Ribaucour transformations revisited

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Abstract

We present a revised definition of a Ribaucour transformation for submanifolds of space forms, with flat normal bundle, motivated by the classical definition and by more recent extensions. The definition introduced in this paper, provides a precise treatment of the geometric aspect of such transformations preserving lines of curvature and it can be applied to submanifolds whose principal curvatures have multiplicities bigger than one. We characterize this transformation in terms of differential equations and we study some of its properties. We show that an n-dimensional sphere or hyperplane can be locally associated by a Ribaucour transformation to any given hypersurface M^n of R^{n+1}, which admits n orthogonal principal direction vector fields. As an application of Ribaucour transformations, we characterize the Dupin hypersurfaces which have a principal curvature of constant multiplicity one, as a manifold foliated by $(n-1)$-dimensional Dupin submanifolds associated by Ribaucour transformations.
Keywords: Ribaucour transformations, Dupin hypersurfaces
MSC: 53C42

1. Introduction.

The classical Ribaucour transformation relates diffeomorphic surfaces M and $\tilde{M}$ of R^3 such that at corresponding points the normal lines intersect at an equidistant point. Moreover, the set of intersection points is also required to describe a surface of R^3 and the diffeomorphism to preserve the lines of curvature of M and $\tilde{M}$. The classical theory (see Bianchi [Bi]) includes the case of hypersurfaces parametrized by lines of curvature, where the principal curvatures of both hypersurfaces have multiplicity one, although this is not stated explicitly. Ribaucour transformations have many applications. In particular they can be applied as a method of construction of surfaces of constant Gaussian curvature, constant mean curvature, and minimal surfaces. Although this was known in the classical literature, it was recently applied for the first time to obtain minimal surfaces [CFT2]. Moreover, Ribaucour transformations for linear Weingarten surfaces and for Dupin hypersurfaces were studied in [CFT3] and [CFT1].

*Partially supported by PROCAD
†Partially supported by CNPq

The extension to submanifolds of higher codimension is quite recent. One of the difficulties relies on extending the condition on the intersection of the normal lines. The first attempt in this direction can be found in Tojeiro [To]. More recently, two distinct extensions were considered by Corro [Co] and Dajczer-Tojeiro [DT1], for submanifolds, with flat normal bundle and parametrized by lines of curvature. We point out that a version of the definition in [DT1], for nonholonomic submanifolds, was introduced in [DT2] and it was used in [BDPT].

While both definitions require the transformation to preserve all lines of curvature, they differ in the extension of the condition on the intersection of the normal lines. One can see in [Co], [DT1] and [DT2] that both are "characterized" by the same integrable system of differential equations. Moreover, solutions of this system provide immersions locally associated by Ribaucour transformation to a given immersion. However, solutions of the system of differential equations do not always preserve multiplicity of principal curvatures. This was observed in [CFT1], where families of Dupin submanifolds were associated to the plane (see also Example 3.7 b). In such cases, the diffeomorphism between the submanifolds does not preserve all lines of curvature. In fact, one can see that the system of differential equations is a necessary but not sufficient condition for the existence of immersions associated to a given one by Ribaucour transformations.

In view of the aspects mentioned above, the definition of a Ribaucour transformation needs to be revised. The definition, that will be introduced in section 2, provides a precise treatment of the geometric aspect of such transformations preserving lines of curvature and it can be applied to submanifolds, of any codimension, whose principal curvatures have multiplicity bigger than one.

Given a submanifold $M^n \subset R^{n+p}$ with flat normal bundle, we will define the submanifolds $\tilde{M}^n$ associated to M by a Ribaucour transformation, with respect to a fixed set of n orthonormal vector fields of principal directions $\{e_i\}$, globally defined on M. Preserving all lines of curvature will be replaced by the requirement of preserving the lines of curvature tangent to $\{e_i\}$. We observe that the existence of such a set of vector fields, for $n \geq 3$, does not imply the existence of a local parametrization such that the coordinate curves are tangent to these vector fields or that the submanifold is holonomic. We will also require the existence of a unit vector field normal to each submanifold such that the corresponding normal lines intersect at an equidistant point as in [Co]. Moreover, the set of intersection points is required to be an n-dimensional submanifold.

We point out that given a submanifold M with a principal curvature with multiplicity bigger than one, the submanifolds $\tilde{M}$ associated to M, by a Ribaucour transformation, may differ depending on the choice of the set of vector fields of principal directions $\{e_i\}$ on M. Moreover, the transformation is invertible in the sense that there exists such a set $\{\tilde{e}_i\}$ on $\tilde{M}$, such that M is associated to $\tilde{M}$ by a Ribaucour transformation with respect to $\{\tilde{e}_i\}$. We will show that the revised defintion is indeed equivalent to the system of differential equations. Moreover, it implies the existence of a normal bundle isometry which commutes with the normal connections, such that the corresponding normal lines are parallel or intersect at an equidistant point as required in [DT1]. We observe that the theory extends obviously to submanifolds and immersions of the sphere or of the hyperbolic space by replacing straight lines by geodesics of the ambient space. We conclude section 2 by showing that an n-dimensional sphere or hyperplane can be locally associated by a Ribaucour transformation to any given hypersurface M^n of R^{n+1}, which admits n orthogonal principal direction vector fields. The results of this section were announced in [T].

In section 3 we consider Dupin submanifolds. Such surfaces were first studied by Dupin in 1822 and more recently by many authors [CC], [CeR1-CeR2], [CJ1-CJ2] [M], [N], [P1-P2], [PT], [Th], which considered several aspects of Dupin hypersurfaces. Since the classification of Dupin hypersurfaces for higher dimensions is far from complete, it is important to study such submanifolds.

We consider Dupin hypersurfaces parametrized by lines of curvature. Hence, we eliminate those which are Lie equivalent to an isoparametric hypersurface contained in S^n. This follows from Pinkall's result [P2], which asserts that an isoparametric hypersurface in S^n with more than two distinct principal curvatures cannot be parametrized by lines of curvature.

The main purpose of section 3 is to characterize the Dupin hypersurfaces which have a principal curvature of constant multiplicity one, as a manifold foliated by $(n-1)$-dimensional Dupin submanifolds associated by Ribaucour transformations. This characterization was obtained in [Co]. More recently, the case of a principal curvature with constant multiplicity bigger than one was considered in [DFT]. The results in this section provide a method of constructing a Dupin hypersurface in R^{n+2}, from a given one in R^{n+1}, via Ribaucour transformations. It might be interesting to relate this method to Pinkall's notion of reducibility [P2].

We conclude section 3 with an example which shows that when a submanifold M has principal curvatures with multiplicity bigger than one, then the submanifolds associated by a Ribaucour transformation to M may differ depending on the choice of the set of principal directions.

2. Ribaucour transformations

In this section, we discuss the classical definition of a Ribaucour transformation for hypersurfaces and the more recent extensions to submanifolds with higher codimension. In view of this discussion, we then introduce a new definition of Ribaucour transformation for submanifolds of the Euclidean space, with flat normal bundle. The revised definition provides a precise treatment of the geometric aspect of such transformations preserving lines of curvature and it extends Ribaucour transformations to submanifolds whose principal curvatures have multiplicity bigger than one. We characterize this transformation in terms of differential equations and we study some of its properties.

The classical Ribaucour transformation relating hypersurfaces parametrized by lines of curvature can be found in Bianchi [Bi]. The extension to submanifolds of higher codimension is more recent. In [Co] and [DT1], two distinct extensions were given for submanifolds, with flat normal bundle, parametrized by lines of curvature.

In [Co], two such manifolds M^n and $\tilde{M}^n$, contained in R^{n+2} are considered to be related by a Ribaucour transformation if there exist a diffeomorphism $\psi : M \to \tilde{M}$, which preserves lines of curvature, a differentiable function $h : M \to R$ and unit normal vector fields N, $\tilde{N}$, parallel in the normal connection of M and $\tilde{M}$, respectively, such that $\forall q \in M$, $q+h(q)N(q) = \psi(q)+h(q)\tilde{N}(\psi(q))$ and the subset $q+h(q)N(q)$ is n-dimensional.

In [DT1], two holonomic isometric immersions $f : M^n \to R^{n+p}$ and $\tilde{f} : \tilde{M}^n \to R^{n+p}$ are said to be related by a Ribaucour transformation when there exist a curvature-lines-preserving diffeomorfism $\psi : M \to \tilde{M}$ with $|f - \tilde{f} \circ \psi| \neq 0$ everywhere, a vector bundle isometry $P : T_f^{\perp} M \to T_{\tilde{f}}^{\perp} \tilde{M}$ covering ψ, and a vector field $\zeta \in T_f^{\perp} M$ that is nowhere a principal curvature normal of f, such that **a)** $P(\xi) - \xi =< \xi, \zeta > (f - \tilde{f} \circ \psi)$ for all $\xi \in T_f^{\perp} M$; and **b)** P is parallel, i.e. P commutes with the normal connection.

Definition in [Co] is quite simple compared to the definition in [DT1]. Before we relate these two definitions, we discuss some basic geometric aspects which motivated the new definition. We start by observing that the definitions above require a Ribaucour transformation to preserve all lines of curvature, This is one of the basic problems that we will treat in this paper.

We first mention that, even in the case of surfaces in R^3, one has to fix a surface and then consider those associated to the given surface by Ribaucour tranformations (instead of considering two surfaces associated by the transformation). An easy example illustrates this observation. Consider in R^3 the following segments: $(1, 0, t)$, $(1 + t, 0, t)$ and $(1+t, 0, 0)$, where $t > 0$. By rotating the segments arround the x_3 axis one gets a half cylinder, a truncated cone and the complement of a unit disc in the x_1, x_2 plane. Let ψ be the diffeomorphism that to each point of the cylinder $(\cos\theta, \sin\theta, t)$ it associates the point $((1+t)\cos\theta, (1+t)\sin\theta, 0)$ on the plane. Then the truncated cone is the set of intersection of the normal lines and ψ preserves the lines of curvature. However, one cannot say that the cylinder is associated to the planar region, since not all lines of curvature of the plane correspond to such curves on the cylinder.

Preserving lines of curvature. It is generally accepted that Ribaucour transformations preserve lines of curvature. In the classical theory and in both [Co] and [DT1], the definition is "characterized" essencially by the same integrable system of differential equations, whose solutions provide immersions locally associated by Ribaucour transformations to a given immersion (see for example Theorem 9 in [DT1]). However, one can show that this procedure does not always preserve multiplicity of principal curvatures. In fact, this was already mentioned in [CFT1] and one can see in Example 3.7 that a torus is locally associated to a plane. In such cases, the requirement of the local diffeomorphism ψ preserving all lines of curvature does not hold. This is due to the fact that the system of differential equations is a necessary condition for the existence of immersions associated to a given one by Ribaucour transformations, but it is not sufficient. We will also show in Corollary 2.10, that given any hypersurface M^n of R^{n+1}, which admits n orthonormal principal direction vector fields, there exists a solution to the system of differential equations so that the associated hypersurface is an open subset of a hyperplane or a sphere.

In the revised definition, we replace the requirement of preserving lines of curvature by the requirement of preserving the lines of curvature tangent to a fixed set of n orthonormal vector fields of principal directions. In that case, the system of equations is indeed equivalent to the definition. Moreover, for submanifolds which admit principal curvatures with multiplicity bigger than one, the choice of distinct set of orthonormal principal directions may provide, by solving the system of equations, distinct families of submanifolds associated by Ribaucour transformations (see Example 3.7).

Holonomic submanifolds. The classical theory and the more recent developments in [Co] and [DT1] require the submanifolds to be holonomic. (i.e. admit a global parametrization by lines of curvature). However, for submanifolds which admit principal curvatures with multiplicity bigger than one, by considering distinct parametrizations by lines of curvatures and solving the system of differential equations one may obtain different associated submanifolds (see Example 3.7). The nonholonomic case was considered in [DT2]. However, it presents the same problem with respect to the transformation preserving all lines of curvature (see Theorem 45 in [DT2] which does not characterize all Ribaucour transformations).

Our definition will not require the manifold to be holonomic. However, it will require the existence of n orthomormal vector fields of principal directions globally defined. If the submanifold is given parametrized by orthogonal lines of curvature, then the vector fields tangent to the coordinate curves provide the principal directions that will be preserved.

Extending the condition on the intersection of the normal lines for higher codimension. Assuming that the submanifolds have flat normal bundle, while [Co] requires the existence of a unit vector field normal to each submanifold such that the corresponding lines intersect at an equidistant point, the definition in [DT1] requires the existence of an isometry of the normal bundles such that the corresponding normal lines are parallel or intersect at an equidistant point.

Our definition will require the existence of a vector field normal to each submanifold such that at corresponding points the lines in these normal directions intersect at an equidistant point. We will show that this condition implies the existence of a correspondence between the normal bundles (resp. tangent bundles) such that the lines at corresponding normal (resp. tangent) vectors are parallel or intersect at an equidistant point. Moreover, the correspondence between the normal bundles can be chosen so that it is an isometry wich commutes with the normal connections.

The set of equidistant points. The classical definitions and the definition in [Co] requires that the set of the intersections of the normal lines is an n-dimensional submanifold. The existence of a normal vector field, which is nowhere a principal curvature normal to the immersion in the definition of [DT1], is equivalent to requiring the existence of a normal vector field for which the set of intersections of the corresponding normal lines is n dimensional.

In view of the aspects mentioned above we propose the following definition.

Definition 2.1 Let M^n be a submanifold of R^{n+p} with flat normal bundle. Assume there exist $e_1, ..., e_n$ orthonormal principal vector fields defined on M. A submanifold $\tilde{M}^n \subset R^{n+p}$, with flat normal bundle, is *associated to* M *by a Ribaucour transformation with respect to* $e_1, ..., e_n$ if there exist a diffeomorphism $\psi : M \to \tilde{M}$, a differentiable function $h : M \to R$ and unit normal vector fields N and $\tilde{N}$, parallel in the normal connection of M and $\tilde{M}$ respectively, such that:
a) $q + h(q)N(q) = \psi(q) + h(q)\tilde{N}\psi(q)$, $\forall q \in M$;
b) the subset $q + h(q)N(q)$, $q \in M$ is n-dimensional;
c) $d\psi(e_i)$ are orthogonal principal directions in $\tilde{M}$.

This transformation is invertible in the sense that there exist orthonormal principal direction vector fields $\tilde{e}_1, ..., \tilde{e}_n$ on $\tilde{M}$ such that M is associated to $\tilde{M}$ by a Ribaucour transformation with respect to these vector fields. One may consider the analogue definition for locally associated submanifolds (or for immersions).

Definition 2.2 Let M^n be a submanifold of R^{n+p} with flat normal bundle. Assume there exist $e_1, ..., e_n$ orthonormal principal vector fields globally defined on M. *A submanifold* $\tilde{M}^n$ *is locally associated to* M *by Ribaucour transformations with respect to* $e_1, ..., e_n$ if for any $\tilde{q} \in \tilde{M}$ there exists a neighborhood $\tilde{V}$ of $\tilde{q}$ in $\tilde{M}$ and an open subset $V \subset M$ such that $\tilde{V}$ is associated to V by a Ribaucour transformation with respect to $e_1, ..., e_n$.

Similar definitions can be considered for immersions in R^{n+p} and also for submanifolds and immersions in the sphere S^{n+p} or the hyperbolic space H^{n+p}. In the latter cases one should replace the straight lines of conditions a) and b) by geodesics of the ambient space.

The definition above reduces to the classical case of surfaces in R^3 or hypersurfaces in R^n, parametrized by lines of curvature, whenever the principal curvatures of the associated submanifolds have multiplicity one and the direction e_i are considered to be tangent to the coordinate curves.

The requirement of ψ being a diffeomorphism implies that both manifolds are topologically equivalent. In general this is a very strong condition. Many interesting applications of this method (see [CFT2,CFT3]) show that in general one has immersions locally associated by Ribaucour transformations to a given one, even when both manifolds are complete.

In what follows we consider a submanifold M^n of R^{n+p}, with flat normal bundle. Let e_i, $1 \leq i \leq n$, be a local orthonormal frame tangent to M and let N_α $1 \leq \alpha \leq p$ be a an orthonormal frame normal to M parallel in the normal connection. We denote by ω_i the one forms dual to the vector fields e_i and by ω_{ij}, $1 \leq i, j \leq n$ the connection forms determined by $d\omega_i = \sum_{j \neq i} \omega_j \wedge \omega_{ji}$, $\omega_{ij} + \omega_{ji} = 0$. The normal connection $\omega_{i\alpha} =< de_i, N_\alpha >$ satisfies $\sum_i \omega_i \wedge \omega_{i\alpha} = 0$. Hence, $\omega_{i\alpha} = \sum_j b^\alpha_{ij} \omega_j$ where $b^\alpha_{ij} = b^\alpha_{ji}$. The Gauss equation is given by

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \sum_\gamma \omega_{i\gamma} \wedge \omega_{\gamma j}$$

and the Codazzi equations are $d\omega_{i\alpha} = \sum_j \omega_{ij} \wedge \omega_{j\alpha}$. If, $e_i, 1 \leq i \leq n$, are orthonormal principal directions, we have

$$dN_\alpha(e_i) = \lambda^{\alpha i} e_i, \qquad \omega_{i\alpha} = -\lambda^{\alpha i}\omega_i. \tag{1}$$

Then the Codazzi equations reduce to

$$d\lambda^{\alpha i}(e_j) = (\lambda^{\alpha i} - \lambda^{\alpha j})\omega_{ij}(e_i) \quad \forall \alpha \text{ and } i \neq j. \tag{2}$$

Theorem 2.3 *Let M^n be an immersed submanifold of R^{n+p}, whose normal bundle is flat and let e_i, $1 \leq i \leq n$ be orthonormal principal vector fields defined on M. A submanifold $\tilde{M}^n$ is locally associated to M by a Ribaucour transformation with respect to the set e_i, if and only if, for any $\tilde{q} \in \tilde{M}$ there exist parametrizations $\tilde{X} : U \subset R^n \to \tilde{M}$ and $X : U \to M$ such that $\tilde{q} \in \tilde{X}(U)$, a differentiable function $h : U \to R$, a $p \times p$ matrix function B defined on U and parallel orthonormal vector fields N_α, $1 \leq \alpha \leq p$, normal to $X(U)$ such that*

$$\tilde{X} = X + h(N_1 - \tilde{N}_1), \tag{3}$$

where $\tilde{N}_1$ is a unit vector field normal to $\tilde{X}(U)$ given by

$$\tilde{N}_1 = (1 - B^{11}) \sum_{i=1}^{n} Z^i e_i + \sum_{\gamma}^{p} B^{1\gamma} N_\gamma, \tag{4}$$

$$Z^i = \frac{dh(e_i)}{1 + h\lambda^{1i}} \qquad \Delta = \sum_{i=1}^{n} \left(Z^i\right)^2, \tag{5}$$

$dN_\alpha(e_i) = \lambda^{\alpha i} e_i$, $1 + h\lambda^i \neq 0$, h and B are generic solutions of the differential equations

$$dZ^j(e_i) + \sum_{k=1}^{n} Z^k \omega_{kj}(e_i) - Z^i Z^j \lambda^{1i} = 0, \quad 1 \leq i \neq j \leq n, \tag{6}$$

$$BB^t + \Delta DD^t = I, \tag{7}$$

$$dB(e_i)B^t - BdB^t(e_i) + \Delta[dD(e_i)D^t - DdD^t(e_i)] + 2Z^i[B\Lambda^i D^t - D(\Lambda^i)^t B^t] = 0 \tag{8}$$

where

$$D^t = (1 - B^{11}, -B^{21}, ..., -B^{p1}) \qquad (\Lambda^i)^t = (\lambda^{1i}, ..., \lambda^{pi}). \tag{9}$$

Proof: Let N_1 be the unit normal of the definition of Ribaucour transformation. We may complete with unit vector fields $N_2, ..., N_p$, normal to M, parallel in the normal connection.

In order to prove the theorem, we will consider $\tilde{N}_\alpha, 1 \leq \alpha \leq p$, to be unit orthonormal vector fields given by

$$\tilde{N}_\alpha = \sum_{i=1}^{n} b^{\alpha i} e_i + \sum_{\gamma=1}^{p} B^{\alpha\gamma} N_\gamma, \tag{10}$$

where

$$\sum_{i=1}^{n} b^{\alpha i} b^{\beta i} + \sum_{\gamma=1}^{p} B^{\alpha\gamma} B^{\beta\gamma} = \delta_{\alpha\beta}. \tag{11}$$

We introduce the following notation

$$d\tilde{N}_\alpha(e_i) = \sum_j L_i^{\alpha j} e_j + \sum_\gamma C_i^{\alpha\gamma} N_\gamma, \tag{12}$$

where, for $1 \leq i \leq n, 1 \leq \alpha \leq p$,

$$L_i^{\alpha j} = db^{\alpha j}(e_i) + \sum_k b^{\alpha k} \omega_{kj}(e_i) + \sum_\gamma B^{\alpha\gamma} \lambda^{\gamma i} \delta_{ij}, \tag{13}$$

$$C_i^{\alpha\gamma} = dB^{\alpha\gamma}(e_i) - b^{\alpha i} \lambda^{\gamma i}. \tag{14}$$

We will show later that the following relations hold

$$b^{\alpha i} = Z^i(\delta_{1\alpha} - B^{\alpha 1}), \qquad 1 \leq i \leq n, 1 \leq \alpha \leq p. \tag{15}$$

In this case, substituting into (11) and using (5) we get

$$\Delta(\delta_{1\alpha} - B^{\alpha 1})(\delta_{1\beta} - B^{\beta 1}) + \sum_\gamma B^{\alpha\gamma} B^{\beta\gamma} = \delta_{\alpha\beta}$$

and the algebraic condition (11) reduces to (7). We will now prove the theorem.

Assume that $\tilde{M}$ is locally associated to M by a Ribaucour transformation. Then by definition there exist local parametrizations X of M, $\tilde{X}$ of $\tilde{M}$ and a function h defined on $U \subset R^n$ such that

$$\tilde{X} + h\,\tilde{N}_1 = X + h\,N_1,$$

where $\tilde{N}_1$ is a unit vector field normal to $\tilde{M}$, which may be considered as in (4). Then

$$< d\tilde{X}(e_i), \tilde{N}_\alpha >= 0, \qquad \text{for all } i,\ 1 \leq i \leq n, 1 \leq \alpha \leq p. \tag{16}$$

Since

$$d\tilde{X} = dX + dh(N_1 - \tilde{N}_1) + h(dN_1 - d\tilde{N}_1) \tag{17}$$

it follows from the relations $dX = \sum_j \omega_j e_j$ and $dN_1(e_i) = \lambda^{1i} e_i$ that

$$d\tilde{X}(e_i) = (1 + h\lambda^{1i})e_i + dh(e_i)(N_1 - \tilde{N}_1) - hd\tilde{N}_1(e_i). \tag{18}$$

Hence, equation (16) implies

$$(1 + h\lambda^{1i})b^{\alpha i} + dh(e_i)(B^{\alpha 1} - \delta_{\alpha 1}) = 0, \qquad 1 \leq i \leq n, 1 \leq \alpha \leq p. \tag{19}$$

We claim that the fact that $\tilde{X}$ is a Ribaucour transformation of X implies that $1 + h\lambda^{1i} \neq 0$ for all i. In fact, consider the center manifold $X^0 = X + hN_1$. Then

$$dX^0(e_i) = (1 + h\lambda^{1i})e_i + dh(e_i)N_1.$$

Assume that $(1+h\lambda^{1i})(u^0)=0$ at a point u^0. Then it follows from (19) that $dh(e_i)(B^{11}-1)(u^0)=0$ and hence $dh(e_i)(u^0)=0$. Otherwise, $B^{11}(u^0)=1$ implies that $\tilde{X}(u^0)=X(u^0)=X^0(u^0)$. Hence $h(u^0)=0$, which is a contradiction, since $(1+h\lambda^{1i})(u^0)=0$. Therefore, we have $dh(e_i)(u^0)=0$ and $dX^0(e_i)(u^0)=0$, which contradicts the fact that the center manifold X^0 is n-dimensional. So $1+h\lambda^{1i}\neq 0$ for all i, therefore we conclude from (19) that the relations (15) hold.

We will show that the differential equation which is satisfied by h, is a consequence of the property

$$< d\tilde{N}_\alpha(e_i), d\tilde{X}(e_j) >= 0 \quad \text{for } i\neq j.$$

Since $d\tilde{X}(e_i)$ are orthogonal principal directions, we have

$$< d\tilde{X}(e_i), d\tilde{X}(e_j) >=< d\tilde{N}_\alpha(e_j), d\tilde{X}(e_i) >=< d\tilde{N}_\alpha(e_j), d\tilde{N}_\beta(e_i) >= 0$$

for all $i\neq j, \alpha\neq\beta$. Hence, using the notation (12) and equation (18), we get

$$\begin{aligned} < d\tilde{N}_\alpha(e_i), d\tilde{X}(e_j) > &= < d\tilde{N}_\alpha(e_i), (1+h\lambda^{1j})e_j + dh(e_j)N_1 >, \\ &= L_i^{\alpha j}(1+h\lambda^{1j}) + C_i^{\alpha 1}dh(e_j) = 0, \end{aligned}$$

for all $i\neq j, \alpha$. By using the relations (13) and (14) in the last equality, we conclude that equation (6) holds. We will show that the differential equation (8), is a consequence of the property

$$< d\tilde{N}_\alpha(e_i), \tilde{N}_\beta >= 0 \quad \text{for all } \alpha, \beta, i.$$

Substituting (10) and (12), we have

$$\sum_j L_i^{\alpha j} b^{\beta j} + \sum_\gamma C_i^{\alpha\gamma} B^{\beta\gamma} = 0.$$

By using (14), (15), (5) and (9) in the last equality, we conclude that equation (8) holds. We observe that h and B are generic solutions of (6)-(8) in the sense that

$$\begin{aligned} d\tilde{X}(e_i) &= \sum_j [(1+h\lambda^{1i})\delta_{ji} - dh(e_i)b^{1j} - hL_i^{1j}]e_j \\ &+ \sum_\gamma [dh(e_i)(\delta_{\gamma 1} - B^{1\gamma}) - hC_i^{1\gamma}]N^\gamma \end{aligned} \tag{20}$$

does not vanish for all i.

Conversely, assume h is a solution of (6) such that $1+h\lambda 1i\neq 0$ and B satisfies (7) and (8). Then we define the functions Z^i and Δ by (5), $b^{\alpha i}$ by (15). We now define vector fields $\tilde{N}_\alpha$ by (10). Since (7) holds, we get that $\tilde{N}_\alpha$ are orthonormal. We consider $d\tilde{N}_\alpha(e_i)$, as a vector of R^{n+p}, given by the expression (12). It follows from the definition of $\tilde{N}_\alpha$ that equations (13) and (14) hold. We now consider $\tilde{N}_\alpha$ and $\tilde{X}$ as in (10) and (3) respectively. We need to verify that $\tilde{X}$ satisfies the conditions of Definition 2.1.

We have seen that $\tilde{N}_\alpha$ are orthonormal. We next verify that $\tilde{N}_\alpha$ is normal to $\tilde{X}$. From the definition of $\tilde{X}$, $d\tilde{X}(e_i)$ is given by (18) and it does not vanish for generic h and B.

Using (15) and the fact that $< d\tilde{N}_\alpha(e_i), \tilde{N}_\beta >= 0$ is equivalent of equations (7) and (8), we conclude that

$$< d\tilde{X}(e_i), \tilde{N}_\alpha >= (1 + h\lambda^{1i})b^{\alpha i} + dh(e_i)(B^{\alpha 1} - \delta_{1\alpha}) - h < d\tilde{N}_1(e_i), \tilde{N}_\alpha >= 0.$$

It follows from (14), (15) and (6), that

$$L_i^{\alpha j} + Z^j C_i^{\alpha 1} = 0, \qquad i \neq j.$$

As a consequence of (18), (12) and $< d\tilde{N}_\alpha(e_i), \tilde{N}_1 >= 0$, we conclude that for $i \neq j$

$$\begin{aligned} < d\tilde{N}_\alpha(e_i), d\tilde{X}(e_j) > &= < d\tilde{N}_\alpha(e_i), (1 + h\lambda^{1j})(e_j) + dh(e_j)N_1 > \\ &= L_i^{\alpha j}(1 + h\lambda^{1j}) + C_i^{\alpha 1} dh(e_j) = 0. \end{aligned}$$

Finally, we conclude that the images of the vector fields e_i, e_j by $d\tilde{X}$ are orthogonal for all $i \neq j$. In fact,

$$\begin{aligned} < d\tilde{X}(e_i), d\tilde{X}(e_j) > &= (1 + h\lambda^{1i}) < e_i, d\tilde{X}(e_j) > + dh(e_i) < N_1, d\tilde{X}(e_j) > \\ &= dh(e_j)[-(1 + h\lambda^{1i})b^{1i} + dh(e_i)(1 - B^{11})] = 0, \end{aligned}$$

where the last equality follows from the definition of $b^{\alpha i}$.

□

Remark. We observe that when $p = 1$ then (7) reduces to $(B^{11})^2 + \Delta(1 - B^{11})^2 = 1$ and (8) is an identity. For any p,introducing the eigenvalues

$$d\tilde{N}_1(e_i) = \tilde{\lambda}^{1i} d\tilde{X}(e_i),$$

from (18) we get

$$(1 + h\tilde{\lambda}^{1i}) d\tilde{X}(e_i) = (1 + h\lambda^{1i})e_i + dh(e_i)(N_1 - \tilde{N}_1).$$

Hence, using the expressions (10) and (15), we obtain

$$|1 + h\tilde{\lambda}^{1i}||d\tilde{X}(e_i)| = |1 + h\lambda^{1i}|. \tag{21}$$

Moreover, it follows from the proof given above that for the submanifold $\tilde{M}$, the normal unit vector fields are given by

$$\tilde{N}_\alpha = \sum_{i=1}^{n} Z^i(\delta_{1\alpha} - B^{\alpha 1})e_i + \sum_{\gamma=1}^{p} B^{\alpha\gamma} N_\gamma, \tag{22}$$

and they are parallel in the normal connection. Moreover, using (12), we have

$$\begin{aligned} < d\tilde{N}_\gamma(e_i), dX(e_i) >= L_i^{\gamma i} &= \tilde{\lambda}^{\gamma i} < d\tilde{X}(e_i), dX(e_i) > &(23)\\ < d\tilde{N}_\gamma(e_i), N_1 >= C_i^{\gamma i} &= \tilde{\lambda}^{\gamma i} < d\tilde{X}(e_i), N_1 > . &(24) \end{aligned}$$

Using (23), (24), (18),(10) and (12) we obtain the principal curvatures given by

$$\tilde{\lambda}^{\gamma i} = \begin{cases} \dfrac{C_i^{\gamma 1}}{dh(e_i)(1-B^{11}) - hC_i^{11}} & \text{if} \quad dh(e_i) \neq 0 \\ \\ \dfrac{L_i^{\gamma i}}{1+h\lambda^{1i} - hL_i^{11}}, & \text{if} \quad dh(e_i) \equiv 0, \end{cases} \tag{25}$$

where $C_i^{\gamma 1}$ and $L_i^{\gamma i}$ are given respectively by (14) and (13).

We conclude this remark by observing that the generic condition, mentioned in Theorem 2.3, will be given more explicitly at the end of this section, in Theorem 2.8.

The following result shows that Definition 2.1 implies that for each point $\tilde{q} \in \tilde{M}$ and any unit vector normal (resp. tangent) at $\tilde{q}$, there exists a unit vector, normal (resp. tangent) at a corresponding point $q \in M$, such that the lines in these directions are parallel or intersect at an equidistant point. Later in Corollary 2.9, we will show that there exists a matrix B, which satisfies (7) and (8), such that the correspondence between the normal bundles is an isometry which commutes with the normal connections (hence it satisfies the condition on the normal bundle isometry of definition in [DT1]).

Proposition 2.4 *Let M^n and $\tilde{M}^n$ be immersed submanifolds of R^{n+p} with flat normal bundle. Suppose that $\tilde{M}$ is associated to M by a Ribaucour transformation with respect to a set of orthonormal principal vectors fields e_i. Then for any $\tilde{q} \in \tilde{M}$ and any unit vector $\tilde{N}(\tilde{q})$ normal (resp. $\tilde{V}(\tilde{q})$ tangent) to $\tilde{M}$, there exists $q \in M$ and a unit vector $N(q)$ normal (resp. $V(q)$ tangent) to M such that the lines of R^{n+p} starting at $\tilde{q}$ and q in the direction of $\tilde{N}(\tilde{q})$ and $N(q)$ (resp. $\tilde{V}(\tilde{q})$ and $V(q)$) respectively are parallel or intersect at a point equidistant to $\tilde{q}$ and q.*

Proof: We will consider the notation of Theorem 2.3.

i) Let $\tilde{N} = \sum_{\alpha=1}^{p} A^\alpha \tilde{N}_\alpha$ be any unit vector field normal to $\tilde{M}$. It follows from (22) that

$$\tilde{N} = \sum_{j=1}^{n} Z^j (A^1 - \sum_{\alpha=1}^{p} A^\alpha B^{\alpha 1}) e_j + \sum_{\gamma=1}^{p} (\sum_{\alpha=1}^{p} A^\alpha B^{\alpha\gamma}) N_\gamma. \tag{26}$$

Hence, if $A^1 - \sum_{\alpha=1}^{p} A^\alpha B^{\alpha 1} = 0$, then there exists $N = \tilde{N}$ such that the lines at corresponding points are parallel. If $A^1 - \sum_{\alpha=1}^{p} A^\alpha B^{\alpha 1} \neq 0$, let

$$\tilde{h} = \frac{h(1-B^{11})}{A^1 - \sum_{\alpha=1}^{p} A^\alpha B^{\alpha 1}}.$$

Using (3), (4), (7) and (26), we get $\tilde{X} + \tilde{h}\tilde{N} = X + \tilde{h}N$, where N is a unit vector given by

$$N = A^1 N_1 + \sum_{\gamma=2}^{p} (\sum_{\alpha=2}^{p} A^\alpha (\frac{B^{\alpha 1} B^{1\gamma}}{1-B^{11}} + B^{\alpha\gamma})) N_\gamma.$$

ii) We will now prove the corresponding result for tangent vectors. It follows from (18), that the unit principal directions of $\tilde{M}$ are given by

$$\tilde{e}_i = e_i + Z^i(N_1 - \tilde{N}_1). \tag{27}$$

Let $\tilde{V} = \sum_{i=1}^n Q^i \tilde{e}_i$ be any unit vector tangent to $\tilde{M}$. It follows from (27), that

$$\tilde{V} = \sum_{i=1}^n Q^i e_i + (N_1 - \tilde{N}_1) \sum_{i=1}^n Q^i Z^i. \tag{28}$$

Hence, if $\sum_{i=1}^n Q^i Z^i = 0$, then there exists $V = \tilde{V}$ such that the lines are parallels. If $\sum_{i=1}^n Q^i Z^i \neq 0$, we consider

$$R = \frac{-h}{\sum_{i=1}^n Q^i Z^i}.$$

It follows from (28) that $\tilde{X} + R\tilde{V} = X + RV$, where V is given by $V = \sum_{i=1}^n Q^i e_i$.

□

The following result linearizes the problem of obtaining the function h,

Proposition 2.5 *If h is a solution of (6) which does not vanish on a simply connected domain, then $h = \Omega/W_1$, where W_1 is a nonvanishing function and the functions Ω, Ω^i, W_1 satisfy*

$$d\Omega^i(e_j) = \sum_{k=1}^n \Omega^k \omega_{ik}(e_j), \qquad \text{for } i \neq j, \tag{29}$$

$$d\Omega = \sum_{i=1}^n \Omega^i \omega_i, \tag{30}$$

$$dW_1 = -\sum_{i=1}^n \Omega^i \lambda^{1i} \omega_i. \tag{31}$$

Conversely, suppose (29)-(31) are satisfied then $h = \Omega/W_1$ is a solution of (6).

Proof: Assume h is a nonvanishing solution of (6), then $\psi = \sum_i Z^i \omega_i / h$, is a closed form. Hence, on a simply connected domain there exists a differentiable function Ω such that $d(\log \Omega) = \psi$. We define $\Omega^i = d\Omega(e_i)$ and $W_1 = \Omega/h$. Then $dh(e_i) = \Omega^i(1 + \Omega\lambda^{1i}/W_1)/W_1$ and (30) holds. Moreover, it follows from (6) that (29) and (31) are satisfied.

Conversely if (29)-(31) hold, considering $Z^i = \Omega^i/W_1$ one concludes that (6) is satisfied. We define $h = \Omega/W_1$, then it follows that $dh(e_i) = Z^i(1 + h\lambda^{1i})$.

□

We observe that it follows from the proof of Proposition 2.5 that

$$dh(e_i) = \frac{\Omega^i}{W_1}(1 + \Omega\lambda^{1i}/W_1) \qquad Z^i = \frac{\Omega^i}{W_1} \qquad \Delta = \frac{1}{(W_1)^2} \sum_j (\Omega^j)^2. \tag{32}$$

Proposition 2.6 *Equation (29) is the integrability condition for (30) and (31). Moreover, (29) implies that there exist functions W_γ, $2 \leq \gamma \leq p$, defined on a simply connected domain such that*

$$dW_\gamma = -\sum_i \Omega^i \lambda^{\gamma i} \omega_i, \quad 2 \leq \gamma \leq p. \tag{33}$$

Proof: The proof follows easily from the fact that the system of equations (29) is equivalent to

$$d\Omega^i \wedge \omega_i - \sum_{j \neq i} \Omega^j \omega_{ij} \wedge \omega_i = 0, \quad 1 \leq i \leq n$$

and from Codazzi equation (2).

□

Our next result provides solutions B for (8).

Proposition 2.7 *Equations (29)-(31) and (33) are the integrability conditions, for the system of equations (7) and (8) for B. Moreover, for a given solution of (29)-(31) and (33), the matrix function*

$$B^{\alpha\beta} = \delta_{\alpha\beta} - 2\frac{W_\alpha W_\beta}{S}, \qquad 1 \leq \alpha, \beta \leq p, \tag{34}$$

where

$$S = \sum_{j=1}^{n} (\Omega^j)^2 + \sum_{\gamma=1}^{p} (W_\gamma)^2. \tag{35}$$

is a solution of (8).

Proof: It follows from (7) that there exists an orthogonal matrix function $A = (a^{\alpha\beta}), 1 \leq \alpha, \beta \leq p$, such that

$$B^{11} = \frac{\Delta + a^{11}}{1 + \Delta}, \quad B^{1\beta} = \frac{a^{1\beta}}{\sqrt{1+\Delta}}, \quad B^{\alpha 1} = \frac{a^{\alpha 1}}{\sqrt{1+\Delta}}, \quad B^{\alpha\beta} = a^{\alpha\beta}, \tag{36}$$

where $2 \leq \alpha, \beta \leq p$.

We introduce the notation $\mathbf{W} = (W_1, ..., W_p)$. It follows from (31)-(33) and (9) that equation (8) is equivalent to

$$dBB^t - BdB^t + \Delta(dDD^t - DdD^t) + \frac{2}{W_1}(Dd\mathbf{W}B^t - Bd\mathbf{W}^t D^t) = 0.$$

As a consequence of (9), (36) and $AA^t = I$ from the last equality, we get

$$da^{11} \;=\; (1 - a^{11}) \left[-(1 + a^{11}) \frac{dH}{2H} - \frac{1}{\sqrt{H}} \sum_{\gamma=2}^{p} a^{1\gamma} dW_\gamma \right], \tag{37}$$

$$da^{1\beta} = a^{11}a^{1\beta}\frac{dH}{2H} - \frac{1-a^{11}}{\sqrt{H}}dW_\beta + \frac{a^{1\beta}}{\sqrt{H}}\sum_{\gamma=2}^{p} a^{1\gamma}dW_\gamma, \tag{38}$$

$$da^{\alpha 1} = a^{\alpha 1}a^{11}\frac{dH}{2H} - \frac{1-a^{11}}{\sqrt{H}}\sum_{\gamma=2}^{p} a^{\alpha\gamma}dW_\gamma, \tag{39}$$

$$da^{\alpha\beta} = a^{\alpha 1}a^{1\beta}\frac{dH}{2H} + \frac{a^{\alpha 1}}{\sqrt{H}}dW_\beta + \frac{a^{1\beta}}{\sqrt{H}}\sum_{\gamma=2}^{p} a^{\alpha\gamma}dW_\gamma, \tag{40}$$

where

$$H = \sum_{j=1}^{n}(\Omega^j)^2 + (W_1)^2. \tag{41}$$

It is a straighthoward computation to see that the compatibility condition for these equations is given by (29)-(31) and (33).

Claim: *For a given solution of (29)-(31)*

a) *The solution to (37) and (38) is given by*

$$a^{11} = 1 - \frac{2H}{S}, \qquad a^{1\gamma} = -\frac{2\sqrt{H}}{S}W_\gamma, \quad 2 \le \gamma \le p, \tag{42}$$

where W_γ are solutions of (33).

b) *There exist solutions to (39) and (40) given by*

$$a^{\alpha 1} = -2\frac{\sqrt{H}}{S}W_\alpha, \qquad a^{\alpha\beta} = -2\frac{W_\alpha W_\beta}{S} \quad 2 \le \alpha, \beta \le p. \tag{43}$$

In fact, we may consider

$$a^{11} = 1 - \frac{2H}{Q}, \qquad a^{1\beta} = -2\frac{\sqrt{H}}{Q}\sigma_\beta, \quad 2 \le \beta \le p. \tag{44}$$

Since the matrix A is orthogonal, it follows from the fact that $\sum_{\gamma=1}^{p}(a^{1\gamma})^2 = 1$ that

$$Q = H + \sum_{\beta=2}^{p}(\sigma_\beta)^2. \tag{45}$$

From (44) and (37) we obtain

$$dQ = dH + \sum_{\gamma=2}^{p} 2\sigma_\gamma dW_\gamma,$$

which, as a consequence of (45), reduces to

$$\sum_{\gamma=2}^{p}\sigma_\gamma d\sigma_\gamma = \sum_{\gamma=2}^{p}\sigma_\gamma dW_\gamma. \tag{46}$$

Similarly, by considering (44), (38), (45) and (46) we conclude that $d\sigma_\beta = dW_\beta$, for $2 \leq \beta \leq p$. Therefore we have $\sigma_\beta = W_\beta$ are solutions of (33) and $Q = S$. This concludes the proof of our claim a).

In order to prove claim b), we first observe that a straightforward computation shows that the matrix A defined by (42) and (43) is orthogonal. Moreover, by differentiating (43), we can easily verify that (39) and (40) are satisfied.

Finally, since the matrix function B is given by (36), where $\Delta = \sum_j (\Omega^j)^2/(W_1)^2$ substituting the expression of the matrix A, and using the fact that $1 + \Delta = H/(W_1)^2$, we conclude that B, given by (34), satisfies (8).

□

If $\tilde{M} \subset R^{n+p}$ is a submanifold locally associated to M by a Ribaucour transformation, Theorem 2.3, shows that the parametrization $\tilde{X}$ of $\tilde{M}$ depends on a function h and a matrix B satisfying the differential equations (7) and (8). However, the expressions of the parametrization $\tilde{X}$ and of the normal vector $\tilde{N}_1$, given by (3) and (4) respectively, depend only on the first row of matrix B. Moreover, Claim a) in the proof of Proposition 2.7 and equation (36) show that all solutions B have the first row given by (34). The other rows of B are related to fixing the remaining unit vector fields $\tilde{N}_\gamma$, normal to $\tilde{X}$, for $\gamma \geq 2$ (see (10)). Considering the matrix function B given by (34), Theorem 2.3 can be rewritten as follows.

Theorem 2.8 *Let M^n be an immersed submanifold of R^{n+p}, with flat normal bundle parametrized by $X : U \subset R^n \to M$. Assume e_i, $1 \leq i \leq n$ are the principal directions, $N_\gamma, 1 \leq \gamma \leq p$, is a parallel orthonormal basis of the normal bundle of $X(U)$ and $\lambda^{\gamma i}$ the corresponding principal curvatures. A submanifold $\tilde{M}^n$ is locally associated to M by a Ribaucour transformation with respect to e_i, if and only if, for each $\tilde{q} \in \tilde{M}$, there exist differentiable functions $W_\gamma, \Omega, \Omega^i : V \subset U \to R$, defined on a simply connected domain V, which satisfy (29)-(31) and (33), such that, for some $1 \leq \alpha \leq p$*

$$W_\alpha S(W_\alpha + \lambda^{\alpha i}\Omega)(S - \Omega T^i) \neq 0, \quad 1 \leq i \leq n, \tag{47}$$

where, S is defined by (35),

$$T^i = 2\left(d\Omega^i(e_i) + \sum_k \Omega^k \omega_{ki}(e_i) - \sum_{\gamma=1}^{p} W_\gamma \lambda^{\gamma i}\right), \tag{48}$$

and $\tilde{X} : V \subset R^n \to \tilde{M}$, is a parametrization of a neighborhood of $\tilde{q}$ in $\tilde{M}$ given by

$$\tilde{X} = X - \frac{2\Omega}{S}\left(\sum_{i=1}^{n} \Omega^i e_i - \sum_{\gamma=1}^{p} W_\gamma N_\gamma\right). \tag{49}$$

Moreover, the unit normal vector fields and the corresponding principal curvatures on $\tilde{M}$ are

$$\tilde{N}_\beta = N_\beta + 2\frac{W_\beta}{S}\left(\sum_{i=1}^{n} \Omega^i e_i - \sum_\gamma W_\gamma N_\gamma\right), \tag{50}$$

$$\tilde{\lambda}^{\beta i} \;=\; \frac{T^i W_\beta + \lambda^{\beta i} S}{S - \Omega T^i}. \tag{51}$$

Proof: Let X and $\tilde{X}$ be parametrizations of M and $\tilde{M}$. We can suppose without loss of generality that $\alpha = 1$. We have already seen that for $1 \leq \beta \leq p$, the normal vector fields $\tilde{N}_\beta$ are given by (22). Hence it follows from (32) and (34) that (50) holds, where $S \neq 0$. The expression (49) follows directly (3) and (50). We will now obtain condition (47) and the expression (51) for the eigenvalues $\tilde{\lambda}^{\beta i}$.

If $\Omega^i \neq 0$, then it follows from (14), (15) and (31)- (34) that

$$C_i^{\gamma 1} = 2\frac{W_1}{S^2}(W_\gamma dS(e_i) + \Omega^i S \lambda^{\gamma i}).$$

Therefore, using (32), (34) and the fact that

$$dS(e_i) = \Omega^i T^i, \tag{52}$$

where T^i is given by (48), we have that

$$dh(e_i)(1 - B^{11}) - hC_i^{11} = 2\frac{W_1 \Omega^i}{S^2}(S - \Omega T^i). \tag{53}$$

Therefore, the left hand side of (53) (which is also $< d\tilde{X}(e_i), N_1 >$) is nonzero if and only if $W_1(S - \Omega T^i) \neq 0$. Moreover, by considering the first expression of (25), we conclude that (51) holds.

If $\Omega^i \equiv 0$, then it follows from (13), (34) and (32) that

$$L_i^{\gamma i} = \frac{1}{S}(S\lambda^{\gamma i} + W_\gamma T^i),$$

where we have used the fact that $db^{\gamma i}(e_i) = 0$. On the other hand,

$$1 + h\lambda^{1i} - hL_i^{11} = \frac{1}{S}(S - \Omega T^i).$$

Hence, the left hand side of this relation (which is $< d\tilde{X}(e_i), e_1 >$) is nonzero if and only if $(S - \Omega T^i) \neq 0$.. Therefore, by considering the second expression of (25), we conclude that (51) holds also in this case.

In order to prove (47), we have seen that one needs $W_1 S(S - \Omega T^i) \neq 0$, $\forall 1 \leq i \leq n$. Requiring $W_1 + \Omega\lambda^{1i} \neq 0$ corresponds to condition b) of Definition 2.1, since $h = \Omega/W_1$.

□

We observe that in (47), $S \neq 0$ determines the domain of $\tilde{X}$. Moreover, a straightforward computation shows that

$$|d\tilde{X}(e_i)| = \frac{|S - \Omega T^i|}{S}.$$

Therefore, the parametrization $\tilde{X}$, given by (49), may extend regularly to points where $W_\alpha(W_\alpha + \lambda^{\alpha i}\Omega) = 0$, whenever $S(S - \Omega T^i) \neq 0$.

From now on, whenever we say that a submanifold $\tilde{M}$ is locally associated by a Ribaucour transformation to M, with respect to a set of principal directions e_i of M, we are assuming that there are functions where Ω^i, Ω and W_γ locally defined, satisfying the system (29)-(31) and (33). Moreover, whenever M is parametrized by orthogonal lines of curvature, we are assuming that e_i are the unit vector fields tangent to the coordinate curves.

As a consequence of the theorem above we show that Definition 2.1 implies the existence of a normal bundle isometry satisfying conditions a) b) and c) of definition [DT1].

Corollary 2.9 *Let M^n and $\tilde{M}^n$ be submanifolds of R^{n+p} with flat normal bundle. Assume that $\tilde{M}$ is associated to M by a Ribaucour transformation with respect to $e_1, ..., e_n$. Then there exists an isometry P of the normal bundles and a normal vector field ζ which is nowhere a principal curvature normal of M, such that $P(N) - N =< N, \zeta > (q - \psi(q))$ for all vector field N normal to M and P commutes with the normal connection.*

Proof: From Theorem 2.8 we have differentiable functions W_γ, Ω, Ω^i which satisfy (29)-(31) and (33) and parametrizations X and $\tilde{X}$ of M and $\tilde{M}$ such that (49) hold, and unit vector fields on $\tilde{M}$ given by (50).

Let P be the normal bundle isometry defined by extending linearly the correspondence $P(N_\gamma) = \tilde{N}_\gamma$, for all $1 \leq \gamma \leq p$. We consider the normal vector field $\zeta = \sum_\gamma W_\gamma N_\gamma / \Omega$. It follows from Definition 2.1 b) that ζ is nowhere a principal curvature normal of M. As a consequence of (49), (50) and the definition of P, we have that $P(N) - N =< N, \zeta > (X - \tilde{X})$. Moreover, since M and $\tilde{M}$ have flat normal bundle, it follows easily that P commutes with the normal connections.

□

We conclude this section by showing that an n-dimensional sphere or hyperplane can be locally associated by a Ribaucour transformation to any given hypersurface M^n of R^{n+1}, which admits n orthogonal principal direction vector fields.

Corollary 2.10 *Let M^n be a hypersurface of R^{n+1}, that admits n orthogonal principal direction vector fields e_i. For any real constants $b_1 \neq 0$ and b_0, the system of equations*

$$\begin{aligned} d\Omega^i &= \sum_k \Omega^k \omega_{ik} + b_0 \omega_i + (b_1 - W)\omega_{in+1} \\ d\Omega &= \sum_{i=1}^n \Omega^i \omega_i, \\ dW_1 &= \sum_{i=1}^n \Omega^i \omega_{in+1} \end{aligned}$$

is integrable. The function $S - 2(b_0\Omega + b_1 W) = c$ is a constant determined by the initial *conditions. Considering $c = 0$, the associated hypersurface is an open subset of a sphere (resp. hyperplane) if $b_0 \neq 0$ (resp. $b_0 = 0$).*

Proof One can easily prove that the system of equations is integrable and as a consequence of (35)that $dS - 2(b_0 d\Omega + b_1 dW) = 0$. Therefore, we can choose the initial conditions so that $S - 2(b_0\Omega + b_1 W) = 0$. Finally, it follows from (51) that any principal curvature of the associated hypersurface is given by b_0/b_1.

□

Observe that the system of equations of Corollary 2.10 contains (29)-(31). Moreover, this Corollary shows that the system of equations (29)-(31) and (33) does not preserve multiplicity of principal directions. This fact had already been observed in [CFT1].

3. Dupin submanifolds

In this section we will consider Dupin submanifolds. A submanifold $M^n \subset R^{n+p}$ with flat normal bundle is a *Dupin submanifold* if for an orthonormal basis of the normal bundle of M, parallel in the normal connection, its principal curvatures are constant along the corresponding lines of curvature, i.e. $d\lambda^{\alpha i}(e_i) = 0$, $\forall 1 \leq i \leq n$. The main purpose of this section is to characterize the Dupin hypersurfaces which have a principal curvature of constant multiplicity one. This characterization was obtained in [Co]. More recently, the case of a principal curvature with constant multiplicity bigger than one was considered in [DFT].

Theorem 3.1 *Let M^n be a Dupin submanifold of R^{n+p}, with flat normal bundle. Let e_i, $1 \leq i \leq n$ be orthonormal principal vector fields, $N_\gamma, 1 \leq \gamma \leq p$ parallel orthonormal basis of the normal bundle of M and $\lambda^{\gamma i}$ the corresponding principal curvatures. Let $\tilde{M}$ be a submanifold of R^{n+p}, locally associated to M by a Ribaucour transformation, with respect to e_i. Then $\tilde{M}$ is a Dupin submanifold, if and only if, the functions Ω^i, Ω and W_γ satisfy the following additional condition for each i $1 \leq i \leq n$,*

$$dT^i(e_i) = 0 \tag{54}$$

where T^i is given by (48).

Proof: Since M is a Dupin submanifold, it follows from (51), (52), (30) and (31) that $d\tilde{\lambda}^{\gamma i}(e_i) = 0$, if and only if,

$$(W_\gamma + \Omega\lambda^{\gamma i})dT^i(e_i) = 0, \quad 1 \leq \gamma \leq p, \ 1 \leq i \leq n.$$

Since $W_1 + \Omega\lambda^{1i} \neq 0$, we conclude the proof.

□

From now on, we will consider submanifolds parametrized by orthogonal lines of curvature, $X(u_1, ..., u_n)$. Then the first fundamental form is of the form $I = \sum_i \omega_i^2$, where

$\omega_i = a_i du_i$ and $a_i = |X_{,i}|$ and the principal directions are the vector fields $e_i = X_{,i}/a_i$, where $X_{,i}$ denotes partial derivative of X with respect to u_i. Then the connection forms are

$$\omega_{ij} = \frac{1}{a_i a_j}\left(-a_{i,j}\omega_i + a_{j,i}\omega_j\right) \tag{55}$$

and the Christoffel symbols are given by

$$\Gamma^i_{ij} = \frac{a_{i,j}}{a_i}, \quad \Gamma^j_{ii} = -\frac{a_i a_{i,j}}{a_j^2}, \quad \Gamma^k_{ij} = 0 \quad \text{for } i, j, k \text{ distinct.} \tag{56}$$

Theorem 3.2 *Let M^n be a Dupin submanifold of R^{n+p}, parametrized by orhogonal lines of curvature $X(u_1, ..., u_n)$ and with flat normal bundle. Let $e_i = X_{,i}/|X_{,i}|$, $1 \leq i \leq n$, and let $N_\gamma, 1 \leq \gamma \leq p$ be parallel orthonormal vector fields normal to M. Consider $\lambda^{\gamma i}$ the corresponding principal curvatures.*
a) If there exists, $\alpha, 1 \leq \alpha \leq p$ such that $\forall 1 \leq i \leq n, \lambda^\alpha = \lambda^{\alpha i}$, on M. Then , up to translations, $M \subset R^{n+p-1}$ or M is a subset of the sphere $S^{n+p-1}(1/\lambda^\alpha)$.
b) If there exists, $\alpha, 1 \leq \alpha \leq p$ and $k, 1 \leq k \leq n$ such that

$$\lambda^{\alpha i} \neq \lambda^{\alpha j}, \quad 1 \leq i \leq k, \quad k+1 \leq j \leq n,$$

then M is foliated by k-dimensional Dupin submanifolds.

Proof: Assume the conditions of item a) are satisfied. Since $d\lambda^{\alpha i}(e_i) = 0$, we conclude that λ^α is constant. If $\lambda^\alpha = 0$, we consider $f(u) =< X(u) - X(u_0), N_\alpha >$. Since $df(e_i) = 0$ and $f(u_0) = 0$ we have that $f(u) = 0$. Therefore up to translation $M \subset R^{n+p-1}$. If $\lambda^\alpha \neq 0$, we consider $Y(u) = X(u) - N_\alpha/\lambda^\alpha$. Since $dY(e_i) = 0$, Y is a constant vector and $|X - Y|^2 = 1/(\lambda^\alpha)^2$, therefore up to translations $M \subset S^{n+p-1}(1/\lambda^\alpha)$.

Assume the conditions of item b) are satisfied. We define

$$Y(u_1, ..., u_k) = X(u_1, ..., u_k, u^0_{k+1}, ..., u^0_n).$$

We will show that Y is a Dupin submanifold with flat normal bundle. In fact, let

$$N_j = \frac{X_{,j}}{a_j}, \quad k+1 \leq j \leq n,$$

Then the set $\{N_j, N_\gamma\}$ is an orthonormal basis for the normal bundle of Y. It follows from (56), that for $1 \leq i \leq k$

$$N_{j,i} = \frac{X_{,ji}}{a_j} - \frac{a_{j,i}}{(a_j)^2}X_{,j} = \frac{\Gamma^i_{ij}}{a_j}Y_{,i}.$$

Hence

$$\lambda^{ji} = \frac{\Gamma^i_{ij}}{a_j}, \quad \lambda^{\gamma i}, \quad 1 \leq i \leq k, \; k+1 \leq j \leq n, \; 1 \leq \gamma \leq p,$$

are the principal curvatures of Y. Using the hypothesis $\lambda^{\alpha i} \neq \lambda^{\alpha j}$, Codazzi equations (2), (55) and (56), we get

$$\lambda^{ji}_{,i} = \left[\frac{1}{a_j}\frac{\lambda^{\alpha i}_{,j}}{\lambda^{\alpha j} - \lambda^{\alpha i}}\right]_i = 0, \quad 1 \leq i \leq k,\ k+1 \leq j \leq n.$$

Therefore Y is a Dupin submanifold.

□

Theorem 3.3 *Let M^n be an orientable immersed hypersurface of R^{n+1} parametrized by orthogonal lines of curvature $X(u_1, ...u_n)$. Then M is a Dupin hypersurface which has a non zero principal curvature of multiplicity one in the direction e_n, if and only if, M is foliated by (n-1)-dimensional Dupin submanifold associated by Ribaucour transformations with respect to e_i, $1 \leq i \leq n-1$.*

Proof: Let $X(u_1, ..., u_n)$ be a local parametrization of M by orthogonal curvature lines. Let $e_i = X_{,i}/|a_i|$, $1 \leq i \leq n$, be the principal directions, N the normal Gauss map and λ^i the principal curvatures, i.e. $dN(e_i) = \lambda^i e_i$. Assume that $\lambda^n \neq 0$ has multiplicity one and define

$$Y = X - \frac{1}{\lambda^n}N.$$

Since $\lambda^n_{,n} = 0$ and $N_{,n} = \lambda^n X_{,n}$, we have that $Y_{,n} = 0$, hence, for $u_n \neq \tilde{u}_n$ we get

$$X(u, u_n) - \frac{N(u, u_n)}{\lambda^n(u)} = X(u, \tilde{u}_n) - \frac{N(u, \tilde{u}_n)}{\lambda^n(u)},$$

where $u = (u_1, ...u_{n-1})$. Using item b) of Theorem 3.2, we conclude that M is foliated by (n-1)-dimensional Dupin submanifolds, associated by Ribaucour transformation with respect to e_i, $1 \leq i \leq n-1$.

Conversely, assume there exists a differentiable function $h(u)$, such that

$$X(u, u_n) + h(u)N(u, u_n) = X(u, \tilde{u}_n) + h(u)N(u, \tilde{u}_n),$$

where $u_n \neq \tilde{u}_n$. Then the map $Y(u) = X(u, u_n) + h(u)N(u, u_n)$ is independent of u_n, hence

$$0 = dY(e_n) = dX(e_n) + hdN(e_n) = dX(e_n)(1 + h\lambda^n),$$

i.e. $\lambda^n = -1/h$ and since $1 + h\lambda^i \neq 0$ for $i \neq n$ we have that $\lambda^n \neq \lambda^i$. Moreover, $N(u, .)$ is a parallel unit vector field normal to $X(u, .)$ and we have that $\lambda^i_{,i} = 0$. Hence M is a Dupin hypersurface wich has a non zero principal curvature of multiplicity one.

□

Proposition 3.4 *Let M^n be a Dupin submanifold of R^{n+2}, with flat normal bundle. Assume that $X : U \subset R^n \to M$ is a parametrization by orthogonal curvature lines, and $\tilde{M}$ is a Dupin submanifold of R^{n+2} locally associated to M by a Ribaucour transformation with* *respect to* $e_i = X_i/|X_{,i}|$, $1 \leq i \leq n$. *Then there exists a 1-parameter family of Ribaucour transformations of* M, *containing* M *and* $\tilde{M}$, *which defines a Dupin hypersurface in* R^{n+2}, *with a non zero principal curvature of multiplicity one. Moreover, a parametrization of the hypersuface* $\tilde{M}$ *is given by*

$$\tilde{X}(u, u_{n+1}) = X - \frac{2\Omega(\sum_{i=1}^{n} \Omega^i e_i - W_1 N_1 - (W_2 + u_{n+1})N_2)}{\sum_{i=1}^{n}(\Omega_i)^2 + (W_1)^2 + (W_2 + u_{n+1})^2},$$

where $(u, u_{n+1}) \in U \times R$.

Proof: Let Ω^i, Ω, W_1, W_2 be a solution of the system (29)-(31) and (33) which associates $\tilde{M}$ to M by a Ribaucour transformation. Then Ω^i, Ω, W_1, $W_2 + u_{n+1}$, $\forall u_{n+1} \in R$ is also a solution of the system. Therefore, there is a 1-parameter family of submanifolds $\tilde{M}_{u_{n+1}}$ associated to M by Ribaucour transformations.

Let T^i and $T^i_{u_{n+1}}$ be the expressions given by (48) corresponding to $\tilde{M}$ and $\tilde{M}_{u_{n+1}}$ respectively. Since M and $\tilde{M}$ are Dupin submanifolds, it follows from (48), that

$$dT^i_{u_{n+1}}(e_i) = dT^i(e_i) = 0.$$

Hence, using Theorem 3.1 we conclude that $\tilde{M}_{u_{n+1}}$ is a Dupin submanifold for each u_{n+1}. We complete the proof by using Theorem 3.3. □

Corollary 3.5 *Let* M^n *and* $\tilde{M}^n$ *be Dupin hypersurfaces of* R^{n+1} *parametrized by orthogonal lines of curvature. Assume that* $\tilde{M}$ *is associated to* M *by a Ribaucour transformation, with respect to the principal directions tangent to the coordinate curves of* M. *Then there exists a Dupin hypersurface of* R^{n+2} *wich has a non zero principal curvature of multiplicity one containing* M *and* $\tilde{M}$.

Proof: We can consider M^n and $\tilde{M}^n$ as submanifolds of R^{n+2} with $N_2 = \tilde{N}_2 = (0, ..., 0, 1) \in R^{n+2}$, normal unit vector fields of M^n and $\tilde{M}^n$. Hence $\lambda^{2i} = \tilde{\lambda}^{2i} = 0, 1 \leq i \leq n$. Therefore, M^n and $\tilde{M}^n$ are associated by Ribaucour transformation as submanifolds of R^{n+2}. Using of Proposition 3.4 we conclude our result.

□

Remark 3.6 As an illustration of the use of Corollary 3.5, we mention [CFT1], where families of Dupin hypersurfaces of R^{n+1} were obtained, by considering hypersurfaces associated by Ribaucour transformations to a hyperplane, to a torus, to $S^1 \times R^{n-1}$ and to $S^2 \times R^{n-2}$. These families generate Dupin hypersurfaces of R^{n+2}, wich have a non zero principal curvature of multiplicity one, by using Corollary 3.5.

Example 3.7 In this example, we will obtain the families of Dupin surfaces in R^3 associated to the plane by Ribaucour transformations, with respect to two distinct pairs

of principal directions tangent to two distinct parametrizations by orthogonal lines of curvature. We will show that the two families do not coincide.
a) Let $X(u_1, u_2) = (u_1, u_2, 0)$ be a parametrization of the plane in R^3 and $e_i = X_{u_i}$, $i = 1, 2$, be the unit tangent vectors. Then the dual and the connection forms are given by $\omega_i = du_i$ and $\omega_{12} = 0$, $\lambda^i = 0$ and the system of equations (29)-(31) reduces $\partial\Omega^i/\partial u_j = 0$, $i \neq j$, $d\Omega = \sum_i \Omega^i du_i$, $dW = 0$. Hence,

$$\Omega^i = f_i'(u_i), \qquad \Omega = f_1 + f_2, \qquad W = c \neq 0,$$

where f_i is a differentiable function of u_i and c is a real number. From Theorem 2.8 we have

$$S = \sum_i (f_i')^2 + c^2, \qquad T^i = 2f_i'' \qquad S - \Omega T^i \neq 0.$$

Now consider the family of surfaces parametrized by $\tilde{X}$ described by (49). Then the principal curvatures are given by $\tilde{\lambda}^i = cT^i/(S - \Omega T^i)$. It follows from Theorem 3.3 that $\tilde{X}$ is a Dupin manifold if and only if $\partial T^i/\partial u_i = 0$. Therefore,

$$f_i = a_i u_i^2 + b_i u_i + c_i, \quad i = 1, 2.$$

We claim that the family $\tilde{X}$ does not contain a parametrization of the torus. In fact, suppose $\tilde{\lambda}^i$ is constant for some $i = 1, 2$, then we either have $a_j = b_j = 0$, $j \neq i$ or $a_2 = a_1$. In both cases, we conclude that all points of $\tilde{X}$ are umbilic. Hence it does not describe a torus.

b) Let $X(u_1, u_2) = (u_1 \cos u_2,\, u_1 \sin u_2, 0)$, $0 < u_1 < \infty$, $0 < u_2 < 2\pi$, be a parametrization of the plane in R^3. We consider the principal directions $e_1 = X_{u_1}$ and $e_2 = X_{u_2}/u_1$. Then $\omega_1 = du_1$, $\omega_2 = u_1 du_2$ and $\omega_{12} = du_2$. By solving the system of equations (29)-(31), we obtain

$$\Omega^1 = f_1' + f_2, \qquad \Omega^2 = f_2', \qquad \Omega = f_1 + u_1 f_2, \qquad W = c \neq 0,$$

where f_i is a differentiable function of u_i and c is a real number. From Theorem 2.8 we have

$$T^1 = 2f_1'', \qquad T^2 = \frac{2}{u_1}(f_1' + f_2 + f_2''), \qquad S - \Omega T^i \neq 0.$$

Consider the family of surfaces described by (49), then these are Dupin surfaces, if and only if,

$$f_1 = a_1 u_1^2 + b_1 u_1 + c_1, \qquad \text{and} \qquad (f_2'' + f_2)' = 0.$$

Now we choose $f_2 = c_2$, $a_1 \neq 0$ and $b_1 + c_2 \neq 0$. Then it is easy to see that $\tilde{\lambda}^1$ is a nonzero constant and $\tilde{\lambda}^2$ is a non constant function of u_1. Hence it is a parametrization of a torus.

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