for minimal surfaces of rotation in R^3, when the Randers metric is Euclidean, i.e. when $b = 0$. The proof of our main result follows from the qualitative study of the ordinary differential equation considering two cases for the parameter b, namely $0 \leq b \leq \sqrt{3}/3$ and $\sqrt{3}/3 < b < 1$. In the second case, besides the forward complete minimal surfaces of rotation, there are non complete ones which include explicit minimal cones (see Proposition 13).

2. Preliminaries

We follow the notation and terminology of [Sh1], and we will make use of the following conventions: we will use Greek letters τ, η, ε for indices running from 1 to n, and Latin letters i, j, k, l for indices running from 1 to $n+1$. We will also use Einstein's convention, i.e., in general we will not write the symbol of the summand to represent the sum on repeated indices.

Let M^n be a C^∞ n-manifold, and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x, y) be a point of TM, $x \in M, y \in T_xM$. We consider local coordinates $(x^1, ..., x^n)$ on an open subset U of M. As usual, $\partial/\partial x^i$ and dx^i are the induced coordinate basis for T_xM and T_x^*M and (x^i, y^i) are local coordinates on $\pi^{-1}(U) \subset TM$, where $y = y^i \partial/\partial x^i$. A function $F : TM \longrightarrow [0, \infty)$ is called a *Finsler metric* on M if F has the following properties: [i] (Regularity) $F \in C^\infty$ in $TM \setminus \{0\}$; [ii] (Positive Homogeneity) $F(x, ty) = tF(x, y)$, $\forall t > 0, (x, y) \in TM$; [iii] (Strong Convexity) $g = (g_{ij}(x, y)) = \left(\frac{1}{2}[F^2(x, y)]_{y_i y_j}\right)$ is positive definite at each point of $TM \setminus \{0\}$. The pair (M, F) is called a *Finsler space*.

Minkowski spaces are the simplest Finsler manifolds. Denote by V^n the standard n-dimensional real vector space. A *Minkowski space* is V^n equipped with a Minkowski norm F (whose indicatrix is strongly convex), i.e., $F(x, y)$ depends only on $y \in T_xV^n$. If F is Euclidean, then $I\!R^n = (V^n, F)$.

A *Randers metric* on M is the Finsler structure F on TM given by

$$F(x, y) = \alpha(x, y) + \beta(x, y),$$

where $\alpha(x, y) = \sqrt{a_{ij}(x) y^i y^j}$, $\beta(x, y) = b_k(x) y^k$, and a_{ij}, a^{ij} are the components of the Riemannian metric and of its inverse matrix respectively and b_k are the components of the 1-form β, whose *norm* $b = \sqrt{a^{ij} b_i b_j}$, satisfies $0 \leq b < 1$.

Let $(\widetilde{M}^m, \tilde{F})$ be a Finsler manifold and let $\varphi : M^n \longrightarrow (\widetilde{M}^m, \tilde{F})$ be an immersion. The induced metric $F = \varphi^* \widetilde{F}$ on M, is a Finsler metric. We have by definition that $F(y) = (\varphi^* \widetilde{F})(y) = \widetilde{F}(\varphi_*(y))$, $\forall\ y \in T_xM$. From now on, we will consider hypersurfaces on Randers spaces $\varphi : M^n \longrightarrow (\widetilde{V}^{n+1}, \widetilde{F})$, where $\widetilde{V}$ is a $n+1$-dimensional real vector space, $\widetilde{F} = \alpha + \beta$, α is the Euclidean metric, and β is a 1-form with norm b, $0 \leq b < 1$. Without loss of generality we will consider $\beta = bdx^{n+1}$. If M^n has local coordinates $x = (x^\varepsilon)$, $\varepsilon = 1, \cdots, n$, and $\varphi(x) = (\varphi^i(x^\varepsilon)) \in \widetilde{V}$, $i = 1, \cdots, n+1$, fix a local frame $e = \{e_a\}$ for TM and define for the Minkowski space (T_xM, F_x), the application

$$\mathcal{F}(x,z) = \frac{vol(B^m)}{vol(D_x^m)}, \tag{1}$$

where $x \in M$,

$$z = (z_a^i) = \left(\frac{\partial \varphi^i}{\partial x^a}\right), \tag{2}$$

and B^m = unitary ball in $I\!R^m$, vol is the Euclidean volume, and

$$D_x^m = \left\{(y^1, y^2, ..., y^m) \in I\!R^m \mid F_x(y^a z_a^i e_{i|_x}) < 1\right\}. \tag{3}$$

The *induced volume element* of (M, F) is given by

$$dV_F = \mathcal{F}(x,z)dx, \tag{4}$$

where $\mathcal{F}(x,z)$ is given by (1).

The Euclidean volume of D_x^m is given by

$$volD_x^m = \frac{\text{vol } B^n}{\left(1 - b^2 A^{\tau\gamma} z_\tau^{n+1} z_\gamma^{n+1}\right)^{\frac{n+1}{2}} \sqrt{detA}}$$

where

$$A = (A_{\tau\gamma}) = \left(\sum_{i=1}^{n+1} z_\tau^i z_\gamma^i\right), \tag{5}$$

and $(A^{\tau\gamma}) = (A_{\tau\gamma})^{-1}$. Then from (4), we have the volume form dV_F given by the following formula ([Sh2])

$$dV_F = \left(1 - b^2 A^{\tau\gamma} z_\tau^{n+1} z_\gamma^{n+1}\right)^{\frac{n+1}{2}} \sqrt{detA}\, dx^1 \cdots dx^n. \tag{6}$$

The *mean curvature* $\mathcal{H}_\varphi$, introduced by ([Sh1]), is given by

$$\mathcal{H}_{|_x}(V) = \frac{1}{\mathcal{F}}\left\{\frac{\partial^2 \mathcal{F}}{\partial z_\varepsilon^i \partial z_\eta^j}\frac{\partial^2 \varphi^j}{\partial \tilde{x}^\varepsilon \partial \tilde{x}^\eta} + \frac{\partial^2 \mathcal{F}}{\partial x^j \partial z_\varepsilon^i}\frac{\partial \varphi^j}{\partial \tilde{x}^\varepsilon} - \frac{\partial \mathcal{F}}{\partial x^i}\right\} V^i.$$

$\mathcal{H}_\varphi(V)$ depends linearly in V and the mean curvature vanishes on $\varphi_*(TM)$ (cf. Lemmas in [Sh1]). Observe that whenever (V, F) is a Minkowsky space, the expression of the mean curvature reduces to

$$\mathcal{H}_{|_x} = \frac{1}{\mathcal{F}}\left\{\frac{\partial^2 \mathcal{F}}{\partial z_\varepsilon^i \partial z_\eta^j}\frac{\partial^2 \varphi^j}{\partial \tilde{x}^\varepsilon \partial \tilde{x}^\eta}\right\}. \tag{7}$$

Now we shall present the concepts and results related to the notion of completeness of a Finsler manifold. We will ommit the proofs and details, which can be found in [BCS].

Let (M, F) be a Finsler manifold, where F is positively homogeneous of degree 1. Let $\sigma : [a, b] \longrightarrow M$ be a piecewise differentiable curve. The *integral length* $L(\sigma)$ is defined as

$$L(\sigma) = \int_a^b F\left(\sigma, \frac{d\sigma}{dt}\right) dt. \tag{8}$$

For $p_0, p_1 \in M$, denote by $\Gamma_{(p_0 p_1)}$ the collection of all piecewise C^∞ curves $\sigma : [a, b] \longrightarrow M$, with $\sigma(a) = p_0$ and $\sigma(b) = p_1$. Define a map $d : M \times M \longrightarrow [0, \infty)$ by

$$d(p_0, p_1) := \inf_{\sigma \in \Gamma_{(p_0 p_1)}} L(\sigma).$$

It can be shown that (M, d) satisfies the first two axioms of a metric space. Namely: **(i)** $d(p_0, p_1) \geq 0$, where the equality holds, if and only if, $p_0 = p_1$ and **(ii)** $d(p_0, p_1) \leq d(p_0, p_2) + d(p_2, p_1)$. If the Finsler structure F is absolutely homogeneous, i.e., $F(x, ty) = |t| F(x, y), \forall\, t \in I\!R$, then one also has the symmetry property: **(iii)** $d(p_0, p_1) = d(p_1, p_0)$. We emphasize that generically, the distance function d on a Finsler manifold does *not* have the symmetry property. In particular, the Randers metric $F = \alpha + \beta$ does not satisfy this property.

A Finsler manifold (M, F) is said to be *forward complete* with respect to the metric distance function d if every forward Cauchy sequence converges in M. A Finsler manifold (M, F) is said to be *forward geodesically complete* if every geodesic $\gamma(t)$, $a \leq t < b$, parametrized to have constant Finsler speed, can be extended to a geodesic defined on $a \leq t < \infty$. Similarly, one defines a backward complete and backward geodesically complete Finsler space.

It can be shown, using the Hopf-Rinow Theorem (see [BCS]), that M is forward complete if and only if the length of any divergent curve is unbounded. If F is absolutely homogeneous of degree one, then forward and backward geodesic completeness either both hold or both fail. This is the case for Riemannian metrics.

3. The differential equation of minimal surfaces of rotation in Randers space.

In this section, we start characterizing the minimal hypersurfaces M^n in Randers space (V^{n+1}, F_b), in terms of differential equations. We then restrict ourselves to surfaces immersed in V^3 and we obtain the ordinary differential equation which characterizes such minimal surfaces of rotation.

Theorem 2 *Let $\varphi : M^n \longrightarrow (V^{n+1}, F_b)$ be an immersion into a Randers space, with local coordinates $(\varphi^j(x_\varepsilon))$. Then φ is minimal if and only if it satisfies the differential equation*

$$\left\{ \frac{(n^2-1)}{4} \frac{\partial B}{\partial z^i_\varepsilon} \frac{\partial B}{\partial z^j_\eta} C - \frac{n+1}{2}(1-B)\left(\frac{\partial^2 B}{\partial z^i_\varepsilon \partial z^j_\eta} C + \frac{\partial B}{\partial z^j_\eta}\frac{\partial C}{\partial z^i_\varepsilon} + \frac{\partial B}{\partial z^i_\varepsilon}\frac{\partial C}{\partial z^j_\eta} \right) + \right.$$
$$\left. + (1-B)^2 \frac{\partial^2 C}{\partial z^i_\varepsilon \partial z^j_\eta} \right\} \frac{\partial^2 \varphi^j}{\partial x^\varepsilon \partial x^\eta} v^i = 0, \qquad \forall v = v^i e_i \in V^{n+1}, \tag{9}$$

where

$$C = \sqrt{detA}, \quad B = b^2 A^{\varepsilon\eta} z_\varepsilon^{n+1} z_\eta^{n+1}, \tag{10}$$

e_i is the canonical basis of V^{n+1}, z_ε^i is given by (2), A is given by (5) and the components of its inverse matrix are denoted by $A^{\varepsilon\eta}$.

Proof: Since C and B are given by (10), we have from (1) and (6) that

$$\mathcal{F} = (1-B)^{\frac{n+1}{2}} C.$$

From (7), the immersion is minimal, if and only if,

$$\frac{\partial^2 \mathcal{F}}{\partial z_\varepsilon^i \partial z_\eta^j} \frac{\partial^2 \varphi^j}{\partial x^\varepsilon \partial x^\eta} v^i = 0.$$

Taking the second order derivates of $\mathcal{F}$ with respect to the variable z_ε^i and substituting in the last equation, we obtain equation (9), which characterizes the minimal immersions in a Randers space. □

From equation (9), it is easy to see that if φ is a minimal immersion into (V^{n+1}, F_b), then the homothetic immersion $\tilde{\varphi} = \lambda\varphi$ is also minimal. In what follows, we will restrict ourselves to studying minimal surfaces in three-dimensional Randers spaces. Therefore, by considering $n = 2$ in Theorem 2, we obtain the following result.

Corollary 3 *Let $\varphi : M^2 \longrightarrow (V^3, F_b)$ be an immersion in a Randers space, with local coordinates $(\varphi^j(x^\varepsilon))$. Then φ is minimal, if and only if, it satisfies the differential equation*

$$\left\{ \frac{12E^2 - (2E + C^2)^2}{C(C^2 - E)} \frac{\partial C}{\partial z_\varepsilon^i} \frac{\partial C}{\partial z_\eta^j} - \frac{3C}{2} \frac{\partial^2 E}{\partial z_\eta^j \partial z_\varepsilon^i} - \frac{3}{2} \left(\frac{2E - C^2}{C^2 - E} \right) \left(\frac{\partial C}{\partial z_\varepsilon^i} \frac{\partial E}{\partial z_\eta^j} + \frac{\partial C}{\partial z_\eta^j} \frac{\partial E}{\partial z_\varepsilon^i} \right) + \right.$$

$$\left. + \frac{3C}{4(C^2 - E)} \frac{\partial E}{\partial z_\varepsilon^i} \frac{\partial E}{\partial z_\eta^j} + \frac{(2E + C^2)}{2C} \frac{\partial^2 C^2}{\partial z_\eta^j \partial z_\varepsilon^i} \right\} \frac{\partial^2 \varphi^j}{\partial x^\varepsilon \partial x^\eta} v^i = 0, \qquad \forall\, v = v^i e_i \in V^3, \tag{11}$$

where $z = (z_\varepsilon^i)$ and C are defined by (2) and (10) respectively,

$$E = b^2 \sum_{k=1}^{3} (-1)^{\gamma+\tau} z_{\tilde{\gamma}}^k z_{\tilde{\tau}}^k z_\gamma^3 z_\tau^3, \qquad \tilde{\tau} = \delta_{\tau 2} + 2\delta_{\tau 1}. \tag{12}$$

Proof: Considering $n = 2$ in Theorem 2, equation (9) reduces to

$$D_{ij}^{\varepsilon\eta} \frac{\partial^2 \varphi^j}{\partial x^\varepsilon \partial x^\eta} v^i = 0,$$

where

$$D_{ij}^{\varepsilon\eta} = \frac{3}{4} \frac{\partial B}{\partial z_\varepsilon^i} \frac{\partial B}{\partial z_\eta^j} C - \frac{3}{2}(1-B) \left(\frac{\partial^2 B}{\partial z_\varepsilon^i \partial z_\eta^j} C + \frac{\partial B}{\partial z_\eta^j} \frac{\partial C}{\partial z_\varepsilon^i} + \frac{\partial B}{\partial z_\varepsilon^i} \frac{\partial C}{\partial z_\eta^j} \right) + (1-B)^2 \frac{\partial^2 C}{\partial z_\varepsilon^i \partial z_\eta^j}.$$

Since $A_{\varepsilon\eta} = \sum_{i=1}^{3} z^i_\varepsilon z^i_\eta$, its inverse matrix is given by

$$A^{\varepsilon\eta} = \sum_{i=1}^{3} \frac{(-1)^{\varepsilon+\eta}}{detA} z^i_{\tilde{\varepsilon}} z^i_{\tilde{\eta}}, \qquad \text{where} \qquad \tilde{\eta} = \delta_{\eta 2} + 2\delta_{\eta 1}.$$

Hence it follows from (10), that

$$B \;=\; b^2 \sum_{k=1}^{3} \frac{(-1)^{\gamma+\tau}}{detA} z^k_{\tilde{\gamma}} z^k_{\tilde{\tau}} z^3_\gamma z^3_\tau \;=\; \frac{1}{C^2} E,$$

where E is given by (12). Taking derivatives of B we get the proof of Corollary 3. □

We observe that in (12) we have introduced a notation $\tilde{\tau}$, which means

$$\tilde{\tau} = \begin{cases} 1 & \text{if } \tau = 2, \\ 2 & \text{if } \tau = 1. \end{cases}$$

In our next result, by considering the immersion to be a surface generated by rotating a plane curve around a fixed axis, we get the ordinary differential equation that characterizes such minimal surfaces.

Theorem 4 *Let $(V^3, F_b = \alpha + \beta)$ be a Randers space, where α is the Euclidean metric, and $\beta = bdx_3$ is a 1-form with norm b, satisfying $0 \leq b < 1$. Let $\varphi : M^2 \longrightarrow V^3$ be an immersion given by $\varphi(t,\theta) = (f_b(t)\cos\theta, f_b(t)\sin\theta, t)$, $\quad f_b > 0$. Then φ is minimal, if and only if, f_b satisfies the equation*

$$\begin{aligned} - \quad & f_b f_b'' \left[\left(1 - b^2 + (f_b')^2\right) \left(1 + 2b^2 + (1 - 3b^3)(f_b')^2\right) + 3b^4 (f_b')^2 \right] + \\ + \quad & \left(1 + (f_b')^2\right) \left(1 - b^2 + (f_b')^2\right) \left[1 - b^2 + \left(1 - 3b^2\right)(f_b')^2\right] = 0. \end{aligned} \tag{13}$$

Before proving this theorem, we observe that whenever $b = 0$, i.e., V^3 is the Euclidean space, equation (13) reduces to the classical differential equation, which characterizes the minimal surfaces of rotation in R^3.

For the proof of Theorem 4, we will need some preliminary observations and a Lemma. We start observing that, since the mean curvature vanishes on tangent vectors of the immersion φ, we only need to consider a vector field v such that the set $\{v, \varphi_t, \varphi_\theta\}$ is linearly independent. Hence, we consider $v = (-\cos\theta, -\sin\theta, f_b'(t))$. For subsequent use, we will express the coordinates of the vector v in the following way

$$v^i = -\delta_{i1}\cos\theta - \delta_{i2}\sin\theta + \delta_{i3} f_b'(t). \tag{14}$$

With the notation $z^i_\varepsilon = \frac{\partial\varphi^i}{\partial x^\varepsilon}$, introduced in (2), in the case of a rotation surface $\varphi(t,\theta)$, we have

$$\begin{aligned} z^i_\varepsilon \quad = \quad & \delta_{\varepsilon 1} \left[\delta_{i1} f_b'(t)\cos\theta + \delta_{i2} f_b'(t)\sin\theta + \delta_{i3}\right] + \\ & + \delta_{\varepsilon 2} \left[-\delta_{i1} f_b(t)\sin\theta + \delta_{i2} f_b(t)\cos\theta\right]. \end{aligned} \tag{15}$$

In particular,

$$z^3_\varepsilon = \delta_{\varepsilon 1}. \tag{16}$$

The second order derivatives of the coordinates of the immersion are given by

$$\begin{aligned} \varphi^j_{x^\varepsilon x^\eta} &= (\delta_{\varepsilon 1}\delta_{\eta 2} + \delta_{\varepsilon 2}\delta_{\eta 1}) f'_b(t)(-\delta_{j1}\sin\theta + \delta_{j2}\cos\theta) + \\ &\quad + [\delta_{\varepsilon 1}\delta_{\eta 1} f''_b(t) - \delta_{\varepsilon 2}\delta_{\eta 2} f_b(t)]\,(\delta_{j1}\cos\theta + \delta_{j2}\sin\theta), \end{aligned} \tag{17}$$

where $x^1 = t$ and $x^2 = \theta$. In particular,

$$\varphi^3_{x^\varepsilon x^\eta} = 0,\ \forall \varepsilon, \eta. \tag{18}$$

Moreover, we obtain from (5), (15) and (16) that

$$A = \begin{pmatrix} 1 + [f'_b(t)]^2 & 0 \\ 0 & [f_b(t)]^2 \end{pmatrix}, \tag{19}$$

$$C^2 = [f_b(t)]^2[1 + [f'_b(t)]^2], \tag{20}$$

$$E = b^2[f_b(t)]^2. \tag{21}$$

In order to prove Theorem 4, we need the expressions

$$\frac{\partial E}{\partial z^i_\varepsilon}, \frac{\partial C}{\partial z^i_\varepsilon}, \frac{1}{2}\frac{\partial^2 C^2}{\partial z^j_\eta \partial z^i_\varepsilon},$$

for the immersion φ. Moreover, we need to obtain the products and sums which appear in (11). This is the aim of the following lemma.

Lemma 5 *Let φ be an immersion in V^3 given by $\varphi(t,\theta) = (f_b(t)\cos\theta, f_b(t)\sin\theta, t)$. Consider the expressions E, C and A, defined by (12), (10) and (5), respectively, then*

$$\frac{\partial C}{\partial z^i_\varepsilon} v^i = 0, \quad \forall \varepsilon, \tag{22}$$

$$\frac{\partial E}{\partial z^i_\varepsilon} v^i = 2b^2 f^2_b(t) f'_b(t)\delta_{\varepsilon 1}, \quad \forall \varepsilon, \tag{23}$$

$$\frac{\partial E}{\partial z^j_\eta} \varphi^j_{x_\varepsilon x_\eta} = 2b^2 f_b(t) f'_b(t)\delta_{\varepsilon 1}, \quad \forall \varepsilon, \tag{24}$$

$$\frac{\partial^2 E}{\partial z^i_\varepsilon \partial z^j_\eta} \varphi^j_{x_\varepsilon x_\eta} v^i = 2b^2 f_b(t)[1 + 2f'^2_b(t)], \tag{25}$$

$$\frac{\partial C}{\partial z^j_\eta} \varphi^j_{x_\varepsilon x_\eta} = \frac{f'_b(t)}{\sqrt{1 + [f'_b(t)]^2}}\delta_{\varepsilon 1}\left[f_b(t) f''_b(t) + 1 + f'^2_b(t)\right], \quad \forall \varepsilon, \tag{26}$$

$$\frac{1}{2}\frac{\partial^2 C^2}{\partial z^j_\eta \partial z^i_\varepsilon} \varphi^j_{x_\varepsilon x_\eta} v^i \quad = \quad f_b(t)[-f_b(t) f''_b(t) + 1 + f'^2_b(t)], \tag{27}$$

where $z^i_\varepsilon = \frac{\partial \varphi^i}{\partial x^\varepsilon}$.

Proof: From (10), we have that $C = \sqrt{\sum_{k \neq l} (z_1^k)^2 (z_2^l)^2 - \sum_{k \neq l} z_1^k z_2^k z_1^l z_2^l}$. Therefore

$$\frac{\partial C}{\partial z_\varepsilon^i} = \frac{1}{C} \sum_{l \neq i} (z_1^i z_2^l - z_2^i z_1^l)(\delta_{\varepsilon 1} z_2^l - \delta_{\varepsilon 2} z_1^l) \tag{28}$$

and

$$\begin{aligned} \frac{1}{2} \frac{\partial C^2}{\partial z_\eta^j \partial z_\varepsilon^i} &= \sum_{l \neq i} \left[\delta_{ij} (\delta_{\eta 1} z_2^l - \delta_{\eta 2} z_1^l) + \delta_{lj} (\delta_{\eta 2} z_1^i - \delta_{\eta 1} z_2^i) \right] (\delta_{\varepsilon 1} z_2^l - \delta_{\varepsilon 2} z_1^l) \\ &\quad + \sum_{l \neq i} (z_1^i z_2^l - z_2^i z_1^l) \delta_{jl} (\delta_{\varepsilon 1} \delta_{\eta 2} - \delta_{\varepsilon 2} \delta_{\eta 1}). \end{aligned} \tag{29}$$

It follows from (15) and (20) that

$$\begin{aligned} \frac{\partial C}{\partial z_\varepsilon^i} &= \frac{1}{\sqrt{1 + f_b'^2(t)}} \Big[\delta_{\varepsilon 1} f_b(t) \left(f_b'(t) (\delta_{i1} \cos\theta + \delta_{i2} \sin\theta) + \delta_{i3} \right) + \\ &\quad + \delta_{\varepsilon 2} (1 + f_b'^2(t)) (-\delta_{i1} \sin\theta + \delta_{i2} \cos\theta) \Big]. \end{aligned} \tag{30}$$

Now, multiplying $\frac{\partial C}{\partial z_\varepsilon^i}$ by v^i given by (14) and adding in i, we obtain equation (22).

Equation (26) is proved by considering (17) and the derivative of C with respect to z_η^j given by (30).

It follows from (12) that the first order derivatives of E are given by

$$\begin{aligned} \frac{\partial E}{\partial z_\varepsilon^i} &= 2b^2 \left[z_{\tilde{\varepsilon}}^3 \sum_\tau (-1)^{\tilde{\varepsilon} + \tau} z_{\tilde{\tau}}^i z_\tau^3 + \delta_{i3} \sum_{\tau, k} (-1)^{\varepsilon + \tau} z_\tau^3 z_{\tilde{\tau}}^k z_{\tilde{\varepsilon}}^k \right] \\ &= 2b^2 f_b(t) \left[\delta_{\tilde{\varepsilon} 1} (-\delta_{i1} \sin\theta + \delta_{i2} \cos\theta) + \delta_{\tilde{\varepsilon} 2} \delta_{i3} f_b(t) \right]. \end{aligned}$$

Multiplying $\dfrac{\partial E}{\partial z_\varepsilon^i}$ by v^i, given by (14), and adding in i we obtain equation (23). Now, multiplying the derivatives of E with respect to the variables z_η^j, given above, by the expression in (17) and adding in j and η, we obtain equation (24).

The second order derivatives of E are given by

$$\begin{aligned} \frac{\partial^2 E}{\partial z_\varepsilon^i \partial z_\eta^j} &= 2b^2 \Big[\delta_{j3} \delta_{\eta \tilde{\varepsilon}} (-1)^{\tilde{\varepsilon} + \tau} z_{\tilde{\tau}}^i z_\tau^3 + z_{\tilde{\varepsilon}}^3 ((-1)^{\tilde{\varepsilon} + \tilde{\eta}} \delta_{ji} z_{\tilde{\eta}}^3 + (-1)^{\tilde{\varepsilon} + \eta} \delta_{j3} z_{\tilde{\eta}}^i) + \\ &\quad + \delta_{i3} \sum_k (-1)^{\varepsilon + \eta} \delta_{j3} z_{\tilde{\eta}}^k z_{\tilde{\varepsilon}}^k + \delta_{i3} (-1)^{\varepsilon + \tilde{\eta}} z_{\tilde{\eta}}^3 z_{\tilde{\varepsilon}}^j + \delta_{i3} (-1)^{\varepsilon + \tau} \delta_{\eta \tilde{\varepsilon}} z_\tau^3 z_{\tilde{\tau}}^j \Big], \end{aligned}$$

using (15) and (16) we obtain

$$\begin{aligned} \frac{\partial^2 E}{\partial z_\varepsilon^i \partial z_\eta^j} &= 2b^2 \Big[\delta_{j3} \delta_{\eta \tilde{\varepsilon}} (-1)^{1 + \tilde{\varepsilon}} f_b(t) (-\delta_{i1} \sin\theta + \delta_{i2} \cos\theta) + \delta_{\tilde{\varepsilon} 1} ((-1)^{1 + \tilde{\eta}} \delta_{ji} \delta_{\tilde{\eta} 1} + \\ &\quad + (-1)^{1 + \eta} \delta_{j3} z_{\tilde{\eta}}^i) + \delta_{i3} \Big(\sum_k (-1)^{\eta + \varepsilon} \delta_{j3} z_{\tilde{\eta}}^k z_{\tilde{\varepsilon}}^k + (-1)^{\varepsilon + \tilde{\eta}} \delta_{\tilde{\eta} 1} \left[\delta_{j1} (-\delta_{\tilde{\varepsilon} 1} f_b'(t) \cos\theta - \right. \\ &\quad - \delta_{\tilde{\varepsilon} 2} f_b(t) \sin\theta) + \delta_{j2} (-\delta_{\tilde{\varepsilon} 1} f_b'(t) \sin\theta + \delta_{\tilde{\varepsilon} 2} f_b(t) \cos\theta) + \left. \delta_{j3} z_{\tilde{\varepsilon}}^j \right] + \\ &\quad + (-1)^{\varepsilon + 1} f_b(t) \delta_{\eta \tilde{\varepsilon}} (-\delta_{j1} \sin\theta + \delta_{j2} \cos\theta) \Big) \Big]. \end{aligned}$$

Multiplying this expression by $\varphi^j_{x_\varepsilon x_\eta} v^i$ and adding in all the indices we have that

$$\begin{aligned}\frac{\partial^2 E}{\partial z^j_\eta \partial z^i_\varepsilon}\varphi^j_{x_\varepsilon x_\eta} v^i &= 2b^2\Big\{\delta_{\varepsilon 2}(-1)^{1+\tilde{\eta}}\delta_{ij}\delta_{\eta 2} + \delta_{i3}\,[\delta_{\eta 2}(-\delta_{j1}(\delta_{\varepsilon 2} f_b'(t)\cos\theta + \delta_{\varepsilon 1} f_b(t)\sin\theta) + \\ &+\delta_{j2}(-\delta_{\varepsilon 2} f_b'(t)\sin\theta + \delta_{\varepsilon 1} f_b(t)\cos\theta)) + (-1)^{1+\varepsilon} f_b(t)\delta_{\tilde{\varepsilon}\eta}(-\delta_{j1}\sin\theta + \\ &+\delta_{j2}\cos\theta)]\Big\}\varphi^j_{x_\varepsilon x_\eta} v^i.\end{aligned}$$

Hence, as a consequence of (14), we obtain equation (25).

In order, to prove equation (27), we consider the expressions given by (14) - (18). Using (29) and considering the sums in all the indices, it follows that

$$\frac{1}{2}\frac{\partial^2 C^2}{\partial z^j_\eta \partial z^i_\varepsilon}\varphi^j_{x_\varepsilon x_\eta} v^i = f_b(t)[1 + f_b'^2(t) - f_b(t) f_b''(t)].$$

This concludes the proof of the lemma. □

We can now prove Theorem 4.

Proof of Theorem 4: We begin obtaining some of the coefficients that appear in equation (11). It follows from (20) and (21) that

$$\frac{3(C^2 - 2E)}{2(C^2 - E)} = \frac{3}{2}\frac{(1 - 2b^2 + f_b'^2)}{1 - b^2 + f_b'^2}; \tag{31}$$

$$-\frac{3C}{2} = -\frac{3 f_b}{2}[1 + f_b'^2]^{\frac{1}{2}}; \tag{32}$$

$$\frac{3C}{4(C^2 - E)} = \frac{3}{4 f_b}\frac{[1 + f_b'^2]^{\frac{1}{2}}}{(1 - b^2 + f_b'^2)}; \tag{33}$$

$$\frac{2E + C^2}{C} = f_b\frac{(1 + 2b^2 + f_b'^2)}{[1 + f_b'^2]^{\frac{1}{2}}}. \tag{34}$$

We observe that the terms in equation (11) which contain $\frac{\partial C}{\partial z^i_\varepsilon} v^i$ vanish due to (22). Hence, it follows from (22) - (27) that equation (11) reduces to

$$\begin{aligned}&-3b^2 C f_b[1 + 2(f_b')^2] - 3\frac{(2E - C^2)}{C^2 - E} b^2 f_b^2 (f_b')^2 \frac{(f_b f_b'' + 1 + (f_b')^2)}{\sqrt{1 + (f_b')^2}} + \frac{3b^4 C}{C^2 - E} f_b^3 (f_b')^2 + \\ &+\frac{2E + C^2}{C} f_b[-f_b f_b'' + 1 + (f_b')^2] = 0.\end{aligned}$$

Substituting the expressions (31) - (34), we obtain equation (13), which proves Theorem 4. □

4. The Minimal Surfaces of Rotation in Randers Space

In this section, we will investigate the existence and uniqueness of the solutions of equation (13), providing a qualitative study of the solutions. Besides, we will obtain the properties of the surfaces generated by the rotation of the solution curves. Before we prove our results, we observe that when $b = 0$, i.e., F is the Euclidean metric, then the classical result shows that the unique minimal surfaces of rotation are the catenoids. In fact, when $b = 0$, the catenary $f_0(t) = c_1 \cosh(\frac{1}{c_1}t + c_2)$ is the only solution of equation (13) and the minimal surface is unique up to homothety. Similarly, in Randers space, if $f_b(t)$ is a solution of (13), then $h(t) = \dfrac{1}{c} f_b(a + ct), c \neq 0$, is also a solution. Therefore, the solutions are unique up to homothety.

We will now start analyzing the existence of solutions of equation (13). Since it is a second order differential equation, it can be written in the form of a system of first order as:

$$\begin{cases} \dot{x}_1 &= x_2 \\ x_1 \dot{x}_2 Q_b(x_2) &= P_b(x_2), \end{cases} \tag{35}$$

where $x_1(t) = f_b(t)$ and $x_2(t) = \dot{x}_1(t)$,

$$P_b(x_2) = (1 + x_2^2)(1 - b^2 + x_2^2)[1 - b^2 + (1 - 3b^2)x_2^2], \tag{36}$$

and

$$Q_b(x_2) = (1 - b^2 + x_2^2)[1 + 2b^2 + (1 - 3b^2)x_2^2] + 3b^4 x_2^2. \tag{37}$$

Remarks:

1) For $0 \leq b \leq \frac{\sqrt{3}}{3}$ the polynomials P_b and Q_b, given by (36) and (37), respectively, are strictly positive.

2) For $\frac{\sqrt{3}}{3} < b < 1$, we have:

i) $P_b(\pm N_1(b)) = 0$, where

$$N_1(b) = \sqrt{\frac{1 - b^2}{3b^2 - 1}}. \tag{38}$$

ii) $P_b(x_2) > 0$ (resp. < 0) if and only if $|x_2| < N_1(b)$, (resp. $|x_2| > N_1(b)$).

iii) $Q_b(\pm N_2(b)) = 0$, where

$$N_2(b) = \sqrt{\frac{1 - b^2 + 3b^4 + b^2\sqrt{12 - 12b^2 + 9b^4}}{3b^2 - 1}}. \tag{39}$$

iv) $Q_b(x_2) > 0$ (resp. < 0) if and only if $|x_2| < N_2(b)$ (resp. $|x_2| > N_2(b)$).

v) $0 < N_1(b) < N_2(b)$ for every b, $\frac{\sqrt{3}}{3} < b < 1$.

3) There are no solutions for equation (13), when $\frac{\sqrt{3}}{3} < b < 1$ and the initial conditions are given by $f_b(0) = a \neq 0$, $f_b'(0) = d$, with $|d| = N_2(b)$.

The last remark follows from the fact that $Q_b(\pm N_2(b)) = 0$ and $P_b(\pm N_2(b)) < 0$. Due to the remarks above, we will consider the study of equation (13) in two cases, depending on the value of b, namely:

Case 1: $0 \le b \le \frac{\sqrt{3}}{3}$, with initial conditions $f_b(0) = a > 0$ and $f_b'(0) = d \in R$;

Case 2: $\frac{\sqrt{3}}{3} < b < 1$, with initial conditions $f_b(0) = a > 0$ and $f_b'(0) = d$ where $|d| \neq N_2(b)$.

Since we are interested in rotational surfaces, without loss of generality we are considering $f_b(0) > 0$. In those conditions, system (35) is equivalent to

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \dfrac{P_b(x_2)}{x_1 Q_b(x_2)}. \end{array} \right.$$

The following lemmas provide information on the solutions of equation (13).

Lemma 6 *Let $0 \le b \le \frac{\sqrt{3}}{3}$, and $f_b(t)$ be the solution of the equation (13) defined on the maximal interval J, satisfying the initial conditions $f_b(0) = a > 0$, $f_b'(0) = d$, where d is an arbitrary constant. Then:*

(i) $f_b(t) f_b''(t) > 0, \forall t \in J$;

(ii) *there exists $t_1 \in J$ such that $f_b'(t_1) = 0$;*

(iii) *The function f_b is symmetric with respect to the straight line $t = t_1$.*

Proof: Since $f_b(t) f_b''(t) = \dfrac{P_b(f_b')}{Q_b(f_b')}$, $\forall\, t \in J$, then property (i) follows from Remark 1. Therefore, $a > 0$ implies that $f_b(t) > 0$ in J and hence $f_b''(t) > 0$, i.e. $f_b'(t)$ is increasing in J. The maximality of J implies the existence of t_1 such that $f_b'(t_1) = 0$.

In order to prove item (iii) we define the functions

$$g_+(t) = f_b(t + t_1), \quad g_-(t) = f_b(-t + t_1).$$

The functions $g_\pm$ satisfy equation (13) with the same initial conditions. By the theorem of existence and uniqueness of solutions of ordinary differential equations, it follows that $g_- = g_+$. Therefore, f_b is symmetric with respect to the straight line $t = t_1$. □

The following lemma folllows immediately from (13):

Lemma 7 Let b, $\frac{\sqrt{3}}{3} < b < 1$, and $f_b(t)$ be the solution of the equation (13), with initial conditions $f_b(0) = a > 0$, $f_b'(0) = \pm N_1(b)$. Then $f_b(t) = \pm N_1(b) t + a$.

Lemma 8 *Let b, $\frac{\sqrt{3}}{3} < b < 1$, and $f_b(t)$ be the solution of the equation (13), defined on the maximal interval J, with initial conditions given by $f_b(0) = a > 0$, $f_b'(0) = d$, where $|d| < N_1(b)$. Then*

(i) $|f_b'(t)| < N_1(b)$, $\forall\, t \in J$;

(ii) $f_b(t) f_b''(t) > 0$, $\forall\, t \in J$;

(iii) $J = (-\infty, \infty)$;

(iv) *there exists $t_1 \in J$ such that $f_b'(t_1) = 0$;*

(v) *$f_b(t)$ is symmetric with respect to the straight line $t = t_1$;*

(vi) *the curve $f_b(t)$ does not intersect the line $sign(d)\, N_1(b) t + a$, $\forall\, t \neq 0$, where sign(d)=1 (resp. -1) when $d > 0$ (resp. $d < 0$).*

Proof: Since $f_b(0) = a > 0$ and $f_b'(0) = d$, with $|d| < N_1(b)$, it follows that in a neighborhood I of $t = 0$, $|f_b'(t)| < N_1(b)$. Therefore, as a consequence of Remark 2 we have

$$P_b(f_b'(t)) > 0 \text{ and } Q_b(f_b'(t)) > 0, \ \forall\, t \in I.$$

We want to prove that $I = J$. Otherwise, there would exist t_0 in the boundary of I, where $f_b'(t_0) = \pm N_1(b)$. By the uniqueness of the solution of the equation (13) f_b would be the linear solution, which contradicts $f_b'(0) = d$, and $|d| < N_1(b)$. This concludes the proof of (i) and (ii).

The properties (i),(ii), Lemma 7 and Remark 2 imply that $J = I\!R$. In fact, one proves easily that the set $J = \{t \in I\!R;\ f_b(t) > 0 \text{ and } |f_b'(t)| < N_1(b)\}$ is a non empty open set whose complement is also open.

The proofs of (iv) and (v) are analogous to the proofs of ii) and iii) of Lemma 6. The property (vi) holds, otherwise there would exist $t_0 \in J\backslash\{0\}$ for which $f_b'(t_0) = \text{sign}(d)\, N_1(b)$ which contradicts (i). □

Lemma 9 *Consider* $\frac{\sqrt{3}}{3} < b < 1$ *and* $f_b(t)$ *the solution of equation (13), defined on the maximal interval* J*, with initial conditions given by* $f_b(0) = a > 0$*,* $f_b'(0) = d$*, with* $N_1(b) < |d| < N_2(b)$*. Then*

(i) $N_1(b) < |f_b'(t)| < N_2(b), \forall\, t \in J$;

(ii) $f_b(t) f_b''(t) < 0, \forall t \in J$;

(iii) *the curve* $f_b(t)$ *for* $t \in J\backslash\{0\}$ *is contained between the lines* $N_1(b)t + a$ *and* $N_2(b)t + a$ *when* $d > 0$ *and between the lines* $-N_2(b)t + a$ *and* $-N_1(b)t + a$ *when* $d < 0$.

(iv) J *is a bounded open interval.*

Proof: (i) follows from Remark 3 and Lemma 7. As a consequence of (i) and Remark 2 ii) and iv) we have that $f_b f_b'' < 0$ in J. When $d > 0$, then the curve $f_b(t)$, $t \in J$, does not intersect the lines $N_i(b)t + a$. In fact, for t close to the origin, we have $N_1(b)t + a < f_b(t) < N_2(b)t + a$ when $t > 0$ and $N_2(b)t + a < f_b(t) < N_1(b)t + a$, when $t < 0$. Since the curve is concave, assume there exists $t_0 > 0$ (resp. $t_0 < 0$) such that $f_b(t_0) = N_1(b)t_0 + a$ (resp. $f_b(t_0) = N_2(b)t_0 + a$), then there exists $t_1 < t_0$ (resp. $t_1 > t_0$), such that $f_b'(t_1) = N_1$ (resp. $f_b'(t_1) = N_2$), which is a contradiction. Similarly, when $d < 0$, one shows that the curve $f_b(t)$, $t \in J$, satisfies $-N_2(b)t + a < f_b(t) < -N_1(b)t + a$, for $t > 0$ and $-N_1(b)t + a < f_b(t) < -N_2(b)t + a$, for $t < 0$.

We now prove that J is a bounded interval for $d > N_1(b) > 0$. Otherwise, since $f_b'' < 0$ there exists $t_0 > 0$ such that $f_b'(t_0) = 0$, hence there exists t_1, $0 < t_1 < t_0$ such that $f_b'(t_1) = N_1(b)$ which is a contradiction, due to Lemma 7. Therefore, J is bounded from above. In order to show that J is bounded from below, we observe that from (iii) we have $f_b(-a/N_1(b)) < 0$, hence there exists t_2, $-a/N_1(b) < t_2 < 0$ such that $f_b(t_2) = 0$ which contradicts ii). Similarly one proves that J is bounded when $d < 0$. □

Lemma 10 *Consider* $\frac{\sqrt{3}}{3} < b < 1$ *and* $f_b(t)$ *the solution of equation (13) defined on the maximal interval* J*, with initial conditions* $f_b(0) = a > 0$*,* $f_b'(0) = d$*, with* $|d| > N_2(b)$*. Then*

(i) $|f_b'(t)| > N_2(b)$, $\forall t \in J$;

(ii) $f_b(t) f_b''(t) > 0$, $\forall t \in J$;

(iii) *the curve* $f_b(t)$ *for* $t \in J \setminus \{0\}$ *does not intersect the line sign(d)* $N_2(b)t + a$*.*
(iv) J *is bounded from below (resp. above) when* $d > N_2(b)$ *(resp.* $d < -N_2(b)$*).*

Proof: We will prove the lemma for $d > 0$. The case $d < 0$ is analogous. Since $d > N_2(b)$, it follows that $f_b'(t) > N_2(b)$ in a neighborhood of the origin and due to Remark 3 it does not reach $N_2(b)$ in J. As a consequence of (i) and Remark 2 ii) and iv) we have that $f_b f_b'' > 0$ in J.

From the initial conditions, we have that $a + N_2(b)t < f_b(t)$ when $t > 0$ and $f_b(t) < a + N_2(b)t$ when $t < 0$ for $t \in J$ sufficiently close to the origin. Assume there exist $t_1 < 0$ in J, such that $f_b(t_1) = N_2(b)t_1 + a$. Then there exists t_0, $t_1 < t_0 < 0$ such that $f_b'(t_0) = N_2(b)$ which contradicts (i). Therefore, property (iii) holds.

Assume J is not bounded below, since $d > N_2(b)$ and $f_b'' > 0$ in J, there exists $t_0 < 0$ such that $f_b'(t_0) = 0$. Hence there exists t_1, $t_0 < t_1 < 0$ with $f_b'(t_1) = N_2(b)$ which again contradicts (i). □

Now we can present the result on the existence and uniqueness of minimal surfaces of rotation, forward complete, in Randers spaces (V^3, F_b), for each b, $0 \leq b \leq \frac{\sqrt{3}}{3}$.

Theorem 11 *Up to homothety, there is a unique minimal surface of rotation in Randers space* (V^3, F_b)*, for* $0 \leq b \leq \frac{\sqrt{3}}{3}$*. Moreover, the minimal surface is embedded, forward complete, symmetric with respect to a plane perpendicular to the rotation axis and it is generated by a concave plane curve.*

Proof: It follows from Theorem 4 that the rotation surface $\varphi_b(t,\theta) = (f_b(t)\cos\theta, f_b(t)\sin\theta, t)$ is minimal if and only if f_b is a solution of the equation (13). Without loss of generality, due to Lemma 6 ii) we may consider initial conditions $f_b(0) = a > 0$ and $f_b'(0) = 0$. Let J be the maximal interval where f_b is defined.

Now, in order to show that φ_b is forward complete we consider an arbitrary divergent curve $\gamma : [0,\infty) \longrightarrow M$, given by $\gamma(s) = (f_b(t(s))\cos\theta(s), f_b(t(s))\sin\theta(s), t(s))$.

Since the curve is divergent, it follows that the length of the vector $\gamma(s)$ in Randers metric is unbounded when $s \to \infty$. Therefore, the euclidean length $|\gamma(s)| \to \infty$ when $s \to \infty$.

If $J = (-\omega, \omega)$, $\omega < \infty$, then $\lim_{t\to\pm\omega} |f_b(t)| = \infty$, and

$$\begin{aligned}
\int_0^r F(\gamma'(s))ds &= \int_0^r \left\{ \sqrt{\left[\frac{d}{ds} f_b(t(s))\right]^2 + [f_b(t(s))\theta'(s)]^2 + t'(s)^2} + bt'(s) \right\} ds \\
&\geq \int_0^r \left\{ \sqrt{\left[\frac{d}{ds} f_b(t(s))\right]^2} + bt'(s) \right\} ds \\
&\geq |f_b(t(r))| + bt(r) - |f_b(t(0))| - bt(0).
\end{aligned}$$

Therefore, it follows from (8) and the unboundedness of $|f_b|$ that $L_F(\gamma) = \lim_{r\to\infty} \int_0^r F(\gamma'(s))ds = \infty$, i.e. the immersion is forward complete.

Similarly, in the case $J = (-\infty, \infty)$ we observe that

$$\begin{aligned} \int_0^r F(\gamma'(s))ds &\geq \left| \int_0^r t'(s)ds \right| + b \int_0^r t'(s)ds \\ &\geq |t(r)| + bt(r) - |t(0)| - bt(0). \end{aligned}$$

We conclude that $L_F(\gamma) = \infty$, since $|t(r)| + bt(r) \to \infty$ when $r \to \infty$. The embeddedness, the symmetry of the surface and the fact that the curve generating the surface is concave follows from (i) and (iii) of Lemma 6.

□

In our next results, we consider the minimal surfaces of rotation in Randers space (V^3, F_b), where $\frac{\sqrt{3}}{3} < b < 1$.

Theorem 12 *Up to homothety, there exists a unique, forward complete, minimal surface of rotation in the Randers space (V^3, F_b) for $\frac{\sqrt{3}}{3} < b < 1$. The surface is embedded, symmetric with respect to a plane perpendicular to the rotation axis and it is generated by a concave plane curve. Moreover, the curve $(0, f_b(t), t)$ generating the surface is such that $|f_b'(t)| < N_1(b), \forall t$.*

The proof of this theorem will be a consequence of two propositions:

Proposition 13 *Let (V^3, F_b) be the Randers space, where $\frac{\sqrt{3}}{3} < b < 1$. The minimal surfaces of rotation generated by the curve $(0, f_b(t), t)$, where f_b is a solution of equation (13) with initial condition $f_b(0) = a > 0$ and $f_b'(0) = d$, with $|d| \geq N_1(b)$ and $|d| \neq N_2(b)$ are not forward complete. In particular, when $d = \pm N_1(b)$, then the minimal surface is a cone generated by the straight line $f_b(t) = \pm N_1(b)t + a$.*

Proof: If $d = \pm N_1(b)$, then it follows from Lemma 7 that the minimal surface is a cone (punctered at the vertex), which is not forward complete.

If $N_1(b) < |d| < N_2(b)$, then Lemma 9 implies that the solution of (13), satisfies $N_1(b) < |f_b'(t)| < N_2(b), \forall t \in J$, where $J = (\omega_-, \omega_+)$ is the bounded maximal interval. Let us consider the divergent curve on the surface $\gamma(t) = (f_b(t), 0, t)$, $t \in (\omega_-, 0)$. The length of γ is given by

$$L_F(\gamma) = \lim_{t \to \omega_-} \int_t^0 \left[\sqrt{1 + f_b'(t)^2} + b \right] dt < -\left[\sqrt{1 + (N_2(b))^2} + b \right] \omega_- < \infty.$$

Therefore, the minimal surface is not forward complete.

If $d > N_2(b)$, let $f_b(t)$ be the solution of (13), defined on the maximal interval J, with initial conditions $f_b(0) = a > 0$ and $f_b'(0) = d$. From Lemma 10, $J = (\omega_-, \omega_+)$ where $-\infty < \omega_- < 0$. Moreover, $f_b''(t) > 0$, hence $N_2(b) < f_b'(t) < f_b'(0) = d$, for all $t < 0$ in J.

Now we consider the divergent curve $\gamma(t) = (f_b(t), 0, t), t \in]\omega_-, 0[$, on the surface generated by γ. Then

$$L_F(\gamma) = \lim_{t \to \omega_-} \int_t^0 [\sqrt{1 + f_b'(t)^2} + b] dt < -[\sqrt{1 + d^2} + b]\omega_- < \infty.$$

We conclude that the surface is not forward complete. Similarly, one proves the non completeness when $d < -N_2(b) < 0$. □

We observe that the minimal cones of the above result converge to a cylinder when $b \to 1$, since $\lim_{b\to 1} N_1(b) = 0$.

Proposition 14 *Let (V^3, F_b) be the Randers space, where $\frac{\sqrt{3}}{3} < b < 1$. The minimal surfaces of rotation generated by the curve $(0, f_b(t), t)$ where f_b is a solution of equation (13), with initial conditions $f_b(0) = a > 0$, $f_b'(0) = d$, where $|d| < N_1(b)$ are forward complete.*

Proof: From Lemma 8, f_b is defined on $(-\infty, \infty)$. Since $\gamma(s)$ is a divergent curve, the length of $\gamma(s)$ in the metric $F_{\alpha+\beta}$ and hence in the Euclidean metric is unbounded. Hence $\lim_{s\to\infty} \sqrt{f_b^2(t(s)) + t^2(s)} = \infty$, therefore $\lim_{s\to\infty} |t(s)| = \infty$. We will compute the length of $\gamma(s)$. Observe that $F(\gamma'(s)) \geq |t'(s)| + bt'(s)$.

Therefore

$$L_F(\gamma) = \lim_{r\to\infty} \int_0^r F(\gamma'(s))ds \geq \lim_{r\to\infty} \left\{ \left| \int_0^r t'(s)ds \right| + bt(s)|_0^r \right\} = \infty.$$

Since γ is an arbitrary curve we conclude that the rotation surface is forward complete. □

Theorem 12 follows from Propositions 13 and 14. The embeddedness and symmetry of the surface, as well as the upper bound for the inclination of the tangent line of the generating curve, follow from Lemma 8.

As a consequence of Theorems 11 and 12, we obtain our main result, Theorem 1 stated in the Introduction, on the existence and uniqueness, up to homothety, of a minimal surface of rotation, forward complete, on a Randers space (V^3, F_b), for each $b, 0 \leq b < 1$.

We conclude by observing that the differential equation which characterizes the minimal surfaces that are graphs of functions on a Randers space were obtained in [So] and it will appear elsewhere. Finally, we want to thank Z. Shen for the series of lectures on Finsler geometry, while he was visiting the University of Brasilia.

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