Profinite and pro-p completions of Poincaré duality groups of dimension 3

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Abstract

We establish some sufficient conditions for the profinite and pro-p completions of an abstract group G of type FP_m (resp of finite cohomological dimension, of finite Euler characteristics) to be of type FP_m over the field $\mathbb{F}_p$ for a fixed natural prime p (resp. of finite cohomological p-dimension, of finite Euler p-characteristics).

We apply our methods for orientable Poincaré duality groups G of dimension 3 and show that the pro-p completion $\widehat{G}_p$ of G is a pro-p Poincaré duality group of dimension 3 if and only if every subgroup of finite index in $\widehat{G}_p$ has deficiency 0 and $\widehat{G}_p$ is infinite. Furthermore if $\widehat{G}_p$ is infinite but not a Poincaré duality pro-p group then either there is a subgroup of finite index in $\widehat{G}_p$ of arbitrary large deficiency or $\widehat{G}_p$ is virtually $\mathbb{Z}_p$. Finally we show that if every normal subgroup of finite index in G has finite abelianization and the profinite completion $\widehat{G}$ of G has an infinite Sylow p-subgroup then $\widehat{G}$ is a profinite Poincaré duality group of dimension 3 at the prime p.

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Introduction

In this paper we study a relation between cohomology and homology of a group G and continuous cohomology and homology of its profinite and pro-p completions. The importance of such a study was observed by J.-P. Serre, who introduced the notion of a good group [17]. A group G is *good* if for every $i \geq 1$ the natural map $G \to \widehat{G}$ induces an isomorphism $H^i(\widehat{G}, M) \to H^i(G, M)$ between the cohomology $H^i(G, M)$ of G with coefficients in any finite G-module M and the (continuous) cohomology $H^i(\widehat{G}, M)$ of the profinite completion $\widehat{G}$ of G. In addition we say that G is *p-good* if for the pro-p completion $\widehat{G}_p$ the natural map $G \to \widehat{G}_p$ induces an isomorphism $H^i(\widehat{G}_p, M) \to H^i(G, M)$ for any finite p-primary G-module M and any $i \geq 1$.

Generally it is hard to check which groups are good. It is known that free groups, surface groups and a succession of extensions of finitely generated free groups are good [17, Chapter 1 §2.6 Exercise 2) (b)]. Recently it was proved that Bianchi groups are good [9]. However the answer to the classical question whether the mapping class groups are good is not known. Arithmetic groups that do not have the congruence subgroup property are not good. One of the main results of this paper (Theorem A) states that an orientable Poincaré duality group G of dimension 3 whose pro-p completion $\widehat{G}_p$ is infinite and all open subgroups of $\widehat{G}_p$ have deficiency 0 is always p-good.

Our methods apply to quite general class of completions $\widehat{G}_{\mathcal{C}} = \varprojlim G/U$ of an abstract group G, where the inverse limit is taken over a directed set $\mathcal{C}$ of normal subgroups of finite index in G. In section 2 we discuss some sufficient conditions for several important homological invariants of G (the homological type FP_m, the Euler characteristics, the cohomological dimension) to be preserved in the completion $\widehat{G}_{\mathcal{C}}$. Our sufficient conditions involve the inverse limits $\varprojlim H_i(U, \mathbb{F}_p)$ over $U \in \mathcal{C}$.

Our main applications are for the class of Poincaré duality groups of dimension 3, but many results hold beyond this class of groups. We have tried to state the results in their most general forms and treat profinite and pro-p completions (Theorem 3.2 and Theorem 4.1), still the results seem to be stronger when pro-p completions are studied. In section 3 we require that G is a group of cohomological dimension 3, and for every subgroup U of finite index $H_3(U, \mathbb{F}_p) \simeq \mathbb{F}_p$ for a fixed prime p, or in some results it will be sufficient that $H_3(U, \mathbb{F}_p) \simeq \mathbb{F}_p$ or 0. In both sections 3 and 4 it is assumed that the profinite completion of G has infinite Sylow p-subgroup or the pro-p completion of G is infinite depending on whether we study profinite or pro-p completions. But we do not require that G is a residually finite group or a residually finite p-group.

We discuss in the preliminaries profinite Poincaré duality group at a prime p together with other important properties as Euler p-characteristics and deficiency. Our main results for pro-p completions of orientable Poincaré duality groups of dimension 3 are the following theorems established in section 4.2.

Theorem A *Let G be an orientable Poincaré duality group of dimension 3. Assume that its pro-p completion $\widehat{G}_p$ is infinite. Then the following conditions are equivalent :*

a) *the homomorphism $\varphi_U : H_2(U, \mathbb{F}_p) \to H_2(\widehat{U}_p, \mathbb{F}_p)$ induced by the homomorphism $U \to \widehat{U}_p$ is an isomorphism for all normal subgroups U of p-power index in G, where $\widehat{U}_p$ is the pro-p completion of U, $H_2(U, \mathbb{F}_p)$ is the abstract and $H_2(\widehat{U}_p, \mathbb{F}_p)$ the profinite homology;*

b) *$\widehat{G}_p$ is an orientable pro-p Poincaré duality group of dimension 3;*

c) *every open subgroup of $\widehat{G}_p$ has deficiency 0;*

d) *G is a p-good group.*

Theorem B *Let G be an orientable Poincaré duality group of dimension 3 and $\widehat{G}_p$ be the pro-p completion of G. Then exactly one of the following conditions holds :*

a) *$\widehat{G}_p$ is finite;*

b) *$\widehat{G}_p$ is an orientable pro-p Poincaré duality group of dim 3;*

c) *there is no upper bound on the deficiency of the subgroups of finite index in $\widehat{G}_p$;*

d) *$\widehat{G}_p$ is infinite and the minimal upper bound on the deficiency of the subgroups of finite index in $\widehat{G}_p$ is one. In this case $\widehat{G}_p$ is virtually $\mathbb{Z}_p$.*

Remark. *If G is non-orientable but $p = 2$ then Theorem B still holds if we delete the condition orientable in* b).

The case of non-orientable Poincaré duality groups G looks much harder than the orientable case for p odd. The following example shows that for p odd the pro-p completion $\widehat{G}_p$ of G can be $\mathbb{Z}_p \times \mathbb{Z}_p$, so a Poincaré duality group of dimension 2, e.g. $G = H \times \mathbb{Z}$, where H is the non-orientable Poincaré duality group of dimension 2 with a presentation $\langle x, y \mid yxy^{-1} = x^{-1} \rangle$. This shows that Theorem B does not hold for non-orientable Poincaré duality groups of dimension 3.

We show in Corollary 4.2 that if G is an orientable Poincaré duality group of dimension 3 with all subgroups of p-power index having finite abelianisations then the pro-p completion $\widehat{G}_p$ is a pro-p Poincaré duality group of dimension 3. This generalizes Reznikov's statement about the pro-p completions of those 3-dimensional cocompact hyperbolic lattices which contradict Thurston conjecture [16]. In contrast to Reznikov's treatment our proofs are homological and much simpler.

Corollary 4.2 is generalized for profinite completions (a case not discussed by Reznikov) in Theorem C. Little is known for the profinite completion of abstract orientable Poincaré duality groups of dimension 3. By [10] the fundamental group of a Haken 3-manifold is residually finite. The group G from Theorem C cannot be the fundamental group of such a manifold.

Theorem C *Let G be an orientable Poincaré duality group of dimension 3. Assume that for a fixed prime p the profinite completion $\widehat{G}$ has an infinite Sylow p-subgroup and that every normal subgroup U of finite index in G has finite abelianization. Then $\widehat{G}$ is an orientable profinite Poincaré duality group of dimension 3 at p.*

In section 5 we discuss more corollaries. Except for Proposition 5.1 from section 5 we do not suppose that G is finitely presented. It is an open question whether there is an abstract Poincaré duality group of dimension 3 that is not finitely presented. But for any $n \geq 4$ there is a Poincaré duality group of dimension n that is not finitely presented [5].

Throughout this paper p always denotes a fixed prime number. If not otherwise stated Ext and Tor are the functors of abstract modules (even if applied to completed group rings). For a group G we denote by $\widehat{G}_p$, $\widehat{G}$ and $\widehat{G}_{\mathcal{C}}$ the pro-p completion, the profinite completion and the inverse limit $\varprojlim G/U$ over $U \in \mathcal{C}$. If not stated otherwise all modules considered are right modules.

Remark. The authors have just learnt that by using different methods Th. Weigel has found an independent proof of Corollary 4.2 and as well of Theorem C under the additional hypothesis that the pro-p completion of G is infinite. We thank him for sending his preprint [21].

1 Preliminaries

1.1 Type FP_m for abstract and profinite modules

We recall the notion of type FP_m for modules and groups. Let G be an abstract group and B a $\mathbb{Z}[G]$-module. For $0 \leq m \leq \infty$ we say that B is of type FP_m if there exists a projective $\mathbb{Z}[G]$-resolution of B

$$\mathcal{R} : \ldots \to R_i \to R_{i-1} \to \ldots \to R_0 \to B \to 0$$

with all R_i finitely generated for $i \leq m$. One says that G is of type FP_m if the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ is of type FP_m.

Now let G be a profinite group and B a profinite $\mathbb{Z}_p[[G]]$-module (resp. $\mathbb{F}_p[[G]]$-module). One says that B is of type FP_m over $\mathbb{Z}_p$ (resp. $\mathbb{F}_p$) if there exists a profinite projective $\mathbb{Z}_p[[G]]$-resolution (resp. $\mathbb{F}_p[[G]]$-resolution) of B

$$\mathcal{R} : \ldots \to R_i \to R_{i-1} \to \ldots \to R_0 \to B \to 0$$

with all R_i finitely generated for $i \leq m$. One says that G is of homological type FP_m over $\mathbb{Z}_p$ (resp. $\mathbb{F}_p$) if the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ (resp. the trivial $\mathbb{F}_p[[G]]$-module $\mathbb{F}_p$) is of type FP_m.

The following simple lemma will be used many times in this paper.

Lemma 1.1. *Let p be a prime number, R the ring $\mathbb{Z}_p$ or $\mathbb{F}_p$ and H a profinite group. Then*

a) every finitely generated abstract projective $R[[H]]$-module P is a profinite projective $R[[H]]$-module under the same action of $R[[H]]$;

b) if the abstract trivial $R[[H]]$-module R is of type FP_m then the profinite group H is of type FP_m over R;

c) if the abstract trivial $\mathbb{Z}_p[[H]]$-module $\mathbb{Z}_p$ has a projective resolution $\mathcal{P}$ of finite length m such that all projective modules are finitely generated then the cohomological p-dimension $cd_p(H) \leq m$.

d) if the abstract trivial $R[[H]]$-module R has type FP_m then for any finite discrete $R[[H]]$-module M and $i \leq m-1$ there is a natural isomorphism between the functor of abstract modules $Ext^i_{R[[H]]}(R, M)$ and the Galois cohomology $H^i(H, M)$;

e) if the abstract trivial $R[[H]]$-module R has type FP_m then for any profinite left $R[[H]]$-module N and $i \leq m-1$ there is a natural isomorphism between the functor of abstract modules $Tor_i^{R[[H]]}(R, N)$ and the profinite homology $H_i(H, N)$.

Proof. a) There is a finitely generated abstract free $R[[H]]$-module F such that P is a direct summand of F as an abstract module i.e. $F = P \oplus P'$. Note that F is also a profinite $R[[H]]$-module because it is a finite direct sum of copies of $R[[H]]$. By [22, Lemma 7.2.2] every abstract homomorphism between profinite finitely generated $R[[H]]$-modules is continuous. Then the map $\varphi : F \to F$, that is identity on P and zero on P', is continuous. In particular $Im(\varphi) = P$ is a profinite $R[[H]]$-module and a direct summand of the free profinite $R[[H]]$-module F, hence P is a profinite projective $R[[H]]$-module.

b) By part a) the m-th skeleton of any projective resolution of R as an abstract $R[[H]]$-module with finitely generated modules in dimensions $\leq m$ has only continuous maps, hence the profinite group H is of type FP_m over R.

c) By part a) the modules in the complex $\mathcal{P}$ are profinite $\mathbb{Z}_p[[H]]$-modules. By [22, Lemma 7.2.2] the homomorphisms of $\mathcal{P}$ are continuous, hence $\mathcal{P}$ is a projective profinite resolution of $\mathbb{Z}_p$ as a profinite $\mathbb{Z}_p[[H]]$-module. The cohomological p-dimension $cd_p(H)$ is at most the length of $\mathcal{P}$ as the proof of [15, Prop. 7.1.4] shows that in [15, Prop. 7.1.4(e)] every appearance of $\mathbb{F}_p$ can be substituted by $\mathbb{Z}_p$.

d),e) Let $\mathcal{P}$ be a projective resolution of the trivial abstract $R[[H]]$-module R with finitely generated projective modules in dimension $\leq m$. By a) and b) the m-skeleton $\mathcal{P}^{(m)}$ of $\mathcal{P}$ is a partial profinite resolution of the trivial profinite $R[[H]]$-module R and can be used to calculate $H_i(H, N)$ and $H^i(H, M)$ for $i \leq m - 1$. In particular $Tor_i^{R[[H]]}(R, N) \simeq H_i(\mathcal{P} \otimes_{R[[H]]} N) \simeq H_i(\mathcal{P} \widehat{\otimes}_{R[[H]]} N) \simeq H_i(H, N)$ for $i \leq m-1$, where the middle isomorphism follows from the fact that the abstract $\otimes_{R[[H]]}$ and complete $\widehat{\otimes}_{R[[H]]}$ tensor products are naturally isomorphic if applied to profinite modules such that at least one of them is finitely generated. As before by [22, Lemma 7.2.2] the set of abstract $R[[H]]$-module homomorphisms from any finitely generated profinite $R[[H]]$-module (in particular P_i for $i \leq m$) to M is the set of all continuous module homomorphisms. Then $Ext^i_{R[[H]]}(R, M) \simeq H^i(Hom_{R[[H]]}(\mathcal{P}, M)) \simeq H^i(H, M)$ for $i \leq m - 1$. $\square$

1.2 Abstract and profinite Poincaré duality groups

There are two (equivalent) ways to define an abstract Poincaré duality group. In this paper we will mainly use Farrell's approach [7] i.e. G is a Poincaré duality group of dimension n if G is a group of type FP_∞, of cohomological dimension $cd(G) = n$ and $H^k(G, \mathbb{Z}[G]) = Ext^k_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}[G]) = 0$ for $k \neq n$ and $\mathbb{Z}$ for $k = n$. If the G-action on $H^n(G, \mathbb{Z}[G])$ is the trivial one, G is called orientable. Otherwise G is non-orientable and G acts on $H^n(G, \mathbb{Z}[G])$ via multiplication with ± 1. Equivalently the condition on $Ext^*(G, \mathbb{Z}[G])$ can be substituted with the existence of an isomorphism $H^i(G, M) \simeq H_{n-i}(G, D \otimes_{\mathbb{Z}} M)$ for all G-modules M and all i, where the dualizing module D is $H^n(G, \mathbb{Z}[G])$ [4, Ch. 8,Prop. 10.1].

There are two definitions of a profinite Poincaré duality group H at a prime p of dimension n [19], [14, 3.4.6]. The definitions differ at the point whether H should be of type FP_∞ over $\mathbb{Z}_p$. Still we do not know an example that satisfies the conditions of [14, 3.4.6] and is not of type FP_∞. In this paper we adopt the approach of [19].

In [19] the profinite duality groups H at p of dimension n are defined as groups of cohomological p-dimension $cd_p(H) = n$, of type FP_∞ over $\mathbb{Z}_p$ (in [19] groups of type FP_∞ over $\mathbb{Z}_p$ are called of type p-FP_∞) and $H^k(H, \mathbb{Z}_p[[H]]) = Ext^k_{\mathbb{Z}_p[[H]]}(\mathbb{Z}_p, \mathbb{Z}_p[[H]])$ is 0 for $k \neq n$ and for $k = n$ is p-torsion free. If in addition $H^n(H, \mathbb{Z}_p[[H]]) \simeq \mathbb{Z}_p$, H is called a Poincaré duality group at p of dimension n. Furthermore if the action of H on $H^n(H, \mathbb{Z}_p[[H]])$ is trivial H is called an orientable Poincaré duality group at p, otherwise it is non-orientable.

1.3 Euler characteristics and deficiency

For a finitely presented pro-p group H the deficiency $def(H)$ is defined as $\mid \widetilde{X} \mid - \mid \widetilde{R} \mid$, where $\langle \widetilde{X} \mid \widetilde{R} \rangle$ is a minimal presentation of H i.e. $\widetilde{X}$ is a minimal set of generators and $\widetilde{R}$ is a minimal set of relations for H such that $\widetilde{R}$ is a subset of a free pro-p group with basis $\widetilde{X}$. Note that the cardinality of $\widetilde{X}$ and $\widetilde{R}$ is $dim_{\mathbb{F}_p} H^1(H, \mathbb{F}_p) = dim_{\mathbb{F}_p} H_1(H, \mathbb{F}_p)$ and $dim_{\mathbb{F}_p} H^2(H, \mathbb{F}_p) = dim_{\mathbb{F}_p} H_2(H, \mathbb{F}_p)$ respectively. Thus $def(H) =$

$$dim_{\mathbb{F}_p} H^1(H, \mathbb{F}_p) - dim_{\mathbb{F}_p} H^2(H, \mathbb{F}_p) = dim_{\mathbb{F}_p} H_1(H, \mathbb{F}_p) - dim_{\mathbb{F}_p} H_2(H, \mathbb{F}_p).$$

Furthermore if $f : F \to H$ is an epimorphism of pro-p groups such that F is a free pro-p group of finite rank then $def(H)$ is the rank of F minus the minimal number of generators of $Ker(f)$ as a closed normal subgroup of F.

Let G be an abstract group of finite cohomological dimension and of type

FP_∞. The Euler characteristics $\chi(G)$ is defined by

$$\chi(G) = \sum_i (-1)^i rk_{\mathbb{Z}} Tor_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}) = \sum_i (-1)^i rank_{\mathbb{Z}} H_i(G, \mathbb{Z})$$

(cf. [4, Ch. IX, Sec. 6], where it is defined for the more general class of groups of finite cohomological type). Furthermore if

$$\mathcal{R} : 0 \to R_m \xrightarrow{\partial_m} R_{m-1} \xrightarrow{\partial_{m-1}} \ldots \xrightarrow{\partial_1} R_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

is a projective resolution of $\mathbb{Z}$ as an abstract $\mathbb{Z}[G]$-module of finite length and with all projective modules R_i finitely generated then

$$\chi(G) = \sum_i (-1)^i rk_{\mathbb{Z}}(R_i \otimes_{\mathbb{Z}[G]} \mathbb{Z}).$$

Since every R_i is finitely generated projective module, there is a free finitely generated $\mathbb{Z}[G]$-module F_i such that R_i is a direct summand of F_i. In particular $R_i \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ is a direct summand of the finite rank free abelian group $F_i \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, hence $R_i \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ is itself finite rank abelian group and $rk_{\mathbb{Z}}(R_i \otimes_{\mathbb{Z}[G]} \mathbb{Z}) = dim_{\mathbb{F}_p}(R_i \otimes_{\mathbb{Z}[G]} \mathbb{F}_p) = dim_{\mathbb{F}_p} Hom_{\mathbb{Z}[G]}(R_i, \mathbb{F}_p)$. Then

$$\chi(G) = \sum_i (-1)^i dim_{\mathbb{F}_p}(R_i \otimes_{\mathbb{Z}[G]} \mathbb{F}_p) =$$

$$\sum_i (-1)^i dim_{\mathbb{F}_p} H_i(G, \mathbb{F}_p) = \sum_i (-1)^i dim_{\mathbb{F}_p} H^i(G, \mathbb{F}_p).$$

If U is a subgroup of finite index in G by [4, Thm. 6.3, Ch. 9] $\chi(U) = (G : U)\chi(G)$.

Let G be an abstract Poincaré duality group of odd dimension n, hence $H^i(G, \mathbb{F}_p) \simeq H_{n-i}(G, D \otimes_{\mathbb{Z}} \mathbb{F}_p)$, where D is the dualizing module $H^n(G, \mathbb{Z}[G]) \simeq \mathbb{Z}$. It is easy to see that $\chi(G) = 0$. Indeed for an abstract orientable Poincaré duality group G_0 (hence $\mathbb{F}_p \simeq H^n(G_0, \mathbb{Z}[G_0]) \otimes_{\mathbb{Z}} \mathbb{F}_p$) of odd dimension n

$$2\chi(G_0) = \sum_i ((-1)^i dim_{\mathbb{F}_p} H_i(G_0, \mathbb{F}_p) + (-1)^{n-i} dim_{\mathbb{F}_p} H^{n-i}(G_0, \mathbb{F}_p)) = 0.$$

Since G has a subgroup G_0 of index ≤ 2 which is an orientable Poincaré duality group, one has $0 = \chi(G_0) = (G : G_0)\chi(G)$ and therefore $\chi(G) = 0$.

For a profinite group H of finite p-cohomological dimension $cd_p(H)$ and type FP_∞ over $\mathbb{Z}_p$ we define the Euler characteristics of H at p as

$$\chi_p(H) = \sum_i (-1)^i rk_{\mathbb{Z}_p} H_i(H, \mathbb{Z}_p) = \sum_i (-1)^i rk_{\mathbb{Z}_p} Tor_i^{\mathbb{Z}_p[[H]]}(\mathbb{Z}_p, \mathbb{Z}_p).$$

Then for a finite length profinite projective resolution $\mathcal{S}$ of a $\mathbb{Z}_p[[H]]$-module $\mathbb{Z}_p$ whose all projective modules are finitely generated

$$\chi_p(H) = \sum_i (-1)^i rk_{\mathbb{Z}_p} (S_i \otimes_{\mathbb{Z}_p[[H]]} \mathbb{Z}_p).$$

As in the abstract case $rk_{\mathbb{Z}_p}(S_i \otimes_{\mathbb{Z}_p[[H]]} \mathbb{Z}_p) = dim_{\mathbb{F}_p}(S_i \otimes_{\mathbb{Z}_p[[H]]} \mathbb{F}_p)$, hence

$$\chi_p(H) = \sum_i (-1)^i dim_{\mathbb{F}_p} (S_i \otimes_{\mathbb{Z}_p[[H]]} \mathbb{F}_p) =$$

$$\sum_i (-1)^i dim_{\mathbb{F}_p} Tor_i^{\mathbb{Z}_p[[H]]}(\mathbb{Z}_p, \mathbb{F}_p) = \sum_i (-1)^i dim_{\mathbb{F}_p} H_i(H, \mathbb{F}_p).$$

If H is a pro-p group the Euler characteristics $\chi(H)$ is defined as $\chi_p(H)$.

2 Completions of abstract groups of type FP_m

Let G be an abstract group of homological type FP_m over the ring $\mathbb{Z}$ for some $m \geq 1$, in particular G is finitely generated. Then there is a projective resolution of the trivial right $\mathbb{Z}[G]$-module $\mathbb{Z}$

$$\mathcal{R} : \ldots \longrightarrow R_i \xrightarrow{\partial_i} R_{i-1} \longrightarrow \ldots \xrightarrow{\partial_1} R_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

with all R_i finitely generated for $i \leq m$. Let $\mathcal{C}$ be a set of normal subgroups U of finite index in G such that $\mathcal{C}$ is directed in the sense that if $U_1, U_2 \in \mathcal{C}$ there is $U_3 \in \mathcal{C}$ with $U_3 \subseteq U_1 \cap U_2$. We define $\widehat{G}_{\mathcal{C}}$ as the inverse limit of G/U and $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$ as the inverse limit of $\mathbb{F}_p[G/U]$, when U runs through $\mathcal{C}$. For $U \in \mathcal{C}$ define $\mathcal{R}_U = \mathcal{R} \otimes_{\mathbb{Z}[U]} \mathbb{F}_p$, thus $\mathcal{R}_U$ is a complex (in general not exact) of projective $\mathbb{F}_p[G/U]$-modules and $\{\mathcal{R}_U\}_{U \in \mathcal{C}}$ is a surjective inverse system of complexes via the surjective maps $G/U_1 \to G/U_2$ for the groups $U_1 \subseteq U_2$ of $\mathcal{C}$. Note that

$$H_0(\mathcal{R}_U) = 0 \text{ and } H_i(\mathcal{R}_U) \simeq Tor_i^{\mathbb{Z}[U]}(\mathbb{Z}, \mathbb{F}_p) \simeq H_i(U, \mathbb{F}_p) \text{ for } i \geq 1.$$

As G is of type FP_m every subgroup of finite index in G is of type FP_m, in particular every $U \in \mathcal{C}$ is of type FP_m. This implies that $H_j(U, \mathbb{F}_p)$ is finite for every $j \leq m$.

Let $\widehat{\mathcal{R}}$ be the inverse limit of the inverse system of complexes $\{\mathcal{R}_U\}_{U \in \mathcal{C}}$. Observe that

$$\widehat{\mathcal{R}}^{(m)} \simeq \mathcal{R}^{(m)} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]], \tag{1}$$

where upper index (m) denotes the m-skeleton of the complex (i.e. all modules and homomorphisms up to dimension m). In dimension -1 the above isomorphism follows from the fact that $\widehat{G}_{\mathcal{C}}$ is topologically finitely generated, say by $x_1, \ldots, x_d$, hence the augmentation ideal of $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$ as an abstract right $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module is $\sum_{1 \leq i \leq d}(x_i - 1)\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$, hence $\widehat{R}_{-1} = \mathbb{F}_p \simeq \mathbb{Z} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$.

Denote by $\{\widehat{\partial}_i\}_{i \geq 0}$ and $\{\widehat{R}_i\}_{i \geq 0}$ the differentials and the modules of $\widehat{\mathcal{R}}$. By [20, Thm. 3.5.8] for every i there is an exact sequence

$$0 \to \varprojlim_{U \in \mathcal{C}}{}^1 H_{i+1}(\mathcal{R}_U) \to H_i(\widehat{\mathcal{R}}) \to \varprojlim_{U \in \mathcal{C}} H_i(\mathcal{R}_U) \to 0.$$

Since G is finitely generated the set of all normal subgroups of finite index in G is countable, so we can replace $\mathcal{C}$ by a totally ordered countable cofinal subset without changing the inverse limits above. By [20, Exer. 3.5.2] or the main result of [8] $\varprojlim^1$ of a tower (i.e. an inverse system indexed by totally ordered countable set) of finite dimensional vector spaces over a fixed field is 0. Applying this for the finite dimensional vector spaces $H_{i+1}(\mathcal{R}_U)$ over $\mathbb{F}_p$

$$\varprojlim_{U \in \mathcal{C}}{}^1 H_{i+1}(\mathcal{R}_U) = 0 \text{ for } i \leq m - 1.$$

Note we have proved the following lemma.

Lemma 2.1. *There is an isomorphism of abstract $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-modules*

$$H_0(\widehat{\mathcal{R}}) = 0 \text{ and } H_i(\widehat{\mathcal{R}}) \simeq \varprojlim_{U \in \mathcal{C}} H_i(\mathcal{R}_U) \simeq \varprojlim_{U \in \mathcal{C}} H_i(U, \mathbb{F}_p) \text{ for } 1 \leq i \leq m - 1,$$

where the G/U-action on $H_i(U, \mathbb{F}_p)$ induced by conjugation induces a $\widehat{G}_{\mathcal{C}}$-action on $\varprojlim H_i(U, \mathbb{F}_p)$.

Theorem 2.2. *Suppose that G is an abstract group of type FP_m for some $m \geq 2$, $\mathcal{C}$ is a directed set of normal subgroups U of finite index in G. Suppose* *further that the inverse limit* $\varprojlim H_i(U, \mathbb{F}_p)$ *over* $U \in \mathcal{C}$ *is of homological type* FP_{m-1-i} *as an abstract* $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$*-module for all* $1 \leq i \leq m-1$*. Then the trivial abstract* $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$*-module* $\mathbb{F}_p$ *is of type* FP_m*.*

Proof. We need only the dimension shifting argument from [1, Prop. 1.4]. More precisely suppose that $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of modules

a) if V is of type FP_∞ and $s \geq 1$ then V'' is of type FP_s if and only if V' is of type FP_{s-1};

b) if V' and V'' are of type FP_s for some $s \geq 0$ then V is of type FP_s.

From now on all modules considered in this proof are abstract $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-modules. Consider the short exact sequences of modules

$$0 \to Ker(\widehat{\partial}_j) \to \widehat{R}_j \to Im(\widehat{\partial}_j) \to 0 \tag{2}$$

and

$$0 \to Im(\widehat{\partial}_j) \to Ker(\widehat{\partial}_{j-1}) \to H_{j-1}(\widehat{\mathcal{R}}) \to 0 \tag{3}$$

We prove by inverse induction on i that $Im(\widehat{\partial}_i)$ is of type FP_{m-i} for all $0 \leq i \leq m$, the case $i = m$ is obvious as $\widehat{R}_m$ is FP_0 (i.e. finitely generated). As $Im(\widehat{\partial}_0) = \mathbb{F}_p$ the case $i = 0$ is exactly what we want to prove.

Suppose $Im(\widehat{\partial}_i)$ is of type FP_{m-i} for some $1 \leq i \leq m$. By Lemma 2.1 $\varprojlim H_{i-1}(U, \mathbb{F}_p) \simeq H_{i-1}(\widehat{\mathcal{R}})$ and by assumption $\varprojlim H_{i-1}(U, \mathbb{F}_p)$ is of type FP_{m-i}. By b) applied to the short exact sequence (3) for $j = i$, $Ker(\widehat{\partial}_{i-1})$ is of type FP_{m-i}. Note that $\widehat{R}_j$ is an abstract finitely generated projective $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module for every $j \leq m$, hence $\widehat{R}_j$ is FP_∞. Applying a) to the short exact sequence (2) for $j = i-1$ we get that $Im(\widehat{\partial}_{i-1})$ is of type FP_{m-i+1}. This completes the inductive step. □

Corollary 2.3. *Under the assumptions of Theorem 2.2 the profinite group* $\widehat{G}_{\mathcal{C}}$ *is of homological type* FP_m *over the ring* $\mathbb{F}_p$*.*

Proof. It follows directly from Theorem 2.2 and Lemma 1.1 b). □

Theorem 2.4. *Suppose that* G *is an abstract group of type* FP_m*,* $\mathcal{C}$ *a directed set of normal subgroups* U *of finite index in* G *and* i_0 *a fixed positive integer such that* $1 \leq i_0 \leq m-1$*. Suppose further that for a fixed prime* p *and for all* $i \in \{1, \ldots, m-1\} \setminus \{i_0\}$

$$\varprojlim_{U \in \mathcal{C}} H_i(U, \mathbb{F}_p) = 0.$$

Then the following conditions are equivalent :

a) $\varprojlim H_{i_0}(U, \mathbb{F}_p)$ *as an abstract* $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$*-module is of homological type* FP_{m-1-i_0}*;*

b) *the trivial abstract* $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$*-module* $\mathbb{F}_p$ *is of type* FP_m*;*

c) $\widehat{G}_{\mathcal{C}}$ *as a profinite group is of type* FP_m *over* $\mathbb{F}_p$*.*

Proof. c) implies b) is obvious and b) implies c) is Lemma 1.1 b). All modules considered in the rest of the proof are abstract $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-modules. By Lemma 2.1 $H_i(\widehat{\mathcal{R}}) = 0$ for $i \in \{1, \ldots, m-1\} \setminus \{i_0\}$, in particular we have the following exact complexes of modules

$$0 \to Ker(\widehat{\partial}_{i_0}) \xrightarrow{\alpha_1} \widehat{R}_{i_0} \xrightarrow{\widehat{\partial}_{i_0}} \widehat{R}_{i_0-1} \xrightarrow{\widehat{\partial}_{i_0-1}} \ldots \to \widehat{R}_0 \xrightarrow{\widehat{\partial}_0} \mathbb{F}_p \to 0 \quad (4)$$

and if $i_0 \neq m-1$

$$0 \to Im(\widehat{\partial}_m) = Ker(\widehat{\partial}_{m-1}) \xrightarrow{\alpha_2} \widehat{R}_{m-1} \xrightarrow{\widehat{\partial}_{m-1}} \ldots \to \widehat{R}_{i_0+1} \xrightarrow{\beta} Im(\widehat{\partial}_{i_0+1}) \to 0 \quad (5)$$

where α_1, α_2 are the inclusion maps, the map β is induced by $\widehat{\partial}_{i_0+1}$. Applying the dimension shifting argument [1, Prop. 1.4] (part (a) of the proof of Theorem 2.2) for the short exact sequences corresponding to the complexes (4) and (5) we get that $Ker(\widehat{\partial}_i)$ is FP_{m-1-i} if and only if $Im(\widehat{\partial}_i)$ is FP_{m-i}. In particular since $Im(\widehat{\partial_m})$ is FP_0 (i.e. finitely generated)

$$Im(\widehat{\partial}_{i_0+1}) \text{ is of type } FP_{m-i_0-1} \quad (6)$$

(note the latter holds even for $i_0 = m-1$) and

$$\mathbb{F}_p \text{ is of type } FP_m \text{ if and only if } Ker(\widehat{\partial}_{i_0}) \text{ is of type } FP_{m-i_0-1}. \quad (7)$$

By (6) and dimension shifting (this time we use both (a) and (b) from the proof of Theorem 2.2) for the short exact sequence $0 \to Im(\widehat{\partial}_{i_0+1}) \to Ker(\widehat{\partial}_{i_0}) \to H_{i_0}(\widehat{\mathcal{R}}) \to 0$

$$Ker(\widehat{\partial}_{i_0}) \text{ is of type } FP_{m-i_0-1} \text{ if and only if } H_{i_0}(\widehat{\mathcal{R}}) \text{ is of type } FP_{m-i_0-1}. \quad (8)$$

By (7) and (8) the items a) and b) are equivalent. □

Theorem 2.5. *Suppose that G is an abstract group of type FP_∞ and finite cohomological dimension, $\mathcal{C}$ a directed set of normal subgroups U of finite index in G. Suppose further that for a fixed prime p and for all $i \geq 1$*

$$\varprojlim_{U \in \mathcal{C}} H_i(U, \mathbb{F}_p) = 0$$

Then for all $m \geq 1$ and $i \geq 1$

$$Tor_i^{\mathbb{Z}[G]}(\mathbb{Z}, (\mathbb{Z}/p^m\mathbb{Z})[[\widehat{G}_{\mathcal{C}}]]) = 0 \text{ and } Tor_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]) = 0.$$

In particular $\widehat{G}_{\mathcal{C}}$ is of type FP_∞ over $\mathbb{Z}_p$.

Proof. Let $\mathcal{R}$ be a projective resolution of $\mathbb{Z}$ as an abstract $\mathbb{Z}[G]$-module such that $\mathcal{R}$ has finite length and all projective modules are finitely generated. By Lemma 2.1 $0 = H_i(\widehat{\mathcal{R}}) \simeq Tor_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]])$ for $i \geq 1$, where by (1) $\widehat{\mathcal{R}} \simeq \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$.

The long exact sequence in homology for the short exact sequence of abstract $\mathbb{Z}[G]$-modules $0 \to (\mathbb{Z}/p^{m-1}\mathbb{Z})[[\widehat{G}_{\mathcal{C}}]] \to (\mathbb{Z}/p^m\mathbb{Z})[[\widehat{G}_{\mathcal{C}}]] \to \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]] \to 0$ implies that if $Tor_i^{\mathbb{Z}[G]}(\mathbb{Z}, (\mathbb{Z}/p^{m-1}\mathbb{Z})[[\widehat{G}_{\mathcal{C}}]]) = 0$ for all $i \geq 1$ then $Tor_i^{\mathbb{Z}[G]}(\mathbb{Z}, (\mathbb{Z}/p^m\mathbb{Z})[[\widehat{G}_{\mathcal{C}}]]) = 0$ for all $i \geq 1$. It follows that for every m the complex $\mathcal{P}_{(m)} := \mathcal{R} \otimes_{\mathbb{Z}[G]} (\mathbb{Z}/p^m\mathbb{Z})[[\widehat{G}_{\mathcal{C}}]]$ is exact.

Let $\mathcal{P}$ be the inverse limit of the tower of exact complexes $\{\mathcal{P}_{(m)}\}_{m \geq 1}$. By [20, Thm. 3.5.8] the complex $\mathcal{P}$ is exact and by construction $\mathcal{P} \simeq \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]$. Then $0 = H_i(\mathcal{P}) = Tor_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]])$ for all $i \geq 1$ and $\mathcal{P}$ is a profinite projective resolution of the trivial profinite $\mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]$-module $\mathbb{Z}_p$ with all projective modules finitely generated, hence $\widehat{G}_{\mathcal{C}}$ is of type FP_∞ over $\mathbb{Z}_p$. $\square$

Theorem 2.6. *Suppose G is an abstract group of finite cohomological dimension $cd(G) = m$ and type FP_∞. Let i_0 be a positive integer such that $1 \leq i_0 \leq m$, p a fixed prime number and $\mathcal{C}$ a directed set of normal subgroups U of finite index in G. Suppose further that for all $i \in \{1, \ldots, m\} \setminus \{i_0\}$*

$$\varprojlim_{U \in \mathcal{C}} H_i(U, \mathbb{F}_p) = 0.$$

Then

a) *the inverse limit $V_{i_0} := \varprojlim H_{i_0}(U, \mathbb{F}_p)$ over $U \in \mathcal{C}$ has a finite projective dimension as an abstract $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module if and only if the profinite group $\widehat{G}_{\mathcal{C}}$ is of finite cohomological p-dimension;*

b) *if* $V_{i_0} = 0$ *then the profinite group* $\widehat{G}_{\mathcal{C}}$ *is of finite cohomological p-dimension* $cd_p(\widehat{G}_{\mathcal{C}}) \leq cd(G)$, *of type* FP_∞ *over* $\mathbb{F}_p$ *and its Euler p-characteristics* $\chi_p(\widehat{G}_{\mathcal{C}}) = \chi(G)$.

Proof. Let $\mathcal{R}$ be a projective resolution

$$\mathcal{R} : 0 \to R_m \to R_{m-1} \to \ldots \to R_0 \to \mathbb{Z} \to 0$$

with all R_i finitely generated for $i \leq m$. Let $\widehat{\mathcal{R}}$ the complex obtained from the inverse limit procedure at the beginning of section 2 i.e. $\widehat{\mathcal{R}} \simeq \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$. To prove a) we note that if there is an exact complex of abstract $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-modules

$$0 \to W' \to Q_j \to Q_{j-1} \to \ldots \to Q_0 \to W'' \to 0$$

with Q_i projective for $0 \leq i \leq j$ then W'' has finite projective dimension if and only if W' has finite projective dimension. One can see it by breaking the complex into short exact sequences and using [14, Prop. 5.2.11] stating that a $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module M has projective dimension n if and only if $Tor_{n+1}^{\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]}(M, N) = 0$ for every simple N (or similarly with Ext).

This applied in the special case $i_0 = m$ for the exact complex of abstract $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-modules

$$0 \to Ker(\partial_m) \to \widehat{R}_m \xrightarrow{\widehat{\partial}_m} \widehat{R}_{m-1} \to \ldots \to \widehat{R}_0 \to \mathbb{F}_p \to 0$$

plus the fact that by Lemma 2.1, $V_{i_0} \simeq H_{i_0}(\widehat{\mathcal{R}}) = Ker(\partial_m)$, shows that a) holds for $i_0 = m$.

Now suppose that $i_0 \leq m - 1$. Then the above argument applied for the exact complex

$$0 \to Ker(\widehat{\partial}_{i_0}) \to \widehat{R}_{i_0} \xrightarrow{\widehat{\partial}_{i_0}} \ldots \to \widehat{R}_0 \to \mathbb{F}_p \to 0$$

shows that $\mathbb{F}_p$ has finite projective dimension as an abstract $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module if and only if $Ker(\widehat{\partial}_{i_0})$ has finite projective dimension. Since $H_i(\mathcal{R})$ is the inverse limit $\varprojlim H_i(U, \mathbb{F}_p)$ over $U \in \mathcal{C}$ and $\varprojlim H_i(U, \mathbb{F}_p) = 0$ for all $i > i_0$, the module $Im(\widehat{\partial}_{i_0+1})$ has a projective resolution as an abstract $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module

$$0 \to \widehat{R}_m \xrightarrow{\widehat{\partial}_m} \widehat{R}_{m-1} \longrightarrow \ldots \longrightarrow \widehat{R}_{i_0+1} \xrightarrow{\widehat{\partial}_{i_0+1}} Im(\widehat{\partial}_{i_0+1}) \to 0,$$

hence $Im(\widehat{\partial}_{i_0+1})$ has finite projective dimension. Finally consider the short exact sequence of $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-modules $0 \to Im(\widehat{\partial}_{i_0+1}) \to Ker(\widehat{\partial}_{i_0}) \to H_{i_0}(\widehat{\mathcal{R}}) \to$

0. Then $Ker(\widehat{\partial}_{i_0})$ has finite projective dimension if and only if $H_{i_0}(\widehat{\mathcal{R}})$ has finite projective dimension. Finally by Lemma 2.1 $H_{i_0}(\widehat{\mathcal{R}}) \simeq V_{i_0}$. This completes the proof of part a).

If $V_{i_0} = 0$ by Theorem 2.5 the complex $\mathcal{S} = \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]$ is a projective resolution of $\mathbb{Z}_p$ as an abstract $\mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]$-module of finite length and all projective modules finitely generated. Then by Lemma 1.1c) $cd_p(\widehat{G}_{\mathcal{C}}) \leq cd(G)$ and

$$\chi_p(\widehat{G}_{\mathcal{C}}) = \sum_i (-1)^i rk_{\mathbb{Z}_p}(S_i \otimes_{\mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]} \mathbb{Z}_p) = \sum_i (-1)^i rk_{\mathbb{Z}_p}(R_i \otimes_{\mathbb{Z}[G]} \mathbb{Z}_p) =$$

$$\sum_i (-1)^i rk_{\mathbb{Z}_p}(R_i \otimes_{\mathbb{Z}[G]} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_p) = \sum_i (-1)^i rk_{\mathbb{Z}}(R_i \otimes_{\mathbb{Z}[G]} \mathbb{Z}) = \chi(G).$$

□

The following corollary follows from Theorem 2.6 b) and Theorem 2.5.

Corollary 2.7. *Suppose G is an abstract group of finite cohomological dimension $cd(G) = m$ and type FP_∞. Let $\mathcal{C}$ be a directed set of normal subgroups U of finite index in G. Suppose further that for a fixed prime p and for all $1 \leq i \leq m$*

$$\lim_{\overleftarrow{U \in \mathcal{C}}} H_i(U, \mathbb{F}_p) = 0.$$

Then the profinite group $\widehat{G}_{\mathcal{C}}$ is of finite cohomological p-dimension $cd_p(\widehat{G}_{\mathcal{C}}) \leq m$, is of type FP_∞ over $\mathbb{F}_p$ and over $\mathbb{Z}_p$ and its Euler p-characteristics $\chi_p(\widehat{G}_{\mathcal{C}}) = \chi(G)$.

Remark Corollary 2.7 can be also deduced from in [17, Complements, p.15] using the Lyndon-Hoschild-Serre spectral sequence.

3 Groups of cohomological dimension 3

3.1 Pro-$\mathfrak{C}$ completions

In this subsection we study pro-$\mathfrak{C}$ completions of groups of cohomological dimension 3 with some additional properties, where $\mathfrak{C}$ is a class of finite groups closed for subgroups, quotients and extensions. In this case our directed set $\mathcal{C}$ is a set of subgroups of G that defines the pro-$\mathfrak{C}$ topology on G. One of the frequent additional property is the infinity of a Sylow p-subgroup for every p. It is worth to observe that the profinite completion of a finitely generated linear group has an infinite Sylow p-subgroup for every p. This follows from the Lubotzky's Alternative see p. 390 in [13].

For $U \in \mathcal{C}$ denote by $\widehat{U}_{\mathcal{C}}$ the inverse limit of U/V over those V in $\mathcal{C}$ that are subgroups of U. Denote by $H_i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p)$ the continuous ith homology of the profinite group $\widehat{U}_{\mathcal{C}}$ with coefficients in $\mathbb{F}_p$ (i.e. calculated using projective resolution of $\mathbb{Z}_p$ in the category of profinite $\mathbb{Z}_p[[\widehat{U}_{\mathcal{C}}]]$-modules). Note that if $\widehat{G}_{\mathcal{C}}$ has an infinite Sylow p-subgroup then $\mathfrak{C}$ contains $\mathbb{F}_p$, and hence $\mathfrak{C}$ contains any finite p-group.

Proposition 3.1. *Let p be a fixed prime, G an abstract group of cohomological dimension 3 and type FP_∞. Furthermore for every $U \in \mathcal{C}$ either $H_3(U, \mathbb{F}_p) \simeq \mathbb{F}_p$ or $H_3(U, \mathbb{F}_p) = 0$. Assume that the profinite group $\widehat{G}_{\mathcal{C}}$ has an infinite Sylow p-subgroup. Then for any projective resolution $\mathcal{R}$ of $\mathbb{Z}$ as an abstract $\mathbb{Z}[G]$-module such that $\mathcal{R}$ has finite length 3 and finitely generated projective modules one has*

$$H_i(\widehat{\mathcal{R}}) = 0 \textit{ for } i = 1 \textit{ and } i = 3$$

for the completed complex $\widehat{\mathcal{R}} \simeq \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[G]]$.

Proof. Since U is finitely generated (because G is), $[U, U]U^p$ has finite index in U, so by in [15, Prop. 3.3.2 (d)] $U/[U, U]U^p \simeq \widehat{U}_{\mathcal{C}}/[\widehat{U}_{\mathcal{C}}, \widehat{U}_{\mathcal{C}}]\widehat{U}_{\mathcal{C}}^p$. Hence the homomorphism $U \to \widehat{U}_{\mathcal{C}}$ induces an isomorphism of $\mathbb{F}_p$-vector spaces

$$H_1(U, \mathbb{F}_p) \simeq U/[U, U]U^p \simeq \widehat{U}_{\mathcal{C}}/[\widehat{U}_{\mathcal{C}}, \widehat{U}_{\mathcal{C}}]\widehat{U}_{\mathcal{C}}^p \simeq H_1(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p).$$

As the continuous homology commutes with inverse limits (see Proposition 6.5.7 in [15]) we get that

$$\varprojlim_{U \in \mathcal{C}} H_1(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \simeq H_1(\varprojlim_{U \in \mathcal{C}} \widehat{U}_{\mathcal{C}}, \mathbb{F}_p) = 0 \text{ since } \varprojlim_{U \in \mathcal{C}} \widehat{U}_{\mathcal{C}} = \cap_{U \in \mathcal{C}} \widehat{U}_{\mathcal{C}} = 1.$$

In particular, by Lemma 2.1

$$H_1(\widehat{\mathcal{R}}) \simeq \varprojlim_{U \in \mathcal{C}} H_1(U, \mathbb{F}_p) = 0.$$

As $H_3(U, \mathbb{F}_p) \simeq \mathbb{F}_p$ or 0, the inverse limit $\varprojlim H_3(U, \mathbb{F}_p)$ is either $\mathbb{F}_p$ or 0. Again using Lemma 2.1

$$H_3(\widehat{\mathcal{R}}) \simeq \varprojlim_{U \in \mathcal{C}} H_3(U, \mathbb{F}_p) = 0 \text{ or } \mathbb{F}_p.$$

In this paragraph we state a homological version of [17, Sec. 3.3, Lemma 4]. Consider two normal subgroups $U_1 \subseteq U_2$ of finite index in G such that the order of U_2/U_1 is divisible by p, then the map

$$Cor_{U_1}^{U_2} \circ Res_{U_2}^{U_1} : H_3(U_2, \mathbb{F}_p) \to H_3(U_2, \mathbb{F}_p) \text{ is zero}$$

since it is the multiplication by $|U_2/U_1|$. Here $Res_{U_2}^{U_1} : H_3(U_2, \mathbb{F}_p) \to H_3(U_1, \mathbb{F}_p)$ is the restriction map, called transfer map in [20, Lemma 6.7.17]. By [20, 6.3.9] the restriction map is the composition $H_3(U_2, \mathbb{F}_p) \to H_3(U_2, Ind_{U_1}^{U_2}(\mathbb{F}_p))$ $\simeq H_3(U_1, \mathbb{F}_p)$, where the last isomorphism is the one given by the Shapiro lemma. As U_2 has cohomological dimension 3 the long exact sequence in homology for the short exact sequence

$$0 \to \mathbb{F}_p \to Ind_{U_1}^{U_2}(\mathbb{F}_p) \to Ind_{U_1}^{U_2}(\mathbb{F}_p)/\mathbb{F}_p \to 0$$

gives an exact sequence

$$H_4(U_2, Ind_{U_1}^{U_2}(\mathbb{F}_p)/\mathbb{F}_p) = 0 \to H_3(U_2, \mathbb{F}_p) \to H_3(U_2, Ind_{U_1}^{U_2}(\mathbb{F}_p)) \to \dots.$$

In particular the map $H_3(U_2, \mathbb{F}_p) \to H_3(U_2, Ind_{U_1}^{U_2}(\mathbb{F}_p))$ is injective and the restriction map $Res_{U_2}^{U_1}$ is injective.

Suppose that $Cor_{U_1}^{U_2} \neq 0$. As $H_3(U_i, \mathbb{F}_p)$ is either $\mathbb{F}_p$ or zero, $H_3(U_1, \mathbb{F}_p) \simeq \mathbb{F}_p \simeq H_3(U_2, \mathbb{F}_p)$ and the injectivity of $Res_{U_2}^{U_1}$ implies that $Res_{U_2}^{U_1}$ is an isomorphism. Finally since $Cor_{U_1}^{U_2} \circ Res_{U_2}^{U_1} = 0$ we obtain $Cor_{U_1}^{U_2} = 0$, a contradiction. Hence $Cor_{U_1}^{U_2} = 0$.

As $\widehat{G}_{\mathcal{C}}$ has an infinite Sylow p-subgroup infinitely many of the corestriction maps in the inverse limit used to calculate $\varprojlim H_3(U, \mathbb{F}_p) \simeq H_3(\widehat{\mathcal{R}})$ are zero, hence $H_3(\widehat{\mathcal{R}}) = 0$. □

Theorem 3.2. *Suppose the hypothesis of the preceding proposition hold and the homomorphism*

$$\varphi_U : H_2(U, \mathbb{F}_p) \to H_2(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p)$$

induced by the canonical homomorphism $U \to \widehat{U}_{\mathcal{C}}$ is an isomorphism for all $U \in \mathcal{C}$. Then

a) $Tor_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]) = 0$ *for all* $i \geq 1$, *the canonical map* $G \to \widehat{G}_{\mathcal{C}}$ *induces an isomorphism* $H_i(G, \mathbb{F}_p) \simeq H_i(\widehat{G}_{\mathcal{C}}, \mathbb{F}_p)$ *for all* i *and* $\chi_p(\widehat{G}_{\mathcal{C}}) = \chi(G)$;

b) *the profinite group* $\widehat{G}_{\mathcal{C}}$ *is of type* FP_∞ *over* $\mathbb{F}_p$ *and over* $\mathbb{Z}_p$ *and has cohomological p-dimension* $cd_p(\widehat{G}_{\mathcal{C}}) \leq 3$, *in particular* $\widehat{G}_{\mathcal{C}}$ *does not have p-torsion. If* $H_3(G, \mathbb{F}_p) \simeq \mathbb{F}_p$ *then* $cd_p(\widehat{G}_{\mathcal{C}}) = 3$;

c) $Tor_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]) = 0$ *for all* $i \geq 1$.

Proof. Let $\mathcal{R}$ be a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module

$$\mathcal{R} : 0 \to R_3 \to R_2 \to R_1 \to R_0 \to \mathbb{Z} \to 0$$

with finitely generated projective modules and $\widehat{\mathcal{R}} \simeq \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$ by (1). As the homomorphism $\varphi_U : H_2(U, \mathbb{F}_p) \to H_2(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p)$ is an isomorphism for all $U \in \mathcal{C}$ and by Lemma 2.1

$$H_2(\widehat{\mathcal{R}}) \simeq \varprojlim_{U \in \mathcal{C}} H_2(U, \mathbb{F}_p) \simeq \varprojlim_{U \in \mathcal{C}} H_2(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \simeq H_2(\varprojlim_{U \in \mathcal{C}} \widehat{U}_{\mathcal{C}}, \mathbb{F}_p) = 0.$$

Combining this with Proposition 3.1 we have $0 = H_i(\widehat{\mathcal{R}}) \simeq Tor_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]])$ for $1 \leq i \leq 3$, hence $\widehat{\mathcal{R}}$ is a finite length projective resolution of $\mathbb{F}_p$ over $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$ with all modules finitely generated and $\mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p \simeq \widehat{\mathcal{R}} \otimes_{\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]} \mathbb{F}_p$. By Corollary 2.7 the profinite group $\widehat{G}_{\mathcal{C}}$ is of type FP_∞ over $\mathbb{Z}_p$ and over $\mathbb{F}_p$ and $cd_p(\widehat{G}_{\mathcal{C}}) \leq 3$. Note that if $\mathcal{F}$ is a projective resolution with all projectives finitely generated of the trivial abstract $\mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]$-module $\mathbb{Z}_p$ then $\mathcal{F} \otimes_{\mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]} \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$ is a projective resolution of the trivial abstract $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module $\mathbb{F}_p$. Therefore, $Tor_i^{\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]}(\mathbb{F}_p, \mathbb{F}_p) \simeq Tor_i^{\mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]}(\mathbb{Z}_p, \mathbb{F}_p)$. Hence

$$H_i(G, \mathbb{F}_p) \simeq H_i(\mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p) \simeq H_i(\widehat{\mathcal{R}} \otimes_{\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]} \mathbb{F}_p) \simeq$$

$$Tor_i^{\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]}(\mathbb{F}_p, \mathbb{F}_p) \simeq Tor_i^{\mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]}(\mathbb{Z}_p, \mathbb{F}_p) \simeq H_i(\widehat{G}_{\mathcal{C}}, \mathbb{F}_p)$$

and (a) is proved.

To finish the proof of item (b) we observe that $H_3(\widehat{G}_{\mathcal{C}}, \mathbb{F}_p) \simeq H_3(G, \mathbb{F}_p) \simeq \mathbb{F}_p \neq 0$ gives $cd_p(\widehat{G}) = 3$. Finally Theorem 2.5 completes the proof of item (c). □

3.2 More on pro-p completions

We shall need the pro-p version of a well-known result for discrete groups.

Lemma 3.3. *Let G be a finitely presented pro-p group and H be an open subgroup of G. Then*

$$def(H) - 1 \geq (G : H)(def(G) - 1).$$

Proof. Denote by $d(G)$ the minimal number of topological generators of G. Let $f : F \longrightarrow G$ be an epimorphism of a free pro-p group F of rank $d(G)$ onto G. Put $U = f^{-1}(H)$. Then U is free pro-p and $d(U) = (F : U)(d(F) - 1) + 1$ (see [15, Thm 3.6.2]). Denote by $r(G)$ and $r(H)$ the minimal number of generators of $\ker(f)$ as a normal subgroup in F and in U respectively. Hence

$$\begin{aligned} def(H) &= d(U) - r(H) \\ &= (F : U)(d(F) - 1) + 1 - r(H) \\ &\geq (F : U)(d(F) - 1) + 1 - (F : U)r(G) \\ &= (F : U)(def(G) - 1) + 1 \end{aligned}$$

as needed. □

Lemma 3.4. *Let G be an abstract finitely generated group of cohomological dimension 3 and Euler characteristics 0 with a subgroup U of G of p-power index such that $H_3(U, \mathbb{F}_p) \simeq \mathbb{F}_p$. Then for the pro-p completion $\widehat{U}_p$ of U the dimension (over $\mathbb{F}_p$) of the kernel of the surjective map*

$$\varphi_U : H_2(U, \mathbb{F}_p) \to H_2(\widehat{U}_p, \mathbb{F}_p)$$

is $def(\widehat{U}_p)$, where

$$def(\widehat{U}_p) = dim_{\mathbb{F}_p} H_1(\widehat{U}_p, \mathbb{F}_p) - dim_{\mathbb{F}_p} H_2(\widehat{U}_p, \mathbb{F}_p)$$

is the deficiency of $\widehat{U}_p$.

Proof. We claim that φ_U is surjective. Indeed let F be a finite rank free group with a normal subgroup R such that $U \simeq F/R$. Then by a p-version of Schur multiplier formula $H_2(U, \mathbb{F}_p) \simeq R \cap [F, F]F^p/[R, F]R^p$ (this formula follows easily from the exact sequence given in [4, Ch. 2,Prop. 5.4]). Let $\widehat{F}$ be the pro-p completion of F (i.e. $\widehat{F}$ is a free pro-p group of rank equal to the rank of F) and $\bar{R}$ be the closure of the image of R in $\widehat{F}$ via the canonical map $\rho : F \to \widehat{F}$. Then $H_2(\widehat{U}_p, \mathbb{F}_p) \simeq \bar{R} \cap [\widehat{F}, \widehat{F}]\widehat{F}^p / [\bar{R}, \widehat{F}]\bar{R}^p$ and the map φ_U is induced by the map ρ.

Note that by [4, Thm. 6.3, Ch. 9] $\chi(U) = (G : U)\chi(G) = 0$ since $\chi(G) = 0$ by the hypothesis of the lemma. Then

$$dim_{\mathbb{F}_p} H_2(U, \mathbb{F}_p) - dim_{\mathbb{F}_p} H_1(U, \mathbb{F}_p) = \chi(U) - dim_{\mathbb{F}_p} H_0(U, \mathbb{F}_p)$$
$$+ dim_{\mathbb{F}_p} H_3(U, \mathbb{F}_p) = \chi(U) = 0,$$
$$dim_{\mathbb{F}_p} H_2(U, \mathbb{F}_p) = dim_{\mathbb{F}_p} H_1(U, \mathbb{F}_p) = dim_{\mathbb{F}_p} U/[U, U]U^p = dim_{\mathbb{F}_p} H_1(\widehat{U}_p, \mathbb{F}_p)$$

and

$$dim_{\mathbb{F}_p} Ker(\varphi_U) = dim_{\mathbb{F}_p} H_2(U, \mathbb{F}_p) - dim_{\mathbb{F}_p} H_2(\widehat{U}_p, \mathbb{F}_p) =$$
$$dim_{\mathbb{F}_p} H_1(\widehat{U}_p, \mathbb{F}_p) - dim_{\mathbb{F}_p} H_2(\widehat{U}_p, \mathbb{F}_p) = def(\widehat{U}_p).$$

□

Corollary 3.5. *Suppose the assumptions of the preceding lemma hold for any subgroup U of p-power index in G. Then the following hold :*

a) *if $U/[U, U]$ is finite then φ_U is an isomorphism;*

b) *let V be a subgroup of p-power index in G such that the map φ_V is not an isomorphism. Then for any proper subgroup W of p-power index in V the map φ_W is not an isomorphism and $def(\widehat{W}_p) \geq def(\widehat{V}_p)$.*

Proof. a) By Lemma 3.4 $def(\widehat{U}_p) = dim_{\mathbb{F}_p} Ker(\varphi_U) \geq 0$. If $def(\widehat{U}_p) \neq 0$ by [13, Lemma 16.4.3,p.370] $\widehat{U}_p$ cannot have finite abelianization, hence U cannot have finite abelianization, a contradiction. Then $def(\widehat{U}_p) = 0$ and φ_U is an isomorphism.

b) By Lemma 3.3 $def(\widehat{W}_p) - 1 \geq (\widehat{V}_p : \widehat{W}_p)(def(\widehat{V}_p) - 1)$. Therefore,

$$def(\widehat{W}_p) - 1 \geq (\widehat{V}_p : \widehat{W}_p)(def(\widehat{V}_p) - 1) \geq 2(def(\widehat{V}_p) - 1).$$

By Lemma 3.4 $def(\widehat{V}_p) = dim_{\mathbb{F}_p} Ker(\varphi_V) \geq 1$. Hence

$$def(\widehat{W}_p) - def(\widehat{V}_p) \geq 2(def(\widehat{V}_p) - 1) + 1 - def(\widehat{V}_p) = def(\widehat{V}_p) - 1 \geq 0$$

and again using Lemma 3.4

$$dim_{\mathbb{F}_p} Ker(\varphi_W) = def(\widehat{W}_p) \geq def(\widehat{V}_p) = dim_{\mathbb{F}_p} Ker(\varphi_V) > 0,$$

in particular $Ker(\varphi_W) \neq 0$. □

Theorem 3.6. *Let G be an abstract group of cohomological dimension 3 such that $H_3(U,\mathbb{F}_p) \simeq \mathbb{F}_p$ for every normal subgroup U of p-power index in G. Assume that the pro-p completion $\widehat{G}_p$ is infinite and that the homomorphism*

$$\varphi_U : H_2(U,\mathbb{F}_p) \to H_2(\widehat{U}_p,\mathbb{F}_p)$$

induced by the homomorphism $U \to \widehat{U}_p$ is an isomorphism for all normal subgroups U of p-power index in G. Then

a) *$Tor_i^{\mathbb{Z}[G]}(\mathbb{Z},\mathbb{F}_p[[\widehat{G}_p]]) = 0$ and $Tor_i^{\mathbb{Z}[G]}(\mathbb{Z},\mathbb{Z}_p[[\widehat{G}_p]]) = 0$ for all $i \geq 1$, $H_i(G,\mathbb{F}_p) \simeq H_i(\widehat{G}_p,\mathbb{F}_p)$ for all i and $\chi(\widehat{G}_p) = \chi(G)$;*

b) *the pro-p group $\widehat{G}_p$ is of type FP_∞ over $\mathbb{F}_p$ and over $\mathbb{Z}_p$ and has cohomological dimension 3, in particular $\widehat{G}_p$ is torsion-free.*

c) *G is a p-good group.*

Proof. Parts a) and b) are specific cases of Theorem 3.2 applied for the set $\mathcal{C}$ of all normal subgroups U of G of p-power index.

To prove c) consider

$$\mathcal{R} : 0 \to R_3 \to R_2 \to R_1 \to R_0 \to \mathbb{Z} \to 0$$

a projective resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ with all modules finitely generated. By part a) $\mathcal{P} = \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{Z}_p[[\widehat{G}_p]]$ is an exact complex. Note that for any finite p-primary G-module M

$$Hom_{\mathbb{Z}[G]}(\mathcal{R}^{del}, M) \simeq Hom_{\mathbb{Z}_p[[\widehat{G}_p]]}(\mathcal{P}^{del}, M),$$

hence

$$H^i(G,M) \simeq H^i(Hom_{\mathbb{Z}[G]}(\mathcal{R}^{del}, M)) \simeq H^i(Hom_{\mathbb{Z}_p[[\widehat{G}_p]]}(\mathcal{P}^{del}, M)) \simeq H^i(\widehat{G}_p, M).$$

□

Corollary 3.7. *Let G be an abstract group of cohomological dimension 3, of type FP_∞, of Euler characteristics $\chi(G) = 0$ and such that $H_3(U,\mathbb{F}_p) \simeq \mathbb{F}_p$ for every normal subgroup U of p-power index in G. Assume that $\widehat{G}_p$ is infinite and that every normal subgroup U of p-power index in G has finite abelianization. Then G is a p-good group.*

Proof. Follows from Corollary 3.5 a) and Theorem 3.6 c). □

3.3 Goodness

Theorem 3.8. *Let G be an abstract group of cohomological dimension 3, of type FP_∞, of Euler characteristics $\chi(G) = 0$ and p a fixed prime number such that $H_3(U, \mathbb{F}_p) \simeq \mathbb{F}_p$ for every normal subgroup U of a finite index in G. Assume that the profinite completion $\widehat{G}$ has an infinite Sylow p-subgroup and that every normal subgroup U of a finite index in G has finite abelianization.*

Then for every finite p-primary discrete G-module M the natural homomorphism $G \to \widehat{G}$ induces an isomorphism $H^i(\widehat{G}, M) \to H^i(G, M)$ for all i.

Proof. We aim to prove that for the set $\mathcal{C}$ of all normal subgroups U of finite index in G

$$\lim_{\overleftarrow{U \in \mathcal{C}}} H_2(U, \mathbb{F}_p) = 0 \tag{9}$$

Note that every $U \in \mathcal{C}$ satisfies the assumptions of Corollary 3.7 except that the pro-p completion of U can be finite. By Corollary 3.5 a) the map $H_2(U, \mathbb{F}_p) \to H_2(\widehat{U}_p, \mathbb{F}_p)$, induced by the natural homomorphism $U \to \widehat{U}_p$, is an isomorphism, where $\widehat{U}_p$ is the pro-p completion of U. Hence for all $U \in \mathcal{C}$ and $\mathcal{C}_U$ the set of all subgroups V of U of p-power index such that V is normal in G

$$\lim_{\overleftarrow{V \in \mathcal{C}_U}} H_2(V, \mathbb{F}_p) = 0.$$

Let

$$\theta : \prod_{U \in \mathcal{C}} H_2(U, \mathbb{F}_p) \to \prod_{U \in \mathcal{C}} (\prod_{V \in \mathcal{C}_U} H_2(V, \mathbb{F}_p))$$

be the injective homomorphism whose restriction on $H_2(U, \mathbb{F}_p)$ is $\prod_{W \in \mathcal{C}} \theta_W$, where $\theta_W : H_2(U, \mathbb{F}_p) \to \prod_{V \in \mathcal{C}_W} H_2(V, \mathbb{F}_p)$ is the natural inclusion of the direct component $H_2(U, \mathbb{F}_p)$ if $U \in \mathcal{C}_W$, and is zero otherwise. Note that θ induces an injective homomorphism

$$\theta^* : \lim_{\overleftarrow{U \in \mathcal{C}}} H_2(U, \mathbb{F}_p) \to \prod_{U \in \mathcal{C}} \lim_{\overleftarrow{V \in \mathcal{C}_U}} H_2(V, \mathbb{F}_p) = \prod 0 = 0$$

hence (9) holds.

Now (9) and Proposition 3.1 combined with Lemma 2.1 show that the hypothesis of Theorem 2.5 are satisfied. Applying it for the set $\mathcal{C}$ of all normal subgroups U of finite index in G, $Tor_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}_p[[\widehat{G}]]) = 0$ for all $i \geq 1$. Let $\mathcal{R} : 0 \to R_3 \to R_2 \to R_1 \to R_0 \to \mathbb{Z} \to 0$ be a projective resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ with all modules finitely generated. Then $\mathcal{P} = \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{Z}_p[[\widehat{G}]]$ is an exact complex and for any finite p-primary G-module M, $Hom_{\mathbb{Z}[G]}(\mathcal{R}^{del}, M) \simeq Hom_{\mathbb{Z}_p[[\widehat{G}]]}(\mathcal{P}^{del}, M)$. Hence for all i

$$H^i(G, M) \simeq H^i(Hom_{\mathbb{Z}[G]}(\mathcal{R}^{del}, M)) \simeq H^i(Hom_{\mathbb{Z}_p[[\widehat{G}]]}(\mathcal{P}^{del}, M)) \simeq H^i(\widehat{G}, M),$$

the last isomorphism follows from [15, Remark 6.2.5]. □

Corollary 3.9. *Let G be an abstract group of cohomological dimension 3, of type FP_∞, of Euler characteristics $\chi(G) = 0$ and such that $H_3(U, \mathbb{F}_p) \simeq \mathbb{F}_p$ for every normal subgroup U of a finite index in G and for all prime numbers p. Assume that the profinite completion $\widehat{G}$ has an infinite Sylow p-subgroup for every prime p and that every normal subgroup U of a finite index in G has finite abelianization. Then G is good.*

Proof. If A is a finite discrete G-module then A is a direct sum of its p-primary submodules $A_{(p)}$, hence we can apply Theorem 3.8 for $M = A_{(p)}$. Then the natural homomorphism $G \to \widehat{G}$ induces an isomorphism $H^i(\widehat{G}, A) \to H^i(G, A)$ for all i. □

4 Poincaré duality groups of dimension 3

4.1 Pro-$\mathfrak{C}$ completions

As in Subsection 3.1 in this subsection we are concerned with pro-$\mathfrak{C}$ completions of groups of cohomological dimension 3, where $\mathfrak{C}$ is a class of finite groups closed for subgroups, quotients and extensions. So our directed set $\mathcal{C}$ in this subsection is a set of normal subgroups of G that defines the pro-$\mathfrak{C}$ topology on G and so the pro-$\mathfrak{C}$ completion $\widehat{G}_{\mathfrak{C}} = \widehat{G}_{\mathcal{C}}$.

Suppose G is an abstract Poincaré duality group of dimension 3 i.e. a PD_3 group. Thus G is an abstract group of cohomological dimension 3, of type FP_∞, of Euler characteristics $\chi(G) = 0$. Note that every subgroup of finite index in a PD_n group is a PD_n group [4, Ch. 8,Prop. 10.2]. Furthermore if a PD_n group G is not orientable (i.e. the G-action on $H^n(G, \mathbb{Z}[G]) \simeq \mathbb{Z}$ is not trivial) then there is a subgroup of index 2 in G that is an orientable PD_n group.

Theorem 4.1. *Let G be an abstract Poincaré duality group of dimension 3 and p be a prime number. Suppose G is orientable if $\mathbb{Z}/2\mathbb{Z} \notin \mathfrak{C}$ and that $\widehat{G}_{\mathfrak{C}}$ has an infinite Sylow p-subgroup. Suppose also that the homomorphism*

$$\varphi_U : H_2(U, \mathbb{F}_p) \to H_2(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p),$$

induced by the canonical homomorphism $U \to \widehat{U}_{\mathcal{C}}$, is an isomorphism for all $U \in \mathcal{C}$. Then

a) *$\chi_p(\widehat{G}_{\mathfrak{C}}) = \chi(G) = 0$, $\widehat{G}_{\mathfrak{C}}$ is of type FP_∞ over $\mathbb{Z}_p$ and has cohomological p-dimension 3, in particular $\widehat{G}_{\mathfrak{C}}$ does not have p-torsion.*

b) *$\widehat{G}_{\mathfrak{C}}$ is a profinite Poincaré duality group at the prime p. If G is orientable then $\widehat{G}_{\mathfrak{C}}$ is orientable.*

Proof. a) Every normal subgroup U of finite index in G is an abstract Poincaré duality group of dimension 3. If U is orientable $H_3(U, \mathbb{F}_p) \simeq H^0(U, \mathbb{F}_p) \simeq \mathbb{F}_p$. If U is not orientable $H_3(U, \mathbb{F}_p) \simeq H^0(U, D \otimes_{\mathbb{Z}} \mathbb{F}_p) \simeq (D \otimes_{\mathbb{Z}} \mathbb{F}_p)^U$ where $D \simeq \mathbb{Z}$ is the orientation module. Hence U acts non-trivially on D and $(D \otimes_{\mathbb{Z}} \mathbb{F}_p)^U = 0$ if $p \neq 2$, $(D \otimes_{\mathbb{Z}} \mathbb{F}_2)^U = \mathbb{F}_2$ if $p = 2$. Then we can apply Theorem 3.2 to obtain that $cd_p(\widehat{G}_{\mathfrak{C}}) \leq 3$ and all the other statements except $cd_p(\widehat{G}_{\mathfrak{C}}) = 3$. It needs only to show this equality.

There is a subgroup G_0 of index ≤ 2 in G such that G_0 is an orientable Poincaré duality group of dimension 3 and if G is orientable $G_0 = G$. Since $\mathfrak{C}$ is extension closed and by assumption if $G \neq G_0$ we have $\mathbb{Z}/2\mathbb{Z} \in \mathfrak{C}$ and $[G : G_0] = 2$, the closure of G_0 in $\widehat{G}_{\mathfrak{C}}$ coincides with $(\widehat{G_0})_{\mathcal{C}}$. Thus it suffices to prove the result for G_0 since $cd_p((\widehat{G_0})_{\mathcal{C}}) \leq cd_p(\widehat{G}_{\mathfrak{C}}) \leq 3$. Let $U_0 \in \mathcal{C}$ be such that $U_0 \subseteq G_0$. As φ_{U_0} is an isomorphism, Theorem 3.2b) holds for G_0, hence $cd_p((\widehat{G_0})_{\mathcal{C}}) = 3$.

b) Let G_0 be as in the proof of part a). Let

$$\mathcal{R} : 0 \to R_3 \to R_2 \to R_1 \to R_0 \to \mathbb{Z} \to 0$$

be a projective resolution of the trivial right $\mathbb{Z}[G_0]$-module $\mathbb{Z}$ with all modules finitely generated (it exists since G_0 is of type FP_∞). Then $H^i(\mathcal{S}) = H^i(G_0, \mathbb{Z}[G_0])$ is 0 for $i \neq 3$ and $\mathbb{Z}$ for $i = 3$, where $\mathcal{S} = Hom_{\mathbb{Z}[G_0]}(\mathcal{R}, \mathbb{Z}[G_0])$ is the dual complex, in particular $\mathcal{S}$ is a complex of left $\mathbb{Z}[G_0]$-modules. Then the complex obtained from $\mathcal{S}$ by adding its unique non-trivial cohomology

$$\mathcal{T} : 0 \to S^0 \to S^1 \to S^2 \to S^3 \to H^3(\mathcal{S}) = \mathbb{Z} \to 0$$

can be viewed as a projective resolution of $\mathbb{Z}$ as a left $\mathbb{Z}[G_0]$-module. By Theorem 3.2c) $Tor_i^{\mathbb{Z}[G_0]}(\mathbb{Z}, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]) = 0$ and similarly $Tor_i^{\mathbb{Z}[G_0]}(\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]], \mathbb{Z}) = 0$. Then

$$\widehat{\mathcal{T}} = \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]] \otimes_{\mathbb{Z}[G_0]} \mathcal{T} : 0 \to T^0 \to T^1 \to T^2 \to T^3 \to \mathbb{Z}_p \to 0$$

is a projective resolution of the trivial abstract left $\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]$-module $\mathbb{Z}_p$. Deleting the term $\mathbb{Z}_p$ of the above resolution we get the deleted complex $\widehat{\mathcal{T}}^{del}$. We claim that

$$\widehat{\mathcal{T}}^{del} \simeq Hom_{\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]}(\mathcal{P}^{del}, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]),$$

where $\mathcal{P} = \mathcal{R} \otimes_{\mathbb{Z}[G_0]} \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]$ is an exact complex of right $\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]$-modules by Theorem 3.2c). Indeed $\widehat{\mathcal{T}}^{del}$ is obtained from the complex $\mathcal{R}^{del}$ of projective finitely generated $\mathbb{Z}[G_0]$-modules by applying first the functor $Hom_{\mathbb{Z}[G_0]}(\ \ , \mathbb{Z}[G_0])$ and then the functor $\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]] \otimes_{\mathbb{Z}[G_0]}$. The composite of these functors is the same as the composite of the functor $\otimes_{\mathbb{Z}[G_0]} \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]$ and then $Hom_{\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]}(\ \ , \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]])$ if applied on a complex of finitely generated, projective $\mathbb{Z}[G_0]$-modules. Hence

$$H^i((\widehat{G_0})_{\mathcal{C}}, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]) = Ext^i_{\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]}(\mathbb{Z}_p, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]) \simeq$$

$$H^i(Hom_{\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]}(\mathcal{P}^{del}, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]])) \simeq H^i(\widehat{\mathcal{T}}^{del})$$

is 0 for $i \neq 3$ and is $\mathbb{Z}_p$ otherwise. Note that $(\widehat{G_0})_{\mathcal{C}}$ is a subgroup of finite index in $\widehat{G}_{\mathfrak{C}}$ and by part a) of Theorem 4.1 $\widehat{G}_{\mathfrak{C}}$ is FP_∞ over $\mathbb{Z}_p$ and $cd_p(\widehat{G}_{\mathfrak{C}}) < \infty$. Then we can apply [19, 4.2.9] to get

$$H^i((\widehat{G_0})_{\mathcal{C}}, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]) \simeq H^i(\widehat{G}_{\mathfrak{C}}, \mathbb{Z}_p[[\widehat{G}_{\mathfrak{C}}]]).$$

Then $H^*(\widehat{G}_{\mathfrak{C}}, \mathbb{Z}_p[[\widehat{G}_{\mathfrak{C}}]])$ is concentrated in dimension 3, where it is $\mathbb{Z}_p$. As discussed in the preliminaries for a profinite group $\widehat{G}_{\mathfrak{C}}$ of type FP_∞ over $\mathbb{Z}_p$ and of finite cohomological p-dimension this condition holds if and only if $\widehat{G}_{\mathfrak{C}}$ is a profinite Poincaré duality group at p of dimension 3. $\square$

4.1.1 Proof of Theorem C

Let $\mathcal{C}$ be the set of all normal subgroups U of finite index in G. The proof of Theorem 4.1 does not need that φ_U is an isomorphism for every $U \in \mathcal{C}$ only its corollary that

$$\lim_{\overleftarrow{U \in \mathcal{C}}} H_2(U, \mathbb{F}_p) = 0$$

The last condition was proved in the proof of Theorem 3.8, where it was denoted by (9). Then Theorem 4.1 completes the proof.

4.2 More on pro-p completions

4.2.1 Proof of Theorem A

a) implies b) is a particular case of Theorem 4.1.

b) implies c) follows from the fact that subgroups of finite index in orientable Poincaré duality pro-p groups of dimension 3 are again orientable Poincaré duality pro-p groups of dimension 3 [14, Exer. 1, p.174], [19, 4.4.1] plus the fact that orientable Poincaré duality pro-p groups of dimension 3 have deficiency 0.

c) implies d) follows from Lemma 3.4 and Theorem 3.6 c).

d) implies a) Any subgroup U of p-power index in G is p-good, hence

$$def(\widehat{U}_p) = -dim_{\mathbb{F}_p} H^2(\widehat{U}_p, \mathbb{F}_p) + dim_{\mathbb{F}_p} H^1(\widehat{U}_p, \mathbb{F}_p) =$$

$$-dim_{\mathbb{F}_p} H^2(U, \mathbb{F}_p) + dim_{\mathbb{F}_p} H^1(U, \mathbb{F}_p).$$

Furthermore as U is an orientable Poincaré duality group of dimension 3 we have $H^i(U, \mathbb{F}_p) \simeq H_{3-i}(U, \mathbb{F}_p)$, hence

$$dim_{\mathbb{F}_p} H^2(U, \mathbb{F}_p) - dim_{\mathbb{F}_p} H^1(U, \mathbb{F}_p) = dim_{\mathbb{F}_p} H_1(U, \mathbb{F}_p) - dim_{\mathbb{F}_p} H_2(U, \mathbb{F}_p)$$

$$+ dim_{\mathbb{F}_p} H_3(U, \mathbb{F}_p) - dim_{\mathbb{F}_p} H_0(U, \mathbb{F}_p) = -\chi(U) = 0.$$

Then by Lemma 3.4 φ_U is an isomorphism. This completes the proof.

Corollary 4.2. *Let G be an orientable Poincaré duality group of dimension 3. Assume that the pro-p completion $\widehat{G}_p$ is infinite and that every normal subgroup U of p-power index in G has finite abelianization. Then $\widehat{G}_p$ is an orientable Poincaré duality pro-p group of dimension 3.*

Proof. Follows directly from Corollary 3.7 and Theorem A. □

Corollary 4.3. *Let G be an orientable Poincaré duality group of dimension 3 and $\widehat{G}_p$ be the pro-p completion of G. Assume that $\widehat{G}_p$ is infinite. Then one of the following holds :*

a) *$\widehat{G}_p$ is a pro-p Poincaré duality group of dim 3;*

b) *there exists a normal subgroup V of p-power index in G such that $def(\widehat{V}_p) \geq 2$. In this case V has $\mathbb{Z} \times \mathbb{Z}$ as a quotient and there is no upper bound on the deficiency of the subgroups of finite index in $\widehat{G}_p$;*

c) *there exists a normal subgroup V of p-power index in G such that $def(\widehat{V}_p) = 1$. In this case V has $\mathbb{Z}$ as a quotient. If furthermore there is an upper bound on the deficiency of the subgroups of finite index in $\widehat{G}_p$ then the minimal such upper bound is 1 and $\widehat{G}_p$ is virtually $\mathbb{Z}_p$.*

Proof. By Theorem A the case a) corresponds to the case when $def(\widehat{U}_p) = 0$ for any normal subgroup U in G of p-power index.

If b) holds the proof of [13, Lemma 3, p.359] shows that the abelianization of $\widehat{V}_p$ has as quotient $\mathbb{Z}_p^{def(\widehat{V}_p)}$. By Lemma 3.3

$$def(A) - 1 \geq (\widehat{V}_p : A)(def(\widehat{V}_p) - 1) \geq (\widehat{V}_p : A)$$

for A an open subgroup in $\widehat{V}_p$, in particular there is no upper bound on the deficiency of the subgroups of finite index in $\widehat{G}_p$.

Suppose that c) holds. Using again that the abelianization of $\widehat{V}_p$ has $\mathbb{Z}_p^{def(\widehat{V}_p)}$ as a quotient we see that $\widehat{V}_p$ has a quotient isomorphic to $\mathbb{Z}_p$. Suppose that there is an upper bound on the deficiency of the subgroups of finite index in $\widehat{G}_p$, then case b) does not hold and the minimal upper bound is 1. By Lemma 3.4 and Corollary 3.5b) the kernel of the map

$$\varphi_U : H_2(U, \mathbb{F}_p) \to H_2(\widehat{U}_p, \mathbb{F}_p)$$

is a $\mathbb{F}_p$-vector space of dimension $def(\widehat{U}_p) = 1$, where U is any normal subgroup of G of p-power index such that $U \subseteq V$.

Let $\mathcal{C}$ be the class of all subgroups of G of p-power index. As the inverse limit of $H_2(\widehat{U}_p, \mathbb{F}_p)$ over $U \in \mathcal{C}$ is 0 and inverse limit is a left exact functor, the inverse limit of $H_2(U, \mathbb{F}_p)$ is isomorphic to the inverse limit of the kernels of φ_U. As $Ker(\varphi_U)$ is a vector space over $\mathbb{F}_p$ of dimension at most 1 the inverse limit $\varprojlim H_2(U, \mathbb{F}_p)$ over $U \in \mathcal{C}$ is either $\mathbb{F}_p$ or 0. If it is 0 then by Lemma

2.1, Theorem 2.5 and Proposition 3.1 the complex $\widehat{\mathcal{R}} = \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[\widehat{G}_p]]$ is exact, where

$$\mathcal{R} : 0 \to R_3 \to R_2 \to R_1 \to R_0 \to \mathbb{Z} \to 0$$

is a projective resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ with all modules finitely generated. Then the proof of Theorem 4.1b) implies that $\widehat{G}_p$ is a pro-p Poincaré group of dimension 3 (the condition that φ_U is an isomorphism does not hold in our case but in the proof of Theorem 4.1 this condition was used only to deduce that $\widehat{\mathcal{R}}$ is exact in dimension 2). Then the subgroup $\widehat{U}_p$ of finite index in $\widehat{G}_p$ is again a pro-p orientable Poincaré duality group, hence has deficiency 0 not 1, a contradiction. Thus

$$H_2(\widehat{\mathcal{R}}) \simeq \varprojlim H_2(U, \mathbb{F}_p) \simeq \mathbb{F}_p$$

is the trivial $\mathbb{F}_p[[\widehat{G}_p]]$-module (any continuous action of a pro-p group on $\mathbb{F}_p$ is trivial). Furthermore by Proposition 3.1

$$H_1(\widehat{\mathcal{R}}) = 0 = H_3(\widehat{\mathcal{R}}).$$

Consider the dual complex $\mathcal{S} = Hom_{\mathbb{Z}[G]}(\mathcal{R}, \mathbb{Z}[G])$, thus $\mathcal{S}$ is a complex of left $\mathbb{Z}[G]$-modules. As G is an orientable Poincaré duality group $H^i(\mathcal{S}) = 0$ for $i \neq 3$ and $H^3(\mathcal{S})$ is the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$. Then the complex obtained from $\mathcal{S}$ by adding its unique non-trivial cohomology

$$\mathcal{T} : 0 \to S^0 \to S^1 \to S^2 \to S^3 \to H^3(\mathcal{S}) = \mathbb{Z} \to 0$$

can be viewed as a projective resolution of $\mathbb{Z}$ as a left $\mathbb{Z}[G]$-module. By the above paragraph applied to complexes of left modules instead of right modules $\widehat{\mathcal{T}} = \mathbb{F}_p[[\widehat{G}_p]] \otimes_{\mathbb{Z}[G]} \mathcal{T}$ is not exact, otherwise using again the proof of Theorem 4.1b) $\widehat{G}_p$ is a pro-p Poincaré duality group, a contradiction. Then $\widehat{\mathcal{T}}$ has only one non-trivial homology group isomorphic to $\mathbb{F}_p$ that would be in dimension 2 if $\mathbb{Z}$ was positioned in dimension -1 and $\widehat{\mathcal{T}}$ was chain complex not a cochain complex, so in our case it is the first cohomology

$$H^1(\widehat{\mathcal{T}}) \simeq \mathbb{F}_p.$$

Observe that $H^1(\widehat{\mathcal{T}}) =$

$$Ker(\mathbb{F}_p[[\widehat{G}_p]] \otimes_{\mathbb{Z}[G]} S^1 \to \mathbb{F}_p[[\widehat{G}_p]]) \otimes_{\mathbb{Z}[G]} S^2) / Im(\mathbb{F}_p[[\widehat{G}_p]] \otimes_{\mathbb{Z}[G]} S^0 \to \mathbb{F}_p[[\widehat{G}_p]] \otimes_{\mathbb{Z}[G]} S^1)$$

$$\simeq H^1(\widehat{G}_p, \mathbb{F}_p[[\widehat{G}_p]]).$$

The last isomorphism follows from the fact that the partial complex

$$\widehat{R}_2 \to \widehat{R}_1 \to \widehat{R}_0 \to \mathbb{F}_p \to 0$$

is exact, where $\widehat{\mathcal{R}} = \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[\widehat{G}_p]]$ and as in the proof of Theorem 4.1b) $\widehat{\mathcal{T}} \simeq Hom_{\mathbb{F}_p[[\widehat{G}_p]]}(\widehat{\mathcal{R}}, \mathbb{F}_p[[\widehat{G}_p]])$. Then

$$H^1(\widehat{G}_p, \mathbb{F}_p[[\widehat{G}_p]]) \simeq \mathbb{F}_p$$

and by [12, Thm. 3] $\widehat{G}_p$ is virtually $\mathbb{Z}_p$. □

4.2.2 Proof of Theorem B

Theorem B follows from Corollary 4.3. The remark to Theorem B follows from the fact that if G is non-orientable and $p = 2$ then G has a subgroup U of index 2 that is orientable Poincaré duality group of dimension 3, the pro-2 completion $\widehat{U}_2$ is a subgroup of index 2 in the pro-2 completion $\widehat{G}_2$ and Theorem B applies for the group U. It remains only to point out that if $\widehat{U}_2$ is a pro-2 Poincaré duality group of dimension 3 then it is FP_∞ over $\mathbb{Z}_2$, hence $\widehat{G}_2$ is FP_∞ over $\mathbb{Z}_2$. Finally by the pro-2 version of [17, Chapter 1 §2.6 Exercise 2) (c)] as G is an extension of U by C_2, both U and C_2 are 2-good and U is FP_∞ we deduce that G is 2-good. In particular $H^4(\widehat{G}_2, \mathbb{F}_2) \simeq H^4(G, \mathbb{F}_2) = 0$, hence $cd(\widehat{G}_2) \leq 3$. As in the last paragraph of the proof of Theorem 4.1b) $H^i(\widehat{G}_2, \mathbb{Z}_2[[\widehat{G}_2]]) \simeq H^i(\widehat{U}_2, \mathbb{Z}_2[[\widehat{U}_2]])$ as profinite abelian groups. In particular $H^*(\widehat{G}_2, \mathbb{Z}_p[[\widehat{G}_2]])$ is concentrated in dimension 3, where it is $\mathbb{Z}_p$, thus $\widehat{G}_2$ is a pro-2 Poincaré duality group.

5 More corollaries

Proposition 5.1. *Let G be an abstract orientable finitely presented Poincaré duality group of dimension 3. Suppose there exists a normal subgroup V of p-power index in G such that $def(\widehat{V}_p) \geq 2$, where $\widehat{V}_p$ is the pro-p completion of V. Then V contains a free subgroup of rank 2 and $\widehat{V}_p$ contains a closed free pro-p subgroup of rank 2.*

Proof. Since $\widehat{V}_p$ has deficiency at least 2, it has a pro-p presentation with $d \geq 2$ generators and r relations such that $r \leq d-2$, hence $r \leq d-2 < d^2/4$. Then by the main result of [23] $\widehat{V}_p$ has a closed free pro-p subgroup of rank 2.

By Corollary 4.3b) $\mathbb{Z} \times \mathbb{Z}$ is a quotient of V. Assume that V does not have a free subgroup of rank 2. Then by [3, Thm. D] there is a finitely generated normal subgroup N of G such that $G/N \simeq \mathbb{Z}$, hence by [11, Cor. 1.1] N is a Poincaré duality group of dimension 2 (the version of [11, Cor. 1.1] for fundamental groups of 3-manifolds can be found in [18]), hence a surface group by [6]. As V does not contain a free subgroup of rank 2 the surface group is $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \rtimes \mathbb{Z}$ with action of $\mathbb{Z}$ on $\mathbb{Z}$ given by multiplication with -1. Then V and its pro-p completion $\widehat{V}_p$ are soluble groups, a contradiction with the fact that $\widehat{V}_p$ has a closed free pro-p subgroup of rank 2. $\square$

Proposition 5.2. *Let G be an abstract orientable Poincaré duality group of dimension 3 and $\widehat{G}_p$ be the pro-p completion of G. Assume that the pro-p completion $\widehat{G}_p$ of G is infinite. Then one of the following or both hold :*

a) *$\widehat{G}_p$ is a pro-p Poincaré duality group of dim 3;*

b) *G has a subgroup of p-power index that is an HNN- extension with finitely generated base and associated subgroups. In general this HNN extension need not be ascending or descending.*

Proof. If a) does not hold by Corollary 4.3 there is a normal subgroup V of p-power index in G such that V maps surjectively to $\mathbb{Z}$. As G is of type FP_2 any subgroup of finite index is FP_2, in particular this holds for V. By the main result of [2] a group of type FP_2 that maps surjectively to $\mathbb{Z}$ is an HNN extension with finitely generated base and associated subgroups. $\square$

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