Ribaucour Transformations for Hypersurfaces in Space Forms

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Abstract

The theory of Ribaucour transformations for hypersurfaces in space forms is established. For any such hypersurface M, that admits orthonormal principal vector fields, it is shown the existence of a totally umbilic hypersurface locally associated to M by a Ribaucour transformation. A method of obtaining linear Weingarten surfaces in a three-dimensional space form is provided. By applying the theory, a new one parameter family of complete *cmc* surfaces in the unit sphere, locally associated to the flat torus, is obtained. The family contains a class of complete *cmc* cylinders in the sphere. In particular, one gets a family of complete minimal surfaces and minimal cylinders, locally associated to the Clifford torus.

0 Introduction

The study of surfaces with constant mean curvature in space forms has been well developed in recent years, principally in two directions: theory and construction of *cmc* surfaces (surfaces with constant mean curvature). The following methods were used in the construction of *cmc* surfaces: the method of perturbation [K1], [K2]; integrable systems [PS], [Bo]; conjugate prime surfaces [La], [Ka]; Weierstrass type representation [D-H] [KMS]. Recently, complete *cmc* surfaces were contructed by applying another method, based on Ribaucour transformations. These transformations for hypersurfaces in space forms were studied by Bianchi in 1918-1919 [Bi]. The classical theory shows that Ribaucour transformations may be used to construct surfaces of constant Gaussian curvature and constant mean curvature surfaces from a given such surface. However, the first application of this method to minimal and *cmc* surfaces in $I\!R^3$ was obtained recently by Corro, Ferreira and Tenenblat in [CFT2] and [CFT3]. They provided new families of complete minimal surfaces associated to the Enneper surface and to the catenoid. Moreover, families of complete *cmc* surfaces were obtained by applying the theory to the cylinder and to the Delaunay surfaces. These families contain the *cmc* n-bubble surfaces described by Sievert [S], [G-B] and [SW].

In [CFT3], the classical theory of Ribaucour transformations was extended to linear Weingarten surfaces in $I\!R^3$, providing a unified version of the classical results. As an application, it was shown the existence of complete hyperbolic linear Weingarten surfaces immersed in $I\!R^3$, in contrast to Hilbert's theorem that asserts the non-existence of complete surfaces of constant negative curvature immersed in $I\!R^3$.

The classical theory of Ribaucour transformations considers hypersurfaces parametrized by lines of curvature, whose principal curvatures have multiplicity one, although this is not stated

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explicitly. In [CT], the definition of a Ribaucour transformation in Euclidean space was revised, extending the concept to submanifolds with flat normal bundle, whose principal curvatures have multiplicity bigger than one.

In this paper, following [CT], we establish the theory of Ribaucour transformations for hypersurfaces M^n in a space form $\overline{M}^{n+1}(\overline{k}), \overline{k} = 0,\ 1,\ -1$. For any hypersurface $M^n \subset \overline{M}^{n+1}$, that admits orthonormal principal vector fields, we show the existence of a totally umbilic hypersurface of $\overline{M}$, locally associated to M by a Ribaucour transformation. We provide a method of obtaining linear Weingarten surfaces in $\overline{M}^3(\overline{k})$ from a given such surface. By using this theory, we construct a 1-parameter family of complete *cmc* surfaces in the unit sphere S^3, associated to the flat torus. For special values of the parameter we obtain a family of complete *cmc* cylinders immersed in S^3. In particular, we get a family of complete minimal surfaces and minimal cylinders immersed in S^3, associated to the Clifford torus by a Ribaucour transformation. In a forthcoming paper, we will give applications of Ribaucour transformations to *cmc* surfaces in the hyperbolic space.

The paper is organized as follows. Section 1 contains the definition of Ribaucour transformations for hypersurfaces of a space form. This definition has also a local form. The first tool in the study of Ribaucour transformations is its characterization in terms of differential equations, given in Teorema 1.5. These equations reduce to a linear system (1.33)-(1.36). We show in Theorem 1.10, for hypersufaces M of S^{n+1} or of H^{n+1} which admits principal orthonormal vector fields, the existence of totally umbilic hypersurfaces, locally associated to M by a Ribaucour transformation. This section extends the results in the Euclidean space obtained in [CFT1, CT].

In section 2, we study Ribaucour transformations for linear Weingarten surfaces in S^3 and H^3 extending the results in $I\!R^3$ obtained in [CFT3]. Recall that (Cf. [T]) *a linear Weingarten surface* in $\overline{M}^3(\overline{k})$ is a surface M whose Gaussian and mean curvatures K and H, satisfy the relation

$$\alpha + \beta H + \gamma(K - \overline{k}) = 0,$$

where $\alpha,\ \beta,\ \gamma \in I\!R,\ \beta^2 - 4\alpha\gamma \neq 0,\ H = -(\lambda_1 + \lambda_2)/2$ and $K = \lambda_1\lambda_2 + \overline{k},\ -\lambda_i, i = 1, 2,$ are the principal curvatures of M. In Theorem 2.1, we get a sufficient condition so that a Ribaucour transformation transforms a linear Weingarten surface into another surface of the same type. This condition provides an integrable system of differential equations (Theorem 2.2) whose solutions enable us to obtain linear Weingarten surfaces in $\overline{M}^3$ from a given such surface. As a particular case of these results, we study Ribaucour transformations between *cmc* surfaces immersed in $\overline{M}^3$.

In Section 3, we construct a one parameter family of surfaces with constant mean curvature in S^3, locally associated to the flat torus by a Ribaucour transformation and we prove in Theorem 3.2 that the surfaces of this family are complete. This family contains complete *cmc* cylinders immersed in S^3. As a particular case, we get a family of complete minimal surfaces and cylinders locally associated to the Clifford torus by a Ribaucour transformation (Corollaries 3.3 e 3.4).

1 Ribaucour Transformations for Hypersurfaces in space forms

This section contains the definitions and the basic theory of Ribaucour transformations for hypersurfaces in a space form, extending the results in [CFT1] and [CFT2] of the theory in the Euclidean space $I\!R^{n+1}$.

Let $M^{n+1}(\overline{k}),\ \overline{k} = \pm 1, 0,$ be the simply connected space form of sectional curvature $\overline{k}$ and let L^{n+2} be the set of $x=(x_0,x_1,...,x_{n+1})\in I\!R^{n+2}$ endowed with the pseudo-inner product

$$<x,y>=-x_0y_0+\sum_{i=1}^{n+1}x_iy_i.$$

A model for the hyperbolic space is the submanifold

$$H^{n+1}=\{x\in L^{n+2}|<x,x>=-1,x_0>0\}.$$

Consider an orientable hypersurface M in $\overline{M}^{n+1}(\overline{k})$, where $\overline{M}^{n+1}(k)=S^{n+1}\subset I\!I\!R^{n+2}$ when $\overline{k}=1$ and $\overline{M}^{n+1}(\overline{k})=H^{n+1}\subset L^{n+2}$ when $\overline{k}=-1$.

Let $e_i, i=1,...,n$, be an orthonormal frame tangent to M and N a unit normal vector field defined on M. Denote by ω_i the 1-forms dual to e_i and $\omega_{ij},\ 1\leq i,j\leq n$, the connexion forms which are defined by

$$d\omega_i=\sum_{j\neq i}\omega_{ij}\wedge w_j,\quad \omega_{ij}+\omega_{ji}=0.$$

The Gauss equations are given by

$$d\omega_{ij}=\sum_{k=1}^{n}\omega_{ik}\wedge\omega_{kj}+\omega_{in+1}\wedge\omega_{n+1j}-\overline{k}\omega_i\wedge\omega_j,$$

where $\omega_{in+1}=-\omega_{n+1i}=<de_i,N>$, and the Codazzi equations are

$$d\omega_{in+1}=\sum_{j=1}^{n}\omega_{ij}\wedge\omega_{jn+1}. \tag{1.1}$$

If the vector fields e_i, defined above are principal directions corresponding to the principal curvatures $-\lambda_i, 1\leq i\leq n$, then

$$\omega_{in+1}=-\lambda^i\omega_i,\quad dN(e_i)=\lambda^ie_i. \tag{1.2}$$

Let us consider a hypersuface M in $I\!R^{n+1}$, S^{n+1} or H^{n+1} parametrized by orthogonal curvature lines, $X(u_1,...,u_n)$, where X denote the position vector in $I\!R^{n+1}$, $I\!R^{n+2}$ or L^{n+2}. The first fundamental form of M is determined by $I=\sum_{i=1}^n\omega_i^2$, where $\omega_i=a_idu_i$ e $a_i=|X_{u_i}|$ are differentiable functions. The principal directions are $e_i=X,_i/a_i$, where $X,_i$ denote the partial derivative with respect to u_i. It is easy to see that

$$\omega_{ij}=\frac{1}{a_ia_j}\left(-\frac{\partial a_i}{\partial u_j}\omega_i+\frac{\partial a_j}{\partial u_i}\omega_j\right),1\leq i,j\leq n. \tag{1.3}$$

Motivated by the classical and modern definitions of the Ribaucour transformation for hypersurfaces in $I\!R^{n+1}$ (Cf. [Bi], [CFT1], [CT], [DT], [T]), we introduce a similar definition for hypersurfaces in a space form.

Definition 1.1. *i) A congruence of geodesic spheres in* $\overline{M}^{n+1}(\overline{k})$ *is a family of* n*-parameter geodesic spheres in* $\overline{M}^{n+1}(\overline{k})$ *such that the set of centers of the geodesic spheres is a hypersurface of* $\overline{M}^{n+1}(\overline{k})$ *and the radii of the geodesic spheres are given by a differentiable function on the hypersurface.*

ii) An involute of a congruence of geodesic spheres is an n*-dimensional submanifold* M *of* $\overline{M}(\overline{k})$*, such that each point of* M *is tangent to a geodesic sphere of the congruence of geodesic spheres.*

iii) Let M and $\tilde{M}$ be hypersurfaces in $\overline{M}^{n+1}(\overline{k})$. We say that M and $\tilde{M}$ are associated by a congruence of geodesic spheres, if there exists a diffeomorfism $\psi : M \to \tilde{M}$ such that, at the corresponding points p and $\psi(p)$, M and $\tilde{M}$ are tangent to a same geodesic sphere of the congruence of geodesic spheres.

An important special case of item *iii*) is when $d\psi$ maps n principal vector fields of M to n principal vector fields of $\tilde{M}$.

Definition 1.2. *Let M be an orientable hypersurface in $\overline{M}^{n+1}(\overline{k})$. Suppose that there exist n orthonormal principal vector fields $e_1, ..., e_n$ defined on M. An orientable hypersurface $\tilde{M} \subset \overline{M}^{n+1}$ is associated to M by a Ribaucour transformation with respect to $e_1, ..., e_n$, if there exist a differentiable function $h : M \to I\!R$, a diffeomorfism $\psi : M \to \tilde{M}$ and unit normal vector fields N and $\tilde{N}$ of M and $\tilde{M}$, respectively, such that*

a) $\exp_p h(p)N(p) = \exp_{\psi(p)} h(p)\tilde{N}(\psi(p))$, $\forall p \in M$, where $\exp$ is the exponential map of $\overline{M}$.

b) the subset $S = \{\exp_p h(p)N(p) | p \in M\}$ is an n-dimensional submanifold of $\overline{M}$;

c) $d\psi(e_i)$, $i = 1, ..., n$, are orthogonal principal directions of $\tilde{M}$.

The Ribaucour transformation above is invertible in the sense that there exist n orthonormal principal vector fields $\tilde{e}_1, ..., \tilde{e}_n$, defined on $\tilde{M}$, such that M is associated to $\tilde{M}$ with respect to $\tilde{e}_1, ..., \tilde{e}_n$. Moreover, one can show [CT] that if M has a principal curvature with multiplicity bigger than one, the submanifolds associated to M, by a Ribaucour transformation may differ depending on the choice of the principal vector fields $\{e_i\}$ on M.

Remark 1.3. Let M and $\tilde{M}$ be hypersurfaces of $\overline{M}^{n+1}(\overline{k})$ as in Definition 1.2, then by considering $\overline{M}$ as $I\!R^{n+1}$, $S^{n+1} \subset I\!R^{n+2}$ or $H^{n+1} \subset L^{n+2}$, we can rewrite condition a) above as

$$p + h(p)N(p) = \psi(p) + h(p)\tilde{N}(\psi(p)), \quad p \in M,$$

and

$$h(p) = \begin{cases} \tan(\phi(p)), & \phi : M \to \left(0, \frac{\pi}{2}\right), \qquad \text{if } \overline{k} = 1, \\ \tanh(\phi(p)), & \phi : M \to I\!R, \qquad \text{if } \overline{k} = -1. \end{cases}$$

We can also formulate a local definition.

Definition 1.4. *Let M be an orientable hypersurface in $\overline{M}^{n+1}(\overline{k}), \overline{k} = 0, 1, -1$. Suppose that there exist n orthonormal principal vector fields $e_1, ..., e_n$ defined on M. We say that a hypersurface $\tilde{M} \subset \overline{M}^{n+1}(\overline{k})$ is locally associated to M by a Ribaucour transformation with respect to $e_1, ..., e_n$, if for any $\tilde{q} \in \tilde{M}$, there exist a neighborhood $\tilde{V}$ of $\tilde{q}$ in $\tilde{M}$ and an open subset $V \subset M$ such that $\tilde{V}$ is associated to V by a Ribaucour transformation with respect to $e_1, ..., e_n$.*

The following theorem, extending the result in the Euclidean space in [CFT1], is a characterization of Ribaucour transformation in terms of differencial equations, when the ambient space is a space form $\overline{M}(\overline{k})$, $\overline{k} = 0, 1, -1$.

Theorema 1.5. *Let M be an orientable hypersurface in $\overline{M}^{n+1}(\overline{k})$. Assume that there exist n orthonormal principal vector fields $e_i, 1 \leq i \leq n$, on M. A hypersurface $\tilde{M} \subset \overline{M}(\overline{k})$ is locally associated to M by a Ribaucour transformation with respect to $e_1, ..., e_n$, if and only if, $\forall p \in \tilde{M}$, there exist parametrization $\tilde{X} : U \subset R^n \to \tilde{M} \subset \overline{M}(\overline{k})$ of a neighborhood of p and a non-vanishing differentiable function h,*

$$\begin{cases} h : U \to I\!R, & \text{when } \overline{k} = 0, 1, \\ h : U \to (-1, 1), & \text{when } \overline{k} = -1, \end{cases}$$

such that

$$\tilde{X} = X + h(N - \tilde{N}), \tag{1.4}$$

where X is a parametrization of an open subset of M, N is a unit vector field of M and the unit normal vector field $\tilde{N}$ of $\tilde{M}$ is given by

$$\tilde{N} = \frac{1}{\Delta + \overline{k}h^2 + 1}\left(\sum_{i=1}^{n} 2Z^i e_i + (\Delta + \overline{k}h^2 - 1)N + 2\overline{k}hX\right), \tag{1.5}$$

with

$$Z^i = \frac{dh(e_i)}{1 + h\lambda^i}, \qquad \Delta = \sum_{i=1}^{n}(Z^i)^2, \tag{1.6}$$

and h satisfies the differential equations

$$dZ^j(e_i) + \sum_{l=1}^{n} Z^l \omega_{lj}(e_i) - Z^i Z^j \lambda^i = 0, \ 1 \le i \ne j \le n. \tag{1.7}$$

Proof. The proof for $\overline{k} = 0$ can be found in [CFT1](see also [CT]). Using similar arguments we sketch the proof for the case $\overline{k} = \pm 1$. Suppose that $\tilde{M}$ is locally associated to M by a Ribaucour transformation with respect to $e_1, ..., e_n$. We can assume, by definition, that for each $p \in \tilde{M}$, there exist local parametrizations $\tilde{X}$ of a neighborhood of p, X of a subset of M, and a differentiable function ϕ defined on $U \subset I\!R^n$ such that

if $\overline{k} = 1$, $(\cos\phi)X + (\sin\phi)N = (\cos\phi)\tilde{X} + (\sin\phi)\tilde{N}$ e $\phi(x) \in (0, \frac{\pi}{2})$, $\forall x \in U$;

if $\overline{k} = -1$, $(\cosh\phi)X + (\sinh\ \phi)N = (\cosh\phi)\tilde{X} + (\sinh\phi)\tilde{N}$,

where N and $\tilde{N}$ are unit normal vector fields of M and $\tilde{M}$, respectively.

We have

$$< d\tilde{X}(e_i), \tilde{N} >= 0, \ 1 \le i \le n. \tag{1.8}$$

Let

$$h = \begin{cases} \tan\phi, & \text{when } \overline{k} = 1, \\ \tanh\phi, & \text{when } \overline{k} = -1. \end{cases}$$

Then $\tilde{X}$ is given by (1.4) and hence

$$d\tilde{X} = dX + dh(N - \tilde{N}) + h(dN - d\tilde{N}). \tag{1.9}$$

Suppose that

$$\tilde{N} = \sum_{k=1}^{n} b^k e_k + b^{n+1}N + \mu X, \tag{1.10}$$

where

$$\sum_{k=1}^{n}(b^k)^2 + (b^{n+1})^2 + \overline{k}\mu^2 = 1. \tag{1.11}$$

Taking the differential of (1.10) and using (1.2), we get

$$(1.12)\qquad d\tilde{N}(e_i) = \sum_{k=1}^{n}\left(db^k(e_i)+\sum_{l=1}^{n}b^l\omega_{lk}(e_i)+\mu\delta_{ik}+b^{n+1}\lambda^k\delta_{ik}\right)e_k + (db^{n+1}(e_i)-\lambda^i b^i)N + (d\mu(e_i)-\overline{k}b^i)X.$$

From (1.9) we have

$$(1.13)\qquad d\tilde{X}(e_i) = (1+h\lambda^i)e_i + dh(e_i)(N-\tilde{N}) - hd\tilde{N}(e_i).$$

Substituting (1.10) and (1.13) into (1.8), we get

$$(1.14)\qquad (1+h\lambda^i)b^i + dh(e_i)(b^{n+1}-1) = 0,\quad 1\le i\le n.$$

We *claim* that $1+h\lambda^i \neq 0$ for all i. Let us prove the *claim* for the case that $\overline{k}=1$, since the case that $\overline{k}=-1$ is similar. Consider the central manifold $X^0=(X+hN)\cos\phi$. Then

$$dX^0(e_i) = \left[(1+h\lambda^i)e_i + dh(e_i)N\right]\cos\phi + (X+hN)(-\sin\phi)d\phi(e_i).$$

We conclude from $h=\tan\phi$ that $d\phi(e_i)=\cos^2\phi\cdot dh(e_i)$. Therefore

$$dX^0(e_i) = \left[(1+h\lambda^i)\cos\phi\right]e_i + (N-(X+hN)\sin\phi\cos\phi)\cos\phi\cdot dh(e_i).$$

Suppose that $(1+h\lambda^i)(u^0)=0$ at a point u^0. Then we get from (1.14) that $(b^{n+1}-1)dh(e_i)(u^0) = 0$ and, therefore, $dh(e_i)(u^0)=0$. Otherwise, $b^{n+1}(u_0)=1$ implies that $\tilde{N}(u^0)=N(u^0)$ and, therefore, $\tilde{X}(u^0)=X(u^0)=X^0(u^0)$, that is, $\phi(u^0)=0$ which is a contradiction. Therefore we have $dh(e_i)(u^0)=0$ and $dX^0(e_i)(u^0)=0$. This is a contradiction since X^0 is an n-dimensional submanifold. Therefore $1+h\lambda^i\neq 0$ for all i. Hence from (1.14), we have

$$(1.15)\qquad b^i = \frac{dh(e_i)}{(1+h\lambda^i)}(1-b^{n+1}) = Z^i(1-b^{n+1}).$$

The condition $<\tilde{X},\tilde{N}>=0$ implies that

$$(1.16)\qquad \mu = h(1-b^{n+1}).$$

Substituting $\Delta=\sum_{k=1}^{n}(Z^i)^2$, (1.15) and (1.16) into (1.11) we obtain

$$(1.17)\qquad b^{n+1} = \frac{\Delta+\overline{k}h^2-1}{\Delta+\overline{k}h^2+1}.$$

It is easy to check from (1.10), (1.15)-(1.17) that $\tilde{N}$ is given by (1.5).

Now we introduce the following notation

$$(1.18)\qquad d\tilde{N}(e_i) = \sum_{k=1}^{n}L_i^k e_k + L_i^{n+1}N + q_iX.$$

By (1.12),

$$(1.19)\qquad L_i^k = db^k(e_i)+\sum_{l=1}^{n}b^l\omega_{lk}(e_i)+\mu\delta_{ik}+b^{n+1}\lambda^k\delta_{ik},$$

$$(1.20)\qquad L_i^{n+1} = db^{n+1}(e_i)-\lambda^i b^i,$$

$$(1.21)\qquad q_i = d\mu(e_i)-\overline{k}b^i.$$

Since $d\tilde{X}(e_i)$ are principal orthogonal directions we have, when $i \neq j$

$$< d\tilde{X}(e_i), d\tilde{X}(e_j) >=< d\tilde{N}(e_i), d\tilde{X}(e_j) >=< d\tilde{N}(e_i), d\tilde{N}(e_j) >= 0.$$

Therefore, we obtain from (1.13) and (1.18) that

$$L_i^j + L_i^{n+1} Z^j = 0, \quad i \neq j. \tag{1.22}$$

Similarly, using

$$< d\tilde{N}(e_i), d\tilde{N}(e_j) >= 0, \quad i \neq j,$$

we get

$$\sum_{k=1}^{n} L_i^k L_j^k + L_i^{n+1} L_j^{n+1} + \overline{k} q_i q_j = 0, \quad i \neq j. \tag{1.23}$$

It follows from (1.15) that

$$db^j = dZ^j(1 - b^{n+1}) - Z^j db^{n+1}, \ 1 \leq j \leq n. \tag{1.24}$$

Substituting (1.15), (1.19), (1.20) and (1.22) into (1.24), we get, when $i \neq j$

$$0 = (1 - b^{n+1}) \left(dZ^j(e_i) + \sum_{k=1}^{n} Z^k \omega_{kj}(e_i) - \lambda^i Z^i Z^j \right)$$

Hence (1.7) holds. This completes the proof of the first part of Theorem 1.5.

Conversely, suppose that h is a non-vanishing solution of (1.7). Define the functions Z^i and Δ by (1.6), b^i, μ, b^{n+1} and $\tilde{N}$ by (1.15), (1.16), (1.17) and (1.9), respectively. From

$$\sum_{k=1}^{n} (b^k)^2 + (b^{n+1})^2 + \overline{k}\mu^2 = \Delta(1 - b^{n+1})^2 + (b^{n+1})^2 + \overline{k}\mu^2 = 1,$$

it follows that $\tilde{N}$ is a unit vector field. We want to show that $\tilde{X}$, given by (1.2), is associated to X by a Ribaucour transformation with respect to $e_1, ..., e_n$. Since h is non-vanishing, we can assume without loss of generality that $h(x) \in (0, +\infty), \ \forall x \in U$. Consider

$$\phi = \begin{cases} \tan^{-1} h, & \text{when } \overline{k} = 1, \\ \tanh^{-1} h, & \text{when } \overline{k} = -1, \end{cases}$$

then (1.4) implies that the condition a) in Definition 1.2 or 1.3 is satisfied. From (1.4), (1.10) and (1.16), we know that $< \tilde{N}, \tilde{X} >= 0$. By definition of $\tilde{X}$, we conclude that $d\tilde{X}(e_i)$ is given by (1.13) and therefore

$$< d\tilde{X}(e_i), \tilde{N} >= (-(1 + h\lambda^i) Z^i + dh(e_i))(b^{n+1} - 1) = 0,$$

which implies that $\tilde{N}$ is a unit normal vector field of $\tilde{X}$ in $\overline{M}^{n+1}(\overline{k})$. By (1.15)-(1.17),

$$db^{n+1}(e_i) = \frac{2d\Delta(e_i) + 4\overline{k} h dh(e_i)}{(\Delta + \overline{k} h^2 + 1)^2}, \tag{1.25}$$

$$db^j(e_i) = \frac{2dZ^j(e_i)}{\Delta + \overline{k} h^2 + 1} - \frac{2Z^j(d\Delta(e_i) + 2\overline{k} h dh(e_i))}{(\Delta + \overline{k} h^2 + 1)^2}, \tag{1.26}$$

$$(1.27)\qquad d\mu(e_i)=\frac{2\overline{k}dh(e_i)}{\Delta+\overline{k}h^2+1}-\frac{2h(d\Delta(e_i)+2\overline{k}hdh(e_i))}{(\Delta+\overline{k}h^2+1)^2}.$$

Therefore, using (1.18)-(1.21), we have

$$(1.28)\qquad \begin{aligned} L_i^i &= \frac{2dZ^i(e_i)}{\Delta+\overline{k}h^2+1}-\frac{2Z^i(d\Delta(e_i)+2\overline{k}hdh(e_i))}{(\Delta+\overline{k}h^2+1)^2}+\frac{2\overline{k}h}{\Delta+\overline{k}h^2+1}\\ &\quad+\frac{\sum_{k=1}^n 2Z^k\omega_{ki}(e_i)}{\Delta+\overline{k}h^2+1}+\frac{(\Delta+\overline{k}h^2-1)\lambda^i}{\Delta+\overline{k}h^2+1},\end{aligned}$$

$$(1.29)\qquad L_i^{n+1}=\frac{2d\Delta(e_i)}{(\Delta+\overline{k}h^2+1)^2}+\frac{2\overline{k}(2hZ^i-\overline{k}(\Delta-\overline{k}h^2+1)\lambda^iZ^i)}{(\Delta+\overline{k}h^2+1)^2}.$$

Hence from (1.21), (1.27) and (1.15) we get

$$(1.30)\qquad q_i=-\overline{k}hL_i^{n+1}.$$

We conclude from (1.7), (1.28) and (1.29) that

$$(1.31)\qquad Z^iL_i^i=\left(\frac{\Delta+\overline{k}h^2+1}{2}-(Z^i)^2\right)L_i^{n+1}.$$

On the other hand, we can deduce from (1.7), (1.19), (1.26) and (1.29) that (1.22) holds. Thus, we obtain from (1.22), (1.30) and (1.31) that for $i\neq j$,

$$\begin{aligned} < d\tilde{N}(e_i),d\tilde{N}(e_j) > &= \left((Z^i)^2+(Z^j)^2-2\cdot\frac{\Delta+\overline{k}h^2+1}{2}\right)L_i^{n+1}L_j^{n+1}\\ &\quad+\sum_{k\neq i,j}(Z^k)^2L_i^{n+1}L_j^{n+1}+L_i^{n+1}L_j^{n+1}+\overline{k}h^2L_i^{n+1}L_j^{n+1}=0.\end{aligned}$$

Now, using (1.13), (1.22) and the above equality, we have, when $i\neq j$

$$< d\tilde{N}(e_i),d\tilde{X}(e_j) >=(1+h\lambda^j)(L_i^j+L_i^{n+1}Z_j)=0.$$

Finally, we conclude that, when $i\neq j$,

$$\begin{aligned} < d\tilde{X}(e_i),d\tilde{X}(e_j) > &= -(1+h\lambda^i)(dh(e_j)Z^i(1-b^{n+1})-hZ^iL_j^{n+1}\\ &\quad -Z^i(1-b^{n+1})dh(e_j)+Z^ihL_j^{n+1})=0.\end{aligned}$$

Define a map $\psi : X(U)\to\tilde{X}(U)$ by

$$\psi : X(p)\to\tilde{X}(p).$$

Then item c) of Definition 1.2 or 1.3 is satisfied.

Let

$$X^0=\begin{cases}(X+hN)\cos\phi, & \text{when } \overline{k}=1,\\ (X+hN)\cosh\phi, & \text{when } \overline{k}=-1.\end{cases}$$

When $\overline{k}=1$,

$$dX^0(e_i)=\left[(1+h\lambda^i)\cos\phi\right]e_i+(N-(X+hN)\sin\phi\cos\phi)\cos\phi\cdot dh(e_i),$$

and so

$$|dX^0(e_i)|^2 = (1+h\lambda^i)^2 + ((\cos\phi - h\sin\phi\cos^2\phi)^2 + \sin^2\phi\cos^4\phi)(dh(e_i))^2 \neq 0,$$

since Z^i is well defined. This implies that $S := \{\cos(\phi(X(p)))X(p) + \sin(\phi(X(p)))N(p)|p \in U\}$ is an n-dimensional submanifold.

On the other hand, when $\overline{k} = -1$,

$$dX^0(e_i) = \left[(1+h\lambda^i)\cosh\phi\right] e_i + (N + (X+hN)\sinh\phi\cosh\phi)\cosh\phi \cdot dh(e_i)$$

and therefore

$$\begin{aligned} |dX^0(e_i)|^2 &= ((\cosh\phi + h\sinh\phi\cosh^2\phi)^2 + \sinh^2\phi\cosh^4\phi)(dh(e_i))^2 \\ &\quad + (1+h\lambda^i)^2 \neq 0. \end{aligned}$$

Hence, the set $\{\cosh(\phi(X(p)))X(p) + \sinh(\phi(X(p)))N(p)|p \in U\}$ is an n-dimensional submanifold. Thus, in both cases, $\tilde{M}$ is locally associated to M by a Ribaucour transformation with respect to $e_1, ..., e_n$. Observe that $\tilde{M}$ is generically n-dimensional. Consequently, $\tilde{M}$ is locally associated to M by a Ribaucour transformation with respect to $e_1, ..., e_n$. □

The following result linearizes the problem of obtaining the function h.

Proposition 1.6. *If h is a solution of (1.7) which does not vanish on a simply connected domain, then $h = \Omega/W$, where W is a nonvanishing function and the functions Ω, Ω^i, W satisfy*

$$d\Omega_i(e_j) = \sum_{k=1}^n \Omega_k \omega_{ik}(e_j), \quad \text{if } i \neq j, \tag{1.32}$$

$$d\Omega = \sum_{i=1}^n \Omega_i \omega_i, \tag{1.33}$$

$$dW = -\sum_{i=1}^n \Omega_i \lambda^i \omega_i. \tag{1.34}$$

Conversely, suppose (1.32)-(1.34) are satisfied then $h = \Omega/W$ is a solution of (1.7).

Proof. Assume h is a non-vanishing solution of (1.7), then $\psi = \frac{1}{h}\sum_{k=1}^n Z^k\omega_k$ is a closed 1-form. Hence, on a simply connected domain there exists a differentiable function Ω such that $d(\log\Omega) = \psi$. We define

$$\Omega_i = d\Omega(e_i), \;\; W = \frac{\Omega}{h}. \tag{1.35}$$

One can verify that that (1.33) holds and

$$dh(e_i) = \frac{\Omega_i}{W}\left(1 + \frac{\Omega\lambda^i}{W}\right), \qquad 1 + h\lambda^i = 1 + \frac{\Omega\lambda^i}{W}. \tag{1.36}$$

$$\Delta = \frac{1}{W}\sum_{k=1}^n \Omega_j^2, \;\; Z^i = \frac{\Omega}{W}. \tag{1.37}$$

Moreover it follows from (1.7) that (1.32) and (1.34) are satisfied.

Conversely, if (1.32)-(1.34) hold, considering $Z^i = \Omega^i/W$, one concludes that (1.7) is satisfied. We define $h = \Omega/W$, then it follows that $dh(e_i) = Z^i(1+h\lambda^i)$. □

From now on, when we say that a hypersurface $\tilde{M}$ in $\overline{M}^{n+1}(\overline{k}), \overline{k} = \pm 1$, or 0, is locally associated by a Ribaucour transformation to M, with respect to a set of principal directions $e_i, i = 1, ..., n$, on M, we are assuming that there exist functions Ω^i, Ω and W, locally defined, satisfying the system (1.32)-(1.34). When all the principal curvatures of M are distinct, we will only say that $\tilde{M}$ is associated to M by a Ribaucour transformation.

We rewrite Theorem 1.5 in the following form for our future use.

Theorem 1.7. *Let $M \subset \overline{M}^{n+1}(\overline{k})$ be a hypersurface parametrized by $X : U \subset R^n \to M$. Assume that $e_i, 1 \leq i \leq n$ are orthonormal vector fields in the principal directions and N is a unit normal vector field of M, $dN(e_i) = \lambda_i e_i, 1 \leq i \leq n$. A hypersurface $\tilde{M} \subset \overline{M}(\overline{k})$ is locally associated to M by a Ribaucour transformation with respect to $e_1, ..., e_n$, if and only if, for any $p \in \tilde{M}$, there exist an open subset $V \subset U$, non-vanishing functions $W, \Omega, \Omega_i : V \subset U \to R$, which are solutions of (1.32)-(1.34) satisfying $WS(W + \lambda^i\Omega)(S - \Omega T^i) \neq 0$, and $\tilde{X} : V \subset R^n \to \tilde{M}$, a parametrization of $\tilde{M}$, given by*

$$\tilde{X} = \left(1 - \frac{2\overline{k}\Omega^2}{S}\right) X - \frac{2\Omega}{S}(\sum_{k=1}^{n} \Omega_i e_i - WN), \tag{1.38}$$

where $S = \sum_{k=1}^{n}(\Omega_i)^2 + \overline{k}\Omega^2 + W^2$.

Proof. Note that

$$\begin{aligned} \tilde{N} &= \frac{1}{\Delta + \overline{k}h^2 + 1}\left[\sum_{k=1}^{n} 2Z^i e_i + (\Delta + \overline{k}h^2 - 1)N + 2\overline{k}hX\right] \\ &= N + \frac{2W}{S}\left(\sum_{k=1}^{n} \Omega_k e_k - WN + \overline{k}\Omega X\right). \end{aligned}$$

Therefore

$$\tilde{X} = X - h(\tilde{N} - N) = \left(1 - \frac{2\overline{k}\Omega^2}{S}\right) X - \frac{2\Omega}{S}(\sum_{k=1}^{n} \Omega_i e_i - WN),$$

which concludes the proof. □

If M is parametrized by orthogonal curvature lines, we will assume that e_i are unit vector fields tangent to the coordinate curves, that is, $e_i = X_{u_i}/a_i$, where $a_i = |X_{u_i}|$. In this case, using (1.3), we can verify that the system (1.32)-(1.34) is given by

$$\frac{\partial \Omega_i}{\partial u_j} = \Omega_j \frac{1}{a_i} \frac{\partial a_j}{\partial u_i}, \quad j \neq i, \tag{1.39}$$

$$\frac{\partial \Omega}{\partial u_i} = a_i \Omega_i \tag{1.40}$$

$$\frac{\partial W}{\partial u_i} = -a_i \lambda^i \Omega_i. \tag{1.41}$$

Proposition 1.8. *The equation (1.32) is the integrability condition of the system (1.33), (1.34) for Ω and W.*

Proof. Consider the idea $\mathcal{I}$ generated by the 1-forms

$$\begin{aligned} \alpha &= d\Omega - \sum_{i=1}^{n} \Omega_i \omega_i, \\ \beta &= dW + \sum_{i=1}^{n} \Omega_i \lambda^i \omega_i = dW - \sum_{i=1}^{n} \Omega_i \omega_{in+1}. \end{aligned}$$

Using Codazzi equation (1.1) and (1.32), a straightforward computation shows that $\mathcal{I}$ is closed under the exterior differenciation.

Theorem 1.9. *Let M be an orientable hypersurface in $\overline{M}^{n+1}(\overline{k})$. Let e_i be orthonormal vector fields in the principal directions defined on M and N be a unit normal vector field of M, $dN(e_i) = \lambda^i e_i, 1 \leq i \leq n$. Suppose that a hypersurface $\tilde{M} \subset \overline{M}^{n+1}(\overline{k})$ is locally associated to M by a Ribaucour transformation with respect to $e_1, ..., e_n$. Then the principal curvatures of $\tilde{M}$ for each $1 \leq i \leq n$ are given by*

$$\tilde{\lambda}^i = \frac{WT^i + \lambda^i S}{S - \Omega T^i}, \tag{1.42}$$

where Ω_i, Ω and W satisfy (1.32)-(1.34) and

$$T^i = 2\left(d\Omega_i(e_i) + \sum_{k=1}^{n} \Omega_k \omega_{ki}(e_i) - W\lambda^i + \overline{k}\Omega\right). \tag{1.43}$$

$$S = \sum_{k=1}^{n} (\Omega_k)^2 + \overline{k}\Omega^2 + W^2, \tag{1.44}$$

Proof. Let X and $\tilde{X}$ be parametrizations of M and $\tilde{M}$, respectively. We have

$$d\tilde{X} = dX + dh(N - \tilde{N}) + h(dN - d\tilde{N}), \tag{1.45}$$

where $h = \Omega/W$ and $\tilde{N}$ is of the form (1.10). The principal curvatures of $\tilde{M}$ are given by

$$\tilde{\lambda}^i = \frac{< d\tilde{N}(e_i), d\tilde{X}(e_i) >}{< d\tilde{X}(e_i), d\tilde{X}(e_i) >}. \tag{1.46}$$

Since $d\tilde{N}(e_i) = \tilde{\lambda}^i d\tilde{X}(e_i)$, $dN(e_i) = \lambda^i dX(e_i)$, we get from (1.45) that

$$(1 + h\tilde{\lambda}^i) d\tilde{X}(e_i) = (1 + h\lambda^i) e_i + dh(e_i)(N - \tilde{N}). \tag{1.47}$$

It then follows from (1.10), (1.15) and (1.6) that

$$(1 + h\tilde{\lambda}^i)^2 < d\tilde{X}(e_i), d\tilde{X}(e_i) >= (1 + h\lambda^i)^2,$$

that is,

$$< d\tilde{X}(e_i), d\tilde{X}(e_i) >= \frac{(1 + h\lambda^i)^2}{(1 + h\tilde{\lambda}^i)^2}. \tag{1.48}$$

On the other hand, we know that

$$< d\tilde{N}(e_i), d\tilde{X}(e_i) >= \frac{1 + h\lambda^i}{1 + h\tilde{\lambda}^i} (L_i^i + Z^i L_i^{n+1}) \tag{1.49}$$

If $\Omega_i \neq 0$, that is, $dh(e_i) \neq 0$, we conclude from (1.31) that

$$L_i^i + Z^i L_i^{n+1} = \frac{\Delta + \overline{k}h^2 + 1}{2Z^i} L_i^{n+1},$$

and therefore,

$$< d\tilde{X}(e_i), d\tilde{N}(e_i) >= \frac{(1+h\lambda^i)(\Delta+\overline{k}h^2+1)}{2(1+h\tilde{\lambda}^i)Z^i}L_i^{n+1}.$$

Combining (1.46), (1.48) and the above equality, we get

$$\tilde{\lambda}^i = \frac{(1+h\tilde{\lambda}_i)(\Delta+\overline{k}h^2+1)}{2Z^i(1+h\lambda^i)}L_i^{n+1}.$$

Since $S-\Omega T^i \neq 0$, we know that

$$2dh(e_i) - h(\Delta+\overline{k}h^2+1))L_i^{n+1} \neq 0, \quad \text{(see (1.51) and (1.53) below)}.$$

Therefore

$$\tilde{\lambda}^i = \frac{(\Delta+\overline{k}h^2+1)L_i^{n+1}}{2dh(e_i) - h(\Delta+\overline{k}h^2+1))L_i^{n+1}}, \tag{1.50}$$

where L_i^{n+1} is given by (1.29).

Observe that $\Delta+\overline{k}h^2+1 = \frac{S}{W^2}$, where S is given by (1.44). We conclude that

$$(\Delta+\overline{k}h^2+1)L_i^{n+1} = 2\frac{dS(e_i)}{S} - 2\frac{dW(e_i)}{W}.$$

Thus it follows from (1.36) that

$$2dh(e_i) - h(\Delta+\overline{k}h^2+1)L_i^{n+1} = 2\frac{\Omega_i}{W} - 2\frac{\Omega dS(e_i)}{WS}, \tag{1.51}$$

and so we can deduce from (1.50) that

$$\tilde{\lambda}^i = \frac{dS(e_i)W + \Omega_i\lambda^i S}{\Omega_i S - \Omega dS(e_i)}, \quad \text{if } \Omega_i \neq 0. \tag{1.52}$$

Observe that, using (1.44), a straightforward computation shows that

$$dS(e_i) = \Omega_i T^i, \tag{1.53}$$

where T^i is given by (1.43). We conclude from (1.52) and (1.53) that the principal curvatures of $\tilde{M}$ are given by

$$\tilde{\lambda}^i = \frac{WT^i + \lambda^i S}{S - \Omega T^i}, i = 1, ..., n, \text{ if } \Omega_i \neq 0. \tag{1.54}$$

If $\Omega_i \equiv 0$, that is, $dh(e_i) \equiv 0$, then $Z^i \equiv 0$ which implies from (1.28) that

$$\begin{aligned} L_i^i &= \frac{1}{\Delta+\overline{k}h^2+1}\left(\sum_{k=1}^n 2Z^k\omega_{ki}(e_i) + 2\overline{k}h + (\Delta+\overline{k}h^2-1)\lambda^i\right) \\ &= \frac{W^2}{S}\left(2\sum_{k=1}^n \frac{\Omega_k}{W}\omega_{ki}(e_i) + 2\frac{\overline{k}\Omega}{W} + \frac{S-2W^2}{W^2}\lambda^i\right) \\ &= \frac{WT^i+\lambda^i S}{S}, \end{aligned}$$

that is,

$$L_i^i = \frac{WT^i + \lambda^i S}{S}. \tag{1.55}$$

From (1.46), (1.48), (1.49) and $Z^i \equiv 0$, it follows that

$$\tilde{\lambda}^i = \frac{1 + h\tilde{\lambda}^i}{1 + h\lambda^i} L_i^i.$$

Therefore, if $\Omega_i \equiv 0$, we have

$$\tilde{\lambda}^i = \frac{L_i^i}{1 + h\lambda^i - hL_i^i} = \frac{WT^i + \lambda^i S}{S - \Omega T^i}. \tag{1.56}$$

□

For hypersurfaces M of the Euclidean space $I\!R^{n+1}$, which admits orthonormal principal vector fields, $e_i, i = 1, ..., n$, it has been shown (see [CT]) that an open subset of $I\!R^n$ or S^n is locally associated to M by a Ribaucour transformation with respect to e_i. The following theorem extends this result to hypersurfaces in S^{n+1} or H^{n+1}.

Theorem 1.10. *Let M^n be a hypersurface of $\overline{M}^{n+1}(\overline{k})$ that admits n orthogonal principal direction vector fields $e_i, i = 1, ..., n$. For any real constants $b_1 \neq 0$ and b_0. The system of equations*

$$d\Omega_i = \sum_k \Omega_k \omega_{ik} + (b_0 - \overline{k}\Omega)\omega_i + (b_1 - W)\omega_{in+1} \tag{1.57}$$

$$d\Omega = \sum_i \Omega_i \omega_i, \tag{1.58}$$

$$dW = \sum_i \Omega_i \omega_{in+1} \tag{1.59}$$

is integrable. The function $S - 2(b_0\Omega + b - 1W) = c$ is a constant determined by the initial conditions. Considering $c = 0$, the associated hypersurface is an open subset of a totally umbilic (resp. totally geodesic) hypersurface if $b_0 \neq 0$(resp $b_0 = 0$).

Proof. One can easily prove that the system of equations is integrable and as a consequence of (1.44) that $dS - 2(b_0 d\Omega + b_1 dW) = 0$. Therefore, we can choose the initial conditions so that $S - 2(b_0\Omega + b_1 W) = 0$. Finally, it follows from (1.42) that any principal curvature of the associated hypersurface is given by b_0/b_1. □

Observe that the system of equations of Theorem 1.10 contains (1.32)-(1.34). Moreover, this theorem shows that this system of equations does not preserve multiplicity of principal directions. This fact had already been observed in [CFT1].

2 Ribaucour Transformations for Linear Weingarten Surfaces in space forms

In this section, we use the theory of section 1, to study Ribaucour transformations for linear Weingarten surfaces in space forms.

Recall from the last section that a surface $\tilde{M}$ in $\overline{M}^3(\overline{k})$ is locally associated by a Ribaucour transformation to M with respect to a set of ortonormal principal vector fields $\{e_i\}_{i=1}^2$ of M, if there exist functions $\Omega^i, i = 1, 2, \Omega$ and W locally defined, satisfying the system (1.35)-(1.37), which for $n = 2$ reduces to

$$d\Omega_i(e_j) = \Omega_j \omega_{ij}(e_j), \quad \text{if } i \neq j, \tag{2.1}$$

$$d\Omega = \sum_{i=1}^{2} \Omega_i \omega_i, \tag{2.2}$$

$$dW = -\sum_{i=1}^{2} \Omega_i \lambda^i \omega_i, \ h = \frac{\Omega}{W}. \tag{2.3}$$

Recall also that (Cf. [T]) a *linear Weingarten surface* in $\overline{M}^3(\overline{k})$ is a surface whose Gaussian and mean curvatures K and H satisfy the relation

$$\alpha + \beta H + \gamma(K - \overline{k}) = 0, \tag{2.4}$$

where $\alpha, \ \beta, \ \gamma \in I\!R, \ \beta^2 - 4\alpha\gamma \neq 0, \ H = -(\lambda^1 + \lambda^2)/2$ and $K = \lambda^1\lambda^2 + \overline{k}$.

The following result provides a sufficient condition for a Ribaucour transformation to transform a linear Weingarten surface to another surface of the same type. The arguments of the proof are similar to the Euclidean case (see [CFT3]) and will be omited.

Theorem 2.1 *Let M be a surface in $\overline{M}^3(\overline{k}), \overline{k} = \pm 1, 0$, which admits orthonormal principal vector fields e_1, e_2. Let $\tilde{M}$ be a regular surface associated to M by a Ribaucour transformation with respect to e_i. Suppose that the functions $\Omega_i \neq 0, \ \Omega$ and W satisfy the algebraic relation*

$$S = 2c(\alpha\Omega^2 + \beta\Omega W + \gamma W^2), \tag{2.5}$$

where

$$S = \Omega_1^2 + \Omega_2^2 + W^2 + \overline{k}\Omega^2, \tag{2.6}$$

$c \neq 0$ and $\alpha, \ \beta, \ \gamma$ are real constant such that $\beta^2 - 4\alpha\gamma \neq 0$. Then $\tilde{M}$ is a linear Weingarten surface satisfying $\alpha + \beta\tilde{H} + \gamma(\tilde{K} - \overline{k}) = 0$ if and only if $\alpha + \beta H + \gamma(K - \overline{k}) = 0$ holds for the surface M, where $K, \tilde{K}$ and $H, \tilde{H}$ are the Gaussian and mean curvatures of M and $\tilde{M}$, respectively. Moreover, $\tilde{M}$ has no umbilic points if and only if M has no umbilic points.

The system (2.1)-(2.3) with the aditional condition (2.5) is always integrable when we start from a linear Weingarten surface. That is, we have the following result:

Theorem 2.2. *Let M be a linear Weingarten surface in $\overline{M}^3(\overline{k})$, which admits orthonormal principal vector fields. Suppose that the Gaussian and mean curvatures of M satisfy $\alpha + \beta H + \gamma(K - \overline{k}) = 0$. Then for any constant $c \neq 0$, the system of the equations*

$$d\Omega = \sum_{i=1}^{2} \Omega_i \omega_i, \tag{2.7}$$

$$dW = \sum_{i=1}^{2} \Omega_i \omega_{i3}, \tag{2.8}$$

$$d\Omega_i = \Omega_j \omega_{ij} + \{(2c\alpha - \overline{k})\Omega - \beta c W\}\omega_i + \{c\beta\Omega + (2c\gamma - 1)W\}\omega_{i3}, \ i \neq j. \tag{2.9}$$

is integrable. Any solution satisfies (2.5) on a simply connected domain U whenever the initial condition satisfies (2.5). Moreover, when $\overline{k} = 1$, or $\overline{k} = 0$ with $\alpha \neq 0$, any solution of the system defined on U is identically zero (hence $S \equiv 0$) or S does not vanish.

Proof. We consider the ideal $\mathcal{I}$ generated by the 1-forms

$$\theta = d\Omega - \sum_{i=1}^{2} \Omega_i \omega_i,$$
$$\phi = dW - \sum_{i=1}^{2} \Omega_i \omega_{i3},$$
$$\theta_i = d\Omega_i - \Omega_j \omega_{ij} - \{(2c\alpha - \overline{k})\Omega - \beta c W\}\omega_i - \{c\beta\Omega + (2c\gamma - 1)W\}\omega_{i3}, \ i \neq j.$$

We have

$$d\theta = -\sum_{i=1}^{2} \theta_i \wedge \omega_i, \ d\phi = -\sum_{i=1}^{2} \theta_i \wedge \omega_{i3},$$

and since M is a linear Weingarten surface

$$\begin{aligned} d\theta_i &= -\theta_j \wedge \omega_{ij} + (c\beta\omega_i + (2c\gamma - 1)\omega_{i3}) \wedge \Delta + ((2c\alpha - \overline{k})\omega_i + c\beta\omega_{i3}) \wedge \theta \\ &\quad -2c\Omega_i(\alpha + \beta H + \gamma(K - \overline{k}))\omega_j \wedge \omega_i \\ &= -\theta_j \wedge \omega_{ij} + (c\beta\omega_i + (2c\gamma - 1)\omega_{i3}) \wedge \Delta + ((2c\alpha - \overline{k})\omega_i + c\beta\omega_{i3}) \wedge \theta, \end{aligned}$$

where $j \neq i$. It follows that $\mathcal{I}$ is closed under the exterior differenciation which implies that the system (2.7)-(2.9) is integrable.

We observe that for any solution of this system $d(S - 2cP) = 0$, where

$$P = \alpha\Omega^2 + \beta\Omega W + \gamma W^2. \tag{2.10}$$

Therefore, when we consider the initial condition at a point p_0 such that $(S - 2cP)(p_0) = 0$, we know that $S = 2cP$ holds identically on a connected domain.

When $\overline{k} = 1$ or $\overline{k} = 0$, assume that $S(p_0) = 0$ for some $p_0 \in U$. It then follows from (1.44) that $\Omega_1, \ \Omega_2, \ W$ and Ω vanish at p_0. Since U is simply connected, the uniqueness of solutions for the system implies that $\Omega \equiv \Omega_1 \equiv \Omega_2 \equiv W \equiv 0$ on U and therefore $S \equiv 0$ on U. This concludes the proof of Theorem 2.2. □

Theorem 2.3. *Let M be a linear Weingarten surface in $\overline{M}^3(\overline{k}), \overline{k} = \pm 1, 0$, satisfying $\alpha + \beta H + \gamma(K - \overline{k}) = 0$, which admits orthonormal principal vector fields $e_1, \ e_2$. Suppose that M is locally parametrized by $X: \ U \subset R^2 \to M \subset \overline{M}^3(\overline{k})$. Then any parametrized linear Weingarten surface in $\overline{M}(\overline{k})$ locally associated to X by a Ribaucour transformation with respect to e_1, e_2, as in Theorem 2.1, is given by*

$$\tilde{X} = \left(1 - \frac{2\overline{k}\Omega^2}{S}\right) X - \frac{2\Omega}{S}(\sum_{i=1}^{2} \Omega_i e_i - WN), \tag{2.11}$$

where $\Omega, \Omega_i, \ i = 1, 2$, and W are solutions of (2.7)-(2.9) and $\tilde{X}$ is defined on

$$\tilde{U} = \{(u_1, u_2) \in U: \ T^2 + 2TQH + Q^2(K - \overline{k}) \neq 0\}, \tag{2.12}$$

where $T = \alpha\Omega^2 - \gamma W^2$ and $Q = 2\gamma\Omega W + \beta\Omega^2$.

Proof. The proof follows the same arguments of Theorem 1.5 in [CFT3]. By Theorem 1.7, it suffices to show that $\tilde{X}$, defined on $\tilde{U}$ by (2.11), is a parametrized surface of $\overline{M}(\overline{k})$.

Since

$$S = 2cP, \tag{2.13}$$

where P is given by (2.10), we have, using (2.2) and (2.3), that

$$d\left(\frac{\Omega}{S}\right) = \frac{1}{2cP^2}\sum_{i=1}^{2}\Omega_i\eta_i,$$

where $\eta_i,\ i=1,2$ are 1-forms, defined by

$$\eta_i = (\gamma W^2 - \alpha\Omega^2)\omega_i - (2\gamma W\Omega + \beta\Omega^2)\omega_{i3},$$

$$d\left(\frac{\Omega^2}{S}\right) = \frac{\Omega}{2cP^2}\sum_{i=1}^{2}\left[(2\gamma W^2 + \beta\Omega W)\omega_i - (2\gamma W\Omega + \beta\Omega^2)\omega_{i3}\right]\Omega_i.$$

Using (2.7)-(2.9), we conclude that

$$d\tilde{X} = \frac{1}{P}\sum_{i=1}^{2}\eta_i\tilde{e}_i,$$

where

$$\tilde{e}_1 = \frac{1}{cP}((-\overline{k}\Omega_1\Omega X + (cP - \Omega_1^2)e_1 - \Omega_1\Omega_2 e_2 + \Omega_1 WN), \tag{2.14}$$

and

$$\tilde{e}_2 = \frac{1}{cP}(-\overline{k}\Omega_2\Omega X + (cP - \Omega_2^2)e_2 - \Omega_1\Omega_2 e_1 + \Omega_2 WN). \tag{2.15}$$

By a simple calculation, we can verify that $\tilde{e}_1$ and $\tilde{e}_2$ are orthonormal. Therefore $\tilde{X}$ is an immersion when $\eta_1 \wedge \eta_2 \neq 0$, that is, on the subset $\tilde{U}$ given by (2.12). □

Remark 2.4. From the proof of Theorem 2.3 we know that the principal directions $\tilde{e}_1,\ \tilde{e}_2$ of $\tilde{X}$ are given by (2.14) and (2.15). Furthermore, their dual forms are given by

$$\tilde{\omega}_i = \frac{1}{P}(\gamma W^2 - \alpha\Omega^2 + (\beta\Omega + 2\gamma W)\Omega\lambda^i)\omega_i,\ i = 1, 2. \tag{2.16}$$

Considering H constant, $\alpha = -H,\ \beta = 1$ and $\gamma = 0$ in Theorems 2.1-2.3, we have the cmc H cases (if $H = 0$, is the case of minimal surface). Observe that $\lambda^1 \neq \lambda^2$ is equivalent to

$$\lambda^1 \neq -H \text{ and } \lambda^2 \neq -H.$$

By (2.9), we have

$$S\Omega^i - \Omega dS(e_i) = 2c\Omega_i\Omega^2(H + \lambda^i).$$

Therefore, the fact that M has no umbilic points with $\Omega_i \neq 0$ implies that $S\Omega_i - \Omega dS(e_i) \neq 0$. We can state the following Corollaries, by considering $\alpha = -H, \beta = 1$ and $\gamma = 0$ on Theorems 2.2 and 2.3.

Corollary 2.5. *Let M be a surface in $\overline{M}^3(\overline{k})$ which admits orthonormal principal vector fields $e_1,\ e_2$. If M is a cmc H surface, then for any constant $c \neq 0$ the system of equations*

$$d\Omega = \sum_{i=1}^{2}\Omega_i\omega_i, \tag{2.17}$$

$$dW = \sum_{i=1}^{2}\Omega_i\omega_{i3}, \tag{2.18}$$

$$d\Omega_i = \Omega_j\omega_{ij} - [(2cH + \overline{k})\Omega - cW]\omega_i + (c\Omega - W)\omega_{i3},\ i \neq j, \tag{2.19}$$

is integrable. If

$$-\overline{k}+c^2-2Hc>0, \tag{2.20}$$

then any solution on a simply connected domain U satisfies

$$S=2c\Omega(-H\Omega+W), \tag{2.21}$$

whenever the initial condition satisfies (2.21). Moreover, when $\overline{k}=1$ or $\overline{k}=0$ with $H\neq 0$, for any nontrivial solution the function S does not vanish on U.

We need to require that $c\neq 0$ also satisfies (2.20), if we want (2.21) to be satisfied by a non-trivial solution. In fact, since S is given by (2.6), the algebraic condition (2.21) reduces to

$$\sum_{i=1}^{2}\Omega_i^2+(W-c\Omega)^2+(\overline{k}-c^2+2cH)\Omega^2=0.$$

Therefore $-\overline{k}+c^2-2Hc>0$.

Corollary 2.6. *Let M be a cmc H surface in $\overline{M}^3(\overline{k})$ parametrized by $X:U\subset R^2\to\overline{M}^3(\overline{k})$, which admits orthonormal principal vector fields $e_1,\ e_2$. Then any cmc H parametrized surface in $\overline{M}^3(\overline{k})$, locally associated to X by a Ribaucour transformation as in Theorem 2.1 is given by*

$$\tilde{X}=\left(1-\frac{\overline{k}\Omega}{c(W-H\Omega)}\right)X-\frac{1}{c(W-H\Omega)}(\sum_{i=1}^{2}\Omega_ie_i-WN), \tag{2.22}$$

where $\Omega,\Omega_i,\ i=1,2,\ W$ are solutions of (2.17)-(2.19) with the constant $c\neq 0$ and $-\overline{k}+c^2-2Hc>0$.

3 A family of *cmc* surfaces associated to the flat torus in S^3

In this section, we construct a family of *cmc* surfaces which are locally associated to the flat torus in S^3 by using the Ribaucour transformations as in Theorem 2.3. As a special case, we obtain a new family of minimal surfaces in S^3.

Theorem 3.1. *Consider the torus T^2 parametrized by*

$$X(u_1,u_2)=(c_1\cos u_1,c_1\sin u_1,c_2\cos u_2,c_2\sin u_2),\ (u_1,u_2)\in R^2.$$

as a cmc H surface, where $c_1,\ c_2$ are positive constants satisfying

$$c_1^2+c_2^2=1,\qquad\text{and}\qquad H=\frac{1}{2}\left(\frac{c_2}{c_1}-\frac{c_1}{c_2}\right). \tag{3.1}$$

A parametrized surface $\tilde{X}_c$ is a cmc H surface locally associated to X by a Ribaucour transformation as in Corollary 2.6, if and only if $c\in(-\infty,-c_1/c_2]\cup[c_2/c_1,\infty)$, and $\tilde{X}_c$ is a torus if $c\in\{-c_1/c_2,c_2/c_1\}$ or

$$\tilde{X}_c=\left(1-\frac{2c_1c_2(c_1^2f-c_2^2g)}{c(c_1^2f+c_2^2g)}\right)X-\frac{2c_1^2c_2^2}{c(c_1^2f+c_2^2g)}\left(\frac{1}{c_1c_2}(f'X_{u_1}-g'X_{u_2})-(f+g)N\right) \tag{3.2}$$

where

$$N = (-c_2 \cos u_1, -c_2 \sin u_1, c_1 \cos u_2, c_1 \sin u_2), \tag{3.3}$$

and $f(u_1)$ *and* $g(u_2)$ *are solutions of the equations*

$$f'' + \left(1 - c\frac{c_1}{c_2}\right) f = 0, \quad g'' + \left(1 + c\frac{c_2}{c_1}\right) g = 0, \tag{3.4}$$

satisfying the initial condition

$$\left(\left(\frac{1}{c_2}f'\right)^2 + \left(\frac{1}{c_1}g'\right)^2 + \frac{1}{c_2^2}\left(1 - c\frac{c_1}{c_2}\right) f^2 + \frac{1}{c_1^2}\left(1 + c\frac{c_2}{c_1}\right) g^2\right)(u_1^0, u_2^0) = 0. \tag{3.5}$$

Moreover, $\tilde{X}_c$ *is a regular surface defined on*

$$\tilde{U} = \{(u_1, u_2) \in R^2 | \frac{c_1}{c_2} f(u_1) - \frac{c_2}{c_1} g(u_2) \neq 0\}. \tag{3.6}$$

Proof. The unit normal vector field of the torus T^2 is given by (3.3). The first and the second fundamental forms of T^2 are given by

$$ds^2 = c_1^2 du_1^2 + c_2^2 du_2^2, \; II = c_1 c_2 (du_1^2 - du_2^2), \tag{3.7}$$

respectively. From (3.6), it follows that the coordinate curves of T^2 are curvature lines and that $\lambda^1 = -c_2/c_1$, $\lambda^2 = c_1/c_2$. Therefore, T^2 has zero Gaussian curvature and the mean curvature of T^2 is given by (3.1). Let $e_i = X_{u_i}/c_i$, $i = 1, 2$. Since $S = 2c\Omega(-H\Omega + W)$, we have

$$\sum_{i=1}^{2} \Omega_i^2 + (W - c\Omega)^2 + (1 - c^2 + 2cH)\Omega^2 = 0,$$

which implies that $1 - c^2 + 2cH \leq 0$, that is, $c \geq \frac{c_2}{c_1}$ or $c \leq -\frac{c_1}{c_2}$. Thus, if $c \in (-c_1/c_2, c_2/c_1)$, $\tilde{X}_c$ does not exist.

Now, suppose that $c \in (-\infty, -c_1/c_2) \cup (c_2/c_1, +\infty)$. In order to obtain the Ribaucour transformation, we need to solve the following system of differential equations derived from (2.1)-(2.3):

$$\frac{\partial \Omega_i}{\partial u_j} = 0, \; i \neq j, \tag{3.8}$$

$$\frac{\partial \Omega}{\partial u_i} = c_i \Omega_i, \; i = 1, 2, \tag{3.9}$$

$$\frac{\partial W}{\partial u_1} = c_2 \Omega_1, \tag{3.10}$$

$$\frac{\partial W}{\partial u_2} = -c_1 \Omega_2, \tag{3.11}$$

with the aditional algebraic condition

$$\Omega_1^2 + \Omega_2^2 + W^2 + \Omega^2 = 2c\Omega(W - H\Omega). \tag{3.12}$$

From (3.8) and (3.10) we know that $\frac{\partial^2 W}{\partial u_1 \partial u_2} = 0$ which implies that

$$W = f(u_1) + g(u_2), \tag{3.13}$$

where f and g are functions of u_1 and u_2, respectively.

Taking the derivatives of (3.11) with respect to u_1 and u_2, respectively and using (3.7)-(3.10), we get

$$\frac{\partial \Omega_1}{\partial u_1} + c_2 W + (1+2cH)c_1\Omega - cc_1 W - cc_2\Omega = 0, \tag{3.14}$$

$$\frac{\partial \Omega_2}{\partial u_2} - c_1 W + (1+2cH)c_2\Omega - cc_2 W + cc_1\Omega = 0. \tag{3.15}$$

The equations (3.10), (3.11) and (3.13) imply

$$\Omega_1 = \frac{1}{c_2}\frac{\partial W}{\partial u_1} = \frac{1}{c_2}f', \qquad \Omega_2 = -\frac{1}{c_1}\frac{\partial W}{\partial u_2} = -\frac{1}{c_1}g', \tag{3.16}$$

and

$$\frac{\partial \Omega_1}{\partial u_1} = \frac{1}{c_2}f'', \quad \frac{\partial \Omega_2}{\partial u_2} = -\frac{1}{c_1}g''. \tag{3.17}$$

On the other hand, we deduce from (3.14) and (3.15) that

$$c_1\frac{\partial \Omega_1}{\partial u_1} + c_2\frac{\partial \Omega_2}{\partial u_2} + (1+2cH)\Omega - cW = 0, \tag{3.18}$$

$$c_2\frac{\partial \Omega_1}{\partial u_1} - c_1\frac{\partial \Omega_2}{\partial u_2} - c\Omega + W = 0. \tag{3.19}$$

Eliminating Ω from (3.18) and (3.19), we get

$$cc_1\frac{\partial \Omega_1}{\partial u_1} + cc_2\frac{\partial \Omega_2}{\partial u_2} + (1+2cH-c^2)W + (1+2cH)\left(c_2\frac{\partial \Omega_1}{\partial u_1} - c_1\frac{\partial \Omega_2}{\partial u_2}\right) = 0.$$

Substituting $W = f + g$ and (3.17) in the above equality, we have

$$\begin{aligned} &\left((1+2cH) + c\frac{c_1}{c_2}\right)f'' + \left((1+2cH) - c^2\right)f \\ &= \left(c\frac{c_2}{c_1} - (1+2cH)\right)g'' + \left(c^2 - (1+2cH)\right)g = d = const. \end{aligned} \tag{3.20}$$

From Corollary 2.5 we know that $1 + 2cH - c^2 < 0$.

It follows from (3.20) that

$$\tilde{f} = f - \frac{d}{1+2cH-c^2}, \quad \tilde{g} = g + \frac{d}{1+2cH-c^2}, \tag{3.21}$$

where $\tilde{f}$ and $\tilde{g}$ are solutions of (3.4), respectively. Therefore

$$W = \tilde{f} + \tilde{g} = f + g, \quad \Omega_1 = \frac{1}{c_2}\tilde{f}', \quad \Omega_2 = -\frac{1}{c_1}\tilde{g}'. \tag{3.22}$$

From (3.13), (3.17) and (3.19) it follows that

$$\Omega = \frac{1}{c}(f'' + g'' + f + g) = \frac{1}{c}(\tilde{f}'' + \tilde{g}'' + \tilde{f} + \tilde{g}). \tag{3.23}$$

Thus, from the expression of $\tilde{X}_c$, we can assume without loss of generality that $d = 0$. Introducing (3.3) into (3.24), we find

$$\Omega = \frac{c_1}{c_2}f - \frac{c_2}{c_1}g. \tag{3.24}$$

Observe that

$$\sum_{i=1}^{2} \Omega_i e_i = \frac{1}{c_1 c_2}(f' X_{u_1} - g' X_{u_2}). \tag{3.25}$$

Hence, we conclude from (3.22) that $\tilde{X}_c$ is given by (3.2).

Substituting (3.13), (3.15), and (3.24) into (3.12), we know that (2.16) reduces to (3.5). From Theorem 2.3 we know that $\tilde{X}_c$ is defined on $\tilde{U} = \{(u_1, u_2) \in R^2 | T^2 + 2TQH + Q^2(K-1) \neq 0\}$. Since $\alpha = -H,\ \beta = 1,\ \gamma = K = 0$, we have $T = -H\Omega^2,\ Q = \Omega^2$, which shows that $\tilde{U}$ reduces to (3.6).

If $1 + 2cH - c^2 = 0$, that is, $c = -c_1/c_2, or\ c = c_2/c_1$, then, $\tilde{X}_c$ is a torus. In fact, let us consider first the case that $c = -\frac{c_1}{c_2}$. It then follows from (3.20) that $g'' = 0$ and so,

$$g(u_2) = au_2 + b, \tag{3.26}$$

where a and b are constants. Since

$$\frac{\partial \Omega}{\partial u_2} = c_2 \Omega_2 = -\frac{c_2}{c_1} g' = -\frac{c_2}{c_1} a,$$

we know that

$$\Omega = l(u_1) - \frac{c_2}{c_1} a u_2 + \tilde{b},\ \ \tilde{b} \in I\!R. \tag{3.27}$$

Using $\frac{c_1}{c_2} f' = \frac{\partial \Omega}{\partial u_1} = l'(u_1)$, we have $l(u_1) = \frac{c_1}{c_2} f(u_1) + \tilde{d}$, which gives

$$\Omega = \frac{c_1}{c_2} f(u_1) - \frac{c_2}{c_1} a u_2 + (\tilde{d} + \tilde{b}). \tag{3.28}$$

Hence, from (3.17) and (3.19),

$$f'' + f + g + \frac{c_1}{c_2}\left(\frac{c_1}{c_2} f(u_1) - \frac{c_2}{c_1} a u_2 + (\tilde{d} + \tilde{b})\right) = 0, \tag{3.29}$$

which implies that

$$f'' + \frac{1}{c_2^2} f = -\frac{c_1}{c_2}(\tilde{d} + \tilde{b}) - b \equiv d_1. \tag{3.30}$$

Therefore,

$$f = b_1 \cos\left(\frac{u_1}{c_2}\right) + b_2 \sin\left(\frac{u_1}{c_2}\right) + c_2^2 d_1, \tag{3.31}$$

$$\Omega = \frac{c_1}{c_2} f - \frac{c_2}{c_1}(g + d_1). \tag{3.32}$$

It is easy to see that Ω satisfies equation (3.8). On the other hand, we deduce from the algebraic condition (2.21) that $a = b_1 = b_2 = 0$, which gives $f = c_2^2 d_1,\ g = b$. Thus (3.13) and (3.32) imply $\Omega = -\frac{c_2^3}{c_1} d_1 - \frac{c_2}{c_1} b$ and $W = c_2^2 d_1 + b$. Therefore, we obtain from (3.13) that

$$\tilde{X}_c = (r_1 \cos u_1, r_1 \sin u_1, r_2 \cos u_2, r_2 \sin u_2), \tag{3.33}$$

where

$$r_i = \left(1 - \frac{\Omega}{c(W - H\Omega)}\right) c_1 + (-1)^i \frac{W c_2}{c(W - H\Omega)},\quad i = 1, 2,$$

are constants. Consequently, $\tilde{X}_c$ is a torus. In the same way, we can show that when $c = \frac{c_2}{c_1}$, $\tilde{X}_c$ is also a torus.

Theorem 3.2. *The surfaces $\tilde{X}_c$ obtained in Theorem 3.1 are complete cmc H surfaces contained in the unit sphere S^3. Excluding the torus and up to an isometry of S^3, $\tilde{X}_c$ is given by (3.1) where $c > c_2/c_1$,*

$$f = \delta\frac{c_2}{c_1}\sqrt{\frac{\nu}{-\eta}}\cosh(\sqrt{-\eta}u_1), \quad g = \sin(\sqrt{\nu}u_2), \tag{3.34}$$

$$\eta(c) = 1 - c\frac{c_1}{c_2}, \quad \nu(c) = 1 + c\frac{c_2}{c_1}, \quad \delta = \pm 1, \ (u_1, u_2) \in I\!R^2. \tag{3.35}$$

Proof. The functions f and g of the surfaces described by (3.2) are given by

$$f = a_1\cos(\sqrt{\eta}u_1) + b_1\sin(\sqrt{\eta}u_1), \quad g = a_2\cosh(\sqrt{\nu}u_2) + b_2\sinh(\sqrt{\nu}u_2), \quad \text{if} \quad c < -\frac{c_1}{c_2}, \tag{3.36}$$

and

$$f = a_1\cosh(\sqrt{-\eta}u_1) + b_1\sinh(\sqrt{-\eta}u_1), \quad g = a_2\cos(\sqrt{\nu}u_2) + b_2\sin(\sqrt{\nu}u_2), \quad \text{if} \quad c > \frac{c_2}{c_1}, \tag{3.37}$$

where ν and η are functions of c defined by (3.35).

From (3.4) it follows that the constants $a_i, \ b_i, \ i = 1, 2$ satisfy the following conditions:

$$\frac{1}{c_2^2}\eta(a_1^2 + b_1^2) + \frac{1}{c_1^2}\nu(a_2^2 - b_2^2) = 0, \quad \text{if } c < -\frac{c_1}{c_2}, \tag{3.38}$$

$$\frac{1}{c_2^2}\eta(b_1^2 - a_1^2) + \frac{1}{c_1^2}\nu(a_2^2 + b_2^2) = 0, \quad \text{if } c > \frac{c_2}{c_1}. \tag{3.39}$$

Consider the case $c < -\frac{c_1}{c_2}$. If $a_1 = b_1 = 0$, then f and g are given by

$$f = 0, \quad g = a_2\left(\cosh(\sqrt{-\nu}u_2) \pm \sinh(\sqrt{-\nu}u_2)\right). \tag{3.40}$$

From (3.2) and (3.40), we get

$$\tilde{X}_c = (c_1\cos u_1, c_1 \sin u_1, c_2\cos(u_2 \pm A), c_2 \sin(u_2 \pm A)),$$

which implies that $\tilde{X}_c$ is a torus. Therefore, excluding the torus, we can assume $a_1^2 + b_1^2 \neq 0$. In this case, using (3.36), (3.38) and (3.2), one can easily see that

$$f = \sin\left(\sqrt{\eta}u_1 + A_1\right), \quad g = \delta\frac{c_1}{c_2}\sqrt{\frac{\eta}{-\nu}}\cosh\left(\sqrt{-\nu}u_2 + B_1\right), \tag{3.41}$$

where

$$\sin A_1 = \frac{a_1}{\sqrt{a_1^2 + b_1^2}}, \quad \cos A_1 = \frac{b_1}{\sqrt{a_1^2 + b_1^2}},$$
$$\cosh B_1 = \frac{\delta a_2}{\sqrt{a_2^2 - b_2^2}}, \quad \sinh B_1 = \frac{\delta b_2}{\sqrt{a_2^2 - b_2^2}},$$

and $\delta = \pm 1$, so that $\delta a_2 > 0$.

Now suppose that $c > \frac{c_2}{c_1}$. By similar arguments, we know that when $a_2 = b_2 = 0$, $\tilde{X}_c$ is a torus and when $a_2^2 + b_2^2 \neq 0$, we have

$$f = \delta \frac{c_2}{c_1} \sqrt{\frac{\nu}{-\eta}} \cosh\left(\sqrt{-\eta} u_1 + A_2\right), \quad g = \sin\left(\sqrt{\nu} u_2 + B_2\right), \tag{3.42}$$

where $\delta = \pm 1$, so that $\delta a_1 > 0$.

Since

$$\Omega = \frac{c_1}{c_2} f - \frac{c_2}{c_1} g,$$

from the expressions of f and g we conclude that, for any $c \in (-\infty, -c_1/c_2) \cup (c_2/c_1, \infty)$, $\tilde{U} = I\!\!R^2$, where $\tilde{U}$ is given by (3.6), that is, $\tilde{X}_c$ is defined on the whole plane.

Observe that when $c < -\frac{c_1}{c_2}$, we have

$$\tilde{X}_{c,A_1,B_1}(u_1, u_2) = R_{\left(-\frac{A_1}{\sqrt{\eta}}, -\frac{B_1}{\sqrt{-\nu}}\right)} \tilde{X}_{c,0,0} \circ h(u_1, u_2),$$

where

$$h(u_1, u_2) = \left(u_1 + \frac{A_1}{\sqrt{\nu}}, u_2 + \frac{B_1}{\sqrt{-\nu}}\right),$$

and $R_{(\theta,\phi)} : I\!\!R^4 \to I\!\!R^4$ is given by

$$R_{(\theta,\phi)} : (x_1, x_2, x_3, x_4) \to$$
$$(x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta, x_3 \cos\phi - x_4 \sin\phi, x_3 \sin\phi + x_4 \cos\phi).$$

Therefore, all the *cmc* H surfaces $\tilde{X}_{c,(A_1,B_1)}$, $c \in (-\infty, -c_1/c_2)$ with different values of A, B are congruent. Similarly, the *cmc* H surfaces $\tilde{X}_{c,A_2,B_2}, c \in (c_2/c_1, \infty)$ with different values of A_2, B_2 are also congruent.

Therefore, without loss of generality we may consider in (3.41) and (3.42)

$$A_1 = B_1 = A_2 = B_2 = 0. \tag{3.43}$$

We will now prove that the surfaces $\tilde{X}_c$ and $\tilde{X}_{2H-c}$ are congruent. From Remark 2.3, the metric of $\tilde{X}_c$ is given by

$$ds_c^2 = \tilde{\omega}_1^2 + \tilde{\omega}_2^2 = \sum_{i=1}^{2} \frac{1}{P^2} (\gamma W^2 - \alpha\Omega^2 + (\beta\Omega + 2\gamma W)\Omega\lambda_i)^2 (|X_{u_i}| du_i)^2,$$

where

$$\alpha = -H = -\frac{1}{2}\left(\frac{c_2}{c_1} - \frac{c_1}{c_2}\right), \; \beta = 1, \gamma = 0,$$
$$P = \alpha\Omega^2 + \Omega W = -H\Omega^2 + \Omega W.$$

Therefore, we deduce from $\lambda_1 = -c_2/c_1, \; \lambda_2 = c_1/c_2$ that

$$ds_c^2 = \frac{(c_1^2 f_c - c_2^2 g_c)^2}{(c_1^2 f_c + c_2^2 g_c)^2} (c_1^2 du_1^2 + c_2^2 du_2^2) \equiv \psi_c^2 (c_1^2 du_1^2 + c_2^2 du_2^2), \tag{3.44}$$

where f_c and g_c are given by (3.41) and (3.42) with (3.43). Moreover, it follows from (1.54) that

$$\tilde{\lambda}_c^1 = -H - \frac{1}{2c_1 c_2 \psi_c^2}, \qquad \tilde{\lambda}_c^2 = -H + \frac{1}{2c_1 c_2 \psi_c^2}. \tag{3.45}$$

Observe that $c > \frac{c_2}{c_1}$, if and only if, $2H - c < -\frac{c_1}{c_2}$ and

$$\eta(2H-c) = \frac{c_1^2}{c_2^2}\nu(c), \qquad \nu(2H-c) = \frac{c_2^2}{c_1^2}\eta(c).$$

Therefore, if $c > \frac{c_2}{c_1}$, we have

$$f_{2H-c}(u_1) = g_c(\tilde{u}_2), \quad g_{2H-c}(u_2) = \frac{c_1^4}{c_2^4} f_c(\tilde{u}_1), \quad \tilde{\lambda}^1_{2H-c} = -\tilde{\lambda}^2_c(\tilde{u}_1, \tilde{u}_2),$$

where $\tilde{u} = \frac{c_2}{c_1}u_2$ and $\tilde{u}_2 = \frac{c_1}{c_2}u_1$. We conclude that the $\tilde{X}_c(u_1, u_2)$ and $\tilde{X}_{2H-c}\left(\frac{c_2}{c_1}u_2, \frac{u_1}{u_2}u_1\right)$ have the same first and second fundamental forms. Therefore they are congruent by an isometry of S^3.

Let us now show each $\tilde{X}_c$ is complete. We only need to consider $c > \frac{c_2}{c_1}$. In this case, it follows from (3.42) and (3.44) that

$$\left|\frac{c_1^2 f - c_2^2 g}{c_1^2 f + c_2^2 g}\right| \quad \geq \quad \frac{c_1^2|f| - c_2^2|g|}{c_1^2|f| + c_2^2|g|} \geq 1 - \frac{2c_2^2|g|}{c_1^2|f| + c_2^2|g|} \geq 1 - \frac{2c_2^2}{c_2^2 + \frac{c_2}{c_1}\sqrt{\frac{\nu(c)}{-\eta(c)}}} > 0.$$

Therefore, there exists an $r > 0$ such that $\psi_c(u_1, u_2) \geq r, \ \forall (u_1, u_2) \in R^2$. We need to show that any divergent curve in the surfaces $\tilde{X}_c$ has infinite length. Such a curve is of the form $\alpha(t) = \tilde{X}_c(u_1(t), u_2(t))$, where $\lim_{t\to\infty}(u_1^2(t) + u_2^2(t)) = \infty$. We assume that $(u_1')^2 + (u_2')^2 = 1$. The length of α is

$$l(\alpha) = \int_0^\infty \psi(\alpha(t))\sqrt{c_1^2(u_1')^2 + c_2^2(u_2')^2} \geq \int_0^\infty \min\{c_1, c_2\} \cdot r = \infty.$$

This shows that $\tilde{X}_c$ is complete and so concludes the proof of Theorem 3.2. □

The following result provides some geometric properties of the surfaces.

Proposition 3.3. *Any cmc surface of the family $\tilde{X}_c$ obtained in Theorem 3.2 has flat total curvature. In particular, for any $c > \frac{c_2}{c_1}$, such that $\sqrt{1 + c\frac{c_2}{c_1}} = \frac{n}{m}$ is an irreducible rational number, the surface corresponds to a cmc immersion of a cylinder in S^3.*

Proof. We know that the principal curvatures $\tilde{\lambda}^i_c, i = 1, 2$, of $\tilde{X}_c$ are given by (3.45). Hence the Gauss curvature of $\tilde{X}_c$ is given by $K_c = 1 + \tilde{\lambda}^1_c\tilde{\lambda}^2_c$, i.e.

$$K_c = \frac{1}{4c_1^2c_2^2}\left(1 - \frac{1}{\psi_c^4}\right), \tag{3.46}$$

where ψ_c^2 is defined in (3.44) as

$$\psi_c^2 = \frac{(c_1^2 f - c_2^2 g)^2}{(c_1^2 f + c_2^2 g)^2}.$$

If c is such that $\sqrt{\nu(c)} = \sqrt{1 + c\frac{c_2}{c_1}}$ is not a rational number we consider the regions

$$U_k = I\!R \times [kT, (k+1)T], \quad T = \frac{2\pi}{\sqrt{\nu(c)}}, \quad k \in \boldsymbol{Z}.$$

Then the total curvature of the surface is given by

$$\int_{I\!R^2} K_c \psi_c^2 c_1 c_2 du_1 du_2$$

which reduces to

$$\sum_{k=-\infty}^{\infty} \int_{U_k} \frac{1}{4c_1c_2}\left(\psi_c^2 - \frac{1}{\psi_c^2}\right) du_1 du_2$$

as a consequence of (3.46). From (3.34) and the expression of ψ_c we have that

$$\int_{kT}^{(k+1)T} \psi_c^2 du_2 = \int_0^T \psi_c^2 du_2 = \int_0^{\frac{T}{2}} \left(\psi_c^2 + \frac{1}{\psi_c^2}\right) du_2.$$

Hence for all k,

$$\int_{U_k} \left(\psi_c^2 - \frac{1}{\psi_c^2}\right) du_2 = 0.$$

Similarly, if c is such that $\sqrt{\nu(c)} = \frac{n}{m}$ ia an irreducible number, the total curvature of the surface is given by

$$\int_{I\!R\times[0,2m\pi]} K_c \psi_c^2 c_1 c_2 du_1 du_2.$$

Since

$$\frac{1}{4c_1c_2}\int_0^{2m\pi} \psi_c^2 du_2 = \int_0^T \psi_c^2 du_2 = \int_0^{\frac{T}{2}} \left(\psi_c^2 + \frac{1}{\psi_c^2}\right) du_2,$$

we conclude that the total curvature is flat.

We observe that whenever $\sqrt{\nu(c)} = \frac{n}{m}$, it follows from (3.34) that

$$\tilde{X}_c(u_1, u_2 + 2m\pi) = \tilde{X}_c(u_1, u_2)$$

and $2m\pi$ is the smallest period. Therefore, the surface corresponds to a *cmc* immersion of a cylinder in S^3. This completes the proof of Proposition 3.3. □

If in Theorems 3.1, 3.2 and 3.3, $c_1 = c_2 = 1/\sqrt{2}$, we have a family of minimal surfaces in S^3 locally associated to the Clifford torus by a Ribaucour transformation. More precisely, we have the following

Corollary 3.4. *Consider the Clifford torus T^2 in S^3 parametrized by*

$$X(u_1, u_2) = \frac{1}{\sqrt{2}}(\cos u_1, \sin u_1, \cos u_2, \sin u_2), \ (u_1, u_2) \in I\!R^2.$$

Excluding the Clifford torus and up to an isometry of S^3, a parametrized surface $\tilde{X}_c$, where $c \neq 0$, is a minimal surface locally associated to X by a Ribaucour transformation, if and only if, $c > 1$ and $\tilde{X}_c$ is given by

$$\begin{aligned}\tilde{X}_c \quad &= \quad \left(1 - \frac{f-g}{c(f+g)}\right) X - \frac{2}{c(f+g)}(f' X_{u_1} - g' X_{u_2}) \\ &\quad + \frac{1}{\sqrt{2}c}(-\cos u_1, -\sin u_1, \cos u_2, \sin u_2),\end{aligned}$$

where

$$f = \delta\sqrt{\frac{c+1}{c-1}}\cosh(\sqrt{c-1}\ u_1), \quad g = \sin(\sqrt{c+1}\ u_2), \ \delta = \pm 1, \ (u_1, u_2) \in I\!R^2.$$

Each surface of the family $\tilde{X}_c$ is a complete minimal surface in S^3, with flat total curvature. Moreover, when $\sqrt{1+c} = \frac{n}{m}$ is an irreducible rational number, the surface corresponds to a minimal immersion of a cylinder in S^3.

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