Profinite HNN-constructions

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Abstract

Let $\mathcal{C}$ be a class of finite groups closed under taking subgroups quotients and extensions. We use a pro-$\mathcal{C}$ analogue of the HNN-construction, to show that every virtually torsion free pro-$\mathcal{C}$ group G can be embedded in a pro-$\mathcal{C}$ group E such that every finite subgroup of E is – up to conjugation – contained in a finite subgroup of E isomorphic to the quotient G/F, where F is an open torsion free normal subgroup of G. Moreover the virtual cohomological dimensions of G and E coincide. As a by-result we provide a structure theorem for cyclic p-extensions of free pro-p groups.

1 Introduction

There are various embedding theorems in profinite group theory. A. Lubotzky and J. Wilson [6] proved a profinite analogue of the Higman, Neumann and Neumann theorem [7] asserting that every topologically countably generated profinite group embeds in a two generated profinite group. However, their construction does not allow to control the torsion. So Z. Chatzidakis [2] returned to the original construction of Higman, Neumann and Neumann to make it work in the profinite and pro-p cases to prove that one can embed a countably generated profinite (respectively, pro-p) group G in a two-generated profinite (respectively, pro-p) group E such that every torsion element in E is conjugate to an element in G. The same construction has been used in [11] to embed any cyclic subgroup separable group in a two generated cyclic subgroup separable group and in [3] to prove the existence of a 2-generated torsion free residually

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p-group whose pro-p completion contains every finite p-group. In the present paper we use an HNN-construction in the category of pro-$\mathcal{C}$ groups with the objective to diminish the torsion in a virtually torsion free pro-$\mathcal{C}$ groups. More precisely, we prove the following

Theorem 1 *Let $\mathcal{C}$ be a class of finite groups closed under forming subgroups, products, and, extensions. Let G be a virtually torsion free pro-$\mathcal{C}$ group and F a torsion free open subgroup of G. Then G can be embedded in a semidirect product $\tilde{G} = E \rtimes G/F$ such that every finite subgroup of $\tilde{G}$ is conjugate to a subgroup of G/F. Moreover, the cohomological dimensions of E and F coincide.*

This result is in the spirit of the Higman, Neumann and Neumann theorem which says that any countable group can be embedded in a countable group in which all elements of the same order are conjugate. However, merely replacing in it "group" by "profinite group" does not yield a profinite analogue. First, every infinite profinite group is non-countable. Secondly, a p-element of infinite order in a profinite group can not be conjugate to its p-power, since its image and the pth power of it in some finite quotient have different orders. So, a profinite version of the Higman Neumann Neumann result can be stated only for elements of finite order. However even then the profinite version of it does not hold in general (see the Example at the end of Section 3). Nevertheless, the profinite analogue of the Higman Neumann Neumann result is valid for virtually torsion free profinite groups.

Corollary 2 *Let G be a virtually torsion free profinite group. Then G embeds into a profinite group $\tilde{G}$ where all elements of the same finite order are conjugate. Moreover, the virtual homological dimensions, $vcd(G)$ and $vcd(\tilde{G})$, coincide.*

Our Theorem has in part been motivated by a result of C. Scheiderer [10] – a homological version reads as follows:

Theorem 3 *Let G be a profinite group of virtual cohomological dimension $d < \infty$ and suppose that G does not contain subgroups isomorphic to $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$. Let T be the set of all finite subgroups of G on which G acts from the right by conjugation. Then*

$$\oplus_{t \in T} H_n(G, \mathbb{F}_p[[tG]]) \longrightarrow H_n(G, \mathbb{F}_p)$$

is an isomorphism for all $n > d$.

Here tG is the set of all conjugates of the cyclic subgroup t isomorphic to $\mathbf{Z}/p\mathbf{Z}$ considered as a subset of the space of all subgroups of G equipped with subspace topology. Now it is desirable to apply Shapiro's lemma to express the homology of G in terms of the homologies of normalizers of finite subgroups, but one needs to do it continuously and that requires a continuous section $T/G \longrightarrow T$. Such a section does not always exists (see [9], example 5.6.9). For virtually free pro-p groups as well as for the Kurosh subgroup theorem the existence of a continuous section is even more important (see [12], [8], [4]). So it appears useful to embed G coherently into a profinite group with similar structure where the corresponding continuous section would exist. As an illustration we apply our embedding result to deduce the following

Theorem 4 *Let G be a pro-p group having a free pro-p subgroup F such that $G/F \cong C_{p^n}$ with n minimal with respect to this property. Then G embeds into a free pro-p product $\tilde{G} = C_{\tilde{G}}(C_p) \amalg H$ of a free pro-p group H and the centralizer $C_{\tilde{G}}(C_p)$ of a group C_p of order p. Moreover,*

(i) $\tilde{G}$ possesses a free pro-p subgroup $\tilde{F}$ such that $\tilde{G}/\tilde{F} \cong C_{p^n}$;

(ii) The quotient group $C_{\tilde{G}}(C_p)/C_p$ is likewise a free product of the centralizer of a subgroup of order p and a free pro-p factor and has a unique conjugacy class of maximal cyclic subgroups.

We shall freely use standard notations from profinite group theory following [9].

2 Preliminaries

In this paper $\mathcal{C}$ denotes a class of finite groups closed under taking subgroups, quotients and extensions.

Definition 5 A *boolean* or *profinite* space is, by definition, an inverse limit of finite discrete spaces, i.e., a compact, Hausdorff, totally disconnected topological space. Morphisms in the category of boolean spaces are continuous maps.

A profinite space X with a profinite group G acting continuously on it will be called a *G-space.*

Definition 6 For a virtually torsion free profinite group G let $Fin(G)$ be the set of its finite subgroups. As G is a projective limit of finite groups, $Fin(G)$ is the projective limit of the respective sets of finite subgroups – hence it carries a natural topology (the *subgroup topology*) – turning it into a boolean space. Equipped with this topology, $Fin(G)$ with G acting by conjugation becomes a G-space.

Definition 7 A *sheaf* of pro-$\mathcal{C}$ groups (over a profinite space X) is a triple $(\mathcal{G}, \gamma, X)$, where $\mathcal{G}$ and X are profinite spaces, and γ is a continuous map from $\mathcal{G}$ onto X, satisfying the following two conditions:

(i) for every $x \in X$, the fiber $\mathcal{G}(x) = \gamma^{-1}(x)$ over x is a pro-$\mathcal{C}$ group;

(ii) if $\mathcal{G}^2$ denotes the subspace of $\mathcal{G} \times \mathcal{G}$ consisting of pairs (g, h) such that $\gamma(g) = \gamma(h)$, then the map $\mu_{\mathcal{G}} : \mathcal{G}^2 \longrightarrow \mathcal{G}$, defined by $\mu_{\mathcal{G}}(g, h) := g^{-1}h \in \mathcal{G}(\gamma(g)) = \mathcal{G}(\gamma(h)) \subseteq \mathcal{G}$, is continuous.

If there is no danger of confusion we shall write $(\mathcal{G}, X)$ instead of $(\mathcal{G}, \gamma, X)$.

A *morphism* of sheaves of pro-$\mathcal{C}$ groups $(\alpha, \bar{\alpha}) : (\mathcal{G}, \gamma, X) \to (\mathcal{H}, \eta, Y)$ is a pair of continuous maps $\alpha : \mathcal{G} \longrightarrow \mathcal{H}$, $\bar{\alpha} : X \longrightarrow Y$ such that the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\alpha"] \arrow[d, "\gamma"] & \mathcal{H} \arrow[d, "\eta"] \\ X \arrow[r, "\bar{\alpha}"] & Y \end{tikzcd}$$

is commutative and, for all $x \in X$, the restriction $\alpha_x := \alpha_{|\mathcal{G}(x)}$ of α to the fiber $\mathcal{G}(x)$ is a homomorphism from $\mathcal{G}(x)$ to $\mathcal{H}(\bar{\alpha}(x))$.

In the special case when $Y = \{y\}$ consists of a single element set, we obtain with $H := \mathcal{H}(y)$ the definition of a *fiber morphism* $\alpha : \mathcal{G} \longrightarrow H$, of the sheaf $\mathcal{G}$ of pro-$\mathcal{C}$ groups to the pro-$\mathcal{C}$ group H. We shall say that α is a *fiber monomorphism* if α_x is injective for every $x \in X$.

The simplest example of a sheaf of pro-$\mathcal{C}$ groups is that of the *constant sheaf* $(G \times X, \mathrm{pr}_X, X)$, where G is some pro-$\mathcal{C}$ group and $\mathrm{pr}_X : G \times X \longrightarrow X$ is the projection. For every $x \in X$, the fiber $(G \times X)(x) = G \times \{x\}$ is isomorphic to G.

Next we introduce a pro-$\mathcal{C}$ analogue of the concept of an HNN-extension, (cf. [7], p. 180), by generalizing the concept of pro-$\mathcal{C}$ HNN-extension as described in 9.4 of [9]. Following R. Bieri [1] we shall term it a *pro-$\mathcal{C}$ HNN-group.*

Definition 8 Let H be a pro-$\mathcal{C}$ group and $\partial_0, \partial_1 : (\mathcal{G}, T) \to H$ fiber monomorphisms. A specialization into K consists of a homomorphism $\beta : H \longrightarrow K$ and a continuous map $\beta_1 : T \longrightarrow K$ such that for all $t \in T$ and $g \in \mathcal{G}(t)$ the equality $\beta(\partial_0(h)) = \beta_1(t)^{-1}\beta(\partial_1(h))\beta_1(t)$ is valid. We denote this situation by writing $(\beta, \beta_1) : (H, \mathcal{G}, T) \to K$.

The pro-$\mathcal{C}$ HNN-group is then a pro-$\mathcal{C}$ group G together with a specialization $(\upsilon, \upsilon_1) : (H, \mathcal{G}, T) \longrightarrow G$, with the following universal property: for every pro-$\mathcal{C}$ group K and every specialization $(\beta, \beta_1) : (H, \mathcal{G}, T) \to K$, there exists a unique homomorphism

$$\omega : G \longrightarrow K,$$

such that $\omega\upsilon_1 = \beta_1$ and $\beta = \omega\upsilon$. We shall denote G by $\mathrm{HNN}_{\mathcal{C}}(H, \mathcal{G}, T)$ or simply by $\mathrm{HNN}(H, \mathcal{G}, T)$ when there is no danger of confusion.

Let us compare our definition with [7], p.180 for injective β_1: First, H is the *base group.* Setting $A_t := \partial_0(\mathcal{G}(t))$ and $B_t := \partial_1(\mathcal{G}(t))$, a family $f := \{f_t :| \; t \in T\}$ of isomorphisms is induced setting $f_t(a_t) := \partial_1(g_t)$ for the unique $g_t \in \mathcal{G}(t)$ with $a_t = \partial_0(g_t)$. Thus, the family f satisfies $f_t(a_t) = a_t^t$ for all $a_t \in A_t$ and $t \in T$, and T plays the role of a space of *stable letters.* In fact, below we shall make use of the abstract HNN-group, and denote it by $\mathrm{HNN}^{abs}(H, \mathcal{A}, f, T)$. For T a singleton set, identifying $\mathcal{G}(t)$ with its image under ∂_0 and setting $f := \partial_1$, the definition of a pro-$\mathcal{C}$-HNN extension given in 9.4 in [9] is recovered.

Proposition 9 *Let H be a pro-$\mathcal{C}$ group, $(\mathcal{G}, T)$ a sheaf of pro-$\mathcal{C}$ groups and $\partial_0, \partial_1 : (\mathcal{G}, T) \to H$ fiber monomorphisms. Then there exists a unique pro-$\mathcal{C}$ HNN-group $G = \mathrm{HNN}(H, \mathcal{G}, T)$.*

Proof: Uniqueness follows easily from the universal property. We give an explicit construction of G to prove its existence. As above consider the family f constituted of isomorphisms $f_t : \partial_0(\mathcal{G}(t)) \to \partial_1(\mathcal{G}(t))$. Form $G^{abs} := \mathrm{HNN}^{abs}(H, \mathcal{A}, f, T)$ and denote by $\varphi^{abs} : H \longrightarrow G^{abs}$ the natural embedding. Let $\mathcal{N}$ be the collection of all normal subgroups N of G^{abs} with $G^{abs}/N \in \mathcal{C}$, the preimage $(\varphi^{abs})^{-1}(N)$ open in H and continuous natural map $T \longrightarrow TN/N$. Define $G = \mathcal{K}_{\mathcal{N}}(G^{abs})$ to be the completion of G^{abs} with respect to $\mathcal{N}$. Let $\iota : G^{abs} \longrightarrow G$ be the natural homomorphism. Put $\varphi = \iota\varphi^{abs}$. We check the universal property for G and φ.

Let $(\beta, \beta_1) : (H, \mathcal{G}, T) \longrightarrow K$ be a specialization to some $K \in \mathcal{C}$. Then, by the universal property for abstract HNN-groups, there is a unique homomorphism $\omega^{abs} : G^{abs} \longrightarrow K$ with $\omega^{abs}(t) = \beta_1(t)$ such that the diagram

$$\begin{tikzcd} G^{abs} \arrow[dr, dotted, "\omega^{abs}"] & \\ H \arrow[u, "\varphi^{abs}"] \arrow[r, "\psi"] & K \end{tikzcd}$$

is commutative. It follows that $(\varphi^{abs})^{-1}(\ker(\omega^{abs})) = \ker(\psi)$ is open in H and ${\omega^{abs}}_{|T}$ is continuous, therefore since $K \in \mathcal{C}$, one has that $\ker(\omega^{abs}) \in \mathcal{N}$; Hence there exists a continuous homomorphism $\omega : G \longrightarrow K$ with $\omega^{abs} = \omega\iota$. Thus the diagram

$$\begin{tikzcd} G \arrow[ddr, dotted, "\omega"] & \\ G^{abs} \arrow[u, "\iota"] \arrow[dr, "\omega^{abs}"] & \\ H \arrow[u, "\varphi^{abs}"] \arrow[uu, bend left, "\varphi"] \arrow[r, "\psi"'] & K \end{tikzcd}$$

is commutative. This means that $\psi = \omega\varphi$ and $\omega(t) = \beta_1(t)$ for all $t \in T$. Uniqueness of ω follows from the fact that $G = \overline{\langle \varphi(H), \iota(T) \rangle}$. $\square$

A pro-$\mathcal{C}$ HNN-group is a special case of the fundamental pro-$\mathcal{C}$ group $\Pi_1(\mathcal{G}, \Gamma)$ of a profinite graph of pro-$\mathcal{C}$ groups $(\mathcal{G}, \Gamma)$ as introduced in [15]. Namely, a pro-$\mathcal{C}$ HNN-group can be thought as $\Pi_1(\mathcal{G}, \Gamma)$, where Γ is a bouquet (i.e., a connected profinite graph having just one vertex – an isolated point of Γ – that serves as a maximal subtree). Note that *acyclicity* and *simply connectivity* do not coincide in the pro-$\mathcal{C}$ situation, though they do when $\mathcal{C}$ consists of soluble groups only. The pro-$\mathcal{C}$ analogue of a maximal subtree is a maximal $\mathcal{C}$-simply connected subgraph. In general a maximal $\mathcal{C}$-simply connected subgraph in a connected profinite graph might not exist. When it exists the definition of the fundamental pro-$\mathcal{C}$ group $\Pi_1(\mathcal{G},\Gamma)$ of a graph of pro-$\mathcal{C}$ groups can be given along the lines of the abstract situation as has been done in [14], Section 3, for finite Γ.

The following is the only result in this note involving fundamental pro-$\mathcal{C}$ groups of profinite graphs of pro-$\mathcal{C}$ groups and is needed in the next section. We use notation from [15] during the proof.

Lemma 10 *Let $G = \mathrm{HNN}(H,\mathcal{G},T)$ be a pro-$\mathcal{C}$ HNN-group and U an open subgroup of G such that $U \cap \upsilon(\mathcal{G}(t)) = 1$ for all $t \in T$. Then U is a free pro-$\mathcal{C}$ product of conjugates $U \cap \upsilon(H)^g$, for certain $g \in G$ and a free pro-$\mathcal{C}$ group. In particular, if $U \cap \upsilon(H)$ is free, then so is U.*

Proof: Repeating the proof of Proposition 4.4 in [15] one obtains that U is the fundamental group $\Pi_1(\mathcal{U},\Delta)$ of a profinite graph of free pro-$\mathcal{C}$ groups with trivial edge groups. Then by the universal property one deduces that $\Pi_1(\mathcal{U},\Delta)$ is a free pro-$\mathcal{C}$ product of the vertex groups and the fundamental group $\pi_1(\Delta)$. The vertex groups are of the form $U \cap \upsilon(H)^g$, $g \in G$ and $\pi_1(\Delta)$ is free pro-$\mathcal{C}$ by Theorem 2.11 in [13]. The result follows.

3 Embedding

We shall need the following criterion for *embedding* a pro-$\mathcal{C}$ group H as a base group into a pro-$\mathcal{C}$ HNN-group $\mathrm{HNN}(H,\mathcal{G},T)$, whose proof is based on Zoé Chatzidakis' ideas [2]. Let $\partial_0, \partial_1 : (\mathcal{G},T) \to H$ be fiber monomorphisms, where the restriction of ∂_0 to A_t is the identity. Recall the family f of isomorphisms $f_t : A_t \to B_t$ as described in connection with Definition 8 and let us write φ for υ. If V is an open normal subgroup of H we write $G_V^{abs} := \mathrm{HNN}^{abs}(H/V, \mathcal{A}_V, f_V, T)$ for the abstract HNN-group, where $A_{tV} = A_tV/V$, $B_{tV} = B_tV/V$ are associated subgroups with isomorphisms $f_{tV} : A_tV/V \longrightarrow B_tV/V$ induced by f_t (and we use this notation omitting V if V is trivial). We also shall use the natural injection $\upsilon_{1V}^{abs} : T \to \mathrm{HNN}^{abs}(H/V, \mathcal{A}_v, T)$ arising from the abstract situation and let $\varphi^{abs} : H \to \mathrm{HNN}^{abs}(H,\mathcal{G},T)$ denote the canonical embedding.

Theorem 11 *The pro-$\mathcal{C}$ HNN-group $G := \mathrm{HNN}(H,\mathcal{G},T)$ is proper, i.e., the natural map $\varphi : H \longrightarrow G$ is mono, if and only if for every open normal subgroup U of H there exists an open normal subgroup V of H contained in U such that*

$$f_t(A_t \cap V) = B_t \cap V$$

and, the intersection of normal subgroups N with $G_V^{abs}/N \in \mathcal{C}$ and $({\upsilon_1}^{abs})^{-1}(gN) \cap T$ clopen in T for all $g \in G_V^{abs}$, is trivial.

Proof: It suffices to prove that $\ker(\varphi) = K$, where

$$K = \bigcap \{U \mid U \lhd_o H, f_t(A_t \cap U) = B_t \cap U, t \in T\}.$$

We adapt the proof of 9.4.3 [9].

Let $G^{abs} = \mathrm{HNN}^{abs}(H, \mathcal{A}, f, T)$ be the abstract HNN-group and identify H with its natural image in G^{abs}. Denote the collection of all normal subgroups N of G^{abs} with $G^{abs}/N \in \mathcal{C}$ such that $(\varphi^{abs})^{-1}(N)$ is open in H and $(\upsilon_1^{abs})^{-1}(gN)$ clopen in T for every $g \in G^{abs}$, by $\mathcal{N}$. From the explicit construction of $G = \mathrm{HNN}(H, \mathcal{A}, f, T)$ (compare the proof of Proposition 9) it follows that

$$\ker(\varphi) = \bigcap_{N \in \mathcal{N}} (N \cap H).$$

Since $N \cap H$ is an open normal subgroup of H for any $N \in \mathcal{N}$, we deduce from $f_t(A_t \cap N) = (A_t \cap N)^t = A^t \cap N = B_t \cap N$, that $K \leq \ker(\varphi)$.

Conversely, pick $1 \neq h \in H$. We shall construct an epimorphism η of G onto a $\mathcal{C}$-group such that $\eta(h) \neq 1$. Let U be an open normal subgroup of H, not containing h, such that $f_t(A_t \cap U) = B_t \cap U$ for all $t \in T$ and such that the set $\mathcal{N}_U$ of all normal subgroups N of $G_U^{abs} := \mathrm{HNN}^{abs}(H/U, \mathcal{A}_U, f_U, T)$ with $G_U^{abs}/N \in \mathcal{C}$ and $(\upsilon_{1U}^{abs})^{-1}(N) \cap T$ clopen in T for all $g \in G_U^{abs}$, intersects trivially. Then there exists a normal subgroup $N \in \mathcal{N}_U$ with $h \notin N$ and we denote by $\pi : G_U^{abs} \to G^{abs}/N$ the canonical epimorphism.

$$\begin{tikzcd}
G & & T \arrow[ll, "\upsilon_1"'] \arrow[rr, "\upsilon_{1U}^{abs}"] \arrow[dr, dashed, "\beta_1"] & & G_U^{abs} \arrow[dl, "\pi"] \\
& H \arrow[ul, "\varphi"] \arrow[rr, dashed, "\beta"'] \arrow[urrr, bend right=50, "\varphi_U^{abs}"'] & & G_U^{abs}/N & \\
\arrow[from=1-1, to=2-4, dotted, "\eta:=\omega_U"]
\end{tikzcd}$$

Define a specialization $(\beta, \beta_1) : (H, \mathcal{G}, T) \to G_U^{abs}/N$ by setting $\beta := \pi\varphi_U^{abs}$ and $\beta_1 := \pi\upsilon_{1U}^{abs}$, where φ_U^{abs} is the natural homomorphism. By the universal property of $G = \mathrm{HNN}(H, \mathcal{G}, T)$ there is continuous homomorphism $\omega_U : G \to G_U^{abs}/N$ which satisfies $(\beta, \beta_1) = \omega_U(\varphi, \upsilon_1)$. Set $\eta := \omega_U$, then, keeping in mind that φ identifies H with its image in G, one can see that $\eta(h) \neq 1$. $\square$

Let G be a pro-$\mathcal{C}$ group and T a closed subset of $Fin(G)$. Define a sheaf $(\mathcal{G}, T)$ putting $\mathcal{G} = \{(g, t) \in G \times T \mid g \in t\}$ and defining $\gamma : \mathcal{G} \longrightarrow T$ to be the restriction to $\mathcal{G}$ of the natural projection $G \times T \longrightarrow T$.

Theorem 12 *Let $G = F \rtimes K$ with F a torsion free normal subgroup and K a finite $\mathcal{C}$-group and let $(\mathcal{G}, T)$ be a sheaf as described above. Define fiber monomorphisms $\partial_0, \partial_1 : (\mathcal{G}, T) \to G$ putting ∂_0 to be the restriction of the natural projection $G \times T \longrightarrow G$ and ∂_1 to be the restriction of the map $G \times T \longrightarrow G$ which sends $((f, k), t) \in (F \rtimes K) \times T$ to k.*

Then $\tilde{G} := \mathrm{HNN}(G, \mathcal{G}, T)$ is virtually free pro-$\mathcal{C}$ and enjoys the following properties:

(i) G canonically embeds in $\tilde{G}$.

(ii) $\tilde{G} = \tilde{F} \rtimes K$ *is again a semidirect product where* $\tilde{F}$ *is a free pro-*$\mathcal{C}$*-product of copies of* F *and some free pro-*$\mathcal{C}$ *group.*

(iii) In $\tilde{G}$ *every finite subgroup is conjugate to a subgroup of* G*. Moreover every* A_t *is* $\tilde{G}$*-conjugate to a subgroup of* K*.*

Proof:

(i). We want to use Theorem 11, let G play the role of H there, and explore its notation. Let $U \leq F$ be an open normal subgroup of G. Since $B_t \leq K$ for all $t \in T$, all groups $\mathcal{G}(t)$ are finite. Then $U \cap A_t = \{1\}$ for all $t \in T$, and hence $\{1\} = f_t(A_t \cap U) = B_t \cap U$ holds. Define the quotient sheaf $(\mathcal{G}_U, \gamma_U, T)$ by setting $\mathcal{G}_U = \mathcal{G}/\partial_0^{-1}(U)$ and γ_U being induced by the projection from $\mathcal{G}$ onto T. Then fiber monomorphisms $\partial_{0U}, \partial_{1U} : (\mathcal{G}_U, T) \to G/U$ can be defined, giving rise to a family f_U of isomorphisms $f_{Ut} : A_{Ut} \longrightarrow B_{Ut}$ of associated subgroups of G/U, with $A_{Ut} := A_tU/U$ and $B_{Ut} := f_{Ut}(A_{Ut}) = B_tU/U$. Denote by G_U^{abs} the abstract HNN-extension $\mathrm{HNN}^{abs}(G/U, \mathcal{A}_U, f_U, T)$. In order to prove the assumptions of Theorem 11 to hold, it suffices to find for given $1 \neq x \in G_U^{abs}$ a normal subgroup N of G_U^{abs} with $x \notin N$, $(\upsilon_{1U}^{abs})^{-1}(gN)$ open in T for all $g \in G_U^{abs}$, and, $G_U^{abs}/N \in \mathcal{C}$.

For every open normal subgroup V of G with $V \leq U$ one can form the sheaf $(\mathcal{G}_{U,V}, T_V)$ with T_V the quotient space mod the relation $t \sim t'$ if and only if $tV = t'V$ and $\mathcal{G}_{U,V}(tV/V) := \mathcal{G}(t)$ (recall that elements of T are finite subgroups of G). Since f_U respects the clopen relation on T, it factors through a map $f_{U,V}$ and one may consider the HNN-extension $G_{U,V}^{abs} := \mathrm{HNN}(G/U, \mathcal{A}_{U,V}, f_{U,V}, T_V)$.

First we claim that there exists an open normal subgroup V of G with $V \leq U$ such that x does not belong to the kernel of χ_V, where χ_V is the canonical epimorphism from G_U^{abs} to $G_{U,V}^{abs}$. In fact, using the normal form theorem for HNN-extensions, one can write

$$x = g_0 t_1^{\epsilon_1} \cdots t_k^{\epsilon_k} g_k$$

with $t_i \in T$, $\epsilon_i = \pm 1$, $k \in \mathbf{N}$ and all g_i belonging to G/U. Now one can find an open normal subgroup V of G with $V \leq U$, such that $t_iV \neq t_jV$ holds, whenever $t_i \neq t_j$. Then, again by the normal form theorem, $\chi_V(x) = 1$ if and only if $x = 1$. Fix V for the rest of the proof.

By the universal property the natural epimorphism from $G \to K$ extends to a unique homomorphism $\omega_{U,V} : G_{U,V}^{abs} \longrightarrow K$. Since the kernel $\ker(\omega_{U,V})$ intersects G/U trivially, it is free, and so residually $\mathcal{C}$. Since $\mathcal{C}$ is extension closed, $G_{U,V}^{abs}$ is residually $\mathcal{C}$ so that Theorem 11 is applicable. In particular, there is a normal subgroup L of $G_{U,V}^{abs}$ with $G_{U,V}^{abs}/L \in \mathcal{C}$ and $\chi_V(x) \notin L$. Set $N := \chi_V^{-1}(L)$, then $G_U^{abs}/N \in \mathcal{C}$, the element x does not belong to N and, by construction, $(\upsilon_U^{abs})^{-1}(gN)$ is clopen in T for every $g \in G_U^{abs}$. Therefore the sheaf $(G, \mathcal{G}, T)$ indeed satisfies the assumptions of Theorem 11 and hence $\mathrm{HNN}(G, \mathcal{A}, f, T)$ is proper, as claimed.

(ii) By the universal property, there exists a unique homomorphism $\omega : \tilde{G} \longrightarrow K$ that extends the natural epimorphism $G \longrightarrow K$. By Lemma 10 $L := \ker(\omega)$ is free pro-$\mathcal{C}$ and therefore it serves as a candidate for $\tilde{F}$.

(iii) follows from the construction of $\tilde{G}$ and Theorem 5.6 in [15]. $\square$

Proof of Theorem 1:

Claim: One can assume G to be a semidirect product of a free pro-$\mathcal{C}$ group F with a finite group $K \in \mathcal{C}$.

Proof of the Claim: Put $K := G/F$, let $\pi : G \to K$ be the canonical projection, and let $L := G \amalg K$ denote the free pro-$\mathcal{C}$ product of the two groups. With $i_G : G \to L$ and $i_K : K \to G$ denoting the canonical embeddings, define the normal subgroup $\Phi := (i_G(g)i_K(\pi(g))^{-1} \mid g \in G)_L$. Observing that Φ intersects $im(i_K)$ trivially and $im(i_G)$ is in the free pro-$\mathcal{C}$ group $i_G(F)$, by the pro-$\mathcal{C}$ version of the Kurosh subgroup Theorem (see Theorem 9.1.9 in [9]) one concludes Φ to be free pro-$\mathcal{C}$, and certainly $L = \Phi \rtimes K$. Whence the Claim holds.

Thus, taking the Claim into account, we assume $G = F \rtimes K$ and keep in mind the meaning of i_G and i_K from its proof. We provide data for constructing a certain HNN-group along Definition 8. Take the boolean space $T := Fin(G)$ and define a sheaf $(\mathcal{G}, T)$ by setting $\mathcal{G}(S) := S$ for every $S \in T$, i.e., every finite subgroup S of G. Define fiber monomorphisms ∂_0, ∂_1 to be induced by the canonical embedding of any finite subgroup S of G in there and $i_K \circ \pi$ respectively. Now use Theorem 12 for providing the desired embedding. $\square$

In the proof to follow we are going to apply Theorem 2.2 in [5] to a pro-p group $\tilde{G}$ using the existence of a continuous section $T_{\tilde{G}}/\tilde{G} \longrightarrow T_{\tilde{G}}$, where $T_{\tilde{G}}$ is the space of subgroups of order p in $\tilde{G}$. In contrast to statement (ii) of the cited theorem a continuous section $T_G/G \longrightarrow T_G$ does not always exists (see [12]).

Proof of Theorem 4:

Let $\mathcal{C}$ be the class of all p-groups. Use Theorem 1 to construct a pro-p group $\tilde{G}$ such that statement (i) of the theorem holds. Then $T_{\tilde{G}}/\tilde{G}$ has cardinality 1 and therefore a continuous section $T_{\tilde{G}}/\tilde{G} \longrightarrow T_{\tilde{G}}$ obviously exists. So one can apply Theorem 2.2 (iii) of [5] to deduce that $\tilde{G} = C_{\tilde{G}}(C_p) \amalg H$ of a free pro-p group F and the centralizer $C_{\tilde{G}}(C_p)$ of a group C_p of order p. Induction applies in the following sense: when $L := C_{\tilde{G}}(C_p)/C_p$ is not free pro-p it has torsion, and hence has a unique conjugacy class of elements of order p. Then one can apply Theorem 2.2 (iii) of [5] to $\tilde{\tilde{G}}$ to see that it is again a free product of a centralizer of a subgroup of order p and some free pro-p factor. Also every finite subgroup of L can be seen to be conjugate of a subgroup of a suitable fixed maximal cyclic subgroup. $\square$

Proof of Corollary 2:

Use Theorem 1 when $\mathcal{C}$ is the class of all finite groups. For a pair of elements g, h of G/F of the same finite order construct the profinite HNN extension $H = \mathrm{HNN}(\tilde{G}, \langle g \rangle, t)$ with stable letter t conjugating g into h. Since associated subgroups $\langle g \rangle$ and $\langle h \rangle$ are finite, the condition of Theorem 11 is satisfied, so the HNN-extension is proper. Note that all torsion elements of H are still conjugate to elements of G/F (since they are conjugate to elements of the base group $\tilde{G}$

(see Theorem 5.6 in [15]). So we can repeat the construction until all pairs of non-conjugate elements of G/F of the same finite order are exhausted. $\square$

Example: Let S be an infinite set and $G = \prod_{s \in S} C_s$ the cartesian product of groups C_s of order 2. Then, when I denotes the set of all non-trivial involutions, observe that the unity element is a cluster point of I. Suppose G could be embedded into a profinite group where all involutions form a single conjugacy class C. Since C is compact and contains I one comes to the contradiction $1 \in C$.

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