Tits alternative for 3-manifold groups

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Abstract

Let M be an irreducible, orientable, closed 3-manifold with fundamental group G. We show that if the pro-p completion $\widehat{G}_p$ of G is infinite then G is either soluble-by-finite or contains a free subgroup of rank 2.

Introduction

In this short note we discuss Tits alternative for the fundamental group of an irreducible, orientable, closed 3-manifold M. As shown in [13, Thm. 2.9] if $dim_{\mathbb{F}_p} H_1(M, \mathbb{F}_p) \geq 4$ for some prime number p, where $\mathbb{F}_p$ is the field with p elements, then the fundamental group $\pi_1(M)$ has a free subgroup of rank 2. This result was further generalized in [12] where it was shown that if $dim_{\mathbb{F}_p} H_1(M, \mathbb{F}_p) \geq 3$ then either $\pi_1(M)$ is soluble-by-finite or contains a free subgroup of rank 2. The proof in [12] uses techniques from p-adic analytic groups [10] and Tits alternative for linear groups over fields of characteristics 0 [14]. The case when $\pi_1(M)$ has infinite abelianization follows from [6, Corollary 4.10] where it is shown that the fundamental group of a sufficiently large, irreducible, 3-manifold satisfies Tits alternative plus the fact that compact 3-manifolds M with infinite $H_1(M, \mathbb{Z})$ are sufficiently large [7, Lemma 6.6].

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Theorem. Let G be the fundamental group of an irreducible, orientable, closed 3-manifold. Assume that for some prime number p the pro-p completion $\widehat{G}_p$ of G is infinite. Then G satisfies Tits alternative, i.e. either G contains a free non-cyclic subgroup or G is soluble-by-finite.

We observe that this theorem can be viewed as a corollary of Thurston geometrization conjecture and Tits original result [14]. We give a short proof of our result following Parry's approach [12] and remark after it that there are alternative approaches one suggested by the referee and the other using some recent results on pro-p completions $\widehat{G}_p$ of orientable Poincare duality groups G of dimension 3 i.e. if every normal subgroup of p-power index in G has finite abelianization then $\widehat{G}_p$ is a pro-p Poincare duality group of dimension 3 (two independent, quite different proofs of this result can be found in [8] and [15]).

Proof.

We start the proof with the following

Lemma. Let G be the fundamental group of an irreducible, orientable, closed 3-manifold whose profinite completion $\widehat{G}$ is infinite. Then G either satisfies Tits alternative or every finitely generated non-trivial subgroup of infinite index in G is infinite cyclic.

Proof. Observe that since $\widehat{G}$ is infinite, the group G is infinite and hence G is a Poincare duality group of dimension 3 and torsion-free.

Suppose first that G does not contain a copy of $\mathbb{Z} \times \mathbb{Z}$. Then by [1, Cor.A1] every finitely generated non-cyclic subgroup of infinite index in G that does not decompose as a non-trivial free product has deficiency at least 2. By [2] every group of deficiency at least 2 has a subgroup of finite index that maps surjectively to a free group of rank 2. Every finitely generated subgroup of G that is a free product of two non-trivial groups contains the free non-cyclic group $\mathbb{Z} * \mathbb{Z}$. So the lemma holds in this case.

Suppose now that G contains a copy of $\mathbb{Z} \times \mathbb{Z}$. By [13, Prop. 2.6] if $\pi_1(M)$ has infinitely many distinct subgroups of finite index (which is the case since $\widehat{G}$ is infinite) and if $\pi_1(M)$ contains a copy of $\mathbb{Z}^2$ then either M contains an incompressible torus or M is a Seifert fibered space. In both cases M

is almost sufficiently large as defined in [13, p. 904] i.e. some finite-sheeted cover of M is sufficiently large. If M is sufficiently large (i.e. containing an incompressible, closed, connected orientable surface that is not a disc or a 2-sphere) with $\pi_2(M) = 0$ by [6, Cor. 4.10] $\pi_1(M)$ is either soluble or contains a free subgroup of rank 2. If M is a Seifert fibered, irreducible, closed 3-manifold the Tits alternative for $\pi_1(M)$ is established in the third paragraph of the proof of [12, Thm. 1.1]. □

Proof of the theorem.

Note that we can assume that for every subgroup U of finite index in G, $dim_{\mathbb{F}_p} H_1(U, \mathbb{F}_p) \leq 2$ and U has finite abelianization otherwise the theorem follows immediately from the results stated in the introduction. In particular the number of generators of any subgroup of finite index in $\widehat{G}_p$ is at most 2. Then $\widehat{G}_p$ is a pro-p group of finite rank at most 2 and $\widehat{G}_p$ has a subgroup of finite index that is uniformly powerful of rank ≤ 2 [10]. By going down to a subgroup of finite index in G we can assume that $\widehat{G}_p$ is uniformly powerful, hence a linear group in characteristic 0 [5, Thm. 8.20] and a pro-p Poincare duality group, so of finite cohomological dimension at most 2 and torsion-free. By the preceding lemma we can assume that every non-trivial subgroup of infinite index in G is isomorphic to $\mathbb{Z}$.

Finally following the last but one paragraph of [12, p.269] it is easy to see that G embeds in $\widehat{G}_p$. Indeed if there are $x, y \in G$ such that $x \neq 1$ is in the kernel of the canonical map $\pi : G \to \widehat{G}_p$, but $y \notin Ker(\pi)$, define D as the subgroup of G generated by x and y. If D has finite index in G then the closure $\overline{D}$ of $\pi(D)$ in $\widehat{G}_p$ has finite index and as $\overline{D}$ is procyclic, non-trivial and torsion-free $\widehat{G}_p$ is virtually $\mathbb{Z}_p$. Since $\widehat{G}_p$ is a torsion-free, virtually free pro-p group by Serre's result [11, Thm. 7.3.7(b)] $\widehat{G}_p \simeq \mathbb{Z}_p$. Hence G has a quotient $\mathbb{Z}$, a contradiction. If D has infinite index in G by the preceding lemma $D \simeq \mathbb{Z}$. Then for some $n > 0$, $y^n \in \langle x \rangle \subseteq Ker(\pi)$ and the image of y in $\widehat{G}_p$ is a non-trivial element of finite order, a contradiction as $\widehat{G}_p$ is torsion-free. Thus G embeds in the linear group $\widehat{G}_p$ and Tits result on linear groups [14] completes the proof.

Remarks. The following remarks give alternative proofs of the theorem, in particular simplifying the last paragraph of the above proof :

1. As mentioned in the introduction some recent results show that $\widehat{G}_p$ is a pro-p Poincare duality group of dimension 3, hence a pro-p group of finite

rank 3. But the first paragraph of the above proof shows that $\widehat{G}_p$ is pro-p group of finite rank ≤ 2, a contradiction.

2. Another approach was pointed out by the referee, who suggested to use the fact that p-adic analytic groups $\widehat{G}_p$ of rank at most 2 are soluble. The case of rank 1 is obvious so we can assume the rank is 2. Furthermore by Lemma the $Ker(\pi)$ is either trivial or infinite cyclic.

We provide the details of this argument and show that $\widehat{G}_p$ is metabelian-by-nilpotent. By going down to a subgroup of a p-power index we can assume that $\widehat{G}_p$ is uniformly powerful, hence a pro-p Poincare duality group of dimension 2. Such a group is a Demushkin group with two generators and by the classification of Demushkin groups [3], [4], [9], $\widehat{G}_p$ has a pro-p presentation with generators x_1, x_2 and one relator $[x_1, x_2]x_1^z$, where $z \in p^{\mathbb{N}=\{1,2,\dots\}} \cup \infty$ and by definition $x_1^\infty = 1$. A simple computation shows that the commutator subgroup of $\widehat{G}_p$ is a subset of the abelian pro-p subgroup of $\widehat{G}_p$ generated by x_1 and $[x_1, x_2]$, so $\widehat{G}_p$ is metabelian.

3. Alternatively the classification of Demushkin groups shows that they always have infinite abelianization, hence G has infinite abelianization, a contradiction.

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