Virtually free pro-p groups whose torsion elements have finite centralizers

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Abstract

It is shown that a finitely generated virtually free pro-p group G with finite centralizers of its torsion elements is the free pro-p product of finite p-groups and a free pro-p factor.

1 Introduction

The objective of this paper is to give a complete description of a finitely generated virtually free pro-p group whose torsion elements have finite centralizers. Our main result is the following

Theorem 1 *Let G be a finitely generated virtually free pro-p group such that the centralizer of every torsion element in G is finite. Then G is a free pro-p product of subgroups which are finite or free pro-p.*

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This is a rather surprising result from a group theory point of view, since the theorem does not hold for abstract groups (as well as for profinite groups): an easy counter example is given in Section 5. However, from a Galois theory point of view it is not so surprising. Indeed, the finite centralizer condition for torsion elements arises naturally in the study of absolute Galois groups. In particular, D. Haran [2] (see also I. Efrat in [1] for a different proof) proved the above theorem for the case when G is an extension of a free pro-2 group with a group of order 2.

The proof of Theorem 1 explores a connection between p-adic representations of finite p-groups and virtually free pro-p groups, which gives a new approach to study virtually free pro-p groups. This connection enables us to use the following beautiful result:

Theorem 2 *([7] A. Weiss) Let G be a finite p-group, N a normal subgroup of G and let M be a finitely generated $\mathbf{Z}_p[G]$-module. Suppose M is a free N-module and M^N is a permutation lattice for G/N. Then M is a permutation lattice for G.*

Here M^N means the fixed submodule for N, and a *permutation lattice* for G means a direct sum of G-modules, each of the form $\mathbf{Z}_p[G/H]$ for some subgroup H of G.

The connection to representation theory can not be used in a straight forward way however. Indeed, if one factors out the commutator subgroup of a free open normal subgroup F then the obtained G/F-module would, in general, not satisfy the hypothesis of Weiss' theorem. In order to make representation theory work, we use pro-p HNN-extensions to embed G into a rather special virtually free pro-p group $\tilde{G}$, in which, after factoring out the commutator of a free open normal subgroup, the hypotheses of Weiss' theorem are satisfied. With its aid we prove Theorem 1 for $\tilde{G}$ and apply the Kurosh subgroup theorem to deduce the result for G.

We use notation for profinite and pro-p groups from [4].

2 Preliminary results

We shall need the following connection between free decompositions and $\mathbf{Z}_p$ representations for free pro-p by C_p groups.

Lemma 3 *Let G be a split extension of a free pro-p group F of finite rank by a group of order p. Then*

(i) *([6]) G has a free decomposition $G = (\coprod_{a\in A} C_a \times H_a) \amalg H$, with $C_a \cong C_p$ and H_a and H free pro-p.*

(ii) *Set $M := F/[F,F]$. Fix $a_0 \in A$ and a generator c of C_{a_0}. Then conjugation with c induces an action of C_{a_0} upon M. The latter module decomposes in the form*

$$M = M_1 \oplus M_p \oplus M_{p-1}$$

such that M_p is a free $\langle c\rangle$-module, on M_{p-1} the equality $1+c+\cdots+c^{p-1}=0$ holds, and c acts trivially on M_1.

Proof: (i) is 1.1 Theorem in [6]. For proving (ii), first pick for each $a \in A$ a generator c_a of C_a and let $c_{a_0} := c$. Apply the Kurosh subgroup theorem, [3] to F and find

$$F = \left(\coprod_{a\in A} H_a\right) \amalg \left(\coprod_{j=0}^{p-1} H^{c^j}\right) \amalg \left(\coprod_{a\in A\setminus\{a_0\}} \langle c_a c^{-1}\rangle\right).$$

Factoring out $[F,F]$ yields the desired decomposition – the images of the three free factors.

□

Lemma 4 *Every finitely generated virtually free pro-p group has, up to conjugation, only a finite number of finite subgroups.*

Proof: Suppose the lemma is false and G is a counter-example possessing a normal free pro-p subgroup F of minimal possible index. When a finite subgroup A is contained in some maximal open subgroup H of G with $F \leq H$ then, as $|H:F| < |G:F|$, the proper subgroup H satisfies the conclusion of the lemma, and so there are, up to conjugation, only finitely many finite subgroups of G. Hence, in order to be a counter-example, G must be of the form $G = F \rtimes K$ for a finite subgroup K of G. The proof is finished, if we can show that up to conjugation, there are only finitely many finite subgroups $L \cong K$ in G. Let t be a central element of order p in K and consider $G_1 := F \rtimes \langle t\rangle$. Certainly G_1 is finitely generated. Hence, as a consequence of Lemma 3 (i), G_1 satisfies the conclusion of the lemma, and so, $G > G_1$. Next observe that any finite subgroup $L \cong K$ of G containing some torsion element $t \in G_1$ is contained in $C_G(t)$. By 1.2 Theorem in [6], $C_F(t)$ is a free factor of F and therefore, since F is finitely generated, $C_F(t)$ is finitely generated as well, and so is $C_G(t)$. Let bar denote passing to the quotient mod the normal subgroup t of $C_G(t)$. Then $|\overline{C_G(t)} : \overline{C_F(t)}| < |G : F|$, so that $\overline{C_G(t)}$ contains only finitely many conjugacy classes of maximal finite subgroups. Since the centralizers of conjugate elements are conjugate, G can, up to conjugation, contain only finitely many maximal finite subgroups, a contradiction. $\square$

We shall frequently use also the following result about free products.

Theorem 5 *([3], Lemma 3.2) Let* $G = \coprod_{i=1}^{n} G_i$ *be a free profinite (pro-p) product. Then* $G_i \cap G_j^g = 1$ *for either* $i \neq j$ *or* $g \notin G_j$.

3 HNN-embedding

We introduce a notion of a pro-p HNN-group as a generalization of pro-p HNN-extension in the sense of [5], page 97. It also can be defined as a sequence of pro-p HNN-extensions. During the definition to follow, i belongs to a finite set I of indices.

Definition 6 Let G be a pro-p group and A_i, B_i be subgroups of G with isomorphisms $\phi_i : A_i \longrightarrow B_i$. The pro-$p$ HNN-group is then a pro-p group $HNN(G, A_i, \phi_i, z_i)$ having presentation $HNN(G, A_i, \phi_i, z_i) = \langle G, z_i \mid rel(G), \forall a_i \in A_i : \quad a_i^{z_i} = \phi_i(a_i)\rangle$. The group G is called the *base group*, A_i, B_i are called *associated subgroups* and z_i are called the *stable letters.*

For the rest of this section let G be a finitely generated virtually free pro-p group, and fix an open free pro-p normal subgroup F of G of minimal index. Also suppose that $C_F(t) = \{1\}$ for every torsion element $t \in G$. Let $K := G/F$ and form $G_0 := G \amalg K$. Let $\psi : G \to K$ denote the canonical projection and I be the set of all G-conjugacy classes of maximal finite subgroups of G. Fix, for every $i \in I$, a maximal finite subgroup K_i of G in the G-conjugacy class i. We define a pro-p HNN-group by considering first $\tilde{G}_0 := G_0 \amalg F(z_i \mid i \in I)$ with z_i constituting a free set of generators, and then taking the normal subgroup R in $\tilde{G}_0$ generated by all elements of the form $k_i^{z_i}\psi(k_i)^{-1}$, with $k_i \in K_i$ and $i \in I$. Finally set

$$\tilde{G} := \tilde{G}_0 R/R,$$

and note that it is an HNN-group $HNN(G_0, K_i, \phi_i, z_i)$, where $\phi_i := \psi_{|K_i}$, G_0 is the base group, the K_i are associated subgroups, and the z_i form a set of stable letters in the sense of Definition 6.

The objective of the section is to show that the centralizers of torsion elements in $\tilde{G}$ are finite. We start with the following

Lemma 7 *Let G be virtually free pro-p and $C_F(t) = \{1\}$ for every torsion element $t \in G$. Then any pair of distinct maximal finite subgroups A, B of G has trivial intersection.*

Proof: Suppose the lemma were false. Then one can pick maximal finite subgroups A and $B \neq A$ such that $1 \neq C := A \cap B$ is of maximal possible cardinality. Then C is a finite normal subgroup of $L := \langle N_A(C), N_B(C)\rangle$, so the latter is itself finite, since $N_G(C)$ must be finite (a finite normal subgroup of a pro-p group intersects the center non-trivially). On the other hand, one must have $L \cap A = C$ due to the maximality assumption. Since $C < A$ one arrives at the contradiction $C < N_A(C) \leq L \cap C = C$. $\square$

Lemma 8 *Let $\tilde{G} = HNN(G_0, K_i, \phi_i, z_i)$ be as explained and $\tilde{F}$ a free pro-p open normal subgroup of minimal index in $\tilde{G}$. Then $C_{\tilde{F}}(t) = 1$ for every torsion element $t \in \tilde{G}$.*

Proof: There is a standard pro-p tree $S := S(\tilde{G})$ associated to $\tilde{G} := HNN(G_0, K_i, \phi_i, z_i)$ on which $\tilde{G}$ acts naturally such that the vertex stabilizers are conjugates of G_0 and each edge stabilizer is a conjugate of some K_i (cf. [5] and §3 in [9]).

Claim: Let e_1, e_2 be two edges of S with a common vertex v. Then the intersection of the stabilizers $\tilde{G}_{e_1} \cap \tilde{G}_{e_2} = 1$.

By translating e_1, e_2, v if necessary we may assume that G_0 is the stabilizer of v. Then, up to orientation, we have two cases:

1) v is initial vertex of e_1 and e_2. Then $\tilde{G}_{e_1} = K_i^g$ and $\tilde{G}_{e_2} = K_j^{g'}$ with $g, g' \in G_0$ and either $i \neq j$ or $g \notin K_i g'$. Suppose $K_i^g \cap K_j^{g'} \neq \{1\}$. Then, since

$G_0 = G \amalg K$, we may apply Theorem 5, in order to deduce the existence of $g_0 \in G_0$ with $K_i^{gg_0} \cap K_j^{g'g_0} \leq G$. Now apply Lemma 7, in order to deduce the contradiction $i = j$ and $gg_0 \in K_i g' g_0$. So we have $K_i^g \cap K_j^{g'} = \{1\}$, as needed.

2) v is the terminal vertex of e_1 and the initial vertex of e_2. Then $\tilde{G}_{e_1} = K^g$ and $\tilde{G}_{e_2} = K_i^{g'}$ for $g, g' \in G_0$ so they intersect trivially by the definition of G_0 and Theorem 5. So the Claim holds.

Now pick a torsion element $t \in \tilde{G}$ and $f \in \tilde{F}$ with $t^f = t$. Let $e \in E(S)$ be an edge stabilized by t. Then fe is also stabilized by t and, as by Theorem 3.7 in [5], the fixed set S^t is a subtree, the path $[e, fe]$ is fixed by t as well. By the above then $fe = e$ contradicting the freeness of the action of $\tilde{F}$ on $E(S)$.

$\square$

4 Proof of the main result

Proposition 9 *Let G be a semidirect product of a free pro-p group F of finite rank with a p-group K and every finite subgroup is conjugate to a subgroup of K. Suppose $C_F(t) = \{1\}$ holds for every torsion element $t \in G$. Then $G = K \amalg F_0$ for a free pro-p factor F_0.*

Proof: When $K \cong C_p$, the proposition is true by virtue of Lemma 3 (i). Assume now that K is of order $\geq p^2$. Let H be any maximal subgroup of K. Then $F \rtimes H$ satisfies the premises of the lemma and hence $F \rtimes H$ is of the form $H \amalg F_1$ for some free factor F_1. Let us denote by bar passing to the quotient mod $(H)_G$. Then Lemma 3 (i) shows $\bar{G} \cong \coprod_{i \in I} (C_i \times C_{\bar{F}}(C_i)) \amalg F_0$ with I finite and F_0 a free factor of $\bar{F}$. Now Proposition 1.7 in [8] implies that torsion maps onto torsion, and therefore, every torsion element in $\bar{G}$ can be lifted to a conjugate of an element in K. Hence I consists of a single element, so that

$$\bar{G} = (\bar{K} \times C_{\bar{F}}(\bar{K})) \amalg F_0. \quad (1)$$

In the sequel we shall use Lemma 3(ii) a couple of times. Consider $M := F/F'$ as a K-module and let J denote the augmentation ideal when M is considered as an H-module. As $F \rtimes H = H \amalg F_1$, as a consequence of Lemma 3 one has that M is a free H-module and the natural homomorphism from

$\bar{F}/\bar{F}' \to M/JM$ is an isomorphism of $\bar{K}$-modules. Note that M^H can be described (in additive notation) as the set of all $\overline{\sum_{h\in H} hf_1 F'/F'}$ for $f_1 \in F_1$, so that factoring the action of H shows $F_1F'/F' \cong M/JM \cong M^H$ as $\bar{K}$-modules. We want to apply Theorem 2. Passing in Eq.(1) to the quotient mod the commutator subgroup of $\bar{F} = (C_{\bar{F}}(\bar{K}), F_0)_{\bar{G}}$, using Lemma 3, and, noting that it coincides with M/JM, one can see that M/JM is indeed a $\bar{K}$-permutation lattice. By the above isomorphism then M^H is a $\bar{K}$-permutation lattice and an application of Theorem 2 shows that M itself is a K-permutation lattice.

We shall show that it is a free K-module. Indeed, if some of the summands is not free, a proper subgroup of K, say S, acts trivially there. Since M is a free H-module , conclude that $S \cap H = \{1\}$. Let us show that $G_1 := F \rtimes S$ satisfies the premises of the lemma. Certainly $C_F(t) = \{1\}$ for every torsion element $t \in G_1$. Pick $x \in Tor(G_1)$. There is $k \in K$ and $f \in F$ with $x = k^f$. Since $k \in (FS) \cap K$ deduce $k \in S$. So there is a single conjugacy class of finite subgroups in G_1. But then, considering the natural homomorphism from $F \rtimes S$ to $M \rtimes S$ and having $F \rtimes S = S \amalg F_S$ in mind, one finds as an application of Lemma 3 that the decomposition of M cannot have direct summands, on which S acts trivially, a contradiction. So M is a free K-module.

Consider $\tilde{G} := K \amalg \tilde{F}_0$ with $\tilde{F}_0 \cong G/\langle Tor(G)\rangle$. By Proposition 1.7 in [8] $G/\langle Tor(G)\rangle$ is free pro-p, so we can fix a section F_0 of $G/\langle Tor(G)\rangle$ inside G, and define an epimorphism $\phi : \tilde{G} \to G$ by sending K to K and $\tilde{F}_0$ onto F_0 and extend it to an epimorphism to $\tilde{G}$ by using the universal property of the free product $\tilde{G} = K \amalg \tilde{F}_0$. By the above the kernel of ϕ must be contained in $[\tilde{F}, \tilde{F}]$. In particular, since the group is finitely generated, one has $\tilde{F} \cong F$, since both groups are free pro-p. Since $K \cap \ker\phi = \{1\}$, conclude that ϕ is an isomorphism, as claimed. □

Proof of Theorem 1: Lemma 4 shows that G can have only a finite number of conjugacy classes of maximal finite subgroups. Therefore one can form $\tilde{G}$ as described before Lemma 8, in order to embed G such that $\tilde{G}$ is both, finitely generated, and, has finite centralizers of its finite subgroups, and, moreover, has a single conjugacy class of maximal finite subgroups. By Proposition 9 the group $\tilde{G}$ is of the form $\tilde{G} = K \amalg F_0$ where K is finite and F_0 is free pro-p. Since G is a finitely generated pro-p subgroup of $\tilde{G}$, the Kurosh subgroup theorem in [3] implies that G must have indeed the form as claimed. □

5 An example

We give an example of a virtually free profinite group that satisfies the centralizer condition of the main theorem but does not satisfy its conclusion. Note that the same example is valid for abstract groups.

Lemma 10 *Let $A \cong B = S_3$ be the symmetric group on a 3-element set and $C := C_2$. Then one can form the amalgamated free profinite product $G = A \amalg_C B$, where C identifies with given 2-Sylow subgroups in A and B respectively.*

Then for every torsion element $t \in G$ its centralizer is finite. Moreover, G cannot be decomposed as a free profinite product with some factor finite.

Proof: It is easy to see that G can be presented in the form $G = N \rtimes C_2$, with $N \cong C_3 \amalg C_3$ and $C_2 = \langle \alpha \rangle$ acting by inverting the generators of the two factors. Then the structure of N, in light of Theorem 5, shows that no element of order 3 can have an infinite centralizer. Since all involutions in G are conjugate, in order to show the first statement of the lemma, it will suffice to show that α acts without fixed points upon $N = \langle a, b \rangle$, where a, b are generators of cyclic free factors of order 3. Since, by Theorem 9.1.6 in [4], N' is freely generated by the commutators $[a^i, b^j]$ with $i, j \in \{1, 2\}$, one can see that α permutes them without fixed points, so that $N' \rtimes \langle \alpha \rangle$ is isomorphic to $F(x, y) \amalg C_2$ with $F(x, y)$ a free profinite group. Thus α has no fixed points in N' and, as an easy consequence, none in N.

Suppose $G = L \amalg K$ with L finite. Then, by Theorem 5, w.l.o.g. we can assume that $A \leq L$ and, since A is a maximal finite subgroup of G, conclude $A = L$. Since the quotient mod the normal closure of L in G is isomorphic to K on the one hand and trivial by construction, find $K = \{1\}$, a contradiction. So G has no finite free factor. $\square$

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