A Bernstein type theorem on a Randers space

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Abstract

We consider Finsler spaces with a Randers metric $F = \alpha + \beta$, on the three-dimensional real vector space, where α is the Euclidean metric and β is a 1-form with norm b, $0 \leq b < 1$. By using the notion of mean curvature for immersions in Finsler spaces, introduced by Z. Shen, we obtain the partial differential equation that characterizes the minimal surfaces which are graphs of functions. For each b, $0 \leq b < 1/\sqrt{3}$, we prove that it is an elliptic equation of mean curvature type. Then the Bernstein type theorem and other properties, such as the nonexistence of isolated singularities, of the solutions of this equation follow from the theory developped by L. Simon. For $b \geq 1/\sqrt{3}$, the differential equation is not elliptic. Moreover, for every b, $1/\sqrt{3} < b < 1$ we provide solutions, which describe minimal cones, with an isolated singularity at the origin.

1. Introduction

A classical result of S. Bernstein states that the plane is the only regular minimal surface of $I\!R^3$, which is the graph of a C^2-function defined on the whole plane. This is a consequence of the partial differential equation which characterizes such surfaces. Many properties of the minimal surfaces can be atributed to the form of this equation. A major contribution to the theory of such equations was given by L. Simon [S1,S2], who considered the class of equations of *mean curvature type*. The pioneering work in this direction was done by R. Finn [F], who considered the *equations of minimal type*. Later other important results were obtained by Jenkins [J], Jenkins-Serrin [JS] and Spruck [Sp].

The theory of minimal surfaces in Finsler spaces is quite recent. Actually, the first non trivial examples of such surfaces were studied in [ST]. The fundamental contribution on this subject was given by Z. Shen in [Sh1]. He introduced the notion of mean curvature for immersions into Finsler manifolds and he established some of its properties. As in the Riemannian case, if the mean curvature is identically zero, then the immersion is said to be minimal.

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The main purpose of this paper is to prove a Bernstein type theorem in a three-dimensional vector space, equipped with a Randers metric. This metric can be viewed as the simplest possible perturbation of the Euclidean metric in a fixed direction. This perturbation has a norm b, where $0 \leq b < 1$ ($b = 0$ being the Euclidean case).

Our main result follows from the partial differential equation that characterizes the minimal surfaces, which are graphs of functions, in this Randers space. We show that for each b, $0 \leq b < 1/\sqrt{3}$, this equation is an elliptic differential equation of mean curvature type. Then the Bernstein type theorem in this Randers space follows from the theory developped by L. Simon for such equations. Similary, as a consequence of this theory, one gets several results for the solutions of the equation such as a-priori gradient estimates, a Bers-type theorem concerning the limiting behaviour of the gradient of solutions defined outside a compact set, a global Hölder continuity estimate for solutions which continuously attain Lipschitz boundary values and a theorem concerning the removability of isolated singularities.

When $b \geq 1/\sqrt{3}$, the differential equation is not elliptic and one does not know if the Bernstein type theorem holds. However, one can show that the property of nonexistence of isolated singularities does not hold. In fact, for each b, $b > 1/\sqrt{3}$, we provide a minimal cone with an isolated singularity at the origin.

We conclude this introduction by pointing out that the differential equation is sensitive to the fixed direction in the Randers space. In fact, a minimal surface which is the graph of a function f over a certain plane may not be minimal as a surface obtained as the graph of f over a different plane (see Examples in Section 3).

2. Preliminaries

We will follow the notation and terminology of [Sh1] and [ST], and we will make use of the following conventions: we will use Greek letters τ, η, ε for indices running from 1 to n, and Latin letters i, j, k, l for indices running from 1 to $n+1$. We will also use Einstein's convention, i.e., in general we will not write the symbol of the summand to represent the sum on repeated indices.

Let M^n be a C^∞ n-manifold, and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x, y) be a point of TM, $x \in M, y \in T_xM$. We consider local coordinates $(x^1, ..., x^n)$ on an open subset U of M. As usual, $\partial/\partial x^i$ and dx^i are the induced coordinate basis for T_xM and T_x^*M and (x^i, y^i) are local coordinates on $\pi^{-1}(U) \subset TM$, where $y = y^i \partial/\partial x^i$. A function $F : TM \longrightarrow [0, \infty)$ is called a *Finsler metric* on M if F has the following properties: [i] (Regularity) $F \in C^\infty$ in $TM \setminus \{0\}$; [ii] (Positive Homogeneity) $F(x, ty) = tF(x, y)$, $\forall t > 0, (x, y) \in TM$; [iii] (Strong Convexity) $g = (g_{ij}(x, y)) = \left(\frac{1}{2}[F^2(x, y)]_{y^i y^j}\right)$ is positive definite at each point of $TM \setminus \{0\}$. The pair (M, F) is called a *Finsler space.*

Examples of Finsler manifolds are Minkowski spaces and Randers spaces. Denote by

V^n the standard n-dimensional real vector space. A *Minkowski space* is V^n equipped with a Minkowski norm F (whose indicatrix is strongly convex), i.e., $F(x,y)$ depends only on $y \in T_xV^n$. A *Randers metric* on M is the Finsler structure F on TM given by

$$F(x,y) = \alpha(x,y) + \beta(x,y),$$

where $\alpha(x,y) = \sqrt{a_{ij}(x)y^iy^j}$, $\beta(x,y) = b_k(x)y^k$, and a_{ij}, a^{ij} are the components of the Riemannian metric and of its inverse matrix respectively and b_k are the components of the 1-form β, whose *norm* $b = \sqrt{a^{ij}b_ib_j}$, satisfies $0 \leq b < 1$.

If (M^n, F) is a Finsler space, then F induces a smooth volume form defined by

$$d\mu_F = \sigma(x)dx^1 \wedge ... \wedge dx^n$$

where

$$\sigma(x) = \frac{vol\,(B^n)}{vol\,\{y \in T_xM;\ F(x,y) \leq 1\}},$$

B^n is a unit ball in $I\!R^n$ and vol is the Euclidean volume.

Let $(\widetilde{M}^m, \tilde{F})$ be a Finsler manifold, with local coordinates $(\tilde{x}^1, ..., \tilde{x}^m)$ and let $\varphi : M^n \longrightarrow (\widetilde{M}^m, \widetilde{F})$ be an immersion. Then there is an induced Finsler metric on M, defined by

$$F(x,y) = (\varphi^*\widetilde{F})(x,y) = \widetilde{F}(\varphi(x), \varphi_*(y)), \quad \forall\ (x,y) \in TM.$$

The notion of mean curvature was introduced by Z. Shen (see [Sh1]) as follows. Let $\varphi : M^n \longrightarrow (\widetilde{M}^m, \widetilde{F})$ be an immersion in a Finsler space and let $\varphi_t : M^n \longrightarrow (\widetilde{M}^m, \widetilde{F})$, $t \in (-\varepsilon, \varepsilon)$ be a variation such that for all t φ_t is an immersion, $\varphi_0 = \varphi$ and $\varphi_t = \varphi$ outside a compact set $\Omega \subset M$. Then $F_t = \varphi_t^*\widetilde{F}$ denotes the induced metric of the variation and $\tilde{X} = \frac{\partial \varphi_t}{\partial t}|_{t=0}$ is the variational vector field. Consider the function $V(t) = \int_\Omega d\mu_{F_t}$. Then

$$V'(0) = \int_M \mathcal{H}_\varphi(\tilde{X})d\mu_F,$$

where $\mathcal{H}_\varphi$ is called the *mean curvature* of the immersion φ. One can show that $\mathcal{H}_\varphi(v)$ depends linearly on v and $\mathcal{H}_\varphi$ vanishes on $\varphi_*(TM)$ (cf. Lemmas in [Sh1]). The immersion φ is said to be *minimal* when $\mathcal{H}_\varphi \equiv 0$.

From now on, we will consider hypersurfaces in a special Randers space $\varphi : M^n \longrightarrow (\widetilde{V}^{n+1}, \widetilde{F}_b)$, where $\widetilde{V}$ is a $n+1$-dimensional real vector space, $\widetilde{F}_b = \alpha + \beta$, where α is the Euclidean metric, and β is a 1-form with norm b, $0 \leq b < 1$. Without loss of generality we will consider $\beta = b\,d\tilde{x}^{n+1}$. If M^n has local coordinates $x = (x^\varepsilon)$, $\varepsilon = 1, \cdots, n$, and $\varphi(x) = (\varphi^i(x^\varepsilon)) \in \widetilde{V}$, $i = 1, \cdots, n+1$, we consider the application

$$\mathcal{F}(x,z) = \frac{vol(B^n)}{vol(D_x^n)}, \tag{1}$$

where $x \in M$,

$$z = (z^i_\alpha) = \left(\frac{\partial \varphi^i}{\partial x^\alpha} \right), \tag{2}$$

B^n = unitary ball in $I\!R^n$ and

$$D^n_x = \left\{ (y^1, y^2, ..., y^n) \in I\!R^n \mid F(x,\, y^\alpha z_{\alpha_{|x}}) < 1 \right\}, \tag{3}$$

where $z_\alpha = \partial\varphi / \partial x^\alpha$.

The *induced volume element* of (M, F) is given by

$$dV_F = \mathcal{F}(x, z) dx, \tag{4}$$

where $\mathcal{F}(x, z)$ is given by (1).

The Euclidean volume of D^n_x is given by

$$vol\; D^n_x = \frac{vol\; B^n}{\left(1 - b^2 A^{\tau\gamma} z^{n+1}_\tau z^{n+1}_\gamma\right)^{\frac{n+1}{2}} \sqrt{detA}},$$

where

$$A = (A_{\tau\gamma}) = \left(\sum_{i=1}^{n+1} z^i_\tau z^i_\gamma \right), \tag{5}$$

and $(A^{\tau\gamma}) = (A_{\tau\gamma})^{-1}$. It follows from (4) that the volume form dV_F is given by the following formula ([Sh2])

$$dV_F = \left(1 - b^2 A^{\tau\gamma} z^{n+1}_\tau z^{n+1}_\gamma\right)^{\frac{n+1}{2}} \sqrt{detA}\, dx^1 \cdots dx^n. \tag{6}$$

The mean curvature $\mathcal{H}_\varphi$ is given by (see [Sh1])

$$\mathcal{H}_\varphi(v) = \frac{1}{\mathcal{F}} \left\{ \frac{\partial^2 \mathcal{F}}{\partial z^i_\varepsilon \partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x^\varepsilon \partial x^\eta} + \frac{\partial^2 \mathcal{F}}{\partial \tilde{x}^j \partial z^i_\varepsilon} \frac{\partial \varphi^j}{\partial x^\varepsilon} - \frac{\partial \mathcal{F}}{\partial \tilde{x}^i} \right\} v^i.$$

Observe that whenever (V, F) is a Minkowsky space, the expression of the mean curvature reduces to

$$\mathcal{H}_\varphi(v) = \frac{1}{\mathcal{F}} \left\{ \frac{\partial^2 \mathcal{F}}{\partial z^i_\varepsilon \partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x^\varepsilon \partial x^\eta} \right\} v^i. \tag{7}$$

3. The differential equation for a minimal surface which is the graph of a function

In this section, we recall the differential equation which characterizes the minimal hypersurfaces M^n in the Randers space (V^{n+1}, F_b). We then restrict ourselves to surfaces immersed in V^3 and we obtain the differential equation which characterizes the minimal surfaces which are the graph of a function.

Theorem 1 [ST] *Let* $\varphi : M^n \longrightarrow (V^{n+1}, F_b)$ *be an immersion into a Randers space, with local coordinates* $(\varphi^j(x))$. *Then* φ *is minimal, if and only if, it satisfies the differential equation*

$$\left\{ \frac{(n^2-1)}{4} \frac{\partial B}{\partial z^i_\varepsilon} \frac{\partial B}{\partial z^j_\eta} C - \frac{n+1}{2}(1-B)\left(\frac{\partial^2 B}{\partial z^i_\varepsilon \partial z^j_\eta} C + \frac{\partial B}{\partial z^j_\eta} \frac{\partial C}{\partial z^i_\varepsilon} + \frac{\partial B}{\partial z^i_\varepsilon} \frac{\partial C}{\partial z^j_\eta} \right) + \right.$$
$$\left. +(1-B)^2 \frac{\partial^2 C}{\partial z^i_\varepsilon \partial z^j_\eta} \right\} \frac{\partial^2 \varphi^j}{\partial x^\varepsilon \partial x^\eta} v^i = 0, \qquad \forall v = v^i e_i \in V^{n+1}, \tag{8}$$

where

$$C = \sqrt{detA}, \quad B = b^2 A^{\varepsilon\eta} z^{n+1}_\varepsilon z^{n+1}_\eta, \tag{9}$$

$\{e_i\}$ *is the canonical basis of* V^{n+1}, z^i_ε *is given by (2),* A *is given by (5).*

In what follows, we will restrict ourselves to studying minimal surfaces in the three-dimensional Randers space. As a consequence of the above theorem one has the following result.

Theorem 2 [ST] *Let* $\varphi : M^2 \longrightarrow (V^3, F_b)$ *be an immersion given in local coordinates by* $(\varphi^j(x))$. *Then* φ *is minimal, if and only if, it satisfies the differential equation*

$$\left\{ \frac{12E^2 - (2E + C^2)^2}{C(C^2 - E)} \frac{\partial C}{\partial z^i_\varepsilon} \frac{\partial C}{\partial z^j_\eta} - \frac{3C}{2} \frac{\partial^2 E}{\partial z^j_\eta \partial z^i_\varepsilon} - \frac{3}{2}\left(\frac{2E - C^2}{C^2 - E} \right) \left(\frac{\partial C}{\partial z^i_\varepsilon} \frac{\partial E}{\partial z^j_\eta} + \frac{\partial C}{\partial z^j_\eta} \frac{\partial E}{\partial z^i_\varepsilon} \right) + \right.$$
$$\left. + \frac{3C}{4(C^2 - E)} \frac{\partial E}{\partial z^i_\varepsilon} \frac{\partial E}{\partial z^j_\eta} + \frac{(2E + C^2)}{2C} \frac{\partial^2 C^2}{\partial z^j_\eta \partial z^i_\varepsilon} \right\} \frac{\partial^2 \varphi^j}{\partial x^\varepsilon \partial x^\eta} v^i = 0, \qquad \forall\, v = v^i e_i \in V^3, \tag{10}$$

where $z = (z^i_\varepsilon)$ *and* C *are defined by (2) and (9) respectively, and*

$$E = b^2 \sum_{k=1}^{3} (-1)^{\gamma+\tau} z^k_{\tilde\gamma} z^k_{\tilde\tau} z^3_\gamma z^3_\tau, \qquad \tilde\tau = \delta_{\tau 2} + 2\delta_{\tau 1}. \tag{11}$$

We observe that in (11) we have introduced the notation $E = C^2 B$ and $\tilde\tau$, which means $\tilde\tau = 1$ if $\tau = 2$ and $\tilde\tau = 2$ if $\tau = 1$.

In our next results, by considering the immersion to be a surface which is the graph of a function f, we obtain the differential equation that characterizes such minimal surfaces. We will first consider the special and important case, obtained in [S], when the surface is a graph over the $x_1 x_2$-plane (observe that we have chosen $\beta = b\, dx_3$ in the Randers metric) and then we will consider the general case when the surface is the graph over any plane.

Theorem 3 *An immersion* $\varphi : U \subset I\!\!R^2 \longrightarrow (V^3, F_b)$ *given by* $\varphi(x_1, x_2) = (x_1,\, x_2,\, f(x_1, x_2))$ *is minimal, if and only if,* f *satisfies*

$$\sum_{i,j=1,2} \left\{ T_b(T_b - 3b^2)\left(\delta_{ij} - \frac{f_{x_i} f_{x_j}}{W^2}\right) + 3b^2(T_b + b^2)\frac{f_{x_i} f_{x_j}}{W^2} \right\} f_{x_i x_j} = 0, \tag{12}$$

where

$$W^2 = 1 + f_{x_1}^2 + f_{x_2}^2, \qquad T_b = b^2 + (1 - b^2)W^2. \tag{13}$$

Proof: In order to obtain equation (12), we need to compute the expressions involved in (10) for the immersion φ. One computes the first and second order derivatives of C, $\det A$ and E with respect to the variables z^i_η, $1 \le i \le 3$, $\eta = 1, 2$ (see also [ST]).

From the expression of φ and (5) we have that

$$A = \begin{pmatrix} 1 + f_{x_1}^2 & f_{x_1} f_{x_2} \\ f_{x_1} f_{x_2} & 1 + f_{x_2}^2 \end{pmatrix}, \qquad C = \sqrt{\det A} = W, \tag{14}$$

where W is given by (13). We now consider the vector field

$$v = (v^1, v^2, v^3) = (-f_{x_1},\, -f_{x_2},\, 1),$$

which is linearly independent with φ_{x_1} and φ_{x_2}.

By using the first and second order derivatives of C, $\det A$, E, and the fact that $\partial^2 \varphi^j / \partial x_\varepsilon \partial x_\eta = \delta_{j3} f_{x_\varepsilon x_\eta}$, a straightforward computation implies that

$$\frac{\partial C}{\partial z^i_\varepsilon} v^i = 0,\ \forall \varepsilon; \tag{15}$$

$$\frac{\partial E}{\partial z^i_\varepsilon} v^i = 2b^2(\delta_{\varepsilon 1} f_{x_1} + \delta_{\varepsilon 2} f_{x_2}),\ \forall \varepsilon; \tag{16}$$

$$\frac{\partial C}{\partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x_\varepsilon \partial x_\eta} = \frac{f_{x_1} f_{x_\varepsilon x_1} + f_{x_2} f_{x_\varepsilon x_2}}{W},\ \forall \varepsilon; \tag{17}$$

$$\frac{\partial E}{\partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x_\varepsilon \partial x_\eta} = 2b^2 \left[f_{x_1} f_{x_\varepsilon x_1} + f_{x_2} f_{x_\varepsilon x_2} \right],\ \forall\, \varepsilon; \tag{18}$$

$$\frac{\partial^2 E}{\partial z^i_\varepsilon \partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x_\varepsilon \partial x_\eta} v^i = 2b^2 \left[(1 + f_{x_1}^2) f_{x_2 x_2} - 2 f_{x_1} f_{x_2} f_{x_1 x_2} + (1 + f_{x_2}^2) f_{x_1 x_1} \right]; \tag{19}$$

$$\frac{1}{2} \frac{\partial^2 detA}{\partial z^j_\eta \partial z^i_\varepsilon} \frac{\partial^2 \varphi^j}{\partial x_\varepsilon \partial x_\eta} v^i = \left[(1 + f_{x_1}^2) f_{x_2 x_2} - 2 f_{x_1} f_{x_2} f_{x_1 x_2} + (1 + f_{x_2}^2) f_{x_1 x_1} \right], \tag{20}$$

where $1 \le i, j \le 3$, $1 \le \varepsilon, \eta \le 2$.

As a consequence of (15) and the definition of E, equation (10) reduces to

$$\begin{aligned}&\left\{-\frac{3C}{2}\frac{\partial^2 E}{\partial z^j_\eta \partial z^i_\varepsilon} - \frac{3}{2}\left(\frac{2E-C^2}{C^2-E}\right)\frac{\partial C}{\partial z^j_\eta}\frac{\partial E}{\partial z^i_\varepsilon} + \frac{3C}{4(C^2-E)}\frac{\partial E}{\partial z^i_\varepsilon}\frac{\partial E}{\partial z^j_\eta} + \right.\\ &\left. + \frac{(2E+C^2)}{2C}\frac{\partial^2 detA}{\partial z^j_\eta \partial z^i_\varepsilon}\right\}\frac{\partial^2 \varphi^j}{\partial x^\varepsilon \partial x^\eta} v^i = 0.\end{aligned} \tag{21}$$

It follows from (5), (9) and (14), that $B = b^2(W^2-1)/W^2$. Therefore,

$$-\frac{2E-C^2}{C^2-E} = \frac{2T_b - W^2}{T_b}, \qquad \frac{3C}{4(C^2-E)} = \frac{3W}{4T_b}, \qquad -\frac{2E+C^2}{2C} = \frac{2T_b - 3W^2}{2W}, \tag{22}$$

where T_b is given by (13). Now it follows, from (15)-(20) and (22), that equation (21) reduces

$$\begin{aligned}&T_b(T_b - 3b^2)\left[(1+f^2_{x_1})f_{x_2x_2} - 2f_{x_1}f_{x_2}f_{x_1x_2} + (1+f^2_{x_2})f_{x_1x_1}\right] + \\ &+3b^2(T_b + b^2)\left[f^2_{x_1}f_{x_1x_1} + 2f_{x_1}f_{x_2}f_{x_1x_2} + f^2_{x_2}f_{x_2x_2}\right] = 0.\end{aligned}$$

This concludes the proof of the theorem, since this equation is equivalent to (12).

□

Our next result will provide the differential equation which must be satisfied for a minimal surface which is the graph of a function over any plane of V^3.

Theorem 4 *An immersion $\varphi : U \subset I\!R^2 \longrightarrow (V^3, F_b)$ which is the graph of a function $f(x_1, x_2)$ over a plane of V^3 is minimal, if and only if, f satisfies*

$$\sum_{i,j=1}^{2}\left\{\tilde{T}_b(\tilde{T}_b - 3b^2w^2)(\delta_{ij} - \frac{f_{x_i}f_{x_j}}{W^2}) + 3b^2W^2(\tilde{T}_b + b^2w^2)(k_i + \frac{f_{x_i}}{W^2}w)(k_j + \frac{f_{x_j}}{W^2}w)\right\} f_{x_ix_j} = 0. \tag{23}$$

where W^2 is defined by (13), k_i are real numbers such that $\sum_{i=1}^{3} k_i^2 = 1$, and

$$w = -k_1 f_{x_1} - k_2 f_{x_2} + k_3, \qquad \tilde{T}_b = b^2w^2 + (1-b^2)W^2. \tag{24}$$

Proof. The proof is similar (although lengthier) to the particular case proved in Theorem 3. Assume that the immersion φ is a graph of a function over an open subset of a plane of V^3. Then φ is a function of the form

$$\varphi(x_1, x_2) = (x_1, x_2, f(x_1, x_2))(m_{ij}), \tag{25}$$

where (m_{ij}) is a 3×3 orthogonal matrix, $(x_1, x_2) \in U \subset I\!R^2$ and the surface is a graph over the plane $m_{31}x + m_{32}y + m_{33}z = 0$.

We need to compute the expressions involved in (10) for the immersion φ. The first and second order derivatives of C, $\det A$ and E with respect to the variables z^i_η, are those computed in the proof of Theorem 3.

From (5) and the expression of φ given by (25), and since the matrix (m_{ij}) is orthogonal, we have that A and C are given by (14). We now consider the vector field $v = (v^1, v^2, v^3)$

$$v^i = -f_{x_1}m_{1i} - f_{x_2}m_{2i} + m_{3i},$$

which is linearly independent with φ_{x_1} and φ_{x_2}.

Observe that

$$z^i_\eta = \frac{\partial \varphi^i}{\partial x_\eta} = m_{\eta i} + f_{x_\eta}m_{3i}, \qquad \frac{\partial^2 \varphi^i}{\partial x_\varepsilon \partial x_\eta} = f_{x_\varepsilon x_\eta}m_{3i}. \tag{26}$$

Moreover, for all $i = 1, 2, 3$ and $\eta, \gamma, \varepsilon = 1, 2$

$$\sum_{i=1}^{3} z^i_\eta v^i = 0, \qquad \sum_{i=1}^{3} v^i m_{3i} = 1, \qquad \sum_{i=1}^{3} z^i_\eta m_{3i} = f_{x_\eta}, \qquad \sum_{i=1}^{3} z^i_\gamma \frac{\partial^2 \varphi^i}{\partial x_\varepsilon \partial x_\eta} = f_{x_\gamma} f_{x_\varepsilon x_\eta}.$$

By using the first and second order derivatives of C, $\det A$ and E, a straightforward computation implies that

$$\frac{\partial C}{\partial z^i_\varepsilon} v^i = 0, \ \forall \varepsilon; \tag{27}$$

$$\frac{\partial E}{\partial z^i_\varepsilon} v^i = 2b^2 (z^3_\varepsilon A_{\tilde{\varepsilon}\tilde{\varepsilon}} - z^3_{\tilde{\varepsilon}} A_{\varepsilon\tilde{\varepsilon}}) w, \ \forall \varepsilon; \tag{28}$$

$$\frac{\partial C}{\partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x_\varepsilon \partial x_\eta} = \frac{f_{x_1} f_{x_\varepsilon x_1} + f_{x_2} f_{x_\varepsilon x_2}}{W}, \ \forall \varepsilon; \tag{29}$$

$$\frac{\partial E}{\partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x_\varepsilon \partial x_\eta} = 2b^2 \left[(f_{x_1} + k_1 w) f_{x_\varepsilon x_1} + (f_{x_2} + k_2 w) f_{x_\varepsilon x_2} \right], \forall \varepsilon; \tag{30}$$

$$\begin{aligned} \frac{\partial^2 E}{\partial z^i_\varepsilon \partial z^j_\eta} \frac{\partial^2 \varphi^j}{\partial x_\varepsilon \partial x_\eta} v^i = 2b^2 \Big\{ & \Big[1 + f^2_{x_1} - k_2 (k_2 W^2 + f_{x_2} w) \Big] f_{x_2 x_2} \\ & - \Big[(1 + k_3^2) f_{x_1} f_{x_2} + k_1 k_2 W^2 + k_1 k_3 f_{x_2} + k_2 k_3 f_{x_1} + k_1 k_2 \Big] f_{x_1 x_2} \\ & + \Big[1 + f^2_{x_2} - k_1 (k_1 W^2 + f_{x_1} w) \Big] f_{x_1 x_1} \Big\} ; \end{aligned} \tag{31}$$

$$\frac{1}{2} \frac{\partial^2 detA}{\partial z^j_\eta \partial z^i_\varepsilon} \frac{\partial^2 \varphi^j}{\partial x_\varepsilon \partial x_\eta} v^i = (1 + f^2_{x_1}) f_{x_2 x_2} - 2 f_{x_1} f_{x_2} f_{x_1 x_2} + (1 + f^2_{x_2}) f_{x_1 x_1}, \tag{32}$$

where $w = v^3$ is given by (24) and we have introduced the notation $k_i = m_{i3}$, for $i = 1, 2, 3$.

As a consequence of (27) and the definition of E, equation (10) reduces to (21). It follows from (5), (9) and (14), that

$$B = \frac{b^2}{W^2}[W^2 - w^2].$$

Therefore,

$$-\frac{2E - C^2}{C^2 - E} = \frac{2\tilde{T}_b - W^2}{\tilde{T}_b}, \qquad \frac{3C}{4(C^2 - E)} = \frac{3W}{4\tilde{T}_b}, \qquad -\frac{2E + C^2}{2C} = \frac{2\tilde{T}_b - 3W^2}{2W}, \tag{33}$$

where $\tilde{T}_b$ is given by (24). Now it follows from (27)-(32), (33) and the fact that $k_3 = w + f_{x_1}k_1 + f_{x_2}k_2$ that equation (21) reduces to (23).

□

Observe that when $k_1 = k_2 = 0$ and $k_3 = 1$, then equation (23) reduces to (12). Moreover, when $b = 0$ both equations reduce to the classical equation of a minimal surface in $I\!R^3$, which is the graph of f.

Examples: A surface which is the graph of a linear function $f(x_1, x_2)$, is minimal $\forall b$, $0 \leq b < 1$. Moreover, one can verify, that for any b such that $1/\sqrt{3} < b < 1$, the cone

$$\left(x_1, x_2, \sqrt{\frac{(3b^2 - 1)(x_1^2 + x_2^2)}{1 - b^2}} \right), \tag{34}$$

where $x_1^2 + x_2^2 \neq 0$ is a minimal surface in (V^3, F_b), since it satisfies (12). In particular, $(x_1, x_2, \sqrt{x_1^2 + x_2^2})$, is a minimal surface, when $b^2 = 1/2$. However, the cone $(x_1, \sqrt{x_1^2 + x_2^2}, x_2)$, is **not** a minimal surface in this Randers space. In fact, by considering $k_1 = k_3 = 0$, $k_2 = 1$ and $b^2 = 1/2$, one shows that the left hand side of (23) is always positive.

For a fixed plane of V^3 of the form $v_1x_1 + v_2x_2 + v_3x_3 = 0$, where $\sum_i v_i^2 = 1$, the minimal graphs over subsets of this plane are the solutions of equation (23), where $k_1 = m_{13}$, $k_2 = m_{23}$, $k_3 = v_3$ and (m_{ij}) is an orthogonal 3×3 matrix such that $m_{31} = v_1$ and $m_{32} = v_2$.

It is not difficult to prove that, for $0 \leq b < 1/\sqrt{3}$, equation (12) is an elliptic equation of mean curvature type, as defined by L. Simon [S1]. In fact, one can show that for such a b, one has $T_b > 0$ and $T_b - 3b^2 > 0$. Hence, (23) can be written as

$$\sum_{i,j=1,2} a_{ij}(x, f, \nabla f) f_{x_i x_j} = 0, \quad \textit{where} \quad a_{ij} = \delta_{ij} + (S_b - 1)\frac{f_{x_i} f_{x_j}}{W^2} \tag{35}$$

and

$$S_b = \frac{3b^2(T_b + b^2)}{T_b(T_b - 3b^2)}.$$

It is simple to verify that for all $\xi \in I\!R^2 \setminus \{0\}$, x, $p \in I\!R^2$ and $z \in R$,

$$0 < \frac{|\xi|^2}{1+|p|^2} \le \sum_{i,j=1,2} a_{ij}(x,z,p)\xi_i\xi_j \le \left(1 + \frac{|p|^2 S_b(p)}{1+|p|^2}\right)|\xi|^2.$$

Therefore, (35) is an elliptic equation. Moreover, one can show that for b fixed, there exists a constant $\mathcal{C} > 0$, such that $S_b(p) \le \mathcal{C}/W^2(p)$ and $\sum_{ij} \ell_{ij}(p)\xi_i\xi_j \ge |\xi|^2/W^2(p)$ where

$$\ell_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{W^2(p)}. \tag{36}$$

Therefore, for all $(x,z,p) \in I\!R^5$ and $\xi \in I\!R^2$

$$|\xi|^2 - \frac{(p\cdot\xi)^2}{1+|p|^2} \le \sum_{i,j=1,2} a_{ij}(x,z,p)\xi_i\xi_j \le (1+\mathcal{C})\left(|\xi|^2 - \frac{(p\cdot\xi)^2}{1+|p|^2}\right),$$

i.e., (12) is an elliptic equation of mean curvature type.

Our next result proves the general case, i.e. that for $0 \le b < 1/\sqrt{3}$, the differential equation of a minimal surface, which is the graph of a function in (V^3, F_b), (23), is an elliptic equation of mean curvature type.

Theorem 5 *Let $\varphi : I\!R^2 \longrightarrow (V^3, F_b)$ be an immersion which is the graph of a function $f(x_1, x_2)$ over a plane. If $0 \le b < 1/\sqrt{3}$, then φ is minimal, if and only if, f satisfies the elliptic differential equation, of mean curvature type, given by*

$$\sum_{i,j=1,2} a_{ij}(x, f, \nabla f) f_{x_i x_j} = 0, \tag{37}$$

where

$$a_{ij} = \delta_{ij} - \frac{f_{x_i} f_{x_j}}{W^2} + Q_b W^2 \left(k_i + \frac{f_{x_i}}{W^2} w\right)\left(k_j + \frac{f_{x_j}}{W^2} w\right), \tag{38}$$

$$Q_b = \frac{3b^2(\tilde{T}_b + b^2)}{\tilde{T}_b(\tilde{T}_b - 3b^2)}, \tag{39}$$

w and $\tilde{T}_b$ are given by (24) and k_i, $i = 1,2,3$, are real numbers such that $\sum_i k_i^2 = 1$.

Proof. From Theorem 4, φ is minimal if and only if f satisfies (23). Since $0 \le b < 1/\sqrt{3}$, it follows that $\tilde{T}_b > 0$ and

$$\tilde{T}_b - 3b^2 w^2 > \tilde{T}_b - w^2 = (1-b^2)(W^2 - w^2).$$

One can see that

$$W^2 - w^2 = (k_2 f_{x_1} - k_1 f_{x_2})^2 + (k_1 + k_3 f_{x_1})^2 + (k_2 + k_3 f_{x_2})^2.$$

Therefore, $\tilde{T}_b - 3b^2 w^2 > 0$. Dividing equation (23) by $-\tilde{T}_b(\tilde{T}_b - 3b^2 w^2)W^2$, we get (37).

With the notation introduced in (36), we observe that for all $\xi \in I\!R^2$

$$\sum_{i,j=1}^{2} \ell_{ij}(p)\xi_i\xi_j = \frac{|\xi|^2}{W^2}(1+|p|^2\sin^2\theta), \tag{40}$$

where θ is the angle function between p and ξ. Moreover,

$$\sum_{i,j=1}^{2} a_{ij}(x,z,p)\xi_i\xi_j = \sum_{i,j=1}^{2} \ell_{ij}\xi_i\xi_j + Q_b W^2 \left[(k_1,k_2)\cdot\xi + \frac{w}{W^2}p\cdot\xi\right]^2, \tag{41}$$

where $\cdot$ is the Euclidean inner product. Therefore, since $Q_b > 0$, for all $\xi \in I\!R^2 \setminus \{0\}$, we have

$$\sum_{i,j=1,2} a_{ij}(x,z,p)\xi_i\xi_j \geq \sum_{i,j=1}^{2} \ell_{ij}\xi_i\xi_j \geq \frac{|\xi|^2}{W^2} > 0, \tag{42}$$

where the second inequality follows from (40). Therefore, (37) is an elliptic equation.

In order to prove that it is a differential equation of mean curvature type, we need to show that there exists a constant $\mathcal{C}$ such that, for all $(x,z,p) \in I\!R^5$ and $\xi \in I\!R^2$,

$$\sum_{i,j=1}^{2} \ell_{ij}(x,z,p)\xi_i\xi_j \leq \sum_{i,j=1}^{2} a_{ij}(x,z,p)\xi_i\xi_j \leq (1+\mathcal{C}) \sum_{i,j=1}^{2} \ell_{ij}(x,z,p)\xi_i\xi_j. \tag{43}$$

The first inequality was obtained in (42). In order to prove the second inequality, it follows from (41), that we only need to prove that there exists a constant $\mathcal{C}$ such that

$$Q_b W^2 \left[(k_1,k_2)\cdot\xi + \frac{w}{W^2}p\cdot\xi\right]^2 \leq \mathcal{C} \sum_{i,j=1}^{2} \ell_{ij}(x,z,p)\xi_i\xi_j, \tag{44}$$

where $w = -k_1p_1 - k_2p_2 + k_3$.

From (24) and (39) we have that

$$Q_b = \frac{3b^2[(1-b^2)W^2 + 2b^2w^2]}{[(1-b^2)W^2 + b^2w^2][(1-b^2)W^2 - 2b^2w^2]} \tag{45}$$

and it follows from (40) that

$$W^2 \left[(k_1,k_2)\cdot\xi + \frac{w}{W^2}p\cdot\xi\right]^2 = \frac{[W^2|(k_1,k_2)|\cos\gamma + w|p|\cos\theta]^2}{1+|p|^2\sin^2\theta} \sum_{i,j=1}^{2} \ell_{ij}(x,z,p)\xi_i\xi_j,$$

where γ is the angle between (k_1,k_2) and ξ. Hence, we need to show that there exists a constant $\mathcal{C}$ such that

$$Q_b \frac{[W^2|(k_1,k_2)|\cos\gamma + w|p|\cos\theta]^2}{1+|p|^2\sin^2\theta} \leq \mathcal{C}. \tag{46}$$

Observe that $W^2 \geq 1$. When $W^2 = 1$, i.e., $p = 0$, the left hand side of (46) is less than or equal to the real number $Q_b(0)(k_1^2 + k_2^2) \geq 0$. Whenever $W^2 > 1$ and $\sin\theta = 0$, then $p \neq 0$ and the vectors p and ξ are parallel. Hence,

$$\left[W^2 \left|(k_1, k_2)\right| \cos\gamma + w|p|\cos\theta\right]^2 = \left[\left|(k_1, k_2)\right| \cos\gamma + k_3|p|\cos\theta\right]^2.$$

Therefore, the left hand side of (46) is a rational function of $|p|$ whose numerator is of degree less than or equal to 4, and denominator is of degree 4 and hence it is a bounded function when $|p|$ (or equivalently W) tends to infinity. Whenever $W^2 > 1$ and $\sin\theta \neq 0$, then $p \neq 0$ and the vectors p and ξ are not parallel. Hence, the left hand side of (46) is a rational function of $|p|$ whose numerator is of degree less than or equal to 6, and denominator is of degree 6 and hence it is a bounded function when $|p|$ (or equivalently W) tends to infinity. This completes the proof of the inequality (46).

$\square$

We observe that when $b = 1/\sqrt{3}$ equation (12) is not elliptic. In fact, in this case, the equation reduces to $\sum_{i,j=1,2} c_{ij} f_{x_i x_j} = 0$, where

$$c_{ij} = \frac{2|\nabla f|^2}{3}\left(1 + \frac{2}{3}|\nabla f|^2\right)\left(\delta_{ij} - \frac{f_{x_i} f_{x_j}}{W^2}\right) + (4 + 2|\nabla f|^2)\frac{f_{x_i} f_{x_j}}{3W^2}.$$

Therefore, $\sum_{i,j} c_{ij}(x, z, p)\xi_i \xi_j$ is a multiple of $|p|^2$ and hence vanishes for $p = 0$.

As an immediate consequence of Theorem 5 and a Bernstein type theorem proved by Simon (see Theorem 4 in [S1], Theorem 4.1 in [S2]), we conclude that

Theorem 6 *A minimal surface in a special Randers Space (V^3, F_b), $0 \leq b < 1/\sqrt{3}$, which is the graph of a function defined on $\mathbb{R}^2$, is a plane.*

Our next results, provide properties of minimal surfaces in the special Randers space, when $0 \leq b < 1/\sqrt{3}$.

Corollary 7 *Assume b is in the interval $[0, 1/\sqrt{3})$. If there exists a solution of the Dirichlet problem for a minimal surface which is the graph of a function f in the special Randers space (V^3, F_b), then it is unique.*

Consider two minimal surfaces in a special Randers space (V^3, F_b), with $0 \leq b < 1/\sqrt{3}$, which are graphs of functions. Assume the surfaces are tangent at a point $p_0 \in V^3$, then both surfaces can be locally considered to be graphs of functions $f(x_1, x_2)$ and $h(x_1, x_2)$ over the same plane of V^3. Let $u = f - h$ be a function defined in the intersection of the domains of f and h. Then

Lemma 8 *The function u satisfies the differential equation*

$$L(u) = \sum_{i,j} a_{ij}(x, f, \nabla f) u_{x_i x_j} + \sum_i c_i u_{x_i} = 0, \tag{47}$$

where a_{ij} is given by (38),

$$c_i = -\sum_{j,\ell=1}^{2}\left[\int_0^1 \frac{\partial \beta_{j\ell}}{\partial p_i}(\nabla f + t(\nabla h - \nabla f))\,dt\right] h_{x_j x_\ell} \tag{48}$$

and for $p = (p_1, p_2) \in I\!R^2$,

$$\beta_{j\ell}(p) = -\frac{p_j p_\ell}{W^2} + Q_b(p) W^2 \left(k_j + \frac{p_j w(p)}{W^2}\right)\left(k_\ell + \frac{p_\ell w(p)}{W^2}\right), \tag{49}$$

where $Q_b(p)$ is given by (39), $W^2 = 1 + |p|^2$ and $w(p) = -k_1 p_1 - k_2 p_2 + k_3$. Moreover, $L(u)$ is an elliptic operator.

Proof. Since the graphs of f and h are minimal surfaces, it follows from Theorem 5 that

$$\sum_{j,\ell} a_{j\ell}(x, f, \nabla f) f_{x_j x_\ell} = 0 \quad \text{and} \quad \sum_{j,\ell} a_{j\ell}(x, h, \nabla h) h_{x_j x_\ell} = 0,$$

where $a_{j\ell}$ is given by (38). Taking the difference of these two equations, adding and subtracting the expression $\sum_{j,\ell} a_{j\ell}(x, f, \nabla f) h_{x_j x_\ell}$, we get

$$0 = \sum_{j,\ell} a_{j\ell}(x, f, \nabla f) u_{x_j x_\ell} + \sum_{j,\ell}\left[\beta_{j\ell}(\nabla f) - \beta_{j\ell}(\nabla h)\right] h_{x_j x_\ell},$$

where $\beta_{j\ell}$ is given by (49). Since

$$\beta_{j\ell}(\nabla h) - \beta_{j\ell}(\nabla f) = \sum_{i=1}^{2}\int_0^1 \frac{\partial \beta_{j\ell}}{\partial p_i}(\nabla f + t(\nabla h - \nabla f))(h_{x_i} - f_{x_i})\,dt,$$

we conclude that

$$\sum_{j,\ell}\left[\beta_{j\ell}(\nabla f) - \beta_{j\ell}(\nabla h)\right] h_{x_j x_\ell} = \sum_{i=1}^{2} c_i u_{x_i},$$

where c_i is given by (48).

□

Since the functions c_i given by (48) are locally bounded, the following theorem follows from Lemma 8 and the maximum principle.

Theorem 9 *Let M_1 and M_2 be minimal surfaces in (V^3, F_b), $0 \leq b < 1/\sqrt{3}$. If M_1 is above M_2 near p_0 and internally tangent at p_0, then M_1 and M_2 coincide in a neighborhood of p_0.*

Besides the Bernstein type theorem given in Theorem 6, as a consequence of Theorem 5 and the theory developped by L. Simon [S1], one gets several results for the solutions of the equation of mean curvature type (37); in particular we will list some of these results such as a-priori gradient estimates, a Bers-type theorem concerning the limiting behaviour of the gradient of solutions defined outside a compact set, a global Hölder continuity estimate for solutions which continuously attain Lipschitz boundary values and a theorem concerning the removability of isolated singularities.

In what follows, it is assumed that $\Omega \subset I\!R^2$, and f is a $C^2(\Omega)$ solution of (37), where we have fixed k_i such that $\sum k_i^2 = 1$. Let x_0 denote a fixed point of Ω, and let p_0 be the corresponding point on the surface M which is the graph of f. Let $D_\rho(x_0) = \{x \in I\!R^2;\ |x - x_0| < \rho\}$ and $S_\rho(p_0) = \{p \in M;\ |p - p_0| < \rho\}$.

Proposition 10 *If $D_\rho(x_0) \subset \Omega$, then*

$$\sup_{S_{\rho/2(p_0)}} \sqrt{1 + |\nabla f|^2} \leq \gamma \inf_{S_{\rho/2(p_0)}} \sqrt{1 + |\nabla f|^2},$$

where $\gamma > 0$ depends only on $\mathcal{C}$ of (43). Moreover, if $f \geq 0$ on $D_\rho(x_0)$, then

$$|\nabla f(x_0)| \geq \gamma_1 exp\{\gamma_2 f(x_0)/\rho\},$$

where γ_1, γ_2 depend only on $\mathcal{C}$.

Proposition 11 *Suppose f is defined outside of a compact subset of $I\!R^2$. Then there is a vector $a \in I\!R^2$ such that $\nabla f(x) \to a$ uniformly for $|x| \to \infty$.*

Proposition 12 *A minimal surface in the Randers space (V^3, F_b), for $0 \leq b < 1/\sqrt{3}$, cannot have an isolated singularity.*

We conclude by observing that the above result fails if the condition on b does not hold. In fact, any minimal cone given by (34), for $1/\sqrt{3} < b < 1$, has an isolated singularity at the origin.

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