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On virtually projective groups

To A. E. Zalesskii on the occasion of his 65-th birthday

By *P. A. Zalesskii**) at Brasília

Abstract. It is proved that the quotient $G/\langle \mathrm{tor}(G) \rangle$ of a virtually projective profinite group G modulo its normal subgroup generated by all torsion of G is projective.

0. Introduction

Let G be a virtually free group. Then by results of Karras, Magnus, Solitar, Cohen and Scott $G = \pi_1(\mathcal{G}, \Gamma)$ is the fundamental group of a graph of finite groups $(\mathcal{G}, \Gamma)$. By the central result of Bass-Serre's theory of groups acting on trees this is equivalent to the fact that G acts on a tree S with finite vertex stabilizers such that $S/G = \Gamma$. Let $\mathrm{tor}(G)$ be the set of all nontrivial torsion elements of G. Since every torsion element must fix a vertex of S, it follows that the group $\langle \mathrm{tor}(G) \rangle$ is generated by the stabilizers of vertices of S and therefore $S/\langle \mathrm{tor}(G) \rangle$ is a tree on which $G/\langle \mathrm{tor}(G) \rangle$ acts freely. Thus $G/\langle \mathrm{tor}(G) \rangle = \pi_1(\Gamma)$ is the fundamental group of the graph Γ and hence is free.

It was proved recently in [HZ] that a finitely generated virtually free pro-p group is the fundamental pro-p group of a finite graph of finite p-groups. Unfortunately, this result does not hold in the infinitely generated case. However, there is still hope that a virtually free pro-p group acts on a pro-p tree with finite vertex stabilizers, because in the pro-p case this is a weaker property than to be the fundamental group of a graph of finite p-groups. Moreover, it is shown in [HZ] that $G/\langle \mathrm{tor}(G) \rangle$ is free pro-p, when G is second countable that would be the consequence of this conjecture if proved in this case.

The situation in the profinite case is more complicated. A virtually free profinite group does not act in general on a profinite tree and so does not have a structure similar to a discrete virtually free group. An example is the semidirect product $\hat{\mathbb{Z}} \rtimes C_2$, where C_2 inverts elements of the 2 component $\mathbb{Z}_2$ of $\hat{\mathbb{Z}}$ and fixes the elements of p components $\mathbb{Z}_p$ for all other primes p.

*) Supported by CNPq.

The objective of the present paper is to show that nevertheless one can obtain quite reasonable information on $G/\langle \mathrm{tor}(G)\rangle$ of the virtually free profinite group G. In fact, our result is even more general.

Theorem. *Let G be a virtually projective profinite group. Then $G/\langle \mathrm{tor}(G)\rangle$ is projective.*

In the case $\langle \mathrm{tor}(G)\rangle = 1$ (i.e. when G is torsion free) the result is due to Serre [S]. Note that free groups, free pro-p groups and projective groups are exactly groups of cohomological dimension 1 in the categories of groups, pro-p groups and profinite groups, respectively. Therefore one could ask as a possible generalization of the theorem above whether for a group G of finite virtual cohomological dimension n one has $\mathrm{cd}\big(G/\langle \mathrm{tor}(G)\rangle\big) \leqq n$. In Section 3 we give an example of a group of virtual cohomological dimension 2 whose quotient $G/\langle \mathrm{tor}(G)\rangle$ is even not torsion free, and so has infinite cohomological dimension. This shows that the situation with groups of (virtual) cohomological dimension 1 is rather special.

The structure of the paper is as follows. In Section 1 the ideas of [RHZ] and [HZ] are used to complete the result in the pro-p case. The main result is proved in Section 2.

In Section 3 besides the example mentioned above we also give an example of a semidirect product $F \rtimes C_2$ of a free pro-2 group F of uncountable rank and a group of order 2 that does not satisfy the Dyer-Scott type decomposition

$$G = \coprod_{x \in X} (H_x \times C_2) \amalg H,$$

where H_x and H are free pro-2 groups. When F is of countable rank the Dyer-Scott decomposition holds (see Theorem 1.2 below).

The necessary material on profinite groups (like a notion of a free profinite group on a topological space) can be found in [RZ] and [W]. The definition of a free pro-p product which is used in the paper can be found in [NSW], Chapter IV, S3, or in [M]. We shall use frequently Serre's result from [S] that states that a virtually projective torsion free profinite group is projective.

Notation. All groups in the paper are profinite, homomorphisms are continuous and subgroups are closed. By p will be denoted usually a prime number. For a pro-p group G we denote the Frattini subgroup of G by $\Phi(G)$. $\mathrm{tor}(G)$ means the subset of torsion elements of G and x^g stands for $x^{-1}gx$. For a profinite space $X = \varprojlim X_i$, $|X_i| < \infty$ and a profinite ring R we denote by $[\![RX]\!] = \varprojlim [RX_i]$ a free profinite module over the space X.

1. The pro-p case

Denote by $\mathbf{n}(G)$ the index of maximal free pro-p normal subgroup of G. The proof of the result in this case uses induction on $\mathbf{n}(G)$. We first formulate a theorem that gives the base of induction.

Let G be a pro-p group having an open free pro-p subgroup F. Then the set $\mathcal{T}$ of all subgroups of order p is a profinite space, since it is a projective limit of corresponding finite discrete spaces of quotients G/U, where U runs through the all open normal subgroups of G which are contained in F. Moreover, G acts continuously on $\mathcal{T}$ by conjugation.

The stabilizer G_T of $T \in \mathcal{T}$ with respect to this action is just the centralizer $G_T = C_G(T)$. We denote by $\varphi_G : \mathcal{T} \to \mathcal{T}/G$ the natural map. Then for $t \in \mathcal{T}/G$ the pre-image $\varphi_G^{-1}(t)$ is the G-orbit. The next lemma is just a homological version of the result of Scheiderer from [Sch], Theorem 12.13.

Lemma 1.1 ([HRZ], Lemma 5). *For any $n \geqq 2$ the canonical homomorphism*

$$\varphi_n : \bigoplus_{t \in \mathcal{T}/G} H_n\big(G, [\![\mathbb{F}_p \varphi_G^{-1}(t)]\!]\big) \to H_n(G, \mathbb{F}_p)$$

is a topological isomorphism.

Now we state a pro-p version of the Dyer-Scott theorem [DS] that was proved in [Sch1] for finitely generated case and in [HRZ] in the form below. Note that the result holds upon the condition of the existence of a continuous section $\mathcal{T}/G \to \mathcal{T}$. Proposition 3.3 shows that this condition is essential.

Theorem 1.2 ([HRZ], Proposition 9). *Let G be a pro-p group having free subgroup F of index p. Suppose there exists a continuous section $\sigma : \mathcal{T}/G \to \mathcal{T}$. Put $T = \sigma(t)$ regarding as a subgroup of G. Then*

$$G = \Big(\coprod_{T \in \operatorname{im}(\sigma)} \big(T \times C_F(T)\big)\Big) \amalg H,$$

is a free pro-p product over the profinite space $\mathcal{T}/G$, where H is a free pro-p subgroup of F.

We note that a section σ always exists if the action is free or if $\mathcal{T}$ is second countable (see [RZ], Lemmas 5.6.5 and 5.6.7).

The next proposition is extracted from the proof of Proposition 13 in [HRZ].

Proposition 1.3. *Let G be a pro-p group having a free pro-p subgroup F of index p. Then G embeds into a free pro-p product*

$$G_0 = (C_p \times H) \amalg H_0 \qquad (6)$$

where H, H_0 are free pro-p groups and C_p is a group of order p.

Proof. If G is free pro-p, there is nothing to prove. So assume that G is not free pro-p; then by Serre's result the torsion $\operatorname{tor}(G) \neq \emptyset$. Let $\varphi : G \to G/F$ be the natural epimorphism. Choose a generator c of $G/F \cong C_p$ and put $C = \operatorname{tor}(G) \cap \varphi^{-1}(c)$. For $T \in \mathcal{T}$ denote by c_T the unique element of $C \cap T$.

For the rest of the proof fix an arbitrary $T_0 \in \mathcal{T}$ and write $c_0 = c_{T_0}$. The set

(2) $$c_0^{-1}C \subset F$$

is naturally homeomorphic to the pointed boolean space $(\mathscr{T}, T_0)$. This way, in the sequel, $(\mathscr{T}, T_0)$ will appear as an indexing profinite space. Let $F(\mathscr{T}, T_0)$ be a free pro-p group over the pointed space $(\mathscr{T}, T_0)$. We shall denote by z_T the image of a point $T \in \mathscr{T}$ under the natural injection $\mathscr{T} \to F(\mathscr{T}, T_0)$. Form the free pro-$p$ product

(3) $$F_0 = F(\mathscr{T}, T_0) \amalg F.$$

Observe that $c_0^{-1}c_T \in F$ for every $T \in \mathscr{T}$. Define an automorphism $\alpha_0 \in \mathrm{Aut}(F_0)$ by putting

(4) $$\alpha_0(z_T) = c_0^{-1}c_T z_T, \quad T \in \mathscr{T},$$
$$\alpha_0(f) = c_0^{-1}fc_0, \quad f \in F.$$

We check that α_0 has order p by looking at generators of F_0. We show first by induction on k that

$$\alpha_0^k(z_T) = c_0^{-k}c_T^k z_T$$

for all $T \in (\mathscr{T}, T_0)$ and $1 \leqq k \leqq p$. The formula follows from the definition of α_0 for $k = 1$. Assuming that the formula holds for $k - 1$, one has

$$\begin{aligned}\alpha_0^k(z_T) &= \alpha_0^{k-1}\alpha_0(z_T) = \alpha_0^{k-1}(c_0^{-1}c_T z_T) = c_0^{-k+1}c_0^{-1}c_T c_0^{k-1}\alpha_0^{k-1}(z_T)\\ &= c_0^{-k}c_T c_0^{k-1}c_0^{-k+1}c_T^{k-1}z_T = c_0^{-k}c_T^k z_T\end{aligned}$$

as required.

Hence $\alpha_0^p(z_T) = z_T$ and certainly, $\alpha_0^p(f) = f^{c_0^p} = f$ for any $f \in F$.

There is a natural embedding of G into $G_0 := F_0 \rtimes \langle \alpha_0 \rangle$ where F is sent to a copy of F in G_0 and c_0 is sent to α_0. We shall identify α_0 and c_0 henceforth.

By construction, the torsion of G_0 coincides with the torsion of G, and since, as a simple consequence of Equation (4) and the identification $\alpha_0 = c_0$

(5) $$z_T c_0 z_T^{-1} = c_T$$

holds for $T \in \mathscr{T}$, G_0 has only one conjugacy class of subgroups of order p. An application of Theorem 1.2 then yields a decomposition

(6) $$G_0 = (C_p \times H) \amalg H_0$$

with H, H_0 suitable free pro-p groups of F_0. □

Corollary 1.4. *Let G be a pro-p group having a free pro-p subgroup F of index p. Then*:

(i) *$G/\langle \mathrm{tor}(G)\rangle$ is free pro-p.*

(ii) *$C_F(c)$ is a free factor of F for any torsion element c of G.*

(iii) *If G is generated by torsion, then there exists a continuous section $\sigma : \mathcal{T}/G \to \mathcal{T}$ and one has*

$$G = \Big(\coprod_{T \in \mathrm{im}(\sigma)} T \Big).$$

Proof. (i) By the preceding theorem G embeds into a free pro-p product

$$G_0 = (C_p \times H) \amalg H_0 \tag{6}$$

where H, H_0 are free pro-p groups. Let X and X_0 be closed bases of H and H_0 respectively. Hence G_0 can be viewed as an HNN-group $\langle C_p, X, X_0 \mid xcx^{-1} = c \text{ for } c \in C_p, x \in X\rangle$. It follows that G_0 acts on a pro-p tree S whose vertex stabilizers are conjugates of C_p (see [ZM], Proposition 3.8). Then G acts on S as well and $\langle \mathrm{tor}(G)\rangle$ is exactly the subgroup of G generated by the vertex stabilizers. So by [RZ1], Corollary 3.6, $G/\mathrm{tor}(G)$ is free pro-p as required.

(ii) Since every torsion element is conjugate in G_0 to some element of C_p, using conjugation if necessary, we may assume that $C_p = \langle c\rangle$. Let $f : G_0 \to H$ be the epimorphism that sends C_p and H_0 to 1 and H identically onto H. The restriction of f to $C_F(c)$ is injective, because $C_{G_0}(C_p) = C_p \times H$ (see [RZ1], Corollary 4). Hence F splits as a semidirect product $F = M \rtimes C_F(c)$. It follows that $\Phi(F) \cap C_F(c) = \Phi\big(C_F(c)\big)$. Then by [RZ], Lemma 9.1.18, $C_F(c)$ is a free factor of F.

(iii) Let T be a subgroup of G of order p. Conjugating it if necessary we may assume that $T = C_p$. Let $\varphi : G_0 \to H \amalg H_0$ be the epimorphism that sends C_p to 1 and H, H_0 identically to their copies in $H \amalg H_0$. As $C_{G_0}(C_p) = C_p \times H$ (see [RZ1], Corollary 4), the restriction of φ to $C_F(T)$ is injective. Since G is generated by torsion and every torsion element is conjugate in G_0 to some element of C_p (cf. [RZ1], Theorem 4.2 (a)), one has $\varphi(G) = 1$. Hence $C_F(T) = 1$ for any subgroup T of G of order p. It follows that F acts freely on $\mathcal{T}$ and so there exists a section $\sigma : \mathcal{T}/F \to \mathcal{T}$ ([RZ], Lemma 5.6.5). But $\mathcal{T}/G = \mathcal{T}/F$, so the result follows from Theorem 1.2. □

A finitely generated version of the next theorem is due to Scheiderer [Sch1].

Theorem 1.5. *Suppose F is a free pro-p group and P is a finite p-group of automorphisms of F. Then the set of fixed points $C_F(P)$ is a free factor of F. In particular, if the rank of F is finite, so is rank of $C_F(P)$.*

Proof. Let P be a nontrivial finite p-group of automorphisms of F of minimal order such that the theorem fails. Consider the holomorph $G = F \rtimes P$. By Corollary 1.4 (ii), $|P| > p$. Pick an element c in the center of P with $c^p = 1$. By the above case $C_F(c)$ is a free factor of F. Therefore $P/\langle c\rangle$ acts on $C_F(c)$, and from the minimality assumption we conclude the result. □

Remark 1.6. If α is an automorphism of order p^∞ of a finitely generated free pro-p group F, then it is not known whether the subgroup of fixed point $C_F(\alpha)$ is finitely generated.

Proposition 1.7. *Let G be any virtually free pro-p group and $N \lhd G$ a normal subgroup of G generated by torsion elements. Then the following statements hold*:

(i) $\mathrm{tor}(G/N) = \mathrm{tor}(G)N/N$ (*torsion from G/N can be lifted*).

(ii) $G/\langle \mathrm{tor}(G)\rangle$ *is free pro-p*.

Proof.

Claim 1. (i) and (ii) are equivalent.

For showing (i) $\Rightarrow$ (ii) pick $\bar{g} \in G/\langle \mathrm{tor}(G)\rangle$ with $\bar{g}^p = 1$. Apply (i) with $N := \langle \mathrm{tor}(G)\rangle$, in order to find $x \in \mathrm{tor}(G)$ with $x\langle \mathrm{tor}(G)\rangle/\langle \mathrm{tor}(G)\rangle = \bar{g}$. Since $x \in \langle \mathrm{tor}(G)\rangle$ conclude $\bar{g} = 1$. So $G/\langle \mathrm{tor}(G)\rangle$ is torsion free. To show that it is free pro-p we use induction on $\mathbf{n}(G)$. Let c be a central element of G/F of order p. Then the preimage G_1 of $\langle c\rangle$ in G satisfies the assumption of Corollary 1.4 and so $G_1/\langle \mathrm{tor}(G_1)\rangle$ is free pro-p. Now from (i) $\mathrm{tor}(G)\langle \mathrm{tor}(G_1)\rangle/\langle \mathrm{tor}(G_1)\rangle = \mathrm{tor}(G/G_1)$ and $\mathbf{n}(G/G_1) < \mathbf{n}(G)$. So from the induction hypothesis we deduce that $G/\langle \mathrm{tor}(G)\rangle = (G/G_1)/\langle \mathrm{tor}(G/G_1)\rangle$ is free pro-p as needed.

Suppose "(ii) $\Rightarrow$ (i)" is false. Then there exists a virtually free pro-p group G having a normal subgroup N generated by torsion and an element $g \in G$ such that $gN/N \in \mathrm{tor}(G/N)$ and $gN \cap \mathrm{tor}(G) = \emptyset$. Then G replaced by $\langle g, N\rangle$ is still a counter example, so we may assume that $G = \langle g, N\rangle$ and denote such a counter example by (g, N). Among the all such counter examples choose one with $[G : N]$ minimal. We prove first that $[G : N] = p$.

Suppose not. Put $M := \langle g^p, N\rangle$ then $g^p \notin N$ and $[M : N] < [G : N]$ so that (g^p, N) cannot be a counter example. Hence $M = \langle \mathrm{tor}(M)\rangle$. On the other hand, $[G : M] < [G : N]$, so that (g, M) is not a counter example either, hence exists $g_0 \in \mathrm{tor}(G)$ with $g_0M/M = gM/M$. Then $\langle g_0, N\rangle = \langle g, N\rangle$, i.e., $g_0 \in gN \cap \mathrm{tor}(G) = \emptyset$, a contradiction. Thus $[G : N] = p$.

For finishing the proof of Claim 1, note that (ii) implies $G = \langle \mathrm{tor}(G)\rangle$. Therefore, there exists a torsion element $g_0 \in G\backslash N$ and, for suitable $1 \leqq k \leqq p-1$, one must have $g_0^k \in gN \cap \mathrm{tor}(G) = \emptyset$, a contradiction. Therefore Claim 1 is established.

We continue the proof of the proposition. Suppose it is false. Then there exists G with $G/\langle \mathrm{tor}(G)\rangle$ not free pro-p. Choose one with $\mathbf{n}(G)$ minimal and let $F \lhd G$ be a free pro-p group with $[G : F] = \mathbf{n}(G)$. Then, in light of Claim 1, there exists $g \in G$ such that $g\langle \mathrm{tor}(G)\rangle/\langle \mathrm{tor}(G)\rangle$ of order p and $g\langle \mathrm{tor}(G)\rangle \cap \mathrm{tor}(G) = \emptyset$. It follows that $\langle g, \mathrm{tor}(G)\rangle$ is still a counter example and since $\big[\langle g, \mathrm{tor}(G)\rangle : \big(\langle g, \mathrm{tor}(G)\rangle \cap F\big)\big] \leqq \mathbf{n}(G)$ it follows that $\mathbf{n}\big(\langle g, \mathrm{tor}(G)\rangle\big) = \mathbf{n}(G)$. So from now on we may assume that $G = \langle g, \mathrm{tor}(G)\rangle$.

Claim 2. G/F is not cyclic.

Suppose it is. Let G_i be the preimage in G of the cyclic subgroup of order p^i in G/F. Choose i maximal such that $G_i = F \rtimes C_{p^i}$. Then $\mathrm{tor}(G) \subseteqq G_i$; indeed, if $g \in \mathrm{tor}(G) \backslash G_i$ then since G/F is cyclic, g has order at least p^{i+1} and $G_{i+1} = F \rtimes \langle g \rangle$ contradicting the choice of i. Put $n = \mathbf{n}(G)$. Then it follows from the minimality assumption on n that $i = n-1$ and $G_{n-1} = \langle \mathrm{tor}(G) \rangle$. Let U be a normal subgroup generated by all p^{n-2} powers of elements of order p^{n-1}. Then by the minimality assumption on n and Claim 1 one has $\mathrm{tor}(G_{n-1}/U) = \mathrm{tor}(G_{n-1})U/U = \mathrm{tor}(G)U/U$. Consequently as it was shown above $G_{n-1}/U = F_0 \rtimes C_{p^{n-2}}$ for some free pro-p group F_0 and so $\mathbf{n}(G_{n-1}/U) = n-2$. Therefore it follows from Claim 1 that to use the minimality assumption on n we have to prove the equality $G/\langle \mathrm{tor}(G) \rangle = (G/U)/\langle \mathrm{tor}(G/U) \rangle$.

Suppose not and $k \in G \backslash \langle \mathrm{tor}(G) \rangle$ such that kU/U is of finite order. Put $K = \langle k, U \rangle$ and let M be a subgroup of G generated by all elements of order p^{n-1}. Since any element of order p^{n-1} centralizes its p^{n-2}-power, $\mathscr{T}_U/M = \mathscr{T}_U/U = \mathscr{T}_U/(F \cap U)$. Since U is generated by its torsion Corollary 1.4 (iii) implies that there exists a continuous section $\sigma_1 : \mathscr{T}_U/U \to \mathscr{T}_U$ with

$$U = \coprod_{T \in \mathrm{im}(\sigma_1)} T.$$

Note that the action of K/M on $\mathscr{T}_U/M = \mathscr{T}_U/U = \mathscr{T}_U/(F \cap U)$ is free. Indeed, if not then there exists $A \in \mathscr{T}_U$ and $f \in F \cap U$ with $A^k = A^f$, because $\mathscr{T}_U/M = T_U/F \cap K$; but then kf centralizes A and, since $C_U(A) = A$ (a free factor is self-centralized, cf. [RZ1], Corollary 4.4), the element kf has to be of finite order; but $F \cap U \leqq \langle \mathrm{tor}(G) \rangle$, so $k \in \langle \mathrm{tor}(G) \rangle$, a contradiction with the choice of k. Therefore there exists a continuous section $\sigma : \mathscr{T}/K \to \mathscr{T}/U = \mathscr{T}/M$ (cf. [RZ], Lemma 5.6.5).

Let c be a generator of K/M. Then the $\mathscr{T}_i := \mathrm{Im}(\sigma)c^i$ $(i = 0, \ldots, p-1)$ form a partition of $\mathscr{T}/M = \mathscr{T}/U$ into p clopen subsets. Define $K_i := \coprod_{t \in \sigma_1(\mathscr{T}_i)} T$ and write $\tilde{F} = (F \cap K_i \mid i = 1, \ldots, p-1)_K$ to be the normal closure of $F \cap K_i$'s. Then $U = \coprod_{i=0}^{p-1} K_i$,

$$(*) \qquad U/\tilde{F} = \coprod_{i=0}^{p-1} C_p$$

is a free pro-p product of groups of order p and the conjugacy classes of the free factors are permuted by the action of K/M. It follows that the abelianization $(K/\tilde{F})/(K/\tilde{F})'$ is of order p^n and since $F \cap K$ contains the commutator subgroup K', the commutator subgroup $(K/\tilde{F})'$ coincides with $F \cap K/\tilde{F}$ showing that $(K/\tilde{F})/(K/\tilde{F})'$ is cyclic of order p^n. But then $K/\tilde{F}$ has to be procyclic, a contradiction to $(*)$.

Thus $G/\langle \mathrm{tor}(G) \rangle = (G/U)/\langle \mathrm{tor}(G/U) \rangle$ and by the minimality assumption on n we deduce that Claim 2 holds.

Claim 3. $G/F \cong C_p \times C_p$ is the direct product of groups of order p.

Since G/F is not cyclic, there exists a normal subgroup M of G containing F such that $G/M \cong C_p \times C_p$. For any $A \subseteqq G$ write $\bar{A} := A\langle \mathrm{tor}(M) \rangle / \langle \mathrm{tor}(M) \rangle$. Let K be an arbitrary normal subgroup of index p in G containing M. Then $\mathbf{n}(K) < \mathbf{n}(G)$

and therefore, by the minimality of $\mathbf{n}(G)$ and Claim 1, $\mathrm{tor}(\bar{K}) = \overline{\mathrm{tor}(K)}$. But every torsion element of G belongs to some K of this sort, showing $\mathrm{tor}(\bar{G}) = \overline{\mathrm{tor}(G)}$ and hence $\bar{G}/\langle \mathrm{tor}(\bar{G})\rangle = \bar{G}/\overline{\langle \mathrm{tor}(G)\rangle}$. If M is non-trivial, then $\mathbf{n}(\bar{G}) < \mathbf{n}(G)$ implying that $G/\langle \mathrm{tor}(G)\rangle \cong \langle \bar{G}/\mathrm{tor}(\bar{G})\rangle$ is free pro-p, a contradiction. Hence $M = 1$ and Claim 3 holds.

Returning to proving the proposition, put $L := \langle g, F\rangle$. Then, as $G/F \cong C_p \times C_p$ deduce $[G : L] = [G : \langle \mathrm{tor}(G)\rangle] = p$ and $L \cap \langle \mathrm{tor}(G)\rangle = F$. On the other hand, L cannot be torsion free, else, by Serre's result [S], it is free pro-p, and, being of index p in G, the group G would be free pro-p by cyclic, contradicting Claim 2. Then $1 \neq \langle \mathrm{tor}(L)\rangle \leqq L \cap \langle \mathrm{tor}(G)\rangle = F$ follows, a clear contradiction. □

2. The profinite case

Lemma 2.1. *Let G be a virtually free pro-p group and $f : G \to H$ a homomorphism to a profinite group H that sends every torsion element to* 1. *Let A be a discrete p-primary $\hat{\mathbb{Z}}[[H]]$-module viewed as a $\hat{\mathbb{Z}}[[G]]$-module via f. Then the induced map $H^2(H, A) \to H^2(G, A)$ is the* 0*-map.*

Proof. Clearly f factors through $G/\langle \mathrm{tor}(G)\rangle$ and so one has the commutative diagram

$$\begin{array}{ccc} G & \longrightarrow & G/\langle \mathrm{tor}(G)\rangle \\ & \searrow & \downarrow \\ & & H \end{array}$$

inducing the commutative diagram

$$\begin{array}{ccc} H^2(G, A) & \longleftarrow & H^2\big(G/\langle \mathrm{tor}(G)\rangle, A\big) \\ & \nwarrow^{\varphi} & \uparrow \\ & & H^2(H, A). \end{array}$$

By Proposition 1.7, $G/\langle \mathrm{tor}(G)\rangle$ is free pro-p and so $H^2\big(G/\langle \mathrm{tor}(G)\rangle, A\big) = 0$. It follows that φ is the 0-map. □

Lemma 2.2. *Let G be a virtually projective group and M a normal subgroup of G generated by elements of order coprime to p and containing all elements of G of p-power order. Then the quotient group S_pM/M is free pro-p for any Sylow p-subgroup S_p of G.*

Proof. It suffices to show that $H^2(G/M, A) = 0$ for any simple p-primary $\hat{\mathbb{Z}}[[G/M]]$-module A (cf. [RZ], Proposition 7.1.4 and Theorem 7.3.1). Define the action of G on A via G/M. Consider the 5-term Hochschild-Serre sequence

$$\begin{aligned} 0 \longrightarrow H^1(G/M, A) &\xrightarrow{\mathrm{Inf}} H^1(G, A) \\ &\xrightarrow{\mathrm{Res}} H^1(M, A)^{G/M} \xrightarrow{\mathrm{tr}} H^2(G/M, A) \xrightarrow{\mathrm{Inf}} H^2(G, A). \end{aligned}$$

Since M is generated by elements of order coprime to p one has

$$H^1(M,A)^{G/M} = \mathrm{Hom}(M,A)^{(G/M)} = 0.$$

So it remains to prove that $H^2(G/M,A) \xrightarrow{\mathrm{Inf}} H^2(G,A)$ is the 0-map.

First note that $H^2(G,A) \to H^2(S_p,A)$ is an injection (cf. [W], Lemma 10.2.1). Moreover, the commutative diagram

$$\begin{array}{ccc} S_p & \longrightarrow & G \\ & \searrow & \downarrow \\ & & G/M \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} H^2(S_p,A) & \longleftarrow & H^2(G,A) \\ & \nwarrow^{\varphi} & \uparrow \\ & & H^2(G/M,A) \end{array}$$

and it suffices to show that φ is the 0-map. However, this is the subject of Lemma 2.1. □

Corollary 2.3. *Let G be a virtually projective group and M be the normal subgroup generated by all elements of order coprime to p. Then* $\mathrm{tor}(G)M/M = \mathrm{tor}(G/M)$.

Proof. For $g \in G$ with gM/M of order p Lemma 2.2 implies that $\langle g, M\rangle$ must have torsion outside of M as needed. □

Lemma 2.4. *Let G be a virtually projective group and $G/\langle \mathrm{tor}(G)\rangle$ is a pro-p group. Then $G/\langle \mathrm{tor}(G)\rangle$ is free pro-p.*

Proof. Let F be a maximal open normal projective subgroup of G. Denote by $O_p(F)$ the kernel of F on its maximal pro-p quotient. Note that $O_p(G) \leqq \langle \mathrm{tor}(G)\rangle$. We show that $\langle \mathrm{tor}(G)\rangle/O_p(F) = \big\langle \mathrm{tor}\big(G/O_p(F)\big)\big\rangle$.

Indeed, let s be an element of $G/O_p(F)$ of prime order q. Denote by S the preimage of $\langle s\rangle$ in G. We need to show that $S \leqq \langle \mathrm{tor}(G)\rangle$. If not then S is torsion free and therefore is projective by Serre's result ([S]). Then $q \neq p$ because otherwise $O_p(F) = O_p(S)$ and so the maximal pro-p quotient $S/O_p(S) = \langle s\rangle$ has to be projective (cf. [RZ], Proposition 7.6.7). So $q \neq p$ and therefore the maximal pro-p quotient of S is trivial. Hence $S \leqq \langle \mathrm{tor}(G)\rangle$.

Thus by factoring out the kernel of the epimorphism of F to its maximal pro-p quotient we may assume that F is free pro-p. Let M be the normal subgroup of G generated by all elements of order coprime to p. For a subset B of G write $\bar{B} := BM/M$ and put $L := FM$.

Claim. $L/\langle \mathrm{tor}(L)\rangle$ is free pro-p.

Consider the 5-term Hochschild-Serre sequence

$$0 \longrightarrow H^1\bigl(\overline{L}/\langle \mathrm{tor}(\overline{L})\rangle, \mathbb{F}_p\bigr) \xrightarrow{\mathrm{Inf}} H^1(\overline{L}, \mathbb{F}_p)$$
$$\longrightarrow H^1\bigl(\langle \mathrm{tor}(\overline{L})\rangle, \mathbb{F}_p\bigr)^{\overline{L}/\langle \mathrm{tor}(\overline{L})\rangle} \xrightarrow{\mathrm{tr}} H^2\bigl(\overline{L}/\langle \mathrm{tor}(\overline{L})\rangle, \mathbb{F}_p\bigr) \xrightarrow{\mathrm{Inf}} H^2(\overline{L}, \mathbb{F}_p).$$

By Corollary 2.3, $\mathrm{tor}(\overline{L}) = \overline{\mathrm{tor}(L)}$. So it suffices to show that $H^2\bigl(\overline{L}/\langle \mathrm{tor}(\overline{L})\rangle, \mathbb{F}_p\bigr) = 0$.

We first show that

$$H^2\bigl(\overline{L}/\langle \mathrm{tor}(\overline{L})\rangle, \mathbb{F}_p\bigr) \xrightarrow{\mathrm{Inf}} H^2(\overline{L}, \mathbb{F}_p)$$

is 0-map. Denote by S_p a Sylow subgroup of L and consider the following commutative diagram:

$$\begin{array}{ccc} H^2\bigl(\overline{L}/\langle \mathrm{tor}(\overline{L})\rangle, \mathbb{F}_p\bigr) & \xrightarrow{\mathrm{Inf}} & H^2(\overline{L}, \mathbb{F}_p) \\ \downarrow & & \downarrow \\ H^2\bigl(S_p/\langle \mathrm{tor}(S_p)\rangle, \mathbb{F}_p\bigr) & \xrightarrow{\mathrm{Inf}} & H^2(S_p, \mathbb{F}_p) \end{array}$$

where the vertical maps are induced by the natural homomorphisms $S_p \to \overline{L}$ and

$$S_p/\langle \mathrm{tor}(S_p)\rangle \to L/\langle \mathrm{tor}(L)\rangle = \overline{L}/\langle \mathrm{tor}(\overline{L})\rangle.$$

By Theorem 1.7 the lower horizontal map is 0-map. Since M is generated by elements of order coprime to p one has

$$H^1(M, \mathbb{F}_p)^{L/M} = \mathrm{Hom}(M, \mathbb{F}_p)^{(L/M)} = 0.$$

So the 5-term Hochschild-Serre sequence relating cohomology of L and $\overline{L}$ implies that

$$\mathrm{Inf} : H^2(\overline{L}, \mathbb{F}_p) \to H^2(L, \mathbb{F}_p)$$

is injective. On the other hand

$$\mathrm{Res} : H^2(L, \mathbb{F}_p) \to H^2(S_p, \mathbb{F}_p)$$

is injective as well (cf. [W], Lemma 10.2.1). Hence the right vertical map is injective. So the commutativity of the diagram implies that

$$H^2\bigl(\overline{L}/\langle \mathrm{tor}(\overline{L})\rangle, \mathbb{F}_p\bigr) \xrightarrow{\mathrm{Inf}} H^2(\overline{L}, \mathbb{F}_p)$$

is 0-map.

Now it suffices to show that

$$H^1(\overline{L}, \mathbb{F}_p) \to H^1\bigl(\langle \mathrm{tor}(\overline{L})\rangle, \mathbb{F}_p\bigr)^{\overline{L}/\langle \mathrm{tor}(\overline{L})\rangle}$$

is surjective. We show that the dual map

$$H_1(\langle \mathrm{tor}(\bar{L})\rangle, \mathbb{F}_p)_{\bar{L}/\langle \mathrm{tor}(\bar{L})\rangle} \to H_1(\bar{L}, \mathbb{F}_p)$$

is injective. Note that the map in question coincides with the natural homomorphism $\langle \mathrm{tor}(\bar{L})\rangle/(\Phi(\langle \mathrm{tor}(\bar{L})\rangle)[\langle \mathrm{tor}(\bar{L})\rangle, \bar{L}]) \to \bar{L}/\Phi(\bar{L})$. Pick $\bar{g} \in \langle \mathrm{tor}(\bar{L})\rangle \cap \Phi(\bar{L})$. We need to show that $\bar{g} \in \Phi(\langle \mathrm{tor}(\bar{L})\rangle)[\langle \mathrm{tor}(\bar{L})\rangle, \bar{L}]$.

As was observed above every torsion element of $\bar{L}$ lifts to some torsion element of L and since M is generated by all elements of p'-order, it also lifts to an element of p-power order. Since all the Sylow subgroups of L are conjugate this implies that $\mathrm{tor}(\bar{L}) = \overline{\mathrm{tor}(S_p)}$; indeed if $s \in L$ is element of p-power order, then $s^k \in S_p$ for some $k \in L$ and so $\bar{s}^{\bar{k}} = \bar{s}^{\bar{l}}$ for some $\bar{l} \in L$ and one can choose an element $l \in S_p$ such that $\overline{s^{kl^{-1}}} = \bar{s}$. It follows that $\bar{g}$ has a preimage in $\langle \mathrm{tor}(S_p)\rangle$ (i.e. lifts to an element $g \in \langle \mathrm{tor}(S_p)\rangle$). Since $\bar{g} \in \Phi(\bar{L})$, $g \in L^p[L,L]M$. As $\bar{L}$ is pro-p, one has $\Phi(S_p)M = L^p[L,L]M$ and so $g \in \Phi(S_p)M$. Write $g = sm$ for some $s \in \Phi(S_p)$, $m \in M \cap S_p$.

Put $\tilde{L} := L/(\Phi(F) \cap M)$ and for $A \leqq L$ write $\tilde{A}$ for the image of A in $\tilde{L}$. Observe that $\Phi(\tilde{F}) = \widetilde{\Phi(F)}$. Then $\tilde{F} \cap \tilde{M}$ is an elementary abelian pro-p normal subgroup of $\tilde{F}$. Moreover, it is central in $\tilde{F}$ since $\tilde{M} \cap \Phi(\tilde{F}) = 1$. Then $\tilde{F} = B \times (\tilde{F} \cap \tilde{M})$, where B is the preimage in $\tilde{F}$ of the direct complement of $(\tilde{F} \cap \tilde{M})\Phi(\tilde{F})/\Phi(\tilde{F})$ in $\tilde{F}/\Phi(\tilde{F})$. It follows that $\tilde{L} = \tilde{F}\tilde{M} = B \times \tilde{M}$, so $\tilde{S}_p = B \times (\tilde{M} \cap \tilde{S}_p)$. Then $\Phi(\tilde{S}_p) = \Phi(B) \times \Phi(\tilde{M} \cap \tilde{S}_p)$ and so we can write $\tilde{s} = \tilde{s}_0\tilde{m}_0$, where $\tilde{s}_0 \in \Phi(B)$, $\tilde{m}_0 \in \Phi(\tilde{M} \cap \tilde{S}_p)$. Then $\tilde{g} = \tilde{s}_0\tilde{m}_0\tilde{m}$. Note that $\langle \mathrm{tor}(\tilde{S}_p)\rangle = (\tilde{M} \cap \tilde{S}_p) \times \langle \mathrm{tor}(B)\rangle$, so $\tilde{s}_0 \in \langle \mathrm{tor}(B)\rangle$. Since $\Phi(F) \cap M \leqq \Phi(S_p)$, using again the equality $\mathrm{tor}(\bar{L}) = \overline{\mathrm{tor}(S_p)}$ one infers that there exist $s_0 \in \Phi(S_p) \cap \langle \mathrm{tor}(S_p)\rangle$, $m' \in M$ such that $g = s_0m'$. But S_p is virtually free pro-p by Theorem 1.7, so $H_2(S_p/\langle \mathrm{tor}(S_p), \mathbb{F}_p\rangle) = 0$. Hence from 5-term Hochschild-Serre exact sequence relating S_p and $S_p/\langle \mathrm{tor}(S_p)\rangle$ one concludes that

$$H_1(\langle \mathrm{tor}(S_p)\rangle, \mathbb{F}_p)_{S_p/\langle \mathrm{tor}(S_p)\rangle} \to H_1(S_p, \mathbb{F}_p)$$

is injective or equivalently

$$\langle \mathrm{tor}(S_p)\rangle/\Phi(\langle \mathrm{tor}(S_p)\rangle)[\langle \mathrm{tor}(S_p)\rangle, S_p] \to S_p/\Phi(S_p)$$

is injective. This means that

$$\Phi(S_p) \cap \langle \mathrm{tor}(S_p)\rangle = \Phi(\langle \mathrm{tor}(S_p)\rangle)[\langle \mathrm{tor}(S_p)\rangle, S_p]$$

and so

$$s_0 \in \Phi(\langle \mathrm{tor}(S_p)\rangle)[\langle \mathrm{tor}(S_p)\rangle, S_p].$$

Therefore $\bar{g} \in \Phi(\langle \mathrm{tor}(\bar{L})\rangle)[\langle \mathrm{tor}(\bar{F})\rangle, \bar{L}]$ and the claim is proved.

Now, the preceding claim shows that $\bar{G} = G/M$ is virtually free pro-p, and since, by Corollary 2.3, $G/\langle \mathrm{tor}(G)\rangle = \bar{G}/\langle \mathrm{tor}(\bar{G})\rangle$, the result follows from Theorem 1.7. □

Theorem 2.5. *Let G be a virtually projective group. Then $G/\langle \mathrm{tor}(G)\rangle$ is projective.*

Proof. It suffices to show that for a Sylow subgroup S_p of G one has $S_p\langle \mathrm{tor}(G)\rangle/\langle \mathrm{tor}(G)\rangle$ is free pro-p for every p. However, this follows from Lemma 2.4 since $S_p\langle \mathrm{tor}(G)\rangle/\langle \mathrm{tor}(G)\rangle$ is a pro-p group. □

3. Examples

We conclude this section with two examples: an example of a pro-2 group G with $\mathrm{vcd}(G) = 2$ such that $G/\langle \mathrm{tor}(G)\rangle$ contains torsion and an example of a pro-2 group H containing a free pro-2 subgroup F of index 2 that does not satisfy the conclusion of Theorem 1.2.

The first example shows that groups of virtual cohomological dimension 1 are exceptional with respect to the property studied in the paper.

The second example shows that the existence of a section $\mathscr{T}/G \to G$ is essential for Theorem 1.2.

Example. Let $G = \mathbb{Z}_2 \amalg_H D_\infty$, where D_∞ is the infinite dihedral pro-2 group and H is the subgroup of order 2 in both factors (note that D_∞ contains a unique subgroup of index 2 isomorphic to $\mathbb{Z}_2$). Then the normal closure N of the first factor is of index 2 in G and isomorphic to the generalized dihedral group $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$ where the action is by invertion. Hence $\mathrm{cd}(N) = 2$. However, the group $\langle \mathrm{tor}(G)\rangle$ is the normal closure of the second factor and $G/\langle \mathrm{tor}(G)\rangle$ has order 2 and so is not torsion free.

Before constructing the second example we need the following

Lemma 3.1. *Let $Z \cong \mathbb{Z}_2$. There exists a profinite Z-space X with one point fixed, all other points having trivial stabilizers and the natural surjection $\pi_Z : X \to X/Z$ does not admit a continuous section.*

Proof. Let X be a direct product of uncountably many copies of $\mathbb{Z}_2$. Define an action of Z on X by coordinatewise multiplication. Then the trivial element of X is the unique fixed point and the stabilizers of all other points of X are trivial. By [CP], Lemma, π does not admit a continuous section. □

Now we start to construct the group H. Let (X, Z) be as in Lemma 3.1 and $\mathscr{H}_0 = X \times C$, where $C \cong C_2$. We are going to use the definition of a free pro-2 product in sense of [M] (see also [NSW], Chapter IV, §3). Denote by $(\mathscr{H}_0, pr_X, X)$ the associated constant sheaf. Put $H_0 = \coprod_X \mathscr{H}_0$. Define an action of Z on $\mathscr{H}_0$ by setting $(x, c)z = (xz, c)$, $x \in X$, $c \in C$. Then the universal property of the free pro-2 product allows one to extend this action canonically to a continuous action of Z on H_0. Put $H = H_0 \rtimes Z$. Using the canonical morphism $\omega : \mathscr{H}_0 \to H$, define $C_x = \omega(\mathscr{H}_0(x)) \cong C_2$, and regard H_0 as the internal free pro-2 product $H_0 = \coprod_{x \in X} C_x$ (see [M], (1.16) and (1.17), or [NSW], Chapter IV, §3). Denote by x_0 a point which is fixed by Z. Let F_0 be the free subgroup of index 2 in H_0. The subset $W = \{c_{x_0}^{-1} c_x \mid x \in X\}$ is clearly a closed subset of F_0 and, in fact, is a pointed basis of F_0. Indeed, $H_0/\Phi(F_0) = F_0/\Phi(F_0) \times C_{x_0}\Phi(F_0)/\Phi(F_0)$ from where it follows that W generates F_0. On the other hand, if Y is a proper closed subset of $\{c_{x_0}^{-1}c_x \mid x \in X\}$, then $\overline{\langle Y, C_{x_0}\rangle} = \coprod_{y \in Y} \langle c_{x_0}y\rangle \amalg H_0$, so Y can not generate F_0. Observe that Z acts on W as follows: $(c_{x_0}^{-1}c_x)^z = c_{x_0}^{-1}(c_x)^z = c_{x_0}^{-1}c_{xz}$.

Lemma 3.2. (a) *For $x_1, x_2 \in X$ one has that C_{x_1} is conjugate in H to C_{x_2} if and only if $x_1 z = x_2$ for some $z \in Z$.*

(b) *H contains a free pro-2 subgroup F of index 2.*

Proof. (a) Suppose $C_{x_1}^h = C_{x_2}$ for some $h \in H$. Let $h_0 \in H_0$, $z \in Z$ be such that $h = zh_0$. Then $C_{x_1}^{zh_0} = (C_{x_1 z})^{h_0} = C_{x_2}$, whence we have $x_1 z = x_2$, as required. Conversely, if $x_1 z = x_2$ then there exists $h_0 \in H_0$ such that $c_{x_1} = c_{x_1 z}^{h_0} = c_{x_1}^{zh_0}$ as needed.

(b) Put $F = F_0 \rtimes Z$. Let $f : F_0 \to F(W/Z)$ be the natural epimorphism induced by the natural surjection $W \to W/Z$. Since $F(W/Z)$ is free, f splits, i.e., there exists a monomorphism $\varphi : F(W/Z) \to F_0$ with $f\varphi = \mathrm{id}$. It suffices to show that the natural homomorphism $F(W/Z) \amalg Z \to F$ induced by φ and by the monomorphism sending Z to its copy in F is an isomorphism. But this is clear since this homomorphism induces an isomorphism on the Frattini quotients (cf. [RZ], Theorem 7.2.7). □

Proposition 3.3. *H cannot be isomorphic to a free product of centralizers of finite groups and a free pro-2 group.*

Proof. Suppose there exists a boolean space T, and a continuous family $\Sigma_H := \{C_t \mid t \in T\}$ of groups of order 2 such that $H = \coprod_{t \in T} C_H(C_t) \amalg L$ for some free pro-2 group L. Note that $\coprod_{t \in T} C_t$ is a subgroup of H_0. For $t \in T$ denote by c_t the generator of C_t and put $S_T = \{c_t \mid t \in T\}$. Similarly for $x \in X$ denote by c_x the generator of C_x and put $S_X = \{c_x \mid x \in X\}$. Let $\overline{S}_T$ and $\overline{S}_X$ be the homeomorphic images of S_T and S_X in $H/\Phi(H_0)$ respectively. By [M], Proposition 4.9, every finite subgroup of H is conjugate to one of its factor C_x. Therefore, $\overline{S}_T \subseteqq \overline{S}_X$. Let $f : H_0/\Phi(H_0) \to H/\Phi(H)$ be the natural homomorphism. Since all C_t are subgroups of free factors $C_H(C_t)$ of H, the restriction $f_{|\overline{S}_T}$ is an injection. It is also a surjection because any finite subgroup is conjugate to a subgroup of a free factor (see [M], Proposition 4.9). Now observe that the Z-set S_X is isomorphic to the Z-set $\overline{S}_X$, where abusing notation we use the same letter for the image of Z in $H/\Phi(H_0)$ and, by Lemma 3.2 (a), it is isomorphic to the Z-set X as well. Also note that the restricition $f_{|\overline{S}_X}$ coincides with the natural quotient map $\overline{S}_X \to \overline{S}_X/Z$. On the other hand $f_{|\overline{S}_T} : \overline{S}_T \to \overline{S}_X/Z$ is also a surjection by Lemma 3.2 (a) taking into account that every C_x is conjugate to some C_t in H (see [M], Proposition 4.9). Thus $f_{|\overline{S}_T} : \overline{S}_T \to \overline{S}_X/Z$ is a homeomorphism. Since, as was mentioned above, Z-set $\overline{S}_X$ is isomorphic to the Z-set X, we obtain a contradiction with the fact established in Lemma 3.1 that a continuous section $X/Z \to X$ does not exist. □

References

[CP] *Z. A. Chatzidakis* and *P. Pappas*, Von Neumann regular rings not representable as rings of continuous functions, Alg. Univ. **29** (1992), 332–337.

[DS] *J. L. Dyer* and *G. P. Scott*, Periodic automorphisms of free groups, Comm. Alg. **3** (1975), 195–201.
[HRZ] *W. N. Herfort*, *L. Ribes* and *P. A. Zalesskii*, *p*-Extensions of Free Pro-*p* Groups, Forum Math. **11** (1998), 49–61.
[HZ] *W. N. Herfort* and *P. A. Zalesskii*, Virtually Free pro-*p* Groups, preprint 2002.
[M] *O. V. Melnikov*, Subgroups and Homology of Free Products of Profinite Groups, Math. USSR Izvest. **34**, 1 (1990), 97–119.
[NSW] *J. Neukirch*, *A. Schmidt* and *K. Wingberg*, Cohomology of Number Fields, Springer, 2000.
[RZ] *L. Ribes* and *P. A. Zalesskii*, Profinite Groups, Springer, 2000.
[RZ1] *L. Ribes* and *P. A. Zalesskii*, Pro-*p* Trees, Chapter 3, M. du Sautoy, A. Shalev, D. Segal, eds., Progr. Math., Birkhäuser, Boston 2000.
[Sch] *C. Scheiderer*, Real and Étale Cohomology, Lect. Notes Math. **1588**, Springer, Berlin 1994.
[Sch1] *C. Scheiderer*, The structure of some virtually free pro-*p* groups, Proc. Amer. Math. Soc. **127** (1999), 695–700.
[S] *J.-P. Serre*, Sur la dimension cohomologique des groupes profinis, Topology **3** (1965), 413–420.
[W] *J. S. Wilson*, Profinite Groups, London Math. Soc. Monogr., Clarendon Press, Oxford 1998.
[ZM] *P. A. Zalesskii* and *O. V. Melnikov*, Fundamental Groups of Graphs of Profinite Groups, Algebra i Analiz **1** (1989); translated in: Leningrad Math. J. **1** (1990), 921–940.

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