Minimal and cmc surfaces obtained by Ribaucour transformations.

K. Tenenblat *

This lecture describes recent results obtained in my joint work with A. Corro and W. Ferreira, contained in [CFT1] and [CFT2], on Ribaucour transformations as a method of obtaining minimal, constant mean curvature (cmc) and linear Weingarten surfaces from a given such surface.

In the last two decades, a great activity in research has been devoted to constructing new complete minimal and cmc surfaces in $\mathbf{R}^3$ (see for example [Co], [HM], [JM]). The main tool in such constructions has been the Weierstrass representation for minimal surfaces, while cmc surfaces have been obtained by using different methods, after the first example of a non totally umbilical compact immersed cmc surface was found in [W1]. These methods include a perturbation approach [K1,K2], integrable systems [PS] and the conjugate cousin method first introduced by B. Lawson [L] and later used by Karcher [Ka].

In our work we considered Ribaucour transformations to construct not only minimal and cmc surfaces, but also linear Weingarten surfaces. Ribaucour transformations for constant Gaussian curvature and constant mean curvature surfaces, including minimal surfaces, were considered at the beginning of the last century [Bi]. However, the first families of complete minimal surfaces based upon the use of the Ribaucour transformation were obtained recently in [CFT1]. Motivated by this work, we proved in [CFT2] that the classical theory of these transformations could be extended to linear Weingarten surfaces and therefore provided a unified version of the classical theory (see [RS] and [BS] for extensions of other results to special elliptic linear Weingarten surfaces).

We observe that linear Weingarten surfaces are locally parallel to surfaces of constant Gaussian curvature or to minimal surfaces. However, the well known Ribaucour transformations for these surfaces cannot be applied to produce complete linear Weingarten surfaces, since these parallel constructions in general produce curves of singularities.

In what follows, we start recalling briefly the classical theory of Ribaucour transformation for surfaces. Imposing a one-parameter algebraic condition on a Ribaucour transformation, one has a correspondence between linear Weingarten surfaces. Starting with such a surface, the system of equations is integrable and provides a family of new Weingarten surfaces. As a consequence of this theory one has the corresponding results for minimal and cmc surfaces. In this report, we state the main results and we include some applications of the theory, which provide interesting families of complete minimal, cmc and linear Weingarten surfaces, associated to the catenoid and to the cylinder. The reader is referred to [CFT1] and [CFT2] for proofs and other applications.

There is a two parameter family of complete minimal surfaces associated to the catenoid. For generic values of the parameter c of the Ribaucour transformation, the minimal surfaces are not periodic and have an infinite number of embedded planar ends. However, special values for c, related to irreducible rational numbers, produce 1-periodic surfaces with any finite number of embedded planar ends and two nonplanar ends.

---

*Partially supported by CNPq and PRONEX

Similarly, associated to the cylinder there is a two parameter family of complete linear Weingarten, immersed surfaces, explicitly given. These are 1-periodic n-bubble surfaces for special values of c and nonperiodic (with infinite number of bubbles) otherwise. We point out that the family associated to the cylinder shows the existence of infinitely many complete hyperbolic, linear Weingarten surfaces in R^3. The existence of such surfaces is unnespected, since these surfaces, as well as the surfaces of constant negative curvature, are in correspondence with solutions of the sine-Gordon equation and Hilbert's theorem proves that there are no complete surfaces of constant negative curvature in R^3.

1. Ribaucour transformations for linear Weingarten surfaces

Let M and $\tilde{M}$ be orientable surfaces of R^3 without umbilic points. We denote by N and $\tilde{N}$ their Gauss map. We say that M and $\tilde{M}$ are *associated by a Ribaucour transformation*, if and only if, there exists a differentiable function h defined on M and a diffeomorphism $\psi : M \to \tilde{M}$ such that: **a)** at corresponding points, the normal lines intersect at an equidistant point i.e. $p + h(p)N(p) = \psi(p) + h(p)\tilde{N}(\psi(p))$, for all $p \in M$; **b)** the center manifold is two dimensional i.e. the subset $p + h(p)N(p)$, $p \in M$, is a surface and **c)** ψ preserves lines of curvature.

We say that M and $\tilde{M}$ are *locally associated by a Ribaucour transformation* if for all $p \in M$ there exists a neighborhood of p in M which is associated by a Ribaucour transformation to an open subset of $\tilde{M}$. Similarly, one may consider the notion of *parametrized surfaces locally associated by a Ribaucour transformation.*

The following results give a characterization of Ribaucour transformations.

Theorem 1.1. *Let M be an orientable surface of R^3 and N its Gauss map. Let e_i, $1 \leq i \leq 2$ be orthonormal principal directions, λ^i the corresponding principal curvatures, i.e. $dN(e_i) = \lambda_i e_i$. A surface $\tilde{M}$ is associated to M by a Ribaucour transformation, if and only if, the function $h : M \to R$ satisfies*

$$dZ^j(e_i) + Z^i\omega_{ij}(e_i) - Z^iZ^j\lambda^i = 0, \quad 1 \leq i \neq j \leq 2. \tag{1}$$

where and ω_{ij} are the connection forms of the frame e_i and $Z^i = dh(e_i)/(1 + h\lambda^i)$.

If h is a nonvanishing function which satisfies equation (1) then, denoting by ω_i the dual frame of e_i, we have that $1/h \sum_{i=1}^{2} Z^i\omega_i$ is a closed 1-form and hence there exists a nonvanishing function Ω, defined on a simply connected domain, such that $d\Omega(e_i) = \Omega Z^i/h$. We define $\Omega_i = d\Omega(e_i)$ and $W = \Omega/h$. With this notation, equation (1) is equivalent to a linear system. In fact, a function h is a solution of (1) defined on a simply connected domain, if and only if $h = \Omega/W$, where Ω and a nonvanishing function W satisfy

$$\begin{aligned} d\Omega_i(e_j) &= \Omega_j\omega_{ij}(e_j), \qquad \text{for } i \neq j, && (2)\\ d\Omega &= \sum_{i=1}^{2} \Omega_i\omega_i, && (3)\\ dW &= -\sum_{i=1}^{2} \Omega_i\lambda^i\omega_i. && (4) \end{aligned}$$

It is easy to see that equation (2) is the integrability condition of the system of equations (3), (4) for Ω and W. Moreover, if a surface in R^3 is parametrized by $X : U \subset R^2 \to M$, one can show that a surface $\tilde{M}$ is locally associated to M, by a Ribaucour transformation, if and only if, there exist differentiable functions $W, \Omega, \Omega_i : V \subset U \to R$, which satisfy (2)-(4) and $\tilde{X} : V \subset R^2 \to \tilde{M}$, is a parametrization of $\tilde{M}$ given by

$$\tilde{X} = X - \frac{2\Omega}{S}\left(\sum_i \Omega_i e_i - WN\right). \tag{5}$$

where

$$S = \sum_i (\Omega_i)^2 + W^2. \tag{6}$$

From now on, whenever we say that two surfaces are locally associated by a Ribaucour transformation we are assuming that there is a local solution of the system (2)-(4), that the surfaces have no umbilic points and they are locally related as in (5).

A *linear Weingarten surface* of R^3 is a surface whose Gaussian and mean curvature K and H satisfy a linear relation $\alpha + \beta H + \gamma K = 0$, where $\alpha, \beta, \gamma \in R$. A sufficient condition for a Ribaucour transformation to transform a linear Weingarten surface into another such surface was given in [CFT2]. This result, which is our next theorem, provides a unified version of the classical results for constant mean and Gaussian curvatures and extends these results to linear Weingarten surfaces.

Theorem 1.2. *Let M and $\tilde{M}$ be regular surfaces of R^3, which are associated by a Ribaucour transformation, such that the normal lines intersect at a distance function h. Assume that $h = \Omega/W$ is not constant along the lines of curvature and the functions Ω_i, Ω and W satisfy the additional relation*

$$S = 2c(\alpha\Omega^2 + \beta\Omega W + \gamma W^2), \tag{7}$$

where S is defined by (6), $c \neq 0$ and α, β, γ are real constants. Then $\tilde{M}$ is a linear Weingarten satisfying $\alpha + \beta\tilde{H} + \gamma\tilde{K} = 0$, if and only if $\alpha + \beta H + \gamma K = 0$ holds for the surface M, where K, H and $\tilde{K}$, $\tilde{H}$ are the Gaussian and mean curvatures of M and $\tilde{M}$ respectively.

One may ask if the system (2)-(4) with the additional condition (7) is integrable, whenever we start with a linear Weingarten surface. The answer to this question is affirmative.

Theorem 1.3. (Integrability) *Let M be a linear Weingarten surface of R^3 satisfying $\alpha + \beta H + \gamma K = 0$. Then the system of equations (2)-(4) and (7) is integrable and the solution is uniquely determined on a simply connected domain U, by any given initial condition satisfying (7). Moreover, whenever $\alpha \neq 0$, any solution of the system defined on U is either identically zero and hence anihilates S or else the function S does not vanish on U.*

As a consequence, one can prove that if M is a linear Weingarten surface locally parametrized by $X : U \subset R^2 \to M \subset R^3$, then any linear Weingarten parametrized surface $\tilde{X}$, locally associated to X by a Ribaucour transformation as above is given by (5), where e_i are orthogonal principal directions, Ω, Ω_i, W are solutions of (2)-(4) and (7), and $\tilde{X}$ is a regular surface defined on $\tilde{U} = \{(u_1, u_2) \in U;\ T^2 + 2TQH + Q^2K \neq 0\}$ where $T = \alpha\Omega^2 - \gamma W^2$ and $Q = 2\gamma\Omega W + \beta\Omega^2$. In general, the surface $\tilde{X}$ depends on 4 parameters. However, in some cases the number of parameters may reduce to one (the parameter c), if one considers surfaces which are congruent by rigid motions of R^3. Special cases of the above results include the minimal surfaces and the cmc surfaces.

The cmc surfaces are obtained by considering $\alpha = -H \neq 0$, $\beta = 1$, $\gamma = 0$ and hence the algebraic condition (7) reduces to $S = 2c\Omega(-H\Omega + W)$, where c satisfies the relation $c(c-2H) > 0$. For any nontrivial solution of the system (2)-(4) and (7), defined on a simply connected domain U, the function S does not vanish. Hence, if $X : U \subset R^2 \to R^3$ is a cmc suface, then a cmc surface $\tilde{X}$ associated to X by Ribaucour transformation is regular on open subsets of U, where X has no umbilic points.

The case of the minimal surfaces is obtained by considering $\alpha = 0$, $\beta = 1$, and $\gamma = 0$ and the algebraic condition reduces to $S = 2c\Omega W$. One can show that the Ribaucour transfomations for minimal surfaces are related to producing embedded planar ends for the new associated minimal surfaces. In fact such ends are produced by the isolated zeros of S, where Ω does not vanish.

The reader is referred to [CFT1] for proofs and details in the case of the minimal surfaces and to [CFT2] for the linear Weingarten and cmc surfaces.

2. Minimal surfaces associated to the catenoid

The first families of complete minimal surfaces obtained by Ribaucour transformations were given in [CFT1], by applying the method to Enneper's surface and to the catenoid. The family of minimal surfaces associated to Enneper's surface is explicitly given and depends on three parameters. Each surface of this family has infinite total curvature and corresponds to a complete immersion of a sphere punctured at an infinite number of points, which are contained on a circle and accumulate at the pole. All except one of the infinite number of ends are embedded planar ends, whose positions are determined by the parameters.

The family associated to the catenoid has more interesting features. Depending on the value of the parameter c of the Ribaucour transformation, the associated surface may have infinitely many embedded planar ends (see [CHM] for minimal surfaces with an infinite number of anular ends) or any finite number of embedded planar ends (see [JM] for minimal surfaces with any finite number of catenoid ends). Moreover, each surface has one or two nonplanar ends. We point out that the family of minimal surfaces associated to the catenoid are of genus zero and contain a special class of 1-periodic surfaces. In the following results we give a brief description of the family associated to the catenoid.

Proposition 2.1. *Consider the catenoid parametrized by*

$$X(u_1, u_2) = (\cos u_2 \cosh u_1,\ \sin u_2 \cosh u_1,\ u_1)\,.$$

Excluding the catenoid and up to rigid motions of R^3, *a parametrized surface* $\tilde{X}_c(u_1, u_2)$ *is a minimal surface locally associated to* X *by a Ribaucour transformation as above, if and only if,*

$$\tilde{X}_c = X - \frac{\cosh u_1}{c}(\cos u_2, \sin u_2, 0) + \frac{1}{c(f+g)}(f' X_{u_1} - g' X_{u_2}), \tag{8}$$

where $c \neq 0$, $f(u_1)$ *and* $g(u_2)$ *are given as follows:*

a) *if* $c = 1/2$, *then* $f = \frac{(c_1 u_1 + b_1)^2}{2c_1}$, $\quad g = \frac{c_1 u_2^2}{2}$, *where* $c_1 \neq 0, b_1 \in R$, *and the function* $\tilde{X}_{1/2}$ *is defined on* $R^2 \setminus \{p_1\}$ *with* $p_1 = -\frac{1}{c_1}(b_1,\ 0)$;

b) *if* $2c - 1 > 0$, *then* $f = \sin(A + \sqrt{2c-1}\,u_1)$, $\quad g = \pm\cosh(\sqrt{2c-1}\,u_2)$, $A \in R$ *and the function* $\tilde{X}_c$ *is defined on* $R^2 \setminus \{p_k,\ k \in Z\}$, *where* $p_k = \frac{1}{\sqrt{2c-1}}(\mp\pi/2 - A + 2k\pi,\ 0)$.

c) *if* $1-2c>0$, *then* $f=\pm\cosh(A+\sqrt{1-2c}\,u_1)$, $g=\sin(\sqrt{1-2c}\,u_2)$, $A\in R$ *and the function* $\tilde{X}_c$ *is defined on* $R^2\setminus\{p_k,\ k\in Z\}$, *where* $p_k=\frac{1}{\sqrt{1-2c}}(-A,\ \mp\pi/2+2k\pi)$.

One can prove

Proposition 2.2. *Any minimal surface locally associated to the catenoid by a Ribaucour transformation given by Proposition 2.1. is complete.*

As one may expect from Proposition 2.1, the geometric properties of the minimal surfaces associated to the catenoid are quite distinct, depending on the value of the parameter c. Our next two results describe these geometric properties. In particular, for special values of c, namely when $\sqrt{1-2c}=n/m$ is a rational number, the associated surface, which will be denoted by $\tilde{X}_{(n,m)}$, is 1-periodic in the variable u_2. Figure 1 contains several examples of such surfaces.

Proposition 2.3. *Let* c *be a real number such that* $0\neq c<1/2$ *and* $\sqrt{1-2c}=n/m$ *is an irreducible rational number, with* $n\neq m$. *Consider* $\tilde{X}_{(n,m)}$ *the family of minimal surfaces associated to the catenoid as in Proposition 2.1.c). Let* $\mathcal{F}^{\pm}$ *be the two ends of any surface of* $\tilde{X}_{(n,m)}$ *corresponding to* $u_1\to\pm\infty$. *Then*

a) *Any surface of* $\tilde{X}_{(n,m)}$ *is a complete minimal surface corresponding to an immersion of a sphere punctured at* $n+2$ *points: the two poles and* n *points contained on a circle.*

b) *Its Gauss map* $\tilde{N}$ *satisfies:* $\lim_{u_1\to\pm\infty}\tilde{N}(u_1,u_2)=(0,0,\pm1)$, $\lim_{(u_1,u_2)\to p_k}\tilde{N}(u_1,u_2)=N(p_k)$, *where* N *is the normal map of the catenoid and* p_k, *for* $k=1,...,n$.

c) *Any surface of* $\tilde{X}_{(n,m)}$ *has* n *embedded planar ends and two ends* $\mathcal{F}^{\pm}$ *of geometric index* m, *that grow asymptotically as the ends of the catenoid. In particular,* $\mathcal{F}^{\pm}$ *are embedded, if and only if* $m=1$; *in this case* $\mathcal{F}^{\pm}$ *are catenoid ends.*

d) *The total curvature of any surface* $\tilde{X}_{(n,m)}$ *is equal to* $-4\pi(n+m)$.

It follows from the above result that for any integer $j\geq 3$, there exists a finite number of two-parameter families of immersed complete minimal surfaces $\tilde{X}_{(n,m)}$ with $j=n+m$, whose total curvature is $-4\pi j$. Any such surface has at least one and at most $j-1$ embedded planar ends and two non planar ends. In particular, any surface of $\tilde{X}_{(j-1,1)}$ has $j-1$ planar ends and two catenoid ends.

Whenever c does not satisfy the conditions of Proposition 2.3, the surface $\tilde{X}$ is not periodic in any variable. These surfaces are treated in the following result.

Proposition 2.4. *Let* $\tilde{X}_c$ *be the family of minimal surfaces locally associated to the catenoid by a Ribaucour transformation given in Proposition 2.1. Assume* $2c-1\geq 0$ *or* $2c-1<0$ *and* $\sqrt{1-2c}$ *is not a rational number. Then*

a) *Any surface of the family* $\tilde{X}_c$ *is a complete minimal surface corresponding to a sphere punctured at two points if* $2c-1=0$ *and punctured at an infinite number of points contained on a circle, otherwise.*

b) *Its normal map* $\tilde{N}$ *satisfies:* $\lim_{(u_1,u_2)\to p_k}\tilde{N}(u_1,u_2)=N(p_k)$, *where* N *is the normal map of the catenoid and* p_k *are given by Proposition 2.1 a) b) and c) respectively.*

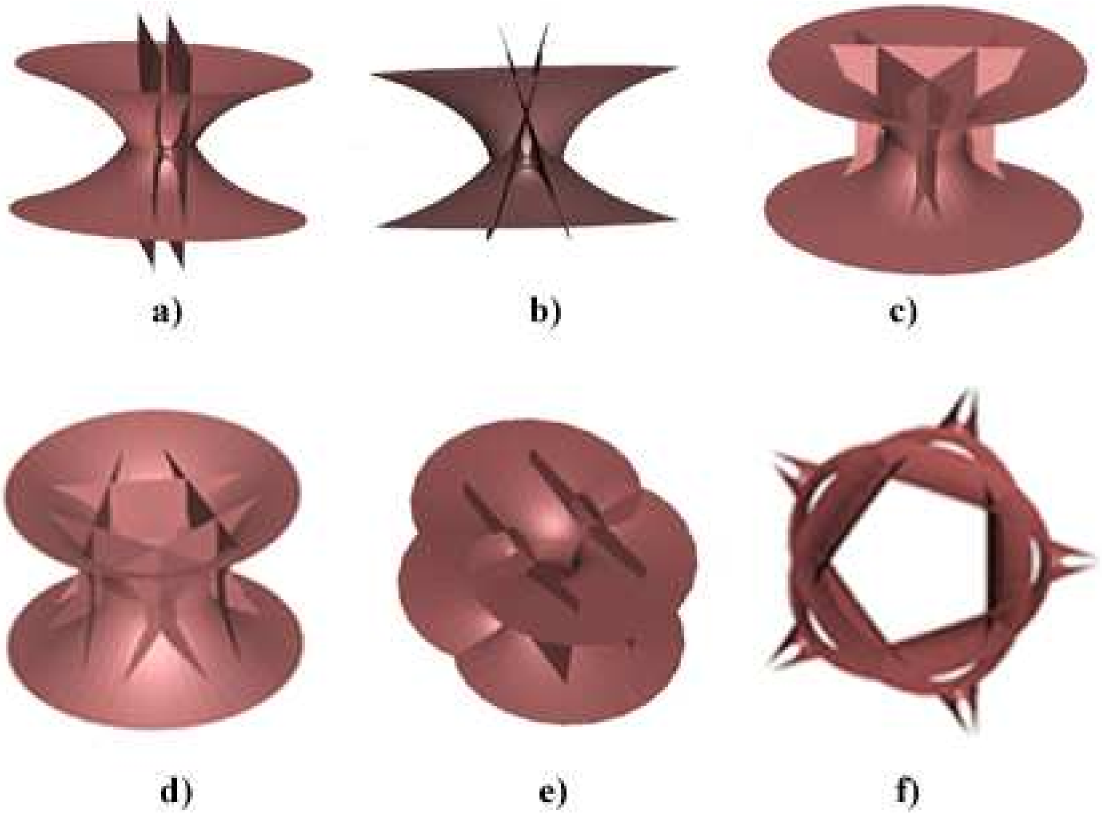

Figure 1: Complete, minimal, 1-periodic surfaces $\tilde{X}_{(n,m)}$ associated to the catenoid by Ribaucour transformations. a) $\tilde{X}_{(2,1)}$ with $A = 0$. b) $\tilde{X}_{(2,1)}$ with $A = 0.5$. c) $\tilde{X}_{(3,1)}$ with $A = 0$. d) $\tilde{X}_{(5,1)}$ with $A = 0$. e) $\tilde{X}_{(2,3)}$ with $A = 0$. f) $\tilde{X}_{(5,2)}$ with $A = 0$ an upper view.

c) *Except one all other ends are planar.*

d) Any surface of $\tilde{X}_c$ *has infinite total curvature.*

We observe that whenever $c = 1/2$ any surface of $\tilde{X}_{1/2}$ is an immersed minimal surface with two ends. One of them, corresponding to p_1, is planar and the other one, corresponding to the pole, is not embedded. Moreover, as a consequence of Propositions 2.3 and 2.4, any minimal surface of the family $\tilde{X}_c$ given by (8) such that $0 \neq c < 1/2$, with infinite total curvature, is the limit of a sequence of minimal surfaces of the family $\tilde{X}_{(n,m)}$, whose total curvature is $-4\pi(n+m)$.

We conclude this section observing that the families of minimal surfaces, associated to the catenoid depend on two parameters. The geometric properties of the surfaces are determined by c while the position of the planar ends is determined by the other parameter.

3. Linear Weingarten and cmc surfaces associated to the cylinder

In this section, we describe a two-parameter family of linear Weingarten surfaces and a one-parameter family of cmc surfaces associated to the cylinder by Ribaucour transformations. Depending on the value of the parameter c of the transformation, the associated surfaces are periodic in one variable or not. In the periodic case, the total absolute curvature of the surfaces is a multiple of 8π and the surfaces have a finite number of bubbles, characterized by the number of points of maximum and minimum for the Gaussian curvature. In the nonperiodic case, the total absolute curvature is infinite and there are infinitely many bubbles.

Proposition 3.1. *Consider the cylinder parametrized by*

$$X(u_1, u_2) = (\cos(u_2), \sin(u_2), u_1) \qquad (u_1, u_2) \in R^2 \tag{9}$$

as a linear Weingarten surface satisfying $-1/2 + H + \gamma K = 0$. *A parametrized surface is a linear Weingarten surface locally associated to* X *by a Ribaucour transformation as in section 1, if and only if, it is given by*

$$\tilde{X}_{c\gamma} = X - \frac{2(f+g)}{c[(2\gamma+1)g^2 - f^2]}(f' X_{u_1} + g' X_{u_2} - gN), \tag{10}$$

where N *is the inner unit normal vector field of the cylinder,* $c \neq 0$ *and* γ *are real constants, such that*

$$\xi(c, \gamma) = 1 - c(2\gamma + 1) \tag{11}$$

and c are not simultaneously positive, and $f(u_1)$, $g(u_2)$ are solutions of the equations

$$f'' + cf = 0, \qquad g'' + \xi g = 0, \tag{12}$$

with initial conditions satisfying

$$\left((f')^2 + (g')^2 + \xi g^2 + cf^2\right)(u_1^0, u_2^0) = 0.$$

Moreover, $\tilde{X}_{c\gamma}$ *is a regular surface defined on the subset of* $U \subset R^2$ *where*

$$\left((f+g)^2 + 2\gamma g^2\right)\left(f^2 + 2(2\gamma+1)fg + (2\gamma+1)g^2\right) \neq 0.$$

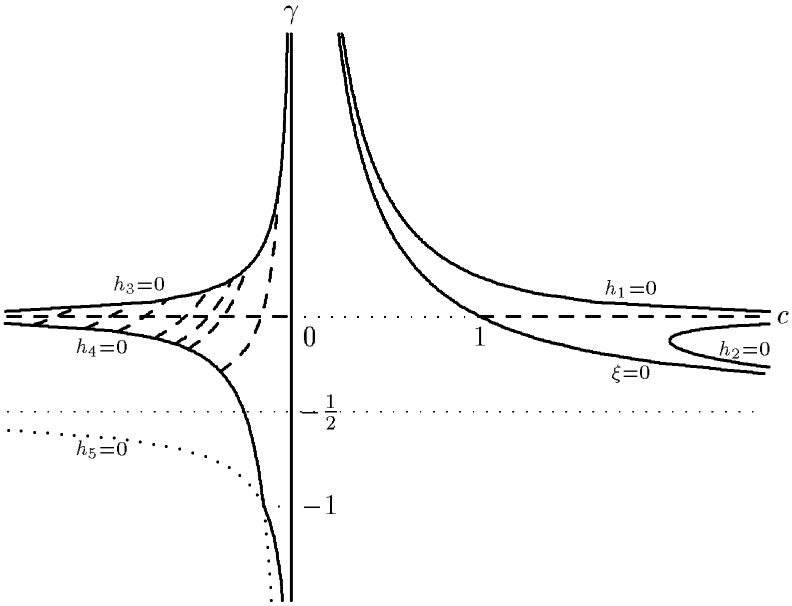

Figure 2: Any pair (c,γ), in each of the two connected components, generates a complete linear Weingarten surface, which satisfies the relation $-1/2+H-\gamma K=0$ and it is cmc when $\gamma=0$. The dashed curves in the left region, given by $1-c(2\gamma+1)=n^2/m^2$, generate 1-periodic n-bubble surfaces with two ends of geometric index m (see figure 3). For other values of (c,γ) the surfaces are not periodic.

One can show that the linear Weingarten surfaces $\tilde{X}_{c\gamma}$ associated to the cylinder and parametrized by (10) (excluding the cylinder), have curves of singularity if $c\xi \geq 0$. Moreover, if $c\xi < 0$ then, up to rigid motions of R^3, the surface $\tilde{X}_{c\gamma}$ is determined by the functions

$$
\begin{array}{lll}
f=\varepsilon_1\sqrt{|\xi|}\sin(\sqrt{c}\,u_1) & g=\varepsilon_2\sqrt{c}\cosh(\sqrt{|\xi|}u_2) & \text{if } c>0, \xi<0 \\
f=\varepsilon_1\sqrt{\xi}\cosh(\sqrt{|c|}u_1) & g=\varepsilon_2\sqrt{|c|}\sin(\sqrt{\xi}\,u_2) & \text{if } c<0, \xi>0
\end{array}
$$

where $\varepsilon_i=\pm 1$, $c\neq 0$ and γ are real numbers and $\xi(c,\gamma)$ is defined by (11). In order to characterize when the surface $\tilde{X}_{c\gamma}$ is complete, we consider the two connected components of R^2 described in Figure 2. The functions which determine the regions of Figure 2 are defined by

$$
\begin{aligned}
h_1(c,\gamma) &= 2c(2\gamma+1)\left(\sqrt{2\gamma(2\gamma+1)}-2\gamma\right)-1 \\
h_2(c,\gamma) &= 2c\left(\sqrt{2|\gamma|}+2\gamma\right)-1 \\
h_3(c,\gamma) &= -2c(2\gamma+1)\left(\sqrt{2\gamma(2\gamma+1)}+2\gamma\right)-1 \\
h_4(c,\gamma) &= 2c\left(\sqrt{2|\gamma|}-2\gamma\right)+1 \\
h_5(c,\gamma) &= -2c(2\gamma+1)\left(\sqrt{2\gamma(2\gamma+1)}-2\gamma\right)+1
\end{aligned}
$$

One can prove that the connected region in Figure 2, where $c < 0$, contains an infinite number of curves such that the corresponding surfaces are complete 1-periodic linear Weingarten surfaces, which have finite total absolute curvature. For pairs (c, γ) not on these curves, the associated surfaces are not periodic in any variable. More precisely

Proposition 3.2. *Any linear Weingarten surface $\tilde{X}_{c\gamma}$, given by Proposition 3.1. is complete, if and only if, $c\xi(c, \gamma) < 0$ and the pair (c, γ) belongs to one of the connected components described in Figure 2.*

a) *If $c < 0$ and $\sqrt{\xi(c, \gamma)} = n/m$ is an irreducible rational number, then $\tilde{X}_{c\gamma}$ is an immersion of a cylinder into R^3, with two ends of geometric index m and n isolated points of maximum (respectively minimum) for the Gaussian curvature. Moreover, the total curvature of $\tilde{X}_{c\gamma}$ is zero, while its total absolute curvature is $8\pi n$. The ends are embedded if and only if $m = 1$; in this case they are cylindrical ends.*

b) *If $c > 0$ or $c < 0$ and $\sqrt{\xi}$ is not a rational number then $\tilde{X}_{c\gamma}$ is an immersion of R^2 into R^3 with an infinite number of isolated critical points of its Gaussian curvature.*

One can also show that the complete linear Weingarten surfaces $\tilde{X}_{c\gamma}$ considered in Proposition 3.2. are asymptotically close to the cylinder. The symmetries of the surfaces are quite distinct for (c, γ) in each connected component of Figure 2 and they are given explicitly (see [CFT2]) in terms of rotations, translations and reflections with respect to certain planes. The lines of curvature $\tilde{X}_{c\gamma}(u_1, u_2^0)$ are planar, while the curves $\tilde{X}_{c\gamma}(u_1^0, u_2)$, when $f'(u_1^0) \neq 0$, are contained on a sphere centered at $(0, 0, f/f'(u_1^0)$ with radius $\sqrt{1 + (f/f')^2}$. Whenever $f'(u_1^0) = 0$ (it can only occur for $c > 0$) then $\tilde{X}_{c\gamma}(u_1^0, u_2)$ are planar lines of curvature. In Figure 3, one can visualize some of the surfaces given by $\tilde{X}_{c\gamma}$.

We observe that the linear Weingarten surfaces given by $\tilde{X}_{c\gamma}$ are tubular surfaces when $\gamma = -1/2$, since they satisfy $\Delta = \beta^2 - 4\alpha\gamma = 0$, and they provide examples of complete surfaces with $\Delta < 0$ and $\Delta > 0$. We point out that the linear Weingarten surfaces with no umbilic points for which $\Delta < 0$ (respectively $\Delta > 0$) correspond (see for example [T]) to solutions of a hyperbolic (resp. elliptic) differential equation, namely the sine-Gordon equation (resp. elliptic sinh-Gordon or cosh-Gordon equations). Hilbert's theorem asserts that there are no complete surfaces of constant negative Gaussian curvature immersed in R^3 and it is well known that such surfaces correspond to solutions of the sine-Gordon equation. However, the examples $\tilde{X}_{c\gamma}$, where $c < 0$ and $\gamma < -1/2$, show that there are complete linear Weingarten surfaces immersed in R^3 for which $\Delta < 0$.

In what follows, we describe a one parameter family of 1/2-cmc surfaces obtained from the cylinder by Ribaucour transformations. The family is contained in the class of linear Weingarten surfaces described in Proposition 3.1. and are explicitly obtained by considering $\gamma = 0$. These surfaces could also be obtained directly from the cylinder by applying the Ribaucour transformation with the appropriate algebraic condition. More precisely

Proposition 3.3. *Excluding the cylinder, any 1/2-cmc parametrized surface locally associated to the cylinder as in section 1, is given, up to rigid motions of R^3, by*

$$\tilde{X}_c = X + \frac{2}{c(f - g)}(f' X_{u_1} + g' X_{u_2} - gN) \tag{13}$$

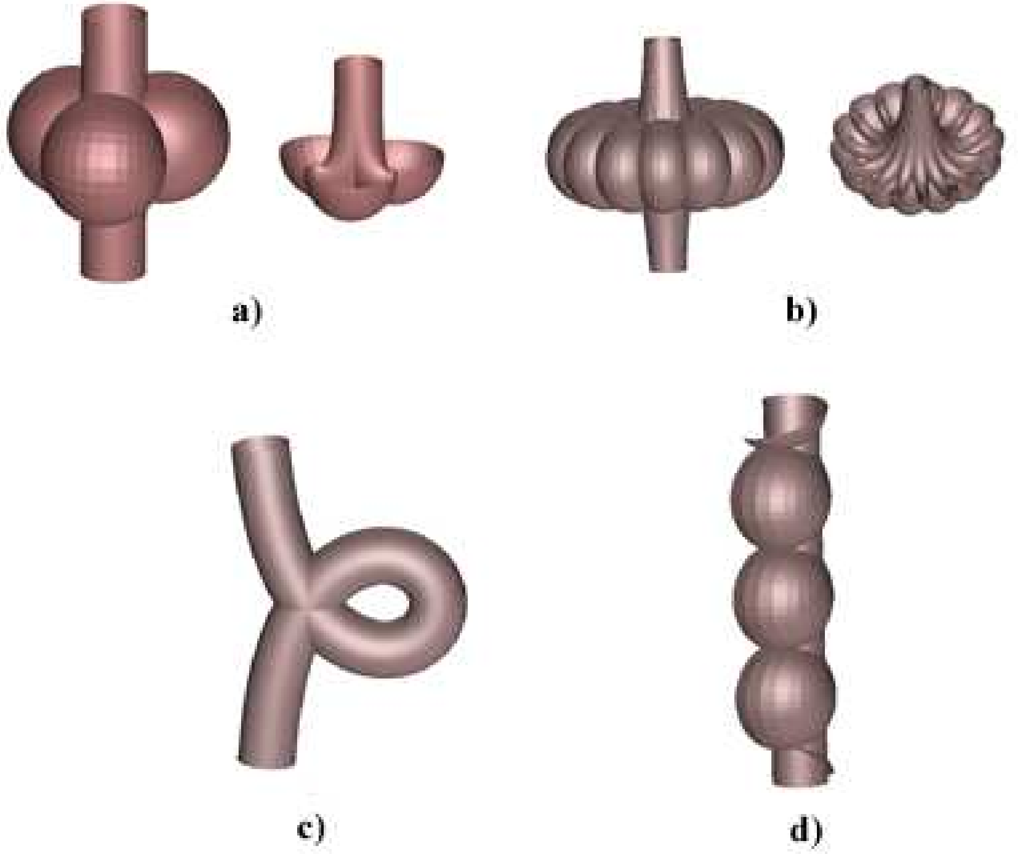

Figure 3: Complete Weingarten surfaces $\tilde{X}_{c\gamma}$ which satisfy the relation $-1/2 + H + \gamma K = 0$ and are associated to the cylinder by Ribaucour transformations. a) 1-periodic cmc surface obtained by considering $\gamma = 0$ and $\sqrt{1-c} = 3/2$; b) 1-periodic Weingarten surfaces for which $\gamma = 0.2$, $\sqrt{1 - c(2\gamma + 1)} = 14/13$; c) tubular surface, $\gamma = -1/2$, $c = -0.1$; d) cmc surface (not periodic) where $c = 2.8$ and $\gamma = 0$.

where N is the inner unit normal field of the cylinder paramerized by (9), c is a real constant such that $c < 0$ or $c > 1$, and the functions $f(u_1)$ and $g(u_2)$ are given by

$$\begin{array}{lll} f = \varepsilon_1\sqrt{c-1}\sin(\sqrt{c}\,u_1) & g = \varepsilon_2\sqrt{c}\cosh(\sqrt{c-1}\,u_2) & \text{if } c > 1, \\ f = \varepsilon_1\sqrt{1-c}\cosh(\sqrt{|c|}\,u_1) & g = \varepsilon_2\sqrt{|c|}\sin(\sqrt{1-c}\,u_2) & \text{if } c < 0. \end{array}$$

Moreover, one can show that these cmc surfaces correspond to solutions of finite type 1 of the equation $w_{z\bar{z}} + \sinh(2w)/2 = 0$, where z is the complex variable $z = u_1 + iu_2$, in the sense introduced by Pinkal and Sterling in [PS]. Some of the properties of these cmc surfaces are given in the follwing result.

Proposition 3.4. *Any 1/2-cmc surfaces $\tilde{X}_c$ given by (13) is a complete surface asymptotically close to the cylinder. Moreover,*

a) *If $c < 0$ and $\sqrt{1-c} = n/m$ is an irreducible rational number, then $\tilde{X}_c$ is an immersion of a cylinder into R^3, with two ends of geometric index m and n isolated points of maximum (respectively minimum) for the Gaussian curvature. The total curvature of $\tilde{X}_c$ is zero, while its total absolute curvature is $8\pi n$. The ends are embedded if and only if $m = 1$. In this case they are cylindrical ends.*

b) *If $c > 1$ or $c < 0$ and $\sqrt{1-c}$ is not a rational number then $\tilde{X}_c$ is an immersion of R^2 into R^3, with an infinite number of isolated critical points of its Gaussian curvature.*

The cmc n-bubble surfaces given explicitly by Proposition 3.4 a) were first described by Sievert [S] for $n = 2$ (see also [PS]) and their existence was proved later in [G-B] and [SW]. Moreover, we observe that the cmc surfaces considered in Proposition 3.3, whenever $c < 0$, are called of Enneper type by Wente [W2], since one family of curvature lines is spherical.

Similarly, by using Ribaucour transformations we obtain families of cmc surfaces associated to the Delaunay surfaces. By restricting the range of the parameter c of the Ribaucour transformation, we provide families of complete cmc surfaces which contain an enumerable subset of one periodic surfaces (see [CFT2]).

References

[Bi] Bianchi, L. Lezioni di Geometria Differenciale, Bologna Nicola Zanichelli Ed. , 1927.

[BS] Brito F.B., Sa Earp, R. *On the structure of certain Weingarten surfaces with boundary a circle*, Ann. Fac. Sci. Toulouse 6 (1997), 243-256.

[CHM] Callahan, M., Hoffman D. Meeks III, W., *Embedded minimal surfaces with an infinite number of ends*, Invent. Math. 96, (1989), 459-505.

[CFT1] Corro, A.V., Ferreira, W., Tenenblat, K., *Minimal surfaces obtained by Ribaucour transformations*, Geometria Dedicata (to appear).

[CFT2] Corro, A.V., Ferreira, W., Tenenblat, K., *Ribaucour transformations for cmc and linear Weingarten surfaces*, to appear.

[Co] Costa, C.J., *Example of a complete minimal immersion in R^3 of genus one and three embedded ends*, Bull. Soc. Bras. Mat. 15, (1984), 47-54.

[G-B] Groβe-Brauckmann, K. *New surfaces of constant mean curvature*, Math. Zeit. 214 (1993), 527-565.

[HM] Hoffman D. Meeks III, W., *Embedded minimal surfaces of finite topology*, Ann. of Math. 131, (1990), 1-34.

[JM] Jorge, L.P., Meeks III, W., *The topology of complete minimal surfaces of finite total Gaussian curvature* Topology, 2 (1983), 203-221.

[K1] Kapouleas, N. *Complete constant mean curvature surfaces in Euclidean three space*, Ann. of Math. 131 (1990), 239-330.

[K2] Kapouleas, N. *Constant mean curvature surfaces constructed by using Wente tori*, Invent. Math. 119, (1995), 443-518.

[Ka] Karcher, H. *The triply periodic minimal surfaces of A. Schoen and their constant mean curvature companions*, Man. math. 64 (1989), 291-357.

[L] Lawson, B. *Complete minimal surfaces in* S^3, Ann. Math. 92 (1970), 335-374.

[PS] Pinkal, U., Sterling, I. *On classification of constant mean curvature mean tori*, Ann. Math. 130, (1989), 407-451.

[RS] Rosenberg, H., Sa Earp, R. *The geometry of properly embedded special surfaces in* R^3*; e.g. surfaces satisfying* $aH + bK = 1$*, where* a *and* b *are positive* Duke Math J. 73 (1994), 291-306.

[S] Sievert, H. *Über die Zentralflächen der Enneperschen Flächen konstanten Krümmungsmasses*, Diss. Tübingen (1886).

[T] Tenenblat, K. Transformations of manifolds and applications to differential equations, Addison Wesley Longman, Pitman Monographs and Surveys in Pure and Applied Mathematics # 93, 1998, 209 pgs.

[SW] Sterling, I., Wente, H.C. *Existence and classification of cmc multibubbleton of finite and infinite type*, Indiana Univ. Math. J. 42 (1993), 1239-1266.

[W1] Wente, H.C. *Counterexample to a conjecture of H. Hopf*, Pac. J. Math. 121, (1986), 193-243.

[W2] Wente, H.C. *Constant mean curvature immersions of Enneper type*, Memoirs of the AMS 478, 1992.

Departamento de Matemática,
Universidade de Brasília
70910-900, Brasília, DF,
Brazil
e-mail: keti@mat.unb.br