On the Ricci and Einstein equations on the pseudo-euclidean and hyperbolic spaces

Romildo Pina* **Keti Tenenblat†**

Abstract

We consider tensors $T = fg$ on the pseudo-euclidean space R^n and on the hyperbolic space H^n, where $n \geq 3$, g is the standard metric and f is a differentiable function. For such tensors, we consider, in both spaces, the problems of existence of a Riemannian metric $\bar{g}$, conformal to g, such that Ric $\bar{g} = T$, and the existence of such a metric which satisfies Ric $\bar{g} - \bar{K}\bar{g}/2 = T$, where $\bar{K}$ is the scalar curvature of $\bar{g}$. We find the restrictions on the Ricci candidate for solvability and we construct the solutions $\bar{g}$ when they exist. We show that these metrics are unique up to homothety, we characterize those globally defined and we determine the singularities for those which are not globally defined. None of the non-homothetic metrics $\bar{g}$, defined on R^n or H^n, are complete. As a consequence of these results, we get positive solutions for the equation $\Delta_g u - \frac{n(n-2)}{4}\lambda u^{\frac{n+2}{n-2}} = 0$, where g is the pseudo-euclidean metric.

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Introduction

Consider the following problems:

> Given a symmetric tensor T, of order two, defined on a manifold M^n, does there exist a Riemannian metric g such that Ric $g = T$? (P_1)

> Find necessary and sufficient conditions on a symmetric tensor T on a manifold so that one can find a metric g to satisfy Ric $g - \frac{K}{2} g = T$ where K is the scalar curvature of g. (P_2)

These are interesting problems which are very difficult to treat in this generality. Solving problem (P_1) or (P_2) is equivalent to studying a system of nonlinear second order partial differential equations.

Different aspects of these problems have been considered in previous articles. Problem (P_1), for $n \geq 3$, was studied by DeTurck [D1], [D2] and Cao [CD1], [CD2]. For compact manifolds, some nonexistence results can be found in [DK] and [H]. Explicit solutions for special tensors on R^n and H^n were obtained in [PT1] and [P].

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With respect to problem (P_2), we observe that if g_{ij} is the Lorentzian metric on a four-dimensional manifold, then it reduces to the Einstein field equation. DeTurck [D3] considered the Cauchy problem for this equation. Whenever the tensor T represents vector fields such as electromagnetic field, perfect fluid, pure radiation or vacuum ($T = 0$), then the equation has been studied by several authors (see references in [SKMHH]). When the metric g is conformal to the $n-$dimensional pseudo-Euclidean space or to the standard metric on the sphere, then solutions for special tensors were also given in [PT2] and [PT3] for $n \geq 3$.

Our purpose in this paper is to study both problems on the pseudo-euclidean space (R^n, g) and on the hyperbolic space (H^n, g), with $n \geq 3$, for tensors $T = fg$, where f is a differentiable function globally defined and g is the standard metric. We want to find necessary and sufficient conditions on T for the existence of metrics $\bar{g}$, conformal to the metric g, which solve problems (P_1) or (P_2), i.e. we want to study the following systems:

$$\begin{cases} \bar{g} = \dfrac{1}{\varphi^2} g \\ \text{Ric } \ \bar{g} = fg. \end{cases} \tag{1}$$

$$\begin{cases} \bar{g} = \dfrac{1}{\varphi^2} g \\ \text{Ric } \bar{g} - \dfrac{\bar{K}}{2} \bar{g} = fg. \end{cases} \tag{2}$$

In our main results, we determine all tensors T and the corresponding metrics $\bar{g}$ which solve these systems for the pseudo-euclidean case and the hyperbolic space. In both cases, we provide necessary and sufficient conditions for solving systems (1) and (2) and all solutions are given explicitly. The corresponding results on the unit sphere S^n were obtained in [PT3].

We observe that problem (1), for a Riemannian manifold (M^n, g), $n \geq 3$, is equivalent to obtaining all Einstein metrics on M, which are conformal to g. This problem is related to a question of Brinkmann of 1925 [Br]: When can an Einstein space be mapped conformally on some Einstein space and in how many ways can it be mapped? This problem was studied by many authors and mainly by Kühnel [Kü]. He proved that, assuming (M, g) is a complete Einstein space, then there exits a non-homothetic, conformal diffeomorphism, globally defined, from (M, g) into $(\bar{M}, \bar{g})$ (possibly non complete), if and only if, (M, g) is isometric to S^n, H^n, R^n or $R \times \tilde{M}$, where $(\tilde{M}, \tilde{g})$ is a complete Ricci flat manifold.

Our results on problem (1) for the euclidean and hyperbolic cases contain those obtained in [Kü], where $\bar{g}$ is considered to be globally defined. We also describe explicitly the singularities of the metric when $\bar{g}$ is not globally defined. We provide solutions for problems (1) and (2), when g is the pseudo-euclidean metric, and for (2) in the hyperbolic space. We find the restrictions on the Ricci candidate for solvability and we construct the solutions $\bar{g}$ when they exist. We show that these metrics are unique up to homothety. We characterize those globally defined and we determine the singularities for those which are not globally defined. We prove that none of the non-homothetic metrics $\bar{g}$, globally defined on R^nor H^n, are complete. As a consequence of the proof of our results, we get explicit positive solutions for the Yamabe problem in the pseudo-Euclidean space, (R^n, g), $n \geq 3$. The solutions are defined on R^n or on subsets of R^n whose boundaries are planes, spheres, quadrics or a point.

In what follows, we state our main results. The proofs will be given in the next section. We will first consider the pseudo-euclidean space (R^n, g) with the usual coordinates $x = (x_1, ..., x_n)$

and the metric $g_{ij} = \epsilon_i \delta_{ij}$, $\epsilon_i = \pm 1$. We will then give our results in the hyperbolic space (H^n, g^*), where we are considering the half space model R^N_+, $x_n > 0$, and $g^*_{ij} = \delta_{ij}/x_n^2$.

Theorem 1.1 *Let (R^n, g), $n \geq 3$ be the pseudo-euclidean space. Then there exists $\bar{g} = \dfrac{1}{\varphi^2} g$ such that Ric $\bar{g} = fg$, if and only if,*

$$\varphi(x) = \sum_{i=1}^{n} (\epsilon_i a x_i^2 + b_i x_i) + c \qquad f(x) = \frac{-(n-1)}{\varphi^2} \lambda, \tag{3}$$

where a, b_i, c are real numbers and $\lambda = \sum_i \epsilon_i b_i^2 - 4ac$. Any such metric $\bar{g}$ is unique up to homothety. Whenever g is the euclidean metric then:

a) If $\lambda < 0$ then $\bar{g}$ is globally defined on R^n and T is positive definite.

b) If $\lambda \geq 0$ then, excluding the homothety, the set of singularity points of $\bar{g}$ consists of

b.1) a point if $\lambda = 0$;

b.2) a hyperplane if $\lambda > 0$ and $a = 0$;

b.3) an $(n-1)$-dimensional sphere if $\lambda > 0$ and $a \neq 0$.

In Theorem 1.1, considering the Euclidean metric g, we observe that for each constant $a \neq 0$, the tensor $T = fg$ and the conformal metric $\bar{g}$ are rotationally symmetric, around the point whose coordinates are $(-b_i/(2a))$. Not all rotationally symmetric tensors T admit (even locally) a conformal metric $\bar{g}$, solving the Ricci equation. In fact, if one considers $T = -(|x|^2 + 1)g$, it is not difficult to see that there is no conformal metric $\bar{g} = g/\varphi^2$, satisfying Ric $\bar{g} = -(|x|^2 + 1)g$ (see equation (8) in the proof of Theorem 1.1).

Theorem 1.2 *Let (R^n, g), $n \geq 3$ be the pseudo-euclidean space. Then there exists $\bar{g} = \dfrac{1}{\varphi^2} g$ such that Ric $\bar{g} - \dfrac{\bar{K}}{2} \bar{g} = fg$, if and only if,*

$$\varphi(x) = \sum_{i=1}^{n} (\epsilon_i a x_i^2 + b_i x_i) + c \qquad f(x) = \frac{(n-1)(n-2)}{2\varphi^2} \lambda, \tag{4}$$

where a, b_i, c are real numbers and $\lambda = \sum_i \epsilon_i b_i^2 - 4ac$. Any such metric $\bar{g}$ is unique up to homothety. Whenever g is the euclidean metric then:

a) If $\lambda < 0$ then $\bar{g}$ is globally defined on R^n and T is negative definite.

b) If $\lambda \geq 0$ then, excluding the homothety, the set of singularity points of $\bar{g}$ consists of

b.1) a point if $\lambda = 0$;

b.2) a hyperplane if $\lambda > 0$ and $a = 0$;

b.3) an $(n-1)$-dimensional sphere if $\lambda > 0$ and $a \neq 0$.

Theorem 1.3 *Let (R^n_+, g^*), $n \geq 3$ be the hyperbolic space. Then there exists $\bar{g} = \dfrac{1}{\varphi^2} g^*$ such that Ric $\bar{g} = f g^*$, if and only if,*

$$\varphi(x) = \frac{1}{x_n}\left[\sum_{i=1}^{n}(a x_i^2 + b_i x_i) + c\right] \qquad f(x) = \frac{-(n-1)}{\varphi^2}\lambda, \tag{5}$$

where a, b_i, c are real numbers and $\lambda = \sum_i b_i^2 - 4ac$. Any such metric $\bar{g}$ is unique up to homothety. Moreover, $\bar{g}$ is globally defined on R^n_+ whenever

a) $\lambda < 0$. In this case T is positive definite.
b) $\lambda = 0$ and $a = 0$ (hence $c \neq 0$).
c) $\lambda = 0$, $a \neq 0$ and $b_n/a \geq 0$.
d) $\lambda > 0$, $a = 0$, $b_i = 0$, $\forall i < n$ and $c/b_n \geq 0$.
d) $\lambda > 0$, $a \neq 0$ and $b_n/a \geq \sqrt{\lambda}/|a|$.

Otherwise, the set of singularity points of $\bar{g}$ consists of a point or the set obtained by intersecting a hyperplane or a sphere with the half space $x_n > 0$.

Teorema 1.4 *Let (R^n_+, g^*), $n \geq 3$, be the hyperbolic space. Then there exists $\bar{g} = \dfrac{1}{\varphi^2} g^*$ such that Ric $\bar{g} - \dfrac{\bar{K}}{2}\bar{g} = f g^*$, if and only if,*

$$\varphi(x) = \frac{1}{x_n}\left[\sum_{i=1}^{n}(a x_i^2 + b_i x_i) + c\right] \qquad f(x) = \frac{(n-1)(n-2)}{2\varphi^2}\lambda, \tag{6}$$

where a, b_i, c are real numbers and $\lambda = \sum_i b_i^2 - 4ac$. Any such metric $\bar{g}$ is unique up to homothety. Moreover, $\bar{g}$ is globally defined on R^n_+ whenever

a) $\lambda < 0$. In this case T is negative definite.
b) $\lambda = 0$ and $a = 0$ (hence $c \neq 0$).
c) $\lambda = 0$, $a \neq 0$ and $b_n/a \geq 0$.
d) $\lambda > 0$, $a = 0$, $b_i = 0$, $\forall i < n$ and $c/b_n \geq 0$.
d) $\lambda > 0$, $a \neq 0$ and $b_n/a \geq \sqrt{\lambda}/|a|$.

Otherwise, the set of singularity points of $\bar{g}$ consists of a point or the set obtained by intersecting a hyperplane or a sphere with the half space $x_n > 0$.

Theorem 1.5. *There are no complete, non-homothetic metrics, $\bar{g}$ as described in Theorems 1.1 - 1.4, globally defined on R^n or H^n.*

As a consequence of Theorem 1.1, considering the change of dependent function $u = \varphi^{-\frac{n-2}{2}}$, our next result provides explicit solutions for the Yamabe problem in the pseudo-Euclidean space, (R^n, g), $n \geq 3$.

Corollary 1.6. *Let g be the pseudo-euclidean metric on R^n. For each $\lambda \in R$ and each integer $n \geq 3$, the differential equation*

$$\Delta_g u - \frac{n(n-2)}{4}\lambda u^{\frac{n+2}{n-2}} = 0, \tag{7}$$

where Δ_g is the Laplacian in the metric g, has infinitely many positive solutions given by

$$u(x) = \left(\sum_{i=1}^{n}(\epsilon_i a x_i^2 + b_i x_i) + c\right)^{-\frac{n-2}{2}}, \qquad \text{with} \qquad a,\, b_i,\, c \in R \quad \lambda = \sum_i \epsilon_i b_i^2 - 4ac.$$

where $g_{ij} = \delta_{ij}\epsilon_i$. Assume u is not constant.

If $a = 0$, let Π be the hyperplane $\sum_{i=1}^{n} b_i x_i + c = 0$. Then u is defined on
a) the half space $\sum_{i=1}^{n} b_i x_i + c > 0$ if n is odd;
b) $R^n \setminus \Pi$, if n is even.

If $a \neq 0$, consider the point $P = -(\epsilon_1 b_1, ..., \epsilon_n b_n)/(2a)$ and the quadric Q (which may be empty) given by $||x - P||_g^2 = \lambda/(4a^2)$. Then u is defined on
c) the region $\{x \in R^n \; ; \; a(||x-P||_g^2 - \lambda/(4a^2)) > 0\}$, if n is odd;
d) $R^n \setminus Q$, if n is even.

In particular, when the metric g is euclidean, and the dimension n is odd, the solution $u(x)$ given above is defined on R^n if $a > 0$ and $\lambda < 0$ and it is defined on $R^n \setminus P$ if $\lambda = 0$. Moreover, when $\lambda > 0$ and $a \neq 0$, $u(x)$ is a radial function. This is the unique positive solution u of (7) on the domain of R^n bounded by the sphere $|x - P|^2 = \lambda/(4a^2)$, such that u tends to $+\infty$ at this boundary. This follows from Theorem 4 in [LN].

We observe that equation (7), when g is restricted to be the euclidean metric, was studied by Gidas, Ni, Nirenberg [GNN], when $\lambda < 0$. The case $\lambda > 0$ was considered by Loewner and Nirenberg [LN] on a domain, bounded or unbounded, of R^n whose boundary is a smooth compact hypersurface. Observe that (7), with $\lambda > 0$ and g euclidean, does not admit positive solutions globally defined on R^n (see [K] and its references).

Proof of the main results

Proof of Theorem 1.1:

The problem of obtaining $\bar{g} = \dfrac{1}{\varphi^2} g$, such that Ric $\bar{g} = fg$, is equivalent to studying the following system of differential equations:

$$\begin{cases} \epsilon_i \varphi_{x_i x_i} = \dfrac{1}{(n-2)}\left(\dfrac{(n-1)|\nabla_g \varphi|^2}{\varphi} + f\varphi - \Delta_g \varphi\right), \\ \\ \varphi_{x_i x_j} = 0, \qquad 1 \le i \neq j \le n. \end{cases} \tag{8}$$

In fact, since $\bar{g} = g/\varphi^2$, we know that

$$\text{Ric } \bar{g} - \text{Ric } g = \frac{1}{\varphi^2}\left\{(n-2)\varphi \text{Hess}_g(\varphi) + (\varphi\Delta_g\varphi - (n-1)|\nabla_g\varphi|^2)g\right\}.$$

From, Ric $g = 0$ and Ric $\bar{g} = fg$, we obtain

$$\frac{1}{\varphi^2}\left\{(n-2)\varphi \text{Hess}_g(\varphi)_{ij} + \left(\varphi\Delta_g\varphi - (n-1)|\nabla_g\varphi|^2\right) g_{ij}\right\} = f g_{ij}. \tag{9}$$

The system of equations (8) is obtained by considering $i = j$ and $i \neq j$ in (9).

It follows, from the second equation of (8), that $\varphi = \sum_{i=1}^{n} \ell_i(x_i)$. Moreover, from the first equation we conclude that $\epsilon_i \ell_i''(x_i) = \epsilon_j \ell_j''(x_j) = a \in R$, for all $i \neq j$. Therefore,

$$\varphi(x) = \sum_{i=1}^{n} (\epsilon_i a x_i^2 + b_i x_i) + c \qquad f(x) = \frac{-(n-1)}{\varphi^2}\lambda, \tag{10}$$

where a, b_i, c are real numbers. The converse is a straightforward computation.

If g is the euclidean metric, i.e. $\epsilon_i = 1 \, \forall i$, then whenever $\lambda < 0$ we conclude that $\bar{g}$ is globally defined and T is positive definite. If $\lambda = 0$ and $a = 0$, then $b_i = 0 \, \forall i$, $\varphi = c \neq 0$ and $\bar{g}$ reduces to a homothety. If $\lambda = 0$ and $a \neq 0$, then $\bar{g}$ has a singularity at the point with coordinates $x_i = -b_i/(2a)$. Whenever $\lambda > 0$, if $a = 0$ then there exists i_0 such that $b_{i_0} \neq 0$ and any point of the hyperplane $\sum_i b_i x_i + c = 0$ is a singularity point of $\bar{g}$. If $\lambda > 0$ and $a \neq 0$ then any point of the $(n-1)$-dimensional sphere, centered at $-b_i/(2a)$ with radius $\sqrt{\lambda}/(2|a|)$, is a singularity point of $\bar{g}$. This concludes the proof of Theorem 1.1.

□

Proof of Theorem 1.2:
The proof is similar to the previous one. The problem of obtaining $\bar{g} = g/\varphi^2$ such that $\mathrm{Ric}\, \bar{g} - \dfrac{\bar{K}}{2}\bar{g} = fg$, is equivalent to solving the system of equations:

$$\begin{cases} \epsilon_i \varphi_{x_i x_i} = \dfrac{f\varphi}{(n-2)} - \dfrac{(n-1)|\nabla_g \varphi|^2}{2\varphi} + \Delta_g \varphi, \\ \varphi_{x_i x_j} = 0, \qquad 1 \leq i \neq j \leq n. \end{cases} \tag{11}$$

In fact, since $\mathrm{Ric}\, g = 0$ and

$$\bar{K} = (n-1)\left(2\varphi \Delta_g \varphi - n|\nabla_g \varphi|^2\right),$$

our problem is equivalent to studying the system

$$\frac{1}{\varphi}(\mathrm{Hess}_g \varphi)_{ij} + \left(\frac{(n-1)|\nabla_g \varphi|^2}{2\varphi^2} - \frac{\Delta_g \varphi}{\varphi}\right) g_{ij} = \left(\frac{f}{n-2}\right) g_{ij}. \tag{12}$$

Considering $i = j$ in (12) we get the first equation of (11). The second equation is obtained by considering $i \neq j$ em (12). The conclusion of the proof follows as in Theorem 1.1.

□

Proof of Theorem 1.3:
Obtaining $\bar{g} = \dfrac{1}{\varphi^2} g^*$, such that $\mathrm{Ric}\, \bar{g} = fg^*$, is equivalent to studying the system

$$\begin{cases} \tilde{\varphi}_{x_i x_i} = \dfrac{1}{(n-2)}\left(\dfrac{f\tilde{\varphi}}{x_n^2} + \dfrac{(n-1)|\nabla_g \tilde{\varphi}|^2}{\tilde{\varphi}} - \Delta_g \tilde{\varphi}\right), \\ \tilde{\varphi}_{x_i x_j} = 0, \end{cases} \tag{13}$$

where $\tilde{\varphi} = x_n \varphi$.

In fact, we observe that

$$\bar{g} = \frac{1}{\varphi^2} g^* = \frac{1}{x_n^2 \varphi^2} g,$$

where g is the euclidean metric. Hence, we have $\bar{g} = \dfrac{1}{\tilde{\varphi}^2} g$ and

$$\mathrm{Ric}\ \bar{g} = \frac{1}{\tilde{\varphi}^2} \left\{ (n-2)\tilde{\varphi}\mathrm{Hess}_g(\tilde{\varphi}) + (\tilde{\varphi}\Delta_g\tilde{\varphi} - (n-1)|\nabla_g\tilde{\varphi}|^2) g \right\}.$$

Since $\mathrm{Ric}\ \bar{g} = \dfrac{f}{x_n^2} g$, our problem is equivalent to

$$\frac{1}{\tilde{\varphi}^2} \left((n-2)\tilde{\varphi}\mathrm{Hess}_g(\tilde{\varphi})_{ij} + (\tilde{\varphi}\Delta_g\tilde{\varphi} - (n-1)|\nabla_g\tilde{\varphi}|^2) g_{ij} \right) = \frac{f}{x_n^2} g_{ij}. \tag{14}$$

Considering $i = j$ and $i \neq j$ in (14) we get both equations of (13).

The second equation implies that $\tilde{\varphi} = \sum_i \ell_i(x_i)$ and the first one implies that $\ell_i''(x_i) = \ell_j''(x_j) = a \in R$. Hence,

$$\varphi(x) = \frac{1}{x_n} \left(\sum_{i=1}^{n} (a x_i^2 + b_i x_i) + c \right) \qquad f(x) = \frac{-(n-1)}{\varphi^2} \lambda, \tag{15}$$

where a, b_i, c are real numbers. The converse is a straightforward computation.

Whenever $\lambda < 0$ then $\bar{g}$ is globally defined on R^n_+ and T is positive definite. If $\lambda = 0$ and $a = 0$, then $b_i = 0 \forall i$, $\varphi = c/x_n \neq 0$ and $\bar{g}$ is globally defined on R^n_+. If $\lambda = 0$ and $a \neq 0$, then if $b_n/(2a) \geq 0$ then $\bar{g}$ is globally defined on R^n_+; otherwise, if $b_n/(2a) < 0$ then $\bar{g}$ has a singularity at the point with coordinates $x_i = -b_i/(2a)$.

Whenever $\lambda > 0$ and $a = 0$, we have two cases. In the first one, we have $b_i = 0$, $\forall i < n$ and $b_n \neq 0$. In this case, if $c/b_n \geq 0$ then $\bar{g}$ is globally defined on R^n_+; otherwise if $c/b_n < 0$, then any point of the hyperplane $x_n = -c/b_n$ is a singularity point of $\bar{g}$. In the second case, we have $b_{i_0} \neq 0$ for some $i_0 < n$, then any point that belongs to the intersection of the hyperplane $\sum_i b_i x_i + c = 0$ with the half space $x_n > 0$, is a singularity point of $\bar{g}$.

When $\lambda > 0$ and $a \neq 0$, if $b_n/a \geq \sqrt{\lambda}/|a|$, then $\bar{g}$ is globally defined on R^n_+. Otherwise, if $b_n/a < \sqrt{\lambda}/|a|$, then any point p of the $(n-1)$-dimensional sphere centered at the point of coordinates $-b_i/(2a)$, with radius $\sqrt{\lambda}/(2|a|)$, such that p is the half space $x_n > 0$, is a point of singularity of $\bar{g}$. This concludes the proof of the theorem.

□

Proof of Theorem 1.4:

The proof is similar to the previous one. We want to obtain $\bar{g} = \dfrac{1}{\varphi^2} g^*$, such that $\mathrm{Ric}\ \bar{g} - \dfrac{\bar{K}}{2}\bar{g} = f g^*$. This is equivalent to studying the following system of equations

$$\begin{cases} \tilde{\varphi}_{x_i x_i} = \Delta_g \tilde{\varphi} - \dfrac{(n-1)||\nabla_g \tilde{\varphi}||^2}{2\tilde{\varphi}} + \dfrac{f\tilde{\varphi}}{(n-2)x_n^2} \\ \tilde{\varphi}_{x_i x_j} = 0, \end{cases} \tag{16}$$

where $\tilde{\varphi} = x_n \varphi$.

In fact, we have $\bar{g} = \dfrac{1}{\tilde{\varphi}^2} g$ where g is the euclidean metric. In this case, we have

$$\bar{K} = (n-1)\left(2\tilde{\varphi}\Delta_g\tilde{\varphi} - n|\nabla_g\tilde{\varphi}|^2\right).$$

Hence our problem is equivalent to solving the system of equations

$$\frac{1}{\tilde{\varphi}^2}\left((n-2)\tilde{\varphi}\mathrm{Hess}_g(\tilde{\varphi})_{ij} + (-(n-2)\tilde{\varphi}\Delta_g\tilde{\varphi} + \frac{(n-1)(n-2)}{2}|\nabla_g\tilde{\varphi}|^2)g_{ij}\right) = \frac{f}{x_n^2}g_{ij}. \tag{17}$$

The system of equations (16) is obtained by considering $i = j$ and $i \neq j$ in (17). We conclude the proof as in Theorem 1.3.

□

Proof of Theorem 1.5:
The metrics $\bar{g}$ given in Theorems 1.3, 1.4 are Einstein metrics. Moreover, in the euclidean case ($\epsilon_i = 1, \forall i$) of Theorems 1.1 and 1.2 the metrics $\bar{g}$ are also Einstein metrics. Therefore, the non-homothetic ones,which are globally defined, are not complete due to the results of [Kü]. One can also verify directly that there are divergent curves with finite length. The metrics $\bar{g}$ given in Theorems 1.1 and 1.2, in the non-euclidean case, if they are non-homothetic then they are not complete. In fact, this follows from a result proved in [KR], where it was shown that if (M, g) is a complete pseudo-Riemannian Einstein manifold, then there is no complete, conformal, non-homothetic diffeomorphism, globally defined, from (M, g) into a semi-Riemannian manifold $(\bar{M}, \bar{g})$.

□

Proof of Corollary 1.6:
In the proof of Theorem 1.1, we have shown that φ, given by (3), satisfies equations (9) when $i = j$. Multiplying the i-th equation in (9) by g^{ii} and adding in i, we have that φ satisfies the equation

$$2\varphi\Delta_g\varphi - n|\nabla_g\varphi|^2 + n\lambda = 0.$$

Substituting $\varphi = u^{-2/(n-2)}$, we conclude the proof of Corollary 1.6, by considering the zeros of φ.

□

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R. Pina
IME, Universidade Federal de Goiás
74001-970 Goiânia, GO, Brazil
romildo@mat.ufg.br

K. Tenenblat
Depto de Matemática, Universidade de Brasília
70910-900, Brasília, DF, Brazil
keti@mat.unb.br