APENDIX

Lemma 0.1. *Let $\Gamma = \Gamma_g = \langle x_i, y_i \mid \prod [x_i, y_i] = 1, i = 1, \dots, g\rangle$ be an orientable profinite surface group of genus g and*

$$\begin{array}{ccc} & & \Gamma \\ & & \downarrow f \\ A & \xrightarrow{\alpha} & B \end{array} \qquad (1),$$

an embedding problem admitting a weak solution $\varphi : U \longrightarrow A$ such that $\varphi(x_1) = \varphi(x_2) = \cdots = \varphi(x_{sn+s-1})$ and $\varphi(y_1) = \varphi(y_2) = \cdots = \varphi(y_{sn+s-1})$, where $n = |K||\varphi(\Gamma)|$, $j_l > j_k$ whenever $l > k$ and s is the minimal number of generators of K. Then (1) admits a proper solution.

Proof. We shall use the notation x^y for $y^{-1}xy$ in the argument to follow. Choose a minimal set of generators $k_1, \dots, k_s$ of K. Let η be a map that sends $x_1, x_2, \dots x_n$ to $\varphi(x_1)k_1$; $x_{n+2}, x_{n+3}, \dots x_{2n+1}$ to $\varphi(x_1)k_2$; $\dots$ $x_{n(s-1)+s-1}, x_{n(s-1)+s}, \dots x_{ns+s-1}$ to $\varphi(x_1)k_s$ and coincides with φ on the other generators. Then η extends to a homomorphism if

$$[\eta(x_1), \eta(y_1)] \dots [\eta(x_{g_i}), \eta(y_{g_i})] = 1$$

(since this would mean that the homomorphism from a free profinite group $F(x_1, y_1, \dots, x_{g_i}, y_{g_i}) \longrightarrow A$ extending η factors through U_i). Now putting

$$k_{10} := k_1^{-\varphi([x_1 y_1])} k_1^{\varphi(y_1)}, \dots k_{s0} := k_s^{-\varphi([x_{n(s-1)+s-1}, y_{n(s-1)+s-1}])} k_s^{\varphi(y_{n(s-1)+s-1})}$$

one has

$$\begin{gathered} [\eta(x_1), \eta(y_1)] \cdots [\eta(x_{g_i}), \eta(y_{g_i})] = \\ ([\varphi(x_1)k_1, \varphi(y_1)])^n [\varphi(x_{n+1}), \varphi(y_{n+1})]([\varphi(x_{n+2})k_2, \varphi(y_{n+2})])^n [\varphi(x_{2n+2}), \varphi(y_{2n+2})] \cdots \\ ([\varphi(x_{n(s-1)+s-1}k_s, \varphi(y_{n(s-1)+s-1}])^n [\varphi(x_{ns+s}), \varphi(y_{ns+s})] \cdots [\varphi(x_{g_i}), \varphi(y_{g_i})] = \\ ([\varphi(x_1), \varphi(y_1)]k_{10})^n, [\varphi(x_{n+1}), \varphi(y_{n+1})][\varphi(x_{n+2}), \varphi(y_{n+2})]k_{20})^n \cdots \\ [\varphi(x_{n(s-1)+s-1}), \varphi(y_{n(s-1)+s-1})]k_{s0})^n [\varphi(x_{sn+s}), \varphi(y_{sn+s})] \cdots [\varphi(x_{g_i}), \varphi(y_{g_i})] \end{gathered}$$

Then putting $b_1 = [\varphi(x_1), \varphi(y_1)]$, $\dots$, $b_s = [\varphi(x_{n(s-1)+s-1}), \varphi(y_{n(s-1)+s-1})]$ and taking into account that $b_1 = [\varphi(x_i), \varphi(y_i)]$,. . ., $b_s = [\varphi(x_{n(s-1)+s-1+i}), \varphi(y_{n(s-1)+s-1+i)})]$ for all $i = 1, \dots, n$ one has

$$\begin{gathered} [\eta(x_1), \eta(y_1)] \cdots [\eta(x_{g_i}), \eta(y_{g_i})] = \\ b_1 k_{10} k_{10}^{b_1^{-1}} k_{10}^{b_1^{-2}} \cdots k_{10}^{b_1^{-n}} b_1^{n-1} [\varphi(x_{n+1}), \varphi(y_{n+1})] \\ b_2 k_{20} k_{20}^{b_2^{-1}} k_{20}^{b_2^{-2}} \cdots k_{20}^{b_2^{-n}} b_2^{n-1} [\varphi(x_{2n+2}), \varphi(y_{2n+2})] \cdots \\ b_s k_{s0} k_{s0}^{b_s^{-1}} k_{s0}^{b_s^{-2}} \cdots k_{s0}^{b_s^{-n}} b_s^{n-1} [\varphi(x_{sn+s}), \varphi(y_{sn+s})] \\ \cdots [\varphi(x_{g_i}), \varphi(y_{g_i})]. \end{gathered}$$

Let $m = |B'|$ and $t = |K|$, so that $n = mt$. Then

$$k_{i0} k_{i0}^{b_i^{-1}} k_{i0}^{b_i^{-2}} \cdots k_{i0}^{b_i^{-n}} = (k_{i0} k_{i0}^{b_i^{-1}} k_{i0}^{b_i^{-2}} \cdots (k_{i0}^{b_i^{-m+1}}))^t = 1$$

so that

$$[\eta(x_1),\eta(y_1)]\cdots[\eta(x_{g_i}),\eta(y_{g_i})]=[\varphi(x_1),\varphi(y_1)]\cdots[\varphi(x_{g_i}),\varphi(y_{g_i})]=$$
$$b_1^n\varphi([x_{n+1},y_{n+1}]b_2^n[\varphi(x_{2n+2}),\varphi(y_{2n+2})]\cdots b_s^n[\varphi(x_{sn+s}),\varphi(y_{sn+s})]\cdots[x_{g_i},y_{g_i}])=$$
$$\varphi([x_1,y_1]\cdots[x_{g_i},y_{g_i}])=1$$

as needed.

Thus η extends to a homomorphism $\psi\colon U\longrightarrow A$ such that $\varphi=\alpha\psi$. But

$$\psi(x_1^{-1}x_{n+1})=k_1,\ldots,\psi(x_{n(s-1)+s-1}^{-1}x_{ns+s})=k_s$$

so ψ is an epimorphism and the lemma is proved. □

Lemma 0.2. *Let $\Gamma=\Gamma_g$ be a profinite surface group of genus g and N a projective subgroup of Γ. Let*

$$\begin{array}{ccc} & & N \\ & & \downarrow f \\ A & \xrightarrow{\alpha} & B \end{array} \qquad (2)$$

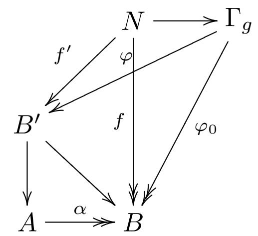
be an embedding problem, where A, B are finite. Then there exists an open subgroup U of Γ containing N and an embedding problem such that

$$\begin{array}{ccc} & & U \\ & & \downarrow \eta \\ A & \xrightarrow{\alpha} & B \end{array} \qquad (1),$$

satisfying hypothesis of Lemma 0.1 such that the restriction $\eta_{|N}=f$. Moreover, if N is accessible U can be chosen normal.

Proof. Since N is projective there exists a homomorphism $f':N\longrightarrow A$ such that $\alpha\varphi(N)=B$. Put $B'=f'(N)$.

By Lemma 8.3.8 in [RZ-2000] there exists an open subgroup U of Γ_g containing N and an epimorphism $\varphi:U\longrightarrow B'$ such that $\varphi_{|N}=f'$. Since an open subgroup of Γ_g is again a profinite surface group, replacing Γ_g by U we may assume the existence of the following commutative diagram:



(2),

where the top horizontal map is the natural inclusion. Moreover, as N is projective, 2 divides $[\Gamma_g:N]$ and so passing to an open subgroup of index 2 containing N if necessary, we may assume to be in oriented case. Let U_i be the family of all open subgroups of Γ_g containing N. Then $\varphi_i:=\varphi_{|U_i}$ is an epimorphism for every i. Note that every U_i is again a profinite surface group and so has a presentation $U_i=\langle x_1,y_1,\ldots x_{g_i},y_{g_i}\mid\prod_{j=1}^{g_i}[x_i,y_i]\rangle$, where the genus g_i of U_i can be computed by the formula $g_i-1=[\Gamma_g:U_i](g-1)$. This means that we can choose i with the number of generators of U_i sufficiently large, so that there exists i such that reordering generators x_j, y_j of U_i if necessary, we have $\varphi(x_1) = \varphi(x_{j_1}) = \cdots = \varphi(x_{j_{sn+s-1}})$ and $\varphi(y_1) = \varphi(y_{j_1}) = \cdots = \varphi(y_{j_{sn+s}})$, where $n = |K||B'|$ and $j_l > j_k$ whenever $l > k$ and s is the minimal number of generators of K. We shall use the notation x^y for $y^{-1}xy$ in the argument to follow. Suppose $j_1 \neq 2$. Then $\prod_{j=1}^{g_i}[x_j, y_j] = [x_1, y_1][x_{j_1}, y_{j_1}]([x_2, y_2] \cdots [x_{j-1}, y_{j-1}])^{[x_{j_1}, y_{j_1}]}[x_{j+1}, y_{j+1}] \cdots [x_{g_i}, y_{g_i}]$ so replacing the generators $x_2, y_2, \ldots, x_{j-1}, y_{j-1}$ by $x_2^{[x_{j_1}, y_{j_1}]}, y_2^{[x_{j_1}, y_{j_1}]}, \ldots x_{j-1}^{[x_{j_1}, y_{j_1}]}, y_{j-1}^{[x_{j_1}, y_{j_1}]}$ we may assume that $j_1 = 2$. Continuing similarly, we in fact may assume that $\varphi(x_1) = \varphi(x_2) = \cdots = \varphi(x_{sn+s})$ and $\varphi(y_1) = \varphi(y_2) = \cdots = \varphi(y_{sn+s})$. □

Theorem 0.3. *2.2 Let $\Gamma = \Gamma_g$ be a profinite surface group of genus g and N a projective accessible subgroup of Γ.*

Then N is isomorphic to an accessible subgroup of infinite index of a free profinite group.

Proof. By Theorem 2.1 we need to solve the following embedding problem for N:

$$\begin{array}{ccc} & & N \\ & & \downarrow f \\ A & \xrightarrow{\alpha} & B \end{array} \qquad (1),$$

where A, B are finite, $K := \mathrm{Ker}(\alpha)$ is minimal normal and $K \leq M(A)$.

By two preceding lemmas we have an open subgroup $N \leq U \leq \Gamma$ and an epimorphism $\psi : U \longrightarrow A$ such that $\alpha(\psi(N) = B$ and so $\psi(N)M(A) = A$. Since $\psi(N)$ is a subnormal subgroup of A by Proposition 8.3.6 in [RZ] $\psi(N) = A$ as needed. □