CONJUGACY SEPARABILITY AND FREE PRODUCTS OF GROUPS WITH CYCLIC AMALGAMATION

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Introduction

A group G is *conjugacy separable* if whenever x and y are non-conjugate elements of G, there exists some finite quotient of G in which the images of x and y are non-conjugate. It is known that free products of conjugacy separable groups are again conjugacy separable [**19**, **12**]. The property is not preserved in general by the formation of free products with amalgamation; but in [**15**] a method was introduced for showing that under certain circumstances, the free product of two conjugacy separable groups G_1 and G_2 amalgamating a cyclic subgroup is again conjugacy separable. The main result of [**15**] states that this is the case if G_1 and G_2 are free-by-finite or finitely generated and nilpotent-by-finite. We show here that the same conclusion holds for groups G_1 and G_2 in a considerably wider class, including, in particular, all polycyclic-by-finite groups. (This answers a question posed by C. Y. Tang, Problem 8.70 of the Kourovka Notebook [**7**], as well as two questions recently asked by Kim, MacCarron and Tang in G. Kim, J. MacCarron and C. Y. Tang, 'On generalised free products of conjugacy separable groups', *J. Algebra* 180 (1996) 121–135.)

Main results

First we recall some definitions. (i) A group R is called *quasi-potent* if each cyclic subgroup H of R contains a subgroup K of finite index with the following property: every subgroup of finite index in K is of the form $H \cap N$ for some normal subgroup N of finite index in R. (ii) A subset X of a group R is said to be *conjugacy distinguished* if whenever $y \in R$ has no conjugate lying in X, there exists a normal subgroup N of finite index in R such that no conjugate of y lies in XN; equivalently, the set

$$\bigcup_{x \in X} x^R$$

is closed in the profinite topology on R. (Thus R is conjugacy separable if and only if every one-element subset of R is conjugacy distinguished.)

Now we define a class of groups $\mathscr{X}$ as follows: a group R is in $\mathscr{X}$ if

(a) R is conjugacy separable;

(b) R is quasi-potent;

(c) whenever A and B are cyclic subgroups of R, the set AB is closed in the profinite topology of R; that is, if $x \in R \setminus AB$ then $x \notin ABN$ for some normal subgroup N of finite index in R;

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(d) every cyclic subgroup of R is conjugacy distinguished;

(e) for any pair of cyclic subgroups C_1 and C_2 of R, one has: $C_1 \cap C_2 = 1$ if and only if $C_1 N \cap C_2 N = N$ for some normal subgroup N of finite index in R; equivalently, $\overline{C_1 \cap C_2} = 1$ if and only if $\overline{C_1} \cap \overline{C_2} = 1$, where $\bar{X}$ denotes the closure of a subset X in the profinite completion $\hat{R}$ of R;

(f) for any element r of infinite order in R and every $\gamma \in \hat{R}$ such that $\gamma \overline{\langle r \rangle} \gamma^{-1} = \overline{\langle r \rangle}$, one has $\gamma r \gamma^{-1} = r$ or $\gamma r \gamma^{-1} = r^{-1}$.

Theorem A. *Let G_1 and G_2 be groups in $\mathscr{X}$. Then their free product $G_1 *_H G_2$ amalgamating a cyclic subgroup H is in $\mathscr{X}$, and, in particular, is conjugacy separable.*

Theorem B. *The class $\mathscr{X}$ contains all polycyclic-by-finite groups, all free-by-finite groups, all Fuchsian groups and all surface groups.*

Putting the two theorems together, we see that a group will be conjugacy separable if it can be obtained from polycyclic-by-finite groups and/or free-by-finite groups by repeatedly forming free products with cyclic amalgamations.

The main new point of Theorem B is the claim regarding polycyclic-by-finite groups; this is proved in Section 3. The fact that free-by-finite groups are in $\mathscr{X}$ was established in [**15**].

A Fuchsian group has a presentation of the form

$$\left\langle a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_n \mid c_1^{e_1} = \ldots = c_n^{e_n} = 1, c_1 \ldots c_n \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle,$$

where each e_i is either a natural number greater than or equal to 2, or possibly ∞ (cf. [**9**, p. 98]). Therefore, a Fuchsian group is a free product of a free product of cyclic groups and a free group, amalgamating a cyclic subgroup. Hence the claim that a Fuchsian group belongs to the class $\mathscr{X}$ follows from Theorem A. The claim that surface groups are in $\mathscr{X}$ follows from the fact that a non-abelian surface group can be expressed as a free product of two free groups with an amalgamated cyclic subgroup (cf. [**24**, p. 71]).

The fact that Fuchsian groups are conjugacy separable was first proved by Fine and Rosenberger in [**2**]. Tang [**21**] has independently proved that the free product of two (non-abelian) surface groups amalgamating a cyclic subgroup is conjugacy separable.

In Section 2 we indicate how the methods developed in [**15**] can be used to prove one part of Theorem A, namely that if G_1 and G_2 are groups in $\mathscr{X}$, then their free product amalgamating a cyclic subgroup is conjugacy separable.

The proof of Theorem A is completed in Section 4. This proof is presented in five propositions corresponding to the properties (b)–(f) in the definition of $\mathscr{X}$. Not all of them require the full strength of the hypotheses of Theorem A, and some of them may be of independent interest.

1. *Notation and preliminaries*

Let R be a group. If $x \in R$, we write $x^R = \{x^r = r^{-1} x r \mid r \in R\}$, as usual. If $n \in \mathbb{N}$, then $R^n = \langle r^n \mid r \in R \rangle$ denotes the subgroup of R generated by the nth powers of the elements of R. If H and K are subgroups of R, we denote by $\mathscr{C}_K(H)$ and $\mathscr{N}_K(H)$ the centralizer and normalizer of H in K, respectively. We use the notation $N \leqslant_f R$ (respectively, $N \lhd_f R$) to indicate that N is a subgroup (respectively, a normal subgroup) of R of finite index. Let $\mathcal{N} = \{N \mid N \lhd_f R\}$. Then the collection $\mathcal{N}$ can be taken as a fundamental system of neighbourhoods of the identity element 1 of R, making R into a topological group. This topology on R is called the *profinite topology* of R. The *profinite completion* $\hat{R}$ of R is the topological completion of R with respect to its profinite topology, that is,

$$\hat{R} = \varprojlim_{N \in \mathcal{N}} R/N.$$

Then $\hat{R}$ becomes a profinite group, that is, a compact Hausdorff totally disconnected topological group; furthermore, there is a natural homomorphism $\iota : R \to \hat{R}$. The map ι is a monomorphism precisely if the group R is residually finite. All groups in this paper are residually finite, and we shall identify R with its image in $\hat{R}$ under ι. The group R is *conjugacy separable* if whenever x and y are non-conjugate elements of R, there exists some finite quotient of R in which the images of x and y are non-conjugate. A conjugacy separable group is necessarily residually finite. If R is residually finite, it is conjugacy separable if and only if whenever two elements of R are conjugate in $\hat{R}$ then they are conjugate in R. More generally, a subset X of R is conjugacy distinguished if and only if $y^R \cap X = \varnothing$ implies $y^{\hat{R}} \cap \bar{X} = \varnothing$ for each $y \in R$, where $\bar{X}$ denotes the closure of X in $\hat{R}$. Similarly, R is called *subgroup conjugacy separable* if whenever H and K are subgroups of R that are non-conjugate in R, then there is some finite quotient of R where the images of H and K are non-conjugate; or, equivalently (for residually finite groups), whenever two subgroups of R have conjugate closures in $\hat{R}$, they are conjugate in R. Finally, R is *subgroup separable* (respectively, *cyclic subgroup separable*) if every finitely generated (respectively, cyclic) subgroup H of R is closed in the profinite topology of R, that is, if

$$H = \bigcap_{N \in \mathcal{N}} HN.$$

Note that if a group R has property (c) then R is certainly cyclic subgroup separable.

If $\phi: R \to S$ is a homomorphism of groups, there is a unique continuous homomorphism $\phi: \hat{R} \to \hat{S}$ that renders the following diagram commutative:

$$\begin{array}{ccc} R & \xrightarrow{\ \phi\ } & S \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ \hat{R} & \xrightarrow{\ \hat{\phi}\ } & \hat{S} \end{array}$$

If H is a subgroup of $\hat{R}$, we denote by $\bar{H}$ the closure of H in $\hat{R}$. It turns out that 'completion' is a right exact functor from the category of groups to the category of profinite groups.

Next we list some well-known facts about polycyclic-by-finite groups, that will be used later in the paper. A self-contained source for these facts is [**16**].

LEMMA 1.1 [**16**, Chapter 10, Corollary 1 or **3**, Theorem 20B]. *The completion functor is exact on the category of polycyclic-by-finite groups. If H is a subgroup of the polycyclic-by-finite group R, then the profinite topology of R induces on H its full profinite topology, and so $\bar{H}$ may be identified with $\hat{H}$.*

LEMMA 1.2 [**4**; **11**; see **16**, Chapter 4, Theorem 3]. *Polycyclic-by-finite groups are conjugacy separable.*

LEMMA 1.3 [**10**; see **16**, Chapter 1, Exercise 11]. *Polycyclic-by-finite groups are subgroup separable.*

LEMMA 1.4 [**6**; see **16**, Chapter 4, Theorem 7]. *Polycyclic-by-finite groups are subgroup conjugacy separable.*

Let G_1 and G_2 be groups with a common subgroup H; then the amalgamated free product of G_1 and G_2 amalgamating H is denoted by $G_1 *_H G_2$, as usual. Let Γ_1 and Γ_2 be profinite groups with a common closed subgroup Δ; consider the push-out Γ of Γ_1 and Γ_2 over Δ in the category of profinite groups; if the canonical homomorphisms $\Gamma_1 \to \Gamma$ and $\Gamma_2 \to \Gamma$ are embeddings, one says that Γ is the *profinite amalgamated free product of* Γ_1 *and* Γ_2 *amalgamating* Δ, and one writes $\Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$. See [**14**] for more details.

If $G = G_1 *_H G_2$ is an amalgamated free product of groups, there is a standard tree $S(G)$ associated with it (cf. [**18**, Theorem I.7]): its vertex set $\mathrm{Ver}\, S(G)$ is the collection of cosets $G/G_1 \cup G/G_2$, and its edge set is the collection of cosets G/H. There exists a natural action of G on the tree $S(G)$. We say that an element $g \in G$ is *hyperbolic* if it acts freely on $S(G)$, that is, if g does not belong to a conjugate in G of either G_1 or G_2. Similarly, there is a standard profinite tree $S(\Gamma)$ associated to a profinite amalgamated free product $\Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$; its profinite space of vertices is $\Gamma/\Gamma_1 \cup \Gamma/\Gamma_2$ and its profinite space of edges is Γ/Δ (cf. [**5**, Section 2]). For convenience we shall think of $S(G)$ as a set, namely the disjoint union of its sets of vertices and edges. Similarly we view $S(\Gamma)$ as the disjoint union of its spaces of vertices and edges.

The following result, due to J. Tits [**18**, Chapter I, Proposition 24], will be used in some of our proofs. We state it here in a form convenient for our purposes.

PROPOSITION 1.5. *Let $G = G_1 *_H G_2$ and assume that $a \in G$ is hyperbolic. Put*

$$m = \min_{v \in \mathrm{Ver}\, S(G)} l[v, av] \quad \text{and} \quad T_a = \{v \in \mathrm{Ver}\, S(G) \mid l[v, av] = m\}.$$

Then T_a is the vertex set of a straight line (that is, a doubly infinite chain of $S(G)$), that we again denote by T_a, on which a acts as a translation of amplitude m; furthermore, every $\langle a \rangle$-invariant subtree of $S(G)$ contains T_a. Finally if $v \in T_a$, then $T_a = \langle a \rangle [v, av[$.

Here, $[v, w]$ denotes the unique path joining vertices v and w, and $l[v, w]$ its length; also $[v, w[= [v, w] \setminus \{w\}$. We shall refer to T_a as the *Tits straight line* corresponding to the hyperbolic element a.

Let k be a positive integer. An element a of a group R will be called *k-potent* if for every natural number n there exists a normal subgroup N of finite index in R such that $\langle a^{kn} \rangle = \langle a \rangle \cap N$. Thus R is quasi-potent if every element of R is k-potent for some positive integer k (depending on the element).

2. *Conjugacy separability of amalgamated free products*

The class $\mathscr{X}$ defined in the Introduction is more or less the largest class of groups for which the methods used in [**15**] apply, so that the amalgamated free product of two groups in that class amalgamating a cyclic subgroup is conjugacy separable.

The proofs of Proposition 3.2 and Lemma 3.3 of [**15**] establish the following.

Lemma 2.1. *Let G_1, G_2 be residually finite groups with a common cyclic subgroup H, and let $G = G_1 *_H G_2$. Assume that*

(a) *G_i is quasi-potent for $i = 1, 2$,*

(b) *H is closed in the profinite topology of G_i for $i = 1, 2$.*

Then

(1) *G is residually finite;*

(2) *the profinite topology of G induces on G_i its full profinite topology* ($i = 1, 2$)*;*

(3) $\hat{G} = \widehat{G_1} \amalg_{\hat{H}} \widehat{G_2}$*;*

(4) *G_1 and G_2 are closed in the profinite topology of G.*

This lemma will be used frequently throughout the paper. If G_1, G_2 and H satisfy the conditions of the lemma, then it follows from Lemma 2.1 that the graph $S(G)$ is naturally embedded in the profinite graph $S(\hat{G})$.

The defining properties (a)–(f) for the class $\mathscr{X}$ have been chosen in such a way that the proof of Theorem 3.8 of [**15**] applies to the groups in $\mathscr{X}$. In fact one does not need the full force of property (e). We define a new property.

Property (e′). *If H is a cyclic subgroup of R, $x \in R$ and $H \cap H^x = 1$, then there exists some $N \lhd_f R$ with $HN \cap H^x N = N$ (equivalently, if $H \cap H^x = 1$ then* $\bar{H} \cap \bar{H}^x = 1$).

Then the proof of Theorem 3.8 of [**15**] yields the following.

Theorem 2.2. *Let G_1 and G_2 be groups having the properties* (a)–(d), (f) *and property* (e′)*, with a cyclic common subgroup H. Then their amalgamated free product $G = G_1 *_H G_2$ is conjugacy separable.*

Since (e′) follows from (e), the theorem applies whenever G_1 and G_2 are in $\mathscr{X}$. In [**15**] it was established that the class $\mathscr{X}$ contains the free-by-finite and the finitely generated nilpotent-by-finite groups. The purpose of the remainder of this paper is to exhibit other classes of groups that belong to $\mathscr{X}$.

3. *Completions of polycyclic-by-finite groups*

In this section we prove that polycyclic-by-finite groups belong to the class $\mathscr{X}$. In fact, in some cases, we shall prove stronger results than are required for that purpose. We consider some properties of the profinite completion functor in the category of polycyclic-by-finite groups; we show in particular that if R is a polycyclic-by-finite group, then the map

$$H \longmapsto \bar{H}$$

that sends a subgroup H of R to its closure in $\hat{R}$ preserves centralizers, normalizers, and intersections. We begin with another property preserved by this map. The following proposition generalizes Lemma 3.5 in [**15**].

PROPOSITION 3.1. *Let R be a polycyclic-by-finite group. Then every cyclic subgroup of R is conjugacy distinguished.*

Proof. Let x, y be elements of R, and suppose that $y^{\hat{R}} \cap \overline{\langle x \rangle} \neq \varnothing$. We shall argue by induction on the Hirsch length $h(R)$ of R to prove that then $y^R \cap \langle x \rangle \neq \varnothing$. If $h(R) = 0$, then R is finite, and the result is obvious. Say $h(R) \geqslant 1$. Note that if either the order of x or the order of y is finite and $y^{\hat{R}} \cap \overline{\langle x \rangle} \neq \varnothing$, then both x and y have finite order; then the result is a consequence of the fact that R is conjugacy separable (Lemma 1.2). So, we assume from now on that both x and y have infinite order. Let A be a non-trivial free-abelian normal subgroup of R (cf. [**16**, Chapter 1, Lemma 6]). Then R/A is polycyclic-by-finite and $h(R/A) < h(R)$. Let m be a natural number, and let $\pi_m \colon R \to R/A^m$ be the canonical epimorphism. Consider the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\pi_m} & R/A^m \\ \downarrow \iota & & \downarrow \iota \\ \hat{R} & \xrightarrow{\widehat{\pi_m}} & \widehat{R/A^m} \end{array}$$

where the maps ι are the canonical injections. Note that $\widehat{\pi_m}(y^{\hat{R}}) = (yA^m)^{\widehat{R/A^m}}$ and $\widehat{\pi_m}(\overline{\langle x \rangle}) = \overline{\langle xA^m \rangle}$; therefore, $(yA^m)^{\widehat{R/A^m}} \cap \overline{\langle xA^m \rangle} \neq \varnothing$. By the induction hypothesis, for each $m \in \mathbb{N}$, there exist some $r(m) \in R$ and $n(m) \in \mathbb{Z}$ such that

$$y^{r(m)} \equiv x^{n(m)} \pmod{A^m}. \tag{1}$$

Without loss of generality, we may replace y by $y^{r(1)}$, and so we have

$$y \equiv x^{n(1)} \pmod{A}. \tag{2}$$

Let $t \in \mathbb{N}$. Then

$$x^{n(t)} \equiv y^{r(t)} \equiv (x^{n(1)})^{r(t)} \pmod{A};$$

hence $x^{n(t)}$ and $x^{n(1)}$ have the same order in any finite quotient of R/A. It follows that

$$\langle x^{n(t)} \rangle N = \langle x^{n(1)} \rangle N$$

whenever $A \leqslant N \lhd_f R$. Since R/A is subgroup separable (see Lemma 1.3), one deduces that

$$\langle x^{n(t)} \rangle A = \langle x^{n(1)} \rangle A = \langle y \rangle A \quad \text{for all } t \in \mathbb{N}.$$

Now we consider two cases.

Case 1. The element yA of R/A has infinite order. Then, so does xA. Hence $n(1) = \pm n(t)$, for all $t \in \mathbb{N}$. In particular, $n(1) = \pm n(t!)$, for all $t \in \mathbb{N}$. If $n(1) = n(t!)$ for infinitely many t, put $k = n(1)$; otherwise, put $k = -n(1)$. Then, according to (1), x^k and y are conjugate modulo $A^{t!}$ for infinitely many t, say, for $t = t_1, t_2, \ldots$. Let $N \lhd_f G$; then there is some $i \in \mathbb{N}$ such that $A^{t_i!} \leqslant A \cap N$; hence x^k and y are conjugate modulo N. Since R is conjugacy separable (Lemma 1.2), we deduce that x^k and y are conjugate in R.

Case 2. The element yA of R/A has finite order. Say the order of yA is f; then xA must also have finite order, say, e. From (2) one obtains that $e = fl$ for some

$l \in \mathbb{Z}$. Since $y^f \in A$ and A is a free abelian group, there is some basis $\{a_1, a_2, \dots\}$ of A and some natural number t such that $a_1^t = y$. So $A^t = \langle y^f \rangle \times C$ for some subgroup C of A. Then yA^{tk} has order fk in the group R/A^{tk}, for every $k \in \mathbb{N}$. Similarly, there exists $s \in \mathbb{N}$ such that xA^{sk} has order ek in R/A^{sk}, for each k. Now let w be any common multiple of t and s. From (1), the order of $x^{n(w)} A^w$ in R/A^w is then fw/t, while the order of xA^w is $ew/s = lfw/s$. Therefore $tl = sq$ for some $q \in \mathbb{N}$, and the order of $x^q A^w$ is fw/t. Since the cyclic subgroup $\langle xA^w \rangle$ of R/A^w has a unique subgroup of order fw/t, we see that

$$\langle x^{n(w)} A^w \rangle = \langle x^q A^w \rangle$$

for all w as above. Therefore, according to (1), the groups

$$\langle x^q A^w \rangle \quad \text{and} \quad \langle yA^w \rangle$$

are conjugate in R/A^w for all such w. It follows that the groups

$$\langle x^q N \rangle \quad \text{and} \quad \langle yN \rangle$$

are conjugate in R/N for all $N \lhd_f R$. Hence

$$\langle x^q \rangle \quad \text{and} \quad \langle y \rangle$$

are conjugate in R, since R is subgroup conjugacy separable (Lemma 1.4), and so $y^r \in \langle x \rangle$ for some $r \in R$. This concludes the proof.

REMARK. Although we have shown that cyclic subgroups are conjugacy distinguished, this is *not* true of arbitrary subgroups in a polycyclic-by-finite group; see the remark at the end of [**17**].

LEMMA 3.2. *Let H and K be subgroups of a polycyclic-by-finite group R. Then for each $U \lhd_f R$ there is some $V \lhd_f R$ with $V \leqslant U$, such that*
(a) $\mathscr{C}_K(HV/V) \leqslant (K \cap U)\mathscr{C}_K(H)$,
(b) $\mathscr{N}_K(HV/V) \leqslant (K \cap U)\mathscr{N}_K(H)$.

Proof. (a) For $n \in \mathbb{N}$, put $R_n = R^{n!}$. Since R is finitely generated, the subgroups

$$R = R_0 \geqslant R_1 \geqslant R_2 \geqslant \dots$$

form a fundamental system of neighbourhoods of 1 in the profinite topology of R. Hence it suffices to show that for each $s \in \mathbb{N}$, there is some natural number $t(s) \geqslant s$ such that $\mathscr{C}_K(HR_{t(s)})/R_{t(s)}) \leqslant (K \cap R_s)\mathscr{C}_K(H)$. Suppose that for every integer $t \geqslant s$, one has $\mathscr{C}_K(HR_t/R_t) \nleqslant (K \cap R_s)\mathscr{C}_K(H)$. Then for each $t \geqslant s$ there exists $y(t) \in K - (K \cap R_s)\mathscr{C}_K(H)$ such that $y(t) \in \mathscr{C}_K(HR_t/R_t)$. Since $K/(K \cap R_s)$ is finite, there exist $x \in K - (K \cap R_s)\mathscr{C}_K(H)$ and natural numbers $t_1 < t_2 < \dots$ such that

$$x = y(t_i)\, z(t_i) \quad \text{with } z(t_i) \in K \cap R_s \quad \text{for all } i.$$

Then for every $h \in H$ we have $x^{-1} hxR_{t_i} = z(t_i)^{-1} hz(t_i)R_{t_i}$. Let $h_1, \dots, h_m$ be a set of generators of H. Then, by Theorem B in [**17**], there exists some $z \in K \cap R_s$ such that

$$x^{-1} h_j x = z^{-1} h_j z \quad \text{for } j = 1, \dots, m.$$

Thus

$$x^{-1} hx = z^{-1} hz \quad \text{for all } h \in H.$$

It follows that $x \in (K \cap R_s)\mathscr{C}_K(H)$, a contradiction.

The proof of (b) is similar.

PROPOSITION 3.3. *Let H and K be subgroups of a polycyclic-by-finite group R. Then*

(a) $\mathscr{C}_{\bar{K}}(\bar{H}) = \overline{\mathscr{C}_K(H)}$,

(b) $\mathscr{N}_{\bar{K}}(\bar{H}) = \overline{\mathscr{N}_K(H)}$,

where the closures $\bar{K}$, $\bar{H}$ etc. are taken in $\hat{R}$. In particular, if H is infinite cyclic, then $\mathscr{N}_{\bar{K}}(\bar{H})/\mathscr{C}_{\bar{K}}(\bar{H})$ has order at most 2.

Proof. We shall prove (a), the proof of (b) being similar. Let $U \lhd_f R$; then, by Lemma 3.2, there is some $V \lhd_f R$ with $V \leqslant U$, such that $\mathscr{C}_K(HV/V) \leqslant (K \cap U)\mathscr{C}_K(H)$. Now, since $\bar{K} = K(\overline{K \cap V})$, we have

$$\mathscr{C}_{\bar{K}}(\bar{H}) \leqslant \mathscr{C}_{\bar{K}}(\bar{H}\bar{V}/\bar{V}) \leqslant \mathscr{C}_K(HV/V)(\overline{K \cap V}) \leqslant \mathscr{C}_K(H)(\overline{K \cap U}).$$

It follows that

$$\mathscr{C}_{\bar{K}}(\bar{H}) \leqslant \bigcap_{U \lhd_f R} \mathscr{C}_K(H)(\overline{K \cap U}) = \overline{\mathscr{C}_K(H)}.$$

The reverse inclusion is clear. The final statement follows, since if H is infinite cyclic then $|\mathscr{N}_K(H) : \mathscr{C}_K(H)| \leqslant 2$.

LEMMA 3.4. *Let $\phi\colon R \to S$ be a homomorphism of polycyclic-by-finite groups, and let $\hat{\phi}\colon \hat{R} \to \hat{S}$ denote the induced homomorphism of the corresponding profinite completions. Let H be a subgroup of S. Then $\overline{\phi^{-1}(H)} = \hat{\phi}^{-1}(\bar{H})$, where the closures $\overline{\phi^{-1}(H)}$ and $\bar{H}$ are taken in $\hat{R}$ and $\hat{S}$ respectively.*

Proof. We use induction on the Hirsch length $h(S)$ of S. Since R and S are subgroup separable (Lemma 1.3), $\phi^{-1}(H) = R \cap \overline{\phi^{-1}(H)}$ and $H = S \cap \bar{H}$; so, $R \cap \overline{\phi^{-1}(H)} = R \cap \phi^{-1}(\bar{H})$. If $h(S) = 0$, it follows that S is finite; then $\overline{\phi^{-1}(H)}$ and $\hat{\phi}^{-1}(\bar{H})$ are open subgroups of $\hat{R}$, and so

$$\overline{\phi^{-1}(H)} = \overline{R \cap \overline{\phi^{-1}(H)}} = \overline{R \cap \hat{\phi}^{-1}(\bar{H})} = \hat{\phi}^{-1}(\bar{H}).$$

Suppose now that $h(S) > 0$ and that the result holds whenever the Hirsch length of the codomain of the homomorphism is smaller than $h(S)$. Let A be an infinite free abelian normal subgroup of S (cf. [**16**, Chapter 1, Lemma 6]). Applying the induction hypothesis to the homomorphism

$$R \xrightarrow{\phi} S \longrightarrow S/A,$$

we infer that $\overline{\phi^{-1}(HA)} = \hat{\phi}^{-1}(\overline{HA})$. We may therefore replace S by HA. Put $N = H \cap A$. If $N \neq 1$, we may repeat the above argument applied now to

$$R \xrightarrow{\phi} S \longrightarrow S/N,$$

to get the desired result $\overline{\phi^{-1}(H)} = \hat{\phi}^{-1}(\bar{H})$. Suppose that $H \cap A = 1$. Then $S = A \rtimes H$, and, by Lemma 1.1, $\hat{S} = \bar{A} \rtimes \bar{H}$. From now on we use additive notation for the group A, but multiplicative notation for S. Consider the free abelian profinite group $M = \hat{A} \oplus u\hat{\mathbb{Z}}$ of rank equal to $1 + \operatorname{rank}(A)$. Define a right $\hat{R}$-module structure on M as follows: if $a \in \hat{A}$, $r \in \hat{R}$, put $a \cdot r = a^{\hat{\phi}(r)}$, and $u \cdot r = u + \delta\hat{\phi}(r)$, where δ is the projection $\hat{S} \to \hat{A}$ (we identify $\hat{A}$ with $\bar{A}$; note that δ is a continuous derivation, that is, $\delta(ss') = \delta(s) \cdot s' + \delta(s'), \forall s, s' \in S$). Observe that $\delta(R) \subseteq A$, and that the profinite completion of $(A \oplus u\mathbb{Z}) \rtimes R$ is $M \rtimes \hat{R}$. Now

$$\begin{aligned} \mathscr{C}_{\hat{R}}(u) &= \{r \in \hat{R} \mid u = u \cdot r = u + \delta\hat{\phi}(r)\} = \{r \in \hat{R} \mid \delta\hat{\phi}(r) = 0\} \\ &= \{r \in \hat{R} \mid \hat{\phi}(r) \in \bar{H}\} = \hat{\phi}^{-1}(\bar{H}), \end{aligned}$$

and similarly

$$\mathscr{C}_R(u) = \phi^{-1}(H).$$

The result therefore follows from Proposition 3.3.

Remark. The last stage of the argument shows the following: if S is a polycyclic-by-finite group, A is an S-module finitely generated over $\mathbb{Z}$, and $\delta : S \to A$ is a derivation, then

$$\ker \hat{\delta} = \overline{\ker \delta},$$

where $\hat{\delta}$ is the extension of δ to a continuous derivation $\hat{S} \to \hat{A}$.

As a consequence of Lemma 3.4 we obtain the following generalization of Lemma 3.6 in [**15**].

Proposition 3.5. *Let R be a polycyclic-by-finite group, and let $H, K \leqslant R$. Then $\overline{H \cap K} = \bar{H} \cap \bar{K}$, where the closures are taken in $\hat{R}$.*

Proof. Let $j : H \to R$ be the inclusion homomorphism. Then we have a commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\ j\ } & R \\ \downarrow & & \downarrow \\ \hat{H} = \bar{H} & \xrightarrow{\ \hat{j}\ } & \hat{R} \end{array}$$

where, by Lemma 1.1, all the homomorphisms are inclusions. Observe that $j^{-1}(K) = H \cap K$ and $\hat{j}^{-1}(\bar{K}) = \bar{H} \cap \bar{K}$. Thus, the result follows from Lemma 3.4.

To show that a polycyclic-by-finite group is in the class $\mathscr{X}$ we still have to prove that such a group is quasi-potent and that condition (c) holds. The second property is a special case of a result due to Lennox and Wilson that we state below. The first property is established in the following lemma which is patterned after Tang's Lemma 3.2 of [**20**].

Lemma 3.6. *Let R be a polycyclic-by-finite group. Then R is quasi-potent.*

Proof. Since R is polycyclic-by-finite, it has a chain of normal subgroups

$$R \geqslant A_0 \geqslant A_1 \geqslant \ldots \geqslant A_{k-1} \geqslant A_k = 1$$

such that R/A_0 is finite, and A_i/A_{i+1} is free abelian ($i = 0, 1, \ldots, k-1$) (cf. [**16**, Chapter 1, Lemma 6]). Let x be an element of R of infinite order, and let s be the smallest positive integer with $x^s \in A_0$. Say $x^s \in A_i - A_{i+1}$. Since A_i/A_{i+1} is free abelian, there is some $y \in A_i$ such that yA_{i+1} forms part of a basis of A_i/A_{i+1} and $x^s A_{i+1} = y^t A_{i+1}$ for some natural number t. Now let n be a natural number. Since A_i^{tn} is a characteristic subgroup of A_i, the subgroup $M_n = A_i^{tn} A_{i+1}$ is normal in R, and the order of $x^s M_n$ in A_i/M_n is n. Therefore the order of xM_n in R/M_n is sn, that is, $|\langle x \rangle : \langle x \rangle \cap M_n| = sn$. Since R is polycyclic-by-finite, M_n is closed in the profinite topology of R (Lemma 1.3); hence there exists $N_n \lhd_f R$ with $M_n \leqslant N_n$ such that $x, x^2, \dots, x^{sn-1} \notin N_n$. Thus $\langle x \rangle \cap N_n = \langle x^{sn} \rangle$, as desired.

PROPOSITION 3.7. ([**8**]; see [**16**, Chapter 4, Exercise 13]). *Let R be a polycyclic-by-finite group, and let H, $K \leqslant R$. Then the set*

$$HK = \{hk \mid h \in H, k \in K\}$$

is closed in the profinite topology of R.

Putting together the results of this section together with Lemma 1.2 we deduce the following.

THEOREM 3.8. *Polycyclic-by-finite groups belong to class $\mathscr{X}$.*

4. *The class $\mathscr{X}$*

The purpose of this section is to show that the class $\mathscr{X}$ is closed under taking free products with cyclic amalgamations. To see this we shall prove that properties (a)–(f) that characterize class $\mathscr{X}$ are preserved under taking such products. Theorem 2.2 ensures that this is the case for property (a).

Throughout the section we shall adopt the following notation and basic assumptions. Let G_1, G_2 be residually finite groups with a common cyclic subgroup H, such that G_i is quasi-potent ($i = 1, 2$), and H is closed in the profinite topology of G_i (for $i = 1, 2$). Put $\Gamma = \hat{G}, \Delta = \hat{H}, \Gamma_i = \widehat{G_i}$ (for $i = 1, 2$), and let $S(G)$ and $S(\Gamma)$ denote the standard tree and profinite tree associated with the amalgamated free product $G = G_1 *_H G_2$ and the profinite amalgamated free product $\Gamma = \Gamma_1 \amalg_\Delta \Gamma_2$, respectively. Since G_1 and G_2 are closed in the profinite topology of G, it follows that $S(G)$ is naturally embedded in $S(\Gamma)$. If A is a subgraph of $S(G)$, then $\bar{A}$ denotes its closure in $S(\Gamma)$.

We begin with the following result whose proof follows closely parts of the proof of Proposition 2.9 of [**15**].

LEMMA 4.1. *Let $B = \langle b \rangle$ be a cyclic subgroup of G. Assume that b is hyperbolic with respect to its action on the standard tree $S(G)$, and let T_b denote the corresponding Tits straight line* (*see Proposition 1.5*). *Then*

(i) *$\bar{B}$ acts freely on the profinite tree $\overline{T_b}$,*

(ii) *$\bar{B} \cong \hat{\mathbb{Z}}$.*

Proof. Observe first that if $b' \in \bar{B}$ fixes one vertex, say v, of $\overline{T_b}$, then it fixes all the vertices of $\overline{T_b}$; indeed, if $w \in \mathrm{Ver}(\overline{T_b})$, then $w \in [b''v, b''bv]$ for some $b'' \in \bar{B}$, since $\overline{T_b} = \bar{B}[v, bv[$. Now, since b' commutes with b'' and b, it follows that b' fixes $b''v$ and $b''bv$, and hence w as well, since $\overline{T_b}$ does not contain cycles. Denote by K the closed subgroup of $\bar{B}$ consisting of those elements that act trivially on $\overline{T_b}$; we must show that $K = 1$. Since B acts freely on T_b, we have $K \neq \bar{B}$. Now, $\bar{B}/K$ acts freely on the profinite tree $\overline{T_b}$ with finite quotient graph $\overline{T_b}/(\bar{B}/K)$ (for T_b/B is finite). Then, according to Theorem 1.7 of [**5**], $\bar{B}/K$ is a free prosolvable group, and, since $\bar{B}$ is procyclic and non-trivial, it must be the free profinite group of rank 1, that is, $\bar{B}/K \cong \hat{\mathbb{Z}}$, and therefore, $\bar{B}$ is also the free profinite group of rank 1. Finally since $\hat{\mathbb{Z}}$ is Hopfian (cf. [**13**, Proposition 7.6]), $K = 1$.

PROPOSITION 4.2. *Let $G = G_1 *_H G_2$ be as above. Then*
(i) *G is quasi-potent,*
(ii) *if, in addition, G_1 and G_2 are cyclic subgroup separable, then G is also cyclic subgroup separable.*

Proof. Let $x \in G$ be an element of infinite order.

Case 1. The element x is not hyperbolic, that is, $x \in G_1^G \cup G_2^G$. Say $x \in gG_1g^{-1}$ for some $g \in G$. Since

$$G = gG_1g^{-1} *_{gHg^{-1}} gG_2g^{-1},$$

we may assume that $x \in G_1$. By Lemma 2.1, G induces on G_1 its full profinite topology; therefore, if G_1 (and G_2) is cyclic subgroup separable, $\langle x \rangle$ is closed in the profinite topology of G_1, and so in the profinite topology of G. This proves part (ii) in this case.

Let $H = \langle h \rangle$. Then there exist natural numbers t_1 and t_2 such that h is t_i-potent in G_i for $i = 1, 2$. Let s be a common multiple of t_1 and t_2. Choose $M_1 \lhd_f G_1$ and $M_2 \lhd_f G_2$ so that $M_1 \cap H = \langle h^s \rangle$ and $M_2 \cap H = \langle h^s \rangle$; consider the natural epimorphism

$$\phi: G \longrightarrow \tilde{G} = G_1/M_1 *_{HM_1/M_1} G_2/M_2;$$

let $M \lhd_f \tilde{G}$ be a normal subgroup of $\tilde{G}$ of finite index with trivial intersection with G_1/M_1 and G_2/M_2; put $N = \phi^{-1}M$, then clearly $N \lhd_f G$ and $N \cap H = \langle h^s \rangle$. Let e be the order of x in G/N. Pick m such that x is m-potent in G_1, and set $k = me$. We claim that x is k-potent in G. For let t be a natural number; choose $T_1 \lhd_f G_1$ so that $T_1 \cap \langle x \rangle = \langle x^{tk} \rangle$; since $x^{tk} \in N$, we have $T_1 \cap N \cap \langle x \rangle = \langle x^{tk} \rangle$. To complete the verification of the claim, we must show that there exists some $S \lhd_f G$ with $S \cap \langle x \rangle = \langle x^{tk} \rangle$; and for this, it suffices to show that there exists such an S with $S \cap G_1 = T_t \cap N$. To see this, let $T_2 \lhd_f G_2$ be such that $T_2 \cap H = T_1 \cap N \cap H$ (here T_2 exists since $N \cap H = \langle h^s \rangle$ and t_2 divides S). Now we proceed as above: consider the natural epimorphism

$$\psi: G \longrightarrow \tilde{\tilde{G}} = G_1/(T_1 \cap N) *_{HT_2/T_2} G_2/T_2;$$

let M' be a normal subgroup of $\tilde{\tilde{G}}$ of finite index with trivial intersection with $G_1/(T_1 \cap N)$ and G_2/T_2; put $S = \psi^{-1}M'$, then clearly $S \lhd_f G$ and $S \cap G_1 = T_1 \cap N$, as needed.

Case 2. The element x is hyperbolic. Then x does not stabilize any vertex or edge of the graph $S(G)$. By Proposition 2.9 of [**15**] x also acts freely on the profinite graph $S(\Gamma)$. For $N \lhd_f G$, write $G_N = G_1N/N *_{HN/N} G_2N/N$. Then (cf. [**14**])

$$\hat{G}_N = G_1N/N \coprod_{HN/N} G_2N/N \quad \text{and} \quad \hat{G} = \varprojlim \hat{G}_N.$$

Hence

$$S(\hat{G}) = \varprojlim S(\hat{G}_N).$$

For each N, let x_N be the image of x under the canonical map $\phi_N: G \to G_N$. Denote by $S(\hat{G})^x$ and $S(\hat{G}_N)^{x_N}$ the sets of fixed points of $S(\hat{G})$ and $S(\hat{G}_N)$ under the actions of x and x_N respectively. Then

$$S(\hat{G})^x = \varprojlim S(\hat{G}_N)^{x_N}.$$

Since x acts freely on $S(\hat{G})$, one has that $S(\hat{G})^x = \varnothing$; therefore there exists some $N \lhd_f G$ such that $S(\hat{G}_N)^{x_N} = \varnothing$, that is, x_N is hyperbolic in

$$\hat{G}_N = G_1 N/N \coprod_{HN/N} G_2 N/N;$$

and in particular x_N has infinite order. Now, G_N is quasi-potent since it is free-by-finite (cf. [**20**, Lemma 3.2]). Therefore $\overline{\langle x_N \rangle} \cong \hat{\mathbb{Z}}$. Since ϕ_N maps $\overline{\langle x \rangle}$ onto $\overline{\langle x_N \rangle}$, we infer that $\overline{\langle x \rangle} \cong \hat{\mathbb{Z}}$, and so ϕ_N sends $\overline{\langle x \rangle}$ isomorphically onto $\overline{\langle x_N \rangle}$. Note that since G_N is free-by-finite, it is subgroup separable (cf. [**10**]); hence $\overline{\langle x_N \rangle} \cap G_N = \langle x_N \rangle$. It follows that $\langle x_N \rangle = \phi_N(\langle x \rangle) \leqslant \phi_N(\overline{\langle x \rangle} \cap G) \leqslant \overline{\langle x_N \rangle} \cap G_N = \langle x_N \rangle$, and so $\phi_N(\langle x \rangle) = \phi_N(\overline{\langle x \rangle} \cap G)$; consequently $\langle x \rangle = \overline{\langle x \rangle} \cap G$, since ϕ_N is injective on $\overline{\langle x \rangle}$. Thus $\langle x \rangle$ is closed in the profinite topology of G. This completes the proof that G is cyclic subgroup separable if each of the groups G_1 and G_2 are cyclic subgroup separable.

Now, from the quasi-potency of G_N, there is some natural number k such that x_N is k-potent. We deduce that x is k-potent, for if m is a natural number and $M \lhd_f G_N$ with $M \cap \langle x_N \rangle = \langle x_N^{km} \rangle$, then $\phi^{-1} M \lhd_f G$ with $\phi^{-1} M \cap \langle x \rangle = \langle x^{km} \rangle$. This completes the proof that G is quasi-potent if each of the groups G_1 and G_2 are quasi-potent.

REMARK. After this paper was written we learnt that part (ii) of the above proposition had been obtained previously by B. Evans [**1**, Lemma 3.2] using very different methods. We have decided to retain our proof because it is part of a uniform treatment that we have tried to maintain for the main results in this section, namely the interplay between groups and trees (both abstract and profinite).

LEMMA 4.3. *Let G_1 and G_2 be quasi-potent, cyclic subgroup separable groups with a common cyclic subgroup H, and set $G = G_1 *_H G_2$. Assume that $a \in G$ is hyperbolic* (*that is, $a \notin G_1^G \cup G_2^G$*), *and let T_a be its corresponding Tits straight line* (*see Proposition* 1.5). *Then*

(i) *$T_a/\langle a^n \rangle = \overline{T_a}/\overline{\langle a^n \rangle}$ for every natural number n,*

(ii) *if $\alpha \in \overline{\langle a \rangle}$, v is a vertex of T_a and $\alpha v \in T_a$, then $\alpha \in \langle a \rangle$,*

(iii) *T_a is a connected component of $\overline{T_a}$ considered as an abstract graph, in other words, the only vertices of $\overline{T_a}$ that are at a finite distance from a vertex of T_a are those of T_a.*

Proof. Let $v \in \mathrm{Ver}(T_a)$. Then $T_a = \langle a \rangle [v, av[$. It follows that $\overline{T_a} = \overline{\langle a \rangle} [v, av[$, and therefore $\overline{T_a}/\overline{\langle a \rangle}$ is a quotient of $T_a/\langle a \rangle = [v, av[$. Observe that if $N \leqslant_o \hat{G}$ is an open subgroup of $\hat{G}$, then the finite graphs $S(G)/(N \cap G)$ and $S(\hat{G})/N$ are naturally isomorphic. By Proposition 4.2, $G = G_1 *_H G_2$ is cyclic subgroup separable. Hence there exists a collection $\{L_i \leqslant_f G \mid i \in I\}$ such that $\langle a^n \rangle = \bigcap_{i\in I} L_i$. Let $D = T_a/\langle a^n \rangle$. Since D is finite, there exists some $i \in I$ such that the restriction to D of the natural epimorphism of graphs $S(G)/\langle a^n \rangle \to S(G)/L_i$, is an injection. Put $N_i = \overline{L_i}$, the closure of L_i in $\hat{G}$; then $L_i = N_i \cap G$. Note that $N_i \geqslant \overline{\langle a^n \rangle}$, and so the image of $\overline{T_a}/\overline{\langle a^n \rangle}$ in $S(\hat{G})/N_i$ coincides with the (isomorphic) image of $D = T_a/\langle a^n \rangle$ in $S(\hat{G})/N_i$:

$$D = T_a/\langle a^n \rangle \hookrightarrow S(G)/L_i \cong S(\hat{G})/N_i \longleftarrow S(\hat{G})/\overline{\langle a^n \rangle} \hookleftarrow \overline{T_a}/\overline{\langle a^n \rangle}.$$

Since $\overline{T_a}/\overline{\langle a^n \rangle}$ is a quotient of $T_a/\langle a^n \rangle$ and both are finite, we infer that $T_a/\langle a^n \rangle = \overline{T_a}/\overline{\langle a^n \rangle}$. This proves (i). To prove (ii), suppose that $\alpha v \in T_a$, then there exists some $g \in \langle a \rangle$ with $g\alpha v \in [v, av[$; so $g\alpha = 1$, and thus $\alpha \in \langle a \rangle$.

To prove (iii), consider a vertex ω of $\overline{T_a}$ which is at a finite distance from v. We need to show that $\omega \in T_a$. Suppose otherwise; then there is a first edge of $[v, \omega]$, say e', which is not in T_a, and we may in fact assume that the initial vertex of e' is v. Since $\overline{T_a} = \overline{\langle a \rangle}[v, av[$, there exists some $\alpha \in \overline{\langle a \rangle}$ such that $\alpha e = e'$, where e is an edge of $[v, av[$. Let w be the origin of e; then $\alpha w = v$, and therefore, by part (ii), $\alpha \in \langle a \rangle$. Thus $e' \in T_a$, a contradiction.

The following result is patterned after Example 1.20A of [**23**].

LEMMA 4.4. *Let T_a be as in Lemma 4.3. Then $\overline{T_a}$ does not have any proper infinite profinite subtrees.*

Proof. Observe that $T_t = T_a/\langle a^t \rangle = [v, a^t v]/(v = a^t v)$ is a cycle of length mt, where m is the length of $[v, av]$. By Lemma 4.3(i),

$$\overline{T_a} = \varprojlim_{t\in\mathbb{N}} T_a/\langle a^t \rangle.$$

Let Δ be an infinite profinite subtree of $\overline{T_a}$; fix $n \in \mathbb{N}$ and denote by Δ_n the image of Δ in $T_a/\langle a^n \rangle$. We must show that $\Delta_n = T_n = T_a/\langle a^n \rangle$. Suppose not, that is, suppose that Δ_n is a proper subgraph of T_n. Since Δ is connected, so is Δ_n, and hence Δ_n is a subtree of T_n (in fact a path which is not a cycle). Let $m > 1$ be a natural number. Then the canonical morphism of graphs $T_{mn} \to T_n$ is a covering. Therefore the preimage of Δ_n in T_{mn} is the disjoint union of m paths isomorphic to Δ_n. Now, since Δ_{mn} is connected and maps onto Δ_n, it follows that Δ_{mn} is one of those paths, and in particular, is isomorphic to Δ_n. Thus $\Delta = \varprojlim_{m>1} \Delta_{mn} = \Delta_n$, contradicting the assumption that Δ is infinite.

PROPOSITION 4.5. *Let G_1 and G_2 be groups in $\mathscr{X}$ with a common cyclic subgroup H. Then $G = G_1 *_H G_2$ has property* (d), *that is, $a^{\hat{G}} \cap \overline{\langle c \rangle} = \varnothing$ whenever a, $c \in G$ and $a^G \cap \langle c \rangle = \varnothing$.*

Proof. Assume that $c^z = \gamma a \gamma^{-1}$, where $\gamma \in \Gamma$ and $z \in \hat{\mathbb{Z}}$. Note that a must have infinite order. Our aim is to show the existence of some $g \in G$ such that $gag^{-1} \in C$.

Case 1. The element a fixes a vertex of $S(G)$ (that is, a is not hyperbolic). First we note that C fixes a vertex of $S(G)$, for otherwise, according to Proposition 2.9 in [**15**],

$\bar{C}$, and hence a, would act freely on $S(\Gamma)$, contradicting our hypothesis. This means that a and c are conjugate in G to elements of G_1 or G_2; so we may assume that $a \in G_1$ and $rcr^{-1} = c' \in G_1 \cup G_2$ for some $r \in G$. Then $(c')^z = r\gamma a \gamma^{-1} r^{-1}$. Now, $a^G \cap C \neq \emptyset$ if and only if $a^G \cap \langle c' \rangle \neq \emptyset$. Thus replacing c by c', we may assume that $a, c \in G_1 \cup G_2$. Say $a \in G_1$. If, in addition, $\gamma \in \Gamma_1$ and $c \in G_1$, then the result follows from property (d) applied to G_1.

For any other case we claim that we may assume that $a \in H$. If $\gamma \in \Gamma_1$ but $c \in G_2$, then $c^z = \gamma^{-1} a \gamma \in \Gamma_1 \cap \Gamma_2 = \Delta$; hence, by property (d) applied to G_1, there exists $g_1 \in G_1$ such that $g_1 a g_1^{-1} \in H$; and so we may assume that $a \in H$. Let now $\gamma \notin \Gamma_1$. Consider the vertices $v_1 = 1\Gamma_1$ and v, the vertex in $S(\Gamma)$ closest to v_1 and fixed by c (note $v = v_1$ or $v = v_2 = 1\Gamma_2$, depending on whether $c \in G_1$ or $c \in G_2$). Then $a = \gamma^{-1} c \gamma$ fixes v_1 and $\gamma^{-1} v$. Note that $\gamma^{-1} v \neq v_1$, for otherwise $v = v_1$ and hence we would have $\gamma \in \Gamma_1$, contrary to our assumption. By Theorem 2.8 in [**23**], the subgraph of $S(\Gamma)$ fixed by a is a profinite subtree T of $S(\Gamma)$; since this subtree contains the two different vertices v_1 and $\gamma^{-1} v$, there exists an edge $e_1 \in T$ whose initial vertex is v_1. Put $e = 1\Delta$; then there exists some $\gamma_1 \in \Gamma_1$ such that $\gamma_1 e_1 = e$; so $\gamma_1 a \gamma_1^{-1} e = e$; therefore $\gamma_1 a \gamma_1^{-1} \in \Delta$, since Δ is the stabilizer of e in Γ. It follows then from property (d) applied to G_1 that $g_1 a g_1^{-1} \in H$, for some $g_1 \in G_1$. Thus we may assume that $a \in H$. This proves the claim. So from now on we assume that a and c are in the same group G_i, say, $a, c \in G_2$, and $a \in H$. Furthermore we may assume $\gamma \notin \Gamma_2$, for if $\gamma \in \Gamma_2$, we simply apply property (d) to G_2.

Note that $c^z = \gamma a \gamma^{-1} \in \Gamma_2$ implies that c^z fixes the distinct vertices v_2 and γv_1. By an argument similar to one used above, there exists some edge e_2 with terminal vertex v_2 such that $c^z e_2 = e_2$, and some $\gamma_2 \in \Gamma_2$ such that $\gamma_2 e_2 = e$. We deduce that $\gamma_2 c^z \gamma_2^{-1} \in \Delta$. Now, observe that a and $\gamma_2 c^z \gamma_2^{-1}$ are elements of Δ and they are conjugate in Γ; therefore, according to Lemma 2.4 in [**15**], $\overline{\langle a \rangle} = \gamma_2 \overline{\langle c^z \rangle} \gamma_2^{-1}$. So $\gamma_2^{-1} a \gamma_2 \in \bar{C}$, and by property (d) applied to G_2, we have that a is conjugate in G_2 to an element of C, as desired.

Case 2. The element a does not fix a vertex of $S(G)$ (in other words, a is hyperbolic). Then by Proposition 2.9 in [**15**], $\bar{C}$ acts freely on $S(\Gamma)$, and therefore C acts freely on $S(G)$. It follows that the generator c of C is hyperbolic as well. Consider the Tits straight lines T_a and T_c corresponding to a and c (see Proposition 1.5), and denote by m_1 and m_2 their respective amplitudes. Let T_1 and T_2 be segments of T_a and T_c of length m_1 and m_2, respectively. Then $T_a = \langle a \rangle T_1$ and $T_c = \langle c \rangle T_2$. Set $e = 1H$, the edge of $S(G)$ stabilized by H. We claim that one may assume that $e \in T_1 \cap T_2$. To see this, consider $g_1, g_2 \in G$ such that $e \in g_1 T_1$ and $e \in g_2 T_2$. Set $a' = g_1 a g_1^{-1}$ and $c' = g_2 c g_2^{-1}$, and remark that $g_2 \gamma g_1^{-1} a' (g_2 \gamma g_1^{-1})^{-1} = (c')^z$. Then a' and c' are also hyperbolic, and one has corresponding straight lines $T_{a'} = g_1 T_a$ and $T_{c'} = g_2 T_c$. Define $T_1' = g_1 T_1$ and $T_2' = g_2 T_2$. Then clearly $T_{a'} = \langle a' \rangle T_1'$, $T_{c'} = \langle c' \rangle T_2'$, and $e \in T_1' \cap T_2'$. Since a is conjugate in G to an element of $\langle c \rangle$ if and only if a' is conjugate in G to an element of $\langle c' \rangle$, the claim follows. So from now on we assume that $e \in T_1 \cap T_2$.

Consider the profinite subgraphs of $S(\Gamma)$ defined as $\overline{T_a} = \overline{\langle a \rangle} T_1$ and $\overline{T_c} = \overline{\langle c \rangle} T_2$. By Proposition 2.9 of [**15**], $\overline{\langle a \rangle}$ and $\overline{\langle c \rangle}$ act freely on $S(\Gamma)$; hence $\overline{T_a}$ is the unique minimal $\overline{\langle a \rangle}$-invariant profinite subtree of $S(\Gamma)$ (cf. [**15**, Lemma 2.2]); similarly, $\overline{T_c}$ is the unique $\overline{\langle c \rangle}$-invariant profinite subtree of $S(\Gamma)$. Since $c^z = \gamma a \gamma^{-1}$, we have that $\gamma \overline{T_a}$ is the minimal $\overline{\langle c^z \rangle}$-invariant profinite subtree of $S(\Gamma)$; therefore $\gamma \overline{T_a} \subseteq \overline{T_c}$. Next we show that $\gamma \overline{T_a} = \overline{T_c}$. By the minimality of $\overline{T_c}$, it suffices to show that $\gamma \overline{T_a}$ is $\overline{\langle c \rangle}$-invariant. Indeed, let $\tilde{c} \in \overline{\langle c \rangle}$, and observe that $\overline{\langle c^z \rangle}$ acts on $\tilde{c} \gamma \overline{T_a}$, since c^z and $\tilde{c}$ commute; so $\tilde{c} \gamma \overline{T_a}$ is a minimal $\overline{\langle c^z \rangle}$-invariant profinite subtree of $S(\Gamma)$, and therefore

$\tilde{c}\gamma \overline{T_a} = \gamma \overline{T_a}$, as desired. From $\gamma \overline{T_a} = \overline{T_c}$ we infer that $\gamma e \in \overline{T_c}$. Choose $c' \in \overline{\langle c \rangle}$ such that $c'\gamma e \in T_2$. Then $c'\gamma e = ge$ for some $g \in G$. Hence $c'\gamma = g\delta$ for some $\delta \in \Delta$. Now, $a = \gamma^{-1} c'^{-1} c^z c' \gamma = \delta^{-1} g^{-1} c^z g \delta$. Therefore, using $g^{-1}Cg$ instead of C, we can assume that $\gamma (= \delta)$ is in Δ.

Next consider the group $R = \Gamma_1 *_\Delta \Gamma_2$ (the amalgamated free product of Γ_1 and Γ_2 amalgamating Δ, as abstract groups). Recall that R is a dense subgroup of Γ (cf. [**14**]). Since a and c^z are hyperbolic and $c^z = \gamma a \gamma^{-1} \in R$, it follows that c^z can be written as a product $c^z = w_1 w_2 \ldots w_m$, where $w_i \in \Gamma_1 \cup \Gamma_2$. This means that the path $[e, c^z e]$ is finite. Furthermore note that $[e, c^z e] \subseteq \overline{T_c}$. Hence, by Lemma 4.3, $c^z \in C$. Using Theorem 2.2, we deduce that a and $c^z \in C$ are conjugate in G, as needed.

LEMMA 4.6. *Let G_1, G_2 be residually finite groups with a common cyclic subgroup H, such that G_i is quasi-potent* $(i = 1, 2)$*, and H is closed in the profinite topology of G_i* (*for* $i = 1, 2$). *Set $G = G_1 *_H G_2$, and let $C = \langle c \rangle$ be a cyclic subgroup of G. Assume that c is hyperbolic with respect to its action on the standard tree $S(G)$, and let T_c be its Tits straight line. Consider the stabilizer*

$$R = \{ g \in G \mid gT_c = T_c \}$$

of T_c in G. Then,

(i) $C \leqslant R$;

(ii) *R is closed in the profinite topology of G;*

(iii) *R is polycyclic-by-finite of Hirsch length at most* 2;

(iv) *the profinite topology of G induces on R its full profinite topology.*

Proof. Statement (i) is obvious. Let $R' = \{ g \in \hat{G} \mid g\overline{T_c} = \overline{T_c} \}$; by Lemma 4.3, $R = R' \cap G$, and so (ii) follows. Consider the natural representation

$$R \longrightarrow \operatorname{Aut}(T_c),$$

and observe that its kernel $K = \langle k \rangle$ is cyclic, for it fixes every edge of T_c and so it is a subgroup of a conjugate of H. On the other hand, $\operatorname{Aut}(T_c)$ is isomorphic to the infinite dihedral group. Therefore R is a polycyclic-by-finite group with Hirsch length at most 2; this proves (iii).

To prove (iv) we proceed in two steps. First we shall show that R contains a free abelian subgroup A of finite index in R and closed in the profinite topology of G. Observe that $K \cap C = 1$, and so R/K is either infinite cyclic or an infinite dihedral group. Let $X \in R$ be such that xK generates a maximal infinite cyclic subgroup of R/K. Then either $\langle xK \rangle$ coincides with R/K or it has index 2 in R/K. Put $M = \langle x, k \rangle$; then M is a torsion-free subgroup of R of index at most 2 in R. Consider the centralizer $A = \mathscr{C}_M(K)$ of K in M. Plainly, either $A = \langle x, k \rangle = M$ or $A = \langle x^2, k \rangle$. Hence A is a free abelian group (of rank 1 or 2) of index at most 2 in M. We claim that $A = \mathscr{C}_G(A)$. Note first that $\mathscr{C}_G(A) \leqslant R$, for let $1 \neq a \in A \cap C$ and note that T_c is also the Tits straight line of a, and so it is the unique minimal $\langle a \rangle$-invariant subtree of $S(G)$. Now, if $z \in \mathscr{C}_G(A)$, then zT_c is a minimal $\langle a \rangle$-invariant subtree of $S(G)$, therefore, $zT_c = T_c$, and hence $z \in R$. If $r \in R - M$, then $rxr^{-1} \equiv x^{-1} \pmod{K}$, so that $r \notin \mathscr{C}_G(A)$; hence $\mathscr{C}_G(A) \leqslant M$. Thus $\mathscr{C}_G(A) = \mathscr{C}_M(A) = A$. This proves the claim. Next note that the centralizer of any element in a topological group is closed; therefore $A = \mathscr{C}_G(A)$ is closed in G.

To complete the proof of (iv), it suffices to show that the profinite topology of G induces on A its full profinite topology. If A has rank 1, this is the case since G is quasi-potent (see Proposition 4.2). Otherwise $A = \langle x, k \rangle$ is free abelian with basis

x, k. Now, since k fixes all elements of T_c, it follows that $\overline{\langle k\rangle}$ also fixes all elements of $\overline{T_c}$; on the other hand, since x acts freely on T_c, the group $\overline{\langle x\rangle}$ acts freely on $\overline{T_c}$ (see Lemma 4.1). We deduce that $\overline{\langle x\rangle} \cap \overline{\langle k\rangle} = 1$, and so $\bar{A} = \overline{\langle x\rangle} \times \overline{\langle k\rangle}$; but according to Proposition 4.2, $\overline{\langle x\rangle} \cong \widehat{\langle x\rangle}$ and $\overline{\langle k\rangle} \cong \widehat{\langle k\rangle}$, and so $\bar{A} \cong \hat{A}$.

PROPOSITION 4.7. *Assume the groups G_1, G_2 are quasi-potent and have property* (*c*) (*the product of two cyclic subgroups of G_i is closed in the profinite topology of G_i* (*for $i = 1, 2$*)), *and let H be a common cyclic subgroup of G_1 and G_2. Then the amalgamated free product $G = G_1 *_H G_2$ has property* (*c*).

Proof. Let C_1, C_2 be cyclic subgroups of G. One must show that $\bar{C}_1 \bar{C}_2 \cap G = C_1 C_2$, where $\bar{C}_1, \bar{C}_2$ denote the closures of C_1, C_2 in $\hat{G} = \hat{G}_1 \coprod_{\hat{H}} \hat{G}_2$, respectively. Consider the standard abstract graph $S(G)$ and the standard profinite graph $S(\hat{G})$ described at the beginning of this section. Let $\gamma_1 \in \overline{C_1}, \gamma_2 \in \overline{C_2}$ and assume that $\gamma_1\gamma_2 = k \in G$. We shall show that $\gamma_1\gamma_2 \in C_1 C_2$. If $\gamma_1 \in C_1$ then $\gamma_2 \in G \cap \overline{C_2} = C_2$, and the result is proved. So from now on we assume that $\gamma_1 \notin C_1$ and $\gamma_2 \notin C_2$. We consider three cases.

Case 1 : C_1 and C_2 are conjugate to subgroups of G_1 or G_2. Say that C_1 is conjugate to a subgroup of G_1; then we may assume that $C_1 \leqslant G_1$. Consider the vertex $v_1 = 1G_1 = 1\hat{G}_1$ and observe that its stabilizer under the action of G on $S(G)$ is G_1, while that under the action of $\hat{G}$ on $S(\hat{G})$ is $\hat{G}_1$. Let w be the closest vertex to v_1 in $S(G)$ which is fixed by C_2; note that $\overline{C_2}$ fixes w as a vertex of $S(\hat{G})$. We use induction on the length l of the path $[v_1, w]$ to prove that $\gamma_1\gamma_2 \in C_1 C_2$. If $l = 0$, then both C_1 and C_2 are subgroups of G_1 and the assertion follows from the fact that G_1 has property (c) and it is closed in G (see Lemma 2.1). Assume now that $l > 0$ and that the result holds whenever the path $[v_1, w]$ has length smaller than l. Let e denote the last edge of the path $[v_1, w]$. Observe that $(\gamma_1\gamma_2)^{-1} v_1 = \gamma_2^{-1} v_1$ and $\gamma_2^{-1} w = w$ are vertices of $S(G)$; hence the path $\gamma_2^{-1}[v_1, w]$ is in $S(G)$, and, in particular, $\gamma_2^{-1} e$ is an edge in $S(G)$. Therefore there exists some $g_w \in G_w$ (the subgroup of G stabilizing w) such that $g_w e = \gamma_2^{-1} e$. We deduce that $g_w \in \overline{C_2}\hat{G}_e$, where $\hat{G}_e$ is the subgroup of $\hat{G}$ stabilizing e; note that $\hat{G}_e = \overline{G_e}$. Since G_e is a conjugate of H, it is cyclic. On the other hand, G_w is a conjugate of either G_1 or G_2, and so it has property (c). Hence there exist $c_2 \in C_2, g_e \in G_e$ such that $g_w = c_2 g_e$. Therefore $c_2 e = g_w e = \gamma_2^{-1} e$. It follows that $\gamma_2 c_2$ fixes e. Denote by w_1 the other vertex of e; then $\gamma_2 c_2$ fixes w_1 and the length of $[v_1, w_1]$ is $l - 1$. We infer from the induction hypothesis that $\gamma_1\gamma_2 c_2$ is in $C_1 C_2$, and thus so is $k = \gamma_1\gamma_2$, as desired.

Case 2: C_2 fixes a vertex v of $S(G)$ and C_1 is hyperbolic. Denote by a a generator of C_1. Let T_{C_1} be the Tits straight line subgraph associated with a (cf. [**18**, Proposition 24]). Then $\gamma_1\gamma_2 v = \gamma_1 v \in S(G)$. Choose a vertex w of T_{C_1}; then $\gamma_1[v, w]$ is a finite path of $S(\hat{G})$, and hence $[\gamma_1 w, w]$ is also finite. Therefore, by Lemma 4.3, $\gamma_1 \in C_1$, a contradiction. Thus this case does not arise under our assumptions.

Case 3: C_1 and C_2 are hyperbolic. Denote by T_{c_1} and T_{c_2} the Tits straight lines subgraphs of $S(G)$ associated with c_1 and c_2, respectively, where $C_1 = \langle c_1\rangle$ and $C_2 = \langle c_2\rangle$ (cf. [**18**, Proposition 24]). Let v be a vertex of T_{c_2}. Note that the profinite paths $[v, \gamma_2 v]$ and $[\gamma_2 v,\ kv]$ in $S(\hat{G})$ have the same image in the profinite tree $S(\hat{G})/[v, kv]$ obtained by collapsing $[v, kv]$ to a vertex (cf. [**23**, Proposition 1.17]). Since $[v, kv]$ is a finite path, it follows that $[v, \gamma_2 v]$ and $[\gamma_2 v, kv]$ differ by at most a finite number of vertices and edges. Now, it follows from Lemma 4.4 that $[v, \gamma_2 v] = \overline{T_{c_2}}$ and that $[v, \gamma_2 v] \cap [\gamma_2 v, kv] = [v, \gamma_2 v] = \overline{T_{c_2}}$, since both $[v, \gamma_2 v]$ and $[v, \gamma_2 v] \cap [\gamma_2 v, kv]$ are infinite profinite subtrees of $\overline{T_{c_2}}$.

Let T be the minimal C_1-invariant subtree of $S(G)$ containing kv. Then $T_{c_1} \subseteq T$ (cf. [**18**, Proposition 24]). Furthermore, $\overline{T}_{c_2} \subseteq \overline{T}$, since $\overline{T}$ contains kv and $\gamma_2 v = \gamma_1^{-1}\gamma_1\gamma_2 v$, and so it contains $[\gamma_2 v, kv]$, which in turn contains $\overline{T_{c_2}}$. Let w be a vertex of T_{c_1}. Then $T_{c_1} = C_1[w, aw]$ and $T = C_1([w, aw] \cup [w, kv])$. Therefore $\overline{T} = \overline{C_1}([w, aw] \cup [w, kv])$, and so one deduces from Lemma 4.3 that $\overline{T} \cap S(G) = T$. Hence both T_{c_1} and T_{c_2} are subgraphs of T. Next we distinguish two possibilities.

If $T_{c_1} = T_{c_2}$, then by Lemma 4.6 $C_1, C_2 \leqslant R \leqslant G$ where R is polycyclic and $\bar{R} \cong \hat{R}$. Therefore C_1C_2 is closed in R (see Proposition 3.7), and so in G.

If, on the other hand, $T_{c_1} \neq T_{c_2}$, then the image of T_{c_2} in the quotient graph $S(G)/T_{c_1}$ is an infinite straight line (for $T_{c_1} \cap T_{c_2}$ is finite by Lemma 4.4), so that its diameter is infinite. However, this image is contained in the image of $T = C_1([w, aw] \cup [w, kv])$ on $S(G)/T_{c_1}$, which clearly has finite diameter (in fact, bounded by the length of $[w, kv]$). This contradiction shows that, in reality, the case $T_{c_1} \neq T_{c_2}$ does not occur.

Proposition 4.8. *Let G_1 and G_2 be quasi-potent, cyclic subgroup separable groups having property* (e). *Suppose that H is a common cyclic subgroup of G_1 and G_2, and let $G = G_1 *_H G_2$. Then for any cyclic subgroups C_1, C_2 of G, we have $\overline{C_1 \cap C_2} = \overline{C_1} \cap \overline{C_2}$.*

Proof. Let $C_1 = \langle c_1 \rangle$, $C_2 = \langle c_2 \rangle$ be cyclic subgroups of G. Plainly, $\overline{C_1 \cap C_2} \leqslant \overline{C_1} \cap \overline{C_2}$; hence the result follows if $\overline{C_1} \cap \overline{C_2} = 1$. So, from now on, we shall assume that $\overline{C_1} \cap \overline{C_2} \neq 1$. First we observe that showing $\overline{C_1 \cap C_2} = \overline{C_1} \cap \overline{C_2}$ is equivalent to showing that $C_1 \cap C_2 \neq 1$. For suppose $C_1 \cap C_2 \neq 1$; then $K = \overline{C_1} \cap \overline{C_2}$ is open in both $\overline{C_1}$ and $\overline{C_2}$; by Proposition 4.2 G is cyclic subgroup separable, and so $C_1 \cap C_2 = \overline{C_1} \cap G \cap \overline{C_2} \cap G = K \cap G = K \cap \overline{C_i} \cap G = K \cap C_i$, and $K \cap C_i \leqslant_f C_i$ for $i = 1, 2$; since $\overline{C_i} = \widehat{C_i}$ (for, according to Proposition 4.2, G_i is quasi-potent), there is a one-to-one correspondence between the open subgroups of $\overline{C_1}$ and the subgroups of finite index of C_1; thus $\overline{C_i} \cap \overline{C_2} = K = \overline{C_1 \cap C_2}$. The opposite implication is obvious.

Consider the tree $S(G)$ and the profinite tree $S(\hat{G})$ associated with $G = G_1 *_H G_2$ and $\hat{G} = \widehat{G_1} *_{\hat{H}} \widehat{G_2}$ respectively. Then according to Proposition 2.9 of [**15**], if an element a of G acts freely on $S(G)$ (that is, a is hyperbolic), then $\overline{\langle a \rangle}$ acts freely on $S(\hat{G})$ as well. Consequently, either c_1 and c_2 are both hyperbolic or both nonhyperbolic.

Case 1. The elements c_1 and c_2 are not hyperbolic. Then c_i (for $i = 1, 2$) is conjugate to an element of G_1 or G_2. Let l be the minimal distance between two vertices u_1 and u_2 of $S(G)$ such that u_i is fixed by c_i $(i = 1, 2)$. We shall prove that $C_1 \cap C_2 \neq 1$ using induction on l. Say $c_1 \in g_1 G_1 g_1^{-1}$. Substituting c_i by $g_1^{-1} c_i g_1$ $(i = 1, 2)$, we may assume that $c_1 \in G_1$. Then c_1 fixes the vertex $v_1 = 1G_1$ of $S(G)$. Let v denote the vertex of $S(G)$ fixed by c_2 and closest to v_1. Then l is the length of $[v_1, v]$. If $l = 0$, then $v_1 = v$ and so $C_1, C_2 \leqslant G_1$, and the result follows by property (e) applied to G_1.

Next we consider the case $l = 1$ separately. In this case $c_2 \in g_1 G_2 g_1^{-1}$ for some $g_1 \in G_1$. Substituting c_i by $g_1^{-1} c_i g_1$ $(i = 1, 2)$, we may assume that $c_2 \in G_2$. Put $v_2 = 1G_2$ and $e = 1H$; then $v = v_2$ and $\overline{C_1} \cap \overline{C_2}$ stabilizes v_1 and v_2, and therefore e. It follows that $\overline{C_1} \cap \overline{C_2} \leqslant \bar{H}$. So $\overline{C_i} \cap \bar{H} \geqslant \overline{C_1} \cap \overline{C_2} \neq 1$ $(i = 1, 2)$. Applying (e) to G_1 and to G_2 we get that $C_1 \cap H \neq 1 \neq C_2 \cap H$. Since H is cyclic, it follows that $C_1 \cap C_2 \neq 1$, as needed.

Assume now that $l > 1$ and that the result holds whenever c_1' and c_2' are nonhyperbolic elements that fix vertices of $S(G)$ that are at a distance smaller than l and such that $\overline{\langle c_1' \rangle} \cap \overline{\langle c_2' \rangle} \neq 1$. Then the path $[v_1, v]$ contains at least two edges. Let $\tilde{e}$, $\tilde{\tilde{e}}$ be the first two edges of $[v_1, v]$. Hence $\tilde{e} = g_1 e$, for some $g_1 \in G_1$. Substituting c_i by $g_1^{-1} c_i g_1$ (for $i = 1, 2$), we may assume that the first edge of $[v_1, v]$ is e. Then the second vertex of $[v_1, v]$ is $v_2 = 1G_2$ and so $\tilde{\tilde{e}} = g_2 e$, for some $g_2 \in G_2$. Note that $\overline{C_1} \cap \overline{C_2}$ stabilizes the vertices and edges of $[v_1, v]$ (for $S(\hat{G})$ is a profinite tree and so it does not contain finite cycles), and in particular it stabilizes e and $g_2 e$. Hence $\overline{C_1} \cap \overline{C_2} \leqslant \Delta_1 = \bar{H} \cap g_2 \bar{H} g_2^{-1}$. By property (e) of G_2, we have that $H \cap g_2 H g_2^{-1} \neq 1$; but since the length of $[g_2 v_1, v]$ is less than l, it follows from the induction hypothesis that $H \cap g_2 H g_2^{-1} \cap C_2 \neq 1$, and from the case $l = 1$ considered above, $H \cap g_2 H g_2^{-1} \cap C_1 \neq 1$. Thus, $C_1 \cap C_2 \neq 1$, since $H \cap g_2 H g_2^{-1}$ is cyclic.

Case 2. The elements c_1 and c_2 are hyperbolic. By Proposition 2.9 of [**15**] $\overline{C_i}$ acts freely on $S(\hat{G})$ (for $i = 1, 2$). Let T_{c_i} be the Tits straight line of $S(G)$ with amplitude m_i corresponding to c_i (for $i = 1, 2$) (see Proposition 1.5). Then $\overline{T_{c_i}} = \overline{\langle c_i \rangle} T_i$, where T_i is a segment of T_{c_i} of length m_i ($i = 1, 2$). Therefore $\overline{T_{c_i}}$ is a minimal $\overline{\langle c_i \rangle}$-invariant subtree of $S(\hat{G})$ ($i = 1, 2$). Since $\overline{C_1} \cap \overline{C_2} \neq 1$, we have that $\overline{T_{c_i}}$ is also a minimal $\overline{C_1} \cap \overline{C_2}$-invariant subtree of $S(\hat{G})$ ($i = 1, 2$) (cf. [**22**, Lemma 2.1]). Since $\overline{C_1} \cap \overline{C_2}$ acts freely on $S(\hat{G})$, one deduces that there is only one minimal $\overline{C_1} \cap \overline{C_2}$-invariant subtree of $S(\hat{G})$ (cf. [**15**, Lemma 2.2]); and so $\overline{T_{c_1}} = \overline{T_{c_2}}$. Then, according to Lemma 4.3,

$$T_{c_i} = \overline{T_{c_i}} \cap S(G) \quad \text{for } i = 1, 2.$$

Hence $T_{c_1} = T_{c_2}$. By Lemma 4.6 there is a polycyclic-by-finite subgroup R of G that contains C_1 and C_2, such that $\bar{R} \cong \hat{R}$. Thus, by Proposition 3.5, $C_1 \cap C_2 \neq 1$, as desired.

Finally we must show that property (f) is preserved under free products with cyclic amalgamation.

Proposition 4.9. *Let G_1 and G_2 be quasi-potent groups having properties* (f) *and* (d). *Suppose that H is a common closed cyclic subgroup of G_1 and G_2, and let $G = G_1 *_H G_2$. Then G has property* (f).

Proof. Let g be an element of G and suppose that $\gamma \in \hat{G}$ satisfies $\gamma \overline{\langle g \rangle} \gamma^{-1} = \overline{\langle g \rangle}$. We need to prove that γ inverts or centralizes g. Consider the tree $S(G)$ and the profinite tree $S(\hat{G})$ associated with $G = G_1 *_H G_2$ and $\hat{G} = \hat{G}_1 \coprod_{\hat{H}} \hat{G}_2$ respectively.

Case 1. The element g is not hyperbolic. Since g is conjugate in G to an element of $G_1 \cup G_2$, we can assume that g is in G_1 or G_2, say in G_1. If $\gamma \in \hat{G}_1$ then the result follows from property (f) for G_1. Otherwise, by Corollary 3.12 of [**23**], $g \in \delta \hat{H} \delta^{-1}$ for some $\delta \in \hat{G}_1$; then, by Proposition 4.5 g is conjugate in G_1 to an element of H, and we may assume that $g \in H$. Hence by Corollary 2.7 of [**15**],

$$\mathcal{N}_{\hat{G}}(\overline{\langle g \rangle}) = \mathcal{N}_{\hat{G}_1}(\overline{\langle g \rangle}) \coprod_{\hat{H}} \mathcal{N}_{\hat{G}_2}(\overline{\langle g \rangle}).$$

Consider the natural homomorphism

$$\phi : \mathcal{N}_{\hat{G}}(\overline{\langle g \rangle}) \longrightarrow \operatorname{Aut} \overline{\langle g \rangle}.$$

Let C denote the subgroup of $\operatorname{Aut} \overline{\langle g \rangle}$ of order 2 consisting of the identity automorphism and the automorphism that inverts g. It follows from the property (f) for G_1 and G_2 that the images of $\mathcal{N}_{\hat{G}_1}(\overline{\langle g\rangle})$ and $\mathcal{N}_{\hat{G}_2}(\overline{\langle g\rangle})$ in $\mathrm{Aut}\,\overline{\langle g\rangle}$ are in C. Hence $\mathrm{Im}\,(\phi)$ is contained in C, as desired.

Case 2. The element g is hyperbolic. Let T_g be the corresponding infinite straight line and $\overline{T_g}$ its closure in $S(\hat{G})$. Since $\overline{T_g}$ is the unique minimal g-invariant subtree of $S(\hat{G})$ (cf. [**15**, Lemma 2.2]), $\mathcal{N}_{\hat{G}}(\overline{\langle g\rangle})$ acts naturally on $\overline{T_g}$. Hence we have the following commutative diagram of natural embeddings

$$\begin{array}{ccc} \langle g\rangle & \longrightarrow & \mathrm{Aut}(T_g) \\ \downarrow & & \downarrow \\ \mathcal{N}_{\hat{G}}(\overline{\langle g\rangle}) & \longrightarrow & \mathrm{Aut}(\overline{T_g}) \end{array}$$

Let $v \in T_g$. By Lemma 4.3(i) there exists $g' \in \overline{\langle g\rangle}$ such that $g'\gamma v \in T_g$. Now, by Lemma 4.3(iii), T_g is a connected component of $\overline{T_g}$ considered as an abstract graph, so $g'\gamma$ acts on T_g. Therefore, the automorphism δ of $\mathrm{Aut}\,(\overline{T_g})$ induced by $g'\gamma$ actually belongs to $\mathrm{Aut}\,(T_g)$. Now, since $\mathrm{Aut}\,(T_g) \cong C_2 * C_2$, the normalizer of every infinite subgroup of $\mathrm{Aut}\,(T_g)$ is the whole group. Thus δ, or equivalently $g'\gamma$, normalizes $\langle g\rangle$ (here g is hyperbolic and so it has infinite order), and so inverts or centralizes g. Finally, since g' centralizes every element of $\overline{\langle g\rangle}$, it follows that γ also inverts or centralizes g, as desired.

We end the paper by indicating that the proof of Theorem A stated in the Introduction follows now from Theorem 2.2 and Propositions 4.2, 4.5, 4.7, 4.8 and 4.9.

In [**15**] it is shown that free-by-finite groups are in the class $\mathcal{X}$. This together with Theorem 3.8 and Theorem A imply, in particular, the following theorem.

THEOREM 4.10. *Let $\mathcal{X}_1$ be the class of all groups that are either free-by-finite or polycyclic-by-finite. For $i > 1$, recursively define the class $\mathcal{X}_i$ to consist of all groups that are free products*

$$G = G_1 *_H G_2$$

of groups G_1, G_2 in $\mathcal{X}_{i-1}$ with a cyclic amalgamated subgroup H. Then every group in the class

$$\mathcal{X}' = \bigcup_{i=1}^{i=\infty} \mathcal{X}_i$$

is conjugacy separable.

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