CORRIGENDUM AND ADDENDUM: VIRTUALLY FREE PRO-p GROUPS

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Abstract. This note corrects Lemma 3.8 in [1] and in a separate section we explain how to modify the proof of Proposition 4.8 ibidem in order to see that the proof of the main result is valid.

The objective of this note is correcting the two statements of [1, Lemma 3.8] in Lemmas 3.8 and 3.13 below. In a separate section we recall Lemma 4.8 from [1] and modify its proofs whereever this lemma had been quoted.

It also appears useful to explain in more detail the connection between the concept of *permutation extension* (of a finitely generated free pro-p group by some finite p-group) and a certain pro-p HNN-extensions.

For easier comparison we keep the relevant section numbers from [1].

Definition 3.6. Given a finite p-group K and a finite K-set X, on which K acts from the right, there is a natural extension of the action of K to the free pro-p group $\tilde{F} = F(X)$. The semidirect product $\tilde{G} := \tilde{F} \rtimes K$ will be called the *permutational extension* of $\tilde{F}$ by K. Now K acts on $\tilde{F}$ from the right by conjugation, i.e., $f \cdot k := f^k$.

Intimately connected with this notion will be a certain type of HNN-extensions:

Definition 3.7. Given a finite group K and a set Z, for a nonempty set I, a collection $\mathcal{A} := \{A_i : i \in I\}$ of pairwise nonconjugate subgroups of K as well as a collection $\mathcal{Z} := \{Z_i : i \in I\}$ of pairwise disjoint subsets of Z with $Z = \bigcup_{i\in I} Z_i$, form the quotient group $\tilde{G}$ of $G := F(Z) \amalg K$ modulo the relations

$$(*)\qquad L := \bigcup_{i\in I} L_i, \quad L_i := \{z_i^{a_i} z_i^{-1} : a_i \in A_i, z_i \in Z_i\}.$$

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Any group arising in this form we call a *central HNN-extension*.

Certainly every central HNN-extension is an HNN-extension in the sense of [1, Definition 3.1]. Namely, for $Z = \bigcup_{i \in I} Z_i$ to be taken as the index set for the presentation of $\tilde{G}$ as an HNN-extension, define $A_z := A_i$ if $z \in Z_i$, and let $\phi_z : A_i \to K$ be the canonical embedding for $z \in Z_i$. Then

$$\tilde{G} = \mathrm{HNN}(K, A_z, \phi_z, z \in Z).$$

We shall find it convenient to denote $\tilde{G}$ by $\mathrm{HNN}(K, A_i, Z_i, i \in I)$ in order to emphasize the role of the sets of *stable letters* Z_i.

We shall abbreviate $\mathrm{HNN}(K, A_i, Z_i, i \in I)$ to $\mathrm{HNN}(K, A, z)$ if I and $Z_i = \{z\}$ are singleton sets.

Lemma 3.8. *Every central HNN-extension* $\tilde{G} = \mathrm{HNN}(K, A_i, Z_i, i \in I)$ *gives rise to a permutation extension* $\tilde{G} = F(X) \rtimes K$ *where*

$$X = \bigcup_{i \in I} X_i, \quad \text{for} \quad X_i := \bigcup_{r \in R, s \in S} Z_i^{sr}.$$

Here S_i *and* R_i *are coset representative sets of respectively* $A_i \backslash N_K(A_i)$ *and* $N_K(A_i) \backslash K$.

Proof. Since every $k \in K$ has a unique decomposition

$$k = a_i s_i r_i, \ a_i \in A_i, \ s_i \in S_i, \ r_i \in R_i,$$

and every element $x \in X$ can be uniquely presented

$$x = z_i^{s_i' r_i'}, \ s_i' \in S_i, \ r_i' \in R_i$$

it turns out that

$$s_i' r_i' a_i s_i r_i = a_i'' x_i'' r_i'', \quad \text{for suitable} \quad a_i'' \in A_i, \ s_i'' \in S_i, \ r_i'' \in R_i,$$

and thus

$$x^k = z_i^{s_i' r_i' a_i s_i r_i} = z_i^{a_i'' s_i'' r_i''} = z_i^{s_i'' r_i''} \in X.$$

Therefore X is K-invariant.

For proving that X is a basis for $F(X) \leq \tilde{G}$ consider the epimorphism

$$\chi : \coprod_{k \in K} F(Z)^k = \coprod_{i \in I} \coprod_{s \in S_i} \coprod_{r \in R_i} \left(\coprod_{a \in A_i} \coprod_{z \in Z_i} F(z)^{asr} \right) \to \coprod_{i \in I} \coprod_{s \in S_i} \coprod_{r \in R_i} F(z)^{rs}$$

that sends for every $i \in I$ and $z \in Z_i$ the free generator $z^{asr} \mapsto z^{sr}$. The kernel of χ is precisely the normal closure in $F(Z) \amalg K$ of the set L in Eq. (*) showing that $\tilde{G} = (F(Z) \amalg K)/\ker(\chi)$. In particular the set X generates $\langle X \rangle$ freely. $\square$

Remark 3.9. A converse of Lemma 3.8 can be formulated by way of applying the following procedure to any given permutation extension

$$\tilde{G} := F(X) \rtimes K.$$

(a) For every $x \in X$ set $X(x) := \{y \in X \mid C_K(y) = C_K(x)\}$.
(b) Then K acts on the set $\Xi := \{X(x) \mid x \in X\}$ and we set $I := \Xi/K$.
(c) Fix a section $\sigma : I \to \Xi$.
(d) Fix any $i \in I$. For $\sigma(i) = X(x)$ in $\sigma(I)$ put $A_i := C_K(x)$. Fix a section $\sigma_i : X(x)/N_K(A_i) \to X(x)$ and denote its image by Z_i.
(e) Fix coset representative sets S_i and R_i of respectively $A_i \backslash N_K(A_i)$ and $N_K(A_i)\backslash K$.
(f) There is a partition $X = \bigcup_{i \in I} X_i$ where $X_i = Z_i S_i R_i$. Each X_i is K-invariant.
(g) The desired central HNN-extension is $\tilde{G} = \mathrm{HNN}(K, Z_i, A_i, i \in I)$.

A brief summary of the findings of Remark 3.9 and Lemma 3.8 reads:

Proposition 3.10. *Every permutation extension gives rise to a central HNN-extension and, conversely, every central HNN-extension arises in this way.*

Lemma 3.11. *Let $F = F_1 \amalg F_2$ be the free product of finitely generated free pro-p groups. Suppose $G = F \rtimes K$ a semidirect product and the free factors F_i are invariant under conjugation with elements in the finite p-group K. Then*

$$C_F(K) = C_{F_1}(K) \amalg C_{F_2}(K).$$

Proof. Let $\phi : F_1 \amalg F_2 \to F_1 \times F_2$ denote the canonical epimorphism. Then certainly, for the induced action of K, $C_{F_1 \times F_2}(K) = C_{F_1}(K) \times C_{F_2}(K)$. Since $C_F(K) \geq C_{F_1}(K) \amalg C_{F_2}(K)$ one has

$$\phi(C_F(K)) \geq C_{F_1}(K) \times C_{F_2}(K).$$

Hence $C_F(K) \subseteq C_{F_1}(K) C_{F_2}(K) \ker(\phi)$ and since the kernel of ϕ is contained in the commutator subgroup F' of F deduce

$$C_F(K) = C_{F_1}(K) C_{F_2}(K) F'.$$

Therefore the $\mathbb{Z}_p$-rank of $C_F(K)$ and $C_{F_1}(K) \amalg C_{F_2}(K)$ agree and since the latter group is a subgroup of $C_F(K)$ this implies equality. □

Lemma 3.12. *Let $G = \mathrm{HNN}(K, B, z)$ be a central HNN-extension and A be a subgroup of K not contained in any conjugate B^k for $k \in K$. Then $N_G(A) = N_K(A)$ and, in particular, $C_G(A) = C_K(A)$.*

Proof. We only prove that $N_G(A) = N_K(A)$. Fix $x \in N_G(A)$. Then, making use of [3, Corollary 7.1.5(c)] where we let the pair $(K, B, \mathbb{I}_B)$ play the role of (H, A, f), we find for some $k \in K$

$$A = A^x \leq K \cap K^x \leq B^k,$$

contradicting our assumptions. □

Lemma 3.13. *Let* $\tilde{G} = \mathrm{HNN}(K, A_i, Z_i, i \in I)$ *be a central HNN-extension. For* A_i *a maximal associated subgroup of* K *(with respect to containment) and* S *a coset representative set of* $A_i \backslash N_K(A_i)$ *its centralizer in* $F(X)$ *is*

$$C_{F(X)}(A_i) = \coprod_{s \in S} F(Z_i)^s.$$

Proof. It will be helpful to consider free product decomposition

$$F(X) = \coprod_{j \in I} \coprod_{z \in Z_j} F_{j,z}, \quad \text{where} \quad F_{j,z} := \coprod_{s \in S_j} \coprod_{r \in R_j} F(z)^{sr},$$

where each factor $F_{j,z}$ is K-invariant and hence also A_i-invariant (see Lemma 3.8). Making use of Lemma 3.11 and induction on the number of A_i-invariant factors one shows

$$C_{F(X)}(A_i) = \coprod_{j \in J} \coprod_{z \in Z_j} C_{F_{j,z}}(A_j).$$

If $j \neq i$ then A_i is not contained in any conjugate of A_j by the maximality condition on A_i showing $C_{F_{j,z}}(A_i) = \{1\}$. Therefore Lemma 3.12 implies $C_{F(X)}(A) \leq C_{F_{j,z}}(A_i)$ and the latter subgroup agrees (by Lemma 3.12 again) with

$$F(z^{sr} : x \in S_i) = \coprod_{s \in S} F(z)^s.$$

□

For the convenience of presenting the previous proofs we had K act on the right upon the set X. When passing from $\tilde{F}$ to the commutator quotient $M := \tilde{F}/[\tilde{F}, \tilde{F}]$ we therefore obtain a K-right module which, in analogy to the definitions in Section 2, will appear to be *permutation* K-right modules.

Effects of Changes

Let us first note that [1, Lemmas 3.11–3.13] are unaffected, as the fact used there is correct and holds by Lemma 3.13. In fact $F - c$-maximal subgroups introduced in [1, Notation 3.9] are conjugate to maximal associated subgroups. Note also that $F - c$ maximality is assumed in the proof and the quotations of [1, Lemma 3.13] but is not explicitly stated as a hypothesis.

Proposition 4.8. *Every PE-group $G = F \rtimes K$ is a permutational extension.*

Proof. Suppose that the proposition is false. Then there is a counter-example with K of minimal order. Among all such counter-examples fix one with $\mathrm{rank}(F)$ minimal. If there is no finite F-$\mathbf{c}$ maximal subgroup $\{1\} \neq L \leq K$ then by [1, Theorem 2.10] we find $G = F_0 \amalg K = \mathrm{HNN}(K, 1, Z, 1)$ where Z is a base of F_0, a contradiction. Therefore, we can fix an F-$\mathbf{c}$ maximal subgroup $\{1\} \neq L \leq K$ and set $Q := \langle C_F(L)^k \mid k \in K \rangle$. Observe that Q is K-invariant.

We claim that Q is a free pro-p factor of F and $Q \rtimes K$ is a permutational extension.

Indeed, if $L \lhd K$ then $Q = C_F(L)$ and hence by [1, Theorem 2.9] Q is a free pro-p factor of F. [1, Lemma 3.12] shows that $Q \rtimes K = N_G(L) = \mathrm{HNN}(K, L, Z_L, \{L\})$ is a permutational extension. If $N_K(L) < K$ fix any maximal subgroup K_0 of K containing $N_K(L)$. By the minimal assumption on $|K|$ we can conclude that $F \rtimes K_0$ is a permutational extension and therefore the claim follows from [1, Lemma 3.13(i)].

Since $Q \rtimes K$ is a permutational extension [1, Proposition 4.1] implies that $\overline{G} := G/(Q)_F = F/(Q)_F \rtimes K$ is a PE-group. As $\mathrm{rank}(\overline{F}) < \mathrm{rank}(F)$ the minimal assumption on $\mathrm{rank}(F)$ implies that

$$\overline{G} = \mathrm{HNN}(K, B_j, Y_j, j \in J) \tag{1}$$

is a permutational (and so central by Proposition 3.10) extension. Hence $Y_j \subset C_{\overline{F}}(B_j)$. Since $C_{\overline{F}}(B_j)$ is free and, by virtue of [1, Proposition 4.1(ii)] $C_{\overline{F}}(B_j) = \overline{C_F(B_j)}$, we can lift Y_j to a subset Z_j of some basis of $C_F(B_j)$.

We devise a "model"-permutational extension $\tilde{G}$ that finally will turn out to be isomorphic to G.

To this end we let $\mathcal{A} = \{(B_j, Y_j) \mid j \in J\} \cup \{L, Z_L)\}$. Form $\tilde{G} := \mathrm{HNN}(K, A, Z_A, (A, Z_A) \in \mathcal{A})$ and consider a bijection ϕ which sends, for all $j \in J$ every $B_j \mapsto B_j$, $Y_j \mapsto Z_j$, $L \mapsto L$ and $Z_L \mapsto Z_L$. Using the universal property of the permutational extension $\tilde{G}$, ϕ extends to an epimorphism from $\tilde{G}$ to G.

Since $\overline{G} = G/(C_F(L)^k \mid k \in K)_F = \mathrm{HNN}(K, B_j, Y_j, j \in J)$ and the latter group is naturally isomorphic to $\tilde{G}/(Z_L)_{\tilde{G}}$, we can conclude that $\ker \phi \leq (Z_L)_{\tilde{G}}$ must hold.

Set $\tilde{F} := \phi^{-1}(F)$ and note that $\tilde{G} = \tilde{F} \rtimes K$. Choose a coset representative set R_L of $K/N_K(L)$ and observe that [1, Proposition 3.11] applied to the family $\{C_{\tilde{F}}(L^r) \mid r \in R_L\}$ yields $\tilde{Q} := \coprod_{r\in R_L} C_{\tilde{F}}(L^r)$. Now choose a coset representative set S_L of $N_K(L)/L$ then Lemma 3.13 shows that $C_{\tilde{F}}(L) = \coprod_{s\in S} F(Z_L^s)$ and so we find

$$\operatorname{rank}(\tilde{Q}) = |Z_L||K : L|. \tag{2}$$

As has been mentioned before $\tilde{F}/(\tilde{Q})_{\tilde{F}} \cong F/(Q)_F$ and so establishing

$$\operatorname{rank}(\tilde{Q}) = \operatorname{rank}(Q) \tag{3}$$

would imply $G \cong \tilde{G}$ giving the final contradiction with $\tilde{G}$ being a permutational extension.

If $N_K(L) < K$, then [1, Lemma 3.13] implies Eq. (3). Otherwise $L \lhd K$ and thus $Q = C_F(L) \cong C_{\tilde{F}}(L)$ because $N_G(L) = \mathrm{HNN}(K, L, Z_L, \{L\}) \cong N_{\tilde{G}}(L)$ (cf. [1, Lemma 3.12]). Hence Eq. (3) holds in this case as well. □

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