Limit Groups are Conjugacy Separable

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Abstract

A limit group is a finitely generated subgroup of a residually free group. We prove the result announced in the tittle.

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1 Introduction

A group G is conjugacy separable if whenever x an y are non-conjugate elements of G, there exists some finite quotient of G in which the images of x and y are non-conjugate. The notion of the conjugacy separability owes its importance to the fact, first pointed out by Mal'cev [M-58], that the conjugacy problem has a positive solution in finitely presented conjugacy separable groups.

The objective of this paper is to prove the conjugacy separability for limit groups, i.e., finitely generated residually free groups.

Theorem 1.1. *A limit group is conjugacy separable.*

Limit groups play a key role in the solution of the Tarski problems ([K-M-06], [K-M-05], [K-M1-05], [S1]-[S6]) that asked whether the theories of free groups of different ranks > 2 are the same and whether this theory is decidable.

Kharlampovich and Myasnikov have studied limit groups extensively under the name fully residually free groups (see [K-M-98] and [K-M2-98]). Remeslennikov [R-89] had previously referred to them as $\exists$-free groups, reflecting the fact

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that these groups have the same existential theory as a free group, or ω-residually free groups.

The Lyndon group plays a very important role in algebraic geometry over groups (see [K-M-98] and [K-M2-98]). It was proved in [K-M2-98]) that a finitely generated group is fully residually free (i.e. a limit group) if and only if it is isomorphic to a subgroup of the Lyndon group. It was proved by Lioutikova [L] that the Lyndon group is conjugacy separable. We give a different proof of this in the paper.

Combined with observation of Mal'cev [M-58] our theorem gives a new proof for the fact that a conjugacy problem admits positive solution for limit groups (cf. [K-M-R-S-2004]).

Our proof is based on the results of the paper [R-S-Z-98], where it was proved that certain residual properties and, in particular, the conjugacy separability, are preserved by free products with cyclic amalgamations. Bass-Serre theory of groups acting on trees and its profinite version are also explored.

2 Preliminaries

The profinite topology on a group G is the topology where the collection of all finite index normal subgroups of G serves as a fundamental system of neighborhoods of the identity element $1 \in G$, turning G into a topological group. Note that for a subgroup H of G, the profinite topology of H can be stronger than the topology induced by the profinite topology of G.

The completion $\widehat{G}$ of G with respect to this topology is called the profinite completion of G and can be expressed as an inverse limit

$$\widehat{G} = \varprojlim_{N} G/N$$

of all finite quotients of G. Thus $\widehat{G}$ is a profinite group. Moreover, there exists a natural homomorphism $\iota : G \longrightarrow \widehat{G}$ that sends $g \mapsto (gN)$; ι is a monomorphism when G is residually finite. If S is a subset of $\widehat{G}$, we denote by $\overline{S}$ its closure in $\widehat{G}$. The profinite topology on G is induced by the topology of $\widehat{G}$.

The next proposition expresses the conjugacy separability property of G in terms of its profinite topology and we shall use it freely in the paper.

Proposition 2.1. *Let G be a group, then the following conditions are equivalent:*

(i) G is conjugacy separable;

(ii) for each $x \in G$, the conjugacy class of x^G of x is closed in the profinite topology. In particular G is residually finite;

(iii) G is residually finite and for each pair of elements $x, y \in G$ such that $y = x^\gamma$, for some $\gamma \in \widehat{G}$, there exists $g \in G$ such that $y = x^g$.

Our main tool are the results in [R-S-Z-98]. To explain these results define the class $\mathcal{X}'$ to be the class of groups obtained by forming successfully free products with cyclic amalgamation starting from free by finite or polycyclic by finite groups. One of the main result in [R-S-Z-98] is the following

Theorem 2.2 (R-S-Z-98)**.** *Any group $G \in \mathcal{X}'$ has the following properties:*

(i) G is conjugacy separable;

(ii) G is quasi-potent, i.e. each cyclic subgroup H of G contains a finite index subgroup K whose every subgroup of finite index is of the form $H \cap N$ for some normal subgroup N of finite index in G;

(iii) the product AB of cyclic subgroups A and B of G is closed in the profinite topology of G;

(iv) every cyclic subgroup of G is conjugacy distinguished, i.e. $\cup_{g \in G} H^g$ is closed in the profinite topology of G.

(v) for any pair of cyclic subgroups C_1 and C_2 of G, one has $C_1 \cap C_2 = 1$ if and only if $\overline{C_1} \cap \overline{C_2} = 1$, where $\overline{X}$ denotes the closure of a subset X in $\widehat{G}$.

(vi) for any element g of infinite order in G and every $\gamma \in \widehat{G}$ such that $\gamma\overline{\langle g \rangle}\gamma^{-1} = \overline{\langle g \rangle}$, one has $\gamma g \gamma^{-1} = g$ or $\gamma g \gamma^{-1} = g^{-1}$.

To each free amalgamated product $G = G_1 *_C G_2$ one can associate a standard tree $S(G)$, constructed as follows: the vertex set is $V(S(G)) = G/G_1 \cup G/G_2$, the edge set is $E(S(G)) = G/H$, and the initial and terminal vertex of an edge gH are respectively gG_1 and gG_2. The group G acts naturally on $S(G)$. Similarly for a profinite amalgamated free product $\widehat{G} = \widehat{G_1} \amalg_{\widehat{H}} \widehat{G_2}$ one can associate a profinite standard tree $S(\widehat{G})$ whose vertex set $V(S(\widehat{G})) = \widehat{G}/\widehat{G}_1 \cup \widehat{G}/\widehat{G}_2$, the edge set is $E(S(\widehat{G})) = \widehat{G}/\widehat{H}$, and the initial and terminal vertex of an edge $g\widehat{H}$ are $g\widehat{G_1}$ and $g\widehat{G_2}$ respectively (see [Z-M-89]). The sets $V(S(\widehat{G})), E(S(\widehat{G}))$ are profinite spaces (i.e, they are compact Hausdorff totally disconnected topological spaces), and the natural action of $\widehat{G}$ on $S(\widehat{G})$ is continuous.

The profinite topology on $G = G_1 *_C G_2$ is called *efficient* if G is residually finite, the profinite topology on G induces the full profinite topology on G_1, G_2 and C, and these subgroups are closed in the profinite topology of G. Note that if the profinite topology on G is efficient, then by the universal property for the profinite amalgamated free product, the profinite completion $\widehat{G}$ of G is the profinite amalgamated free product $\widehat{G} = \widehat{G_1} \amalg_{\widehat{C}} \widehat{G_2}$ of the profinite completions of the factors.

The following remark allows to use the profinite version of the Bass-Serre theory of groups acting on trees.

Remark 2.3. *If G belongs to the class $\mathcal{X}'$, then the properties (ii) and (iv) in Theorem 2.2 imply that the profinite topology on G is efficient (see Lemma 2.1 in [R-Z-96]). The efficience of the profinite topology on G implies in turn that $S(G)$ embeds naturally in $S(\widehat{G})$. This follows from the fact that G/G_i embeds in $\widehat{G}/\widehat{G}_i$ because G_i are closed in G for $i = 1, 2$. Moreover, $S(G)$ is dense in $S(\widehat{G})$.*

3 Proofs

We apply Theorem 2.2 to give another proof of conjugacy separability of the Lyndon group. The construction of the Lyndon group can be given as follows (see [M-R-96], Theorem 8): Let F be a free group and put $\mathcal{Y}_1 = F$. For $i > 1$, define the class $\mathcal{Y}_i$ to consist of all groups that are free products $G_i = G_{i-1} *_C A$ of a group $G_{i-1} \in \mathcal{Y}_{i-1}$ and a free abelian group A of finite rank amalgamating maximal cyclic subgroup of G_{i-1} with a subgroup of A generated by a generater of A (this construction is known as an extension of the centralizer). Let $\mathcal{Y} = \bigcup_{n\in\mathbb{N}} \mathcal{Y}_n$. Clearly, the groups of $\mathcal{Y}$ constitute an inductive system with respect to inclusions. The Lyndon group L is defined to be the inductive limit $L = \varinjlim_{G\in\mathcal{Y}} G$.

The class $\mathcal{Y}$ is a subclass of $\mathcal{X}'$ and so as an immediate consequence of Theorem 2.2 we have the following

Proposition 3.1. *Each group from $\mathcal{Y}$ enjoys the properties (i)-(iv) of Theorem 2.2. In particular, every group of $\mathcal{Y}$ is conjugacy separable.*

Theorem 3.2. *The Lyndon group is conjugacy separable.*

Proof. Let a, b be elements of the Lyndon group L which are conjugate in $\widehat{L}$. Then there exists $G_i \in \mathcal{Y}_i$ such that $a, b \in G_i$. We claim that there exists an epimorphism f of L onto G_i such that $f_{|G_i}$ is id.

First note that for any group $G_j = G_{j-1} *_C A$ from $\mathcal{Y}_j$ there is an epimorphism $f_j : G_j \longrightarrow G_{j-1}$ constructed as follows: choose a direct complement to C in A and send it to 1; send the elements of G_{j-1} identically to G_{j-1} and extend this map to a homomorphism f_j by the universal property for amalgamated free products. Put $f_{ji} = f_{i+1}f_{i+2}\cdots f_{j-1}f_j$ for $j > i$. By the universal property of a direct limit there exists $f : L \longrightarrow G_i$ that extends f_{ji} for all $j > i$. Note that $\varphi_i f = \mathrm{id}$, where $\varphi_i : G_i \longrightarrow L$ is the natural embedding.

Extend f to $\hat{f} : \widehat{L} \longrightarrow \widehat{G}_i$. Since a and b are conjugate in the completion of the Lyndon group, their images in G_i conjugate in $\widehat{G}_i$. Then, by Proposition 3.1, they are conjugate in G_i as needed. □

Lemma 3.3. *Let G be a limit group and H a cyclic subgroup of G. Then $N_G(H) = C_G(H)$.*

Proof. Pick $n \in N_G(H) \setminus H$. By Lemma 1 in [B-62] a 2-generated residually free group is either free or abelian, hence so is the subgroup $\langle n, H\rangle$. Since the normalizer of every cyclic subgroup of a free group coincides with the centralizer, the result follow. □

Since a limit group G is a finitely generated subgroup of the Lyndon group L (see Theorem 4 in [K-M2-98]), there exists n such that G embeds in some $G_n \in \mathcal{Y}_n$.

Proposition 3.4. *Let G be a limit group and H a cyclic subgroup of G. Then $N_{\widehat{G}}(\bar{H}) = C_{\widehat{G}}(\bar{H})$.*

Proof. Since G is a subgroup of the group G_n it suffices to prove the proposition assuming that $G = G_n$. Let h be a generator of H and $\gamma \in N_{\widehat{G}}(\bar{H})$. Then by Proposition 3.1 and Theorem 2.2 either γ centralizers h or $h^\gamma = h^{-1}$. Since $G = G_n$ is conjugacy separable (see Proposition 3.1 again) there exists $g \in G$ with $h^g = h^{-1}$ contradicting the preceding lemma. □

Lemma 3.5. *Let $G \in \mathcal{Y}_n$ and g be an element of G. Then*

(a) $\overline{\langle g\rangle} \cap \overline{\langle g\rangle}^x \neq 1$ *implies* $\overline{\langle g\rangle} = \overline{\langle g\rangle}^x$ *for any* $x \in \widehat{G}$.

(b) $\overline{C_G(\langle g\rangle)} = C_{\bar{G}}(\overline{\langle g\rangle})$

Proof. We use induction on n. Without loss of generality we may assume that $\langle g\rangle$ is maximal cyclic.

Let $n = 1$. Then G is free. Consider its action on its Cayley graph $\Gamma(G)$ and the action of $\widehat{G}$ on its Cayley graph $\Gamma(\widehat{G})$. We think of $\Gamma(G)$ as a dense subgraph of $\Gamma(\widehat{G})$.

Put $Z = \langle g \rangle$. By Proposition 3.4 in [S-77] there exists the infinite straight line T_g on which g acts. The intersection $\widehat{Z} \cap \widehat{Z}^x$ is non-trivial and acts on $\overline{T_g} \cap x^{-1}\overline{T_g}$ and since $\overline{T_g}$ have no nontrivial infinite closed subgraphs (see Lemma 4.4 in [R-S-Z-98]) $\overline{T_g} = x^{-1}\overline{T_g}$. It follows that x acts on $\overline{T_g}$. Let H be a closed subgroup of $\widehat{G}$ leaving $\overline{T_g}$ invariant. Then $H/\bar{Z}$ acts freely on a circuit $\overline{T_z}/\bar{Z}$. Thus H is the profinite fundamental group of a circuit $\overline{T}_z/H = (\overline{T_z}/\bar{Z})/(H/\bar{Z})$ and so is procyclic. Since $\overline{\langle g \rangle}$, $\overline{\langle g \rangle}^x$ are subgroup of H by Lemma 2.2 in [R-Z-96] $\overline{\langle g \rangle} = \overline{\langle g \rangle}^x$. This proves (a).

Since $\overline{T_g}$ is the unique minimal g-invariant subtree of $S(\widehat{G})$ (cf. [R-Z-96], Lemma 2.2), $C_{\widehat{G}}(g)$ acts naturally on $\overline{T_g}$ and so is contained in H. But $\overline{T_g}/H = T_g/(G \cap H)$ because for $h \in H$ and $m, hm \in \Gamma(G)$ one has obligatory that $h \in G \cap H$. Hence $H = \overline{H \cap G}$. Since Z is maximal abelian $Z = H \cap G$ and so $H = C_{\widehat{G}}(g) = \bar{Z} = \widehat{Z}$ and $x \in \widehat{Z}$ follows.

Suppose now $n > 1$ and for $n - 1$ the proposition holds. Recall that $G = G_{n-1} *_C A$, where $G_{n-1} \in \mathcal{Y}_{n-1}$, A is free abelian of finite rank and C is infinite cyclic. Let $S(G)$ and $S(\widehat{G})$ be the trees associated with decompositions of G and $\widehat{G}$. Since the profinite topology on G is efficient, $S(G)$ is embedded in $S(\widehat{G})$ (see Remark 2.3).

Claim 1. Let $g \in \widehat{C}$. Then $C_{\widehat{G}}(g) = \widehat{A}$.

By Corollary 2.7 in [R-Z-96] combined with Proposition 3.4

$$C_{\widehat{G}}(g) = N_{\widehat{G}}(\overline{\langle g \rangle}) = N_{\widehat{G}_{n-1}}(\overline{\langle g \rangle}) \amalg_{\widehat{C}} N_{\widehat{A}}(\overline{\langle g \rangle}) = C_{\widehat{G}_{n-1}}(g) \amalg_{\widehat{C}} C_{\widehat{A}}(g).$$

By the induction hypothesis $C_{\widehat{G}_{n-1}}(g) = \overline{C_{G_{n-1}}(g)} = \widehat{C}$. So $C_{\widehat{G}}(g) = \widehat{C} \amalg_{\widehat{C}} \widehat{A} = \widehat{A}$ as required.

Claim 2.

- If $g \in G_{n-1} \setminus A^{G_{n-1}}$. Then $C_{\hat{G}}(g) \cap \widehat{G}_e = 1$, for all $e \in E(S(\widehat{G}))$.
- If $g \in A$, then $C_{\hat{G}}(g) = \widehat{A}$.

Suppose first that $g \in G_{n-1} \setminus A^{G_{n-1}}$. Let $1 \neq z \in C_{\hat{G}}(g) \cap \widehat{G}_e$. Since $g \in G_{n-1}$, g stabilizes a vertex v.

By Proposition 2.8 in [Z-M-89] $g = g^z$ stabilizes the geodesic $[v, x^{-1}v]$. If $v = z^{-1}v$ it follows that $z \in \widehat{G}_{n-1}$ and so the result follows from the fact that C is conjugacy distinguished in G_{n-1}. Otherwise, since $\emptyset \neq E[v, z^{-1}v] = E(S(\widehat{G})) \cap [v, z^{-1}v]$ is compact, by Proposition 2.15 in [Z-M-89] there exist an edge $e \in [v, z^{-1}v]$ whose vertex is v. It follows that g stabilizes e. Since C is conjugugacy distinguished in $\widehat{G}_{n-1}$, g is conjugate to an element of C contradicting the hypothesis.

Suppose now that $g \in A$. Let $h \in \hat{G}$ with $[h, g] = 1$ and w a vertex whose stabilizer is equal to A. Then $gw = w$ and $ghw = hw$ so by Theorem 2.8 in [Z-M-89] g stabilizes the geodesic $[w, hw]$. If $w = hw$, then $h \in \widehat{A}$ and there is nothing to prove. Otherwise, as before there exist the edge e in $[w, hw]$ that have w as a vertex. Then conjugating g if necessary we may assume that $g \in \widehat{G}_e = \widehat{C}$, and by Claim 1 $C_{\hat{G}}(g) = \hat{A}$. The claim is proved.

Case 1 (non-hyperbolic). g stabilizes a vertex v.

(a) Put $Z = \langle g \rangle$. If x centralizes Z there is nothing to prove. Without loss of generality may assume that $g \in G_{n-1} \cup A$.

Let v be a vertex stabilized by G_{n-1} or A. If $v = xv$, then $x \in G_{n-1}$ or $x \in A$, so by induction hypothesis the result follows. Suppose $v \neq xv$. Then by Theorem 2.8 in [Z-M-89] Z stabilizes a geodesic $[v, xv]$ in $S(\widehat{G})$. Hence Z stabilize an edge and so is conjugate in $\widehat{G}$ to a subgroup of $\widehat{C}$. Since C is conjugacy distinguished (see Proposition 3.1 and Theorem 2.2) Z is conjugate in G to a subgroup of C and hence is conjugate of C since is maximal cyclic. Thus we may assume that $Z = C$.

Since $C \cap C^x$ and $(C \cap C^x)^x$ are subgroups of C^x, by Lemma 2.4 (ii) in [R-Z-96] they are equal, and so $C \cap C^x$ is normalized by x. Then by Claim 1 $x \in \widehat{A}$, and so $C^x = C$.

(b) Since g is conjugate to an element of $G_{n-1} \cup A$, we can assume that g is in G_{n-1} or A, say in G_{n-1}. Let g be an element of G and suppose that $\gamma \in \widehat{G} = \widehat{G_{n-1}} \amalg_{\widehat{C}} \widehat{A}$ satisfies $\gamma g \gamma^{-1} = g$. If $\gamma \in \widehat{G}_{n-1}$ then the result follows from the induction hypothesis. Otherwise, by Theorem 3.12 in [Z-M-89], $g \in \delta \widehat{C} \delta^{-1}$ for some $\delta \in \widehat{G}_{n-1}$. By Proposition 3.1 and Theorem 2.2 C is conjugacy distinguished so g is conjugate in G_{n-1} to an element of C, and therefore we may assume that $g \in C$. By Corollary 2.7 in [R-Z-96] $N_{\widehat{G}}(\overline{\langle g \rangle}) = N_{\widehat{G}_{n-1}}(\overline{\langle g \rangle}) \amalg_{\widehat{C}} N_{\widehat{A}}(\overline{\langle g \rangle})$ and so by Proposition 3.4,

$$C_{\widehat{G}}(g) = N_{\widehat{G}}(\overline{\langle g \rangle}) = N_{\widehat{G}_{n-1}}(\overline{\langle g \rangle}) \amalg_{\widehat{C}} N_{\widehat{A}}(\overline{\langle g \rangle}) = C_{\widehat{G}_{n-1}}(g) \amalg_{\widehat{C}} C_{\widehat{A}}(g).$$

Since $C_{\widehat{G}_{n-1}}(g) = \overline{C_{G_{n-1}}(g)}$ by induction hypothesis, the result follows in this case.

Case 2. g does not stabilize any vertex of $S(G)$.

(b) By Proposition 3.4 in [S-77] there exists the infinite straight lines T_g on which g acts. Since $\overline{T_g}$ is the unique minimal g-invariant subtree of $S(\widehat{G})$ (cf. [R-Z-96], Lemma 2.2), $C_{\widehat{G}}(g)$ acts naturally on $\overline{T_g}$. Moreover, the kernel of this action is trivial. Indeed, if not then by Case 1 (a) all edge stabilizers of $\overline{T_g}$ are equal and so g normalizes an edge stabilizer $\widehat{G}_e$. But $N_{\widehat{G}}(\widehat{G}_e) = C_{\widehat{G}}(\widehat{G}_e)$ by Lemma 3.4. Therefore, by Case 1 (b) applied to a generated of G_e, so $g \in \widehat{G}_e$, contradicting to g being hyperbolic.

We prove that $\overline{T_g}/C_{\widehat{G}}(g) = T_g/C_G(g)$. Indeed, for $e \in E(T_g)$ and $z \in C_{\widehat{G}}(g)$ suppose $ze \in T_g$. Translating e and conjugating g correspondingly we may assume that e is the edge stabilized by C. Choose $h \in G$ such that $he = ze$. Then there exists $\hat{c} \in \widehat{C}$ such that $h\hat{c} = z$. Let e_1 be the third edge of $[e, ze]$ (the geodesic $[e, ze]$ has more then two edges since two adjacent edges in $S(\widehat{G})$ have opposite orientation and therefore can not be translations of each other).

We show that if $\hat{c} \notin C$ then $\hat{c}e_1 \notin S(G)$. Indeed, otherwise $\hat{c}\hat{g}_{e_1} \in G$ for some $\hat{g}_{e_1} \in \widehat{G}_{e_1}$ and since by Proposition 3.1 G satisfies property (iii) of Theorem 2.2, CG_{e_1} is closed in the profinite topology of G; therefore $\hat{c}\hat{g}_{e_1} = cg_{e_1}$ for some $c \in C, g_{e_1} \in G_{e_1}$, so that $c^{-1}\hat{c} = g_{e_1}\hat{g}_{e_1}^{-1}$.

To arrive at contradiction we show that $c^{-1}\hat{c} = 1$. Choose two edges from $[e, e_1]$ that have the common vertex v whose stabilizer is a conjugate of $\widehat{G}_{n-1}$, say e, e_0. If $1 \neq c^{-1}\hat{c} \in \bigcap_{e\in[e,e_1]} \widehat{G}_e$, then by Case 1 (a) $\widehat{C} = \widehat{G}_{e_0} = \widehat{C}^{g_v}$ for some $g_v \in \widehat{G}_v$. Hence $g_v \in N_{\widehat{G}}(\widehat{C}) = C_{\widehat{G}}(\widehat{C})$ (see Proposition 3.4) and $g_v \notin \widehat{C}$. Since C is self centralized, this contradicts maximality of the abelian group $\widehat{C}$ proved in Case 1 (b). Therefore $\hat{c} = c$ contradicting $\hat{c} \notin C$.

Now $\hat{c}e_1 \notin S(G)$ implies $h\hat{c}e_1 = ze_1 \notin S(G)$ because h leaves $S(G)$ invariant and so by Lemma 4.3 (iii) in [R-S-Z-98] $h\hat{c}e = ze$ can not be in $S(G)$. This contradiction shows that $z \in G \cap C_{\widehat{G}}(g) = C_G(g)$ as required.

Since the action of $C_{\widehat{G}}(g)$ on $\overline{T_g}$ is free and $\overline{T_g}/C_{\widehat{G}}(g) = T_g/C_G(g) = \overline{T_g}/\overline{C_G(g)}$ we deduce $C_{\widehat{G}}(g) = \overline{C_G(g)}$.

(a) Put $Z = \langle g \rangle$. By Proposition 3.4 in [S-77] there exists the infinite straight lines T_g on which g acts. The intersection $\widehat{Z} \cap \widehat{Z}^x$ is non-trivial it acts on $\overline{T_g} \cap x\overline{T_g}$ and since $\overline{T_g}$ have no nontrivial infinite closed subgraphs (see Lemma 4.4 in [R-S-Z-98]) $\overline{T_g} = x\overline{T_g}$. It follows that x acts on $\overline{T_g}$. Let H be the maximal closed subgroup of $\widehat{G}$ leaving $\overline{T_g}$ invariant. By Case 1 (a) applied to the stabilizer of an edge in T_g, the kernel of the action of H on $\overline{T_g}$ is trivial. Indeed, if not then by Case 1 (a) all edge stabilizers of $\overline{T_g}$ are equal and so g normalizes an edge stabilizer $\widehat{G}_e$. But $N_{\widehat{G}}(\widehat{G}_e) = C_{\widehat{G}}(\widehat{G}_e)$ by Lemma 3.4. Therefore, by Case 1 (b) applied to a generate of G_e, you have $g \in \widehat{G}_e$, contradicting to g being hyperbolic. Therefore, the H-stabilizers of vertices in $\overline{T_g}$ are of order at most 2 and since $\widehat{G}$ is torsion free, H acts freely on $\overline{T_g}$. Then $H/\bar{Z}$ acts freely on a circuit $\overline{T_g}/\bar{Z}$. Thus H is the profinite fundamental group of a circuit $\overline{T}_g/H = (\overline{T_g}/\bar{Z})/(H/\bar{Z})$ and so is procyclic. Therefore x centralize Z and the result follows. □

In the next proposition we prove the conjugacy separability for a subgroup of finite index of $G_n \in \mathcal{Y}_n$. We note that in general it is an open question whether a subgroup of finite index of a conjugacy separable group is conjugacy separable.

Remark: The statement (b) of Lemma 3.5 is valid in fact for Limit group L. Indeed, L is a subgroup of some $G_n \in \mathcal{Y}_n$. By Theorem 3.7 below there exist a subgroup of finite index H in G that contain L such that L is semi direct factor of H. Therefore it suffices to prove the result for H.

Since $C_H(g) = C_G(g) \cap H$ and $C_{\widehat{H}}(g) = C_{\widehat{G}}(g) \cap \widehat{H}$, then $C_H(g)$ is dense in $C_{\widehat{H}}(g)$ by Exercise 3 on page 9 in [W-1998].

Proposition 3.6. *Let H be a finitely generated finite index subgroup of a group $G = G_n \in \mathcal{Y}_n$. Then H is conjugacy separable.*

Proof. Let $h_1, h_2 \in H$ be elements such that $h_1 = h_2^\gamma$, where $\gamma \in \widehat{H}$. We show that h_1 and h_2 are conjugate in H.

By Proposition 3.1 G_n is conjugacy separable, so there exists $g \in G_n$ such that $h_1^g = h_2$. Then $\delta := g\gamma \in C_{\widehat{G}}(h_1)$. It follows that $\gamma^{-1}\delta \in C_{\widehat{G}}(h_1)\widehat{H} \cap G$. Since H is of finite index in G the set $C_G(h_1)H$ is closed in the profinite topology, i.e. $\overline{C_G(h_1)H} \cap G = C_G(h_1)H$. By Lemma 3.5 $\overline{C_G(h_1)H} = C_{\widehat{G}}(h_1)\widehat{H}$, so $C_{\widehat{G}}(h_1)\widehat{H} \cap G = C_G(h_1)H$ and therefore $g = ch$ for some $c \in C_G(h_1), h \in H$. Hence $h_1^g = h_1^h = h_2$ as needed. □

Theorem 3.7 (W-2006)**.** *Let G be a limit group and H a finitely generated subgroup of G. Then there exists a finite index subgroup K of G containing H and a epimorphism $\varphi : K \to H$, such that $\varphi_{|H} = id$.*

Theorem 3.8. *A limit group is conjugacy separable.*

Proof. Let G be a limit group and $h_1, h_2 \in G$ elements such that $h_1 = h_2^\gamma$ for some $\gamma \in \widehat{G}$. We show that h_1 and h_2 are conjugate in G. Pick G_n such that $G \leq G_n$.

Since every finitely generated subgroup of a Lyndon group is a limit group, G_n is a limit group. Then by Theorem 3.7 there exists a finite index subgroup U of G_n and an epimorphism $f : U \longrightarrow G$ such that $f_{|G} = id$. By Proposition 3.6 U is conjugacy separable, so h_1 and h_2 are conjugate in U. It follows that $h_1^{f(u)} = h_2$ as needed. □

References

[B-62] Baumslag G., *On generalised free products*, Math. Zeitschr., **78** (1962), 423–438.

[K-M-98] Kharlampovich O., Myasnikov A., *Irreducible affine varieties over a free group. I. Irreducibility of quadratic extensions and Nullstellensatz*, J. Algebra **200** (1998) 472–516.

[K-M2-98] Kharlampovich O., Myasnikov A., *Irreducible affine varieties over a free group. II. Systems in triangular quasi quadratic form and description of resudually free groups*, J. Algebra **200** (1998) 517–570.

[K-M-06] Kharlampovich O., Miasnikov A., *Elementary theory of free non-abelian groups* Journal of Algebra **302** (2006) 451–552.

[K-M-05] Kharlampovich O., Myasnikov A., *Implicit function theorem over free groups* Journal of Algebra, **290** (2005), 1–203.

[K-M1-05] O. Kharlampovich, A. Miasnikov Effective JSJ decompositions Contemp.Math. AMS, Algorithms,Languages, Logic (Borovik, ed.), CONM/ 378, 2005, 87–212.

[L] Lioutikova, E., *Lyndon's group is conjugately residually free*, Internat. J. Algebra Comput. **13** (2003), no. 3, 255–275.

[M-58] Mal'cev A. I., *On Homormorphisms onto finite groups*, Uchen. Zap. Ivanovskogo Gos. Ped. Inst., **18** (1958) 40-60.

[K-M-R-S-2004] Kharlampovich O., Myasnikov A. G., Remeslennikov V. N., and Serbin D. E., *Regular Free Length Functions on Lyndon's Free $Z[t]$-group $F^{Z[t]}$*, Contemporary Mathematics, **360** (2004) 63–101.

[M-R-96] Myasnikov A. G., Remeslennikov, V. N., *Exponential groups 2: extension of centralizers and tensor completion of CSA-groups*. Intern. Journal of Algebra and Computation, **6** (1996) 687-711.

[R-89] Remeslennikov V. N., *$\exists$-free groups*, Sibirsk. Mat. Zh., **30** (1989), no. 6, 193-197.

[R-S-Z-98] Ribes L., Segal D., Zalesskii P. A., *Conjugacy separability and free products of groups with cyclic amalgamation*, J. London Math. Soc., 2, **57** (1998) 609-628.

[R-Z-96] Ribes, L., Zalesskii, P. A., *Conjugacy separability of amalgamated free products of groups*, Journal of Algebra, **179** (1996) 751-774.

[S1] Z. Sela, *Diophantine geometry over groups. I: Makanin-Razborov diagrams*, Publ. Inst. Hautes Études Sci. **93** (2001) 31-105.

[S2] Sela, Z. *Diophantine geometry over groups. II. Completions, closures and formal solutions.* Israel J. Math. **134** (2003), 173–254.

[S3] Sela, Z. *Diophantine geometry over groups. III. Rigid and solid solutions.* Israel J. Math. **147** (2005) 1–73.

[S4] Sela, Z. *Diophantine geometry over groups. IV. An iterative procedure for validation of a sentence.* Israel J. Math. **143** (2004) 1–130.

[S5] Sela, Z. *Diophantine geometry over groups V_2: quantifier elimination. II.* Geom. Funct. Anal. **16** (2006) 537–706.

[S6] Sela, Z., *Diophantine geometry over groups. VI. The elementary theory of a free group*, Geom. Funct. Anal. **16** (2006) 707–730.

[S-77] Serre, J. P., *Trees*, Springer Monographs in Mathematics, Springer, 1977.

[Z-M-89] Zalesskii, P. A., Mel'nikov, O. V., *Subgroups of profinite groups acting on trees*, Math. Sb., **63** (1989) 405–424.

[W-1998] Wilson, J. S., *Profinite Groups*, 1998. Clarendon Press, Oxford.

[W-2006] Wilton, H., *Hall's theorems for limit groups*, 2006. Preprint. ArXiv: math.GR/0605546.