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Nonautonomous and non periodic Schrödinger equation with asymptotic growth in $\mathbb{R}^N$

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Resumo

Neste trabalho, consideramos a equação de Schödinger não-autônoma e não periódica com crescimento assintótico no $\mathbb{R}^N$

$$\begin{cases} -div(\xi(x)\nabla u)+V(x)u=f(x,u), & \text{em} \quad \mathbb{R}^N, \\ u(x)\to 0, \quad \text{quando} & |x|\to\infty, \end{cases} \tag{P}$$

com $N\geq 3$, $\xi:\mathbb{R}^N\to\mathbb{R}^+$ e $V:\mathbb{R}^N\to\mathbb{R}$ satisfazendo algumas condições e a não linearidade f assintoticamente linear no infinito e assumimos ser de classe $C^1(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$. Na primeira parte mostramos a existência de solução positiva com $V(x)\equiv 1$ no primeiro capítulo e $V(x)$ positiva no segundo capítulo.

Em seguida, estamos em busca de solução nodal. Para tanto, assumimos algum tipo de simetria para o problema. Mais especificamente, consideramos o problema

$$\begin{cases} -div(\xi(x)\nabla u)+V(x)u=f(x,u), & \text{em } \mathbb{R}^N, \\ u(\tau x)=-u(x), & \\ u(x)\to 0, \quad \text{quando} & |x|\to\infty, \end{cases} \tag{P_τ}$$

com $N\geq 3$ e $\tau:\mathbb{R}^N\to\mathbb{R}^N$ uma involução ortogonal não trivial que é uma tranformação ortogonal em $\mathbb{R}^N$ tal que $\tau\neq Id$ e $\tau^2=Id$, sendo Id o operador identidade em $\mathbb{R}^N$. Uma solução u do problema (P_τ) é chama $\tau-$ antissimétrica. Assim como na primeira parte, consideramos $V(x)\equiv 1$ no primeiro capítulo e $V(x)$ positiva no segundo capítulo.

Finalmente, buscamos a existência de uma solução não trivial para o problema (P) com o potencial V mudando de sinal. Estabelemos que V possui um limite positivo no infinito e que o espectro do operador $Lu=-div(\xi(x)\nabla u)+V(x)u$ tem ínfimo negativo. Com isso, e com base nas interações entre soluções transladadas do problema no infinito associado, é possível mostrar que tal problema satisfaz a geometria do Teorema de Linking e garantir a existência de uma solução fraca não trivial.

Abstract

In this work, we consider the nonautonomous and non periodic Schördinger equation with asymptotic growth in $\mathbb{R}^N$

$$\begin{cases} -div(\xi(x)\nabla u) + V(x)u = f(x,u), & \text{in} \quad \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as} & |x| \to \infty, \end{cases} \tag{P}$$

where $N \geq 3$, $\xi : \mathbb{R}^N \to \mathbb{R}^+$ and $V : \mathbb{R}^N \to \mathbb{R}$ satisfying some conditions and the nonlinearity f being asymptotically linear at infinity and is assumed to be a $C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. In the first part, we show the existence of a positive solution with $V(x) \equiv 1$ in the first chapter and $V(x)$ positive in the second chapter.

In the second part, we look for a nodal solution. In this case, we assume some type of symmetric for the problem. More specifically, we consider the problem

$$\begin{cases} -div(\xi(x)\nabla u) + V(x)u = f(x,u), & \text{em } \mathbb{R}^N, \\ u(\tau x) = -u(x), & \\ u(x) \to 0, \quad \text{quando} & |x| \to \infty, \end{cases} \tag{P_τ}$$

where $N \geq 3$ and $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is a nontrivial orthogonal involution, in other words, it is a linear orthogonal in $\mathbb{R}^N$ such that $\tau \neq Id$ and $\tau^2 = Id$, with Id being the identity operator in $\mathbb{R}^N$. As in the first part, we consider $V(x) \equiv 1$ in the first chapter and $V(x)$ positive in the second chapter.

Finally, we look the existence of a nontrivial solution to problem (P) with the potential V changing sign. We establish that V has a positive limit at infinity and that the spectrum of the operator $Lu = -div(\xi(x)\nabla u) + V(x)u$ has a negative infimum. With this, and based on interactions between translated solutions of the associated infinite problem, it is possible to show that such problem satisfies the geometry of the Linking Theorem and ensure the existence of a nontrivial solutions.

Notation

$B_R(x)$	open ball of radius R centered in x;
$u_n \to u$	strong convergence (in norm);
$u_n \rightharpoonup u$	weak convergence;
$u_n \to u$, a.e. in Ω	convergence almost everywhere in Ω;
$\nabla u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N}\right)$	gradiente of u;
$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$	Laplacian of u;
$A \subset\subset B$	$\overline{A}$ is compact and it is a subset of Ω;
$\|\Omega\|$	measure of Ω;
$\overline{\Omega}$	closure of Ω;
$\partial\Omega$	boundary of f;
$suppf$	support of f;
$C(X;Y)$	continuous functions from X to Y;
$C^1(X;Y)$	continuously differentiable functions from X to Y;
X^*	dual space of X;
$L^p := L^p(\mathbb{R}^N)$	Lebesgue functions $p-$ integrable;
$L^p_{loc}(\Omega)$	$L^p_{loc}(\Omega) = \{u \in L^p(\Omega^*),\ \forall\ \Omega^* \subset\subset \Omega\}$;
$W^{k,p}(\mathbb{R}^N)$	$W^{k,p}(\mathbb{R}^N) = \{u \in L^p; D^\alpha u \in L^p,\ \forall\ \|\alpha\| \leq k\}$;
$H^1(\mathbb{R}^N)$	Sobolev space $W^{1,2}(\mathbb{R}^N)$
$H^{-1}(\mathbb{R}^N)$	dual space of $H^1(\mathbb{R}^N)$;
$H^2(\mathbb{R}^N)$	Sobolev space $W^{2,2}(\mathbb{R}^N)$;
$\\|u\\|_{H^1(\mathbb{R}^N)} = \left(\\|\nabla u\\|^2_{L^2} + \\|u\\|^2_{L^2}\right)^{1/2}$	usual norm of $H^1(\mathbb{R}^N)$;
$\\|u\\|_{L^p} = \left(\int_{\mathbb{R}^N} \|u\|^p dx\right)^{1/p}$	usual norm of L^p;
$\\|u\\|_{L^\infty} = \sup_{x \in \mathbb{R}^N} ess\|u(x)\|$	usual norm of L^∞.

Table of contents

Introduction

This thesis is divided into three chapters that deal with the Schrödinger equation

$$\begin{cases} -div(\xi(x)\nabla u) + V(x)u = f(x,u), & \text{in} \quad \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as} & |x| \to \infty, \end{cases} \tag{P}$$

where $N \geq 3$, $\xi : \mathbb{R}^N \to \mathbb{R}^+$ and $V : \mathbb{R}^N \to \mathbb{R}$ satisfying some conditions and the nonlinearity f is asymptotically linear at infinity and is assumed to be a $C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

About the function ξ we have that the operator $-div(\xi(x)\nabla u)$ is known as the divergence operator of a tensor field $u(x)$. This operator appears in various areas of physics and engineering, especially in problems involving diffusion and transport of physical quantities such as heat, mass, and electric charge.

If $-div(\xi(x)\nabla u)$ is a symmetric and positive definite matrix, the operator represents anisotropic diffusion, where the diffusion rate varies according to the direction of the flow. This is crucial in physical phenomena where conductivity is not uniform in all directions, such as in porous media or anisotropic materials. The physical motivation for considering this operator can be found in diffusive processes in heterogeneous media, such as the transport of substances in non-homogeneous soil or the diffusion of heat in materials with variable thermal properties. Additionally, in fluid mechanics, this operator appears in the Navier-Stokes equation to model fluid viscosity.

Understanding this operator is fundamental for solving a variety of physical and engineering problems, allowing the analysis and prediction of how physical quantities diffuse and distribute in complex systems. Studying its properties and behaviors is essential for understanding a wide range of natural and industrial phenomena.

For more information about the operator $-div(\xi(x)\nabla u)$ and its applications in physics and engineering, you can refer to [15]. This book provides a comprehensive introduction to partial differential equations, including a detailed discussion on differential operators, such as the divergence operator, and their applications in various physical contexts. Another reference is [18], this book is a classic reference in the study of elliptic partial differential equations, addressing in detail the theory and applications of these equations, including the divergence operator. We also have the book [22] which is an excellent source for learning about numerical methods to solve partial differential equations, including approaches for dealing with differential operators like the divergence operator in physical problems. These references provide a solid foundation for understanding the theory and applications of the operator $-div(\xi(x)\nabla u)$ in physical and engineering problems.

In [26], Maia and Ruviaro worked with the equation

$$-\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N$$

where V is bounded and invariant under an orthogonal and converges to a positive constant as $|x| \to +\infty$, and f is asymptotic linear at infinity. The structure of the first two chapters were based to obtain the positive and nodal solutions.

In [9], Chabrowski studied the problem

$$-div(a(x)\nabla u) + \lambda u = K(x)|u|^{q-2}u, \quad \text{in } \mathbb{R}^N \tag{0.0.1}$$

with $N \geq 3$, $\lambda > 0$, $2 < q < 2N/(N-2)$ and $a \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying $0 \leq a(x) \leq \lim_{|x|\to\infty} a(x)$, supposing additionally that a is positive in some exterior ball $B_R(0)$. The author showed an existence result using the minimization method, assuming an integrability condition for a and requiring that $K \in L^\infty(\mathbb{R}^N)$ verifies either a is periodic or $K(x) \geq \lim_{|x|\to\infty} K(x)$. Furthermore, weighted Sobolev's space is used with the following assumptions: $\{x : a(x) = 0\} \subset B_{R_0}(0)$ and $1/a \in L^q(B_{R_0}(0))$.

Another paper in this class of problems was treated by Lazzo in [21]. She studied the problem (0.0.1) with $K \equiv 1$, and the function a satisfying

$$0 < a_0 := \inf_{x\in\mathbb{R}^N} a(x) < a_\infty := \liminf_{|x|\to\infty} a(x). \tag{0.0.2}$$

Using the minimization method, it was proved that there exists $\lambda^* > 0$ such that the problem (0.0.1) has a positive solution for $\lambda > \lambda^*$. It was also proved that for λ sufficiently large, the number of solutions of (0.0.1) is bounded below by the Ljusternick-Schrinelmann category. Furthermore, she studied the asymptotic behavior of such minimizers as λ goes to infinity and proved that they concentrate around the global minimum point of a using techniques based on [34].

In [10], Cingolani and Lazzo studied the multiplicity of solutions to the problem $\varepsilon^2\Delta u + V(x)u = |u|^{p-2}u,\ x \in \mathbb{R}^N$, where $V(x) = u(x) + \lambda$. The main result proved the existence and multiplicity of solutions to the problem under the hypothesis $\liminf\limits_{|x|\to\infty} V(x) > V_0 > 0$.

The next papers we will cite here were written by Figueiredo and Furtado, whose results also guided this thesis. In [16], they studied the problem

$$-\varepsilon^p div(a(x)|\nabla u|^{p-2}\nabla u) + u^{p-1} = f(u), \quad \text{in } \mathbb{R}^N \tag{0.0.3}$$

with f being a superlinear function and a satisfying (0.0.2). They showed the existence of a ground state solution using minimax theorems and a result of the existence of multiple solutions.

In [17], they obtained the multiplicity of positive solutions to quasilinear equation (0.0.3) with $\varepsilon > 0$ as a small parameter, f being supercritical linearity, and a a positive potential, considering a weaker condition than (0.0.2), namely $0 < a_0 = \inf\limits_{x\in\Lambda} a(x) < \inf\limits_{x\in\partial\Lambda} a(x)$ where Λ is a bounded domain in $\mathbb{R}^N$. The main result is proved using the Lusternik–Schnirelmann theory. To show the existence of a solution, they considered a penalized problem, and the solution will belong to the Nehari manifold, using the minimization theory. In this type of problem, we can not apply the Maximum Principle, and because of that, it is necessary to use a different technique based on the work of [23] to show that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{loc}(\mathbb{R}^N)$, a technique that will also be used by us.

In the first chapter, we study the problem (P) with $V \equiv 1$ with the functions $\xi \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfying:

(ξ_1) there exists $\xi_0 > 0$ such that $\xi(x) \geq \xi_0$;

(ξ_2) $\lim\limits_{|x|\to\infty} \xi(x) = \xi_\infty$;

(ξ_3) $\xi(x) \lneqq \xi_\infty$;

(f_1) $\lim\limits_{s\to 0^+} \dfrac{f(x,s)}{s} = 0$, uniformly for $x \in \mathbb{R}^N$;

(f_2) there exist $a \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $h \in C(\mathbb{R}, \mathbb{R}^+)$ an even function satisfying $h(s) > 0$ for all $s > 0$, $h(0) = 0$ and

$$\lim_{s\to\infty}\frac{f(x,s)}{s}=a(x), \qquad \lim_{|x|\to\infty}\frac{f(x,s)}{s}=h(s),$$

$$\lim_{|x|\to\infty,\, s\to\infty}\frac{f(x,s)}{s}=\lim_{s\to\infty}h(s)=\lim_{|x|\to\infty}a(x)=a_\infty\ ;$$

(f_3) $\dfrac{f(x,s)}{s}\geq h(s)$, for all $x\in\mathbb{R}^N$ and all $s\in\mathbb{R}^+$ and $\dfrac{f(x,s)}{s}>h(s)$ for all x in subset Ω of positive Lebesgue measure and all $s\in\mathbb{R}^+$;

(f_4) $1<a_\infty\lneqq a(x)$, for all $x\in\mathbb{R}^N$;

(f_5) if we set $F(x,s)=\int_0^s f(x,t)dt$ and $Q(x,s)=\dfrac{1}{2}f(x,s)s-F(x,s)$, then

$$\lim_{s\to+\infty}Q(x,s)=+\infty$$

and there exists $D\geq 1$ such that

$$Q(x,s)<DQ(x,t), \quad \text{for all } x\in\mathbb{R}^N \ \text{ and } \ 0\leq s<t.$$

The first result of this chapter can be stated as follows.

Theorem 0.0.1. *Suppose f satisfies $(f_1)-(f_5)$ and ξ satisfies $(\xi_1)-(\xi_3)$. Then problem (P) has a positive solution $u\in H^1(\mathbb{R}^N)$.*

In the second part of this chapter, we look for a nodal solution. In this case, we assume some type of symmetry for the problem. More specifically, we consider the problem

$$\begin{cases} -div(\xi(x)\nabla u)+u=f(x,u), & \text{in } \mathbb{R}^N,\\ u(\tau x)=-u(x), & \\ u(x)\to 0, \quad \text{as} \quad |x|\to\infty, & \end{cases} \tag{P_τ}$$

where $N\geq 3$ and $\tau:\mathbb{R}^N\to\mathbb{R}^N$ is a nontrivial orthogonal involution, in other words, it is a linear orthogonal transformation in $\mathbb{R}^N$ such that $\tau\neq Id$ and $\tau^2=Id$ with Id being the identity operator in $\mathbb{R}^N$. A solution u of (P_τ) is called a τ-antisymmetric solution. Let $x=(x_1,x_2)$, an example of function τ is given by $\tau(x_1,x_2)=(-x_1,-x_2)$.

In this new setting, we need some technical assumptions. So we shall suppose that ξ and f satisfies:

(ξ_4) $\xi(\tau x)=\xi(x)$, for all $x\in\mathbb{R}^N$;

(f_6) $f(\tau x,s)=-f(x,-s)$, for all $x\in\mathbb{R}^N,\ s\in\mathbb{R}$;

(f_7) there exists $C_1>1$ such that $f(x,s)\le C_1 f(x,t)$ with $0\le s\le t$, for all $x\in\mathbb{R}^N$.

Our result concerning nodal solution is stated next.

Theorem 0.0.2. *Assume that ξ satisfy the hypotheses $(\xi_1)-(\xi_4)$ and f satisfies $(f_1)-(f_7)$. Then problem (P_τ) has a sign-changing solution provided one of the following conditions holds:*

$$\xi(x)\le\xi_\infty - Ce^{-\beta_1|x|},\ \textit{for all } x\in\mathbb{R}^N \tag{0.0.4}$$

or

$$F(x,s)\ge H(s)+Ce^{-\beta_2|x|}|s|^2,\ \textit{for all } x\in\mathbb{R}^N,\ s\in\mathbb{R}, \tag{0.0.5}$$

for constants $C>0$ and $0<\beta_1,\beta_2<\beta$.

In the second chapter we have results similar to those in the first chapter, but with the potential V being positive. Thus, we have another norm associated with $H^1(\mathbb{R}^N)$, which is equivalent to the first one found. For the main results, we will have the conditions on V which are:

(V_1) there exists $V_0>0$ such that $V(x)\ge V_0$;

(V_2) $\lim\limits_{|x|\to\infty} V(x)=V_\infty$;

(V_3) $V(x)\lneqq V_\infty$;

(ξ_4) $\xi(\tau x)=\xi(x)$, for all $x\in\mathbb{R}^N$;

and the hypothesis (f_4) is adapted to

(f_4') $V_\infty<a_\infty\lneqq a(x)$, for all $x\in\mathbb{R}^N$.

The first result of this chapter can be stated as follows.

Theorem 0.0.3. *Suppose f satisfy $(f_1)-(f_3),(f_4'),(f_5)$ and ξ satisfies $(\xi_1)-(\xi_3)$. Then problem (P) has a positive solution $u\in H^1(\mathbb{R}^N)$.*

In the second part of this chapter, we look for a nodal solution to the problem (P_τ) with V positive is given by

$$\begin{cases} -div(\xi(x)\nabla u) + V(x)u = f(x,u), & \text{in } \mathbb{R}^N, \\ u(\tau x) = -u(x), \\ u(x) \to 0, \qquad \text{as} \quad |x| \to \infty, \end{cases} \tag{P'_τ}$$

In this new setting, in addition to the hypotheses $(\xi_4), (f_6), (f_7)$ we shall suppose that V satisfies:

(V_4) $V(\tau x) = V(x)$, for all $x \in \mathbb{R}^N$.

Our result concerning the nodal solution in the second chapter is stated next.

Theorem 0.0.4. *Assume that ξ and V satisfy the hypotheses $(\xi_1)-(\xi_4)$ and $(V_1)-(V_4)$, respectively, and f satisfy $(f_1)-(f_3), (f_4'), (f_5)-(f_7)$. Then problem ($P'_\tau$) has a sign-changing solution provided one of the following conditions holds:*

$$\xi(x) \le \xi_\infty - Ce^{-\beta_1|x|}, \textit{ for all } x \in \mathbb{R}^N \tag{0.0.6}$$

or

$$V(x) \le V_\infty - Ce^{-\beta_2|x|}, \textit{ for all } x \in \mathbb{R}^N \tag{0.0.7}$$

or

$$F(x,s) \ge H(s) + Ce^{-\beta_3|x|}|s|^2, \textit{ for all } x \in \mathbb{R}^N, \ s \in \mathbb{R}, \tag{0.0.8}$$

for constants $C > 0$ and $0 < \beta_i < \beta$, with $i = 1,2,3$.

To prove the results from this chapter, since f is not homogeneous and $f(x,s)/s$ for $s > 0$ is not necessarily, the appropriate minimization process is to use the Pohozaev manifold. We work with the difference of two solutions u ground state $z_y = u(x-y) - u(x-\tau x)$ without making any truncation.

The fact that the functions ξ and V are bounded allows us to define a norm in $H^1(\mathbb{R}^N)$ and consider the appropriate space of function to obtain solutions of (P), in the first chapter only with ξ and in the second chapter with ξ and V. Since the embedded of $H^1(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$, $2 \le p < 2^*$ is not compact, the main problem consists of the fact that the associated functional does not satisfy a compactness condition. To overcome this difficulty, we will present and prove a version of the concentration compactness theorem of P.L. Lions [24], as presented by M. Struwe in [30] so-called the Splitting Lemma. Therefore, we can describe for which energy levels our associated functional, restricted to the manifold considered, satisfies the compactness condition.

It is important to highlight that our operator does not admit Maximum Principle. To prove the Theorems 0.0.1 and 0.0.3, we need to adjust a result from Li and Wang [23], which ensures that the solutions we discover belong to $L^\infty(\mathbb{R}^N)\cap C^{1,\alpha}_{loc}(\mathbb{R}^N)$. Additionally, we utilize the Harnack inequality to ensure the positivity of the solution obtained the Mountain Pass Theorem.

The third chapter was inspired on the work of Junior, Maia and Ruviaro in [25]. They worked on the problem

$$-\Delta u + V(x)u = f(x,u)$$

in $\mathbb{R}^N$ under the condition of non-periodicity in V e f, where the potential V changes sign. In this framework, it is not possible to apply the Mountain Pass Theorem. Therefore, the authors employed spectral theory. As a consequence, a new norm was introduced, allowing the application of the Linking Theorem of Rabinowitz [29] with Cerami condition to obtain a positive solution.

In [33], Stuart and Zhou proved the existence of a radial and positive solution of the asymptotically linear problem with radially symmetric V. By leveraging the radial symmetric of the working set, they managed to recover the compactness of the problem in an unbounded domain.

Another important paper was addressed by Kryszew and Szulkin [20] and Pankov [27], who demonstrated the existence of a nontrivial solution to the nonautonomous problem. They considered superquadratic nonlinearity in s, incorporating the periodicity hypotheses of Jeanjean and Tanaka in [19]. This study established the existence of a positive solution under the specified condition: $V(x) \geq \alpha > 0$ and f asymptotically linear at infinity, with $f(s)/s \to a > 0$ as $s \to \infty$ where $a > \inf \sigma(-\Delta + V)$. Here, $\sigma(-\Delta+V)$ denotes the spectrum of operator $-\Delta + V$.

In Chapter 3 we consider ξ positive and the potential V with a negative part, on the other hand, it can change sign and satisfy the following hypotheses:

(ξ_1) there exists $\xi_0 > 0$ such that $\xi(x) \geq \xi_0$;

(ξ_2) $\lim_{|x|\to\infty} \xi(x) = \xi_\infty$;

(ξ_3) $\xi(x) \lneqq \xi_\infty$;

(V_1) there exists $V_0 > 0$ such that $V(x) \geq -V_0$;

(V_2) $\lim_{|x|\to\infty} V(x) = V_\infty$;

(V_3) $V(x) \leq V_\infty$;

(V_4) $0 \notin \sigma(L)$ and $\inf \sigma(L) < 0$, where $\sigma(L)$ is the spectrum of the operator $L(\cdot) = -div(\xi(x)\nabla(\cdot)) + V(x)(\cdot)$.

Under the nonlinear function $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ we have the following hypotheses:

(f_1) $\lim_{s\to 0^+} \dfrac{f(x,s)}{s} = 0$, uniformly in $x \in \mathbb{R}^N$;

(f_2) there exist $a \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $h \in C(\mathbb{R}, \mathbb{R}^+)$ a even function satisfying $h(s) > 0$ for all $s > 0$, $h(0) = 0$ and such that

$$\lim_{s\to\infty} \frac{f(x,s)}{s} = a(x), \qquad \lim_{|x|\to\infty} \frac{f(x,s)}{s} = h(s),$$

$$\lim_{|x|\to\infty,\, s\to\infty} \frac{f(x,s)}{s} = \lim_{s\to\infty} h(s) = \lim_{|x|\to\infty} a(x) = a_\infty \ ,$$

uniformly in $x \in \mathbb{R}^N$. Moreover, $\dfrac{|f(x,s)|}{|s|} \leq a(x)$ and $a(x) \geq a_0 > V_\infty$, for all $s \neq 0$ and all $x \in \mathbb{R}^N$;

(f_3) $h(s) < a_\infty$, for all $s \in \mathbb{R}$;

(f_4) if $F(x,s) := \int_0^s f(x,t)dt$, $H(s) := \int_0^s h(t)tdt$, $G(s) := \dfrac{1}{2}h(s)s^2 - H(s)$ and $Q(x,s) := \dfrac{1}{2}f(x,s)s - F(x,s)$, then, for all $s \in \mathbb{R}\setminus\{0\}$ and all $x \in \mathbb{R}^N$,

$$G(s) > 0,\ F(x,s) \geq 0,\ Q(x,s) > 0 \text{ and } \lim_{s\to+\infty} Q(x,s) = +\infty;$$

(f_5) the function $s \mapsto f(x,s)/s$ is increasing in $s \in (0,+\infty)$ for all $x \in \mathbb{R}^N$.

And the main result of this chapter is the following:

Theorem 0.0.5. *Assume that* ξ *and* V *satisfy the hypotheses* $(\xi_1)-(\xi_3)$ *and* $(V_1)-(V_4)$*, respectively, and the function* f *satisfies* $(f_1)-(f_6)$*. Then problem* (P_2) *has a nontrivial weak solution* $u \in H^1(\mathbb{R}^N)$ *provided one of the followings conditions holds:*

$$\xi(x) \leq \xi_\infty - C_1 e^{-\gamma_1 |x|}, \quad \textit{for all} \;\; x \in \mathbb{R}^N \tag{0.0.9}$$

or

$$V(x) \leq V_\infty - C_2 e^{-\gamma_2 |x|}, \quad \textit{for all} \;\; x \in \mathbb{R}^N \tag{0.0.10}$$

for constants $C_1,\ C_2 > 0$ *and* $0 < \gamma_1,\ \gamma_2 < \sqrt{V_\infty/\xi_\infty}$.

One difficulty encountered in this type of problem is that the associated functional is strongly undefined. To overcome this challenge, the space $H^1(\mathbb{R}^N)$ is decomposed into a direct sum of two subspaces E^+ and E^-, one of which has finite dimension, and assumes the condition of non-quadraticity in F, the primitive of f. In this context, it is not possible to apply the Mountain Pass Theorem. Hence, we employ the Linking Theorem under the Cerami condition to obtain a non-trivial solution to the problem.

An additional challenge arose with the operator spectral theory $L(u) = -\nabla(\xi u) + V(x)u$. Since the function ξ is not constant, we do not immediately have the operator being self-adjoint to apply the spectral theory. Therefore, we use the Fourier Transform on the function ξ to circumvent this obstacle and ensure self-adjointness of the operator.

As previously mentioned, we also cannot apply the Maximum Principle to guarantee the non-triviality and positivity of the solution found. To address this, we adapt, once again, the results of Li and Wang [23]. Together with the Harnack inequality, these results assure us that our solution is non-trivial and positive.

Chapter 1

Problem with ξ positive and $V \equiv 1$

1.1 Variational Setting

We consider the following problem

$$\begin{cases} -div(\xi(x)\nabla u) + u = f(x,u), & \text{in} \quad \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{P_1}$$

with $N \geq 3$, under the following assumptions on $\xi \in C(\mathbb{R}^N, \mathbb{R}^+)$:

(ξ_1) there exists $\xi_0 > 0$ such that $\xi(x) \geq \xi_0$;

(ξ_2) $\lim_{|x|\to\infty} \xi(x) = \xi_\infty$;

(ξ_3) $\xi(x) \lneqq \xi_\infty$.

The hypotheses on the nonlinearity $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ are the following:

(f_1) $\lim_{s\to 0^+} \frac{f(x,s)}{s} = 0$, uniformly for $x \in \mathbb{R}^N$;

(f_2) there exist $a \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $h \in C(\mathbb{R}, \mathbb{R}^+)$ an even function satisfying $h(s) > 0$ for all $s > 0$, $h(0) = 0$ and

$$\lim_{s\to\infty} \frac{f(x,s)}{s} = a(x), \qquad \lim_{|x|\to\infty} \frac{f(x,s)}{s} = h(s),$$

$$\lim_{|x|\to\infty,\ s\to\infty} \frac{f(x,s)}{s} = \lim_{s\to\infty} h(s) = \lim_{|x|\to\infty} a(x) = a_\infty \ ;$$

(f_3) $\dfrac{f(x,s)}{s} \geq h(s)$, for all $x \in \mathbb{R}^N$ and all $s \in \mathbb{R}^+$ and $\dfrac{f(x,s)}{s} > h(s)$ for all $x \in \Omega$, where Ω is a subset of positive Lebesgue measure and for all $s \in \mathbb{R}^+$;

(f_4) $1 < a_\infty \lneqq a(x)$, for all $x \in \mathbb{R}^N$;

(f_5) if we set $F(x,s) = \int_0^s f(x,t)dt$ and $Q(x,s) = \frac{1}{2}f(x,s)s - F(x,s)$, then

$$\lim_{s \to +\infty} Q(x,s) = +\infty$$

and there exists $D \geq 1$ such that

$$Q(x,s) < DQ(x,t), \quad \text{for all } x \in \mathbb{R}^N \text{ and } 0 \leq s < t.$$

An example that $f(s)/s$ that is non-increasing and satisfies assumptions $(f_1) - (f_5)$:

$$f(s) = \frac{s^7 - 1,5s^5 + 2s^3}{1+s^6}.$$

The first result of this chapter can be stated as follows.

Theorem 1.1.1. *Suppose f satisfies $(f_1) - (f_5)$ and ξ satisfies $(\xi_1) - (\xi_3)$. Then problem (P_1) has a positive solution $u \in H^1(\mathbb{R}^N)$.*

Remark 1.1.1. *Hypothesis (f_2) implies that there exists a constant $a_0 > 0$ such that*

$$a(x) \leq a_0, \quad \textit{for all} \quad x \in \mathbb{R}^N. \tag{1.1.1}$$

Remark 1.1.2. *Note that conditions (f_1), (f_2) and (1.1.1) imply that for a given $\varepsilon > 0$ and $2 \leq p \leq 2^*$, there exists $0 < C = C(\varepsilon, p)$ such that*

$$|f(x,s)| \leq \varepsilon s + C|s|^{p-1} \tag{1.1.2}$$

and

$$|F(x,s)| \leq \frac{\varepsilon}{2}s^2 + C|s|^p. \tag{1.1.3}$$

Indeed, using (f_1), there exists $0 < r < 1$ such that $|s| < r$. Thus, we obtain

$$\lim_{s \to 0^+} \frac{f(x,s)}{s} = 0 \Rightarrow \left|\frac{f(x,s)}{s}\right| \leq \varepsilon \Rightarrow |f(x,s)| \leq \varepsilon |s|.$$

For $|s|> r$, applying (f_2) and Remark 1.1.1 we have

$$\left|\frac{f(x,s)}{s}\right| \leq a_0.$$

Therefore,

$$|f(x,s)|\leq a_0|s|.$$

Note that

$$|s|= \frac{1}{|s|^{p-2}}|s|^{p-1}\leq \overline{C}|s|^{p-1},$$

where $\overline{C} = \overline{C}(\varepsilon,p) = \max\limits_{r\leq s\leq 1}\left\{\frac{1}{|s|^{p-2}}\right\}$. Hence, we obtain

$$|f(x,s)| \leq \varepsilon|s|+a_0|s|\leq \varepsilon|s|+a_0\overline{C}|s|^{p-1}= \varepsilon|s|+C|s|^{p-1}.$$

It follows from the definition of $F(x,\cdot)$ that

$$|F(x,s)| \leq \int_0^s |f(x,t)|dt \leq \int_0^s (\varepsilon|t|+C|t|^{p-1})dt = \frac{\varepsilon}{2}|s|^2+C|s|^p.$$

Remark 1.1.3. *By (f_1) and (f_5) we obtain that $Q(x,s) > 0$ for $s > 0$ and $x \in \mathbb{R}^N$. Moreover, by (f_2) and (f_5) it follows that $0 \leq \frac{1}{2}h(s)s^2 - H(s) \leq D\left(\frac{1}{2}h(t)t^2 - H(t)\right)$ for $0 \leq s \leq t$, if $H(s) = \int_0^s h(\zeta)\zeta d\zeta$ and by assumptions (f_1) and (f_3) we have $\frac{1}{2}h(s)s^2 - H(s) > 0$ for $s > 0$.*

Let us show the second statement. Using the definition of $Q(x,\cdot)$ and the hypothesis (f_5), we have

$$\begin{aligned} Q(x,s) &\leq DQ(x,t) \\ \frac{1}{2}f(x,s)s - F(x,s) &\leq D\left(\frac{1}{2}f(x,t)t - F(x,t)\right) \\ \frac{1}{2}\frac{f(x,s)}{s}s^2 - \int_0^s \frac{f(x,\zeta)}{\zeta}\zeta d\zeta &\leq D\left(\frac{1}{2}\frac{f(x,t)}{t}t^2 - \int_0^t \frac{f(x,\zeta)}{\zeta}\zeta d\zeta\right). \end{aligned}$$

Applying the limit when $|x|$ goes to infinity on both sides and using Lebesgue's dominated convergence theorem, we obtain

$$\frac{1}{2}h(s)s^2 - \int_0^s h(\zeta)\zeta d\zeta \;\leq\; D\left(\frac{1}{2}h(t)t^2 - \int_0^t h(\zeta)\zeta d\zeta\right)$$
$$\frac{1}{2}h(s)s^2 - H(s) \;\leq\; D\left(\frac{1}{2}h(t)t^2 - H(t)\right)$$

for $0 \leq s \leq t$ as claimed.

In the second part of this chapter, we look for a nodal solution. In this case, we assume some type of symmetry for the problem. More specifically, we consider the problem

$$\begin{cases} -div(\xi(x)\nabla u) + u = f(x,u), & \text{in } \mathbb{R}^N, \\ u(\tau x) = -u(x), & \\ u(x) \to 0, \quad \text{as} & |x| \to \infty, \end{cases} \tag{P_τ}$$

where $N \geq 3$ and $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is a nontrivial orthogonal involution, in other words, it is a linear orthogonal transformation in $\mathbb{R}^N$ such that $\tau \neq Id$ and $\tau^2 = Id$, with Id being the identity operator in $\mathbb{R}^N$. A solution u of (P_τ) is called a τ-antisymmetric solution.

In this new setting, we need some technical assumptions. So we shall suppose that ξ and f satisfies:

(ξ_4) $\xi(\tau x) = \xi(x)$, for all $x \in \mathbb{R}^N$;

(f_6) $f(\tau x, s) = -f(x, -s)$, for all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$;

(f_7) there exists $C_1 > 1$ such that $f(x,s) \leq C_1 f(x,t)$ with $0 \leq s \leq t$, for all $x \in \mathbb{R}^N$.

Remark 1.1.4. *We do not assume that $f(x,s)/s$ for $s > 0$ is increasing in s.*

Consider $H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in (L^2(\mathbb{R}^N))^N\}$ equipped with the norm $\|u\|^2 = \int_{\mathbb{R}^N} (\xi_\infty |\nabla u|^2 + u^2) dx$ and the limit problem

$$-div(\xi_\infty \nabla u) + u = h(u)u, \;\text{ in } \mathbb{R}^N. \tag{1.1.4}$$

The functional associated with the equation (1.1.4) is given by

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} (\xi_\infty |\nabla u|^2 + u^2) dx - \int_{\mathbb{R}^N} H(u) dx. \tag{1.1.5}$$

It is well defined and in $C^1(H^1(\mathbb{R}^N),\mathbb{R})$ with

$$I'_\infty(u)\varphi = \int_{\mathbb{R}^N}(\xi_\infty \nabla u \nabla \varphi + u\varphi)dx - \int_{\mathbb{R}^N} h(u)u\varphi dx, \quad \text{for all} \quad u,\ \varphi \in H^1(\mathbb{R}^N).$$

Critical points of the functional I_∞ are weak solutions of problem (1.1.4). The functional I_∞ is continuous, $I_\infty(0)=0$ and if ω is the positive solution of (1.1.4), the maximum of $I_\infty\left(\omega\left(\frac{\cdot}{t}\right)\right)>0$ holds on $t=1$. Furthermore, there exists a real number $L>0$, large sufficiently such that $I_\infty\left(\omega\left(\frac{\cdot}{t}\right)\right)<0$ for all $t\geq L$. Thus, there exists $L_0>1$ such that

$$I_\infty\left(\omega\left(\frac{\cdot}{L_0}\right)\right)=0 \tag{1.1.6}$$

and

$$I_\infty\left(\omega\left(\frac{\cdot}{t}\right)\right)<0, \text{ if } t\geq L_0. \tag{1.1.7}$$

Therefore, consider

$$\beta \in \left(0, \sqrt{\frac{1}{\xi_\infty}}\right). \tag{1.1.8}$$

Our result concerning nodal solution is stated next.

Theorem 1.1.2. *Assume that ξ satisfies the hypotheses $(\xi_1)-(\xi_4)$ and f satisfies $(f_1)-(f_7)$. Then problem (P_τ) has a sign-changing solution provided one of the following conditions holds:*

$$\xi(x)\leq \xi_\infty - Ce^{-\beta_1|x|}, \textit{ for all } x\in\mathbb{R}^N \tag{1.1.9}$$

or

$$F(x,s)\geq H(s)+Ce^{-\beta_2|x|}|s|^2, \textit{ for all } x\in\mathbb{R}^N,\ s\in\mathbb{R}, \tag{1.1.10}$$

for constants $C>0$ and $0<\beta_1,\beta_2<\beta$.

We will state and prove some preliminary results essential for the development of this chapter and for the proof of the main results.

Any solution u of the limit problem (1.1.4) satisfies Pohozaev identity (see [28])

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx = N\int_{\mathbb{R}^N} G_\infty(u)dx, \tag{1.1.11}$$

where $G_\infty(u)=\frac{1}{\xi_\infty}\left(H(u)-\frac{1}{2}u^2\right)$. We define the Pohozaev manifold as

$$\mathcal{P} = \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : J(u) = 0 \right\}, \tag{1.1.12}$$

where

$$J(u) := \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - N \int_{\mathbb{R}^N} G_\infty(u) dx \tag{1.1.13}$$

and denote

$$m_\infty := \inf_{u \in \mathcal{P}} I_\infty(u). \tag{1.1.14}$$

Remark 1.1.5. *Note that*

$$G_\infty(\zeta) = \frac{1}{\xi_\infty} \int_0^\zeta (h(s)s - s) ds > 0 \tag{1.1.15}$$

implies $\mathcal{P} \neq \emptyset$*.*

Lemma 1.1.1. *Let* $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ *be the functional* (1.1.13)*. Then*

(i) $\mathcal{P} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : J(u) = 0\}$ *is closed;*

(ii) $\mathcal{P}$ *is a manifold of class* C^1*;*

(iii) there exists $\sigma > 0$ *such that* $\|u\|_{H^1(\mathbb{R}^N)} > \sigma$ *for all* $u \in \mathcal{P}$*.*

Proof. We first verify items (i) and (ii). By definition of J, we have

$$J(u) = \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - N \int_{\mathbb{R}^N} G_\infty(u) dx,$$

which is a functional of class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$. Thus

$$\mathcal{P} \cup \{0\} = J^{-1}(\{0\}).$$

Then, it follows that $\mathcal{P}$ is a closed set since $\{0\}$ is an isolated point. Furthermore, using the Remark 1.1.4 and $g_\infty(u) := \frac{1}{\xi_\infty}(h(u)u - u)$, we obtain

$$\begin{aligned} J'(u)u &= 2N \int_{\mathbb{R}^N} G_\infty(u) dx - N \int_{\mathbb{R}^N} g_\infty(u) u dx \\ &= 2N \int_{\mathbb{R}^N} \left(H(u) - \frac{1}{2}u^2 - \frac{1}{2}h(u)u^2 + \frac{1}{2}u^2 \right) dx \\ &= 2N \int_{\mathbb{R}^N} \left(H(u) - \frac{1}{2}h(u)u^2 \right) dx < 0, \end{aligned}$$

which implies $J'(u) \neq 0$ and hence $\mathcal{P}$ is a C^1 manifold. Finally, for the proof of item (iii), let $u \in \mathcal{P}$ and $2^* = 2N/(N-2)$, then we have

$$\begin{aligned} \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 dx \; - \; N\int_{\mathbb{R}^N} G_\infty(u)dx = 0 \\ \int_{\mathbb{R}^N}\xi_\infty|\nabla u|^2 dx \; = \; \frac{2N}{N-2}\int_{\mathbb{R}^N} H(u)dx - \frac{N}{N-2}\int_{\mathbb{R}^N} u^2 dx \\ \int_{\mathbb{R}^N}\Big(\xi_\infty|\nabla u|^2 \; + \; \frac{N}{N-2}u^2\Big)dx = 2^*\int_{\mathbb{R}^N} H(u)dx. \end{aligned}$$

Then, taking $M := \min\left\{1, \dfrac{N}{N-2}\right\}$ and using (f_3), we obtain

$$M\|u\|^2_{H^1(\mathbb{R}^N)} \leq 2^*\int_{\mathbb{R}^N} H(u)dx \leq 2^*\int_{\mathbb{R}^N} F(x,u)dx.$$

From (1.1.3) and using Sobolev's embedding with $2 \leq p \leq 2^*$ it follows

$$M\|u\|^2_{H^1(\mathbb{R}^N)} \leq 2^*\int_{\mathbb{R}^N}\left(\frac{\varepsilon}{2}|u|^2 + C|u|^p\right)dx \leq \frac{2^*\varepsilon}{2}\|u\|^2_{H^1(\mathbb{R}^N)} + 2^*C\|u\|^p_{H^1(\mathbb{R}^N)}.$$

Now, taking ε small sufficiently we obtain $\dfrac{M}{2}\|u\|^2_{H^1(\mathbb{R}^N)} \leq 2^*C\|u\|^p_{H^1(\mathbb{R}^N)}$ and hence there exists $\sigma > 0$, such that $\sigma \leq \|u\|^{p-2}_{H^1(\mathbb{R}^N)}$. □

Lemma 1.1.2. *If f satisfies $(f_1)-(f_3)$, (u_n) is a bounded sequence and $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$, then*

$$f(x,u_n) - f(x,u_n - u_0) \to f(x,u_0), \quad \textit{in } H^{-1}(\mathbb{R}^N) \tag{1.1.16}$$

and

$$\int_{\mathbb{R}^N}|F(x,u_n) - F(x,u_n - u_0) - F(x,u_0)|dx \to 0. \tag{1.1.17}$$

Furthermore,

$$h(u_n)u_n - h(u_n - u_0)(u_n - u_0) \to h(u_0)u_0, \quad \textit{in } H^{-1}(\mathbb{R}^N) \tag{1.1.18}$$

and

$$\int_{\mathbb{R}^N}|H(u_n) - H(u_n - u_0) - H(u_0)|dx \to 0. \tag{1.1.19}$$

To demonstrate (1.1.17), we will use the following result.

Lemma 1.1.3 (Brezis-Lieb [8])**.** *Consider a continuous function, $j : \mathbb{C} \to \mathbb{C}$ with $j(0) = 0$. Furthermore, consider the following hypotheses: for each small enough, $\varepsilon > 0$ there exists* *two continuous and non-negative functions* φ_ε *and* ψ_ε *such that*

$$|j(a+b)-j(a)| \leq \varepsilon\varphi_\varepsilon(a)+\psi_\varepsilon(b) \tag{1.1.20}$$

for all $a,b \in \mathbb{C}$. *Consider* $f_n = f+g_n$ *a sequence of measurable function from* Ω *in* $\mathbb{C}$ *such that:*

(i) $g_n \to 0$ *a.e.;*

(ii) $j(f) \in L^1$;

(iii) $\int \varphi_\varepsilon(g_n(x))d\mu(x) \leq C < \infty$ *for some constant* C *independent of* ε *and* n;

(iv) $\int \psi_\varepsilon(f(x))d\mu(x) < \infty$ *for all* $\varepsilon > 0$.

Then, if $n \to \infty$,

$$\int |j(f+g_n)-j(g_n)-j(f)|d\mu \to 0. \tag{1.1.21}$$

Proof of Lemma 1.1.2: By the mean value theorem, there exists $0 < \theta < 1$ such that

$$|f(x,u_n)-f(x,u_n-u_0)| = |f'(x,u_n-u_0+\theta u_0)u_0| = |f'(x,u_n-(1-\theta)u_0)||u_0|.$$

Thus, fixed $R > 0$ and $\omega \in H^1(\mathbb{R}^N)$, we obtain by Hölder inequality and Sobolev's embedding, that

$$\begin{aligned}\left|\int_{|x|>R}|f(x,u_n)\right. &- \left. f(x,u_n-u_0)|\omega dx\right| \\ &\leq \int_{|x|>R}|f'(x,u_n-(1-\theta)u_0)||u_0||\omega|dx \\ &\leq \|f'(x,u_n-(1-\theta)u_0)\|_{L^2}\|\omega\|_{H^1(\mathbb{R}^N)}\left[\int_{|x|>R}|u_0|^2dx\right]^{1/2}.\end{aligned}$$

Again by Hölder inequality, by Sobolev's embedding, and using (1.1.3),

$$\begin{aligned}\left|\int_{|x|>R} f(x,u_0)\omega dx\right| &\leq \int_{|x|>R}|f(x,u_0)||\omega|dx \\ &\leq \varepsilon\int_{|x|>R}|u_0||\omega|dx + C\int_{|x|>R}|u_0|^{p-1}|\omega|dx \\ &\leq \varepsilon\|\omega\|_{L^2}\left(\int_{|x|>R}|u_0|^2dx\right)^{1/2} + C\|\omega\|_{L^{2-p}}\left(\int_{|x|>R}(|u_0|^{p-1})^{\frac{p-2}{p-1}}dx\right)^{\frac{p-1}{p-2}}\end{aligned}$$

$$\leq \quad \varepsilon\|\omega\|_{H^1(\mathbb{R}^N)}\left(\int_{|x|>R}|u_0|^2dx\right)^{1/2}+C\|\omega\|_{H^1(\mathbb{R}^N)}\left(\int_{|x|>R}|u_0|^{p-2}dx\right)^{\frac{p-1}{p-2}}.$$

Since for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{|x|>R}|u_0|^2dx,\ \int_{|x|>R}|u_0|^{p-2}dx < \varepsilon.$$

Then, for all $\omega \in H^1(\mathbb{R}^N)$, using the above inequalities, we obtain

$$\begin{aligned}
&\left|\int_{|x|>R}(f(x,u_n)-f(x,u_n-u_0)-f(x,u_0))wdx\right| \\
&\leq \int_{|x|>R}|f(x,u_n)-f(x,u_n-u_0)||w|dx+\int_{|x|>R}|f(x,u_0)||w|dx \\
&\leq C\|w\|_{H^1(\mathbb{R}^N)}\left[\int_{|x|>R}|u_0|^2dx\right]^{1/2}+\varepsilon\|w\|_{H^1(\mathbb{R}^N)}\left[\int_{|x|>R}|u_0|^2dx\right]^{1/2} \\
&+C\|w\|_{H^1(\mathbb{R}^N)}\left[\int_{|x|>R}|u_0|^{p-1}dx\right]^{\frac{p-1}{p-2}} \\
&\leq C\varepsilon\|w\|_{H^1(\mathbb{R}^N)}.
\end{aligned}$$

We claim that

$$f(x,u_n)-f(x,u_n-u_0)\to f(x,u_0),\quad \text{in } L^r(B_R(0)) := L^r(B), \tag{1.1.22}$$

with $r := \dfrac{p}{p-1}$. Assuming our statement above, we obtain that

$$\begin{aligned}
&\left|\int_{|x|<R}(f(x,u_n)-f(x,u_n-u_0)-f(x,u_0))wdx\right| \\
&\leq \|w\|_{L^{p+1}}\|f(x,u_n)-f(x,u_n-u_0)-f(x,u_0)\|_{L^r} \\
&\leq C\varepsilon\|w\|_{H^1(\mathbb{R}^N)}.
\end{aligned}$$

It remains to check (1.1.22). In fact, we have that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$, thus $u_n \to u_0$ in $L^q_{loc}(\mathbb{R}^N)$, for $1 \leq q < 2^*$. Therefore,

$$u_n-u_0\to 0,\ \text{in } L^q(B)\ \text{ and }\ u_n(x)\to u_0(x),\ \text{a.e. } x\in B_R(0).$$

It follows that

$$u_n(x) - (u_n - u_0)(x) \to u_0(x), \text{ a.e. } x \in B_R(0). \tag{1.1.23}$$

Also,

$$|u_n(x)|, |u_0(x)| \leq g(x), \quad g \in L^q_{loc}(\mathbb{R}^N)$$

and

$$|(u_n - u_0)(x)| \leq h(x), \quad h \in L^q_{loc}(\mathbb{R}^N).$$

Thus,

$$\begin{aligned}\Big|f(x,u_n) \; - \; & f(x,u_n-u_0) - f(x,u_0)|^{\frac{p}{p-1}} \leq \Big[\varepsilon|u_n(x)| + C|u_n(x)\Big|^{p-1} \\ & +\varepsilon|(u_n-u_0)(x)| + C|(u_n-u_0)(x)|^{p-1} + \varepsilon|u_0(x)| + C|u_0(x)|^{p-1}\Big]^{\frac{p}{p-1}} \\ \leq \; & 2^{\frac{p}{p-1}}\Big(\varepsilon|u_n(x)|^{\frac{p}{p-1}} + C|u_n(x)|^p + \varepsilon|(u_n-u_0)(x)|^{\frac{p}{p-1}} \\ & +C|(u_n-u_0)(x)|^p + \varepsilon|u_0(x)|^{\frac{p}{p-1}} + C|u_0(x)|^p\Big) \\ \leq \; & 2^{\frac{p}{p-1}}\Big(\varepsilon g(x)^{\frac{p}{p-1}} + Cg(x)^p + \varepsilon h(x)^{\frac{p}{p-1}} + Ch(x)^p + \varepsilon g(x)^{\frac{p}{p-1}} + Cg(x)^p\Big).\end{aligned}$$

If $1 < p < 2^* - 1$, then $g, h \in L^p(B)$. Therefore $g^{\frac{p}{p-1}}, g^p, h^{\frac{p}{p-1}}, h^p \in L^1(B)$. Combining this conclusion with (1.1.23) and the Lebesgue's Dominated Convergence Theorem, it follows that

$$f(x,u_n) - f(x,u_n-u_0) \to f(x,u_0), \text{ in } L^r(B),$$

where $r = \dfrac{p}{p-1}$ and the proof of (1.1.16) is complete.

Next, the main object is to apply the Brezis-Lieb Lemma with $j(s) = F(x,s)$. Since F is continuous and $F(0) = 0$, we will show that given $\varepsilon > 0$, there exist φ_ε and ψ_ε such that, $\varphi_\varepsilon(a) = C(|a|^2 + |a|^p)$ and $\psi_\varepsilon(b) = (C_\varepsilon + 1)(|b|^2 + |b|^p)$. In fact $0 \leq t \leq 1$, using (1.1.3) we have

$$\begin{aligned}|F(x,a-b) - F(x,a)| &= \left|\int_0^1 \frac{d}{dt}F(x,a-tb)dt\right| \\ &= \left|\int_0^1 f(x,a-tb)(-b)dt\right| \\ &\leq \int_0^1 |f(x,a-tb)||b|dt \\ &\leq \varepsilon\int_0^1 |a-tb||b|dt + C\int_0^1 |a-tb|^{p-1}|b|dt\end{aligned}$$

$$\begin{aligned} &\leq \varepsilon\int_0^1|a||b|dt+\varepsilon\int_0^1 t|b|^2dt+C\int_0^1|a|^{p-1}|b|dt+C\int_0^1 t^{p-1}|b|^p dt \\ &\leq \varepsilon|a||b|+\varepsilon|b|^2+C|a|^{p-1}|b|+C|b|^p \\ &\leq \varepsilon C(|a|^2+|a|^p)+(C_\varepsilon+1)(|b|^2+|b|^p). \end{aligned}$$

Then

$$\int_{\mathbb{R}^N}\Big(F(x,f+g_n)-F(x,g_n)-F(x,f)\Big)dx=o_n(1), \tag{1.1.24}$$

where $g_n=f_n-f\to 0$ a.e., with F,g_n and f is satisfying the items $(i),(iii)$ and (iv). Thus, we can rewrite (1.1.24) as

$$\int_{\mathbb{R}^N}\Big(F(x,f_n)-F(x,f)-F(x,f_n-f)\Big)dx=o_n(1).$$

Now considering $g_n=u_n-u_0$, with $f_n=u_n$ and $f=u_0$, we have

$$\int_{\mathbb{R}^N}\Big(F(x,u_n)-F(x,u_n-u_0)-F(x,u_0)\Big)dx\to 0.$$

The results (1.1.18) and (1.1.19) follow as an immediate consequence of (1.1.16) and (1.1.17). □

Let E be the Hilbert space $H^1(\mathbb{R}^N)$ with the inner product $\langle\cdot,\cdot\rangle$ given by the expression

$$\langle u,v\rangle=\int_{\mathbb{R}^N}(\xi(x)\nabla u\nabla v+uv)dx$$

and the norm by

$$\|u\|^2=\int_{\mathbb{R}^N}(\xi(x)|\nabla u|^2+u^2)dx, \tag{1.1.25}$$

which is equivalent to the usual norm because of (ξ_1) and (ξ_3). The functional $I:E\to\mathbb{R}$ associated with (P_1) is given by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}(\xi(x)|\nabla u|^2+u^2)dx-\int_{\mathbb{R}^N}F(x,u)dx \tag{1.1.26}$$

is well defined, belongs to $C^1(E,\mathbb{R})$ and

$$I'(u)\varphi=\int_{\mathbb{R}^N}(\xi(x)\nabla u\nabla\varphi+u\varphi)dx-\int_{\mathbb{R}^N}f(x,u)\varphi dx,\ \text{for all } u,\varphi\in E.$$

Hypotheses (ξ_3) and (f_3) imply

$$I(u) \leq I_\infty(u), \text{ for all } u \in E. \tag{1.1.27}$$

Indeed,

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(\xi(x)|\nabla u|^2+u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx \\
&\leq \frac{1}{2}\int_{\mathbb{R}^N}(\xi_\infty|\nabla u|^2+u^2)dx - \int_{\mathbb{R}^N}\left(\int_0^u \frac{f(x,s)}{s} s ds\right)dx \\
&\leq \frac{1}{2}\int_{\mathbb{R}^N}(\xi_\infty|\nabla u|^2+u^2)dx - \int_{\mathbb{R}^N}\left(\int_0^u h(s)sds\right)dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N}(\xi_\infty|\nabla u|^2+u^2)dx - \int_{\mathbb{R}^N} H(u)dx \\
&= I_\infty(u), \quad \text{for all } \ u \in E.
\end{aligned}$$

Now, let $z_0 = 0$ and fix $L > L_0$ such that $z_1 := w\left(\frac{\cdot}{L}\right)$ and $I_\infty(z_1) < 0$. Define also

$$c := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \tag{1.1.28}$$

where $\Gamma = \{\gamma \in C([0,1],E),\ \gamma(0) = z_0 \text{ and } \gamma(1) = z_1\}$.

Definition 1.1.1. *A functional $I \in C^1(E,\mathbb{R})$ in a Hilbert space E satisfies the Palais-Smale condition, denoted (PS) condition, if given any sequence $(u_n) \subset E$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$, has a convergent subsequence. We say that (u_n) is a Cerami sequence at level c, denoted by $(Ce)_c$, if*

$$I(u_n) \to c \quad \textit{and} \quad (1 + \|u_n\|)\|I'(u_n)\| \to 0. \tag{1.1.29}$$

Moreover, I satisfies the Cerami sequence condition at level c, shortly $(Ce)_c$, if any Cerami sequence $(u_n) \subset E$ at level c has a convergent subsequence.

Lemma 1.1.4. *If (u_n) is a $(Ce)_c$ sequence of the functional I_∞, then (u_n) is bounded.*

Proof. This proof will be postponed to Lemma 1.3.1. □

Remark 1.1.6. *If (u_n) is a Cerami sequence $(Ce)_c$ and is a bounded, then (u_n) is bounded (PS) sequence.*

Lemma 1.1.5 (Splitting)**.** *Let* $(u_n) \subset E$ *be a sequence such that* $I(u_n) \to c$ *and* $I'(u_n) \to 0$ *in* E^**. Then there exists* $u_0 \in E$ *such that* $u_n \rightharpoonup u_0$*,* $I'(u_0) = 0$ *and either*

(a) $u_n \to u_0$ *strongly in* E*, or*

(b) there exist $k \in \mathbb{N}$*,* $(y_n^j) \in \mathbb{R}^N$ *with* $|y_n^j| \to \infty$ *and* $|y_n^j - y_n^{j'}| \to \infty$*, for* $j \neq j'$*,* $j = 1, ..., k$*, and nontrivial solutions* $u^1, ..., u^k$ *of problem* (1.1.4)*, such that*

$$I(u_n) \to I(u_0) + \sum_{j=1}^{k} I_\infty(u^j) \quad \textit{and} \quad \left\| u_n - u_0 - \sum_{j=1}^{k} u^j(\cdot - y_n^j) \right\| \to 0. \tag{1.1.30}$$

Proof. Step 1) Since (u_n) is bounded, then there exists $u_0 \in E$ such that $u_n \rightharpoonup u_0$. Let us prove that $I'(u_0) = 0$. In fact, $E \hookrightarrow L^p_{loc}(\mathbb{R}^N)$ is compactly embedded if $1 \leq p < 2^* - 1$. Using the continuity of f, the weak convergence $u_n \rightharpoonup u_0$ in E and the Lebesgue dominated convergence theorem, it follows that $\lim_{n\to\infty} I'(u_n)\varphi = I'(u_0)\varphi$, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. The hypothesis that $\lim_{n\to\infty} I'(u_n)\varphi = 0$, for all $\varphi \in C_0^\infty(\mathbb{R}^N)$ and due to the uniqueness of the limit, we have $I'(u_0)\varphi = 0$, for all $\varphi \in C_0^\infty(\mathbb{R}^N)$.

Step 2) Define now $u_n^1 := u_n - u_0 \in H^1(\mathbb{R}^N)$. If $n \to \infty$, then:

(i) $\|u_n^1\|^2 = \|u_n\|^2 - \|u_0\|^2 + o_n(1)$;

(ii) $I_\infty(u_n^1) \to c - I(u_0)$;

(iii) $I'_\infty(u_n^1) \to 0$.

To prove (i), note that $u_n^1 + u_0 = (u_n - u_0) + u_0 = u_n$. Therefore,

$$\|u_n^1 + u_0\|^2 = \langle u_n^1 + u_0, u_n^1 + u_0 \rangle = \|u_n\|^2 + \|u_0\|^2 + 2\langle u_n^1, u_0 \rangle.$$

Since $u_n^1 \rightharpoonup 0$ and using the Riez Representation theorem [7], it follows that $\langle u_n^1, u_0 \rangle = g(u_n^1) \to 0$ for all $g \in H^{-1}(\mathbb{R}^N)$. Hence

$$\|u_n^1\|^2 = \|u_n\|^2 - \|u_0\|^2 + 2g(u_n)$$

implies that

$$\|u_n^1\|^2 = \|u_n\|^2 - \|u_0\|^2 + o_n(1).$$

To prove item (ii), note that the weak convergence of (u_n) for u_0 implies $u_n^1 \rightharpoonup 0$,

$$\int_{\mathbb{R}^N}\Big(\xi_\infty|\nabla(u_n-u_0)|^2-\xi(x)|\nabla u_n|^2+\xi(x)|\nabla u_0|^2\Big)dx$$
$$=\int_{\mathbb{R}^N}(\xi_\infty-\xi(x))(|\nabla u_n|^2-|\nabla u_0|^2)dx+o_n(1) \tag{1.1.31}$$

and

$$\int_{\mathbb{R}^N}\Big((u_n-u_0)^2-u_n^2+u_0^2\Big)\,dx=o_n(1). \tag{1.1.32}$$

From (1.1.31) and (1.1.32), it follows that

$$\begin{aligned}
I_\infty(u_n^1) \; - \; I(u_n)+I(u_0) &= \frac{1}{2}\int_{\mathbb{R}^N}\xi_\infty|\nabla u_n^1|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}(u_n^1)^2dx-\int_{\mathbb{R}^N}H(u_n^1)dx\\
&-\frac{1}{2}\int_{\mathbb{R}^N}\xi(x)|\nabla u_n|^2dx-\frac{1}{2}\int_{\mathbb{R}^N}u_n^2dx+\int_{\mathbb{R}^N}F(x,u_n)dx\\
&+\frac{1}{2}\int_{\mathbb{R}^N}\xi(x)|\nabla u_0|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}u_0^2dx-\int_{\mathbb{R}^N}F(x,u_0)dx\\
&= \frac{1}{2}\int_{\mathbb{R}^N}\Big(\xi_\infty|\nabla u_n-\nabla u_0|^2-\xi(x)|\nabla u_n|^2+\xi(x)|\nabla u_0|^2\Big)dx\\
&+\frac{1}{2}\int_{\mathbb{R}^N}\Big((u_n-u_0)^2-u_n^2+u_0^2\Big)dx+\int_{\mathbb{R}^N}\Big(F(x,u_n)-F(x,u_0)-H(u_n^1)\Big)dx\\
&= \int_{\mathbb{R}^N}\Big(F(x,u_n^1)-H(u_n^1)\Big)dx+o_n(1).
\end{aligned} \tag{1.1.33}$$

Since (u_n) is bounded, using the hypothesis (f_2) we have

$$\int_{\mathbb{R}^N}\Big(H(u_n^1)-F(x,u_n^1)\Big)dx=o_n(1).$$

Replacing in (1.1.33) we obtain

$$I_\infty(u_n^1)-I(u_n)+I(u_0)=o_n(1). \tag{1.1.34}$$

To verify (iii), consider $\varphi\in C_0^\infty(\mathbb{R}^N)$. Applying (f_1), (f_2), (1.1.16) and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
o_n(1) &= \Big\langle I'(u_n),\varphi\Big\rangle=\Big\langle I'(u_0+u_n^1),\varphi\Big\rangle\\
&= \int_{\mathbb{R}^N}(\xi(x)\nabla(u_0+u_n^1)\nabla\varphi+(u_0+u_n^1)\varphi)dx-\int_{\mathbb{R}^N}f(x,u_0+u_n^1)(u_0+u_n^1)\varphi dx\\
&= \int_{\mathbb{R}^N}(\xi(x)\nabla u_0\nabla\varphi+u_0\varphi)dx-\int_{\mathbb{R}^N}f(x,u_0)u_0\varphi dx+\int_{\mathbb{R}^N}(\xi(x)\nabla u_n^1\nabla\varphi
\end{aligned}$$

$$
\begin{aligned}
& -u_n^1\varphi)dx - \int_{\mathbb{R}^N} h(u_n^1)u_n^1\varphi dx + \int_{\mathbb{R}^N} f(x,u_0)u_0\varphi dx + \int_{\mathbb{R}^N} h(u_n^1)u_n^1\varphi dx \\
& - \int_{\mathbb{R}^N} f(x,u_0+u_n^1)(u_0+u_n^1)\varphi dx \\
= \;& \langle I'(u_0),\varphi\rangle + \int_{\mathbb{R}^N}(\xi_\infty \nabla u_n^1 \nabla\varphi + u_n^1\varphi)dx - \int_{\mathbb{R}^N} h(u_n^1)u_n^1 dx \\
= \;& \langle I'_\infty(u_n^1),\varphi\rangle - \int_{\mathbb{R}^N} f(x,u_n^1)\varphi dx + o_n(1) + \int_{\mathbb{R}^N} h(u_n^1)u_n^1\varphi dx \\
= \;& \langle I'_\infty(u_n^1),\varphi\rangle + \left[\int_{\mathbb{R}^N}\Big(h(u_n^1)u_n^1\varphi - f(x,u_n^1)\varphi\Big)dx\right] + o_n(1),
\end{aligned}
$$

since φ has compact support, $u_n^1 \to 0$ in the support and then $I'_\infty(u_n^1)\to 0$ in E^* when $n\to\infty$. And then, (u_n^1) is a $(PS)_c$ sequence of I_∞.

Step 3) Consider

$$\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n^1(x)|^2dx.$$

If $\delta = 0$, it follows from Lions' Lemma [24] that

$$u_n^1 \to 0, \quad \text{in} \quad L^p(\mathbb{R}^N), \quad \text{for any} \quad 2 < p < 2^*. \tag{1.1.35}$$

On the other hand, since (u_n^1) is bounded, item (iii) implies that

$$I'_\infty(u_n^1)u_n^1 = \int_{\mathbb{R}^N}\Big(\xi_\infty|\nabla u_n^1|^2 + (u_n^1)^2 - h(u_n^1)(u_n^1)^2\Big)dx \to 0, \quad \text{if } n\to\infty. \tag{1.1.36}$$

From (1.1.35) and (1.1.3), we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N}\Big(\xi_\infty|\nabla u_n^1|^2 + (u_n^1)^2\Big)dx \;&=\; \int_{\mathbb{R}^N} h(u_n^1)(u_n^1)^2dx + o_n(1) \\
&\le\; \varepsilon\int_{\mathbb{R}^N}|u_n^1|^2dx + C\int_{\mathbb{R}^N}|u_n^1|^pdx.
\end{aligned} \tag{1.1.37}
$$

Therefore, (1.1.35) and (1.1.37) give us that $\left\|u_n^1\right\| \to 0$. In other words, $u_n \to u_0$ strongly in E, and this proof the item (a).

Step 4) Now, if $\delta > 0$, there is a sequence $(y_n)\subset\mathbb{R}^N$ such that

$$\int_{B_1(y_n)}|u_n^1(x)|^2dx > \frac{\delta}{2}. \tag{1.1.38}$$

Define a new sequence $(v_n^1)\subset E$ by $v_n^1 := u_n^1(\cdot + y_n^1)$. Since (u_n^1) is bounded, then (v_n^1) is also bounded $u^1\in E$ such that $v_n^1 \rightharpoonup u^1$ in E and $v_n^1(x)\to u^1(x)$ at almost every point in $x \in \mathbb{R}^N$. Making a change of variable, we obtain

$$\frac{\delta}{2} < \int_{B_1(y_n^1)} |u_n^1(x)|^2 dx = \int_{B_1(0)} |u_n^1(x+y_n^1)|^2 dx = \int_{B_1(0)} |v_n^1(x)|^2 dx. \tag{1.1.39}$$

Applying Fatou's Lemma [5],

$$\frac{\delta}{2} \le \int_{B_1(0)} \liminf_{n\to\infty} |v_n^1(x)|^2 dx = \int_{B_1(0)} |u^1(x)|^2 dx.$$

Thus, $u^1 \neq 0$. Moreover, since $u_n^1 \rightharpoonup 0$ in E, it follows that up to a subsequence, we can assume that $|y_n^1| \to \infty$. Now, we will show that $I'_\infty(u^1) = 0$. In fact, take $\phi \in C_0^\infty(\mathbb{R}^N)$. Since $|y_n^1| \to \infty$, then we can find n_0 such that $\phi_n := \phi(x - y_n^1)$ in $C_0^\infty(\mathbb{R}^N)$ for all $n \ge n_0$. Besides that, $\|\phi_n\| = \|\phi\|$. As a consequence of item (iii),

$$\begin{aligned}
\sup_{\|\phi\|\le 1} \left| \left\langle I'_\infty(v_n^1), \phi \right\rangle \right| &= \sup_{\|\phi\|\le 1} \left| \left\langle I'_\infty(u_n^1(x+y_n^1), \phi \right\rangle \right| \\
&= \sup_{\|\phi\|\le 1} \left| \left\langle I'_\infty(u_n^1(x)), \phi(x-y_n^1) \right\rangle \right| \\
&\le \sup_{\|\phi\|\le 1} \left| \left\langle I'_\infty(u_n^1), \phi \right\rangle \right| = o_n(1).
\end{aligned}$$

Therefore, using the fact that $u_n^1(\cdot + y_n^1) \rightharpoonup u^1$, for all $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$o_n(1) = I'_\infty(u_n^1(\cdot + y_n^1))\phi = I'_\infty(u^1)\phi + o_n(1)\,. \tag{1.1.40}$$

Define $u_n^2(x) := u_n^1(x) - u^1(x - y_n^1)$, and $u_n^2(\cdot + y_n^2) = v_n^1 + u^1$, then (u_n^2) is a (PS) sequence of I_∞. Indeed, making a change of variables,

$$\begin{aligned}
I_\infty(u_n^2) &= \frac{1}{2}\int_{\mathbb{R}^N} \left[\xi_\infty |\nabla u_n^2|^2 + (u_n^2)^2\right] dx - \int_{\mathbb{R}^N} H(u_n^2) dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \left[\xi_\infty |\nabla (u_n^1(x) - u^1(x-y_n^1))|^2 + |u_n^1(x) - u^1(x-y_n^1)|^2\right] dx \\
&\quad - \int_{\mathbb{R}^N} H(u_n^1(x) - u^1(x-y_n^1)) dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \left[\xi_\infty |\nabla (u_n^1(x+y_n^1) - u^1(x))|^2 + |u_n^1(x+y_n^1) - u^1(x)|^2\right] dx \\
&\quad - \int_{\mathbb{R}^N} H(u_n^1(x+y_n^1) - u^1(x)) dx.
\end{aligned}$$

On the other hand,

$$\|u_n^1(\cdot+y_n^1)-u^1\|^2 = \|u_n^1(\cdot+y_n^1)\|^2 - 2\langle u_n^1(\cdot+y_n^1),u^1\rangle + \|u^1\|^2. \tag{1.1.41}$$

Since $u_n^1(\cdot+y_n^1) \rightharpoonup u^1$ in E, $\langle u_n^1(\cdot+y_n^1),\varphi\rangle \to \langle u^1,\varphi\rangle$, for all $\varphi\in E$. In particular, if $\varphi=u^1$, we have $\langle u_n^1(\cdot+y_n^1),u^1\rangle \to \langle u^1,u^1\rangle$, which it follows that $\langle u_n^1(\cdot+y_n^1),u^1\rangle = \|u^1\|^2+o_n(1)$. Replacing in (1.1.41), we obtain

$$\|u_n^1(\cdot+y_n^1)-u^1\|^2 = \|u_n^1(\cdot+y_n^1)\|^2 - 2\|u^1\|^2 + o_n(1) + \|u^1\|^2 = \|u_n^1\|^2 - \|u^1\|^2 + o_n(1). \tag{1.1.42}$$

Therefore,

$$\begin{aligned} I_\infty(u_n^1)-I_\infty(u_n^2)-I_\infty(u^1) &= \frac{1}{2}\left(\left\|u_n^1\right\|^2 - \left\|u_n^1-u^1\right\|^2 - \left\|u^1\right\|^2\right) \\ &\quad - \int_{\mathbb{R}^N}\Big(H(u_n^1)-H(u_n^2)-H(u^1)\Big)dx, \end{aligned}$$

and using (f_3), (1.1.42) and Lemma 1.1.2, it follows

$$I_\infty(u_n^2) = I_\infty(u_n^1) - I_\infty(u^1) + o_n(1). \tag{1.1.43}$$

By (ii) and (iii), (u_n^1) is a (PS) sequence of I_∞, hence $I_\infty(u_n^2)$ converges to a constant.

Finally, using (f_2), (f_3) and Lemma 1.1.2, from (iii) and (1.1.40), we obtain

$$\begin{aligned} |I'_\infty(u_n^2)\varphi| &= \Bigg| \int_{\mathbb{R}^N}(\xi_\infty\nabla u_n^1\nabla\varphi + u_n^1\varphi)dx - \int_{\mathbb{R}^N}(\xi_\infty\nabla u^1\nabla\varphi + u^1\varphi)dx \\ &\quad - \int_{\mathbb{R}^N} h(u^1)u_n^1\varphi dx + \int_{\mathbb{R}^N} h(u^1)u^1\varphi dx - \int_{\mathbb{R}^N} h(u_n^1-u^1)(u_n^1-u^1)\varphi dx \\ &\quad + \int_{\mathbb{R}^N} h(u_n^1)u_n^1\varphi dx + \int_{\mathbb{R}^N} h(u^1)u^1\varphi dx \Bigg| \\ &= o_n(1) + \int_{\mathbb{R}^N} |h(u_n^1)u_n^1 - h(u_n^1-u^1)(u_n^1-u^1) - h(u^1)u^1||\varphi|dx \\ &= o_n(1), \end{aligned} \tag{1.1.44}$$

for all $\varphi\in C_0^\infty(\mathbb{R}^N)$. Therefore (u_n^2) is a (PS) sequence of I_∞.

Step 5) Now we proceed by iteration. Note that if u is a nontrivial critical point of I_∞ and ω is the solution (1.1.4), then

$$I_\infty(u) \geq I_\infty(\omega) > 0. \tag{1.1.45}$$

Furthermore, by (1.1.34) and (1.1.43),

$$I_\infty(u_n^2) = c - I(u_0) - I_\infty(u^1) + o_n(1). \tag{1.1.46}$$

Applying (1.1.45) and (1.1.46) the iteration must be terminated at some index $k \in \mathbb{N}$. Therefore, there exist k solutions to the problem (1.1.4), thus satisfying the second part of the lemma. □

1.2 Existence of a positive solution

Lemma 1.2.1. *The functional I satisfies $(Ce)_c$ for all $0 \leq c < m_\infty$.*

Proof. Consider $(u_n) \subset E$ and $0 \leq c < m_\infty$ such that

$$I(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|) \left\| I'(u_n) \right\| \to 0.$$

By Lemma 1.1.4, (u_n) is bounded in E and taking a subsequence if necessary, $u_n \rightharpoonup u_0$ in E. Lemma 1.1.5 gives $I'(u_0) = 0$ and by condition (f_5)

$$\begin{aligned} I(u_0) &= \frac{1}{2}\int_{\mathbb{R}^N} (\xi(x)|\nabla u_0|^2 + u_0^2)dx - \int_{\mathbb{R}^N} F(x,u_0)dx \\ &= \int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u_0)u_0 - F(x,u_0)\right) dx \\ &= \int_{\mathbb{R}^N} Q(x,u_0)dx \geq 0. \end{aligned} \tag{1.2.1}$$

If u_n does not converge to u_0 in E, applying Lemma 1.1.5 we find $k \in \mathbb{N}$ and nontrivial solutions $u^1, \ldots, u^k$ of (1.1.4) satisfying

$$c = \lim_{n \to \infty} I(u_n) = I(u_0) + \sum_{j=1}^{k} I_\infty(u^j) \geq k m_\infty \geq m_\infty,$$

thus contradicting the assumption. Therefore $u_n \to u_0$ in E. □

Remark 1.2.1. *For each $u \in E\backslash\{0\}$ such that $\int_{\mathbb{R}^N} G_\infty(u)dx > 0$, there exists a unique real number $t > 0$ such that $u\left(\frac{\cdot}{t}\right) \in \mathcal{P}$ and $I_\infty\left(u\left(\frac{\cdot}{t}\right)\right)$ is the maximum of the function*

$$t \mapsto I_\infty\left(u\left(\frac{\cdot}{t}\right)\right), \quad t > 0.$$

In fact, consider the function g defined by

$$g(t) := I_\infty\left(u\left(\frac{\cdot}{t}\right)\right) = \frac{1}{2}\int_{\mathbb{R}^N}\left(\xi_\infty\left|\nabla u\left(\frac{\cdot}{t}\right)\right|^2 + \left(u\left(\frac{\cdot}{t}\right)\right)^2\right)dx - \int_{\mathbb{R}^N} H\left(u\left(\frac{\cdot}{t}\right)\right)dx$$

making changes of variable

$$\begin{aligned} f : &\mathbb{R}^N \to \mathbb{R}^N \\ &x \mapsto tx, \end{aligned}$$

the determinant of the Jacobian of this change of variable is $|J(x_1, \cdots, x_N)| = t^N$. Thus, by the change of variable theorem

$$\begin{aligned} \int_{\mathbb{R}^N}\left|\nabla u\left(\frac{x}{t}\right)\right|^2 dx &= \int_{\mathbb{R}^N}\left|\frac{1}{t}\nabla u(x)\right|^2 t^N dx = \int_{\mathbb{R}^N}\frac{1}{t^2}t^N|\nabla u(x)|^2 dx \\ &= t^{N-2}\int_{\mathbb{R}^N}|\nabla u(x)|^2 dx, \\ \int_{\mathbb{R}^N}\left|u\left(\frac{x}{t}\right)\right|^2 dx &= \int_{\mathbb{R}^N}|u(x)|^2 t^N dx = t^N\int_{\mathbb{R}^N}|u(x)|^2 dx, \\ \int_{\mathbb{R}^N} H\left(u\left(\frac{x}{t}\right)\right)dx &= \int_{\mathbb{R}^N} H(u(x))t^N dx = t^N\int_{\mathbb{R}^N} H(u(x))dx. \end{aligned}$$

It follows from this that the function g can be rewritten as

$$g(t) = \frac{t^{N-2}}{2}\int_{\mathbb{R}^N}\xi_\infty|\nabla u|^2 dx + \frac{t^N}{2}\int_{\mathbb{R}^N}|u|^2 dx - t^N\int_{\mathbb{R}^N} H(u)dx.$$

Then $g'(t) = 0$ if and only if, $t = 0$ or

$$\begin{aligned} 0 = g'(t) = \frac{N-2}{2}t^{N-3}\int_{\mathbb{R}^N}\xi_\infty|\nabla u|^2 dx &+ \frac{N}{2}t^{N-1}\int_{\mathbb{R}^N}|u|^2 dx - Nt^{N-1}\int_{\mathbb{R}^N} H(u)dx \\ t^{N-1}N\int_{\mathbb{R}^N}\left(H(u) - \frac{1}{2}|u|^2\right)dx &= \frac{N-2}{2}t^{N-3}\int_{\mathbb{R}^N}\xi_\infty|\nabla u|^2 dx \end{aligned}$$

$$t^2 = \frac{N-2\int_{\mathbb{R}^N}\xi_\infty|\nabla u|^2dx}{2N\int_{\mathbb{R}^N}G_\infty(u)dx}.$$

Let $\omega \in \mathcal{P}$ be a positive, radial, ground state solution of equation (1.1.4) and

$$\omega_y(x) := \omega(x-y), \tag{1.2.2}$$

for some $y \in \mathbb{R}^N$ fixed.

Remark 1.2.2. *The inequality*

$$\int_{\mathbb{R}^N} G_\infty(\omega_y)dx > 0, \tag{1.2.3}$$

if $|y| > 0$ *is big enough. Indeed,*

$$\int_{\mathbb{R}^N} G_\infty(\omega_y(x))dx = \int_{\mathbb{R}^N} G_\infty(\omega(x-y))dx = \int_{\mathbb{R}^N} G_\infty(\omega(x))dx > 0,$$

where we have used the translation invariance of the integrals and that the solution ω *of* (1.1.4) *satisfies Pohozaev identity and so* $\int_{\mathbb{R}^N} G_\infty(\omega(x))dx > 0$.

Lemma 1.2.2. *Suppose* (ξ_3) *and* (f_3)*, then* c *defined as in* (1.1.28) *satisfies*

$$0 < c < m_\infty.$$

Proof. From Remark 1.2.2, $\int_{\mathbb{R}^N} G_\infty(\omega_y)dx > 0$, follows from Remark 1.2.1, (1.1.6) and (1.1.1) that there exists $0 \le t_y \le L_0$ such that

$$\max_{0\le t\le L_0} I\left(\omega_y\left(\frac{\cdot}{t}\right)\right) = I\left(\omega_y\left(\frac{\cdot}{t_y}\right)\right) = I\left(\omega\left(\frac{\cdot}{t_y}-y\right)\right).$$

Furthermore, using (ξ_3), (f_3), (1.1.27) and the translation invariance of the integral

$$\begin{aligned} I\left(\omega_y\left(\frac{\cdot}{t_y}\right)\right) &< I_\infty\left(\omega_y\left(\frac{\cdot}{t_y}\right)\right) = I_\infty\left(\omega\left(\frac{\cdot}{t_y}-y\right)\right) \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left(\xi_\infty\left|\nabla\omega\left(\frac{\cdot}{t_y}-y\right)\right|^2 + \left|\omega\left(\frac{\cdot}{t_y}-y\right)\right|^2\right)dx \\ &- \int_{\mathbb{R}^N} H\left(\omega\left(\frac{\cdot}{t_y}-y\right)\right)dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^N}\left(\xi_\infty\left|\nabla\omega\left(\frac{\cdot}{t_y}\right)\right|^2+\left|\omega\left(\frac{\cdot}{t_y}\right)\right|^2\right)dx-\int_{\mathbb{R}^N}H\left(\omega\left(\frac{\cdot}{t_y}\right)\right)dx\\
&= I_\infty\left(\omega\left(\frac{\cdot}{t_y}\right)\right)\le I_\infty(\omega)=m_\infty.
\end{aligned}
$$

In order to conclude, we construct a path $\gamma\in\Gamma$ such that

$$\max_{0\le t\le L_0} I(\gamma(t))=I\left(\omega_y\left(\frac{\cdot}{t_y}\right)\right)<m_\infty,$$

for Γ defined in (1.1.28). Since we assumed that $L>L_0$, then we have

$$I\left(\omega_y\left(\frac{\cdot}{L}\right)\right)<I_\infty\left(\omega_y\left(\frac{\cdot}{L}\right)\right)=I_\infty\left(\omega\left(\frac{\cdot}{L}\right)\right)=I_\infty(z_1)<0.$$

Consider

$$\kappa(t):=\omega\left(\frac{\cdot}{L}t+(1-t)\left(\frac{\cdot}{L}-y\right)\right).$$

Then $\kappa(0)=\omega_y\left(\frac{\cdot}{L}\right)$ and $\kappa(1)=\omega\left(\frac{\cdot}{L}\right)=z_1$ and hence $\kappa(t)$ is a path which connects $\omega_y\left(\frac{\cdot}{L}\right)$ to z_1. Furthermore, using (ξ_3), (f_3) and the translation invariance of I_∞, we obtain

$$
\begin{aligned}
I(\kappa(t)) &= I\left(\omega\left(\frac{\cdot}{L}t+(1-t)\left(\frac{\cdot}{L}-y\right)\right)\right)\\
&= I\left(\omega\left(\frac{\cdot}{L}+y(t-1)\right)\right)\\
&< I_\infty\left(\omega\left(\frac{\cdot}{L}+y(t-1)\right)\right)\\
&= I_\infty\left(\omega\left(\frac{\cdot}{L}\right)\right)\\
&= I_\infty(z_1)<0.
\end{aligned}
$$

Thus, the functional I is negative along the path $\kappa(t)$. Consider $\bar\phi$ the path given by

$$\bar\phi(t):=\begin{cases} z_0=0, & if\ \ t=0,\\ \omega_y\left(\frac{\cdot}{t}\right), & if\ \ 0<t\le L,\end{cases}$$

then $\bar\phi$ is a path connecting $z_0=0$ to $\omega_y\left(\frac{\cdot}{L}\right)$, trough $\omega_y\left(\frac{\cdot}{t_y}\right)$, because $0<t_y\le L_0<L$. Take $\gamma(t)$ the succession of the paths $\bar\phi(t)$ and $\kappa(t)$, then $\gamma(t)\in\Gamma$ and by (ξ_3) and (f_3) it

follows

$$\max_{0\le t\le L_0} I(\gamma(t)) = I\left(\omega_y\left(\frac{\cdot}{t_y}\right)\right) < I_\infty\left(\omega_y\left(\frac{\cdot}{t_y}\right)\right) \le I_\infty(\omega) = m_\infty,$$

which yields

$$c < m_\infty.$$

□

Lemma 1.2.3. *If F satisfies (1.1.3), then there exists $\rho > 0$ and $\alpha > 0$ such that $I(u) \ge \alpha > 0$, for all $u \in E$ with $\|u\| = \rho$.*

Proof. From (1.1.3), Sobolev's embedding for $2 < p < 2^*$, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(\xi(x)|\nabla u|^2+|u|^2)dx - \int_{\mathbb{R}^N} F(x,u)dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^N}|u|^2dx - C\int_{\mathbb{R}^N}|u|^p dx \\ &\ge \left(\frac{1}{2}-\frac{\varepsilon}{2}\right)\|u\|^2 - C\|u\|^p. \end{aligned}$$

For $\|u\| = \rho$ we obtain

$$I(u) \ge \left(\frac{1}{2}-\frac{\varepsilon}{2}\right)\rho^2 - C\rho^p = \alpha > 0,$$

for $\rho = \|u\|$ small enough. □

Remark 1.2.3. *Since $I(u) \le I_\infty(u)$ for all $u \in E$, then there exists $z_1 \in E\setminus B_\rho(0)$ such that $I(z_1) \le I_\infty(z_1) < 0$.*

Lemma 1.2.4. *Let v_n be a solution of the following problem*

$$\begin{cases} -div(\xi(x)\nabla v_n) + v_n = f(x,v_n), & in \quad \mathbb{R}^N, \\ v_n \in H^1(\mathbb{R}^N), \quad with \quad N \ge 3, \\ v_n(x) \ge 0, \quad for\ all \quad x \in \mathbb{R}^N. \end{cases}$$

Assuming that $(\xi_1)-(\xi_4)$, $(f_1)-(f_5)$ holds and that $v_n \to v$ in $H^1(\mathbb{R}^N)$ with $v \not\equiv 0$, then $v_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $\|v_n\|_{L^\infty} \le C$ for all $n \in \mathbb{N}$. Furthermore,

$$\lim_{|x|\to\infty} v_n(x) = 0, \text{ uniformly in } n.$$

Proof. For any $R > 0$, $0 < r \le R/2$, let $\eta \in C^\infty(\mathbb{R}^N)$, $0 \le \eta \le 1$ with $\eta(x) = 1$ if $|x| \ge R$ and $\eta(x) = 0$ if $|x| \le R - r$ and $|\nabla\eta| \le 2/r$. Note that, by Remark 1.1.2 and by Sobolev's embedding for $2 \leq p \leq 2^*$, we obtain the following growth condition for f:

$$f(x,s) \leq \varepsilon|s| + C_\varepsilon |s|^{p-1} \leq \varepsilon|s| + C_\varepsilon |s|^{2^*-1}. \tag{1.2.4}$$

For each $n \in \mathbb{N}$ and for $L > 0$, let

$$v_{L,n}(x) = \begin{cases} v_n(x), & v_n(x) \leq L, \\ L, & v_n(x) \geq L, \end{cases}$$

$z_{L,n} = \eta^2 v_{L,n}^{2(\beta-1)} v_n$ and $w_{L,n} = \eta v_n v_{L,n}^{\beta-1}$ with $\beta > 1$ to be determined later. Taking $z_{L,n}$ as a test function, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 dx &= -2(\beta-1)\int_{\mathbb{R}^N} \xi(x) v_{L,n}^{2\beta-3}\eta^2 v_n \nabla v_n \nabla v_{L,n} dx \\ &\quad + \int_{\mathbb{R}^N} f(x,v_n)\eta^2 v_n v_{L,n}^{2(\beta-1)} dx - \int_{\mathbb{R}^N} v_n^2 \eta^2 v_{L,n}^{2(\beta-1)} dx \\ &\quad -2\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla \eta dx. \end{aligned}$$

Note that, $-2(\beta-1)\int_{\mathbb{R}^N} \xi(x) v_{L,n}^{2\beta-3}\eta^2 v_n \nabla v_n \nabla v_{L,n} dx \leq 0$, then

$$\begin{aligned} \int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 dx &\leq -2\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla \eta dx - \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^2 dx \\ &\quad + \int_{\mathbb{R}^N} f(x,v_n)\eta^2 v_n v_{L,n}^{2(\beta-1)} dx. \end{aligned}$$

By (1.2.4) and for ε sufficiently small, we have the following inequality

$$\begin{aligned} \int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 dx &\leq -2\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla \eta dx - \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^2 dx \\ &\quad + \varepsilon\int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^2 dx + C_\varepsilon \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^{2^*} dx \\ &\leq -2\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla \eta dx + C_\varepsilon \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^{2^*} dx \\ &\leq C_\varepsilon \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^{2^*} dx + 2\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla \eta dx. \end{aligned}$$

For each $\varepsilon > 0$, using the Young's inequality we get

$$\begin{aligned} \int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 dx &\leq C_\varepsilon \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^{2^*} dx + 2\varepsilon \int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 dx \\ &\quad + 2C_\varepsilon \int_{\mathbb{R}^N} \xi(x) v_n^2 v_{L,n}^{2(\beta-1)} |\nabla \eta|^2 dx. \end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small,

$$\int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)}|\nabla v_n|^2 dx \leq C\int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^{2^*} dx + C\int_{\mathbb{R}^N} \xi(x) v_n^2 v_{L,n}^{2(\beta-1)}|\nabla\eta|^2 dx. \tag{1.2.5}$$

Now, from Sobolev's embedding, by (1.2.5) and by (ξ_1) we have

$$\begin{aligned} \xi_0\|w_{L,n}\|_{L^{2^*}}^2 &\leq \int_{\mathbb{R}^N} \xi(x)\eta^2 v_n^2 v_{L,n}^{2(\beta-1)} dx \leq \int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)}|\nabla v_n|^2 dx \\ &\leq C\Big[\int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^{2^*} dx + \int_{\mathbb{R}^N} \xi(x) v_n^2 v_{L,n}^{2(\beta-1)}|\nabla\eta|^2 dx\Big]. \end{aligned} \tag{1.2.6}$$

We claim that $v_n \in L^{\frac{2^{*2}}{2}}(|x|\geq R)$ for R large enough and uniformly in n. In fact, $\beta = 2^*/2$, from (1.2.6), we have

$$\xi_0\|w_{L,n}\|_{L^{2^*}}^2 \leq C\Big[\int_{\mathbb{R}^N} \eta^2 v_{L,n}^{(2^*-2)} v_n^{2^*} dx + \int_{\mathbb{R}^N} \xi(x) v_n^2 v_{L,n}^{(2^*-2)}|\nabla\eta|^2 dx\Big] \tag{1.2.7}$$

or equivalently, using (ξ_3) we obtain

$$\xi_0\|w_{L,n}\|_{L^{2^*}}^2 \leq C\Big[\int_{\mathbb{R}^N} \xi(x) v_n^2 v_{L,n}^{(2^*-2)}|\nabla\eta|^2 dx + \int_{\mathbb{R}^N} \eta^2 v_n^2 v_{L,n}^{(2^*-2)} v_n^{(2^*-2)} dx\Big].$$

Using the Hölder inequality with exponent $2^*/2$ and $2^*/(2^*-2)$

$$\begin{aligned} \xi_0\|w_{L,n}\|_{L^{2^*}}^2 &\leq C\int_{\mathbb{R}^N} v_n^2 v_{L,n}^{(2^*-2)}|\nabla\eta|^2 dx \\ &\quad + C\Big(\int_{\mathbb{R}^N} \Big[v_n \eta v_{L,n}^{\frac{(2^*-2)}{2}}\Big]^{2^*} dx\Big)^{2/2^*} \Big(\int_{|x|\geq R/2} v_n^{2^*} dx\Big)^{(2^*-2)/2^*}. \end{aligned}$$

By definition of $w_{L,n}$, we have

$$\begin{aligned} \Big(\int_{\mathbb{R}^N} \Big[v_n \eta v_{L,n}^{\frac{(2^*-2)}{2}}\Big]^{2^*} dx\Big)^{2/2^*} &\leq C\beta^2 \int_{\mathbb{R}^N} v_n^2 v_{L,n}^{(2^*-2)}|\nabla\eta|^2 dx \\ &\quad + C\beta^2 \Big(\int_{\mathbb{R}^N} \Big[v_n \eta v_{L,n}^{\frac{(2^*-2)}{2}}\Big]^{2^*} dx\Big)^{2/2^*} \Big(\int_{|x|\geq R/2} v_n^{2^*} dx\Big)^{(2^*-2)/2^*}. \end{aligned}$$

Since $v_n \to v$ in $H^1(\mathbb{R}^N)$, for R sufficiently large, we conclude

$$\int_{|x|\geq R/2} v_n^{2^*} dx \leq \varepsilon, \text{ uniformly in } n.$$

Hence

$$\Big(\int_{|x|\geq R}\Big[v_n v_{L,n}^{\frac{(2^*-2)}{2}}\Big]^{2^*}dx\Big)^{2/2^*}\leq C\beta^2\int_{\mathbb{R}^N} v_n^2 v_{L,n}^{(2^*-2)}dx$$

or equivalently

$$\Big(\int_{|x|\geq R}\Big[v_n v_{L,n}^{\frac{(2^*-2)}{2}}\Big]^{2^*}dx\Big)^{2/2^*}\leq C\beta^2\int_{\mathbb{R}^N} v_n^{2^*}dx\leq K<\infty.$$

Using the Fatou's Lemma in the variable L, we have

$$\int_{|x|\geq R} v_n^{\frac{2^{*2}}{2}}\,dx<\infty$$

and therefore the claim holds.

Next, we note that if $\beta=\dfrac{2^*(t-1)}{2t}$, with $t=\dfrac{2^{*2}}{2(2^*-2)}$, then $\beta>1$, $\dfrac{2t}{t-1}<2^*$ and $v_n\in L^{(\beta 2t)/t-1}(|x|\geq R-r)$. Returning to inequality (1.2.6), using the hypothesis (ξ_3), we obtain

$$\|w_{L,n}\|^2_{L^{2^*}}\leq C\beta^2\Big[\int_{R\geq|x|\geq R-r} v_n^2 v_{L,n}^{2(\beta-1)}dx+\int_{|x|\geq R-r} v_n^{2^*}v_{L,n}^{2(\beta-1)}dx\Big]$$

or equivalently

$$\|w_{L,n}\|^2_{L^{2^*}}\leq C\beta^2\Big[\int_{R\geq|x|\geq R-r} v_n^{2\beta}dx+\int_{|x|\geq R-r} v_n^{2^*-2}v_n^{2\beta}dx\Big].$$

Using the Hölder's inequality with exponent $t/(t-1)$ and t, we get

$$\begin{aligned}\|w_{L,n}\|^2_{L^{2^*}} &\leq C\beta^2\Big\{\Big[\int_{R\geq|x|\geq R-r} v_n^{2\beta t/(t-1)}dx\Big]^{(t-1)/t}\Big[\int_{R\geq|x|\geq R-r} 1dx\Big]^{1/t}\\ &+\Big[\int_{|x|\geq R-r} v_n^{(2^*-2)t}dx\Big]^{1/t}\Big[\int_{|x|\geq R-r} v_n^{2\beta t/(t-1)}dx\Big]^{t/(t-1)}\Big\}.\end{aligned}$$

Since that $(2^*-2)t=2^{*2}$, we conclude

$$\|w_{L,n}\|^2_{L^{2^*}}\leq C\Big(\int_{|x|\geq R-r} v_n^{2\beta t/(t-1)}dx\Big)^{(t-1)/t}.$$

Note that

$$\begin{aligned} \|v_{L,n}\|^{2\beta}_{L^{2^*\beta}(|x|\geq R)} &\leq \Big(\int_{|x|\geq R-r} v_{L,n}^{2^*\beta} dx\Big)^{2/2^*} \leq \Big(\int_{\mathbb{R}^N} \eta^{2^*} v_n^{2^*} v_{L,n}^{2^*(\beta-1)} dx\Big)^{2/2^*} \\ &= \|w_{L,n}\|^2_{L^{2^*}} \leq C\beta^2\Big(\int_{|x|\geq R-r} v_n^{2\beta t/(t-1)} dx\Big)^{(t-1)/t} \\ &= C\|v_n\|^{2\beta}_{L^{2\beta t/(t-1)}(|x|\geq R-r)}. \end{aligned}$$

Applying Fatou's Lemma

$$\|v_n\|^{2\beta}_{L^{2^*\beta}(|x|\geq R)} \leq C\|v_n\|^{2\beta}_{L^{2\beta t/(t-1)}(|x|\geq R-r)}.$$

Considering $\chi = \dfrac{2^*(t-1)}{2t}$, $s = \dfrac{2t}{t-1}$ and the last inequality, we can prove

$$\|v_n\|_{L^{\beta\chi s}(|x|\geq R)} \leq C^{1/2\beta}\|v_n\|_{L^{\beta s}(|x|\geq R-r)} \leq C^{1/\beta}\|v_n\|_{L^{2^*}(|x|\geq R-r)}.$$

Let $\beta = \chi^m$, $(m = 1, 2, \cdots)$, then we get

$$\|v_n\|_{L^{\chi^{m+1}s}(|x|\geq R)} \leq C^{\chi^{-m}}\|v_n\|_{L^{2^*}(|x|\geq R-r)} \leq C^{\sum_{i=1}^{m}\chi^{-i}}\|v_n\|_{L^{2^*}(|x|\geq R-r)}.$$

Letting $m \to +\infty$ in the last inequality, we obtain

$$\|v_n\|_{L^{\infty}(|x|\geq R)} \leq C\|v_n\|_{L^{2^*}(|x|\geq R-r)}.$$

Using again the convergence of (v_n) to v in $H^1(\mathbb{R}^N)$, for $\varepsilon > 0$ fixed there exists $R > 0$ such that

$$\|v_n\|_{L^{\infty}(|x|\geq R)} < \varepsilon, \quad \text{for all } n \in \mathbb{N}. \tag{1.2.8}$$

Thus,

$$\lim_{|x|\to\infty} v_n(x) = 0, \text{ uniformly in n}$$

and the proof of lemma is finish. □

Proof of Theorem 1.1.1. By Lemma 1.2.3 and Remark 1.2.3, the functional I satisfies the geometry of the Mountain Pass Theorem [4], then by Ekeland Variational Principle [13] and considering c defined by (1.1.28) there exists a sequence $(u_n) \subset E$

satisfying

$$I(u_n) \to c \qquad \text{and} \qquad (1+\|u_n\|)\left\|I'(u_n)\right\| \to 0.$$

Using the Lemma 1.2.2, we obtain that c satisfies $0<c<m_\infty$ and, up to a subsequence, (u_n) converges strongly to $u \in E$, by Lemma 1.2.1. Moreover, since $I \in C^1(E,\mathbb{R})$, then $I(u)=c$ and $I'(u)=0$. It follows that u is a solution of problem (P_1).

To show that u is nonnegative we can assume in the beginning that $f(x,s)=0$ for all $s \leq 0$. Thus, $I'(u)u^- = 0$, and so

$$\begin{aligned} 0 = I'(u)u^- &= \int_{\mathbb{R}^N} (\xi(x)\nabla u \nabla u^- + uu^-)dx - \int_{\mathbb{R}^N} f(x,u)u^- dx \\ &= \int_{\{x:\ u(x)<0\}} (\xi(x)|\nabla u^-|^2 + |u^-|^2)dx \\ &= \|u^-\|^2, \end{aligned} \tag{1.2.9}$$

implies that $u^- \equiv 0$. Hence, $u \geq 0$ in $\mathbb{R}^N$. Since u is solution of the problem (P_1) and nonnegative, by Lemma 1.2.4 we have that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{loc}(\mathbb{R}^N)$ for some $0 < \alpha < 1$. Then, by Harnack's inequality [2] we obtain

$$\sup_{u \in B_R} I(u) \leq C \inf_{u \in B_R} I(u). \tag{1.2.10}$$

Suppose that there exists a point $x_0 \in \mathbb{R}^N$ such that $u(x_0) = 0$ in $B_R(x_0)$, thus $\inf_{u \in B_R(x_0)} I(u) = 0$. On other hand, $\sup_{u \in B_R} I(u) \geq 0$. We conclude that $u \equiv 0$ in $B_R(x_0)$. However, since $\mathbb{R}^N$ is path-connected we have $u \equiv 0$ in $\mathbb{R}^N$, which is absurd. Therefore, $u(x) > 0$ for all $x \in \mathbb{R}^N$. In other words, u is a nontrivial and positive solution of (P_1).

1.3 Nodal Solution

A nontrivial orthogonal involution $\tau : \mathbb{R}^N \to \mathbb{R}^N$ induces an involution $T_\tau : E \to E$ defined by

$$T_\tau(u(x)) := -u(\tau x). \tag{1.3.1}$$

Consider

$$E^\tau := \{u \in E : T_\tau(u(x)) = u(x)\} \tag{1.3.2}$$

the subspace of $\tau-$invariant in E and consider the following $\tau-$ invariant Pohozaev manifold

$$\mathcal{P}^{\tau} := \{u \in \mathcal{P} : T_{\tau}(u(x)) = u(x)\} = \mathcal{P} \cap E^{\tau}. \tag{1.3.3}$$

Lemma 1.3.1. *If $c > 0$ and (u_n) is a $(Ce)_c$ sequence of the functional I restricted to E^{τ}, then (u_n) is a bounded sequence.*

Proof. Suppose by contradiction that $\|u_n\| \to \infty$. Define $\tilde{u}_n = \dfrac{2\sqrt{c}u_n}{\|u_n\|}$, then $(\tilde{u}_n)$ is a bounded sequence with $\|\tilde{u}_n\| = 2\sqrt{c}$ and consequently $\tilde{u}_n \rightharpoonup \tilde{u}$ in E. One of the two following cases occurs:

Case 1) $\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\tilde{u}_n|^2 dx > 0$;

Case 2) $\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\tilde{u}_n|^2 dx = 0$.

Consider that *Case* 2 occurs. Without loss of generality, suppose $L > 1$ and

$$\begin{aligned} I\left(\frac{L}{\|u_n\|}2\sqrt{c}u_n\right) &= \frac{1}{2}\left(\frac{L^2 4c}{\|u_n\|^2}\right)\int_{\mathbb{R}^N}(\xi(x)|\nabla u_n|^2 + u_n^2)dx - \int_{\mathbb{R}^N} F\left(x, \frac{L}{\|u_n\|}2\sqrt{c}u_n\right)dx \\ &= 2L^2 c - \int_{\mathbb{R}^N} F\left(x, \frac{L}{\|u_n\|}2\sqrt{c}u_n\right)dx. \end{aligned}$$

Given $\varepsilon > 0$ and $2 < p < 2^*$, from (1.1.4) we have

$$\begin{aligned} \int_{\mathbb{R}^N}\left|F\left(x, \frac{L}{\|u_n\|}2\sqrt{c}u_n\right)\right|dx &\le \frac{\varepsilon}{2}\int_{\mathbb{R}^N}\left|\frac{L}{\|u_n\|}2\sqrt{c}u_n\right|^2 dx + C\int_{\mathbb{R}^N}\left|\frac{L}{\|u_n\|}2\sqrt{c}u_n\right|^p dx \\ &= \frac{2\varepsilon c L^2}{\|u_n\|^2}\int_{\mathbb{R}^N}|u_n|^2 dx + cL^p\int_{\mathbb{R}^N}|\tilde{u}_n|^p dx. \end{aligned}$$

By Lions' Lemma [24], we obtain

$$\int_{\mathbb{R}^N}|\tilde{u}_n|^p dx \to 0, \quad \text{for} \quad 2 < p < 2^*$$

thus,

$$\int_{\mathbb{R}^N}\left|F\left(x, \frac{L}{\|u_n\|}2\sqrt{c}u_n\right)\right|dx < 2\varepsilon c L^2 + o_n(1).$$

Taking $\varepsilon = 1/2$ we obtain

$$I\left(\frac{L}{\|u_n\|}2\sqrt{c}u_n\right) > 2L^2 c - (cL^2 + o_n(1)) = L^2 c - o_n(1).$$

Since $\|u_n\| \to \infty$, then $\dfrac{2L\sqrt{c}}{\|u_n\|} \in (0,1)$ for $n > 0$ sufficiently large, so

$$\max_{t\in[0,1]} I(tu_n) \geq I\left(\frac{L}{\|u_n\|}2\sqrt{c}u_n\right) > L^2c - o_n(1).$$

Consider $t_n \in (0,1)$ such that $I(t_nu_n) = \max_{t\in[0,1]} I(tu_n)$. Then

$$I(t_nu_n) > L^2c - o_n(1). \tag{1.3.4}$$

On other hand, $t_n < 1$ because $I(u_n) = c + o_n(1)$, $I'(t_nu_n)u_n = 0$ and by (f_5)

$$\begin{aligned} I(t_nu_n) &= \frac{1}{2}\int_{\mathbb{R}^N}(\xi(x)|\nabla(t_nu_n)|^2 + |t_nu_n|^2)dx - \int_{\mathbb{R}^N} F(x,t_nu_n)dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} f(x,t_nu_n)(t_nu_n)dx - \int_{\mathbb{R}^N} F(x,t_nu_n)dx \\ &= \int_{\mathbb{R}^N}\left(\frac{1}{2}f(x,t_nu_n)(t_nu_n) - F(x,t_nu_n)\right)dx \qquad (1.3.5) \\ &< D\int_{\mathbb{R}^N}\left(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right)dx \\ &= D\left[\frac{1}{2}\int_{\mathbb{R}^N}(\xi(x)|\nabla u_n|^2 + u_n^2)dx - \int_{\mathbb{R}^N} F(x,u_n)dx\right] \\ &= DI(u_n) = Dc + o_n(1). \qquad (1.3.6) \end{aligned}$$

From (1.3.4) and (1.3.6) it follows that

$$c - o_n(1) < I_\infty(t_nu_n) < Dc + o_n(1)$$

and making $L > 0$ sufficiently large we arrive at a contradiction.

In *Case* 1, if (y_n) is such that $|y_n| \to \infty$ and

$$\int_{B_1(y_n)} |\tilde{u}_n|^2 dx > \frac{\delta}{2},$$

then

$$\int_{B_1(0)} |\tilde{u}_n(x+y_n)|^2 dx > \frac{\delta}{2},$$

and knowing that $\tilde{u}_n(\cdot + y_n) \rightharpoonup \tilde{v}$, we have

$$\int_{B_1(0)} |\tilde{v}(x)|^2 dx > \frac{\delta}{2}$$

thus obtaining that $\tilde{v} \not\equiv 0$. Therefore there exists $\Omega \subset B_1(0)$ subset of positive Lebesgue measure such that

$$0 < \tilde{v}(x) = \lim_{n\to\infty} \tilde{u}_n(x+y_n) = \lim_{n\to\infty} \frac{u_n(x+y_n)2\sqrt{c}}{\|u_n\|}, \text{ for all } x \in \Omega.$$

Recalling the assumption that $\|u_n\| \to \infty$, then necessarily

$$u_n(x+y_n) \to \infty, \text{ for all } x \in \Omega \subset B_1(0)$$

and so from (f_5) and Fatou's Lemma, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right) dx \\
&\quad = \liminf_{n\to\infty} \int_{\Omega} \left(\frac{1}{2} f(x+y_n, u_n(x+y_n))u_n(x+y_n) - F(x+y_n, u_n(x+y_n))\right) dx \\
&\quad \geq \int_{\Omega} \liminf_{n\to\infty} \left(\frac{1}{2} f(x+y_n, u_n(x+y_n))u_n(x+y_n) - F(x+y_n, u_n(x+y_n))\right) dx \\
&\quad = +\infty. \qquad (1.3.7)
\end{aligned}$$

On other hand, by (1.1.29) we have that

$$|I'|_{E^\tau}(u_n)u_n| \leq \|I'|_{E^\tau}(u_n)\| \|u_n\| \to 0,$$

and so

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right) dx &= \frac{1}{2} \int_{\mathbb{R}^N} (\xi(x)|\nabla u_n|^2 + u_n^2) dx - \int_{\mathbb{R}^N} F(x,u_n) dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^N} (\xi(x)|\nabla u_n|^2 + u_n^2) dx - \frac{1}{2} \int_{\mathbb{R}^N} f(x,u_n)u_n dx \\
&= I|_{E^\tau}(u_n) - \frac{1}{2} I'|_{E^\tau}(u_n)u_n \\
&\leq c + o_n(1). \qquad (1.3.8)
\end{aligned}$$

From (1.3.7) and (1.3.8) we obtain a contradiction in *Case* 1, under the assumption that $|y_n| \to +\infty$.

Now, if we have $|y_n| \leq R$ with $R > 1$, then

$$\frac{\delta}{2} \leq \int_{B_1(0)} |\tilde{u}_n(x+y_n)|^2 dx \leq \int_{B_{2R}(0)} |\tilde{u}_n(x+y_n)|^2 dx$$

and since $\tilde{u}_n(\cdot + y_n) \to \tilde{v}$ strongly in $L^2(B_{2R}(0))$, it follows that

$$\frac{\delta}{2} \leq \int_{B_1(0)} |\tilde{v}(x)|^2 dx.$$

Hence, as in the previous case, there exists a $\Omega \subset B_1(0)$ such that $|\Omega| > 0$ and

$$\lim_{n\to\infty} \frac{u_n(x+y_n)2\sqrt{c}}{\|u_n\|} = \lim_{n\to\infty} \tilde{u}_n(x+y_n) = \tilde{v}(x) \neq 0, \text{ for all } x \in \Omega.$$

Following the previous arguments, by (1.3.7) and (1.3.8) again a contradiction follows. We conclude that (u_n) is a bounded sequence. □

Remark 1.3.1. *The proof of Lemma 1.1.4 is analogous to that just presented for I, using Lions' Lemma, hypothesis (f_3), as well Fatou's Lemma and (f_5) for the function h.*

Remark 1.3.2. *If (u_n) a Cerami $(Ce)_c$ sequence restricted to E^τ, then (u_n) is bounded (PS) sequence of I restricted to E^τ.*

Lemma 1.3.2. *If $u, \nabla u \in L^2(\mathbb{R}^N)$, $|y| \to \infty$ and $|y - \tau y| \to \infty$, then*

$$\int_{\mathbb{R}^N} u(x-y)u(\tau x - y)dx = o_y(1) \tag{1.3.9}$$

and

$$\int_{\mathbb{R}^N} \nabla u(x-y).\nabla u(\tau x - y)dx = o_y(1). \tag{1.3.10}$$

Proof. Indeed, making a change of variable, we obtain

$$\int_{\mathbb{R}^N} u(x-y)u(\tau x - y)dx = \int_{\mathbb{R}^N} u(z)u(\tau z \ + \tau y - y)dz.$$

Since $u \in L^2(\mathbb{R}^N)$, given $\varepsilon > 0$ there exists $R > 0$ independent of y such that

$$\int_{\mathbb{R}^N \setminus B_R(0)} |u(z)|^2 dz < \frac{\varepsilon}{2}.$$

Thus, using Hölder's inequality

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R(0)} u(z)u(\tau z + \tau u - y)dz &\leq \left(\int_{\mathbb{R}^N \setminus B_R(0)} |u(z)|^2 dz\right)^{1/2} \left(\int_{\mathbb{R}^N} |u(\tau z + \tau y - y)|^2 dz\right)^{1/2} \\ &\leq \frac{\varepsilon}{2} \|u\|_{L^2}. \end{aligned}$$

For $\varepsilon > 0$ and $R > 0$ fixed as previously, $|y - \tau y| \to \infty$ and $u \in L^2(\mathbb{R}^N)$, we obtain

$$\int_{B_R(0)} u(z)u(\tau z + \tau u - y)dz \leq \left(\int_{B_R(0)} |u(z)|^2 dz\right)^{1/2} \left(\int_{\mathbb{R}^N} |u(\tau z + \tau y - y)|^2 dz\right)^{1/2}$$
$$\leq \frac{\varepsilon}{2} \|u\|_{L^2}.$$

We conclude that

$$\int_{\mathbb{R}^N} u(x-y)u(\tau x - y)dx = o_y(1),$$

as $|y| \to \infty$ and $|y - \tau y| \to \infty$.

Using the same arguments,

$$\int_{\mathbb{R}^N} \nabla u(x-y).\nabla u(\tau x - y)dx = \int_{\mathbb{R}^N} \nabla u(z).\nabla u(\tau z \ + \tau y - y)dz.$$

Since $\nabla u \in L^2(\mathbb{R}^N)$, given $\varepsilon > 0$ there exists $R_1 > 0$ independent of y such that

$$\int_{\mathbb{R}^N \setminus B_{R_1}(0)} |\nabla u(z)|^2 dz < \frac{\varepsilon}{2}.$$

Thus, using Hölder's inequality

$$\begin{aligned}\int_{\mathbb{R}^N \setminus B_{R_1}(0)} \nabla u(z) \nabla u(\tau z \ &+ \ \tau y - y)dz \\ &\leq \left(\int_{\mathbb{R}^N \setminus B_{R_1}(0)} |\nabla u(z)|^2 dz\right)^{1/2} \left(\int_{\mathbb{R}^N} |\nabla u(\tau z + \tau y - y)|^2 dz\right)^{1/2} \\ &\leq \frac{\varepsilon}{2} \|\nabla u\|_{L^2}.\end{aligned}$$

For $\varepsilon > 0$ and $R_1 > 0$ fixed as before, $|y - \tau y| \to \infty$ and $u \in L^2(\mathbb{R}^N)$, we obtain

$$\begin{aligned}\int_{B_{R_1}(0)} \nabla u(z) \nabla u(\tau z \ &+ \ \tau y - y)dz \\ &\leq \left(\int_{B_{R_1}(0)} |\nabla u(z)|^2 dz\right)^{1/2} \left(\int_{\mathbb{R}^N} |\nabla u(\tau z + \tau y - y)|^2 dz\right)^{1/2} \\ &\leq \frac{\varepsilon}{2} \|\nabla u\|_{L^2}.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^N} \nabla u(x-y) \nabla u(\tau x - y)dx = o_y(1),$$

when $|y| \to \infty$ and $|y - \tau y| \to \infty$. And we conclude the proof of the lemma. $\square$

Now, we define $G(x,u)$ for $u \in E^\tau$ by

$$G(x,u) := \frac{1}{\xi(x)}\left(F(x,u) - \frac{1}{2}u^2\right).$$

Consider ω the ground state radial positive solution of equation (1.1.4) and define

$$z_y(x) := \omega(x-y) - \omega(x-\tau y) \in E^\tau. \tag{1.3.11}$$

Remark 1.3.3. *If we fix $y \in \mathbb{R}^N$, $|y| > 0$ sufficiently large, from (ξ_3) and (f_3) it follows*

$$\int_{\mathbb{R}^N} G(x, z_y)dx \geq \int_{\mathbb{R}^N} G_\infty(z_y)dx > 0. \tag{1.3.12}$$

Therefore, there exists $t > 0$ such that $u\left(\frac{\cdot}{t}\right) \in \mathcal{P}$. Moreover, there exists t_{z_y} such that

$$I\left(z_y\left(\frac{\cdot}{t_{z_y}}\right)\right) = \max_{t>0} I\left(z_y\left(\frac{\cdot}{t}\right)\right). \tag{1.3.13}$$

Indeed,

$$\int_{\mathbb{R}^N} G(x, z_y)dx \geq \int_{\mathbb{R}^N} \frac{1}{\xi_\infty}\left(H(z_y) - \frac{1}{2}z_y^2\right)dx = \int_{\mathbb{R}^N} G_\infty(z_y)dx.$$

In what follows consider $z_0 = 0$, and

$$\overline{z}_1 := \omega\left(\frac{\cdot}{L} - y\right) - \omega\left(\frac{\cdot}{L} - \tau y\right), \text{ in } E^\tau$$

for a fixed $L > L_0$, $|y| > 0$ and $|y - \tau y|$ large enough, such that $I_\infty(\overline{z}_1) < 0$. This is possible by (1.1.6), (1.1.7) and by Lemma 1.3.2. Now define

$$c^\tau := \inf_{\gamma \in \Gamma_\tau} \max_{0 \leq t \leq 1} I(\gamma(t)), \tag{1.3.14}$$

where $\Gamma_\tau = \{\gamma \in C([0,1], E^\tau) : \gamma(0) = z_0 \text{ and } \gamma(1) = \overline{z}_1\}$.

Remark 1.3.4. $\mathcal{P} \cap E^\tau \neq \emptyset$.

Lemma 1.3.3. *There exists a sequence* $(u_n) \subset E^\tau$ *satisfying*

$$I(u_n) \to c^\tau \quad \text{and} \quad (1+\|u_n\|)\left\|I'|_{E^\tau}(u_n)\right\| \to 0.$$

Proof. The existence of $(Ce)_{c^\tau}$ sequence will be guaranteed if we can apply the Ghoussoub-Preiss Theorem. To show the existence of a Cerami sequence converging to c^τ as defined in (1.3.14) we need to show that $\mathcal{F} \cap I_{c^\tau}$ separates $z_0 = 0$ and z_1 where

$$I_{c^\tau} = \{u \in E^\tau : I(u) \geq c^\tau\}$$

is a closed subset of E^τ.

Given the definition of z_y in (1.3.11), define also

$$z_y\left(\frac{x}{t}\right) = \begin{cases} 0, & t = 0, \\ \omega\left(\frac{x}{t} - y\right) - \omega\left(\frac{x}{t} - \tau y\right), & t > 0. \end{cases}$$

Since $I(u) \leq I_\infty(u)$ for $u \in E^\tau$ we have

$$I\left(z_y\left(\frac{\cdot}{t}\right)\right) < I_\infty\left(z_y\left(\frac{\cdot}{t}\right)\right), \quad \text{if} \quad t > 0. \tag{1.3.15}$$

Consider

$$\mathcal{F} = \{u \in E^\tau : I_\infty(u) \geq 0\}$$

which is a closed subset of E^τ. Since I satisfies the Mountain Pass geometry, by Lemma 1.2.3 and Remark 1.2.3, then there exists $\rho > 0$ such that

$$0 < I(u), \quad \text{if} \quad 0 < \|u\| < \rho. \tag{1.3.16}$$

Therefore, from (1.1.27) and (1.3.16), if $u \in B_\rho(0)$ then $u \in \mathcal{F}$, but $u \notin I_{c^\tau}$. Moreover, we will check that if $u \in B_\rho(0)$, then

$$0 \leq I(u) < c^\tau. \tag{1.3.17}$$

In fact, by Mountain Pass geometry, we have that

$$I(u) = \frac{1}{2}\|u\|^2 - o(\|u\|^2) < \frac{3}{2}\|u\|^2, \quad \text{if} \quad \|u\| < \rho.$$

Therefore, if we consider $3\rho^2/2 < c^\tau$ we have (1.3.17). This way, if $\|u\| < \rho$, then $u \notin \mathcal{F} \cap I_{c^\tau}$, such that $z_0 \in B_\rho(0) \not\subset \mathcal{F} \cap I_{c^\tau}$. Furthermore, by (1.3.14) and (1.3.15) we have that

$$I(z_1) < I_\infty(z_1) < 0$$

implying that $z_1 \notin \mathcal{F} \cap I_{c^\tau}$.

We conclude that the closed subset $\mathcal{F} \cap I_{c^\tau}$ separates z_0 and z_1, and thus we can apply the Ghoussoub-Preis Theorem with $X = E^\tau$, $\phi = I|_{E^\tau}$ and $F = \mathcal{F} \cap I_{c^\tau}$ such that $I(u_n) \to c^\tau$ and $(1 + \|u_n\|) \left\| I'|_{E^\tau}(u_n) \right\| \to 0$. □

Lemma 1.3.4. *If $(u_n) \subset E^\tau$ is a (PS) sequence of the functional I restricted to E^τ, then (u_n) is a (PS) sequence of I.*

Proof. Using that the action T_τ is isometric, we will prove that

$$T_\tau I'(u_n) = I'(u_n). \tag{1.3.18}$$

It follows from the (f_6) hypothesis that F is even and that $F(\tau x, s) = F(x, -s) = F(x, s)$ and using the hypothesis (ξ_4) we have

$$\begin{aligned} I(T_\tau(u_n)) &= I(-u_n(\tau x)) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} (\xi(\tau x)|\nabla(-u_n(\tau x))|^2 + |-u_n(\tau x)|^2)dx - \int_{\mathbb{R}^N} F(\tau x, -u_n(\tau x))dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} (\xi(x)|\nabla u_n(x)|^2 + |u_n(x)|^2)dx - \int_{\mathbb{R}^N} F(x, u_n(x))dx \\ &= I(u_n). \end{aligned} \tag{1.3.19}$$

In addition, using the hypothesis (f_6) and making change of variables, we obtain

$$\begin{aligned} I'(T_\tau u_n(x))v(x) = I'(-u_n(\tau(x)))v(x) \\ &= \int_{\mathbb{R}^N} \Big(\xi(\tau x)\nabla(-u_n(\tau x))\nabla v(x) + (-u_n(\tau x))v(x)\Big)dx \\ &- \int_{\mathbb{R}^N} f(\tau x, -u_n(\tau x))v(x)dx \\ &= \int_{\mathbb{R}^N} \Big(\xi(y)\nabla u_n(y)\nabla(-v(\tau y)) + u_n(y)(-v(\tau y))\Big)dy \\ &- \int_{\mathbb{R}^N} f(y, u_n(y))(-v(\tau y))dy \\ &= \int_{\mathbb{R}^N} \Big(\xi(y)\nabla u_n \nabla(T_\tau(v)) + u_n(T_\tau(v))\Big)dy \end{aligned}$$

$$-\int_{\mathbb{R}^N} f(y,u_n)(T_\tau(v))dy$$
$$= I'(u_n)(T_\tau(v)), \quad \text{for all} \quad v \in E. \tag{1.3.20}$$

Since T_τ is isometric, then

$$\Big\langle I'(u_n), T_\tau(v) \Big\rangle = \Big\langle T_\tau(I'(u_n)), T_\tau(T_\tau(v)) \Big\rangle = \Big\langle T_\tau(I'(u_n)), v \Big\rangle. \tag{1.3.21}$$

It follows from (1.3.20) and (1.3.21) that

$$I'(T_\tau(u_n)) = T_\tau(I'(u_n)). \tag{1.3.22}$$

Since $(u_n) \subset E^\tau$, then by (1.3.22) we obtain

$$T_\tau(I'(u_n)) = I'(T_\tau(u_n)) = I'(u_n) \tag{1.3.23}$$

and hence $I'(u_n) \in E^\tau$, implying that $I'(u_n)v = 0$ for all $v \in (E^\tau)^\perp$. On ther hand, since (u_n) is a (PS) sequence of the functional I restricted to E^τ, then $I'(u_n)v_1 \to 0$ for all $v_1 \in E^\tau$. Denoting $v = v_1 + v_2$ with $v_1 \in E^\tau$ and $v_2 \in (E^\tau)^\perp$, it follows that $I'(u_n)v = I'(u_n)v_1 \to 0$. Therefore $I'(u_n)v \to 0$ for all $v \in E$. □

Next we present a version of the Concentration Compactness Lemma of Lions for I restricted to E^τ.

Lemma 1.3.5. *Let $(u_n) \subset E^\tau$ be a bounded sequence, such that*

$$I(u_n) \to c \qquad \textit{and} \qquad I'(u_n) \to 0.$$

Then, there exists $u_0 \in E^\tau$ such that, up to a subsequence, $u_n \rightharpoonup u_0$, $I'(u_0) = 0$ and there exist two integers k_1, $k_2 \geq 0$, $k_1 + k_2$ sequences (y_n^j), a $\tau-$antisymmetric solution u_0 of problem (P_τ), k_1 solutions u^j, $j = 1, \cdots, k_1$ and k_2 $\tau-$antisymmetric solutions u^j, $j = k_1+1, \cdots, k_1+k_2$ of the equation (1.1.4), that is, $-div(\xi_\infty \nabla u^j) + u^j = h(u^j)u^j$ in $\mathbb{R}^N$ and $u^j(\tau x) = -u^j(x)$, $u^j(x) \to 0$ as $|x| \to \infty$ such that, either:

1. *$u_n \to u_0$ strongly in E, or the following statement holds;*
2. *if $j = 1, ..., k_1$, then $\tau y_n^j \neq y_n^j$, and $|y_n^j| \to \infty$ when $n \to \infty$;*
3. *if $j = k_1+1, ..., k_1+k_2$, then $\tau y_n^j = y_n^j$, and $|y_n^j| \to \infty$ when $n \to \infty$;*

4. $u_n(x) = u_0(x) + \sum_{j=1}^{k_1}[u^j(x-y_n^j) + T_\tau u^j(x-y_n^j)] + \sum_{j=k_1+1}^{k_1+k_2} u^j(x-y_n^j) + o_n(1)$;

5. $I(u_n) \to I(u_0) + 2\sum_{j=1}^{k_1} I_\infty(u^j) + \sum_{j=k_1+1}^{k_1+k_2} I_\infty(u^j)$.

Proof. Step 1) By Lemma 1.3.3, if $(u_n) \subset E^\tau$ is a (PS) sequence of the functional I restricted to E^τ, $I|_{E^\tau}$, then (u_n) is a (PS) sequence of I.

Step 2) From the hypothesis that (u_n) is bounded, then $u_n \rightharpoonup u_0$ in E. We show now that $I'(u_0) = 0$. Using the compact embedding $E \hookrightarrow L^p_{loc}(\mathbb{R}^N)$ for $1 \le p < 2^*$, then $u_n \to u_0$ in $L^p_{loc}(\mathbb{R}^N)$, for $1 \le p < 2^*$. The continuity of f, the weak convergence $u_n \rightharpoonup u_0$ in E and Lebesgue dominated convergence theorem imply

$$\lim_{n\to\infty} I'(u_n)\varphi = I'(u_0)\varphi, \quad \text{for all} \;\; \varphi \in C_0^\infty(\mathbb{R}^N).$$

Moreover, since (u_n) is a (PS) sequence of I, then

$$I'(u_0)\varphi = 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{1.3.24}$$

Step 3) Now we verify that $u_0 \in E^\tau$. Since $u_n(x) \to u_0(x)$ a.e. $x \in \mathbb{R}^N$. Furthermore, $u_n \in E^\tau$, implies that $T_\tau(u_n(x)) = u_n(x)$, thus

$$\begin{aligned} T_\tau(u_0(x)) &:= -u_0(\tau x) = -\lim_{n\to\infty} u_n(\tau x) = \lim_{n\to\infty} -u_n(\tau x) \\ &= \lim_{n\to\infty} T_\tau(u_n(x)) = \lim_{n\to\infty} u_n(x) = u_0(x). \end{aligned}$$

Therefore, $u_0 \in E^\tau$.

Step 4) Let $u_n^1 := u_n - u_0$. Then, if $n \to \infty$, we have:

(i) $\|u_n^1\|^2 = \|u_n\|^2 - \|u_0\|^2 + o_n(1)$;

(ii) $I_\infty(u_n^1) \to c - I(u_0)$;

(iii) $I'_\infty(u_n^1) \to 0$.

Indeed, since $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$, then

$$\langle u_n, u_0\rangle \to \langle u_0, u_0\rangle = \|u_0\|^2.$$

Thus,

$$\|u_n^1\| = \|u_n - u_0\|^2 = \|u_n\|^2 - 2\langle u_n, u_0\rangle + \|u_0\|^2 = \|u_n\|^2 - \|u_0\|^2 + o_n(1),$$

as claimed. The proof of (ii) and (iii) is similar to Step 2 in Lemma 1.1.5. By (ii) and (iii), (u_n^1) is a (PS) sequence of I_∞ and

$$\left\langle I'_\infty(u_n^1), \varphi\right\rangle = \left\langle I'(u_n), \varphi\right\rangle - \left\langle I'(u_0), \varphi\right\rangle = o_n(1).$$

Furthermore, since u_n, $u_0 \in E^\tau$ and T_τ is linear, it follows that $T_\tau(u_n^1)(x) = T_\tau(u_n - u_0)(x) = T_\tau(u_n)(x) - T_\tau(u_0)(x) = u_n(x) - u_0(x) = u_n^1(x)$ and $u_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$.

Consider

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n^1(x)|^2 dx.$$

Step 5) If $\delta = 0$, it follows from Lions' lemma that

$$u_n^1 \to 0, \quad \text{in } L^p(\mathbb{R}^N), \text{ for all } 2 < p < 2^*. \tag{1.3.25}$$

On the other hand, since (u_n^1) is a bounded sequence and (iii) holds, then

$$I'_\infty(u_n^1)u_n^1 = \int_{\mathbb{R}^N} \left(\xi_\infty|\nabla u_n^1|^2 + (u_n^1)^2 - h(u_n^1)(u_n^1)^2\right) dx \to 0. \tag{1.3.26}$$

Using the estimate (1.1.3) we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} (\xi_\infty|\nabla u_n^1|^2 + (u_n^1)^2) dx &= \int_{\mathbb{R}^N} h(u_n^1)(u_n^1)^2 dx + o_n(1)\\ &< \varepsilon \int_{\mathbb{R}^N} (u_n^1)^2 dx + C\int_{\mathbb{R}^N} |u_n^1|^p dx.\end{aligned} \tag{1.3.27}$$

Thus, by (1.3.25) and (1.3.27) we have $\|u_n^1\| \to 0$, that is, $u_n \to u_0$ and u_0 is a τ-antisymmetric solution of problem (1.1.4) which completes the proof of the item 1.

Step 6) Now, if $\delta > 0$, there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_n)} |u_n^1(x)|^2 dx > \frac{\delta}{2}. \tag{1.3.28}$$

Define a new sequence of functions $(v_n^1) \subset E$ by $v_n^1 := u_n^1(\cdot + y_n)$. Since (u_n^1) is bounded then (v_n^1) is also bounded, and thus we can assume that $v_n^1 \rightharpoonup u^1$, in E and $v_n^1(x) \to u^1(x)$ a.e. $x \in \mathbb{R}^N$. From (1.3.28) we have

$$\int_{B_1(0)} |v_n^1(x)|^2 dx > \frac{\delta}{2}. \tag{1.3.29}$$

The weak convergence implies that $v_n^1 \to u^1$ strongly in $L^2(B_1(0))$ and hence

$$\int_{B_1(0)} |u^1(x)|^2 dx \geq \frac{\delta}{2},$$

from which $u^1 \not\equiv 0$. Since $u_n^1 \rightharpoonup 0$ in E, we have that $|y_n|$ is a unbounded sequence. Therefore, up to a subsequence, we can assume that $|y_n| \to \infty$. Finally, we obtain as in (1.1.40) that $I'_\infty(u^1) = 0$. Consider now $\mathbb{R}^N = \Gamma \oplus \Gamma^\perp$, where $\Gamma := \{x \in \mathbb{R}^N : \tau(x) = x\}$, and consider P_Γ the projection on the subspace Γ. We can distinguish two cases:

Case I: If $|y_n - \tau y_n|$ is bounded, we define $y_n^1 := P_\Gamma(y_n)$;

Case II: If $|y_n - \tau y_n|$ is unbounded, we define $y_n^1 := y_n$.

Let us study each of these cases. In *Case I*, first note that $|y_n^1| \to \infty$. In fact, the orthogonal linear transformation $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is diagonalizable and without loss of generality, we may assume that

$$\tau(x_1, ..., x_k, x_{k+1}, ..., x_N) = (x_1, ..., x_k, -x_{k+1}, ..., -x_N). \tag{1.3.30}$$

Denoting y_n by

$$y_n = P_\Gamma(y_n) + w_n = y_n^1 + w_n,$$

then $y_n^1 := P_\Gamma(y_n)$ implies $\tau(y_n^1) = y_n^1$. Let $y_n = (x_1^n, ..., x_k^n, x_{k+1}^n, ..., x_N^n)$, where $y_n^1 = (x_1^n, ..., x_k^n, 0, ..., 0)$ and $w_n = (0, ..., 0, x_{k+1}^n, ..., x_N^n)$. We have

$$\tau(y_n) = (x_1^n, ..., x_k^n, -x_{k+1}^n, ..., -x_N^n),$$

and

$$|y_n - \tau y_n| = |(0, ..., 0, 2x_{k+1}^n, ..., 2x_N^n)| = 2|w_n|.$$

Thus, in the new basis we have that $|y_n - \tau y_n|$ is bounded, that is, there exists $M > 0$ such that $|y_n - \tau y_n| \leq 2M$, which gives $|w_n| \leq M$. Since $y_n = y_n^1 + w_n$, $|y_n| \to \infty$

when $n \to \infty$ and $|w_n| \leq M$, then $|y_n^1| \to \infty$ when $n \to \infty$. Furthermore, we consider the sequence $(u_n^1(\cdot + y_n^1))$, which is bounded, so up to a subsequence, $u_n^1(\cdot + y_n^1) \rightharpoonup u^1$ in E, and $u^1 \not\equiv 0$ is a solution of the limit problem (1.1.4). Moreover, since $\tau(y_n^1) = y_n^1$ then

$$\begin{aligned} T_\tau(u^1(x)) &:= -u^1(\tau x) = -\lim_{n\to\infty} u_n^1(\tau x + y_n^1) \\ &= \lim_{n\to\infty} -u_n^1(\tau(x + y_n^1)) \\ &= \lim_{n\to\infty} u_n^1(x + y_n^1) = u^1(x). \end{aligned} \tag{1.3.31}$$

We continue by considering

$$u_n^2(x) := u_n^1(x) - u^1(x - y_n^1)$$

and verify that (u_n^2) is a (PS) sequence of I_∞. In fact, we have that

$$\begin{aligned} I_\infty(u_n^2) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(\xi_\infty|\nabla u_n^2|^2 + (u_n^2)^2\right) dx - \int_{\mathbb{R}^N} H(u_n^2)dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \left(\xi_\infty|\nabla(u_n^1(x) - u^1(x - y_n^1))|^2 + |u_n^1(x) - u^1(x - y_n^1)|^2\right) dx \\ &\quad - \int_{\mathbb{R}^N} H(u_n^1(x) - u^1(x - y_n^1))dx. \end{aligned}$$

If $z = x - y_n^1$, then $x = z + y_n^1$ and $dx = dz$. Renaming z by x when changing variables, we obtain

$$\begin{aligned} I_\infty(u_n^2) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(\xi_\infty|\nabla(u_n^1(x + y_n^1) - u^1(x))|^2 + |u_n^1(x + y_n^1) - u^1(x)|^2\right) dx \\ &\quad - \int_{\mathbb{R}^N} H(u_n^1(x + y_n^1) - u^1(x))dx. \end{aligned}$$

Hence we have that

$$\|u_n^1(\cdot + y_n^1) - u^1\|^2 = \|u_n^1(\cdot + y_n^1)\|^2 - 2\langle u_n^1(\cdot + y_n^1), u^1\rangle + \|u^1\|^2. \tag{1.3.32}$$

Since $u_n^1(\cdot + y_n^1) \rightharpoonup u^1$ in E, by weak convergence and Riez Representation Theorem, we obtain

$$\langle u_n^1(\cdot + y_n^1), \varphi\rangle \to \langle u^1, \varphi\rangle, \quad \text{for all } \varphi \in E.$$

In particular, if $\varphi = u^1$, then

$$\langle u_n^1(\cdot + y_n^1), u^1 \rangle \to \langle u^1, u^1 \rangle,$$

it follows that

$$\langle u_n^1(\cdot + y_n^1), u^1 \rangle = \|u^1\|^2 + o_n(1).$$

Replacing in (1.3.32) we obtain

$$\begin{aligned} \|u_n^1(\cdot + y_n^1) - u^1\|^2 &= \|u_n^1(\cdot + y_n^1)\|^2 - 2\|u^1\|^2 + o_n(1) + \|u^1\|^2 \\ &= \|u_n^1\|^2 - \|u^1\|^2 + o_n(1). \end{aligned} \tag{1.3.33}$$

On the other hand, we note that

$$\begin{aligned} I_\infty(u_n^1) - I_\infty(u_n^2) - I_\infty(u^1) &= \frac{1}{2}\Big(\|u_n^1\|^2 - \|u_n^1 - u^1\|^2 - \|u^1\|^2\Big) \\ &\quad - \int_{\mathbb{R}^N} \Big(H(u_n^1) - H(u_n^2) - H(u^1)\Big)\, dx. \end{aligned}$$

Now, using (1.3.33) and (1.1.19), we have that

$$I_\infty(u_n^2) = I_\infty(u_n^1) - I_\infty(u^1) + o_n(1).$$

Since (u_n^1) is a (PS) sequence for I_∞, we know that $I_\infty(u_n^1)$ converges to a constant, and thus $I_\infty(u_n^2)$ also converge. Finally, we show that

$$I'_\infty(u_n^2)\varphi \to 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{1.3.34}$$

We know that (u_n^1) is a (PS) sequence for I_∞, then

$$I'_\infty(u_n^1)\varphi = o_n(1), \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{1.3.35}$$

Furthermore, u^1 is a solution of equation (1.1.4) we have

$$I'_\infty(u^1)\varphi = 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{1.3.36}$$

Thus, with a change of variable, by (1.3.35) and (1.3.36) and by Lemma 1.1.4, we obtain that

$$\begin{aligned}
|I'_\infty(u_n^2)\varphi| &= \left|\int_{\mathbb{R}^N}\Big(\xi_\infty\nabla(u_n^1-u^1)\nabla\varphi+(u_n^1-u^1)\varphi\Big)\,dx-\int_{\mathbb{R}^N}h(u_n^1-u^1)(u_n-u^1)\varphi dx\right|\\
&= \left|I'_\infty(u_n^1)\varphi-I'_\infty(u^1)\varphi+\int_{\mathbb{R}^N}[h(u_n^1)(u_n^1)-h(u_n^1-u^1)(u_n-u^1)-h(u^1)(u^1)]\varphi dx\right|\\
&\leq o_n(1)+\int_{\mathbb{R}^N}|h(u_n^1)(u_n^1)-h(u_n^1-u^1)(u_n-u^1)-h(u^1)(u^1)||\varphi|dx\\
&\leq C_\varepsilon\|\varphi\|_{H^1(\mathbb{R}^N)}.
\end{aligned}$$

Thus (1.3.34) holds. Therefore, (u_n^2) is a (PS) sequence for I_∞ and *Case I* is complete.

Case II: Here we have that $|y_n-\tau y_n|$ is unbounded and we define $y_n^1=y_n$. Moreover, we know that $u^1\not\equiv 0$ is a weak solution of the equation (1.1.4). Let $u_n^2:=u_n^1-\gamma_n$, where

$$\gamma_n(x):=u^1(x-y_n^1)-u^1(\tau x-y_n^1). \tag{1.3.37}$$

Note that since T_τ is an orthogonal linear transformation, it follows that

$$\begin{aligned}
T_\tau(\gamma_n(x)):=-\gamma_n(\tau x)&=-u^1(\tau x-y_n^1)+u^1(x-y_n^1)\\
&=u^1(x-y_n^1)-u^1(\tau x-y_n^1)=\gamma_n(x).
\end{aligned}$$

Thus, $u_n^2\in E^\tau$ because

$$\begin{aligned}
T_\tau(u_n^2(x))=T_\tau(u_n^1(x)-\gamma_n(x))&=T_\tau(u_n^1(x))-T_\tau(\gamma_n(x))\\
&=u_n^1(x)-\gamma_n(x)=u_n^2(x).
\end{aligned}$$

In this case we must show that (u_n^2) is a (PS) sequence of I_∞. We will show that

$$I_\infty(u_n^2)=I_\infty(u_n^1)-2I_\infty(u^1)+o_n(1) \tag{1.3.38}$$

using the fact that (u_n^1) is a (PS) sequence of I_∞. We have that

$$\|u_n^2\|^2=\|u_n^1-\gamma_n\|^2=\|u_n^1\|^2-2\langle u_n^1,\gamma_n\rangle+\|\gamma_n\|^2, \tag{1.3.39}$$

such that

$$\begin{aligned}\langle u_n^1, \gamma_n\rangle &= \int_{\mathbb{R}^N}(\xi_\infty \nabla u_n^1 \nabla \gamma_n + u_n^1\gamma_n)dx\\ &= \int_{\mathbb{R}^N}(\xi_\infty \nabla u_n^1 \nabla \{u^1(x-y_n^1) - u^1(\tau x - y_n^1)\})dx\\ &\quad + \int_{\mathbb{R}^N}(u_n^1\{u^1(x-y_n^1) - u^1(\tau x - y_n^1)\})dx\\ &= \int_{\mathbb{R}^N}\xi_\infty \nabla u_n^1 \nabla u^1(x-y_n^1)dx + \int_{\mathbb{R}^N}\xi_\infty \nabla u_n^1 \nabla u^1(\tau x-y_n^1)dx\\ &\quad + \int_{\mathbb{R}^N} u_n^1 u^1(x-y_n^1)dx + \int_{\mathbb{R}^N} u_n^1 u^1(\tau x-y_n^1)dx.\end{aligned}$$

Firstly, we claim that

$$\langle u_n^1, \gamma_n\rangle = 2\|u^1\|^2 + o_n(1). \tag{1.3.40}$$

Indeed, let

$$A_n^1 = \int_{\mathbb{R}^N}(\xi_\infty \nabla u_n^1 \nabla u^1(x-y_n^1) + u_n^1 u^1(x-y_n^1))dx$$

and

$$A_n^2 = \int_{\mathbb{R}^N}(\xi_\infty \nabla u_n^1 \nabla u^1(\tau x-y_n^1) + u_n^1 u^1(\tau x-y_n^1))dx.$$

We show that

$$A_n^1 \to \left\{\int_{\mathbb{R}^N}(\xi_\infty|\nabla u^1|^2 + (u^1)^2)dx\right\}, \quad \text{when} \quad n\to\infty,$$

and

$$A_n^2 \to -\left\{\int_{\mathbb{R}^N}(\xi_\infty|\nabla u^1|^2 + (u^1)^2)dx\right\}, \quad \text{when} \quad n\to\infty. \tag{1.3.41}$$

Let $z = x - y_n^1$, thus $x = z + y_n^1$ and $dx = dz$. Combining this with the fact $u_n^1(\cdot + y_n^1) \rightharpoonup u^1(\cdot)$, we have

$$\int_{\mathbb{R}^N}(\xi_\infty \nabla u_n^1(z+y_n^1)\nabla u^1(z) + u_n^1(z+y_n^1)u^1(z))dz \to \int_{\mathbb{R}^N}(\xi_\infty|\nabla u^1|^2 + (u^1)^2)dz.$$

To evaluate A_n^2, let us consider the following change of variables $\tau x - y_n^1 = z$, then $x = \tau(z + y_n^1)$ and $dx = dz$. Thus,

$$A_n^2 = \int_{\mathbb{R}^N}(\xi_\infty \nabla u_n^1(\tau(z+y_n^1))\nabla u^1(z) + u_n^1(\tau(z+y_n^1))u^1(z)dz.$$

Since u_n^1 is τ-antisymmetric, we have

$$A_n^2 = -\left\{\int_{\mathbb{R}^N} \left(\xi_\infty \nabla u_n^1(z+y_n^1)\nabla u^1(z) + u_n^1(z+y_n^1)u^1(z)\right) dz\right\}.$$

Therefore, in a similar way to A_n^1, we obtain (1.3.41) and thus prove (1.3.40). Now, we claim

$$\|\gamma_n\|^2 = 2\|u^1\|^2 + o_n(1). \tag{1.3.42}$$

In fact, from (1.3.9) and (1.3.10) we have that

$$\begin{aligned}
\|\gamma_n\|^2 &= \int_{\mathbb{R}^N}(\xi_\infty|\nabla\gamma_n|^2+\gamma_n^2)dx \\
&= \int_{\mathbb{R}^N}\xi_\infty|\nabla(u^1(x-y_n^1)-u^1(\tau x-y_n^1))|^2dx + \int_{\mathbb{R}^N}|u^1(x-y_n^1)-u^1(\tau x-y_n^1)|^2dx \\
&= \int_{\mathbb{R}^N}\xi_\infty|\nabla u^1(x-y_n^1)|^2dx - 2\int_{\mathbb{R}^N}\xi_\infty\nabla u^1(x-y_n^1)\nabla u^1(\tau x-y_n^1)dx \\
&\quad + \int_{\mathbb{R}^N}\xi_\infty|\nabla u^1(\tau x-y_n^1)|^2dx + \int_{\mathbb{R}^N}|u^1(x-y_n^1)|^2dx \\
&\quad -2\int_{\mathbb{R}^N}u^1(x-y_n^1)u^1(\tau x-y_n^1)dx + \int_{\mathbb{R}^N}|u^1(\tau x-y_n^1)|^2dx \\
&= 2\|u^1\|^2 - 2\int_{\mathbb{R}^N}\xi_\infty\nabla u^1(x-y_n^1)\nabla u^1(\tau x-y_n^1)dx - 2\int_{\mathbb{R}^N}u^1(x-y_n^1)u^1(\tau x-y_n^1)dx \\
&= 2\|u^1\|^2 + o_n(1).
\end{aligned}$$

Thus obtaining (1.3.42).

Finally, replacing (1.3.39) and (1.3.40) in (1.3.38)

$$\|u_n^2\|^2 = \|u_n^1\|^2 - 2\|u^1\|^2 + o_n(1). \tag{1.3.43}$$

To conclude (1.3.38) we need to verify the following equality

$$\int_{\mathbb{R}^N} H(u_n^2)dx = \int_{\mathbb{R}^N} H(u_n^1)dx - 2\int_{\mathbb{R}^N} H(u^1)dx + o_n(1). \tag{1.3.44}$$

Define $\rho := \dfrac{|y_n^1 - \tau y_n^1|}{2}$, $S_n = \mathbb{R}^N \setminus B_{\rho_n}(0) \cup B_{\rho_n}(\tau y_n^1 - y_n^1)$ and using the fact that $u^1(\tau x - y_n^1) = u^1(\tau(x-\tau y_n^1)) = -u^1(x-\tau y_n^1)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} H(u_n^2)dx &= \int_{\mathbb{R}^N} H(u_n^1-\gamma_n)dx = \int_{\mathbb{R}^N} H(u_n^1(x)-u^1(x-y_n^1)-u^1(\tau x-y_n^1))dx \\
&= \int_{B_{\rho_n}(0)} H(u_n^1(z+y_n^1)-u^1(z)-u^1(z+y_n^1-\tau y_n^1))dz
\end{aligned}$$

$$+\int_{B_{\rho_n}(\tau y_n^1-y_n^1)} H(u_n^1(z+y_n^1)-u^1(z)-u^1(z+y_n^1-\tau y_n^1))dz$$
$$+\int_{S_n} H(u_n^1(z+y_n^1)-u^1(z)-u^1(z+y_n^1-\tau y_n^1))dz$$
$$=\int_{B_{\rho_n}(0)} H(u_n^1(z+y_n^1)-u^1(z+y_n^1-\tau y_n^1))dz-\int_{B_{\rho_n}(0)} H(u^1(z))dz$$
$$+\int_{B_{\rho_n}(\tau y_n^1-y_n^1)} H(u_n^1(z+y_n^1)-u^1(z))dz-\int_{B_{\rho_n}(\tau y_n^1-y_n^1)} H(u^1(z+y_n^1-\tau y_n^1))dz$$
$$+\int_{S_n} H(u_n^1(z+y_n^1)-u^1(z+y_n^1-\tau y_n^1))dz-\int_{S_n} H(u^1(z))dz+o_n(1).$$

Under the assumptions that $u_n^1(z+y_n^1)-u^1(z)\to 0$ if $|y_n^1|\to\infty$ a.e. $z\in\mathbb{R}^N$ and that $u^1(z+y_n^1+\tau y_n^1)\to 0$ a.e. $z\in\mathbb{R}^N$, together with the Brezis-Lieb Lemma, we verify the following statements:

(A) $\int_{B_{\rho_n}(0)} H(u_n^1(z+y_n^1)-u^1(z+y_n^1-\tau y_n^1))dz-\int_{B_{\rho_n}(0)} H(u^1(z))dz=o_n(1);$

(B) $\int_{B_{\rho_n}(\tau y_n^1-y_n^1)} H(u_n^1(z+y_n^1)-u^1(z))dz-\int_{B_{\rho_n}(\tau y_n^1-y_n^1)} H(u^1(z+y_n^1-\tau y_n^1))dz=o_n(1);$

(C) $\int_{S_n} H(u_n^1(z+y_n^1)-u^1(z+y_n^1-\tau y_n^1))dz-\int_{S_n} H(u^1(z))dz=o_n(1);$

(D) $\int_{B_{\rho_n}(0)} H(u^1(z))dz=\int_{\mathbb{R}^N} H(u^1(z))dz+o_n(1);$

(E) $\int_{B_{\rho_n}(\tau y_n^1-y_n^1)} H(u^1(z+y_n^1-\tau y_n^1))dz=\int_{\mathbb{R}^N} H(u^1(z))dz+o_n(1);$

(F) $\int_{S_n} H(u^1(z))dz=o_n(1).$

First, we will verify that condition (A) is true. By (1.1.3) with $0\le p\le 2^*-2$ and by mean value theorem, there exists $0\le\theta\le 1$, such that

$$\int_{B_{\rho_n}(0)}\Big(H(u_n^1(z+y_n^1)-u^1(z+y_n^1-\tau y_n^1))-H(u_n^1(z+y_n^1))\Big)dz$$
$$\le\int_{B_{\rho_n}(0)} h(u_n^1(z+y_n^1)+\theta(z)u^1(z+y_n^1-\tau y_n^1))(u_n^1(z+y_n^1)$$
$$+\theta(z)u^1(z+y_n^1-\tau y_n^1)).u^1(z+y_n^1-\tau y_n^1)dz$$
$$\le\varepsilon\int_{B_{\rho_n}(0)}(u_n^1(z+y_n^1)+\theta(z)u^1(z+y_n^1-\tau y_n^1))u^1(z+y_n^1-\tau y_n^1)dz$$
$$+C\int_{B_{\rho_n}(0)}|u_n^1(z+y_n^1)+\theta(z)u^1(z+y_n^1-\tau y_n^1)|^{p-1}u^1(z+y_n^1-\tau y_n^1)dz$$
$$\le\varepsilon\|u_n^1\|^2_{H^1(\mathbb{R}^N)}\left(\int_{B_{\rho_n}(0)}|u^1(z+y_n^1-\tau y_n^1)|^2dz\right)^{1/2}+\varepsilon\int_{B_{\rho_n}(0)}|u^1(z+y_n^1-\tau y_n^1)|^2dz$$

$$+C\|u_n^1\|^2_{H^1(\mathbb{R}^N)}\left(\int_{B_{\rho n}(0)}|u^1(z+y_n^1-\tau y_n^1)|^{p-2}dz\right)^{\frac{p-1}{p-2}}+C\int_{B_{\rho n}(0)}|u^1(z+y_n^1-\tau y_n^1)|^p dz.$$

Consider the following change of variables $x=z+y_n^1-\tau y_n^1$. Thus, if $|z|<\rho=\dfrac{|y_n^1-\tau y_n^1|}{2}$, we have $|z+y_n^1-\tau y_n^1|>|y_n^1-\tau y_n^1|-|z|>\dfrac{|y_n^1-\tau y_n^1|}{2}=\rho_n\to\infty$. Therefore, given $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that, for all $n\geq n_0$, we have

$$\int_{B_{\rho n}(0)}|u^1(z+y_n^1-\tau y_n^1)|^2dz\leq\int_{\mathbb{R}^N\setminus B_{\rho n}(0)}|u^1(z)|^2dz<\varepsilon$$
$$\int_{B_{\rho n}(0)}|u^1(z+y_n^1-\tau y_n^1)|^pdz\leq\int_{\mathbb{R}^N\setminus B_{\rho n}(0)}|u^1(z)|^pdz<\varepsilon.$$

Therefore, we show (A) and in an entirely analogous way we show (B), because using the mean value theorem again, there exists $0\leq\theta\leq1$, such that

$$\begin{aligned}
&\int_{B_{\rho n}(\tau y_n^1-y_n^1)}\Big(H(u_n^1(z+y_n^1)-u^1(z))-H(u_n^1(z+y_n^1))\Big)dz\\
&\leq\int_{B_{\rho n}(\tau y_n^1-y_n^1)}h(u_n^1(z+y_n^1)+\theta u^1(z))(u_n^1(z+y_n^1)+\theta u^1(z)).u^1(z)dz\\
&\leq\varepsilon\int_{B_{\rho n}(\tau y_n^1-y_n^1)}|u_n^1(z+y_n^1)+\theta u^1(z)||u^1(z)|dz\\
&+C\int_{B_{\rho n}(\tau y_n^1-y_n^1)}|u_n^1(z+y_n^1)+\theta u^1(z)|^{p-1}|u^1(z)|dz\\
&\leq\varepsilon\|u_n^1\|^2_{H^1(\mathbb{R}^N)}\left(\int_{B_{\rho n}(\tau y_n^1-y_n^1)}|u^1(z+y_n^1-\tau y_n^1)|^2dz\right)^{1/2}+\varepsilon\int_{B_{\rho n}(\tau y_n^1-y_n^1)}|u^1(z+y_n^1-\tau y_n^1)|^2dz\\
&+C\|u_n^1\|^2_{H^1(\mathbb{R}^N)}\left(\int_{B_{\rho n}(\tau y_n^1-y_n^1)}|u^1(z+y_n^1-\tau y_n^1)|^{p-2}dz\right)^{\frac{p-1}{p-2}}+C\int_{B_{\rho n}(\tau y_n^1-y_n^1)}|u^1(z+y_n^1-\tau y_n^1)|^pdz,
\end{aligned}$$

and the result (B) follows as made in (A). Next, we will check (C). In fact, we first consider $w^1(z)=u^1(z+y_n^1-\tau y_n^1)$. So, we have to

$$\begin{aligned}
&\int_{S_n}\Big(H(u_n^1(z+y_n^1)-u^1(z+y_n^1-\tau y_n^1))-H(u^1(z))\Big)dz\\
&\quad\leq\int_{S_n}h(u_n^1(z+y_n^1)+\theta(z)w^1(z))(u_n^1(z+y_n^1)+\theta(z)w^1(z))w^1(z)dz\\
&\quad\leq\varepsilon\|u_n^1\|^2_{H^1(\mathbb{R}^N)}\left(\int_{S_n}|w^1(z)|^2dz\right)^{1/2}+\varepsilon\int_{S_n}|w^1(z)|^2dz\\
&\quad+C\|u_n^1\|^{p-1}_{H^1(\mathbb{R}^N)}\left(\int_{S_n}|w^1(z)|^{p-2}dz\right)^{\frac{p-2}{p-1}}+C\int_{S_n}|w^1(z)|^pdz.
\end{aligned}$$

We claim that

$$\int_{S_n} |w^1(z)|^2 dz, \int_{S_n} |w^1(z)|^p dz = o_n(1).$$

Indeed, making a change of variable $x = z - (\tau y_n^1 - y_n^1)$ together with $|z + y_n^1 - \tau y_n^1| > |y_n^1 - \tau y_n^1| - |z| > \frac{|y_n^1 - \tau y_n^1|}{2} = \rho_n \to \infty$ when $n \to \infty$ and that $u^1 \in L^p(\mathbb{R}^N),\ 2 \le p < 2^*$, we have

$$\begin{aligned}
\int_{S_n} |w^1(z)|^2 dz &= \int_{S_n} |w^1(z + y_n^1 - \tau y_n^1)|^2 dz \\
&\le \int_{\mathbb{R}^N \setminus B_{\rho_n}(\tau y_n^1 - y_n^1)} |w^1(z + y_n^1 - \tau y_n^1)|^2 dz \\
&= \int_{\mathbb{R}^N \setminus B_{\rho_n}(0)} |w^1(z)|^2 dz < \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\int_{S_n} |w^1(z)|^p dz &= \int_{S_n} |w^1(z + y_n^1 - \tau y_n^1)|^p dz \\
&\le \int_{\mathbb{R}^N \setminus B_{\rho_n}(\tau y_n^1 - y_n^1)} |w^1(z + y_n^1 - \tau y_n^1)|^p dz \\
&\le \int_{\mathbb{R}^N \setminus B_{\rho_n}(0)} |w^1(z)|^p dz < \varepsilon.
\end{aligned}$$

We check item (C). In an entirely analogous way as done in item (C) and using the growth of H from (1.1.4), we show (F). Next we will verify (D). In fact, using $u^1 \in L^p(\mathbb{R}^N),\ 2 \le p < 2^*$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus B_{\rho_n}(0)} H(u^1(z)) dz &= \int_{0<|z|<\rho_n} H(u^1(z)) dz + \int_{|z|>\rho_n} H(u^1(z)) dz \\
&\quad - \int_{0<|z|<\rho_n} H(u^1(z)) dz = o_n(1).
\end{aligned}$$

Similarly, (E) also holds. Therefore, the proof of all six statements are complete and (1.3.44) is holds.

From (1.3.43) and (1.3.44) we obtain that $I_\infty(u_n^2) = I_\infty(u_n^1) - 2I_\infty(u^1) + o_n(1)$ which complete the proof of (1.3.38).

Since (u_n^1) is a (PS) sequence of I_∞, then $I_\infty(u_n^2)$ converges to a constant. To complete the prove we will show that if $n \to \infty$, then (1.3.34) is hold. Indeed,

$$|I'_\infty(u_n^2)\varphi| = \left| \int_{\mathbb{R}^N} (\xi_\infty \nabla(u_n^1 - \gamma_n) \nabla\varphi + (u_n^1 - \gamma_n)\varphi) dx - \int_{\mathbb{R}^N} h(u_n^1 - \gamma_n)(u_n^1 - \gamma)\varphi dx \right|$$

$$\leq \left|\int_{\mathbb{R}^N}(\xi_\infty \nabla u_n^1 \nabla \varphi + u_n^1 \varphi)dx - \int_{\mathbb{R}^N} h(u_n^1)u_n^1\varphi dx + \int_{\mathbb{R}^N}(\xi_\infty \nabla \gamma_n \nabla \varphi + \gamma_n \varphi)dx \right.$$
$$- \int_{\mathbb{R}^N} h(\gamma_n)\gamma_n \varphi dx - \int_{\mathbb{R}^N} h(u_n^1 - \gamma_n)(u_n^1 - \gamma)\varphi dx$$
$$\left. + \int_{\mathbb{R}^N} h(u_n^1)u_n^1 \varphi dx + \int_{\mathbb{R}^N} h(\gamma_n)\gamma_n \varphi dx \right|.$$

And since (u_n^1) is a (PS) sequence of I_∞ we have that

$$\int_{\mathbb{R}^N}(\xi_\infty \nabla u_n^1 \nabla \varphi + u_n^1 \varphi)dx - \int_{\mathbb{R}^N} h(u_n^1)u_n^1 \varphi dx = o_n(1). \tag{1.3.45}$$

From (1.3.45), using the definition of γ_n and from the triangular inequality we obtain that

$$|I'_\infty(u_n^2)\varphi| \leq K_n^1 + K_n^2 + o_n(1), \tag{1.3.46}$$

where

$$K_n^1 := \int_{\mathbb{R}^N}(\xi_\infty \nabla \gamma_n \nabla \varphi + \gamma_n \varphi)dx$$
$$= \int_{\mathbb{R}^N}(\xi_\infty \nabla (u^1(x - y_n^1) - u^1(\tau x - y_n^1))\nabla \varphi + (u^1(x - y_n^1) - u^1(\tau x - y_n^1))\varphi)dx$$

and

$$K_n^2 := \int_{\mathbb{R}^N} |h(\gamma_n)||\gamma_n||\varphi| dx$$
$$= \int_{\mathbb{R}^N} |h(u^1(x - y_n^1) - u^1(\tau x - y_n^1))||u^1(x - y_n^1) - u^1(\tau x - y_n^1)||\varphi| dx.$$

We will first show that $K_n^1 = o_n(1)$. In fact, let us consider $\varphi \in C_0^\infty(\mathbb{R}^N)$, with $\Omega = supp(\varphi)$, $|y_n^1| \to \infty$, $|\nabla u^1| \in L^2(\mathbb{R}^N)$ and using Hölder's inequality we have

$$\int_{\mathbb{R}^N} |\nabla u^1(x - y_n^1)||\nabla \varphi| dx = \int_\Omega |\nabla u^1(x - y_n^1)||\nabla \varphi| dx$$
$$\leq \left(\int_\Omega |\nabla u^1(x - y_n^1)|^2 dx\right)^{1/2} \|\varphi\|_{H^1(\mathbb{R}^N)} < \varepsilon,$$

when $n \to \infty$. Similarly

$$\int_{\mathbb{R}^N} |\nabla u^1(\tau x - y_n^1)||\nabla \varphi| dx < \varepsilon \text{ and } \int_\Omega |u^1(\tau x - y_n^1)||\varphi| dx < \varepsilon,$$

thus implying that $K_n^1 = o_n(1)$. The next step is to also show that $K_n^2 = o_n(1)$. Using the growth of h from (1.1.3) and an argument analogous to the previous one we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |h(u^1(x-y_n^1)-u^1(\tau x-y_n^1))(u^1(x-y_n^1)-u^1(\tau x-y_n^1))\varphi|dx \\
&\quad \le \varepsilon \int_{\mathbb{R}^N} |u^1(x-y_n^1)-u^1(\tau x-y_n^1)||\varphi|dx + C\int_{\mathbb{R}^N} |u^1(x-y_n^1)-u^1(\tau x-y_n^1)|^{p-1}|\varphi|dx \\
&\quad \le C_1 \int_{\mathbb{R}^N} |u^1(x-y_n^1)||\varphi|dx + C_1 \int_{\mathbb{R}^N} |u^1(\tau x-y_n^1)||\varphi|dx \\
&\quad + C_2 \int_{\mathbb{R}^N} |u^1(x-y_n^1)|^{p-1}|\varphi|dx + C_2 \int_{\mathbb{R}^N} |u^1(\tau x-y_n^1)|^{p-1}|\varphi|dx \\
&\quad < \varepsilon.
\end{aligned}
$$

Therefore we conclude that $K_n^2 = o_n(1)$. In this way, (1.3.34) holds, and thus we verify that (u_n^2) is a (PS) sequence of I_∞, also in *Case II*.

Now proceeding by iteration, we note that if u is a non-trivial critical point of I_∞ and ω is a minimum energy solution of the equation (1.1.4) given by Berestycki and Lions, then we have that

$$I_\infty(u) \ge I_\infty(\omega) > 0. \tag{1.3.47}$$

On the other hand, from (1.3.38) and item (ii) we obtain

$$
\begin{aligned}
I_\infty(u_n^2) &= I_\infty(u_n^1) - 2I_\infty(u^1) + o_n(1) \\
&= I(u_n) - I(u_0) - 2I_\infty(u^1) + o_n(1) \\
&= c - I(u_0) - 2I_\infty(u^1) + o_n(1).
\end{aligned} \tag{1.3.48}
$$

From (1.3.45) and (1.3.46) the iteration must end at some index $k \in \mathbb{N}$ and the proof of lemma is complete. □

In the next result, we verify that the functional I restricted to E^τ, associated with the problem (1.1.4), satisfying $(Ce)_c$ for c below the level $2m_\infty$.

Lemma 1.3.6. *The functional I restricted to E^τ satisfies $(Ce)_c$ for any $c < 2m_\infty$.*

Proof. Let $(u_n) \subset E^\tau$ such that

$$I(u_n) \to c < 2m_\infty \qquad \text{and} \qquad (1+\|u_n\|)\left\|I'|_{E^\tau}(u_n)\right\| \to 0.$$

This imply that $I'|_{E^\tau}(u_n) \to 0$ and by Lemma 1.3.4 we have $I'(u_n) \to 0$. Moreover, by Lemma 1.3.1, (u_n) is a bounded sequence, up to a subsequence, $u_n \rightharpoonup u_0$ in E and $I'(u_0)\varphi = 0$, for all $\varphi \in E$. In particular,

$$I'(u_0)u_0 = \int_{\mathbb{R}^N} \left(\xi(x)|\nabla u_0|^2 + u_0^2\right) dx - \int_{\mathbb{R}^N} f(x,u_0)u_0 dx = 0. \tag{1.3.49}$$

It follows from the hypothesis (f_5) and (1.3.49) that

$$\begin{aligned} I(u_0) &= \frac{1}{2}\|u_0\|^2 - \int_{\mathbb{R}^N} F(x,u_0)dx \\ &= \int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u_0)u_0 - F(x,u_0)\right) dx \geq 0. \end{aligned} \tag{1.3.50}$$

If (u_n) does not converge strongly to u_0 in the norm of E then, by Lemma 1.3.5 there exists two integers $k_1 \geq 1$ or $k_2 \geq 1$, k_1 solutions u^j, $j = 1, \ldots, k_1$ and k_2 τ-antisymmetric solutions u^j, $j = k_1+1, \ldots, k_1+k_2$ of equation (1.1.4), satisfying

$$\begin{aligned} c = \lim_{n\to\infty} I(u_n) &= I(u_0) + 2\sum_{j=1}^{k_1} I_\infty(u^j) + \sum_{j=k_1+1}^{k_1+k_2} I_\infty(u^j) \\ &\geq I(u_0) + 2k_1 m_\infty + \sum_{j=k_1+1}^{k_1+k_2} I_\infty(u^j) \geq 2m_\infty, \end{aligned} \tag{1.3.51}$$

since $I_\infty(u^j) \geq 2m_\infty$ for all nontrivial τ-antisymmetric solution u^j of (1.1.4), which contradicts our assumption. Therefore, up to a subsequence, $u_n \to u_0 \in E^\tau$ and the lemma is proved. □

Lemma 1.3.7. *Let* $m_\infty^\tau := \inf_{u\in\mathcal{P}} I_\infty(u)$, *then*

$$2m_\infty \leq m_\infty^\tau.$$

Proof. Let us show first that if $u \in \mathcal{P}$, then u^+, $u^- \in \mathcal{P}$. Using a change of variables and that $G(s)$ is an even fuction and defining $A^\tau := \{x : -u(\tau x) \geq 0\}$, we obtain

$$\begin{aligned} J(u^+) &= \int_{\{x:u(x)\geq 0\}} |\nabla u|^2 dx - 2^* \int_{\{x:u(x)\geq 0\}} G_\infty(u)dx \\ &= \int_{A^\tau} |\nabla(-u(\tau x))|^2 dx - 2^* \int_{A^\tau} G_\infty(-u(\tau x))dx \\ &= \int_{\{z:u(z)\leq 0\}} |\nabla u|^2 dz - 2^* \int_{\{z:u(z)<0\}} G_\infty(-u(z))dz \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N} |\nabla u^-|^2 dz - 2^* \int_{\mathbb{R}^N} G_\infty(u^-) dz \\
&= J(u^-).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
0 = J(u) &= \int_{\{x:u(x)\geq 0\}} |\nabla u|^2 dx - 2^* \int_{\{x:u(x)\geq 0\}} G_\infty(u) dx \\
&\quad + \int_{\{x:u(x)<0\}} |\nabla u|^2 dx - 2^* \int_{\{x:u(x)<0\}} G_\infty(u) dx \\
&= \int_{\mathbb{R}^N} |\nabla u^+|^2 dx - 2^* \int_{\mathbb{R}^N} G_\infty(u^+) dx \\
&\quad + \int_{\mathbb{R}^N} |\nabla u^-|^2 dx - 2^* \int_{\mathbb{R}^N} G_\infty(u^-) dx \\
&= J(u^+) + J(u^-) = 2J(u^+) = 2J(u^-).
\end{aligned}
$$

Therefore $u^+,\ u^- \in \mathcal{P}$. Now, since H is even we have

$$
\begin{aligned}
I_\infty(u^+) &= \int_{\{x:u(x)\geq 0\}} (\xi_\infty |\nabla u|^2 + u^2) dx - \int_{\{x:u(x)\geq 0\}} H(u) dx \\
&= \int_{A^\tau} (\xi_\infty |\nabla(-u(\tau x))|^2 + |-u(\tau x)|^2) dx - \int_{A^\tau} H(-u(\tau x)) dx \\
&= \int_{\{z:u(z)\leq 0\}} (\xi_\infty |\nabla u|^2 + u^2) dz - \int_{\{z:u(z)\leq 0\}} H(-u) dz \\
&= \int_{\mathbb{R}^N} (\xi_\infty |\nabla u^-|^2 + (u^-)^2) dz - \int_{\mathbb{R}^N} H(u^-) dz \\
&= I_\infty(u^-).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
I_\infty(u) &= \int_{\{x:u(x)\geq 0\}} (\xi_\infty |\nabla u|^2 + u^2) dx - \int_{\{x:u(x)\geq 0\}} H(u) dx \\
&\quad + \int_{\{x:u(x)<0\}} (\xi_\infty |\nabla u|^2 + u^2) dx - \int_{\{x:u(x)<0\}} H(u) dx \\
&= \int_{\mathbb{R}^N} (\xi_\infty |\nabla u^+|^2 + |u^+|^2) dx - \int_{\mathbb{R}^N} H(u^+) dx \\
&\quad + \int_{\mathbb{R}^N} (\xi_\infty |\nabla u^-|^2 + |u^-|^2) dx - \int_{\mathbb{R}^N} H(u^-) dx \\
&= I_\infty(u^+) + I_\infty(u^-).
\end{aligned}
$$

Therefore, for all $u \in \mathcal{P}$ we have

$$I_\infty(u) = I_\infty(u^+) + I_\infty(u^-) = 2I_\infty(u^+) \geq 2m_\infty,$$

thus,

$$m_\infty^\tau = \inf_{u\in\mathcal{P}} I_\infty(u) \geq 2m_\infty.$$

□

Remark 1.3.5. *If $z_y(x) = \omega(x-y) - \omega(x-\tau y)$, then t_{z_y} as in (1.3.13) is bounded when $|y| \to \infty$ and $|y-\tau y| \to \infty$.*

Lemma 1.3.8. *Suppose ξ satisfies $(\xi_1)-(\xi_4)$ and either (1.1.9) or (1.1.10). Then*

$$c^\tau < 2m_\infty.$$

Proof. Denote $t = t_{z_y}$, for simplicity of notation. Since I_∞ is translation invariance we obtain

$$\begin{aligned}
I\left(z_y\left(\frac{\cdot}{t}\right)\right) &= \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} \xi(tx)|\nabla\omega(x-y)|^2dx + \frac{t^{N-2}}{2}\int_{\mathbb{R}^N}\xi(tx)|\nabla\omega(x-\tau y)|^2dx \\
&\quad -2\frac{t^{N-2}}{2}\int_{\mathbb{R}^N}\xi(tx)\nabla\omega(x-y)\nabla\omega(x-\tau y)dx + \frac{t^N}{2}\int_{\mathbb{R}^N}\omega^2(x-y)dx \\
&\quad +\frac{t^N}{2}\int_{\mathbb{R}^N}\omega^2(x-\tau y)dx - 2\frac{t^N}{2}\int_{\mathbb{R}^N}\omega(x-y)\omega(x-\tau y)dx \\
&\quad -t^N\int_{\mathbb{R}^N} F(tx,\omega(x-y)-\omega(x-\tau y))dx \\
&= I_\infty\left(\omega\left(\frac{\cdot}{t}-y\right)\right) + I_\infty\left(\omega\left(\frac{\cdot}{t}-\tau y\right)\right) \\
&\quad +\frac{t^{N-2}}{2}\int_{\mathbb{R}^N}(\xi(tx)-\xi_\infty)|\nabla\omega(x-y)|^2dx \\
&\quad +\frac{t^{N-2}}{2}\int_{\mathbb{R}^N}(\xi(tx)-\xi_\infty)|\nabla\omega(x-\tau y)|^2dx \\
&\quad -t^{N-2}\int_{\mathbb{R}^N}\xi(tx)\nabla\omega(x-y)\nabla\omega(x-\tau y)dx \\
&\quad -t^N\int_{\mathbb{R}^N}\omega(x-y)\omega(x-\tau y)dx \\
&\quad +t^N\int_{\mathbb{R}^N}\Big(H(\omega(x-y)) - F(tx,\omega(x-y))\Big)dx \\
&\quad +t^N\int_{\mathbb{R}^N}\Big(H(\omega(x-\tau y)) - F(tx,\omega(x-\tau y))\Big)dx \\
&\quad -t^N\int_{\mathbb{R}^N}F(tx,\omega(x-y)-\omega(x-\tau y))dx + t^N\int_{\mathbb{R}^N}F(tx,\omega(x-y))dx
\end{aligned}$$

$$
\begin{aligned}
& +t^N \int_{\mathbb{R}^N} F(tx, \omega(x-\tau y))dx \\
= \quad & I_\infty\left(\omega\left(\frac{\cdot}{t}\right)\right) + I_\infty\left(\omega\left(\frac{\cdot}{t}\right)\right) + R(\xi, \xi_\infty, |y|, |y-\tau y|), \qquad (1.3.52)
\end{aligned}
$$

where

$$
\begin{aligned}
R(\xi,\xi_\infty,|y|,|y-\tau y|) &= \frac{t^{N-2}}{2}\int_{\mathbb{R}^N}(\xi(tx)-\xi_\infty)|\nabla\omega(x-y)|^2dx \\
&+ \frac{t^{N-2}}{2}\int_{\mathbb{R}^N}(\xi(tx)-\xi_\infty)|\nabla\omega(x-\tau y)|^2dx - t^{N-2}\int_{\mathbb{R}^N}\xi(tx)\nabla\omega(x-y)\nabla\omega(x-\tau y)dx \\
&- t^N\int_{\mathbb{R}^N}\omega(x-y)\omega(x-\tau y)dx + t^N\int_{\mathbb{R}^N}\Big(H(\omega(x-y)) - F(tx,\omega(x-y))\Big)dx \\
&+ t^N\int_{\mathbb{R}^N}\Big(H(\omega(x-\tau y)) - F(tx,\omega(x-\tau y))\Big)dx - t^N\int_{\mathbb{R}^N}F(tx,\omega(x-y)-\omega(x-\tau y))dx \\
&+ t^N\int_{\mathbb{R}^N}F(tx,\omega(x-y))dx + t^N\int_{\mathbb{R}^N}F(tx,\omega(x-\tau y))dx. \qquad (1.3.53)
\end{aligned}
$$

In order to evaluate the sum

$$
\int_{\mathbb{R}^N}F(tx,\omega(x-y)-\omega(x-\tau y))dx - \int_{\mathbb{R}^N}F(tx,\omega(x-y))dx - \int_{\mathbb{R}^N}F(tx,\omega(x-\tau y))dx,
$$

we use hypothesis (f_7). The Theorem A, (1.1.3) with $\varepsilon > 0$ and $2 < p < 2^*$, give us

$$
\begin{aligned}
\Big|F(tx,\omega(x-y) \quad &- \quad \omega(x-\tau y)) - F(tx,\omega(x-y)) - F(tx,\omega(x-\tau y))\Big| \\
&\leq \quad 2\Big[|f(tx,\omega(x-y))|\,|\omega(x-\tau y)| + |f(tx,\omega(x-\tau y))|\,|\omega(x-y)|\Big] \\
&\leq \quad 2\varepsilon\,|\omega(x-y)|\,|\omega(x-\tau y))| + C\,|\omega(x-y)|^{p-1}\,|\omega(x-\tau y)| \\
&\qquad +2\varepsilon\,|\omega(x-\tau y)|\,|\omega(x-y)| + C\,|\omega(x-\tau y)|^{p-1}\,|\omega(x-y)|\,.
\end{aligned}
$$

It follows from the above estimate and the invariance of translation of the integral that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\Big|F(tx,\omega(x-y)-\omega(x-\tau y)) - F(tx,\omega(x-y)) - F(tx,\omega(x-\tau y))\Big|dx \\
&\leq 4\varepsilon\int_{\mathbb{R}^N}\Big|\omega(x-y)\Big|\Big|\omega(x-\tau y)\Big|dx + C\int_{\mathbb{R}^N}\Big|\omega(x-y)\Big|^{p-1}\Big|\omega(x-\tau y)\Big|dx \\
&+ C\int_{\mathbb{R}^N}\Big|\omega(x-\tau y)\Big|^{p-1}\Big|\omega(x-y)\Big|dx \\
&= 4\varepsilon\int_{\mathbb{R}^N}\Big|\omega(z)\Big|\Big|\omega(z+y-\tau y)\Big|dz + C\int_{\mathbb{R}^N}\Big|\omega(z)\Big|^{p-1}\Big|\omega(z+y-\tau y)\Big|dz \\
&+ C\int_{\mathbb{R}^N}\Big|\omega(\hat{z})\Big|^{p-1}\Big|\omega(\hat{z}-(y-\tau y)\Big|d\hat{z}
\end{aligned}
$$

$$= 4\varepsilon\int_{\mathbb{R}^N}\Big|\omega(z)\Big|\Big|\omega(z+y-\tau y)\Big|dz + 2C\int_{\mathbb{R}^N}\Big|\omega(z)\Big|^{p-1}\Big|\omega(z+y-\tau y)\Big|dz.$$

Now we estimate the integrals above. Let $0<\delta<1/2$ to be chosen later, define $A_y := B_{\frac{|y-\tau y|}{p}(1-\delta)}(0)\subset\mathbb{R}^N$ and $R_y := \dfrac{|y-\tau y|}{p}(1-\delta)$. Since ω is solution of (1.1.4), we have $|\omega(x)|\le Ce^{-\beta|x|}$ for all $\beta\in\left(0,\sqrt{1/\xi_\infty}\right)$ and

$$\begin{aligned}
\int_{A_y}|\omega(x-y)|^{p-1}|\omega(x-\tau y)|dx &= \int_{A_y}|\omega(z)|^{p-1}|\omega(z+y-\tau y)|dz\\
&\le \left(\int_{\mathbb{R}^N}(|\omega(z)|^{p-1})^{\frac{p}{p-1}}dz\right)^{\frac{p-1}{p}}\left(\int_{A_y}|\omega(z+y-\tau y)|^p dz\right)^{1/p}\\
&= \left(\int_{\mathbb{R}^N}|\omega(z)|^p dz\right)^{\frac{p-1}{p}}\left(\int_{A_y}|\omega(z+y-\tau y)|^p dz\right)^{1/p}\\
&= \|\omega\|_{L^p}^{p-1}\left(\int_{A_y}|\omega(z+y-\tau y)|^p dz\right)^{1/p}\\
&\le C\|\omega\|_{L^p}^{p-1}\left(\int_{A_y}e^{-\beta p|z+y-\tau y|}dz\right)^{1/p}\\
&\le C\left(e^{-\beta p|y-\tau y|}\int_{A_y}e^{-\beta p|z|}dz\right)^{1/p}\\
&= Ce^{-\beta|y-\tau y|}\left(\int_{A_y}e^{-\beta p|z|}dz\right)^{1/p}, \qquad (1.3.54)
\end{aligned}$$

making change of variable $\tilde{f}:\mathbb{R}^N\to\mathbb{R}^N,\ z\mapsto -r$ with determinant of the Jacobian given by $det(J(z_1,\cdots,z_N)) = r^{N-1}$, and by change of variable theorem , we have that

$$\begin{aligned}
\int_{A_y}e^{-\beta p|z|}dz &= \int_0^{\frac{|y-\tau y|}{p}(1-\delta)}e^{\beta pr}det(J(z_1,\cdots,z_N))dr\\
&= \int_0^{\frac{|y-\tau y|}{p}(1-\delta)}e^{\beta pr}r^{N-1}dr.
\end{aligned}$$

Replacing in (1.3.54)

$$\int_{A_y}|\omega(x-y)|^{p-1}|\omega(x-\tau y)|dx \le Ce^{-\beta|y-\tau y|}\left(\int_0^{\frac{|y-\tau y|}{p}(1-\delta)}e^{\beta pr}r^{N-1}dr\right)^{1/p}$$

$$
\begin{aligned}
&\leq \; Ce^{-\beta|y-\tau y|}\left(e^{\beta p\frac{|y-\tau y|}{p}(1-\delta)}\int_0^{\frac{|y-\tau y|}{p}(1-\delta)} r^{N-1}dr\right)^{1/p}\\
&= \; Ce^{-\beta|y-\tau y|\frac{p}{p}}e^{\beta\frac{|y-\tau y|}{p}(1-\delta)}\left(\frac{|y-\tau y|}{p}(1-\delta)\right)^{N/p}\\
&\leq \; C(\delta)e^{-\beta|y-\tau y|\frac{p-1}{p}}e^{-\beta|y-\tau y|\frac{\delta}{p}}|y-\tau y|^{N/p}\\
&\leq \; C(\delta)e^{-\beta|y-\tau y|\frac{p-1}{p}}, \qquad (1.3.55)
\end{aligned}
$$

since $1 < p-1$ and $0 < \delta < 1/2$. Moreover,

$$
\begin{aligned}
\int_{\mathbb{R}^N\setminus A_y}|\omega(x-y)|^{p-1}|\omega(x-\tau y)|dx &= \int_{\mathbb{R}^N\setminus A_y}|\omega(z)|^{p-1}|\omega(z+y-\tau y)|dz\\
&\leq \left(\int_{\mathbb{R}^N\setminus A_y}(|w(z)|^{p-1})^{\frac{p}{p-1}}dz\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}|w(z+y-\tau y)|^p dz\right)^{1/p}\\
&= \left(\int_{\mathbb{R}^N\setminus A_y}|w(z)|^p dz\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}|w(z)|^p dz\right)^{1/p}\\
&\leq C\,\|\omega\|_{L^p}\left(\int_{\mathbb{R}^N\setminus A_y}e^{-\beta p|z|}dz\right)^{\frac{p-1}{p}}\\
&= C\,\|\omega\|_{L^p}^{p-1}\left(\int_{\frac{|y-\tau y|}{p}(1-\delta)}^{\infty}e^{-\beta pr}r^{N-1}dr\right)^{\frac{p-1}{p}}.
\end{aligned}
$$

Now, using integration by parts, for any $k > 0$ we have

$$\int e^{-kr}r^{N-1}dr = e^{-kr}P(r),$$

where

$$P(r) := \frac{r^{N-1}}{k} - \frac{(N-1)}{k^2}r^{N-2} + \frac{(N-1)(N-2)}{k^3}r^{N-3} + \ldots + (-1)^{N+1}\frac{(N-1)!}{k^N}.$$

Thus,

$$\int_{R_y}^{\infty} e^{-kr}r^{N-1}dr = e^{-kr}P(r)\Big|_{R_y}^{\infty} = e^{-kR_y}P(R_y). \qquad (1.3.56)$$

Therefore, taking $k := \beta p$, we obtain that

$$\int_{\mathbb{R}^N\setminus A_y}|\omega(x-y)|^{p-1}|\omega(x-\tau y)|dx$$

$$\begin{aligned}
&\leq C\,\|\omega\|_{L^p}\left[e^{-\beta p|y-\tau y|\frac{1-\delta}{p}}P\left(|y-\tau y|\frac{1-\delta}{p}\right)\right]^{\frac{p-1}{p}}\\
&= C\|w\|_{L^p}e^{-\beta p|y-\tau y|(1-\delta)\frac{p-1}{p}}P\left(|y-\tau y|\frac{1-\delta}{p}\right)^{\frac{p-1}{p}}\\
&= C\|w\|_{L^p}e^{-\beta p|y-\tau y|(1-2\delta)\frac{p-1}{p}}\left[e^{\beta p|y-\tau y|\delta}P\left(|y-\tau y|\frac{1-\delta}{p}\right)\right]^{\frac{p-1}{p}}\\
&\leq C(\delta)\,\|\omega\|_{L^p}\,e^{-\beta|y-\tau y|\frac{p-1}{p}(1-2\delta)}.
\end{aligned}$$

Hence, taking δ sufficiently small such that $0<(1-2\delta)<1$, we obtain

$$\int_{\mathbb{R}^N\setminus A_y}|\omega(x-y)|^{p-1}|\omega(x-\tau y)|dx\leq C(\delta)e^{-\beta|y-\tau y|\frac{p-1}{p}(1-2\delta)}. \tag{1.3.57}$$

Thus, from (1.3.55) and (1.3.57) we have

$$\int_{\mathbb{R}^N}|\omega(x-y)|^{p-1}|\omega(x-\tau y)|dx\leq Ce^{-\beta|y-\tau y|\frac{p-1}{p}(1-2\delta)}. \tag{1.3.58}$$

For $p=2$ we argue similarly and define $A_y=B_{\frac{|y-\tau y|}{2}(1-\delta)}(0)\subset\mathbb{R}^N$. Choosing $R_y:=\dfrac{|y-\tau y|}{2}(1-\delta)$ and using Hölder's inequality we obtain

$$\begin{aligned}
\int_{A_y}\omega(z)\omega(z+y-\tau y)dz &= \int_{A_y}\omega(z)\omega(z+y-\tau y)dz\\
&\leq \left(\int_{\mathbb{R}^N}|\omega(z)|^2dz\right)^{1/2}\left(\int_{A_y}|\omega(z+y-\tau y)|^2dz\right)^{1/2}\\
&\leq C\|\omega\|_{L^2}\left(\int_{A_y}e^{-\beta 2|z+y-\tau y|}dz\right)^{1/2}\\
&\leq Ce^{-\beta|y-\tau y|}\left(\int_0^{\frac{|y-\tau y|}{2}(1-\delta)}e^{\beta 2r}r^{N-1}dr\right)^{1/2}\\
&\leq Ce^{-\beta|y-\tau y|}\left(e^{2\beta\frac{|y-\tau y|}{2}(1-\delta)}\int_0^{\frac{|y-\tau y|}{2}(1-\delta)}r^{N-1}dr\right)^{1/2}\\
&= Ce^{-\beta|y-\tau y|}e^{\beta\frac{|y-\tau y|}{2}(1-\delta)}\left(\frac{|y-\tau y|}{2}(1-\delta)\right)^{N/2}\\
&\leq C(\delta)e^{-\beta\frac{|y-\tau y|}{2}}.
\end{aligned} \tag{1.3.59}$$

On the other hand, using Hölder's inequality and (1.3.56), it follows

$$\begin{aligned}
\int_{\mathbb{R}^N\setminus A_y} \omega(x-y)\omega(x-\tau y)dx &= \int_{\mathbb{R}^N\setminus A_y} \omega(z)\omega(z+y-\tau y)dz \\
&\leq \left(\int_{\mathbb{R}^N\setminus A_y} |\omega(z)|^2 dz\right)^{1/2} \left(\int_{\mathbb{R}^N} |\omega(z+y-\tau y)|^2 dz\right)^{1/2} \\
&\leq C\left(\int_{\mathbb{R}^N\setminus A_y} |\omega(z)|^2 dz\right)^{1/2} \left(\int_{\mathbb{R}^N} e^{-2\beta|z|} dz\right)^{1/2} \\
&\leq C\|\omega\|_{L^2}\left(\int_{|y-\tau y|\frac{1-\delta}{2}}^{\infty} e^{-2\beta r} r^{N-1} dr\right)^{1/2} \\
&\leq C\|\omega\|_{L^2} e^{-\beta|y-\tau y|\frac{1-2\delta}{2}} \left(e^{\beta|y-\tau y|\delta} P\left(|y-\tau y|\frac{1-\delta}{2}\right)\right)^{1/2} \\
&\leq C(\delta) e^{-\beta|y-\tau y|\frac{1-2\delta}{2}}. \qquad (1.3.60)
\end{aligned}$$

By (1.3.59), (1.3.60) and $0<(1-2\delta)<1$ it holds that

$$\begin{aligned}
\int_{\mathbb{R}^N} \omega(x-y)\omega(x-\tau y)dx &= \int_{\mathbb{R}^N} \omega(z)\omega(z+y-\tau y)dz \\
&= \int_{A_y} \omega(z)\omega(z+y-\tau y)dz \\
&\quad + \int_{\mathbb{R}^N\setminus A_y} \omega(z)\omega(z+y-\tau y)dz \\
&\leq C(\delta) e^{-\beta\frac{|y-\tau y|}{2}} + C(\delta) e^{-\beta|y-\tau y|\frac{(1-2\delta)}{2}} \\
&\leq C(\delta) e^{-\beta|y-\tau y|1/2(1-2\delta)}. \qquad (1.3.61)
\end{aligned}$$

Arguing as in the proof of inequality (1.3.61), we obtain

$$\int_{\mathbb{R}^N} \nabla\omega(x-y)\nabla\omega(x-\tau y)dx \leq Ce^{-\beta|y-\tau y|\frac{1}{2}(1-2\delta)}. \qquad (1.3.62)$$

We consider $\beta_1<\beta<\sqrt{1/\xi_\infty}$ or $\beta_2<\beta<\sqrt{1/\xi_\infty}$. By (1.1.9) and a change of variable, there exists a positive constant C such that

$$\begin{aligned}
\int_{\mathbb{R}^N} (\xi(x)-\xi_\infty)|\nabla\omega(x-y)|^2 dx &< -C\int_{\mathbb{R}^N} e^{-\beta_1|x|}|\nabla\omega(x-y)|^2 dx \\
&= -C\int_{\mathbb{R}^N} e^{-\beta_1|z+y|}|\nabla\omega(z)|^2 dz \\
&\leq -Ce^{-\beta_1|y|}\int_{\mathbb{R}^N} e^{-\beta_1|z|}|\nabla\omega(z)|^2 dz
\end{aligned}$$

$$\leq -Ce^{-\beta_1|y|}. \tag{1.3.63}$$

Similarly, we obtain

$$\int_{\mathbb{R}^N}(\xi(x)-\xi_\infty)|\nabla\omega(x-\tau y)|^2dx \leq -Ce^{-\beta_1|\tau y|} = -Ce^{-\beta_1|y|}. \tag{1.3.64}$$

Or else by (1.1.10), there exists a positive constant C such that

$$\begin{aligned}\int_{\mathbb{R}^N}|H(\omega(x-y))-F(tx,\omega(x-y))|dx &\leq -C\int_{\mathbb{R}^N}e^{-\beta_2|x|}|\omega(x-y)|^2dx\\ &= -C\int_{\mathbb{R}^N}e^{-\beta_2|z+y|}|\omega(z)|^2dz\\ &\leq -Ce^{-\beta_2|y|}\int_{\mathbb{R}^N}e^{-\beta_2|z|}|\omega(z)|^2dz\\ &\leq -Ce^{-\beta_2|y|}. \end{aligned} \tag{1.3.65}$$

In an analogous way, we have

$$\int_{\mathbb{R}^N}|H(\omega(x-\tau y))-F(tx,\omega(x-\tau y))|dx \leq -Ce^{-\beta_2|\tau y|} = -Ce^{-\beta_2|y|}. \tag{1.3.66}$$

Now we study the sign of $R(\xi,\xi_\infty,|y|,|y-\tau y|)$. If we consider the inequalities from (1.3.54) to (1.3.66) in the definition of $R(\xi,\xi_\infty,|y|,|y-\tau y|)$ in (1.3.53), then

$$\begin{aligned}R(\xi,\xi_\infty,|y|,|y-\tau y|) &\leq -Ce^{-\beta_1|y|}-Ce^{-\beta_1|y|}+C(\delta)e^{-\beta|y-\tau y|\frac{(1-2\delta)}{2}}\\ &\quad +C(\delta)e^{-\beta|y-\tau y|\frac{(1-2\delta)}{2}}-Ce^{-\beta_2|y|}-Ce^{-\beta_2|y|}+C(\delta)e^{-\beta|y-\tau y|\frac{p-1}{p}(1-2\delta)}\\ &\quad -Ce^{-\beta_2|y|}+Ce^{-\beta|y-\tau y|(1-2\delta)}+Ce^{-\beta|y-\tau y|\frac{1}{2}(1-2\delta)}.\end{aligned}$$

Let $\tilde{y}=(y_1,\ldots,y_k,\ldots,y_n)$, $\tau\tilde{y}=(y_1,\ldots,y_k,-y_{k+1},\ldots,-y_n)$, the projection $P_k\tilde{y}=(y_1,\ldots,y_k,0,\ldots,0)$ and $|\tilde{y}-\tau\tilde{y}|=|(0,\ldots,0,2y_{k+1},\ldots,2y_n)|=2|(0,\ldots,0,y_{k+1},\ldots,y_n)|$ be such that $|(0,\ldots,0,y_{k+1},\ldots,y_n)|\to\infty$. If we choose $y:=P_\Gamma^\perp\tilde{y}=(0,\ldots,0,y_{k+1},\ldots,y_n)$, such that $2|y|=|y-\tau y|$, since $t=t_{z_y}$ is bounded and $\frac{1}{2}<\frac{p-1}{p}$, we obtain for $|y|$ sufficiently large

$$R(\xi,\xi_\infty,|y|,|y-\tau y|) \leq -Ce^{-\beta_1|y|}-Ce^{-\beta_2|y|}+Ce^{-\beta(1-2\delta)|y|}<0. \tag{1.3.67}$$

Replacing (1.3.67) in (1.3.52) we obtain that $I\left(z_y\left(\frac{\cdot}{t_{z_y}}\right)\right) < 2m_\infty$.

To finish the proof of the lemma, let us fix any $y \in \mathbb{R}^N$, $|y| > 0$ sufficiently large and consider $n \in \mathbb{N}$, $n > 1$ and ny, such that

$$I\left(z_{ny}\left(\frac{\cdot}{t_{ny}}\right)\right) = \max_{t>0} I\left(z_{ny}\left(\frac{\cdot}{t}\right)\right).$$

Thus $0 < t_{ny} < L_0$ and for n sufficiently large, by (1.3.52) and (1.3.67)

$$I\left(z_{ny}\left(\frac{\cdot}{t_{ny}}\right)\right) < 2m_\infty. \tag{1.3.68}$$

On the other hand, by Remark 1.3.3, t_{ny} is such that

$$z_{ny}\left(\frac{\cdot}{t_{ny}}\right) \in \mathcal{P} \tag{1.3.69}$$

and there exists $L_{ny} > 0$ such that

$$I_\infty\left(z_{ny}\left(\frac{\cdot}{L_{ny}}\right)\right) < 0. \tag{1.3.70}$$

Now fix $n \in \mathbb{N}$, $n > 1$, let $L = \max\{L_{ny}, L_y\}$ and for $s \in [0,1]$ define the path

$$\gamma_n(s) = \omega\left(\frac{\cdot}{L} - (sy + (1-s)ny)\right) - \omega\left(\frac{\cdot}{L} - \tau(sy + (1-s)ny)\right),$$

$\gamma_n(s)$ belongs to E^τ,

$$\gamma_n(0) = \omega\left(\frac{\cdot}{L} - ny\right) - \omega\left(\frac{\cdot}{L} - \tau(ny)\right) = z_{ny}\left(\frac{\cdot}{L}\right),$$

and

$$\gamma_n(1) = \omega\left(\frac{\cdot}{L} - y\right) - \omega\left(\frac{\cdot}{L} - \tau y\right) = z_y\left(\frac{\cdot}{L}\right).$$

If we denote $X_s(n) := sy + (1-s)ny$, $0 \le s \le 1$, use the translation invariance of I_∞, then we obtain

$$I_\infty(\gamma_n(s)) = I_\infty\left(\omega\left(\frac{\cdot}{L} - X_s(n)\right) - \omega\left(\frac{\cdot}{L} - \tau X_s(n)\right)\right)$$

$$= I_\infty\left(\omega\left(\frac{\cdot}{L} - X_s(n)\right)\right) + I_\infty\left(\omega\left(\frac{\cdot}{L} - \tau X_s(n)\right)\right) + o_n(1)$$

$$= I_\infty\left(\omega\left(\frac{\cdot}{L}\right)\right) + I_\infty\left(\omega\left(\frac{\cdot}{L}\right)\right) + o_n(1) < 0, \tag{1.3.71}$$

for $0 \leq s \leq 1$ and all $n > 1$

$$|X_s(n)| = |sy + (1-s)ny| = |(s - sn + n)y| \geq |y|,$$

and

$$\begin{aligned} |\tau X_s(n) - X_s(n)| &= |sy + (1-s)ny - \tau(sy + (1-s)ny)| \\ &= |s(y - \tau y) + (1-s)n(y - \tau y)| \\ &\geq |y - \tau y|. \end{aligned}$$

For each $n > 1$ we consider the paths

$$\gamma_0(t) := \begin{cases} z_0 = 0, & \text{if} \quad t = 0, \\ z_{ny}\left(\frac{\cdot}{t}\right), & \text{if} \quad 0 < t \leq L, \end{cases}$$

and $\gamma_n(s)$, which respectively link the pairs of vectors $\left\{z_0, z_{ny}\left(\frac{\cdot}{L}\right)\right\}$ and $\left\{z_{ny}\left(\frac{\cdot}{L}\right), z_y\left(\frac{\cdot}{L}\right)\right\}$, and denote by γ_1 the path connects the pair $\left\{z_y(\frac{\cdot}{L}), z_y\left(\frac{\cdot}{L_y}\right)\right\}$ given by

$$\gamma_1(t) := z_y\left(\frac{\cdot}{tL_y + (1-t)L}\right).$$

The succession of these paths $\gamma_1 \circ \gamma_n \circ \gamma_0$, belongs to set Γ and connects z_0 to $z_1 = z_y(\frac{\cdot}{L_y})$. Furthermore, $I(\gamma_1(t)) \leq I_\infty(\gamma_1(t)) < 0$, and $I(\gamma_n(s)) \leq I_\infty(\gamma_n(s)) < 0$, thus

$$\max_{0 \leq t \leq 1} I(\gamma_1 \circ \gamma_n \circ \gamma_0(t)) = I\left(z_{n\bar{y}}\left(\frac{\cdot}{t_{n\bar{y}}}\right)\right). \tag{1.3.72}$$

Finally, if we take $n > 1$ sufficiently large, from (1.3.68), (1.3.72) and the definition of c^τ, we obtain

$$c^\tau \leq I\left(z_{n\bar{y}}\left(\frac{\cdot}{t_{n\bar{y}}}\right)\right) < 2m_\infty,$$

and the proof of Lemma 1.3.8 is complete. □

Proof of Theorem 1.1.2. Let $(u_n) \subset E^\tau$ be the sequence given by Ghoussoub-Priess Theorem in Lemma 1.3.3. By Lemma 1.3.1 this sequence is bounded, by Remark 1.3.2

$$I(u_n) \to c^\tau \quad \text{and} \quad I'(u_n) \to 0 \ \text{ in } (E^\tau)^*.$$

Up to a subsequence, $u_n \rightharpoonup u_0$ weakly in E and $I'(u_0) = 0$. By Lemma 1.3.5 we have that either $u_n \to u_0$ strongly in E or there exists two integers $k_1, k_2 \geq 0$, k_1 solutions u^j, $j = 1, ..., k_1$ and k_2 τ-antisymmetric solutions u^j, $j = k_1 + 1, ..., k_1 + k_2$ of equation (1.1.4), satisfying the conclusions of Lemma 1.3.5. Suppose the second case is holds. It follows from Lemma 1.3.8 that $c^\tau < 2m_\infty$ and hence in Lemma 1.3.5 item 5 we must have $k_1, k_2 = 0$. Otherwise, without loss of generality, if $k_1 \geq 1$ then by Lemma 1.3.7 we get

$$\begin{aligned} c^\tau &= I(u_0) + 2\sum_{j=1}^{k_1} I_\infty(u^j) + \sum_{j=k_1+1}^{k_1+k_2} I_\infty(u_j) \\ &\geq 2k_1 m_\infty + (k_1 + k_2) m_\infty^\tau \\ &\geq 2k_1 m_\infty + 2(k_1 + k_2) m_\infty \geq 2m_\infty, \end{aligned}$$

contrary our assumption that $c^\tau < 2m_\infty$. Therefore, $k_1 = k_2 = 0$, $u_n \to u_0$ strongly in E and $c^\tau = I(u_0)$. Moreover, since $I(u_0) = c^\tau > 0$, it follows that $u_0 \not\equiv 0$, u_0 is τ-antisymmetric and hence it is a sing-changing solution of (P_τ). □

Chapter 2

Problem with ξ and V positive

In this chapter, we will deal with the problem (P) considering ξ and V as positive functions, where V will assume some conditions and hypotheses that will be detailed below. Additionally, within this chapter, our focus will extend to investigating the nonautonomous and non-periodic Shrödinger equation exhibiting asymptotic growth in $\mathbb{R}^N$.

2.1 Variational Setting

We consider the following problem

$$\begin{cases} -div(\xi(x)\nabla u) + V(x)u = f(x,u), & \text{in} \quad \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{P_2}$$

with $N \geq 3$, under the following assumptions on ξ, $V \in C(\mathbb{R}^N, \mathbb{R}^+)$:

(ξ_1) there exists $\xi_0 > 0$ such that $\xi(x) \geq \xi_0$;

(ξ_2) $\lim_{|x|\to\infty} \xi(x) = \xi_\infty$;

(ξ_3) $\xi(x) \lneqq \xi_\infty$;

(V_1) there exists $V_0 > 0$ such that $V(x) \geq V_0$;

(V_2) $\lim_{|x|\to\infty} V(x) = V_\infty$;

(V_3) $V(x) \lneqq V_\infty$.

The hypotheses on the nonlinearity $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ are the following:

(f_1) $\lim_{s\to 0^+} \dfrac{f(x,s)}{s} = 0$, uniformly in $x \in \mathbb{R}^N$;

(f_2) there exist $a \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $h \in C(\mathbb{R}, \mathbb{R}^+)$ an even function satisfying $h(s) > 0$ for all $s > 0$, $h(0) = 0$ and

$$\lim_{s\to\infty} \frac{f(x,s)}{s} = a(x), \qquad \lim_{|x|\to\infty} \frac{f(x,s)}{s} = h(s),$$

$$\lim_{|x|\to\infty,\, s\to\infty} \frac{f(x,s)}{s} = \lim_{s\to\infty} h(s) = \lim_{|x|\to\infty} a(x) = a_\infty \ ;$$

(f_3) $\dfrac{f(x,s)}{s} \geq h(s)$, for all $x \in \mathbb{R}^N$ and all $s \in \mathbb{R}^+$ and $\dfrac{f(x,s)}{s} > h(s)$ for all $x \in \Omega$, where Ω is a subset of positive Lebesgue measure and for all $s \in \mathbb{R}^+$;

(f_4) $V_\infty < a_\infty \lneqq a(x)$, for all $x \in \mathbb{R}^N$;

(f_5) if we set $F(x,s) = \int_0^s f(x,t)dt$ and $Q(x,s) = \dfrac{1}{2} f(x,s)s - F(x,s)$, then

$$\lim_{s\to+\infty} Q(x,s) = +\infty$$

and there exists $D \geq 1$ such that

$$Q(x,s) < DQ(x,t), \quad \text{for all } x \in \mathbb{R}^N \text{ and } 0 \leq s < t.$$

The first result of this chapter can be stated as follows:

Theorem 2.1.1. *Suppose f satisfies $(f_1)-(f_5)$, ξ and V satisfy $(\xi_1)-(\xi_3)$ and $(V_1)-(V_3)$, respectively. Then problem (P_2) has a positive solution $u \in H^1(\mathbb{R}^N)$.*

The next remarks are the same as those of the first chapter here repeat for completeness, and their proofs will be omitted.

Remark 2.1.1. *Hypothesis (f_2) implies that there exists a constant $a_0 > 0$ such that*

$$a(x) \leq a_0, \quad \textit{for all} \ \ x \in \mathbb{R}^N. \tag{2.1.1}$$

Remark 2.1.2. *Note that conditions* (f_1), (f_2) *and* (2.1.1) *imply that for a given* $\varepsilon > 0$ *and* $2 \leq p \leq 2^*$, *there exists* $0 < C = C(\varepsilon, p)$ *such that*

$$|f(x,s)| \leq \varepsilon s + C|s|^{p-1} \tag{2.1.2}$$

and

$$|F(x,s)| \leq \frac{\varepsilon}{2}s^2 + C|s|^p. \tag{2.1.3}$$

Remark 2.1.3. *By* (f_1) *and* (f_5) *we obtain that* $Q(x,s) > 0$ *for* $s > 0$ *and* $x \in \mathbb{R}^N$. *Moreover, by* (f_2) *and* (f_5) *it follows that* $0 \leq \frac{1}{2}h(s)s^2 - H(s) \leq D\left(\frac{1}{2}h(t)t^2 - H(t)\right)$ *for* $0 \leq s \leq t$, *if* $H(s) = \int_0^s h(\zeta)\zeta d\zeta$ *and by assumptions* (f_1) *and* (f_5) *we have* $\frac{1}{2}f(x,s)s^2 - H(s) > 0$ *for* $s > 0$.

In the second part of this chapter, we look for a nodal solution. In this case, we assume some type of symmetry for the problem. More specifically, we consider the problem

$$\begin{cases} -div(\xi(x)\nabla u) + V(x)u = f(x,u), & \text{in} \quad \mathbb{R}^N, \\ u(\tau x) = -u(x), & \\ u(x) \to 0, & \text{as} \quad |x| \to \infty, \end{cases} \tag{P'_τ}$$

where $N \geq 3$ and $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is a nontrivial orthogonal involution, in other words, it is a linear orthogonal transformation in $\mathbb{R}^N$ such that $\tau \neq Id$ and $\tau^2 = Id$, with Id being the identity operator in $\mathbb{R}^N$. A solution u of (P'_τ) is called a τ-antisymmetric solution.

In this new setting, we need some technical assumptions. So we shall suppose that ξ, V and f satisfies:

(ξ_4) $\xi(\tau x) = \xi(x)$, for all $x \in \mathbb{R}^N$;

(V_4) $V(\tau x) = V(x)$, for all $x \in \mathbb{R}^N$;

(f_6) $f(\tau x, s) = -f(x,-s)$, for all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$;

(f_7) there exists $C > 1$, such that $f(x,s) \leq Cf(x,t)$, with $0 \leq s \leq t$, for all $x \in \mathbb{R}^N$.

Remark 2.1.4. *We do not assume that* $f(x,s)/s$ *for* $s > 0$ *is increasing in* s.

Consider the space $H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in (L^2(\mathbb{R}^N))^N\}$ equipped with the norm $\|u\|^2 = \int_{\mathbb{R}^N}(\xi_\infty|\nabla u|^2 + V_\infty u^2)dx$ and the limit problem

$$-div(\xi_\infty \nabla u) + V_\infty u = h(u)u, \quad \text{in } \mathbb{R}^N. \tag{2.1.4}$$

The functional associated with the equation (2.1.4) is given by

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N}(\xi_\infty|\nabla u|^2 + V_\infty u^2)dx - \int_{\mathbb{R}^N} H(u)dx. \tag{2.1.5}$$

It is well defined and in $C^1(H^1(\mathbb{R}^N),\mathbb{R})$ with

$$I'_\infty(u)\varphi = \int_{\mathbb{R}^N}(\xi_\infty \nabla u \nabla\varphi + V_\infty u\varphi)dx - \int_{\mathbb{R}^N} h(u)u\varphi dx, \text{ for all } u,\ \varphi \in H^1(\mathbb{R}^N).$$

Hence, critical points of the functional I_∞ are weak solutions of problem (2.1.4). The functional I_∞ is continuous, $I_\infty(0) = 0$ and if ω is a positive solutions of (2.1.4), the maximum of $I_\infty\left(\omega\left(\frac{\cdot}{t}\right)\right) > 0$ holds on $t = 1$. Furthermore, there exists a real number $L > 0$, large enough such that $I_\infty\left(\omega\left(\frac{\cdot}{t}\right)\right) < 0$ for all $t \geq L$. Thus, there exists $L_0 > 1$ such that

$$I_\infty\left(\omega\left(\frac{\cdot}{L_0}\right)\right) = 0 \tag{2.1.6}$$

and

$$I_\infty\left(\omega\left(\frac{\cdot}{t}\right)\right) < 0, \text{ if } t \geq L_0. \tag{2.1.7}$$

Therefore, consider

$$\beta \in \left(0, \sqrt{\frac{V_\infty}{\xi_\infty}}\right). \tag{2.1.8}$$

Our result concerning nodal solution is stated next.

Theorem 2.1.2. *Assume that ξ and V satisfy the hypotheses $(\xi_1)-(\xi_4)$ and $(V_1)-(V_4)$, respectively, and f satisfies $(f_1)-(f_7)$. Then3 problem (P'_τ) has a sign-changing solution provided one of the following conditions holds:*

$$\xi(x) \leq \xi_\infty - Ce^{-\beta_1|x|}, \text{ for all } x \in \mathbb{R}^N \tag{2.1.9}$$

or

$$V(x) \leq V_\infty - Ce^{-\beta_2|x|}, \text{ for all } x \in \mathbb{R}^N \tag{2.1.10}$$

or

$$F(x,s) \geq H(s) + Ce^{-\beta_3|x|}|s|^2, \text{ for all } x \in \mathbb{R}^N,\ s \in \mathbb{R}, \tag{2.1.11}$$

for constants $C > 0$ and $0 < \beta_i < \beta$, with $i = 1,2,3$.

Less than from equivalences and similarities in Chapter 1 we will use the same notations. Any solution u of the limit problem (2.1.4) satisfies Pohozaev identity

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx = N\int_{\mathbb{R}^N}G_\infty(u)dx, \tag{2.1.12}$$

where $G_\infty(u) = \dfrac{1}{\xi_\infty}\left(H(u) - \dfrac{V_\infty}{2}u^2\right)$. We define the Pohozaev manifold as

$$\mathcal{P} = \left\{u \in H^1(\mathbb{R}^N)\setminus\{0\} : J(u) = 0\right\}, \tag{2.1.13}$$

where

$$J(u) := \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx - N\int_{\mathbb{R}^N}G_\infty(u)dx, \tag{2.1.14}$$

and denote

$$m_\infty := \inf_{u\in\mathcal{P}} I_\infty(u). \tag{2.1.15}$$

Remark 2.1.5. *Note that*

$$G_\infty(\zeta) = \frac{1}{\xi_\infty}\int_0^\zeta (h(s)s - V_\infty s)ds > 0, \tag{2.1.16}$$

implies $\mathcal{P} \neq \emptyset$.

Lemma 2.1.1. *Let* $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ *be the functional* (2.1.14)*. Then*

(i) $\mathcal{P} = \{u \in H^1(\mathbb{R}^N)\setminus\{0\} : J(u) = 0\}$ *is closed;*

(ii) $\mathcal{P}$ *is a manifold of class* C^1*;*

(iii) there exists $\sigma > 0$ *such that* $\|u\| > \sigma$ *for all* $u \in \mathcal{P}$.

Proof. Although the proof follows the same way as the previous chapter, we will show the necessary adaptations. The first follows exactly as the proof of item (i) of Lemma 1.1.1. Using the Remark 2.1.3 and $g_\infty(u) := \dfrac{1}{\xi_\infty}(h(u)u - V_\infty u)$, we obtain

$$J'(u)u = 2N\int_{\mathbb{R}^N}\left(H(u) - \frac{h(u)u^2}{2}\right)dx < 0,$$

which implies $J'(u) \neq 0$ and hence $\mathcal{P}$ is a C^1 manifold, and we prove item (ii). Finally, for the proof of item (iii), let $u \in \mathcal{P}$ and $2^* = 2N/(N-2)$, then we have

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx - N\int_{\mathbb{R}^N} G_\infty(u)dx = 0$$
$$\int_{\mathbb{R}^N}\left(\xi_\infty|\nabla u|^2 + \frac{N}{N-2}V_\infty u^2\right)dx = 2^*\int_{\mathbb{R}^N} H(u)dx.$$

Then, taking $M := \min\left\{\xi_\infty, \dfrac{V_\infty N}{N-2}\right\}$ and using (f_3), we obtain

$$M\|u\|^2 \leq 2^*\int_{\mathbb{R}^N} H(u)dx \leq 2^*\int_{\mathbb{R}^N} F(x,u)dx.$$

And we finish the proof the same way as the proof of Lemma 1.1.1. □

The next result is the same as Lemma 1.1.2 which will be stated for completeness.

Lemma 2.1.2. *If f satisfies $(f_1)-(f_3)$, (u_n) is a bounded sequence and $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$, then*

$$f(x,u_n) - f(x,u_n-u_0) \to f(x,u_0), \quad \textit{in } H^{-1}(\mathbb{R}^N) \tag{2.1.17}$$

and

$$\int_{\mathbb{R}^N}|F(x,u_n) - F(x,u_n-u_0) - F(x,u_0)|dx \to 0. \tag{2.1.18}$$

Furthermore,

$$h(u_n)u_n - h(u_n-u_0)(u_n-u_0) \to h(u_0)u_0, \ \textit{in } H^{-1}(\mathbb{R}^N) \tag{2.1.19}$$

and

$$\int_{\mathbb{R}^N}|H(u_n) - H(u_n-u_0) - H(u_0)|dx \to 0. \tag{2.1.20}$$

Let E be the Hilbert space $H^1(\mathbb{R}^N)$ with the inner product $\langle\cdot,\cdot\rangle$ given by the expression

$$\langle u,v\rangle = \int_{\mathbb{R}^N}(\xi(x)\nabla u\nabla v + V(x)uv)dx$$

and the norm by

$$\|u\|^2 = \int_{\mathbb{R}^N}(\xi(x)|\nabla u|^2 + V(x)u^2)dx, \tag{2.1.21}$$

which is equivalent to the usual norm and the norm (1.1.25) because of (ξ_1), (ξ_3), (V_1) and (V_3). The functional $I : E \to \mathbb{R}$ associated with (P_2) is given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(\xi(x)|\nabla u|^2+V(x)u^2)dx - \int_{\mathbb{R}^N}F(x,u)dx, \tag{2.1.22}$$

is well defined, belongs to $C^1(E,\mathbb{R})$ and

$$I'(u)\varphi = \int_{\mathbb{R}^N}(\xi(x)\nabla u\nabla\varphi + V(x)u\varphi)dx - \int_{\mathbb{R}^N}f(x,u)\varphi dx, \text{ for all } u,\varphi \in E.$$

The hypotheses (ξ_3), (V_3) and (f_3) implies

$$I(u) \le I_\infty(u), \text{ for all } u \in E. \tag{2.1.23}$$

Indeed,

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(\xi(x)|\nabla u|^2+V(x)u^2)dx - \int_{\mathbb{R}^N}F(x,u)dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N}(\xi_\infty|\nabla u|^2+V_\infty u^2)dx - \int_{\mathbb{R}^N}H(u)dx \\ &= I_\infty(u), \quad \text{for all } u \in E. \end{aligned}$$

Let $z_0 = 0$ and fix $L > L_0$ such that $z_1 := w\left(\frac{\cdot}{L}\right)$ and $I_\infty(z_1) < 0$. Define also

$$c := \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I(\gamma(t)), \tag{2.1.24}$$

where $\Gamma = \{\gamma \in C([0,1],E), \gamma(0) = z_0 \text{ and } \gamma(1) = z_1\}$.

Lemma 2.1.3. *If (u_n) is a $(Ce)_c$ sequence of the functional I_∞ then (u_n) is bounded.*

Proof. This proof will be postponed to Lemma 2.3.1. □

Lemma 2.1.4 (Splitting)**.** *Let $(u_n) \subset E$ be a sequence such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in E^*. Then there exists $u_0 \in E$ such that $u_n \rightharpoonup u_0$, $I'(u_0) = 0$ and either*
(a) $u_n \to u_0$ strongly in E, or
(b) there exist $k \in \mathbb{N}$, $(y_n^j) \in \mathbb{R}^N$ with $|y_n^j| \to \infty$ and $|y_n^j - y_n^{j'}| \to \infty$, for $j \ne j'$, $j = 1,...,k$,

and nontrivial solutions $u^1,....,u^k$ *of problem* (2.1.4)*, such that*

$$I(u_n)\to I(u_0)+\sum_{j=1}^{k}I_\infty(u^j)\quad and\quad \left\|u_n-u_0-\sum_{j=1}^{k}u^j(\cdot-y_n^j)\right\|\to 0. \tag{2.1.25}$$

Proof. Step 1) Since (u_n) is bounded, it follows the same way as step 1 of Lemma 1.1.5.

Step 2) Define now $u_n^1:=u_n-u_0\in H^1(\mathbb{R}^N)$. If $n\to\infty$, then:

(*i*) $\|u_n^1\|^2=\|u_n\|^2-\|u_0\|^2+o_n(1)$;

(*ii*) $I_\infty(u_n^1)\to c-I(u_0)$;

(*iii*) $I'_\infty(u_n^1)\to 0$.

The proof of item (*i*) can be done using the steps of the proof of item (*i*) of Lemma 1.1.5. To prove item (*ii*), note that the weak convergence of (u_n) for u_0 implies $u_n^1\rightharpoonup 0$, with the same calculation to obtain (1.1.31)

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big(\xi_\infty|\nabla(u_n-u_0)|^2-\xi(x)|\nabla u_n|^2+\xi(x)|\nabla u_0|^2\Big)dx\\ &\quad=\int_{\mathbb{R}^N}(\xi_\infty-\xi(x))(|\nabla u_n|^2-|\nabla u_0|^2)dx+o_n(1)\end{aligned} \tag{2.1.26}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big(V_\infty|u_n-u_0|^2-V(x)u_n^2+V(x)u_0^2\Big)dx\\ &\quad=\int_{\mathbb{R}^N}\Big((V_\infty-V(x))(u_n^2-u_0^2)dx+o_n(1).\end{aligned} \tag{2.1.27}$$

From (2.1.26) and (2.1.27), it follows that

$$\begin{aligned}I_\infty(u_n^1)\ -\ I(u_n)+I(u_0)&=\frac{1}{2}\int_{\mathbb{R}^N}\xi_\infty|\nabla u_n^1|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V_\infty(u_n^1)^2dx-\int_{\mathbb{R}^N}H(u_n^1)dx\\ &\quad-\frac{1}{2}\int_{\mathbb{R}^N}\xi(x)|\nabla u_n|^2dx-\frac{1}{2}\int_{\mathbb{R}^N}V(x)u_n^2dx+\int_{\mathbb{R}^N}F(x,u_n)dx\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^N}\xi(x)|\nabla u_0|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u_0^2dx-\int_{\mathbb{R}^N}F(x,u_0)dx\\ &=\ \frac{1}{2}\int_{\mathbb{R}^N}\Big(\xi_\infty|\nabla u_n-\nabla u_0|^2-\xi(x)|\nabla u_n|^2+\xi(x)|\nabla u_0|^2\Big)dx\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^N}\Big(V_\infty|u_n-u_0|^2-V(x)u_n^2+V(x)u_0^2\Big)dx\\ &\quad+\int_{\mathbb{R}^N}\Big(F(x,u_n)-F(x,u_0)-H(u_n^1)\Big)dx\end{aligned}$$

$$= \int_{\mathbb{R}^N} \Big(F(x,u_n^1) - H(u_n^1)\Big)dx + o_n(1). \tag{2.1.28}$$

Since (u_n) is bounded, using the hypothesis (f_2) we have $\int_{\mathbb{R}^N} \Big(H(u_n^1) - F(x,u_n^1)\Big)dx = o_n(1)$. Replacing in (2.1.28) we obtain

$$I_\infty(u_n^1) - I(u_n) + I(u_0) = o_n(1). \tag{2.1.29}$$

To verify (iii), consider $\varphi \in C_0^\infty(\mathbb{R}^N)$. Applying (f_1), (f_2), (2.1.17), (2.1.19) and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
o_n(1) &= \Big\langle I'(u_n),\varphi\Big\rangle = \Big\langle I'(u_0+u_n^1),\varphi\Big\rangle \\
&= \int_{\mathbb{R}^N}(\xi(x)\nabla(u_0+u_n^1)\nabla\varphi + V(x)(u_0+u_n^1)\varphi)dx - \int_{\mathbb{R}^N} f(x,u_0+u_n^1)(u_0+u_n^1)\varphi dx \\
&= \int_{\mathbb{R}^N}(\xi(x)\nabla u_0\nabla\varphi + V(x)u_0\varphi)dx - \int_{\mathbb{R}^N} f(x,u_0)u_0\varphi dx \\
&\quad + \int_{\mathbb{R}^N}(\xi(x)\nabla u_n^1\nabla\varphi - V(x)u_n^1\varphi)dx - \int_{\mathbb{R}^N} h(u_n^1)u_n^1\varphi dx + \int_{\mathbb{R}^N} f(x,u_0)u_0\varphi dx \\
&\quad + \int_{\mathbb{R}^N} h(u_n^1)u_n^1\varphi dx - \int_{\mathbb{R}^N} f(x,u_0+u_n^1)(u_0+u_n^1)\varphi dx \\
&= \langle I'(u_0),\varphi\rangle + \int_{\mathbb{R}^N}(\xi_\infty\nabla u_n^1\nabla\varphi + V_\infty u_n^1\varphi)dx - \int_{\mathbb{R}^N} h(u_n^1)u_n^1\varphi dx \\
&= \langle I'_\infty(u_n^1),\varphi\rangle - \int_{\mathbb{R}^N} f(x,u_n^1)u_n^1\varphi dx + o_n(1) + \int_{\mathbb{R}^N} h(u_n^1)u_n^1\varphi dx \\
&= \langle I'_\infty(u_n^1),\varphi\rangle + \left[\int_{\mathbb{R}^N}\Big(h(u_n^1)u_n^1\varphi - f(x,u_n^1)u_n^1\varphi\Big)dx\right] + o_n(1),
\end{aligned}$$

since φ has compact support, $u_n^1 \to 0$ in the support and then $I'_\infty(u_n^1) \to 0$ in E^* when $n \to \infty$. Therefore, (u_n^1) is a $(PS)_c$ sequence of I_∞.

Consider

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n^1(x)|^2 dx.$$

Step 3) If $\delta = 0$, it follows from Lions' Lemma [24] that

$$u_n^1 \to 0 \quad \text{in} \quad L^p(\mathbb{R}^N), \quad \text{for any } 2 < p < 2^*. \tag{2.1.30}$$

On the other hand, since (u_n^1) is bounded, item (iii) implies that

$$I'_\infty(u_n^1)u_n^1 = \int_{\mathbb{R}^N} \Big(\xi_\infty|\nabla u_n^1|^2 + V_\infty(u_n^1)^2 - h(u_n^1)(u_n^1)^2\Big)dx \to 0, \quad \text{if} \quad n \to \infty. \tag{2.1.31}$$

From (2.1.14) and (2.1.30), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\left(\xi_\infty|\nabla u_n^1|^2+V_\infty(u_n^1)^2\right)dx &= \int_{\mathbb{R}^N}h(u_n^1)(u_n^1)^2dx+o_n(1)\\ &\leq \varepsilon\int_{\mathbb{R}^N}(u_n^1)^2dx+C\int_{\mathbb{R}^N}|u_n^1|^pdx. \qquad (2.1.32)\end{aligned}$$

Therefore, (2.1.30) and (2.1.32) give us that $\left\|u_n^1\right\|\to 0$, that is, $u_n\to u_0$ strongly in E, and this proof the item (a).

Step 4) If $\delta>0$, we follow the calculations made in Step 4 of Lemma 1.1.5 of the previous chapter and using the fact that $u_n^1(\cdot+y_n^1)\rightharpoonup u^1$, for all $\phi\in C_0^\infty(\mathbb{R}^N)$, we obtain

$$o_n(1)=I'_\infty(u_n^1(\cdot+y_n^1))\phi=I'_\infty(u^1)\phi+o_n(1)\,. \qquad (2.1.33)$$

Step 5) Define $u_n^2(x):=u_n^1(x)-u^1(x-y_n^1)$, and $u_n^2(\cdot+y_n^2)=v_n^1+u^1$, then (u_n^2) is a $(PS)_c$ sequence of I_∞. Indeed, making a change of variables,

$$\begin{aligned}I_\infty(u_n^2) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(\xi_\infty|\nabla u_n^2|^2+V_\infty(u_n^2)^2\right)dx-\int_{\mathbb{R}^N}H(u_n^2)dx\\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left(\xi_\infty|\nabla(u_n^1(x)-u^1(x-y_n^1))|^2+V_\infty|u_n^1(x)-u^1(x-y_n^1)|^2\right)dx\\ &\quad -\int_{\mathbb{R}^N}H(u_n^1(x)-u^1(x-y_n^1))dx\\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left(\xi_\infty|\nabla(u_n^1(x+y_n^1)-u^1(x))|^2+V_\infty|u_n^1(x+y_n^1)-u^1(x)|^2\right)dx\\ &\quad -\int_{\mathbb{R}^N}H(u_n^1(x+y_n^1)-u^1(x))dx.\end{aligned}$$

On the other hand,

$$\|u_n^1(\cdot+y_n^1)-u^1\|^2=\|u_n^1(\cdot+y_n^1)\|^2-2\langle u_n^1(\cdot+y_n^1),u^1\rangle+\|u^1\|^2. \qquad (2.1.34)$$

Since $u_n^1(\cdot+y_n^1)\rightharpoonup u^1$ in E, $\langle u_n^1(\cdot+y_n^1),\varphi\rangle\to\langle u^1,\varphi\rangle$, for all $\varphi\in E$. In particular, if $\varphi=u^1$, we have $\langle u_n^1(\cdot+y_n^1),u^1\rangle\to\langle u^1,u^1\rangle$, which it follows that $\langle u_n^1(\cdot+y_n^1),u^1\rangle=\|u^1\|^2+o_n(1)$. Replacing in (2.1.34), we obtain

$$\|u_n^1(\cdot+y_n^1)-u^1\|^2=\|u_n^1\|^2-\|u^1\|^2+o_n(1). \qquad (2.1.35)$$

Therefore,

$$\begin{aligned} I_\infty(u_n^1) - I_\infty(u_n^2) - I_\infty(u^1) &= \frac{1}{2}\left(\left\|u_n^1\right\|^2 - \left\|u_n^1 - u^1\right\|^2 - \left\|u^1\right\|^2\right) \\ &\quad - \int_{\mathbb{R}^N}\Big(H(u_n^1) - H(u_n^2) - H(u^1)\Big)dx\,, \end{aligned}$$

and using (f_3), (2.1.35) and Lemma 2.1.2, it follows

$$I_\infty(u_n^2) = I_\infty(u_n^1) - I_\infty(u^1) + o_n(1). \tag{2.1.36}$$

By (ii) and (iii), (u_n^1) is a $(PS)_c$ sequence of I_∞, hence $I_\infty(u_n^2)$ converges to a constant.

Finally, using (f_2), (f_3) and Lemma 2.1.2, from (iii) and (2.1.33), we obtain

$$\begin{aligned} |I'_\infty(u_n^2)\varphi| &= \Bigg|\int_{\mathbb{R}^N}(\xi_\infty\nabla u_n^1\nabla\varphi + V_\infty u_n^1\varphi)dx - \int_{\mathbb{R}^N}(\xi_\infty\nabla u^1\nabla\varphi + V_\infty u^1\varphi)dx \\ &\quad - \int_{\mathbb{R}^N} h(u_n^1)u_n^1\varphi dx + \int_{\mathbb{R}^N} h(u^1)u^1\varphi dx - \int_{\mathbb{R}^N} h(u_n^1 - u^1)(u_n^1 - u^1)\varphi dx \\ &\quad + \int_{\mathbb{R}^N} h(u_n^1)u_n^1\varphi dx - \int_{\mathbb{R}^N} h(u^1)u^1\varphi dx\Bigg| \\ &= o_n(1) + \int_{\mathbb{R}^N}|h(u_n^1)u_n^1 - h(u_n^1 - u^1)(u_n^1 - u^1) - h(u^1)u^1||\varphi|dx \\ &= o_n(1), \end{aligned} \tag{2.1.37}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Therefore (u_n^2) is a $(PS)_c$ sequence of I_∞.

Step 6) Now we proceed by iteration. Note that if u is a nontrivial critical point of I_∞ and ω is the solution of (2.1.15), then

$$I_\infty(u) \geq I_\infty(\omega) > 0. \tag{2.1.38}$$

Therefore, by (2.1.29) and (2.1.36),

$$I_\infty(u_n^2) = c - I(u_0) - I_\infty(u^1) + o_n(1). \tag{2.1.39}$$

Applying (2.1.38) and (2.1.39) the iteration must be terminated at some index $k \in \mathbb{N}$. Therefore, there exist k solutions to the problem (2.1.4), thus satisfying the second part of the lemma. $\square$

2.2 Existence of a positive solution

Lemma 2.2.1. *The functional I satisfies $(Ce)_c$ for all $0 \leq c < m_\infty$.*

Proof. Consider $(u_n) \subset E$ and $0 \leq c < m_\infty$ such that

$$I(u_n) \to c \quad \text{and} \quad (1+\|u_n\|)\left\|I'(u_n)\right\| \to 0.$$

By Lemma 2.1.3, (u_n) is a bounded sequence in E and taking a subsequence if necessary, $u_n \rightharpoonup u_0$ in E. Lemma 2.1.4 give us $I'(u_0) = 0$ and by condition (f_5)

$$\begin{aligned} I(u_0) &= \frac{1}{2}\int_{\mathbb{R}^N} (\xi(x)|\nabla u_0|^2 + V(x)u_0^2)dx - \int_{\mathbb{R}^N} F(x,u_0)dx \\ &= \int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u_0)u_0 - F(x,u_0)\right) dx \\ &= \int_{\mathbb{R}^N} Q(x,u_0)dx \geq 0. \end{aligned} \tag{2.2.1}$$

If u_n does not converge to u_0 in E, applying the Lemma 2.1.4 we find $k \in \mathbb{N}$ and nontrivial solutions $u^1,\ldots,u^k$ of (2.1.4) satisfying

$$c = \lim_{n\to\infty} I(u_n) = I(u_0) + \sum_{j=1}^{k} I_\infty(u^j) \geq k m_\infty \geq m_\infty,$$

which contradicting the assumption. Therefore, $u_n \to u_0$ in E. $\square$

Remark 2.2.1. *For each $u \in E\backslash\{0\}$ such that $\int_{\mathbb{R}^N} G_\infty(u)dx > 0$, there exists a unique real number $t > 0$ such that $u\left(\frac{\cdot}{t}\right) \in \mathcal{P}$ and $I_\infty\left(u\left(\frac{\cdot}{t}\right)\right)$ in the maximum of the function*

$$t \mapsto I_\infty\left(u\left(\frac{\cdot}{t}\right)\right), \quad t > 0.$$

In fact, consider the function g defined by

$$\begin{aligned} g(t) &:= I_\infty\left(u\left(\frac{\cdot}{t}\right)\right) \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left(\xi_\infty\left|\nabla u\left(\frac{\cdot}{t}\right)\right|^2 + V_\infty\left(u\left(\frac{\cdot}{t}\right)\right)^2\right) - \int_{\mathbb{R}^N} H\left(u\left(\frac{\cdot}{t}\right)\right)dx \end{aligned}$$

making changes of variable, the function g can be rewritten as

$$g(t) = \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} \xi_\infty |\nabla u|^2 dx + \frac{t^N}{2}\int_{\mathbb{R}^N} V_\infty u^2 dx - t^N \int_{\mathbb{R}^N} H(u)dx.$$

Then $g'(t) = 0$ *if and only if* $t = 0$ *or*

$$\begin{aligned} 0 = g'(t) = \frac{N-2}{2}t^{N-3}\int_{\mathbb{R}^N} \xi_\infty |\nabla u|^2 dx \; + \; \frac{N}{2}t^{N-1}\int_{\mathbb{R}^N} V_\infty u^2 dx - N t^{N-1}\int_{\mathbb{R}^N} H(u)dx \\ t^{N-1} N \int_{\mathbb{R}^N}\left(H(u) - \frac{V_\infty}{2}u^2\right)dx \;=\; \frac{N-2}{2}t^{N-3}\int_{\mathbb{R}^N}\xi_\infty|\nabla u|^2 dx \\ t^2 \;=\; \frac{N-2\displaystyle\int_{\mathbb{R}^N}\xi_\infty|\nabla u|^2 dx}{2N\displaystyle\int_{\mathbb{R}^N} G_\infty(u)dx}. \end{aligned}$$

Let $\omega \in \mathcal{P}$ be a positive, radial, ground state solution of equation (2.1.4) and

$$\omega_y(x) := \omega(x - y), \tag{2.2.2}$$

for some $y \in \mathbb{R}^N$ fixed.

Remark 2.2.2. *The inequality*

$$\int_{\mathbb{R}^N} G_\infty(\omega_y)dx > 0, \tag{2.2.3}$$

if $|y| > 0$ *is large enough. This follows from the translation invariance of the integral and by Pohozaev identity.*

Lemma 2.2.2. *Suppose* (ξ_3), (V_3) *and* (f_3), *then c defined as in* (2.1.24) *satisfies*

$$0 < c < m_\infty.$$

Proof. From Remark 2.2.2, $\int_{\mathbb{R}^N} G_\infty(\omega_y)dx > 0$, follows from Remark 2.2.1, from (2.1.6) and (2.1.1) that there exists $0 \leq t_y \leq L_0$ such that

$$\max_{0<t\leq L_0} I\left(\omega_y\left(\frac{\cdot}{t}\right)\right) = I\left(\omega_y\left(\frac{\cdot}{t_y}\right)\right) = I\left(\omega\left(\frac{\cdot}{t_y} - y\right)\right).$$

Furthermore, using (ξ_3), (V_3), (f_3), (2.1.23) and the translation invariance of the integral

$$\begin{aligned} I\left(\omega_y\left(\frac{\cdot}{t_y}\right)\right) &< I_\infty\left(\omega_y\left(\frac{\cdot}{t_y}\right)\right) = I_\infty\left(\omega\left(\frac{\cdot}{t_y}-y\right)\right) \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left(\xi_\infty\left|\nabla\omega\left(\frac{\cdot}{t_y}\right)\right|^2 + V_\infty\left|\omega\left(\frac{\cdot}{t_y}\right)\right|^2\right)dx - \int_{\mathbb{R}^N} H\left(\omega\left(\frac{\cdot}{t_y}\right)\right)dx \\ &= I_\infty\left(\omega\left(\frac{\cdot}{t_y}\right)\right) \le I_\infty(\omega) = m_\infty. \end{aligned}$$

The conclusion of the lemma follows the steps of the proof of Lemma 1.2.2. □

Lemma 2.2.3. *If F satisfies (2.1.3), then there exists $\rho > 0$ and $\alpha > 0$ such that $I(u) \ge \alpha > 0$, for all $u \in E$ with $\|u\| = \rho$.*

Proof. Using the norm of space, by (2.1.3), Sobolev's embedding for $2 < p < 2^*$, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(\xi(x)|\nabla u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^N} u^2 dx - C\int_{\mathbb{R}^N}|u|^p dx \\ &\ge \left(\frac{1}{2}-\frac{\varepsilon}{2}\right)\|u\|^2 - C\|u\|^p. \end{aligned}$$

For $\|u\| = \rho$ we obtain

$$I(u) \ge \left(\frac{1}{2}-\frac{\varepsilon}{2}\right)\rho^2 - C\rho^p = \alpha > 0,$$

for $\rho = \|u\|$ small enough. □

Remark 2.2.3. *Since $I(u) \le I_\infty(u)$ for all $u \in E$, then there exists $z_1 \in E \setminus B_\rho(0)$ such that $I(z_1) \le I_\infty(z_1) < 0$.*

The next lemma will be stated by the completeness of the work and the proof is analogous to the proof of Lemma 1.2.4 using the hypothesis (V_1).

Lemma 2.2.4. *Let v_n be a solution of the following problem*

$$\begin{cases} -div(\xi(x)\nabla v_n) + V(x)v_n = f(x,v_n), & in\ \mathbb{R}^N, \\ v_n \in H^1(\mathbb{R}^N), \quad with \quad N \ge 3, \\ v_n(x) \ge 0, \quad for\ all \quad x \in \mathbb{R}^N. \end{cases}$$

Assuming that $(\xi_1)-(\xi_3)$, $(V_1)-(V_4)$, $(f_1)-(f_5)$ *holds and that* $v_n \to v$ *in* $H^1(\mathbb{R}^N)$ *with* $v \not\equiv 0$, *then* $v_n \in L^\infty(\mathbb{R}^N)$ *and there exists* $C>0$ *such that* $\|v_n\|_{L^\infty} \leq C$ *for all* $n \in \mathbb{N}$. *Furthermore,*

$$\lim_{|x|\to\infty} v_n(x) = 0, \text{ uniformly in } n.$$

Proof. For any $R>0$, $0<r\leq R/2$, let $\eta \in C^\infty(\mathbb{R}^N)$, $0\leq\eta\leq 1$ with $\eta(x)=1$ if $|x|\geq R$ and $\eta(x)=0$ if $|x|\leq R-r$ and $|\nabla\eta|\leq 2/r$. Note that, by Remark 2.1.2 and by Sobolev's embedding for $2\leq p\leq 2^*$, we obtain the following growth condition for f:

$$f(x,s) \leq \varepsilon|s| + C_\varepsilon|s|^{p-1} \leq \varepsilon|s| + C_\varepsilon|s|^{2^*-1}. \tag{2.2.4}$$

For each $n\in\mathbb{N}$ and for $L>0$, let

$$v_{L,n}(x) = \begin{cases} v_n(x), & v_n(x)\leq L, \\ L, & v_n(x)\geq L, \end{cases}$$

$z_{L,n} = \eta^2 v_{L,n}^{2(\beta-1)} v_n$ and $w_{L,n} = \eta v_n v_{L,n}^{\beta-1}$ with $\beta>1$ to be determinated later. Taking $z_{L,n}$ as a test function, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)}|\nabla v_n|^2 dx &= -2(\beta-1)\int_{\mathbb{R}^N}\xi(x)v_{L,n}^{2\beta-3}\eta^2 v_n\nabla v_n\nabla v_{L,n}dx \\ &\quad +\int_{\mathbb{R}^N} f(x,v_n)\eta^2 v_n v_{L,n}^{2(\beta-1)}dx - \int_{\mathbb{R}^N} V(x)v_n^2\eta^2 v_{L,n}^{2(\beta-1)}dx \\ &\quad -2\int_{\mathbb{R}^N}\xi(x)\eta v_{L,n}^{2(\beta-1)} v_n\nabla v_n\nabla\eta dx.\end{aligned}$$

Note that, $-2(\beta-1)\int_{\mathbb{R}^N}\xi(x)v_{L,n}^{2\beta-3}\eta^2 v_n\nabla v_n\nabla v_{L,n}dx \leq 0$, then

$$\begin{aligned}\int_{\mathbb{R}^N}\xi(x)\eta^2 v_{L,n}^{2(\beta-1)}|\nabla v_n|^2dx &\leq -2\int_{\mathbb{R}^N}\xi(x)\eta v_{L,n}^{2(\beta-1)}v_n\nabla v_n\nabla\eta dx - \int_{\mathbb{R}^N}V(x)\eta^2 v_{L,n}^{2(\beta-1)}v_n^2dx \\ &\quad +\int_{\mathbb{R}^N} f(x,v_n)\eta^2 v_n v_{L,n}^{2(\beta-1)}dx.\end{aligned}$$

By (2.2.4), hypothesis (V_1) and for ε sufficiently small, we have the following inequality

$$\begin{aligned}\int_{\mathbb{R}^N}\xi(x)\eta^2 v_{L,n}^{2(\beta-1)}|\nabla v_n|^2dx &\leq -2\int_{\mathbb{R}^N}\xi(x)\eta v_{L,n}^{2(\beta-1)}v_n\nabla v_n\nabla\eta dx - V_0\int_{\mathbb{R}^N}\eta^2 v_{L,n}^{2(\beta-1)}v_n^2dx \\ &\quad +\varepsilon\int_{\mathbb{R}^N}\eta^2 v_{L,n}^{2(\beta-1)}v_n^2dx + C_\varepsilon\int_{\mathbb{R}^N}\eta^2 v_{L,n}^{2(\beta-1)}v_n^{2^*}dx \\ &\leq -2\int_{\mathbb{R}^N}\xi(x)\eta v_{L,n}^{2(\beta-1)}v_n\nabla v_n\nabla\eta dx + C_\varepsilon\int_{\mathbb{R}^N}\eta^2 v_{L,n}^{2(\beta-1)}v_n^{2^*}dx\end{aligned}$$

$$\leq \quad C_\varepsilon \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^{2^*}\, dx + 2\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla \eta dx.$$

For each $\varepsilon > 0$, using the Young's inequality we get

$$\begin{aligned}\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)}|\nabla v_n|^2 dx \quad &\leq \quad C_\varepsilon \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^{2^*}\, dx + 2\varepsilon \int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)}|\nabla v_n|^2 dx \\ &\quad +2C_\varepsilon \int_{\mathbb{R}^N} \xi(x) v_n^2 v_{L,n}^{2(\beta-1)}|\nabla \eta|^2 dx.\end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small,

$$\int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)}|\nabla v_n|^2 dx \leq C\int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^{2^*}\, dx + C\int_{\mathbb{R}^N} \xi(x) v_n^2 v_{L,n}^{2(\beta-1)}|\nabla \eta|^2 dx. \tag{2.2.5}$$

Now, from Sobolev's embedding, by (2.2.5) and by (ξ_1) we have

$$\begin{aligned}\xi_0 \|w_{L,n}\|_{L^{2^*}}^2 \quad &\leq \quad \int_{\mathbb{R}^N} \xi(x)\eta^2 v_n^2 v_{L,n}^{2(\beta-1)} dx \leq \int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)}|\nabla v_n|^2 dx \\ &\leq \quad C\Big[\int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^{2^*}\, dx + \int_{\mathbb{R}^N} \xi(x) v_n^2 v_{L,n}^{2(\beta-1)}|\nabla \eta|^2 dx\Big].\end{aligned} \tag{2.2.6}$$

To complete the proof, follow the same steps from (1.2.7) to (1.2.8) as in proof in the proof of Lemma 1.2.4. □

Proof of Theorem 2.1.1. By Lemma 2.2.3 and Remark 2.2.3, the functional I satisfies of the Mountain Pass Theorem, then by Ekeland Variational and consider c defined by (2.1.24) there exists a sequence $(u_n) \subset E$ satisfies

$$I(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\left\|I'(u_n)\right\| \to 0.$$

Using the Lemma 2.2.2, we obtain that c satisfies $0 < c < m_\infty$ and, up to a subsequence, (u_n) converge strongly to $u \in E$, by Lemma 2.2.1. Moreover, since $I \in C^1(E,\mathbb{R})$, then $I(u) = c$ and $I'(u) = 0$. It follows that u is a solution of problem (P_2).

Consider $f(x,s) = 0$ for all $s \leq 0$ in the beginning, then $I'(u)u^- = 0$ and with the same calculations done in (1.2.9) we obtain $u^- \equiv 0$. Hence $u \geq 0$ in $\mathbb{R}^N$. By Lemma 2.2.4 we have that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{loc}(\mathbb{R}^N)$ for some $0 < \alpha < 1$. Then, Harnarck's inequality [2] guarantees that $u > 0$ for all $u(x) > 0$ for all $x \in \mathbb{R}^N$. Therefore, u is a nontrivial and positive solution of (P_2). □

2.3 Nodal Solution

A nontrivial orthogonal $\tau : \mathbb{R}^N \to \mathbb{R}^N$ induce an involution $T_\tau : E \to E$ defined by

$$T_\tau(u(x)) := -u(\tau(x)). \tag{2.3.1}$$

Consider

$$E^\tau := \{u \in E : T_\tau(u(x)) = u(x)\} \tag{2.3.2}$$

the subspace of $\tau-$invariant in E and consider the following $\tau-$ invariant Pohozaev manifold

$$\mathcal{P}^\tau := \{u \in \mathcal{P} : T_\tau(u(x)) = u(x)\} = \mathcal{P} \cap E^\tau. \tag{2.3.3}$$

Lemma 2.3.1. *If $c > 0$ and (u_n) is a $(Ce)_c$ sequence of the functional I restricted to E^τ, then (u_n) is a bounded sequence.*

Proof. Suppose by contradiction that $\|u_n\| \to \infty$. Define a new sequence $\tilde{u}_n = \dfrac{2\sqrt{c}u_n}{\|u_n\|}$, then $(\tilde{u}_n)$ is a bounded sequence with $\|\tilde{u}_n\| = 2\sqrt{c}$ and consequently $\tilde{u}_n \rightharpoonup \tilde{u}$ in E. One of the two following cases occurs:

Case 1) $\limsup\limits_{n\to\infty} \sup\limits_{y\in\mathbb{R}^N} \displaystyle\int_{B_1(y)} |\tilde{u}_n|^2 dx > 0,$

Case 2) $\limsup\limits_{n\to\infty} \sup\limits_{y\in\mathbb{R}^N} \displaystyle\int_{B_1(y)} |\tilde{u}_n|^2 dx = 0.$

Consider the *Case* 2 occurs. Without loss of generality, suppose $L > 1$ and

$$\begin{aligned} I\left(\frac{L}{\|u_n\|}2\sqrt{c}u_n\right) &= \frac{1}{2}\left(\frac{L^2 4c}{\|u_n\|^2}\right)\int_{\mathbb{R}^N}\left(\xi(x)|\nabla u_n|^2 + V(x)u_n^2\right)dx \\ &\quad - \int_{\mathbb{R}^N} F\left(x, \frac{L}{\|u_n\|}2\sqrt{c}u_n\right)dx \\ &= 2L^2 c - \int_{\mathbb{R}^N} F\left(x, \frac{L}{\|u_n\|}2\sqrt{c}u_n\right)dx. \end{aligned}$$

Given $\varepsilon > 0$ and $2 < p < 2^*$, from (2.1.4) we have

$$\int_{\mathbb{R}^N}\left|F\left(x, \frac{L}{\|u_n\|}2\sqrt{c}u_n\right)\right|dx \le \frac{2\varepsilon c L^2}{\|u_n\|^2}\int_{\mathbb{R}^N} u_n^2 dx + cL^p \int_{\mathbb{R}^N} |\tilde{u}_n|^p dx.$$

Now, by Lions' Lemma, we obtain

$$\int_{\mathbb{R}^N} |\tilde{u}_n|^p dx \to 0, \quad \text{for} \quad 2 < p < 2^*$$

thus,

$$\int_{\mathbb{R}^N}\left|F\left(x,\frac{L}{\|u_n\|}2\sqrt{c}u_n\right)\right|dx<2\varepsilon cL^2+o_n(1).$$

Taking $\varepsilon=1/2$ we obtain

$$I\left(\frac{L}{\|u_n\|}2\sqrt{c}u_n\right)>2L^2c-(cL^2+o_n(1))=L^2c-o_n(1).$$

Since $\|u_n\|\to\infty$, then $\dfrac{2L\sqrt{c}}{\|u_n\|}\in(0,1)$ for $n>0$ sufficiently large, so

$$\max_{t\in[0,1]}I(tu_n)\geq I\left(\frac{L}{\|u_n\|}2\sqrt{c}u_n\right)>L^2c-o_n(1).$$

Consider $t_n\in(0,1)$ such that $I(t_nu_n)=\max\limits_{t\in[0,1]}I(tu_n)$. Then

$$I(t_nu_n)>L^2c-o_n(1). \tag{2.3.4}$$

On the other hand, $t_n<1$ because $I(u_n)=c+o_n(1)$, $I'(t_nu_n)u_n=0$ and by hypothesis (f_5), we obtain

$$\begin{aligned}
I(t_nu_n) &< D\int_{\mathbb{R}^N}\left(\frac{1}{2}f(x,u_n)u_n-F(x,u_n)\right)dx\\
&= D\left[\frac{1}{2}\int_{\mathbb{R}^N}\left(\xi(x)|\nabla u_n|^2+V(x)u_n^2\right)dx-\int_{\mathbb{R}^N}F(x,u_n)dx\right]\\
&= DI(u_n)=Dc+o_n(1).
\end{aligned} \tag{2.3.5}$$

From (2.3.4) and (2.3.5) it follows that

$$c-o_n(1)<I_\infty(t_nu_n)<Dc+o_n(1)$$

and making $L>0$ sufficiently large we arrive at the contradiction in *Case* 2.

In *Case* 1, if (y_n) is such that $|y_n|\to\infty$, and $\int_{B_1(y_n)}|\tilde{u}_n|^2dx>\delta/2$, then we get $\int_{B_1(y_n)}|\tilde{u}_n(x+y_n)|^2dx>\delta/2$, and knowing that $\tilde{u}_n(\cdot+y_n)\rightharpoonup\tilde{v}$, we have

$$\int_{B_1(0)}|\tilde{v}(x)|^2dx>\frac{\delta}{2}$$

thus obtaining that $\tilde{v} \not\equiv 0$. Therefore there exists $\Omega \subset B_1(0)$ subset of positive Lebesgue measure such that

$$0 < \tilde{v}(x) = \lim_{n\to\infty} \tilde{u}_n(x+y_n) = \lim_{n\to\infty} \frac{u_n(x+y_n)2\sqrt{c}}{\|u_n\|}, \quad \text{for all} \quad x \in \Omega.$$

Recalling the assumption that $\|u_n\| \to \infty$, then necessarily

$$u_n(x+y_n) \to \infty, \quad \text{for all} \quad x \in \Omega \subset B_1(0)$$

and so from (f_5) and Fatou's Lemma [5], we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right) dx \\
&\geq \int_{\Omega} \liminf_{n\to\infty} \left(\frac{1}{2} f(x+y_n, u_n(x+y_n))u_n(x+y_n) - F(x+y_n, u_n(x+y_n))\right) dx \\
&= +\infty. \qquad (2.3.6)
\end{aligned}$$

On other hand, by (1.1.29) we have that

$$|I'|_{E^\tau}(u_n)u_n| \leq \|I'|_{E^\tau}(u_n)\| \|u_n\| \to 0,$$

and so,

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u_n)u_n \right. &- \left. F(x,u_n)\right) dx = \frac{1}{2}\int_{\mathbb{R}^N} (\xi(x)|\nabla u_n|^2 + V(x)u_n^2) dx - \int_{\mathbb{R}^N} F(x,u_n) dx \\
&- \left(\frac{1}{2}\int_{\mathbb{R}^N} (\xi(x)|\nabla u_n|^2 + V(x)u_n^2) dx - \frac{1}{2}\int_{\mathbb{R}^N} f(x,u_n)u_n dx\right) \\
&= I|_{E^\tau}(u_n) - \frac{1}{2} I'|_{E^\tau}(u_n)u_n \\
&\leq c + o_n(1). \qquad (2.3.7)
\end{aligned}$$

From (2.3.6) and (2.3.7) we obtain a contradiction in *Case* 1, under the assumption that $|y_n| \to +\infty$.

Now, if we have $|y_n| \leq R$ with $R > 1$, then

$$\frac{\delta}{2} \leq \int_{B_1(0)} |\tilde{u}_n(x+y_n)|^2 dx \leq \int_{B_{2R}(0)} |\tilde{u}_n(x+y_n)|^2 dx$$

and since $\tilde{u}_n(\cdot + y_n) \to \tilde{v}$ strongly in $L^2(B_{2R}(0))$, it follows that

$$\frac{\delta}{2} \le \int_{B_1(0)} |\tilde{v}(x)|^2 dx.$$

Hence, as in the previous case there exists a $\Omega \subset B_1(0)$ such that $|\Omega| > 0$ and

$$\lim_{n\to\infty} \frac{u_n(x+y_n)2\sqrt{c}}{\|u_n\|} = \lim_{n\to\infty} \tilde{u}_n(x+y_n) = \tilde{v}(x) \neq 0, \text{ for all } x \in \Omega.$$

Following the previous arguments by (2.3.6) and (2.3.7) again a contradiction follows. We conclude that (u_n) is a bounded sequence. □

Lemma 2.3.2. *If* $u, |\nabla u| \in L^2(\mathbb{R}^N)$, $|y| \to \infty$ *and* $|y - \tau y| \to \infty$, *then*

$$\int_{\mathbb{R}^N} u(x-y)u(\tau x - y)dx = o_y(1) \tag{2.3.8}$$

and

$$\int_{\mathbb{R}^N} \nabla u(x-y).\nabla u(\tau x - y)dx = o_y(1). \tag{2.3.9}$$

Proof. See proof of Lemma 1.3.2. □

Now, we define $G(x,u)$ for $u \in E^\tau$ by

$$G(x,u) := \frac{1}{\xi(x)}\left(F(x,u) - \frac{V(x)}{2}u^2\right).$$

Consider ω the ground state radial positive solution of equation (2.1.4) and define

$$z_y(x) := \omega(x-y) - \omega(x - \tau y) \in E^\tau. \tag{2.3.10}$$

Remark 2.3.1. *If we fix* $y \in \mathbb{R}^N$, $|y| > 0$ *sufficiently large, from* (ξ_3), (V_3) *and* (f_3) *it follows*

$$\int_{\mathbb{R}^N} G(x, z_y)dx \ge \int_{\mathbb{R}^N} G_\infty(z_y)dx > 0. \tag{2.3.11}$$

Therefore, there exists $t > 0$ *such that* $u\left(\frac{\cdot}{t}\right) \in \mathcal{P}$. *Moreover, there exists* t_{z_y} *such that*

$$I\left(z_y\left(\frac{\cdot}{t_{z_y}}\right)\right) = \max_{t>0} I\left(z_y\left(\frac{\cdot}{t}\right)\right). \tag{2.3.12}$$

Indeed,

$$\begin{aligned}\int_{\mathbb{R}^N} G(x,z_y)dx &= \int_{\mathbb{R}^N} \frac{1}{\xi(x)}\left(F(x,z_y)-\frac{V(x)}{2}z_y^2\right)dx \\ &\geq \int_{\mathbb{R}^N} \frac{1}{\xi_\infty}\left(\left(\int_0^{z_y}\frac{f(x,s)}{s}sds\right)-\frac{V_\infty}{2}z_y^2\right)dx \\ &\geq \int_{\mathbb{R}^N} G_\infty(z_y)dx.\end{aligned}$$

In what follows consider $z_0 = 0$, and

$$\overline{z}_1 := \omega\left(\frac{\cdot}{L}-y\right)-\omega\left(\frac{\cdot}{L}-\tau y\right) \text{ in } E^\tau$$

for a fixed $L > L_0$, $|y| > 0$ and $|y-\tau y|$ sufficiently large, such that $I_\infty(\overline{z}_1) < 0$. This is possible by (2.1.6), (2.1.7) and by Lemma 2.3.2. Now, define

$$c^\tau := \inf_{\gamma\in\Gamma_\tau}\max_{0\leq t\leq 1} I(\gamma(t)), \tag{2.3.13}$$

where $\Gamma_\tau = \{\gamma \in C([0,1],E^\tau) : \gamma(0) = z_0 \text{ and } \gamma(1) = \overline{z}_1\}$.

Remark 2.3.2. $\mathcal{P}\cap E^\tau \neq \emptyset$.

Lemma 2.3.3. *There exists a sequence* $(u_n) \subset E^\tau$ *satisfying*

$$I(u_n) \to c^\tau \quad \textit{and} \quad (1+\|u_n\|)\left\|I'|_{E^\tau}(u_n)\right\| \to 0.$$

Proof. See proof of Lemma 1.3.3. □

Lemma 2.3.4. *If* $(u_n) \subset E^\tau$ *is a* (PS) *sequence of the functional* I *restricted to* E^τ, *then* (u_n) *is a* (PS) *sequence of* I.

Proof. Using the fact that the action T_τ is isometric, we will prove that

$$T_\tau I'(u_n) = I'(u_n). \tag{2.3.14}$$

It follows from the (f_6) and hypothesis that F is even and that $F(\tau x,s) = F(x,-s) = F(x,s)$ and using the hypotheses (ξ_4) and (V_4), we have

$$I(T_\tau u_n) \;=\; I(-u_n(\tau x))$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^N}\Big(\xi(\tau x)|\nabla(-u_n(\tau x))|^2+V(\tau x)(-u_n(\tau x))^2\Big)dx-\int_{\mathbb{R}^N}F(\tau x,-u_n(\tau x))dx\\
&= \frac{1}{2}\int_{\mathbb{R}^N}\Big(\xi(x)|\nabla u_n(x)|^2+V(x)u_n^2(x)\Big)dx-\int_{\mathbb{R}^N}F(x,u_n(x))dx\\
&= I(u_n). \qquad (2.3.15)
\end{aligned}
$$

In addition, using the hypothesis (f_6) and making change of variables, we obtain

$$
\begin{aligned}
I'(T_\tau u_n(x))v(x) &= I'(-u_n(\tau x))v(x)\\
&= \int_{\mathbb{R}^N}\Big(\xi(\tau x)\nabla u_n(\tau x)\nabla(-v(x))+V(\tau x)u_n(\tau x)(-v(x))\Big)dx\\
&\quad -\int_{\mathbb{R}^N}f(\tau x,u_n(\tau x))(-v(x))dx\\
&= \int_{\mathbb{R}^N}\Big(\xi(y)\nabla u_n(y)\nabla(-v(\tau y))+V(y)u_n(y)(-v(\tau y))\Big)dy\\
&\quad -\int_{\mathbb{R}^N}f(y,u_n(y))(-v(\tau y))dy\\
&= I'(u_n)(T_\tau(v)), \quad \text{for all } v\in E.
\end{aligned}
$$

Then we finish as the proof of Lemma 1.3.4. □

Next, we present a version of the concentration compactness I restricted to E^τ.

Lemma 2.3.5. *Let $(u_n)\subset E^\tau$ be a bounded sequence such that*

$$I(u_n)\to c \quad \text{and} \quad I'(u_n)\to 0.$$

Then, there exists $u_0\in E^\tau$ such that, up to a subsequence, $u_n \rightharpoonup u_0$, $I'(u_0)=0$ and there exist two integers $k_1,\ k_2\geq 0$, k_1+k_2 sequences (y_n^j), a $\tau-$antisymmetric solution u_0 of problem (P'_τ), k_1 solutions u^j, $j=1,\cdots,k_1$ and k_2 $\tau-$ antisymmetric solutions u^j, $j=k_1+1,\cdots,k_1+k_2$ of the equation (2.1.4), that is, $-div(\xi_\infty\nabla u^j)+V_\infty u^j=h(u^j)u^j$ in $\mathbb{R}^N$ and $u^j(\tau x)=-u^j(x)$, $u^j(x)\to 0$ as $|x|\to\infty$ such that, either

1. *$u_n\to u_0$ strongly in E, or the following statements are hold;*
2. *if $j=1,...,k_1$, then $\tau y_n^j\neq y_n^j$, and $|y_n^j|\to\infty$ when $n\to\infty$;*
3. *if $j=k_1+1,...,k_1+k_2$, then $\tau y_n^j=y_n^j$, and $|y_n^j|\to\infty$ when $n\to\infty$;*
4. *$u_n(x)=u_0(x)+\sum_{j=1}^{k_1}[u^j(x-y_n^j)+T_\tau u^j(x-y_n^j)]+\sum_{j=k_1+1}^{k_1+k_2}u^j(x-y_n^j)+o_n(1)$;*

5. $I(u_n) \to I(u_0) + 2\sum_{j=1}^{k_1} I_\infty(u^j) + \sum_{j=k_1+1}^{k_1+k_2} I_\infty(u^j)$.

Proof. Step 1) By Lemma 2.3.3, if $(u_n) \subset E^\tau$ is a (PS) sequence of the functional I restricted to E^τ, $I|_{E^\tau}$, then (u_n) is a (PS) sequence of I.

Step 2) It follows exactly the same way as Step 2 of Lemma 1.3.5. As (u_n) is bounded, then $u_n \rightharpoonup u_0$ in E and $I'(u_0) = 0$.

Step 3) Now we verify that $u_0 \in E^\tau$. Since $u_n(x) \to u_0(x)$ a.e. $x \in \mathbb{R}^N$. Furthermore, $u_n \in E^\tau$, implies that $T_\tau(u_n(x)) = u_n(x)$, thus

$$\begin{aligned} T_\tau(u_0(x)) &:= -u_0(\tau x) = -\lim_{n\to\infty} u_n(\tau x) = \lim_{n\to\infty} -u_n(\tau x) \\ &= \lim_{n\to\infty} T_\tau(u_n(x)) = \lim_{n\to\infty} u_n(x) = u_0(x). \end{aligned}$$

Therefore, $u_0 \in E^\tau$.

Step 4) Let $u_n^1 := u_n - u_0$. Then, if $n \to \infty$, we have:

(i) $\|u_n^1\|^2 = \|u_n\|^2 - \|u_0\|^2 + o_n(1)$;

(ii) $I_\infty(u_n^1) \to c - I(u_0)$;

(iii) $I'_\infty(u_n^1) \to 0$.

The proof of $(i), (ii)$ and (iii) is similar to Step 2 in Lemma 1.1.5. By (ii) and (iii), (u_n^1) is a (PS) sequence of I_∞ and

$$\left\langle I'_\infty(u_n^1), \varphi \right\rangle = \left\langle I'(u_n), \varphi \right\rangle - \left\langle I'(u_0), \varphi \right\rangle = o_n(1).$$

Furthermore, since $u_n, u_0 \in E^\tau$ and the operator T_τ is linear, it follows that $T_\tau(u_n^1)(x) = T_\tau(u_n - u_0)(x) = T_\tau(u_n)(x) - T_\tau(u_0)(x) = u_n(x) - u_0(x) = u_n^1(x)$ and $u_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$.

Consider

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n^1(x)|^2 dx.$$

Step 5) If $\delta = 0$, it follows from Lions' Lemma that

$$u_n^1 \to 0 \quad \text{in } L^p(\mathbb{R}^N), \quad \text{for all } 2 < p < 2^*. \tag{2.3.16}$$

On the other hand, since (u_n^1) is a bounded sequence and (iii) holds, then

$$I'_\infty(u_n^1)u_n^1 = \int_{\mathbb{R}^N} \left(\xi_\infty|\nabla u_n^1|^2 + V_\infty(u_n^1)^2 - h(u_n^1)(u_n^1)^2\right) dx \to 0. \tag{2.3.17}$$

Using the estimate (2.1.3) we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (\xi_\infty|\nabla u_n^1|^2 + V_\infty(u_n^1)^2) dx &= \int_{\mathbb{R}^N} h(u_n^1)(u_n^1)^2 dx + o_n(1) \\ &< \varepsilon \int_{\mathbb{R}^N} (u_n^1)^2 dx + C\int_{\mathbb{R}^N} |u_n^1|^p dx \end{aligned} \tag{2.3.18}$$

Thus, by (2.3.16) and (2.3.18) we have $\|u_n^1\| \to 0$, that is, $u_n \to u_0$ and u_0 is a τ-antisymmetric solution of problem (2.1.4) which completes the proof of item 1.

Step 6) Just as in Step 6 of proof of Lemma 1.3.5 of Chapter 1, if $\delta > 0$, define a new sequence $v_n^1 := u_n^1(\cdot + y_n)$ bounded because (u_n^1) is bounded, we have the same result in a previous chapter. Consider now $\mathbb{R}^N = \Gamma \oplus \Gamma^\perp$, where $\Gamma := \{x \in \mathbb{R}^N : \tau(x) = x\}$, and consider P_Γ the projection on the subspace Γ. We can distinguish two cases:

Case I: If $|y_n - \tau y_n|$ is bounded, we define $y_n^1 := P_\Gamma(y_n)$;

Case II: If $|y_n - \tau y_n|$ is unbounded, we define $y_n^1 := y_n$.

Let us study each of these cases. In *Case I*, first note that $|y_n^1| \to \infty$. In fact, the orthogonal linear transformation $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is diagonalizable and without loss of generality, we may assume that

$$\tau(x_1, \ldots, x_k, x_{k+1}, \ldots, x_N) = (x_1, \ldots, x_k, -x_{k+1}, \ldots, -x_N). \tag{2.3.19}$$

Denoting by y_n by $y_n = P_\Gamma(y_n) + w_n = y_n^1 + w_n$, then $y_n^1 := P_\Gamma(y_n)$ implies $\tau(y_n^1) = y_n^1$. Let $y_n = (x_1^n, \ldots, x_k^n, x_{k+1}^n, \ldots, x_N^n)$, where $y_n^1 = (x_1^n, \ldots, x_k^n, 0, \ldots, 0)$ and $w_n = (0, \ldots, 0, x_{k+1}^n, \ldots, x_N^n)$. We have

$$\tau(y_n) = (x_1^n, \ldots, x_k^n, -x_{k+1}^n, \ldots, -x_N^n),$$

and

$$|y_n - \tau y_n| = |(0, \ldots, 0, 2x_{k+1}^n, \ldots, 2x_N^n)| = 2|w_n|.$$

Thus, is the new basis we have that $|y_n - \tau y_n|$ is bounded, that is, there exists $M > 0$ such that $|y_n - \tau y_n| \leq 2M$, which gives $|w_n| \leq M$. Since $y_n = y_n^1 + w_n$, $|y_n| \to \infty$ when $n \to \infty$ and $|w_n| \leq M$, then $|y_n^1| \to \infty$ when $n \to \infty$. Futhermore, we consider the sequence $\{u_n^1(\cdot + y_n^1)\}$, which is bounded, so up to a subsequence, $u_n^1(\cdot + y_n^1) \rightharpoonup u^1$ in E, and $u^1 \not\equiv 0$

is a solution of the limit problem (2.1.4). Moreover, since $\tau(y_n^1) = y_n^1$ then

$$\begin{aligned} T_\tau(u^1(x)) &:= -u^1(\tau x) = -\lim_{n\to\infty} u_n^1(\tau x + y_n^1) = \lim_{n\to\infty} -u_n^1(\tau(x+y_n^1)) \\ &= \lim_{n\to\infty} u_n^1(x+y_n^1) = u^1(x). \end{aligned} \tag{2.3.20}$$

We continue by considering

$$u_n^2(x) := u_n^1(x) - u^1(x-y_n^1)$$

and verify that (u_n^2) is a (PS) sequence of I_∞. In fact, we have that

$$\begin{aligned} I_\infty(u_n^2) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(\xi_\infty|\nabla u_n^2|^2 + V_\infty(u_n^2)^2\right)dx - \int_{\mathbb{R}^N} H(u_n^2)dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \left(\xi_\infty|\nabla(u_n^1(x) - u^1(x-y_n^1))|^2 + V_\infty|u_n^1(x) - u^1(x-y_n^1)|^2\right)dx \\ &\quad - \int_{\mathbb{R}^N} H(u_n^1(x) - u^1(x-y_n^1))dx. \end{aligned}$$

If $z = x - y_n^1$ then $x = z + y_n^1$ and $dx = dz$. Renaming z by x when changing variables, we obtain

$$\begin{aligned} I_\infty(u_n^2) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(\xi_\infty|\nabla(u_n^1(x+y_n^1) - u^1(x))|^2 + V_\infty|u_n^1(x+y_n^1) - u^1(x)|^2\right)dx \\ &\quad - \int_{\mathbb{R}^N} H(u_n^1(x+y_n^1) - u^1(x))dx. \end{aligned}$$

Hence we have that

$$\|u_n^1(\cdot + y_n^1) - u^1\|^2 = \|u_n^1(\cdot + y_n^1)\|^2 - 2\langle u_n^1(\cdot + y_n^1), u^1\rangle + \|u^1\|^2. \tag{2.3.21}$$

Since $u_n^1(\cdot + y_n^1) \rightharpoonup u^1$ in E, by weak convergence and Riez Representation, we obtain

$$\langle u_n^1(\cdot + y_n^1), \varphi\rangle \to \langle u^1, \varphi\rangle, \quad \text{for all} \quad \varphi \in E.$$

In particular, if $\varphi = u^1$, then $\langle u_n^1(\cdot + y_n^1), u^1\rangle \to \langle u^1, u^1\rangle$, it follows that $\langle u_n^1(\cdot + y_n^1), u^1\rangle = \|u^1\|^2 + o_n(1)$. Replacing in (2.3.21) we obtain

$$\|u_n^1(\cdot + y_n^1) - u^1\|^2 = \|u_n^1\|^2 - \|u^1\|^2 + o_n(1). \tag{2.3.22}$$

On the other hand, we observe that

$$I_\infty(u_n^1) - I_\infty(u_n^2) - I_\infty(u^1) = \frac{1}{2}\left(\|u_n^1\|^2 - \|u_n^1 - u^1\|^2 - \|u^1\|^2\right) - \int_{\mathbb{R}^N}\Big(H(u_n^1) - H(u_n^2) - H(u^1)\Big)dx.$$

Now, using (2.3.22) and (2.1.20), we have that

$$I_\infty(u_n^2) = I_\infty(u_n^1) - I_\infty(u^1) + o_n(1).$$

Since (u_n^1) is a (PS) sequence for I_∞, we know that $I_\infty(u_n^1)$ converges to a constant, and thus $I_\infty(u_n^2)$ also converge. Finally, we will show that

$$I'_\infty(u_n^2)\varphi \to 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N); \tag{2.3.23}$$

We know that (u_n^1) is a (PS) sequence for I_∞, then

$$I'_\infty(u_n^1)\varphi = o_n(1), \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{2.3.24}$$

Furthermore, u^1 is a solution of equation (2.1.4) we have

$$I'_\infty(u^1)\varphi = 0, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{2.3.25}$$

Thus, with a change of variable, by (2.3.24), (2.3.25) and by Lemma 2.1.3, we obtain that

$$\begin{aligned}
|I'_\infty(u_n^2)\varphi| &= \left| I'_\infty(u_n^1)\varphi - I'_\infty(u^1)\varphi + \int_{\mathbb{R}^N}\Big(h(u_n^1)u_n^1 - h(u_n^1 - u^1)(u_n^1 - u^1) - h(u^1)u^1\Big)\varphi dx \right| \\
&\le o_n(1) + \int_{\mathbb{R}^N}\left|h(u_n^1)u_n^1 - h(u_n^1 - u^1)(u_n^1 - u^1) - h(u^1)u^1\right||\varphi| dx \\
&\le C_\varepsilon \|\varphi\|_{H^1(\mathbb{R}^N)}.
\end{aligned}$$

Thus (2.3.23) holds. Therefore, (u_n^2) is a (PS) sequence for I_∞ and *Case I* is complete.

Case II: Here we have that $|y_n - \tau y_n|$ is unbounded and we define $y_n^1 = y_n$. Moreover, we know that $u^1 \not\equiv 0$ is a weak solution of the equation (2.1.4). Let $u_n^2 := u_n^1 - \gamma_n$, where

$$\gamma_n(x) := u^1(x - y_n^1) - u^1(\tau x - y_n^1). \tag{2.3.26}$$

Note that, since T_τ is an orthogonal linear transformation, it follows that

$$\begin{aligned} T_\tau(\gamma_n(x)) &:= -\gamma_n(\tau x) = -u^1(\tau x - y_n^1) + u^1(x - y_n^1) \\ &= u^1(x - y_n^1) - u^1(\tau x - y_n^1) = \gamma_n(x). \end{aligned}$$

Thus, $u_n^2 \in E^\tau$, because

$$\begin{aligned} T_\tau(u_n^2(x)) &= T_\tau(u_n^1(x) - \gamma_n(x)) = T_\tau(u_n^1(x)) - T_\tau\gamma_n(x)) \\ &= u_n^1(x) - \gamma_n(x) = u_n^2(x). \end{aligned}$$

In this case we must show that (u_n^2) is a (PS) sequence of I_∞. We will show that

$$I_\infty(u_n^2) = I_\infty(u_n^1) - 2I_\infty(u^1) + o_n(1) \tag{2.3.27}$$

using the fact that (u_n^1) is a (PS) sequence of I_∞. We have that

$$\|u_n^2\|^2 = \|u_n^1 - \gamma_n\|^2 = \|u_n^1\|^2 - 2\langle u_n^1, \gamma_n\rangle + \|\gamma_n\|^2, \tag{2.3.28}$$

such that

$$\begin{aligned} \langle u_n^1, \gamma_n\rangle &= \int_{\mathbb{R}^N} \xi_\infty \nabla u_n^1 \nabla u^1(x - y_n^1)dx + \int_{\mathbb{R}^N} \xi_\infty \nabla u_n^1 \nabla u^1(\tau x - y_n^1)dx \\ &\quad + \int_{\mathbb{R}^N} V_\infty u_n^1 u^1(x - y_n^1)dx + \int_{\mathbb{R}^N} V_\infty u_n^1 u^1(\tau x - y_n^1)dx. \end{aligned}$$

Firstly, we claim that

$$\langle u_n^1, \gamma_n\rangle = 2\|u^1\|^2 + o_n(1). \tag{2.3.29}$$

Indeed, let

$$A_n^1 = \int_{\mathbb{R}^N} \Big(\xi_\infty \nabla u_n^1 \nabla u^1(x - y_n^1) + V_\infty u_n^1 u^1(x - y_n^1)\Big)dx$$

and

$$A_n^2 = \int_{\mathbb{R}^N} \Big(\xi_\infty \nabla u_n^1 \nabla u^1(\tau x - y_n^1) + V_\infty u_n^1 u^1(\tau x - y_n^1)\Big)dx.$$

We will show that

$$A_n^1 \to \left\{\int_{\mathbb{R}^N} \Big(\xi_\infty |\nabla u^1|^2 + V_\infty (u^1)^2\Big)dx\right\}, \quad \text{when} \quad n \to \infty,$$

and

$$A_n^2 \to -\left\{\int_{\mathbb{R}^N}\left(\xi_\infty|\nabla u^1|^2+V_\infty(u^1)^2\right)dx\right\}, \quad \text{when} \quad n\to\infty. \tag{2.3.30}$$

Let $z=x-y_n^1$, thus $x=z+y_n^1$ and $dx=dz$, combining this with $u_n^1(\cdot+y_n^1)\rightharpoonup u^1(\cdot)$, we have

$$\int_{\mathbb{R}^N}\left(\xi_\infty\nabla u_n^1(z+y_n^1)\nabla u^1(z)+V_\infty u_n^1(z+y_n^1)u^1(z)\right)dx \to \int_{\mathbb{R}^N}\left(\xi_\infty|\nabla u^1|^2+V_\infty(u^1)^2\right)dx$$

To evaluate A_n^2, let us consider the following change of variables $\tau x-y_n^1=z$, then $x=\tau(z+y_n^1)$ and $dx=dz$. Thus,

$$A_n^2=\int_{\mathbb{R}^N}\left(\xi_\infty\nabla u_n^1(\tau(z+y_n^1))\nabla u^1(z)+V_\infty u_n^1(\tau(z+y_n^1))u^1(z)\right)dx.$$

Since u_n^1 is τ-antisymmetric, we have

$$A_n^2=-\left\{\int_{\mathbb{R}^N}\left(\xi_\infty\nabla u_n^1(\tau(z+y_n^1))\nabla u^1(z)+V_\infty u_n^1(\tau(z+y_n^1))u^1(z)\right)dx\right\}.$$

Therefore, in a similar way to A_n^1, we obtain (2.3.30) and thus prove (2.3.29). Now, we claim

$$\|\gamma_n\|^2=2\|u^1\|^2+o_n(1). \tag{2.3.31}$$

In fact, from (2.3.8) and (2.3.9) we have that

$$\begin{aligned}\|\gamma_n\|^2 &= \int_{\mathbb{R}^N}(\xi_\infty|\nabla\gamma_n|^2+V_\infty\gamma_n^2)dx\\ &= 2\|u^1\|^2-2\int_{\mathbb{R}^N}\xi_\infty|\nabla u^1(x-y_n^1)\nabla u^1(\tau x-y_n^1)dx-2\int_{\mathbb{R}^N}V_\infty u^1(x-y_n^1)u^1(\tau x-y_n^1)dx\\ &= 2\|u^1\|^2+o_n(1).\end{aligned}$$

Thus, obtaining (2.3.31).

Finally, replacing (2.3.29) and (2.3.31) in (2.3.27), then

$$\|u_n^2\|^2=\|u_n^1\|^2-2\|u^1\|+o_n(1). \tag{2.3.32}$$

To conclude (2.3.27) we need to verify the following equality

$$\int_{\mathbb{R}^N}H(u_n^2)dx=\int_{\mathbb{R}^N}H(u_n^1)dx-2\int_{\mathbb{R}^N}H(u^1)dx+o_n(1). \tag{2.3.33}$$

Define $\rho := \dfrac{|y_n^1 - \tau y_n^1|}{2}$, $S_n = \mathbb{R}^N \setminus B_{\rho_n}(0) \cup B_{\rho_n}(\tau y_n^1 - y_n^1)$ and using the fact that $u^1(\tau x - y_n^1) = u^1(\tau(x - \tau y_n^1)) = -u^1(x - \tau y_n^1)$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} H(u_n^2)dx &= \int_{\mathbb{R}^N} H(u_n^1 - \gamma_n)dx = \int_{\mathbb{R}^N} H(u_n^1(x) - u^1(x - y_n^1) - u^1(\tau x - y_n^1))dx \\
&= \int_{B_{\rho_n}(0)} H(u_n^1(z + y_n^1) - u^1(z) - u^1(z + y_n^1 - \tau y_n^1))dz \\
&\quad + \int_{B_{\rho_n}(\tau y_n^1 - y_n^1)} H(u_n^1(z + y_n^1) - u^1(z) - u^1(z + y_n^1 - \tau y_n^1))dz \\
&\quad + \int_{S_n} H(u_n^1(z + y_n^1) - u^1(z) - u^1(z + y_n^1 - \tau y_n^1))dz \\
&= \int_{B_{\rho_n}(0)} H(u_n^1(z + y_n^1) - u^1(z + y_n^1 - \tau y_n^1))dz - \int_{B_{\rho_n}(0)} H(u^1(z))dz \\
&\quad + \int_{B_{\rho_n}(\tau y_n^1 - y_n^1)} H(u_n^1(z + y_n^1) - u^1(z))dz - \int_{B_{\rho_n}(\tau y_n^1 - y_n^1)} H(u^1(z + y_n^1 - \tau y_n^1))dz \\
&\quad + \int_{S_n} H(u_n^1(z + y_n^1) - u^1(z + y_n^1 - \tau y_n^1))dz - \int_{S_n} H(u^1(z))dz + o_n(1).
\end{aligned}
$$

Under the assumptions that $u_n^1(z + y_n^1) - u^1(z) \to 0$ if $|y_n^1| \to \infty$ a.e. $z \in \mathbb{R}^N$ and that $u^1(z + y_n^1 + \tau y_n^1) \to 0$ a.e. $z \in \mathbb{R}^N$, together with the Brezis-Lieb Lemma, we have the $(A) - (F)$ statements of proof of Lemma 1.3.5. Then using (2.3.32) and (2.3.33) we have

$$
I_\infty(u_n^2) = I_\infty(u_n^1) - 2I_\infty(u^1) + o_n(1).
$$

which completes the proof of (2.3.27).

Since (u_n^1) is a (PS) sequence of I_∞, then $I_\infty(u_n^2)$ converges to a constant. To complete the proof we will show that if $n \to \infty$, then (2.3.23) holds. Indeed

$$
\begin{aligned}
|I'_\infty(u_n^2)\varphi| &= \left| \int_{\mathbb{R}^N} \Big(\xi_\infty \nabla(u_n^1 - \gamma_n)\nabla\varphi + V_\infty(u_n^1 - \gamma_n)\varphi \Big) dx - \int_{\mathbb{R}^N} h(u_n^1 - \gamma_n)(u_n^1 - \gamma)\varphi dx \right| \\
&\le \left| \int_{\mathbb{R}^N} \Big(\xi_\infty \nabla u_n^1 \nabla\varphi + V_\infty u_n^1 \varphi \Big) dx - \int_{\mathbb{R}^N} h(u_n^1) u_n^1 \varphi dx + \int_{\mathbb{R}^N} \Big(\xi_\infty \nabla\gamma_n \nabla\varphi + V_\infty \gamma_n \varphi \Big) dx \right. \\
&\quad - \int_{\mathbb{R}^N} h(\gamma_n)\gamma_n \varphi dx - \int_{\mathbb{R}^N} h(u_n^1 - \gamma_n)(u_n^1 - \gamma)\varphi dx \\
&\quad \left. + \int_{\mathbb{R}^N} h(u_n^1) u_n^1 \varphi dx + \int_{\mathbb{R}^N} h(\gamma_n)\gamma_n \varphi dx \right|.
\end{aligned}
$$

And since (u_n^1) is a (PS) sequence of I_∞ we have that

$$
\int_{\mathbb{R}^N} \Big(\xi_\infty \nabla u_n^1 \nabla\varphi + V_\infty u_n^1 \varphi \Big) dx - \int_{\mathbb{R}^N} h(u_n^1) u_n^1 \varphi dx = o_n(1). \tag{2.3.34}
$$

From (2.3.34), using the definition of γ_n and from the triangular inequality we obtain that

$$|I'_\infty(u_n^2)\varphi| \leq K_n^1 + K_n^2 + o_n(1), \tag{2.3.35}$$

where

$$\begin{aligned} K_n^1 &:= \int_{\mathbb{R}^N} \Big(\xi_\infty \nabla\gamma_n \nabla\varphi + V_\infty \gamma_n \varphi\Big) dx \\ &= \int_{\mathbb{R}^N} \Big(\xi_\infty \nabla(u^1(x-y_n^1) - u^1(\tau x - y_n^1))\nabla\varphi + V_\infty(u^1(x-y_n^1) - u^1(\tau x - y_n^1))\varphi\Big) dx \end{aligned}$$

and

$$\begin{aligned} K_n^2 &:= \int_{\mathbb{R}^N} |h(\gamma_n)||\gamma_n||\varphi| dx \\ &= \int_{\mathbb{R}^N} |h(u^1(x-y_n^1) - u^1(\tau x - y_n^1))||u^1(x-y_n^1) - u^1(\tau x - y_n^1)||\varphi| dx. \end{aligned}$$

we have that $k_n^1 = o_n(1)$ and $k_n^2 = o_n(1)$. The proof once again follows as in Lemma 1.3.5 using Hölder's inequality and the growth of h, this completes the proof of (2.3.34) and thus we verify that $\{u_n^2\}$ is a (PS) sequence of I_∞, also in *Case II*.

Now proceeding by iteration, we note that if u is a non-trivial critical point of I_∞ and ω is a minimum energy solution of the equation (2.1.4) given by Berestycki and Lions, then we have that

$$I_\infty(u) \geq I_\infty(\omega) > 0. \tag{2.3.36}$$

On the other hand, from (2.3.27) and item (ii) we obtain

$$I_\infty(u_n^2) = c - I(u_0) - 2I_\infty(u^1) + o_n(1). \tag{2.3.37}$$

From (2.3.34) and (2.3.35) the iteration must end at some index $k \in \mathbb{N}$ and the proof of the lemma is complete. □

In the next result, we verify that the functional I restricted to E^τ, associated with the problem (2.1.4), satisfying $(Ce)_c$ for c below the level $2m_\infty$.

Lemma 2.3.6. *The functional I restricted to E^τ satisfies $(Ce)_c$ condition for any $c < 2m_\infty$.*

Proof. Let (u_n) be a sequence in E^τ such that

$$I(u_n) \to c < 2m_\infty \quad \text{and} \quad (1+\|u_n\|)\left\|I'|_{E^\tau}(u_n)\right\| \to 0.$$

This imply that $I'|_{E^\tau}(u_n) \to 0$, namely, (u_n) is a (PS) sequence of I restricted to E^τ and by Lemma 2.3.4 we have $I'(u_n) \to 0$. Moreover, by Lemma 2.3.1, (u_n) is bounded sequence, up to a subsequence, $u_n \rightharpoonup u_0$ in E and $I'(u_0)\varphi = 0$, for all $\varphi \in E$. In particular,

$$I'(u_0)u_0 = \int_{\mathbb{R}^N} \left(\xi(x)|\nabla u_0|^2 + V(x)u_0^2\right) dx - \int_{\mathbb{R}^N} f(x,u_0)u_0 dx = 0. \tag{2.3.38}$$

It follows from the hypothesis (f_5), the definition of norm in E and (2.3.38) that

$$I(u_0) = \frac{1}{2}\|u_0\|^2 - \int_{\mathbb{R}^N} F(x,u_0)dx = \int_{\mathbb{R}^N} \left(\frac{1}{2}f(x,u_0)u_0 - F(x,u_0)\right) dx \geq 0. \tag{2.3.39}$$

If (u_n) does not converge strongly to u_0 in the norm of E then, by Lemma 2.3.5 there exists two integers $k_1 \geq 1$ or $k_2 \geq 1$, k_1 solutions u^j, $j = 1, ..., k_1$ and k_2 τ-antisymmetric solutions u^j, $j = k_1+1, ..., k_1+k_2$ of equation (2.1.4), satisfying

$$c = \lim_{n\to\infty} I(u_n) \geq I(u_0) + 2k_1 m_\infty + \sum_{j=k_1+1}^{k_1+k_2} I_\infty(u^j) \geq 2m_\infty, \tag{2.3.40}$$

since $I_\infty(u^j) \geq 2m_\infty$ for all nontrivial τ-antisymmetric solution u^j of (2.1.4), which contradicts our assumption. Therefore, up to a subsequence, $u_n \to u_0 \in E^\tau$ and the lemma is proved. □

Lemma 2.3.7. *Let* $m_\infty^\tau := \inf_{u\in\mathcal{P}} I_\infty(u)$, *then*

$$2m_\infty \leq m_\infty^\tau.$$

Proof. Let us show first that if $u \in \mathcal{P}$ then u^+, $u^- \in \mathcal{P}$. Using a change of variable and that $G(s)$ is an even function and defining $A^\tau := \{x : -u(\tau x) \geq 0\}$, we obtain

$$J(u^+) = \int_{\mathbb{R}^N} |\nabla u^-|^2 dz - 2^* \int_{\mathbb{R}^N} G_\infty(u^-)dz = J(u^-).$$

On the other hand,

$$0 = J(u) = J(u^+) + J(u^-) = 2J(u^+) = 2J(u^-).$$

Therefore $u^+,\ u^- \in \mathcal{P}$. Now, taking into account that H is even we have

$$I_\infty(u^+) = \int_{\mathbb{R}^N} (\xi_\infty|\nabla u^-|^2 + V_\infty(u^-)^2)dz - \int_{\mathbb{R}^N} H(u^-)dz = I_\infty(u^-).$$

Finally,

$$I_\infty(u) = I_\infty(u^+) + I_\infty(u^-).$$

Therefore, for all $u \in \mathcal{P}$ we have

$$I_\infty(u) = I_\infty(u^+) + I_\infty(u^-) = 2I_\infty(u^+) \geq 2m_\infty,$$

thus,

$$m_\infty^\tau = \inf_{u\in\mathcal{P}} I_\infty(u) \geq 2m_\infty.$$

□

Remark 2.3.3. *If $z_y(x) = \omega(x-y) - \omega(x-\tau y)$, then t_{z_y} as in (2.3.12) is bounded when $|y| \to \infty$ and $|y-\tau y| \to \infty$.*

Lemma 2.3.8. *Suppose ξ, V satisfies $(\xi_1)-(\xi_4)$, $(V_1)-(V_4)$, respectively and either (2.1.9) or (2.1.10) or (2.1.11). Then*

$$c^\tau < 2m_\infty.$$

Proof. Denote $t = t_{z_y}$, for simplicity of notation. Since I_∞ is translation invariance we obtain

$$\begin{aligned}
I\left(z_y\left(\frac{\cdot}{t}\right)\right) &= \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} \xi(tx)|\nabla\omega(x-y)|^2dx + \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} \xi(tx)|\nabla\omega(x-\tau y)|^2dx \\
&\quad -2\frac{t^{N-2}}{2}\int_{\mathbb{R}^N}\xi(tx)\nabla\omega(x-y)\nabla\omega(x-\tau y)dx + \frac{t^N}{2}\int_{\mathbb{R}^N} V(tx)(\omega(x-y))^2dx \\
&\quad +\frac{t^N}{2}\int_{\mathbb{R}^N} V(tx)(\omega(x-\tau y))^2dx - 2\frac{t^N}{2}\int_{\mathbb{R}^N} V(tx)\omega(x-y)\omega(x-\tau y)dx \\
&\quad -t^N\int_{\mathbb{R}^N} F(tx,\omega(x-y)-\omega(x-\tau y))dx \\
&= I_\infty\left(\omega\left(\frac{\cdot}{t}-y\right)\right) + I_\infty\left(\omega\left(\frac{\cdot}{t}-\tau y\right)\right) + \frac{t^{N-2}}{2}\int_{\mathbb{R}^N}(\xi(tx)-\xi_\infty)|\nabla\omega(x-y)|^2dx \\
&\quad +\frac{t^{N-2}}{2}\int_{\mathbb{R}^N}(\xi(tx)-\xi_\infty)|\nabla\omega(x-\tau y)|^2dx - t^{N-2}\int_{\mathbb{R}^N}\xi(tx)\nabla\omega(x-y)\nabla\omega(x-\tau y)dx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{t^N}{2}\int_{\mathbb{R}^N}(V(tx)-V_\infty)(\omega(x-y))^2dx+\frac{t^N}{2}\int_{\mathbb{R}^N}(V(tx)-V_\infty)(\omega(x-\tau y))^2dx\\
&-t^N\int_{\mathbb{R}^N}V(tx)\omega(x-y)\omega(x-\tau y)dx+t^N\int_{\mathbb{R}^N}H(\omega(x-y))-F(tx,\omega(x-y))dx\\
&+t^N\int_{\mathbb{R}^N}H(\omega(x-\tau y))-F(tx,\omega(x-\tau y))dx-t^N\int_{\mathbb{R}^N}F(tx,\omega(x-y)-\omega(x-\tau y))dx\\
&+t^N\int_{\mathbb{R}^N}F(tx,\omega(x-y))dx+t^N\int_{\mathbb{R}^N}F(tx,\omega(x-\tau y))dx\\
=\;&I_\infty\left(\omega\left(\frac{\cdot}{t}\right)\right)+I_\infty\left(\omega\left(\frac{\cdot}{t}\right)\right)+R(\xi,\xi_\infty,V,V_\infty,|y|,|y-\tau y|),
\end{aligned}
\tag{2.3.41}
$$

where

$$
\begin{aligned}
R(\xi,\xi_\infty,V,V_\infty,|y|,|y-\tau y|)&=\frac{t^{N-2}}{2}\int_{\mathbb{R}^N}(\xi(tx)-\xi_\infty)|\nabla\omega(x-y)|^2dx\\
&+\frac{t^{N-2}}{2}\int_{\mathbb{R}^N}(\xi(tx)-\xi_\infty)|\nabla\omega(x-\tau y)|^2dx-t^{N-2}\int_{\mathbb{R}^N}\xi(tx)\nabla\omega(x-y)\nabla\omega(x-\tau y)dx\\
&+\frac{t^N}{2}\int_{\mathbb{R}^N}(V(tx)-V_\infty)(\omega(x-y))^2dx+\frac{t^N}{2}\int_{\mathbb{R}^N}(V(tx)-V_\infty)(\omega(x-\tau y))^2dx\\
&-t^N\int_{\mathbb{R}^N}\omega(x-y)\omega(x-\tau y)dx+t^N\int_{\mathbb{R}^N}H(\omega(x-y))-F(tx,\omega(x-y))dx\\
&+t^N\int_{\mathbb{R}^N}H(\omega(x-\tau y))-F(tx,\omega(x-\tau y))dx-t^N\int_{\mathbb{R}^N}F(tx,\omega(x-y)-\omega(x-\tau y))dx\\
&+t^N\int_{\mathbb{R}^N}F(tx,\omega(x-y))dx+t^N\int_{\mathbb{R}^N}F(tx,\omega(x-\tau y))dx.
\end{aligned}
\tag{2.3.42}
$$

To evaluate the sum

$$
\int_{\mathbb{R}^N}F(tx,\omega(x-y)-\omega(x-\tau y))dx-\int_{\mathbb{R}^N}F(tx,\omega(x-y))dx-\int_{\mathbb{R}^N}F(tx,\omega(x-\tau y))dx,
$$

we use hypothesis (f_7). The Lemma A.2, (2.1.3) with $\varepsilon>0$ and $2<p<2^*$, give us

$$
\begin{aligned}
&\Big|F(tx,\omega(x-y)-\omega(x-\tau y))-F(tx,\omega(x-y))-F(tx,\omega(x-\tau y))\Big|\\
&\le\varepsilon\Big|\omega(x-y)\Big|\Big|\omega(x-\tau y))\Big|+C\Big|\omega(x-y)\Big|^{p-1}\Big|\omega(x-\tau y)\Big|\\
&+\varepsilon\Big|\omega(x-\tau y)\Big|\Big|\omega(x-y)\Big|+C\Big|\omega(x-\tau y)\Big|^{p-1}\Big|\omega(x-y)\Big|.
\end{aligned}
$$

It follows from the above estimate and the invariance of translation of the integral that

$$
\int_{\mathbb{R}^N}\Big|F(tx,\omega(x-y)-\omega(x-\tau y))-F(tx,\omega(x-y))-F(tx,\omega(x-\tau y))\Big|dx
$$

$$\leq 4\varepsilon\int_{\mathbb{R}^N}\Big|\omega(z)\Big|\Big|\omega(z+y-\tau y)\Big|dz+2C\int_{\mathbb{R}^N}\Big|\omega(z)\Big|^{p-1}\Big|\omega(z+y-\tau y)\Big|dz.$$

Now we estimate the integrals above. Let $0<\delta<1/2$ to be chosen later, define $A_y := B_{\frac{|y-\tau y|}{p}(1-\delta)}(0)\subset\mathbb{R}^N$ and $R_y := \dfrac{|y-\tau y|}{p}(1-\delta)$. Since ω is solution of (2.1.4), we have $|\omega(x)|\leq Ce^{-\beta|x|}$ for all $\beta\in\left(0,\sqrt{V_\infty/\xi_\infty}\right)$ and

$$\begin{aligned}\int_{A_y}|\omega(x\,-\,y)|^{p-1}|\omega(x-\tau y)|dx &= \int_{A_y}|\omega(z)|^{p-1}|\omega(z+y-\tau y)|dx\\ &\leq \left(\int_{\mathbb{R}^N}(|\omega(z)|^{p-1})^{\frac{p}{p-1}}dz\right)^{(p-1)/p}\left(\int_{A_y}|\omega(z+y-\tau y)|^p dz\right)^{1/p}\\ &\leq C\|\omega\|_{L^p}^{p-1}\left(\int_{A_y}e^{-\beta p|z+y-\tau y|}dz\right)^{1/p}\\ &\leq C\left(e^{-\beta p|y-\tau y|}\int_{A_y}e^{-\beta p|z|}dz\right)^{1/p}\\ &= Ce^{-\beta|y-\tau y|}\left(\int_{A_y}e^{-\beta p|z|}dz\right)^{1/p}, \end{aligned}\tag{2.3.43}$$

making change of variable $\tilde{f}:\mathbb{R}^N\to\mathbb{R}^N,\ z\mapsto -r$ with determinant of the Jacobian given by $det(J(z_1,\cdots,z_N))=r^{N-1}$, and by change of variable theorem, we have that

$$\int_{A_y}e^{-\beta p|z|}dz=\int_0^{\frac{|y-\tau y|}{p}(1-\delta)}e^{\beta pr}det(J(z_1,\cdots,z_N))dr=\int_0^{\frac{|y-\tau y|}{p}(1-\delta)}e^{\beta pr}r^{N-1}dr.$$

Replacing in (2.3.43)

$$\begin{aligned}\int_{A_y}|\omega(x\,-\,y)|^{p-1}|\omega(x-\tau y)|dx &\leq Ce^{-\beta|y-\tau y|}\left(\int_0^{\frac{|y-\tau y|}{p}(1-\delta)}e^{\beta pr}r^{N-1}dr\right)^{1/p}\\ &\leq C(\delta)e^{-\beta|y-\tau y|\frac{p-1}{p}}e^{-\beta|y-\tau y|\frac{\delta}{p}}|y-\tau y|^{N/p}\\ &\leq C(\delta)e^{-\beta|y-\tau y|\frac{p-1}{p}}, \end{aligned}\tag{2.3.44}$$

since $1<p-1$ and $0<\delta<1/2$. Moreover

$$\begin{aligned}\int_{\mathbb{R}^N\setminus A_y}|\omega(x-y)|^{p-1}|\omega(x-\tau y)|dx &= \int_{\mathbb{R}^N\setminus A_y}|\omega(z)|^{p-1}|\omega(z+y-\tau y)|dx\\ &\leq C\,\|\omega\|_{L^p}^{p-1}\left(\int_{\mathbb{R}^N\setminus A_y}e^{-\beta p|z|}dz\right)^{\frac{p-1}{p}}\end{aligned}$$

$$= C\,\|\omega\|_{L^p}^{p-1}\left(\int_{\frac{|y-\tau y|}{p}(1-\delta)}^{\infty} e^{-\beta p r} r^{N-1} dr\right)^{\frac{p-1}{p}}.$$

Now, using integration by parts, for any $k > 0$ we have

$$\int e^{-kr} r^{N-1} dr = e^{-kr} P(r),$$

where

$$P(r) := \frac{r^{N-1}}{k} - \frac{(N-1)}{k^2} r^{N-2} + \frac{(N-1)(N-2)}{k^3} r^{N-3} + \ldots + (-1)^{N+1} \frac{(N-1)!}{k^N}.$$

Thus,

$$\int_{R_y}^{\infty} e^{-kr} r^{N-1} dr = e^{-kr} P(r)\Big|_{R_y}^{\infty} = e^{-kR_y} P(R_y). \tag{2.3.45}$$

Therefore, taking $k := \beta p$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N \setminus A_y} &|\omega(x - y)|^{p-1} |\omega(x-\tau y)| dx \\ &\leq C\|w\|_{L^p} e^{-\beta p|y-\tau y|(1-2\delta)\frac{p-1}{p}} \left[e^{\beta p|y-\tau y|\delta} P\left(|y-\tau y|\frac{1-\delta}{p}\right)\right]^{\frac{p-1}{p}} \\ &\leq C(\delta)\,\|\omega\|_{L^p}\, e^{-\beta|y-\tau y|\frac{p-1}{p}(1-2\delta)}.\end{aligned}$$

Hence, taking δ sufficiently small such that $0 < (1-2\delta) < 1$, we obtain

$$\int_{\mathbb{R}^N \setminus A_y} |\omega(x-y)|^{p-1} |\omega(x-\tau y)| dx \leq C(\delta) e^{-\beta|y-\tau y|\frac{p-1}{p}(1-2\delta)}. \tag{2.3.46}$$

Thus, from (2.3.44) and (2.3.46) we have

$$\int_{\mathbb{R}^N} |\omega(x-y)|^{p-1} |\omega(x-\tau y)| dx \leq C e^{-\beta|y-\tau y|\frac{p-1}{p}(1-2\delta)}. \tag{2.3.47}$$

For $p = 2$ we argue similarly and define $A_y = B_{\frac{|y-\tau y|}{2}(1-\delta)}(0) \subset \mathbb{R}^N$. Choosing $R_y := \dfrac{|y-\tau y|}{2}(1-\delta)$ and using Hölder's inequality we obtain

$$\int_{A_y} \omega(z)\omega(z+y-\tau y) dz \leq C e^{-\beta|y-\tau y|} e^{\beta\frac{|y-\tau y|}{2}(1-\delta)} \left(\frac{|y-\tau y|}{2}(1-\delta)\right)^{N/2} \leq C(\delta) e^{-\beta\frac{|y-\tau y|}{2}}. \tag{2.3.48}$$

On the other hand, using Hölder's inequality and (2.3.45), it follows

$$\int_{\mathbb{R}^N\setminus A_y}\omega(x-y)\omega(x-\tau y)dz \le C\|\omega\|_{L^2}e^{-\beta|y-\tau y|\frac{1-2\delta}{2}}\left(e^{\beta|y-\tau y|\delta}P\left(|y-\tau y|\frac{1-\delta}{2}\right)\right)^{1/2}$$
$$\le C(\delta)e^{-\beta|y-\tau y|\frac{1-2\delta}{2}}. \quad (2.3.49)$$

By (2.3.48), (2.3.49) and $0<(1-2\delta)<1$ it holds that

$$\int_{\mathbb{R}^N}\omega(x-y)\omega(x-\tau y)dx \le C(\delta)e^{-\beta\frac{|y-\tau y|}{2}}+C(\delta)e^{-\beta|y-\tau y|\frac{(1-2\delta)}{2}}$$
$$\le C(\delta)e^{-\beta|y-\tau y|\frac{1}{2}(1-2\delta)}. \quad (2.3.50)$$

Arguing as in the proof of inequality (2.3.50), we obtain

$$\int_{\mathbb{R}^N}\nabla\omega(x-y)\nabla\omega(x-\tau y)dx \le Ce^{-\beta|y-\tau y|\frac{1}{2}(1-2\delta)}. \quad (2.3.51)$$

We consider $\beta_1<\beta<\sqrt{V_\infty/\xi_\infty}$ or $\beta_2<\beta<\sqrt{V_\infty/\xi_\infty}$ or $\beta_3<\beta<\sqrt{V_\infty/\xi_\infty}$. By (2.1.9) and a change of variable, there exists a positive constant C such that

$$\int_{\mathbb{R}^N}(\xi(x)-\xi_\infty)|\nabla\omega(x-y)|^2dx \le -Ce^{-\beta_1|y|}. \quad (2.3.52)$$

We also have

$$\int_{\mathbb{R}^N}(\xi(x)-\xi_\infty)|\nabla\omega(x-\tau y)|^2dx \le -Ce^{-\beta_1|y|}. \quad (2.3.53)$$

Or else by (2.1.10), there exists a positive constant C such that

$$\int_{\mathbb{R}^N}(V(x)-V_\infty)|\omega(x-y)|^2dx < -Ce^{-\beta_2|y|}\int_{\mathbb{R}^N}e^{-\beta_2|z|}|\omega(z)|^2dz \le -Ce^{-\beta_2|y|}. \quad (2.3.54)$$

Similarly, we obtain

$$\int_{\mathbb{R}^N}(V(x)-V_\infty)|\nabla\omega(x-\tau y)|^2dx \le -Ce^{-\beta_2|\tau y|} = -Ce^{-\beta_2|y|}. \quad (2.3.55)$$

Or else by (2.1.11), as well as in (4.0.63) from the previous chapter, there exists a positive constant $C>0$ such that

$$\int_{\mathbb{R}^N}|H(\omega(x-y))-F(tx,\omega(x-y))|dx \le -Ce^{-\beta_3|y|}. \quad (2.3.56)$$

Analogously, we have

$$\int_{\mathbb{R}^N} |H(\omega(x-\tau y)) - F(tx,\omega(x-\tau y))|dx \leq -Ce^{-\beta_3|y|}. \tag{2.3.57}$$

Now we study the sign of $R(\xi,\xi_\infty,V,V_\infty,|y|,|y-\tau y|)$. If we consider the inequalities from (2.3.43) to (2.3.57) in the definition of $R(\xi,\xi_\infty,V,V_\infty,|y|,|y-\tau y|)$ in (2.3.42), then

$$\begin{aligned} R(\xi,\xi_\infty,V,V_\infty,|y|,|y-\tau y|) &\leq -Ce^{-\beta_1|y|} - Ce^{-\beta_1|y|} - C(\delta)e^{-\beta|y-\tau y|\frac{(1-2\delta)}{2}} \\ &- Ce^{-\beta_2|y|} - Ce^{-\beta_2|y|} - C(\delta)e^{-\beta|y-\tau y|\frac{(1-2\delta)}{2}} - Ce^{-\beta_3|y|} - Ce^{-\beta_3|y|} \\ &+ C(\delta)e^{-\beta|y-\tau y|\frac{p-1}{p}(1-2\delta)} - Ce^{-\beta 3|y|} + Ce^{-\beta|y-\tau y|(1-2\delta)} + Ce^{-\beta|y-\tau y|\frac{1}{2}(1-2\delta)}. \end{aligned}$$

Let $\tilde{y} = (y_1,\dots,y_k,\dots,y_n)$, $\tau\tilde{y} = (y_1,\dots,y_k,-y_{k+1},\dots,-y_n)$, the projection $P_k\tilde{y} = (y_1,\dots,y_k,0,\dots,0)$ and $|\tilde{y}-\tau\tilde{y}| = |(0,\dots,0,2y_{k+1},\dots,2y_n)| = 2|(0,\dots,0,y_{k+1},\dots,y_n)|$ be such that $|(0,\dots,0,y_{k+1},\dots,y_n)| \to \infty$. If we choose $y := P_\Gamma^\perp\tilde{y} = (0,\dots,0,y_{k+1},\dots,y_n)$, such that $2|y| = |y-\tau y|$, since $t = t_{z_y}$ is bounded and $\dfrac{1}{2} < \dfrac{p-1}{p}$, we obtain for $|y|$ sufficiently large

$$R(\xi,\xi_\infty,V,V_\infty,|y|,|y-\tau y|) \leq -Ce^{-\beta_1|y|} - Ce^{-\beta_2|y|} - Ce^{-\beta_3|y|} + Ce^{-\beta(1-2\delta)|y|} < 0. \tag{2.3.58}$$

Replacing (2.3.58) in (2.3.41) we obtain that $I\left(z_y\left(\dfrac{\cdot}{t_{z_y}}\right)\right) < 2m_\infty$.

To finish the proof of the lemma, see Lemma 1.3.8. □

Proof of Theorem 2.1.2. Let $(u_n) \subset E^\tau$ be the sequence given by Ghoussou-Priess Theorem in Lemma 2.3.3. By Lemma 2.3.1 this sequence is bounded, and

$$I(u_n) \to c^\tau \quad \text{and} \quad I'(u_n) \to 0 \ \text{ in } (E^*)^\tau.$$

Up to a subsequence, $u_n \rightharpoonup u_0$ weakly in E and $I'(u_0) = 0$. By Lemma 2.3.5 we have either $u_n \to u_0$ strongly in E or there exists two integers $k_1,k_2 \geq 0$, k_1 solutions u^j, $j = 1,...,k_1$ and k_2 τ-antisymmetric solution u^j, $j = k_1+1,...,k_1+k_2$ of equation (2.1.4), satisfying the conclusion of Lemma 2.3.5. Suppose that the second case is holds. It follows from Lemma 2.3.8 that $c^\tau < 2m_\infty$ and hence by Lemma 2.3.5 item 5 we must have $k_1,k_2 = 0$.

Otherwise, without loss of generality, if $k_1 \geq 1$ then by Lemma 2.3.7, we get

$$c^{\tau} \geq 2k_1 m_{\infty} + 2(k_1 + k_2)m_{\infty} \geq 2m_{\infty},$$

contrary the assumption that $c^{\tau} < 2m_{\infty}$. Therefore, $k_1 = k_2 = 0$, $u_n \to u_0$ strongly in E and $c^{\tau} = I(u_0)$. Moreover, since $I(u_0) = c^{\tau} > 0$, it follows that $u_0 \not\equiv 0$, u_0 is τ-antisymmetric and hence it is a sing-changing solution u_0 of (P_{τ}). □

Chapter 3

Problem with ξ positive and V sign-change

3.1 Spectral Theory

In this section, we present some definitions and results on spectral theory, the proof will be omitted and can be found in [12] and [31].

Definition 3.1.1. *Let H be a Hilbert space and let $A : D(A) \subset H \to H$ be a linear operator whose domain $D(A)$ is a dense subspace of H. Its adjoint operator $A^* : D(A^*) \subset H \to H$ is defined by*

$$v \in D(A^*) \Longleftrightarrow \begin{cases} v \in H \quad \text{and there exists an element} \quad w \in H, \\ \text{such that,} \ \langle Au, v\rangle = \langle u, w\rangle, \quad \text{for all} \quad u \in D(A), \end{cases}$$

and

$$Av = w, \quad \text{for all} \quad v \in D(A^*)$$

so that, by the density of $D(A)$ in H, w is the only element associated with v by definition of $D(A^*)$.

The operator A is said symmetric when $\langle Au, v\rangle = \langle u, Av\rangle$, for all $u, v \in D(A)$, and if, in addition, $D(A) = D(A^*)$, the operator is called self-adjoint.

Definition 3.1.2. *An operator B is an extension of the operator A when $D(A) \subset D(B)$ and $A = B$ in $D(A)$. When the extension is unique, the operator is said to be essentially self-adjoint.*

Lemma 3.1.1. *Let $A : D(A) \subset H \to H$ be a self-adjoint in a real Hilbert space. For $\lambda \in \mathbb{R}$, we have that $A - \lambda I : D(A) \subset H \to H$ is an isomorphism if only if there exists a positive constant $c > 0$ such that $\|(A-\lambda I)u\| \geq c\|u\|$, for all $u \in D(A)$.*

Definition 3.1.3. *Let $A : D(A) \subset H \to H$ be a self-adjoint operator. A resolvent set $\rho(A)$ of an operator A is a set*

$$\rho(A) = \Big\{\lambda \in \mathbb{R} : \ A - \lambda I : D(A) \to H \text{ is an isomorphism}\Big\}$$

and the spectrum of A is the set

$$\sigma(A) = \mathbb{R} \setminus \rho(A).$$

The elements of $\rho(A)$ are called regular values for A. The point spectrum is given by the set

$$\sigma_p(A) = \Big\{\lambda \in \mathbb{R} : \ \ker(A - \lambda I) \neq \{0\}\Big\}$$

and its elements are called eigenvalues of A. The discrete spectrum is the set

$$\sigma_d(A) = \Big\{\lambda \in \mathbb{R} : \dim\ker(A - \lambda I) < \infty \text{ and } \lambda \text{ is an isolated point of } \sigma_p(A)\Big\}$$

and its complement in $\sigma(A)$ is called the essential spectrum

$$\sigma_e(A) = \sigma(A) \setminus \sigma_d(A),$$

and it consists of $\lambda \in \sigma(A)$ that not isolate eigenvalues of a finite multiplicity.

3.1.1 The Schrödinger operator

Definition 3.1.4. *Given the functions $\xi, V \in L^\infty(\mathbb{R}^N)$, we define the Schrödinger operator $L : D(L) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ generated by the potential V and by ξ given by*

$$D(L) = H^2(\mathbb{R}^N) \text{ and } Lu = -div(\xi(x)\nabla u) + V(x)u, \quad \text{for all } \ u \in H^2(\mathbb{R}^N).$$

To show that the operator L is self-adjoint, we will use the Fourier transform. For this purpose, it will be necessary to hypothesize that the Fourier transformation of the function ξ is ξ itself. Furthermore, we must use a complex-value function. The corresponding function spaces will be distinguished from the use of italics. Thus, $\mathcal{L}^p = L^p(\mathbb{R}^N, \mathbb{C})$ and $L^p = L^p(\mathbb{R}^N, \mathbb{R})$, etc. The Schwartz space of smooth rapidly decreasing functions will be denoted by

$$\begin{aligned}\mathcal{S} &= \mathcal{S}(\mathbb{R}^N, \mathbb{C})\\ &= \{v \in C^\infty : |x|^j D^\alpha v(x) \in \mathcal{L}^\infty, \text{ for all } j \in \mathbb{N} \text{ and multi-indices } \alpha \in \mathbb{N}^N\}.\end{aligned}$$

For $v \in \mathcal{S}$ (or more generally $v \in \mathcal{L}^1$), its Fourier transform $\hat{v}$ is defined by

$$\hat{v}(\zeta) = (2\pi)^{N/2} \int v(x) e^{-i\zeta.x} dx, \text{ for all } x \in \mathbb{R}^N.$$

We have the following properties

- $\hat{v} \in \mathcal{S}$, for all $v \in \mathcal{S}$, it holds the Parseval's identity

$$\int v\overline{w} dx = \int \hat{v}\overline{\hat{w}} dx, \quad \text{for all } v, w \in \mathcal{S}.$$

- For $v \in \mathcal{S}$, $\partial_j v \in \mathcal{S}$ for all $j = 1, \cdots, N$ and

$$\widehat{\partial_j v}(\zeta) = i\zeta_i \hat{v}(\zeta), \text{ for all } \zeta \in \mathbb{R}^N.$$

More generally,

$$\widehat{D^\alpha v}(\zeta) = (i\zeta)^\alpha \hat{v}(\zeta), \quad \text{for all } \zeta \in \mathbb{R}^N. \tag{3.1.1}$$

Lemma 3.1.2. *Let $v, w \in L^2(\mathbb{R}^N)$ and $\xi \in C(\mathbb{R}^N, \mathbb{R}^+)$ with $\hat{\xi}(x) = \xi(x)$, such that*

$$\int_{\mathbb{R}^N} v div(\xi(x)\nabla z) dx = \int_{\mathbb{R}^N} wz dx, \quad \textit{for all } z \in C_0^\infty(\mathbb{R}^N). \tag{3.1.2}$$

Then $v \in H^2(\mathbb{R}^N)$, $i\zeta(\xi(x) i\zeta \hat{v}(\zeta)) = \hat{w}(\zeta)$, for almost all $\zeta \in \mathbb{R}^N$ and $div(\xi(x)\nabla v) = w$.

Proof. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^2(\mathbb{R}^N)$ and using the Riesz Representation Theorem and Divergence Theorem, we have

$$\int_{\mathbb{R}^N} v div(\xi(x)\nabla z) dx = \int_{\mathbb{R}^N} \xi(x)\nabla v \nabla z dx = \int_{\mathbb{R}^N} wz dx, \quad \text{for all } \; z \in H^2(\mathbb{R}^N). \tag{3.1.3}$$

For $\varphi \in \mathcal{S}$, the real and imaginary part of φ belong to $H^2(\mathbb{R}^N)$ and so

$$\int_{\mathbb{R}^N} v\overline{div(\xi(x)\nabla\varphi)}dx = \int_{\mathbb{R}^N} w\overline{\varphi}dx, \text{ for all } \varphi \in \mathcal{S}. \tag{3.1.4}$$

Furthermore, using $\overline{\xi}(x) = \xi(x)$, since $\xi(x) \geq 0$ for all $x \in \mathbb{R}^N$, from equality (3.1.1) and from Parseval's identity, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} v\overline{div(\xi(x)\nabla\varphi)}dx &= \int_{\mathbb{R}^N} \hat{v}\overline{div(\widehat{\xi(x)}\nabla\varphi)}dx \\ &= \int_{\mathbb{R}^N} \hat{v}i\zeta(\hat{\xi}(x)i\zeta\overline{\hat{\varphi}})d\zeta = \int_{\mathbb{R}^N} \hat{v}i\zeta(\xi(x)i\zeta\overline{\hat{\varphi}})d\zeta\end{aligned} \tag{3.1.5}$$

and

$$\int_{\mathbb{R}^N} w\overline{\varphi}dx = \int_{\mathbb{R}^N} \hat{w}\overline{\hat{\varphi}}d\zeta. \tag{3.1.6}$$

Replacing (3.1.5) and (3.1.6) in (3.1.4) we have

$$\int_{\mathbb{R}^N} \hat{v}i\zeta(\xi(x)i\zeta\overline{\hat{\varphi}})d\zeta = \int_{\mathbb{R}^N} \hat{w}\overline{\hat{\varphi}}d\zeta, \quad \text{for all } \varphi \in \mathcal{S}.$$

Since $\overline{\hat{\mathcal{S}}} = \mathcal{S}$, this means that

$$\int_{\mathbb{R}^N} \hat{v}i\zeta(\xi(x)i\zeta\eta)d\zeta = \int_{\mathbb{R}^N} i\zeta(\xi(x)i\zeta\hat{v})\eta d\zeta = \int_{\mathbb{R}^N} \hat{w}\eta d\zeta, \text{ for all } \eta \in \mathcal{S}. \tag{3.1.7}$$

In particular,

$$\left|\int_{\mathbb{R}^N} i\zeta(\xi(x)i\zeta\hat{v})\eta d\zeta\right| \leq \|\hat{w}\|_{L^2}\|\eta\|_{L^2}, \quad \text{for all } \eta \in \mathcal{S},$$

and since $\mathcal{S}$ is dense in $\mathcal{L}^2$, it follows that $i\zeta(\xi(x)i\zeta\hat{v}(\zeta)) \in \mathcal{L}^2$ and

$$i\zeta(\xi(x)i\zeta\hat{v}(\zeta)) = \hat{w}(\zeta), \quad \text{for almost all } \zeta \in \mathbb{R}^N. \tag{3.1.8}$$

Thus $v \in \{v \in \mathcal{L}^2 : |\eta|^2\hat{v}(\eta) \in \mathcal{L}^2\} = \mathcal{H}^2$. Then, by (3.1.8) and (3.1.7) we obtain

$$\int_{\mathbb{R}^N} \hat{v}i\zeta(\xi(x)i\zeta\eta)d\zeta = \int_{\mathbb{R}^N} i\zeta(\xi(x)i\zeta\hat{v})\eta d\zeta. \tag{3.1.9}$$

Using again the equality (3.1.1) we have from (3.1.9) and (3.1.2) that

$$\int_{\mathbb{R}^N} vdiv(\xi(x)\nabla z)dx = \int_{\mathbb{R}^N} div(\xi(x)\nabla v)zdx,$$

for all $z \in C_0^\infty$, which it follows that $div(\xi(x)\nabla v) = w$ and we finish the proof. $\square$

An example of a bounded function that the Fourier transformation is the identity of is $\xi(x) = exp\left(\frac{-|x|^2}{2}\right) + V_0$, with $V_0 > 0$.

Theorem 3.1.1. *For $\xi, V \in L^\infty(\mathbb{R}^N)$ with $\hat{\xi}(x) = \xi(x)$, for all $x \in \mathbb{R}^N$, the Schrödinger operator $L : D(L) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ generated by ξ and by the potential V is self-adjoint.*

Proof. Note that $H^2(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$ so the adjoint Schrödinger operator $L^* : D(L^*) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is well defined. Furthermore, for all $u, v \in H^2(\mathbb{R}^N)$ and by Lemma 3.1.2

$$\begin{aligned}
\int_{\mathbb{R}^N} (Lu)v dx &= \int_{\mathbb{R}^N} \Big(-div(\xi(x)\nabla u) + V(x)u\Big) v dx \\
&= -\int_{\mathbb{R}^N} div(\xi(x)\nabla u) v dx + \int_{\mathbb{R}^N} V(x) uv dx \\
&= -\int_{\mathbb{R}^N} div(\xi(x)\nabla v) u dx + \int_{\mathbb{R}^N} V(x) vu dx \\
&= \int_{\mathbb{R}^N} \Big(-div(\xi(x)\nabla v) + V(x)v\Big) u dx
\end{aligned}$$

where $-div(\xi(x)\nabla v) + V(x)v \in L^2(\mathbb{R}^N)$. This shows that $H^2(\mathbb{R}^N) \subset D(L^*)$ and that $L^*v = -div(\xi(x)\nabla v) + V(x)v = Lv$ for all $v \in H^2(\mathbb{R}^N)$.

On the other hand, if $v \in D(L^*)$, then $v \in L^2(\mathbb{R}^N)$ and there exists an element $w \in L^2(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (Lu)v dx = \int_{\mathbb{R}^N} uw dx, \quad \text{for all} \quad u \in D(L) = H^2(\mathbb{R}^N).$$

Thus,

$$\int_{\mathbb{R}^N} \Big(-div(\xi(x)\nabla u) + V(x)u\Big) v dx = \int_{\mathbb{R}^N} uw dx, \quad \text{for all} \quad u \in C_0^\infty(\mathbb{R}^N)$$

and so

$$\int_{\mathbb{R}^N} (w - V(x)v) u dx = -\int_{\mathbb{R}^N} div(\xi(x)\nabla u) v dx, \quad \text{for all} \quad u \in C_0^\infty(\mathbb{R}^N)$$

where v and $(V(x)v - w) \in L^2(\mathbb{R}^N)$. By Lemma 3.1.2, $div(\xi(x)\nabla v) = w - V(x)v$ and $v \in H^2(\mathbb{R}^N)$. This shows that $D(L^*) \subset H^2(\mathbb{R}^N)$, completing the proof. $\square$

Now, we define the number Λ that characterizes the smallest value in the spectrum of L. For any $\xi, V \in L^\infty(\mathbb{R}^N)$, consider

$$\Lambda = \inf\left\{\int_{\mathbb{R}^N} \Big(\xi(x)|\nabla u|^2 + V(x)u^2\Big)dx : u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u^2 dx = 1\right\}.$$

The next result shows that the spectrum of the operator L is never empty and characterizes its infinity related to the number Λ.

Theorem 3.1.2. *Let $\xi, V \in L^\infty(\mathbb{R}^N)$. Then,*

(i) $\sigma(L) \subset [\Lambda, +\infty)$*;*

(ii) $\Lambda \in \sigma(L)$*.*

In particular $\Lambda = \inf \sigma(L)$*.*

The following results will be needed to prove the theorem:

Lemma 3.1.3. *Let $L : H \to H$ be a self-adjoint and let*

$$m = \inf\{\langle Lu, u\rangle : u \in H \text{ and } \|u\| = 1\},$$
$$M = \sup\{\langle Lu, u\rangle : u \in H \text{ and } \|u\| = 1\}.$$

Then,

(i) $\sigma(L) \subset [m, M]$*;*

(ii) $\|L\| = \sup\{|\lambda| : \lambda \in \sigma(L)\} = \max\{|m|, |M|\}$*;*

(iii) $m, M \in \sigma(L)$*.*

Lemma 3.1.4. *Let $\xi, V \in L^\infty(\mathbb{R}^N)$. Then,*

(1) $\Lambda \geq -\|V\|_\infty > -\infty$*;*

(2) $\Lambda = \inf\left\{\int_{\mathbb{R}^N} (\xi(x)|\nabla u|^2 + V(x)u^2)dx : u \in C_0^\infty(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u^2 dx = 1\right\}$
and so we also have,

$$\Lambda = \inf\left\{\int_{\mathbb{R}^N} (Lu)u dx : u \in H^2(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} u^2 dx = 1\right\};$$

(3) If $u \in H^1(\mathbb{R}^N)$ *with* $\int_{\mathbb{R}^N} u^2 dx = 1$ *and* $\int_{\mathbb{R}^N} (\xi(x)|\nabla u|^2 + V(x)u^2)dx = \Lambda$, *then* $u \in H^2(\mathbb{R}^N)$, $u \in \ker(L - \Lambda I)$ *and* $\Lambda \in \sigma_p(L)$.

Proof of Theorem 3.1.2. (i) By item (2) of Lemma 3.1.4 we have for all $u \in H^1(\mathbb{R}^N)$

$$\Lambda \int_{\mathbb{R}^N} u^2 dx \leq \int_{\mathbb{R}^N} (Lu)u dx$$

and so, for all $\lambda \in \mathbb{R}$,

$$(\Lambda - \lambda)\|u\|_{L^2}^2 \leq \int_{\mathbb{R}^N} [(L - \lambda I)u]u dx \leq \|(L - \lambda I)u\|_{L^2} \|u\|_{L^2}.$$

Thus,

$$\|(L - \lambda I)u\|_{L^2} \geq (\Lambda - \lambda)\|u\|_{L^2}, \quad \text{for all} \quad u \in C_0^\infty(\mathbb{R}^N)$$

and follows from the Lemma 3.1.1 that $\lambda \in \rho(L)$ if $\Lambda - \lambda > 0$.

(ii) From part (i) we know that $\sigma(L) \subset [\Lambda, \infty)$. Let $m \geq \Lambda$ be such that $\sigma(L) \subset [m, \infty)$. To complete the proof we have to show that $m \leq \Lambda$. We choose any $\eta \in (-\infty, m)$. Since $\eta \in \rho(L)$, we set

$$A = (L - \eta I)^{-1}$$

and we have that $A : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is linear, bounded and selfadjoint operator. Furthermore, $0 \in \sigma(A)$ since $R(A) = D(L) = H^2(\mathbb{R}^N) \neq L^2(\mathbb{R}^N)$. For $\lambda \neq 0$,

$$A - \lambda I = \lambda \left\{ \frac{1}{\lambda} I - (L - \eta I) \right\} A = \lambda \left\{ \left(\frac{1}{\lambda} - \eta \right) I - L \right\} A$$

and so

$$\begin{aligned} &A - \lambda I : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N) \text{ is an isomorphism} \\ &\Longleftrightarrow L - \left(\frac{1}{\lambda} + \eta \right) : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N) \\ &\Longleftrightarrow \left(\frac{1}{\lambda} + \eta \right) \in \rho(L). \end{aligned}$$

Therefore, we see that

$$\sigma(A) = \{0\} \cup \left\{ \frac{1}{\mu - \eta} : \mu \in \sigma(A) \right\}$$

and hence $\sigma(A) \subset \left[0, \dfrac{1}{m-\eta}\right]$. By Lemma 3.1.1 implies that

$$\int_{\mathbb{R}^N} (Av)v dx \geq 0, \quad \text{for all} \quad v \in L^2(\mathbb{R}^N).$$

For any $u \in H^2(\mathbb{R}^N)$, we consider $v = (L - \eta I)u$ and we obtain that

$$\int_{\mathbb{R}^N} [(L-\eta I)u] u dx = \int_{\mathbb{R}^N} (Av) v dx \geq 0$$

this shows that $\int_{\mathbb{R}^N} (Lu) u dx \geq \eta \int_{\mathbb{R}^N} u^2 dx$ for all $u \in H^2(\mathbb{R}^N)$ and it follows from this item (ii) of Lemma 3.1.4 that $\eta \leq \Lambda$. But η is an arbitrary number smaller than m. We can conclude that $m \leq \Lambda$, completing the proof. □

Lemma 3.1.5. *Let $\xi, V \in L^\infty(\mathbb{R}^N)$. For $\varepsilon > 0$, let X be a closed subspace of $H^1(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N} (\xi(x)|\nabla u|^2 + V(x)u^2) dx \leq (l - \varepsilon) \int_{\mathbb{R}^N} u^2 dx, \quad \textit{for all} \quad u \in X, \tag{3.1.10}$$

with $l = \liminf_{|x| \to \infty} V(x)$. Then, $\dim X < \infty$.

Proof. Observe that $\int_{\mathbb{R}^N} (\xi(x)|\nabla u|^2 + V(x)u^2) dx$ and $\int_{\mathbb{R}^N} u^2 dx$ are both continuous functions of u in $H^1(\mathbb{R}^N)$. And so from (3.1.10) it holds for all u in the closure of X. Therefore, we can assume X that is a closed subspace of $H^1(\mathbb{R}^N)$. Consider a sequence $(u_n) \subset X$ such that $\|u_n\|_{H^1(\mathbb{R}^N)} = 1$ for all $n \in \mathbb{N}$. We need only show that (u_n) has a subsequence that converges strongly in $H^1(\mathbb{R}^N)$. Passing to a subsequence we can assume that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ for some element of $H^1(\mathbb{R}^N)$. If Pu denotes the orthogonal projection of u onto X in $H^1(\mathbb{R}^N)$ then

$$\|u - Pu\|^2_{H^1(\mathbb{R}^N)} = \langle (I-P)u, u \rangle_{H^1(\mathbb{R}^N)} = \langle (I-P)u, u - u_n \rangle \to 0,$$

thus $Pu = u \in X$. For definition of l, there exists $R > 0$ such that

$$V(x) \geq l - \frac{\varepsilon}{2}, \text{ for almost all } |x| \geq R. \tag{3.1.11}$$

Then by compact embedding of $H^1(B_R(0))$ in $L^2(B_R(0))$, it follows that

$$\int_{|x|\leq R}(u_n-u)^2dx \to 0. \tag{3.1.12}$$

From (3.1.12) we have that $\int_{\mathbb{R}^N}\xi(x)|\nabla u|^2dx \leq (l-\varepsilon)\int_{\mathbb{R}^N}u^2dx - \int_{\mathbb{R}^N}V(x)u^2dx$ and using (3.1.11) we obtain that

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_{|x|\leq R}(u_n-u)^2dx + \int_{\mathbb{R}^N}\xi(x)|\nabla(u_n-u)|^2dx\\
&\leq \frac{\varepsilon}{2}\int_{|x|\leq R}(u_n-u)^2dx + (l-\varepsilon)\int_{\mathbb{R}^N}(u_n-u)^2dx - \int_{\mathbb{R}^N}V(x)(u_n-u)^2dx\\
&= \int_{|x|\leq R}\left(\frac{\varepsilon}{2}+l-\varepsilon-V(x)\right)(u_n-u)^2dx + \int_{|x|\geq R}(l-\varepsilon-V(x))(u_n-u)^2dx\\
&= \int_{|x|\leq R}\left(l-\frac{\varepsilon}{2}-V(x)\right)(u_n-u)^2dx + \int_{|x|\geq R}(l-\varepsilon-V(x))(u_n-u)^2dx\\
&\leq (l+\|V\|_{L^\infty})\int_{|x|\geq R}(u_n-u)^2dx \to 0.
\end{aligned}$$

It follows that $\int_{|x|\leq R}(u_n-u)^2dx \to 0$ and $\int_{\mathbb{R}^N}\xi(x)|\nabla(u_n-u)|^2dx \to 0$ when $n\to\infty$ that combining with (3.1.12) give us $\|u_n-u\|_{H^1(\mathbb{R}^N)}\to 0$, which completes the proof. □

Theorem 3.1.3. *Let $\xi, V \in L^\infty(\mathbb{R}^N)$ and consider $\eta < l$ where $l = \lim_{R\to\infty} ess \inf_{|x|\geq R} V(x)$. For each $\mu \in (0,\sqrt{l-\eta})$, there exists a constant C, depending on η and μ, such that*

$$|u(x)| \leq C\|u\|_{L^\infty}e^{-\mu|x|}$$

for all $x\in\mathbb{R}^N$ since $u\in \ker(L-\lambda I)$ for some $\lambda\leq\eta$.

Proof. Consider $r=|x|$ we obtain

$$\Delta e^{-\mu r} = (e^{-\mu r})'' + \frac{N-1}{r}(e^{-\mu r})' = \left\{\mu^2 - \frac{N-1}{N}\mu\right\}e^{-\mu r}, \quad \text{for} \quad x\neq 0.$$

Since $0<\mu^2<l-\eta$, there exists $R=R(\eta,\mu)>0$ such that

$$V(x)\geq \eta+\mu^2, \quad \text{for} \quad |x|\geq R,$$

and then, for all $\lambda \leq \eta$, we also have

$$V(x) > \lambda$$

and $\lambda - V(x) + \mu^2 - \dfrac{N-1}{N} < 0$, for all $|x| \geq R$.

Now, consider $C = e^{\mu R}$ and, for any $u \in \ker(L - \lambda I) \setminus \{0\}$ with $\lambda \leq \eta$, consider the function w defined by

$$w(x) = u(x) - C\|u\|_{L^\infty} e^{-\mu|x|}, \quad \text{for all } x \in \mathbb{R}^N.$$

By [Theorem 3.18, [31]] we have that

$$w \in C^0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \quad \text{and} \quad \lim_{|x| \to \infty} w(x) = 0.$$

The definition of $C^0(\mathbb{R}^N)$ guarantees us that

$$w \leq 0, \quad \text{for all } |x| \leq R.$$

Therefore, by [Lemma 7.6, [18]], $w^+ \in C^0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, $\lim_{|x| \to \infty} w^+(x) = 0$ and $w^+ \equiv 0$ for $|x| \leq R$. Let

$$\Omega = \{x \in \mathbb{R}^N : \; w^+ > 0\}.$$

The set Ω is open and $\Omega \subset \mathbb{R}^N \setminus \overline{B_R(0)} \equiv E(R)$. Suppose that $\Omega \neq \emptyset$, $w \in H^2(E(R))$ and $w^+ = 0$ on $\partial E(R)$. Then

$$\begin{aligned}
\int_{\mathbb{R}^N} \xi(x) |\nabla w^+|^2 dx &= \int_{E(R)} \xi(x) \nabla w \nabla w^+ dx \\
&\leq -\xi_\infty \int_{E(R)} (\Delta w) w^+ dx \\
&= -\xi_\infty \int_{E(R)} (\Delta w) w dx \\
&= \xi_\infty \int_\Omega \{(\lambda - V(x)) u + C\|u\|_{L^\infty} \Delta(e^{-\mu|x|})\} w dx \\
&\leq \xi_\infty \int_\Omega \Big\{\lambda - V(x) + \mu^2 - \frac{N-1}{r}\mu\Big\} C\|u\|_{L^\infty} e^{-\mu|x|} w dx
\end{aligned}$$

since $\lambda - V(x) \leq 0$ and $u(x) > C\|u\|_{L^\infty} e^{-\mu|x|}$ onto Ω. But $w > 0$ in $\Omega \subset E(R)$ and R was chosen so that $\lambda - V(x) + \mu^2 - \dfrac{N-1}{r}\mu < 0$ in $E(R)$. Thus, we saw

$$0 \leq \int_{\mathbb{R}^N} |\nabla(w^+)|^2 dx < 0$$

if $\Omega \neq \emptyset$. Therefore, we must have $\Omega = \emptyset$ and $w \leq 0$ in $\mathbb{R}^N$. Hence $u(x) \leq C\|u\|_{L^\infty} e^{-\mu|x|}$ for all $x \in \mathbb{R}^N$. Replacing u by $-u$ we complete the proof. □

Theorem 3.1.4. *If a mensurable locally bounded functions V, ξ such that $\liminf_{|x|\to\infty} V(x) \geq l$ and $\liminf_{|x|\to\infty} \xi(x) \geq \delta$, then the operator $L = -div(\xi(x)\nabla\ ) + V(x)$ is semi-bounded from below and has a discrete spectrum on $(-\infty, l)$, so that for any $\varepsilon > 0$ the spectrum of L in $(-\infty, l-\varepsilon)$ consists of a finite number of eigenvalues of finite multiplicities..*

To prove this theorem it is necessary to state the following lemma:

Lemma 3.1.6. *If $\liminf_{|x|\to\infty} V(x) \geq a$, $\liminf_{|x|\to\infty} \xi(x) \geq b$ and $u \in D(L)$, then*

$$\langle Lu, u\rangle = \int_{\mathbb{R}^N} (\xi(x)|\nabla u|^2 + V(x)u^2) dx < \infty.$$

Proof. Since $V(x) \geq C$ and $\xi(x) \geq D$, we can substitute L by $L-(C-1)I$ and assume the following estimates $V(x) \geq 1$ and $\xi(x) \geq 1$ such that $\int_{\mathbb{R}^N} V(x)u^2 dx \geq \int_{\mathbb{R}^N} u^2(x)dx$ and $\int_{\mathbb{R}^N} \xi(x)|\nabla u|^2 dx \geq \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx$. Let us introduce in $D(L)$ the following norm $\|u\|_\Gamma = \left(\int_{\mathbb{R}^N} (u^2 + |Lu|^2) dx\right)^{1/2}$. If $u \in C_0^\infty(\mathbb{R}^N)$, then

$$\langle Lu, u\rangle = \int_{\mathbb{R}^N} (\xi(x)|\nabla u|^2 + V(x)u^2) dx.$$

The convergence of the sequence (u_k) is the graph norm $\|\cdot\|_\Gamma$ implies its convergence in $H^1(\mathbb{R}^N)$ and in the space $L^2(\mathbb{R}^N)$ with the weight function V and the function ξ. Therefore, for the limit function u the integral $\int_{\mathbb{R}^N} (\xi(x)|\nabla u|^2 + V(x)u^2) dx$ has a finite value and is equal to $\langle Lu, u\rangle$. □

Proof of Theorem 3.1.4. We will prove that, for $\varepsilon > 0$, the dimension of the subspace

$$S := \left\{u \in D(L);\ \langle Lu, u\rangle \leq (l-\varepsilon)\|u\|^2\right\}$$

is finite. By Lemma 3.1.6, this inequality is equivalent to the following one

$$\int_{\mathbb{R}^N}(\xi(x)|\nabla u|^2+V(x)u^2)dx \leq (l-\varepsilon)\int_{\mathbb{R}^N}u^2dx$$
$$\int_{\mathbb{R}^N}(\xi(x)|\nabla u|^2+V(x)u^2)dx-(l-\varepsilon)\int_{\mathbb{R}^N}u^2dx \leq 0$$
$$\int_{\mathbb{R}^N}(\xi(x)|\nabla u|^2+(V(x)-l+\varepsilon)u^2)dx \leq 0, \quad u \in S.$$

Let $R>0$ be such that $V(x) \geq l-\varepsilon/2$ for $|x| \geq R$ and $V(X) \geq m$ for all $x \in \mathbb{R}^N$. Then,

$$\begin{aligned}0 &\geq \int_{|x|\leq R}(\xi(x)|\nabla u|^2+(V(x)-l+\varepsilon)u^2)dx+\int_{|x|>R}(\xi(x)|\nabla u|^2+(V(x)-l+\varepsilon)u^2)dx \\ &\geq \int_{|x|\leq R}(\xi(x)|\nabla u|^2+(m-l+\varepsilon)u^2)dx+\int_{|x|>R}\Big(\xi(x)|\nabla u|^2+\Big(l-\frac{\varepsilon}{2}-l+\varepsilon\Big)u^2\Big)dx \\ &= \int_{|x|\leq R}(\xi(x)|\nabla u|^2+(m-l+\varepsilon)u^2)dx+\int_{|x|>R}\Big(\xi(x)|\nabla u|^2+\frac{\varepsilon}{2}u^2\Big)dx.\end{aligned}$$

Therefore,

$$\int_{|x|\leq R}\xi(x)|\nabla u|^2dx+\int_{|x|>R}\Big(\xi(x)|\nabla u|^2+\frac{\varepsilon}{2}u^2\Big)dx \leq C\int_{|x|\leq R}u^2dx, \quad u \in S,$$

if $C \geq l-m+\varepsilon \geq 0$. Let B be the operator of restriction of functions from S on the ball $K_R := \{x : |x| \leq R\}$, that is, $B : S \subset L^2(\mathbb{R}^N) \to L^2(K_R)$. This operator is continuous in $L^2(\mathbb{R}^N)$ and injective in virtue of the latter estimate. To prove that S has finite dimension we will show that the subset BS, which is the operator B applied to the set S, has finite dimension. However, by the same estimate we have $\|u\|_{H^1(K_R)} \leq C\|u\|_{L^2(K_R)}$, $u \in BS$. Furthermore, $H^1(K_R) \subset\subset L^2(K_R)$. Therefore, the unit ball in the space $BS \cap L^2(K_R)$ is compact. And so, BS has finite-dimensional and, since B is injective, we can conclude that S is finite-dimensional. □

3.2 Variational Setting

In this chapter, we consider the following problem

$$\begin{cases} -div(\xi(x)\nabla u)+V(x)u = f(x,u), & \text{in } \mathbb{R}^N, \\ u(x) \to 0, & \text{as } \quad |x| \to \infty, \end{cases} \tag{P_3}$$

with $N \geq 3$, under the following assumptions in $\xi \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $V \in C(\mathbb{R}^N, \mathbb{R})$:

(ξ_1) there exists $\xi_0 > 0$ such that $\xi(x) \geq \xi_0$;

(ξ_2) $\lim\limits_{|x| \to \infty} \xi(x) = \xi_\infty$;

(ξ_3) $\xi(x) \lneqq \xi_\infty$;

(V_1) there exists $V_0 > 0$ such that $V(x) \geq -V_0$;

(V_2) $\lim\limits_{|x| \to \infty} V(x) = V_\infty$;

(V_3) $V(x) \leq V_\infty$;

(V_4) $0 \notin \sigma(L)$ and $\inf \sigma(L) < 0$, where $\sigma(L)$ is the spectrum of the operator $L(\cdot) = -div(\xi(x)\nabla(\cdot)) + V(x)(\cdot)$.

The conditions that we consider on the nonlinearity $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ are the following:

(f_1) $\lim\limits_{s \to 0^+} \dfrac{f(x,s)}{s} = 0$, uniformly in $x \in \mathbb{R}^N$;

(f_2) there exist $a \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $h \in C(\mathbb{R}, \mathbb{R}^+)$ a even function satisfying $h(s) > 0$ for all $s > 0$, $h(0) = 0$ and such that

$$\lim_{s \to \infty} \frac{f(x,s)}{s} = a(x), \qquad \lim_{|x| \to \infty} \frac{f(x,s)}{s} = h(s),$$

$$\lim_{|x| \to \infty,\, s \to \infty} \frac{f(x,s)}{s} = \lim_{s \to \infty} h(s) = \lim_{|x| \to \infty} a(x) = a_\infty,$$

uniformly in $x \in \mathbb{R}^N$. Moreover, $\dfrac{|f(x,s)|}{|s|} \leq a(x)$ and $a(x) \geq a_0 > V_\infty$, for all $s \neq 0$ and all $x \in \mathbb{R}^N$;

(f_3) $h(s) < a_\infty$ for all $s \in \mathbb{R}$;

(f_4) if $F(x,s) := \int_0^s f(x,t)dt$, $H(s) := \int_0^s h(t)tdt$, $G(s) := \dfrac{1}{2}h(s)s^2 - H(s)$ and $Q(x,s) := \dfrac{1}{2}f(x,s)s - F(x,s)$, then, for all $s \in \mathbb{R} \setminus \{0\}$ and all $x \in \mathbb{R}^N$,

$$G(s) > 0, \; F(x,s) \geq 0, \; Q(x,s) > 0 \text{ and } \lim_{s \to +\infty} Q(x,s) = +\infty;$$

(f_5) there exist $C_2 > 0$ and $1 < p_1 \leq p_2$ such that $p_1, p_2 < 2^* - 1$ and

$$|f^{(k)}(x,s)| \leq C_2\Big(|s|^{p_1-k} + |s|^{p_2-k}\Big)$$

for $k \in \{0,1\},\ s \in \mathbb{R}$ and $x \in \mathbb{R}^N$;

(f_6) the function $s \mapsto f(x,s)/s$ is increasing in $s \in (0,+\infty)$ for all $x \in \mathbb{R}^N$.

Consider the function $f(x,s) = \dfrac{|s|^3}{1+c(x)s^2}$ for $x \in \mathbb{R}^N$, where $c \in C(\mathbb{R}^N, \mathbb{R})$ is a positive function, $c(x) \to c_\infty > 0$ when $|x| \to \infty$ and $0 < c_0 \leq c(x) < c_\infty$, is an example of a function that satisfies the assumptions $(f_1)-(f_6)$, with $a(x) = \dfrac{1}{c(x)}$ and $h(s) = \dfrac{s^2}{1+c_\infty s^2}$.

The main result of this chapter is the following theorem.

Theorem 3.2.1. *Assume that ξ and V satisfy the hypotheses $(\xi_1)-(\xi_3)$ and $(V_1)-(V_4)$, respectively, and the function f satisfies $(f_1)-(f_6)$. Then problem (P_3) has a nontrivial weak solution $u \in H^1(\mathbb{R}^N)$ provided one of the followings conditions holds:*

$$\xi(x) \leq \xi_\infty - C_1 e^{-\gamma_1|x|}, \quad \textit{for all} \ \ x \in \mathbb{R}^N \tag{3.2.1}$$

or

$$V(x) \leq V_\infty - C_2 e^{-\gamma_2|x|}, \quad \textit{for all} \ \ x \in \mathbb{R}^N \tag{3.2.2}$$

for constants $C_1,\ C_2 > 0$ and $0 < \gamma_1,\ \gamma_2 < \sqrt{V_\infty/\xi_\infty}$.

Remark 3.2.1. *The condition (f_2) implies that $h(s) \leq a_\infty$ for all $s \in \mathbb{R}$. However, we will need the strict inequality (f_3) forward.*

Consider the space $H^1(\mathbb{R}^N)$ equipped with the norm

$$\|u\|_\infty^2 = \int_{\mathbb{R}^N} \Big(\xi_\infty|\nabla u|^2 + V_\infty u^2\Big)dx \tag{3.2.3}$$

and the limit problem

$$-div(\xi_\infty \nabla u) + V_\infty u = h(u)u, \ \ \text{in} \ \ \mathbb{R}^N. \tag{3.2.4}$$

The functional associated with the equation (3.2.4) is given by

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N}(\xi_\infty|\nabla u|^2 + V_\infty u^2)dx - \int_{\mathbb{R}^N} H(u)dx, \tag{3.2.5}$$

for $u \in H^1(\mathbb{R}^N)$. Since $V_\infty < a_\infty$, is proved by Berestick-Lions in [6] that the problem (3.2.4) has a symmetric and positive classical solution $u \in H^1(\mathbb{R}^N)$.

Let $E := H^1(\mathbb{R}^N)$ be the space equipped with the norm established later. The functional $I : E \to \mathbb{R}$ associated with the problem (P_3) is given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(\xi(x)|\nabla u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx,$$

with $u \in E$. From hypotheses $(\xi_2),(V_2)$ and (V_3), the eigenvalue problem

$$-div(\xi(x)\nabla u) + V(x)u = \lambda u, \; u \in L^2(\mathbb{R}^N) \tag{3.2.6}$$

has a sequence of eigenvalues $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots$. Making $\varepsilon = V_\infty > 0$ in Theorem 3.1.4 we have the spectrum of $-div(\xi(x)\nabla(\cdot)) + V(x)(\cdot)$ in $(-\infty, 0)$ has a finite number of eigenvalues. In other words, the eigenvalue problem (3.2.6) has a finite sequence of eigenvalues $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k < 0$, with finite multiplicity.

Denote by ϕ_i the eigenfunction corresponding to λ_i, $i = \{1,2,\cdots,k\}$, in $H^1(\mathbb{R}^N)$. Setting

$$E^- := span\{\phi_i, i = 1,2,\cdots,k\} \;\text{ and }\; E^+ = (E^-)^\perp,$$

we see that $E = E^+ \oplus E^-$. By Theorem 3.1.4 the essential spectrum of $-div(\xi(x)\nabla(\cdot)) + V(x)(\cdot)$ is the interval $[V_\infty, +\infty)$ and this implies that $dim E^- < \infty$, because for each $\lambda_i < 0$ it has a finite multiplicity. Having made theses considerations, every function $u \in E$ may be written as $u = u^+ + u^-$ uniquely, where $u^+ \in E^+$ and $u^- \in E^-$. By condition (V_3) we have that $0 \notin \sigma(-div(\xi(x)\nabla(\cdot)) + V(x)(\cdot))$, thus, using the arguments in Lemma 1.2 of Costa-Tehrani [11], we can introduce the new inner product $\langle \cdot,\cdot \rangle$ in E, namely

$$\langle u,v\rangle = \begin{cases} \displaystyle\int_{\mathbb{R}^N}(\xi(x)\nabla u\nabla v + V(x)uv)dx, & \text{if } u,v \in E^+,\\ \displaystyle -\int_{\mathbb{R}^N}(\xi(x)\nabla u\nabla v + V(x)uv)dx, & \text{if } u,v \in E^-,\\ 0, & \text{if } u \in E^+ \text{ and } v \in E^-, \end{cases}$$

such that corresponding norm $\|\cdot\|$ is equivalent the usual norm in standard space $H^1(\mathbb{R}^N)$ by hypotheses (ξ_3) and (V_1). In addition, the functional I may be written as

$$I(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \tag{3.2.7}$$

for every function $u = u^+ + u^- \in E$. We call attention to the fact, since $\lambda_i \neq 0$ for all $i = \{1, 2, \cdots, k\}$ it follows from (3.2.6) and by definition of ϕ_i that

$$\int_{\mathbb{R}^N} u^+(x)v^-(x)dx = 0$$

for every function $u^+ \in E^+$ and $v^- \in E^-$. Indeed, for all $u^+ \in E^+$ and $v^- \in E^-$ we have

$$\int_{\mathbb{R}^N} (\nabla u^+ \nabla u^- + u^+ u^-)dx = 0$$

because $E^+ = (E^-)^\perp$. If $u^+ \in E^+$ and $v^- \in E^-$ we get

$$\|u^+\|^2 - \|v^-\|^2 = \int_{\mathbb{R}^N} (\xi(x)|\nabla(u^+ + v^-)|^2 + V(x)|u^+ + v^-|^2)dx.$$

Developing the right side of this equality, we obtain

$$\begin{aligned} \|u^+\|^2 - \|v^-\|^2 &= \int_{\mathbb{R}^N} (\xi(x)|\nabla u^+|^2 + V(x)(u^+)^2)dx + 2\int_{\mathbb{R}^N} (\xi(x)\nabla u^+ \nabla v^- + V(x)u^+ v^-)dx \\ &- \int_{\mathbb{R}^N} (\xi(x)|\nabla v^-|^2 + V(x)(v^-)^2)dx \end{aligned}$$

and this implies that

$$\int_{\mathbb{R}^N} (\xi(x)\nabla u^+ \nabla v^- + V(x)u^+ v^-)dx = 0. \tag{3.2.8}$$

From equation (3.2.6) we have that

$$-div(\xi(x)\nabla\phi_i) + V(x)\phi_i = \lambda_i\phi_i$$

if and only if

$$\int_{\mathbb{R}^N} (\xi(x)\nabla\phi_i \nabla u^+ + V(x)\phi_i u^+)dx = \lambda_i \int_{\mathbb{R}^N} \phi_i u^+ dx, \quad \text{for all} \quad u^+ \in E^+.$$

From equality (3.2.8), for $\lambda_i \neq 0$ we have $\int_{\mathbb{R}^N} \phi_i u^+ dx = 0$ and thus, by linearity,

$$\int_{\mathbb{R}^N} u^+ v^- dx = \int_{\mathbb{R}^N} V(x)u^+ v^- dx = \int_{\mathbb{R}^N} \xi(x)\nabla u^+ \nabla v^- dx = 0,$$

and this completes our claim.

3.3 Boundedness of a Cerami Sequence

Lemma 3.3.1. *Under the assumptions* (f_1) *and* (f_2)*, given* $\varepsilon > 0$*, there exists* $C_\varepsilon > 0$ *such that, for* $2 \le p \le 2^*$*,*

$$|f(x,s)| \le \varepsilon|s| + C_\varepsilon|s|^{p-1}$$

and

$$|F(x,s)| \le \frac{\varepsilon}{2}|s|^2 + C_\varepsilon|s|^p$$

for all $s \in \mathbb{R}$ *and all* $x \in \mathbb{R}^N$*.*

Proof. From hypotheses (f_1) and (f_2), given $\varepsilon > 0$, there exist $R, \delta > 0$ such that $R > \delta$ with

$$|f(x,t)| \le \varepsilon|t|, \text{ whenever } |t| < \delta, \text{ and for all } x \in \mathbb{R}^N \tag{3.3.1}$$

and

$$|f(x,t) - a(x)t| \le \varepsilon|t|, \text{ whenever } |t| > R, \text{ and for all } x \in \mathbb{R}^N. \tag{3.3.2}$$

The inequality (3.3.1) and the hypothesis (f_2) imply that

$$|f(x,t)| \le \varepsilon|t| + a_0|t|, \text{ whenever } |t| > R, \text{ and for all } x \in \mathbb{R}^N \tag{3.3.3}$$

where $a_0 = \sup_{\mathbb{R}^N} |a(x)|$. For values of t such that $|t| > R$ holds $|t| < \dfrac{|t|^{p-1}}{R^{p-2}}$. Thus (3.3.3) becomes

$$|f(x,t)| \le \varepsilon|t| + \frac{a_0}{R^{p-2}}|t|^{p-1}, \text{ whenever } |t| > R \text{ and for all } x \in \mathbb{R}^N. \tag{3.3.4}$$

By hypothesis (f_2), we have

$$|f(x,t)| \le |a(x)t| \le a_0|t|, \text{ whenever } \delta \le |t| \le R, \text{ and for all } x \in \mathbb{R}^N.$$

Therefore, for values of t so that $\delta \le |t| \le R$, we obtain

$$|f(x,t)| \le \frac{a_0}{\delta^{p-2}}|t|^{p-1}, \text{ whenever } \delta \le |t| \le R, \text{ and for all } x \in \mathbb{R}^N. \tag{3.3.5}$$

It follows from (3.3.1), (3.3.4) and (3.3.5) that

$$|f(x,t)| \le \varepsilon|t| + \left(\frac{a_0}{R^{p-2}} + \frac{a_0}{\delta^{p-1}}\right)|t|^{p-1}, \text{ for all } t \in \mathbb{R}, \text{ and all } x \in \mathbb{R}^N. \tag{3.3.6}$$

Taking $C_\varepsilon := \left(\dfrac{a_0}{R^{p-2}} + \dfrac{a_0}{\delta^{p-1}}\right)$ and replacing in (3.3.6) we obtain $|f(x,t)| \leq \varepsilon|t| + C_\varepsilon|t|^{p-1}$, next integrating this inequality of 0 to s, we obtain

$$|F(x,s)| \leq \frac{\varepsilon}{2}|s|^2 + C_\varepsilon|s|^p, \text{ for all } x \in \mathbb{R}^N,$$

and we conclude the proof of lemma. □

We note that, if (v_n) is a bounded sequence in E, then (v_n) satisfies one the following cases:

(i) *vanishing*: for all $r > 0$,

$$\limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |v_n|^2 dx = 0.$$

(ii) *or nonvanishing*: there exist $r,\ \eta > 0$ and a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$\limsup_{n\to+\infty} \int_{B_r(y_n)} |v_n|^2 dx > \eta.$$

Lemma 3.3.2. *Let $(u_n) \subset E$ be a sequence such that*

$$I(u_n) \to c > 0 \text{ and } \|I'(u_n)\|_{E^*}(1 + \|u_n\|) \to 0 \text{ as } n \to \infty.$$

Then, (u_n) has a bounded subsequence.

Proof. Let us assume $\|u_n\| \to +\infty$ and obtain a contradiction. To this end, we consider $v_n = \dfrac{u_n}{\|u_n\|}$ and observe that $\|v_n\| = 1$. The sequence (v_n) is bounded, however, we will show that neither (i) or (ii) is true. First, suppose that (ii) holds for the sequence (v_n). Write $f(x,s) = a(x)s + (f(x,s) - a(x)s) = a(x)s + f_\infty(x,s)$ and consider $\varphi \in C_0^\infty(\mathbb{R}^N)$. By equivalence of the norms in E and the standard in $H^1(\mathbb{R}^N)$, there exist constant $C_1,\ C_2 > 0$ such that

$$\|w\| \leq C_1\|w\|_E \leq C_2\|w\|, \text{ for all } w \in E. \tag{3.3.7}$$

Let $(y_n) \subset \mathbb{R}^N$ be the sequence given by hypothesis (ii). Since the sequence (u_n) is a Cerami sequence, and considering $\varphi_n(x) = \varphi(x - y_n)$ we have from (3.3.7)

$$|I'(u_n)\varphi_n| \leq \|I'(u_n)\|_{E^*}\|\varphi_n\| \leq C_1\|I'(u_n)\|_{E^*}\|\varphi_n\|_E = C_1\|I'(u_n)\|_{E^*}\|\varphi\|_E \to 0.$$

Therefore,

$$\begin{aligned}
o_n(1) &= \frac{1}{\|u_n\|} I'(u_n)\varphi_n \\
&= \frac{1}{\|u_n\|}\left(\langle u_n^+ - u_n^-, \varphi_n\rangle - \int_{\mathbb{R}^N} f(x,u_n)\varphi_n dx\right) \\
&= \langle v_n^+ - v_n^-, \varphi_n\rangle - \int_{\mathbb{R}^N} \frac{f(x,u_n)}{\|u_n\|}\varphi_n dx \\
&= \langle v_n^+ - v_n^-, \varphi_n\rangle - \int_{\mathbb{R}^N} \frac{a(x)u_n + f_\infty(x,u_n)}{\|u_n\|}\varphi_n dx \\
&= \langle v_n^+ - v_n^-, \varphi_n\rangle - \int_{\mathbb{R}^N} a(x)v_n\varphi_n dx - \int_{\mathbb{R}^N} \frac{f_\infty(x,u_n)}{\|u_n\|}\varphi_n dx \qquad (3.3.8)\\
&= \langle v_n^+ - v_n^-, \varphi_n\rangle - \int_{\mathbb{R}^N} a(x)v_n\varphi_n dx - \int_{\mathbb{R}^N} \frac{f_\infty(x,u_n)}{u_n} v_n\varphi_n dx.
\end{aligned}$$

Consider $\tilde{v}_n(x) = v_n(x + y_n)$ and $\tilde{u}_n(x) = u_n(x + y_n)$. Note that $(\tilde{v}_n)$ is bounded in E. Thus, up to a subsequence,

$$\begin{cases} \tilde{v}_n \rightharpoonup \tilde{v}, & \text{in } E, \\ \tilde{v}_n \to \tilde{v}, & \text{in } L^2_{loc}(\mathbb{R}^N), \\ |\tilde{v}_n(x)| \leq h_0(x), & \text{a.e. in } K, \end{cases} \qquad (3.3.9)$$

for some function $h_0 \in L^1(K)$, where $K = supp(\varphi)$. By hypotheses (f_1) and (f_2) we remember that $f_\infty(x,s) = f(x,s) - a(x)s$, we have

$$\left|\frac{f_\infty(x + y_n, \tilde{u}_n)}{\tilde{u}_n}\tilde{v}_n\varphi\right| \leq Ch_0(x)\varphi \ \in L^1(K). \qquad (3.3.10)$$

Note that $\tilde{v} \neq 0$, from item (ii) and estimates in (3.3.9) we get

$$\begin{aligned}
\int_{B_r(0)} \tilde{v}^2 dx &= \lim_{n\to\infty}\int_{B_r(0)} \tilde{v}_n^2 dx = \limsup_{n\to\infty}\int_{B_r(0)} v_n^2(x + y_n)dx \\
&= \limsup_{n\to\infty}\int_{B_r(y_n)} v_n^2 dx > \eta > 0.
\end{aligned}$$

By hypothesis (f_2), (3.3.10) and the Lebesgue Dominated Convergence Theorem, it follow that

$$\int_{\mathbb{R}^N} \frac{f_\infty(x,\tilde{u}_n)}{\tilde{u}_n}\tilde{v}_n\varphi_n dx \to 0 \text{ as } n\to\infty. \tag{3.3.11}$$

Thus, from (3.3.8), (3.3.11) and the change of variables theorem we get

$$\begin{aligned} o_n(1) &= \int_{\mathbb{R}^N}(\xi(x+y_n)\nabla\tilde{v}_n^+\nabla\varphi + V(x+y_n)\tilde{v}_n^+\varphi)dx \\ &\quad - \int_{\mathbb{R}^N}(\xi(x+y_n)\nabla\tilde{v}_n^-\nabla\varphi + V(x+y_n)\tilde{v}_n^-\varphi)dx - \int_{\mathbb{R}^N} a(x+y_n)\tilde{v}_n\varphi dx. \end{aligned} \tag{3.3.12}$$

Case 1: $|y_n| \to \infty$. In this case, hypotheses (ξ_2), (V_2) and (f_2) ensures that $\xi(x+y_n)$ converges to ξ_∞, $V(x+y_n)$ converges to V_∞ and $a(x+y_n)$ converges to a_∞ almost everywhere in $\mathbb{R}^N$, when $n\to\infty$. Thus,

$$\begin{aligned} o_n(1) &= \int_K \Big[(\xi_\infty + o_n(1))\nabla\tilde{v}_n^+\nabla\varphi + (V_\infty + o_n(1))\tilde{v}_n^+\varphi\Big]dx + \int_K \Big[(\xi_\infty + o_n(1))\nabla\tilde{v}_n^-\nabla\varphi \\ &\quad + (V_\infty + o_n(1))\tilde{v}_n^-\varphi\Big]dx - \int_K (a_\infty + o_n(1))\tilde{v}_n\varphi dx. \end{aligned} \tag{3.3.13}$$

Therefore, for every function $\varphi\in C_0^\infty(\mathbb{R}^N)$, taking $n\to+\infty$ in (3.3.12), we obtain

$$\int_{\mathbb{R}^N}\Big(\xi_\infty\nabla(\tilde{v}^+ + \tilde{v}^-)\nabla\varphi + V_\infty(\tilde{v}^+ + \tilde{v}^-)\varphi\Big)dx - \int_{\mathbb{R}^N} a_\infty\tilde{v}\varphi dx = 0,$$

that is, $\tilde{v}\neq 0$ is weak solution of problem

$$-div(\xi_\infty\nabla\tilde{v}) + V_\infty\tilde{v} = a_\infty\tilde{v}, \quad \text{in } \mathbb{R}^N.$$

Since $V_\infty < a_\infty$ and there is no Laplacian eigenvalue in $\mathbb{R}^N$, this is absurd. Therefore, (ii) is not valid when $|y_n|\to+\infty$.

Case 2: (y_n) is a bounded sequence. From estimate (3.3.7) and translation invariance of integration we have

$$\|\tilde{u}_n\| \geq \frac{C_1}{C_2}\|\tilde{u}_n\|_E = \frac{C_1}{C_2}\|u_n\|_E \geq \frac{1}{C_2}\|u_n\|,$$

which goes to infinite as $n\to\infty$. It follows from (3.3.9) that

$$0\neq|\tilde{v}(x)| = \lim_{n\to\infty}|\tilde{v}_n(x)| = \lim_{n\to\infty}\frac{|\tilde{u}_n(x)|}{\|\tilde{u}_n\|}, \quad \text{a.e in } \Omega$$

for some $\Omega \subset B_1(0)$, with $\mu(\Omega) > 0$. Since $\|\tilde{u}_n\| \to \infty$, we have $\tilde{u}_n(x) \to \infty$ a.e. in Ω. Thus, Fatou's lemma and hypothesis (f_4) yield

$$\begin{aligned}
&\liminf_{n\to+\infty} \int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right) dx \\
&\qquad \geq \int_{\mathbb{R}^N} \liminf_{n\to+\infty} \left(\frac{1}{2} f(x+y_n,\tilde{u}_n)\tilde{u}_n - F(x+y_n,\tilde{u}_n)\right) dx = +\infty.
\end{aligned}$$

However, this contradicts the fact

$$\int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right) dx = I(u_n) - \frac{1}{2} I'(u_n)u_n = c + o_n(1).$$

Hence, *Case* 2 is not valid when the sequence (y_n) is bounded. This shows that hypothesis (ii) does not hold for the sequence (v_n).

Now, suppose that the hypothesis (i) holds for the sequence (v_n). Since (u_n) is a Cerami sequence, we have $I'(u_n)u_n^- \to 0$ and $I'(u_n)u_n^+ \to 0$. Thus

$$o_n(1) = I'(u_n)\frac{u_n^+}{\|u_n\|^2} = \|v_n^+\|^2 - \int_{\mathbb{R}^N} \big(\frac{f(x,u_n)}{u_n} v_n v_n^+\big) dx \tag{3.3.14}$$

and, similarly,

$$o_n(1) = I'(u_n)\frac{u_n^-}{\|u_n\|^2} = \frac{1}{\|u_n\|} I'(u_n)v_n^- = -\|v_n^-\|^2 - \int_{\mathbb{R}^N} \left(\frac{f(x,u_n)}{u_n} v_n v_n^-\right) dx. \tag{3.3.15}$$

Subtracting the equation (3.3.14) from (3.3.15), we have

$$\begin{aligned}
o_n(1) &= \|v_n^+\|^2 - \int_{\mathbb{R}^N} \left(\frac{f(x,u_n)}{u_n} v_n v_n^+\right) dx + \|v_n^-\|^2 + \int_{\mathbb{R}^N} \left(\frac{f(x,u_n)}{u_n} v_n v_n^-\right) dx \\
&= \|v_n\|^2 - \int_{\mathbb{R}^N} \left(\frac{f(x,u_n)}{u_n} v_n (v_n^+ - v_n^-)\right) dx \\
&= 1 - \int_{\mathbb{R}^N} \left(\frac{f(x,u_n)}{u_n} v_n (v_n^+ - v_n^-)\right) dx.
\end{aligned}$$

Thus, necessarily, when $n \to +\infty$

$$\int_{\mathbb{R}^N} \left(\frac{f(x,u_n)}{u_n} v_n (v_n^+ - v_n^-)\right) dx \to 1. \tag{3.3.16}$$

By Sobolev's embedding, there exists a constant $\mu_0 > 0$ such that

$$\|w\|^2 \geq \mu_0 \|w\|_{L^2}^2 \tag{3.3.17}$$

for any $w \in E$. Given $0 < \varepsilon < \mu_0/2$, by hypothesis (f_1), there exists $\delta > 0$ satisfying

$$\frac{|f(x,s)|}{|s|} < \varepsilon, \quad \text{for} \quad 0 \neq |s| < \delta \quad \text{and for all } x \in \mathbb{R}^N.$$

For each $n \in \mathbb{N}$, consider the set

$$\tilde{\Omega}_n = \{x \in \mathbb{R}^N; |u_n(x)| < \delta\}.$$

Thus, from (3.3.17) and by Hölder's inequality

$$\begin{aligned}
\int_{\tilde{\Omega}_n} \left(\frac{f(x,u_n)}{u_n} v_n (v_n^+ - v_n^-) \right) dx &\leq \varepsilon \int_{\tilde{\Omega}_n} |v_n| |v_n^+ - v_n^-| dx \\
&\leq \varepsilon \Big(\|v_n\|_{L^2} \|v_n^+\|_{L^2} + \|v_n\|_{L^2} \|v_n^-\|_{L^2} \Big) \\
&\leq 2\varepsilon \|v_n\|_{L^2}^2 \leq \frac{2\varepsilon}{\mu_0} \|v_n\|^2 = \frac{2\varepsilon}{\mu_0} < 1.
\end{aligned}$$

From the convergence given in (3.3.16) we conclude that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N \setminus \tilde{\Omega}_n} \left(\frac{f(x,u_n)}{u_n} v_n (v_n^+ - v_n^-) \right) dx > 0. \tag{3.3.18}$$

Since $\frac{|f(\cdot,\cdot)|}{|\cdot|}$ is bounded, by Hölder's inequality with exponent $p > 2$, we obtain a constant $C > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus \tilde{\Omega}_n} \left(\frac{f(x,u_n)}{u_n} v_n (v_n^+ - v_n^-) \right) dx &\leq \int_{\mathbb{R}^N \setminus \tilde{\Omega}_n} \left| \frac{f(x,u_n)}{u_n} v_n (v_n^+ - v_n^-) \right| dx \\
&\leq C \int_{\mathbb{R}^N \setminus \tilde{\Omega}_n} |v_n| |v_n^+ - v_n^-| dx \\
&\leq C \int_{\mathbb{R}^N \setminus \tilde{\Omega}_n} |v_n|^2 dx \\
&\leq C \mu(\mathbb{R}^N \setminus \tilde{\Omega}_n)^{(p-2)/p} \|v_n\|_{L^p}^{2/p}. \qquad (3.3.19)
\end{aligned}$$

Assumption (i) and Lion's Lemma ensure that $\|v_n\|_{L^p}\to 0$. Therefore, up to a subsequence, it follows from (3.3.18) that

$$\mu(\mathbb{R}^N \setminus \tilde{\Omega}_n) \to \infty, \quad \text{as} \quad n \to \infty. \tag{3.3.20}$$

Now, we consider two disjoint subsets of $\mathbb{R}^N/\tilde{\Omega}_n$. Hypothesis (f_3) implies there exists $R>0$ such that, if $|s|> R$, for all $x\in\mathbb{R}^N$,

$$\frac{1}{2}f(x,s)s - F(x,s) > 1.$$

Without loss of generality, we assume $0<\delta<R$. For each $n\in\mathbb{N}$, consider the set $A_n := \{x\in\mathbb{R}^N : |u_n(x)|> R\}$. Thus, by hypothesis (f_4)

$$\begin{aligned}
c+o_n(1) &= I(u_n) - \frac{1}{2}I'(u_n)u_n \\
&= \int_{\mathbb{R}^N}\left(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right)dx \\
&\geq \int_{A_n}\left(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right)dx \\
&> \mu(A_n),
\end{aligned}$$

which implies that the sequence $(\mu(A_n))$ is bounded. Also consider the set $B_n := \{x\in\mathbb{R}^N : \delta \leq |u_n(x)|\leq R\}$. Since $B_n = (\mathbb{R}^N\setminus\tilde{\Omega}_n)\setminus A_n$, we have

$$\mu(\mathbb{R}^N\setminus\tilde{\Omega}) = \mu(A_n)+\mu(B_n).$$

It follows from (3.3.20) and the boundedness of the sequence $(\mu(A_n))$ that

$$\mu(B_n) \to +\infty. \tag{3.3.21}$$

We claim that $\overline{\delta} := \inf_{s\in[\delta,R],\ x\in\mathbb{R}^N}\left(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right) > 0$. In fact, let $(x_n,s_n)\in\mathbb{R}^N\times[\delta,R]$ be a sequence satisfying

$$\lim_{n\to\infty}\left(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right) = \overline{\delta}.$$

Since the interval $[\delta,R]$ is compact, we can assume that $s_n \to s_0 \in [\delta,R]$. If $x_n \to x_0$, from the continuity of functions f and F, we have by assumption (f_4) that $\overline{\delta} > 0$. On the other hand, if $|x_n| \to \infty$, writing

$$\frac{1}{2}f(x_n,s_n)s_n - F(x_n,s_n) = \frac{1}{2}\left(\frac{f(x_n,s_n)}{s_n} - h(s_n)\right)s_n^2 - \Big(F(x_n,s_n) - H(s_n)\Big) + G(s_n)$$

where $G(s_n) = \dfrac{1}{2}h(s_n)s_n^2 - H(s_n)$, it follows from the uniform limits in (f_2) and (f_4) that

$$\overline{\delta} = \lim_{n\to\infty}\Big(\frac{1}{2}f(x_n,s_n)s_n - F(x_n,s_n)\Big) = G(s_0) > 0,$$

as claimed. Thus, from (3.3.21) and hypothesis (f_4)

$$\begin{aligned}\int_{\mathbb{R}^N}\left(\frac{1}{2}f(x_n,u_n)u_n - F(x_n,u_n)\right)dx &\geq \int_{B_n}\left(\frac{1}{2}f(x_n,u_n)u_n - F(x_n,u_n)\right)dx\\ &\geq \overline{\delta}\mu(B_n) \to +\infty.\end{aligned}$$

We have again a contradiction in the fact that

$$\int_{\mathbb{R}^N}\Big(\frac{1}{2}f(x_n,u_n)u_n - F(x_n,u_n)\Big)dx = I(u_n) - \frac{1}{2}I'(u_n) = c + o_n(1).$$

Therefore, (i) does not hold either for the sequence (v_n). We conclude that, up to a subsequence, (u_n) is bounded. □

3.4 A nontrivial solution

In this section we will prove our main result, however, first, let us verify that the functional I satisfies the geometry of the classical linking theorem in [29] and proved in [23] under the Cerami condition.

Theorem 3.4.1 (Linking Theorem under the $(Ce)_c$ condition)**.** *Let $E = E^+ \oplus E^-$ be a Banach space with $\dim E^- < \infty$. Let $R > \rho > 0$ and let $u \in E^+$ be a fixed element such that $\|u\| = \rho$. Define*

$$M := \{w = tu + v^- : \|w\| \leq R,\ t \geq 0,\ v^- \in E^-\},$$
$$M_0 := \{w = tu + v^- :\ v^- \in E^-, \|w\| = R,\ t \geq 0 \ or\ \|w\| \leq R,\ t = 0\},$$

$$N_\rho := \{w \in E^+ : \|w\| = \rho\}.$$

Let $I \in C^1(E,\mathbb{R})$ *be such that*

$$b := \inf_{N_\rho} I > a := \max_{M_0} I.$$

Then, $c \geq b$ *and there exists a Cerami sequence at level* c *for the functional* I*, where*

$$c := \inf_{\gamma\in\Gamma}\max_{w\in M} I(\gamma(w))$$

with $\Gamma := \{\gamma \in C(M,E) : \gamma|_{M_0} = Id\}$.

To simplify the notation, given $w \in E$ and $y \in \mathbb{R}^N$, we write $w^+(\cdot - y)$ (or $w^-(\cdot - y)$) referring to the projection in E^+ (respectively, in E^-) of the translated function $w(\cdot - y)$.

Remark 3.4.1. *If* w *and* v *are function in* $L^2(\mathbb{R}^N)$*, it holds*

$$\int_{\mathbb{R}^N} w(x-y)v(x)dx \to 0, \quad \text{if} \quad |y| \to \infty.$$

Indeed, given $\varepsilon > 0$ and v, w functions in $L^2(\mathbb{R}^N)$ there exist C, $k > 0$ such that $\|v\|_{L^2} < C$, $\|w\|_{L^2} < \infty$ which imply $\int_{B_k(0)^c} w(x)dx < \varepsilon/2C$. We can rewrite the above integral as

$$\int_{\mathbb{R}^N} w(x-y)v(x)dx = \int_{B_k(0)^c} w(x-y)v(x)dx + \int_{B_k(0)} w(x-y)v(x)dx.$$

Analyzing each integral, using the estimates above and Hölder's inequality, we have

$$\begin{aligned}\int_{B_k(0)^c} w(x-y)v(x)dx &\leq \|w(x-y)\|_{L^2(B_k(0)^c)}\|v\|_{L^2(B_k(0)^c)} \\ &= \|w\|_{L^2(B_k(0)^c)}\|v\|_{L^2(B_k(0)^c)} \\ &< \frac{\varepsilon}{2}\end{aligned}$$

and

$$\int_{B_k(0)} w(x-y)v(x)dx \leq \|w(x-y)\|_{L^2(B_k(0))}\|v\|_{L^2(B_k(0))}.$$

Note that, for y big enough, we obtain $\|w(x-y)\|_{L^2(B_k(0))}< \dfrac{\varepsilon}{2C}$. In fact, for y sufficiently large we have that $B_k(y) \subset B_k(0)^c$. It follows that

$$\begin{aligned}\|w(x-y)\|^2_{L^2(B_k(0))} &= \int_{B_k(0)} w^2(x-y)dx = \int_{B_k(y)} w^2(x)dx \\ &\leq \int_{B_k(0)^c} w^2(x)dx < \frac{\varepsilon}{2C}.\end{aligned}$$

Hence, $\int_{B_k(0)} w(x-y)v(x)dx < \varepsilon/2$. Thus, $\int_{\mathbb{R}^N} w(x-y)v(x)dx < \varepsilon$ for $|y|$ big enough and this proof of remark.

For $R>0$ and $y \in \mathbb{R}^N$, consider

$$M = \{w = tu_0^+(\cdot - y) + v^- : \|w\| \leq R,\ t \geq 0,\ v^- \in E^-\}$$

and

$$M_0 = \{w = tu_0^+(\cdot - y) + v^- :\ v^- \in E^-,\ \|w\| = R,\ t \geq 0 \text{ or } \|w\| \leq R,\ t = 0\}.$$

Lemma 3.4.1. *There exist $R>0$ and $y \in \mathbb{R}^N$, with R and $|y|$ sufficiently large, such that*

$$I|_{M_0} \leq 0.$$

Proof. The subset M_0 is equal to a disjoint union of M_1 and M_2, where

$$\begin{aligned}M_1 &= \{w = tu_0^+(\cdot - y) + v^-;\ v^- \in E^-,\ \|w\| \leq R,\ t = 0\},\\ M_2 &= \{w = tu_0^+(\cdot - y) + v^-;\ v^- \in E^-,\ \|w\| = R,\ t > 0\}.\end{aligned}$$

Since $M_1 \subset E^-$, we have that $I(w) \leq 0$ for each $w \in M_1$. Indeed, since $w \in E^-$, it follows that

$$\begin{aligned}I(w) &= -\frac{1}{2}\int_{\mathbb{R}^N} (\xi(x)|\nabla v^-|^2 + V(x)(v^-)^2)dx - \int_{\mathbb{R}^N} F(x,w)dx \\ &= -\frac{1}{2}\|v^-\|^2 - \int_{\mathbb{R}^N} F(x,w)dx \leq 0.\end{aligned}$$

Now, let us show that given $R>0$ and $w \in M_2$ with $\|w\| = R$ we have that $I(w) \leq 0$. Writing

$$w = \|w\|\frac{w}{\|w\|} = \|w\|u(w) = \|w\|(\lambda(w)u_0^+(\cdot - y) + v^-(w)).$$

So, we obtain

$$\begin{aligned} I(w) &= \|w\|^2\left[\frac{1}{2}\lambda^2(w)\|u_0^+(\cdot-y)\|^2-\frac{1}{2}\|v^-(w)\|^2\right]-\int_{\mathbb{R}^N}\frac{F(x,\|w\|u(w))}{u(w)^2}u(w)^2dx \\ &= \frac{1}{2}\|w\|^2\left\{\lambda^2(w)\|u_0^+(\cdot-y)\|^2-\|v^-(w)\|^2\ -2\int_{\mathbb{R}^N}\frac{F(x,Ru(w))}{(Ru(w))^2}u(w)^2dx\right\}. \end{aligned}$$

To simplify the notation, we write λ, u and v^- instead of $\lambda(w)$, $u(w)$ and $v^-(w)$, respectively.

Claim 3.4.1.

$$\lim_{s\to\infty}\frac{F(x,s)}{s^2}=\frac{1}{2}a(x)\ \textit{and}\ \frac{F(x,s)}{s^2}\le\frac{1}{2}a(x)$$

for all $s\neq 0$ *and all* $x\in\mathbb{R}^N$.

Indeed, by the L'Hopital rule and the hypothesis (f_2) we have

$$\lim_{s\to\infty}\frac{F(x,s)}{s^2}=\lim_{s\to\infty}\frac{F'(x,s)}{(s^2)'}=\lim_{s\to\infty}\frac{f(x,s)}{2s}=\frac{1}{2}\lim_{s\to\infty}\frac{f(x,s)}{s}=\frac{1}{2}a(x).$$

Also from hypothesis (f_2) we have that $|f(x,s)|/|s|\le a(x)$ and hence $|f(x,s)|\le a(x)|s|$. Thus,

$$\begin{aligned} \left|\frac{F(x,s)}{s^2}\right| &= \left|\frac{1}{s^2}\int_0^s f(x,t)dt\right|\le\frac{1}{|s|^2}\int_0^s|f(x,t)|dt \\ &< \frac{1}{|s|^2}\int_0^s a(x)|t|dt=\frac{1}{|s|^2}a(x)\frac{|s|^2}{2} \\ &= \frac{1}{2}a(x). \end{aligned}$$

which concludes our claim.

From Claim 3.4.1 and (f_2) the following inequality

$$\left|\frac{F(x,Ru)}{(Ru)^2}u^2\right|\le\frac{1}{2}a(x)|u|^2\le\frac{a_\infty}{2}|u|^2\ \in L^1(\mathbb{R}^N)$$

and by Lebesgue Dominated Convergence Theorem

$$\lim_{R\to\infty}\int_{\mathbb{R}^N}\left(\frac{F(x,Ru)}{(Ru)^2}-\frac{a(x)}{2}\right)dx=0 \tag{3.4.1}$$

for all $u \in E$ such that $\|u\|=1$. Since M_2 is contained in a finite dimensional subspace of E, $w=\|w\|Ru \in M_2$ with $\|u\|=1$, then the limit (3.4.1) is uniform in u, see Lemma A.3 in Appendix A. It follows from the fact $a(x) \leq a_\infty$ and $\int_{\mathbb{R}^N} u_0^+(x-y)v^-(x)dx=0$, that

$$\begin{aligned} I(w) &\leq \frac{1}{2}\|w\|^2\left\{\lambda^2\|u_0^+(\cdot-y)\|^2-\|v^-\|^2-a_\infty\int_{\mathbb{R}^N}(\lambda u_0^+(x-y)+v^-)^2dx+o_R(1)\right\}\\ &= \frac{1}{2}\|w\|^2\left\{\lambda^2\|u_0^+(\cdot-y)\|^2-\|v^-\|^2-a_\infty\int_{\mathbb{R}^N}\lambda(u_0^+)^2(x-y)dx\right.\\ &\quad\left.-a_\infty\int_{\mathbb{R}^N}(v^-)^2dx+o_R(1)\right\}\\ &\leq \frac{1}{2}\|w\|^2\left\{\lambda^2\|u_0^+(\cdot-y)\|^2-a_\infty\int_{\mathbb{R}^N}\lambda(u_0^+)^2(x-y)dx+o_R(1)\right\}. \end{aligned} \tag{3.4.2}$$

By hypotheses (ξ_1), $\xi(x)\leq\xi_\infty$ and (V_1), $V(x)\leq V_\infty$, for all $x\in\mathbb{R}^N$, and it follows that

$$\begin{aligned} \|u_0^+(\cdot-y)\|^2 &= \int_{\mathbb{R}^N}(\xi(x)|\nabla u_0^+(x-y)|^2+V(x)(u_0^+)^2(x-y))dx\\ &\leq \int_{\mathbb{R}^N}(\xi_\infty|\nabla u_0^+(x-y)|^2+V_\infty(u_0^+)^2(x-y))dx\\ &= \|u_0^+(\cdot-y)\|_\infty^2\leq\|u_0(\cdot-y)\|_\infty^2. \end{aligned} \tag{3.4.3}$$

Since I_∞ is translation invariant, then u_0 and $u_0(\cdot-y)$ are critical points of the functional I_∞. Therefore, $I'_\infty(u_0(\cdot-y))u_0(\cdot-y)=0$, that is,

$$\|u_0(\cdot-y)\|_\infty^2=\int_{\mathbb{R}^N}h(u_0(x-y))u_0^2(x-y)dx. \tag{3.4.4}$$

From (3.4.3) and (3.4.4),

$$\|u_0(\cdot-y)\|^2\leq\int_{\mathbb{R}^N}h(u_0(x-y))u_0^2(x-y)dx. \tag{3.4.5}$$

Replacing (3.4.5) in (3.4.2) and, after, using the term $a_\infty\int_{\mathbb{R}^N}u_0^2(x-y)dx$, we obtain

$$\begin{aligned} I(w) &\leq \frac{1}{2}\|w\|^2\left\{\lambda^2\left[\int_{\mathbb{R}^N}h(u_0(x-y))u_0^2(x-y)dx-a_\infty\int_{\mathbb{R}^N}(u_0^+)^2(x-y)dx\right]+o_R(1)\right\}\\ &= \frac{1}{2}\|w\|^2\left\{\lambda^2\left[\int_{\mathbb{R}^N}h(u_0(x-y))u_0^2(x-y)dx-a_\infty\int_{\mathbb{R}^N}u_0^2(x-y)dx\right.\right. \end{aligned}$$

$$
\begin{aligned}
&+a_\infty \int_{\mathbb{R}^N}[u_0^2(x-y)-(u_0^+)^2(x-y)]dx\Big]+o_R(1)\Big\}\\
=\ &\frac{1}{2}\|w\|^2\Big\{\lambda^2\Big[\int_{\mathbb{R}^N} h(u_0(z))u_0^2(z)dz - a_\infty\int_{\mathbb{R}^N} u_0^2(z)dz\\
&+a_\infty \int_{\mathbb{R}^N}[u_0^2(x-y)-(u_0^+)^2(x-y)]dx\Big]+o_R(1)\Big\}. \qquad (3.4.6)
\end{aligned}
$$

We estimate the following integrals

$$\int_{\mathbb{R}^N}\Big(h(u_0(z))-a_\infty\Big)u_0^2(z)dz \qquad (3.4.7)$$

and

$$\int_{\mathbb{R}^N} a_\infty\Big[u_0^2(x-y)+(u_0^+)^2(x-y)\Big]dx. \qquad (3.4.8)$$

Since u_0 is radial and continuous, the function $h(u_0(\cdot))$ assumes its maximum at $x_0\in\mathbb{R}^N$. It follows by hypothesis (f_3) we have $h(s)<a_\infty$ for all $s\in\mathbb{R}^N$, that

$$
\begin{aligned}
\int_{\mathbb{R}^N}\Big(h(u_0(z))-a_\infty\Big)u_0^2(z)dz &\le \int_{\mathbb{R}^N}\Big(h(u_0(x_0))-a_\infty\Big)u_0^2(z)dz\\
&= \Big(h(u_0(x_0))-a_\infty\Big)\int_{\mathbb{R}^N} u_0^2(z)dz\\
&= \Big(h(u_0(x_0))-a_\infty\Big)\|u_0\|^2_{L^2(\mathbb{R}^N)}<-\gamma,
\end{aligned}
$$

where, $\frac{1}{2}\Big(a_\infty - h(u_0(x_0))\Big)\|u_0\|^2_{L^2}>0$. In other words, there exists $\gamma>0$ such that

$$\int_{\mathbb{R}^N}\Big(h(u_0(z))-a_\infty\Big)u_0^2(z)dz<-\gamma. \qquad (3.4.9)$$

To estimate the integral (3.4.8) some statements will be necessary. Before that, since $\int_{\mathbb{R}^N} u_0^+(x-y)u_0^-(x-y)dx=0$ and $u_0=u_0^+ + u_0^-$, we have that

$$
\begin{aligned}
\int_{\mathbb{R}^N}\Big(u_0^2(x-y)-(u_0^+)^2(x-y)\Big)dx &= \int_{\mathbb{R}^N}\Big[\Big(u_0^+(x-y)+u_0^-(x-y)\Big)^2-(u_0^+)^2(x-y)\Big]dx\\
&= \int_{\mathbb{R}^N}\Big[(u_0^+)^2(x-y)+(u_0^-)^2(x-y)-(u_0^+)^2(x-y)\Big]dx\\
&= \int_{\mathbb{R}^N}(u_0^-)^2(x-y)dx. \qquad (3.4.10)
\end{aligned}
$$

Claim 3.4.2. *The integral* $\int_{\mathbb{R}^N}(u_0^-)^2(x-y)dx\to 0$ *as* $|y|\to\infty$.

Indeed, since $\{\phi_1, \cdots, \phi_k\}$ is a basis of eigenfuctions for the subspace E^-, Remark 3.4.1 and hypotheses (V_1) and (ξ_3) ensure that, given $\varepsilon > 0$, for each $i \in \{1, \cdots, k\}$ there exists $M_i > 0$, then

$$\langle u_0(x-y), \phi_i \rangle = \int_{\mathbb{R}^N} (\xi(x)\nabla u_0(x-y)\nabla\phi_i(x) + V(x)u_0(x-y))\phi_i(x)dx < \varepsilon.$$

Taking $\overline{M} = \max\{M_1, \cdots, M_k\}$ it follows that, for all $i \in \{1, \cdots, k\}$

$$\langle u_0(x-y), \phi_i \rangle < \varepsilon \quad \text{if} \quad |y| \geq \overline{M}. \tag{3.4.11}$$

Since $u_0^-(\cdot - y) \in E^-$ is a linear combination of the vectors $\phi_1, \cdots, \phi_k$, we get

$$u_0^-(x-y) = \sum_{i=1}^{k} \eta_i(y)\phi_i(x),$$

it follows from (3.4.11) that there exists $\tilde{M} > 0$ such that if $|y| \geq \tilde{M}$, then

$$\|u_0^-(\cdot - y)\|^2 = \langle u_0(\cdot - y), \sum_{i=1}^{k} \eta_i(y)\phi_i(x) \rangle < \varepsilon k\Big(\max\{|\eta_1(y), \cdots, |\eta_k(y)|\}\Big). \tag{3.4.12}$$

Claim 3.4.3. *There exists a constant $C > 0$, which does not depend on y such that*

$$\max\{|\eta_1(y)|, \cdots, |\eta_k(y)|\} < C, \quad \textit{for all} \quad y \in \mathbb{R}^N. \tag{3.4.13}$$

To show the claim, we remember that $\dim E^- < \infty$, by the equivalence of the norms in a finite dimensional space, there exists $D > 0$, which does not depend on y such that

$$\left\| \sum_{i=1}^{k} \eta_i(y)\phi_i(x) \right\|_\infty^2 \geq D\bigg(\max\{|\eta_1(y)|, \cdots, |\eta_k(y)|\}\bigg)^2.$$

Therefore,

$$\|u_0\|_\infty^2 \geq \|u_0^-(\cdot - y)\|_\infty^2 = \left\| \sum_{i=1}^{k} \eta_i(y)\phi_i(x) \right\|_\infty^2 \geq D\Big(\max\{|\eta_1(y)|, \cdots, |\eta_k(y)|\}\Big)^2. \tag{3.4.14}$$

This proves Claim 3.4.3, choosing $C = \|u_0\|_\infty^2 / \sqrt{D} > 0$.

Now, replacing (3.4.13) in (3.4.12), we obtain

$$\|u_0^-(\cdot - y)\|^2 < \varepsilon k C, \quad \text{for} \quad |y| \geq \tilde{M}.$$

Since the norm $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent in E, it follows that $\|u_0^-(\cdot - y)\|_\infty \to 0$ as $|y| \to \infty$. Thus,

$$\int_{\mathbb{R}^N} (u_0^-)^2(x-y)dx \leq C\|u_0^-(\cdot - y)\|_\infty^2 \to 0, \quad \text{as} \quad |y| \to \infty. \tag{3.4.15}$$

concluding the proof of Claim 3.4.2.

Substituting (3.4.9), (3.4.10) and (3.4.15) in (3.4.6), we obtain

$$\begin{aligned} I(w) \leq & \frac{1}{2}\|w\|^2 \Big\{ \lambda^2 \Big[\int_{\mathbb{R}^N} h(u_0(z))u_0^2(z)dz - a_\infty \int_{\mathbb{R}^N} u_0^2(z)dz \\ & + a_\infty \int_{\mathbb{R}^N} [u_0^2(x-y) - (u_0^+)^2(x-y)]dx \Big] + o_R(1) \Big\} \\ \leq & \frac{1}{2}\|w\|^2 \Big\{ \lambda^2 [-\gamma + o_{|y|}(1)] + o_R(1) \Big\} \end{aligned} \tag{3.4.16}$$

for $|y|$ and R sufficiently large.

To conclude the proof of this lemma, we will analyze the following cases for values λ:

Case 1: Consider $\lambda^2 < 1/(C\|u_0\|_\infty^2)$, where $C > 0$ is a constant that does not depend on y. Since $w = \|w\|(\lambda u_0^+(\cdot - y) + v^-)$ and F is a nonnegative function, by hypothesis (f_4), we have

$$\begin{aligned} I(w) &= \frac{1}{2}\|w\|^2 \Big(\lambda^2 \|u_0^+(\cdot - y)\|^2 - \|v^-\|^2\Big) - \int_{\mathbb{R}^N} F(w)dx \\ &\leq \frac{1}{2}\|w\|^2 \Big(\lambda^2 \|u_0^+(\cdot - y)\|^2 - \|v^-\|^2\Big). \end{aligned} \tag{3.4.17}$$

It follows from the fact $\|\lambda u_0^+(\cdot - y) + v^-\|^2 = 1$ that $\lambda^2\|u_0^+(\cdot - y)\|^2 + \|v^-\|^2 = 1$. The the equation (3.4.17) becomes

$$\begin{aligned} I(w) &\leq \frac{1}{2}\|w\|^2 \Big(\lambda^2 \|u_0^+(\cdot - y)\|^2 + \lambda^2 \|u_0^+(\cdot - y)\|^2 - \lambda^2 \|u_0^+(\cdot - y)\|^2 - \|v^-\|^2\Big) \\ &= \frac{1}{2}\|w\|^2 \Big(2\lambda^2 \|u_0^+(\cdot - y)\|^2 - 1\Big). \end{aligned}$$

By the equivalence of the norm and the translation invariance of the norm $\|\cdot\|_\infty$, there exists $C>0$, which does not depend on y, such that

$$2\|u_0^+(\cdot-y)\|^2 \le C\|u_0(\cdot-y)\|_\infty^2 = C\|u_0\|_\infty^2.$$

Thus, for

$$\lambda^2 < \frac{1}{C\|u_0\|_\infty^2} < \frac{1}{2\|u_0^+(\cdot-y)\|^2},$$

we have $I(w)<0$ and the lemma is proved for such values of λ.

Case 2: $\lambda^2 \ge 1/(C\|u_0\|_\infty^2)$.

Denote by $\lambda^2 \ge 1/(C\|u_0\|_\infty^2) =: K_0 > 0$. We choose $y\in\mathbb{R}^N$ with $|y|$ sufficiently large such that

$$-\gamma + o_{|y|}(1) < -\gamma/2.$$

Then, we can rewrite the inequality (3.4.16) as

$$\begin{aligned} I(w) &\le \frac{1}{2}\|w\|^2\Big[-\lambda^2\frac{\gamma}{2} + o_R(1)\Big] \\ &\le \frac{1}{2}\|w\|^2\Big[-K_0\frac{\gamma}{2} + o_R(1)\Big] \le 0 \end{aligned}$$

thus the lemma is proved for the values λ such that $\lambda^2 \ge K_0$ and R sufficiently large. This concludes the proof of the lemma. □

Lemma 3.4.2. *Suppose* ξ, V *satisfies* $(\xi_1)-(\xi_3)$ *and* $(V_1)-(V_4)$ *respectively, and either* (3.2.1) *or* (3.2.2). *Then, it holds that*

$$c < c_\infty := \inf\Big\{I_\infty(w):\ w\in H^1(\mathbb{R}^N)\setminus\{0\},\ I'_\infty(w)=0\Big\}.$$

To prove these results, we will need some auxiliary lemmas. The first two may be found in [1] and [25]. For the sake of completeness, we will present the proof of each of them.

Lemma 3.4.3. *There exists* $\mu\in(1,2]$ *with the following property: for any* $\rho>0$ *there exists a constant* $C_\rho>0$ *such that the inequality*

$$F(x,u+v)-F(x,u)-F(x,v)-f(x,u)v-f(x,v)u \ge -C_\rho|uv|^\mu$$

is true for all $x\in\mathbb{R}^N$ *and* $u,\ v\in\mathbb{R}$ *with* $|u|,\ |v|\le\rho$.

Proof. Let $p = p_1$ and $\mu := \min\left\{\frac{p+1}{2}, 2\right\}$. By hypothesis (f_6), f is increasing, which yields

$$F(x,u+v) - F(x,u) = \int_u^{u+v} f(x,w)dw \geq f(x,u)v.$$

Moreover, by hypothesis (f_5), for every $1 < \mu \leq 2$ we have $f(x,s) = o(|s|^\mu)$, as $|s| \to 0$ and then $\tilde{C}_\rho := \sup_{0<u\leq\rho} \frac{f(x,u)}{u^\mu} < \infty$. Now, for $0 < v \leq u \leq \rho$, we deduce

$$\begin{aligned}
F(x,u+v) &- F(x,u) - F(x,v) - f(x,u)v - f(x,v)u \geq -F(x,v) - f(x,v)u \\
&= \int_0^v -\frac{f(x,w)}{w^\mu} w^\mu dw - \frac{f(x,v)}{v^\mu} uv^\mu \\
&\geq -\tilde{C}_\rho \frac{v^{\mu+1}}{\mu+1} - \tilde{C}_\rho uv^\mu \\
&\geq -\left[\left(\left(\frac{u}{v}\right)^{\mu+1} + \frac{u}{v}\right)\right] \tilde{C}_\rho (uv)^\mu \\
&\geq -C_\rho (uv)^\mu.
\end{aligned}$$

By the symmetry in u and v, the same estimate holds for $0 < u \leq v$ and the proof is complete. □

Lemma 3.4.4. *If $\mu_2 > \mu_1 \geq 0$ then, there exists $C > 0$ such that, for all $x_1,\ x_2 \in \mathbb{R}^N$,*

$$\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} dx \leq Ce^{-\mu_1|x_1-x_2|}.$$

Proof. Observe that

$$\mu_1|x_1 - x_2| + (\mu_2 - \mu_1)|x - x_2| \leq \mu_1|x - x_1| + \mu_2|x - x_2|.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} dx &\leq \int_{\mathbb{R}^N} e^{-\mu_1|x_1-x_2|} e^{-(\mu_2-\mu_1)|x-x_2|} dx \\
&\leq \int_{\mathbb{R}^N} e^{-\mu_1|x_1-x_2|} \frac{1}{e^{(\mu_2-\mu_1)|x-x_2|}} dx \\
&\leq e^{-\mu_1|x_1-x_2|}
\end{aligned}$$

and the lemma follows. □

We note that the set M defined in the Theorem 3.4.1 is closed, bounded and it is contained in a finite-dimensional space, namely, in the space $\mathbb{R}u_0^+(\cdot - y) \oplus E^-$. Therefore, M is a compact set, which implies that for all $y \in \mathbb{R}^N$, there exists $w_y = v_y^- + t_y u_0^+(\cdot - y) \in M$ satisfying

$$\max_{w \in M} I(w) = I(v_y^- + t_y u_0^+(\cdot - y))$$

since I is a continuous functional.

The following results show that the values t_y are uniformly bounded on y by positive constants if $|y|$ is sufficiently large.

Lemma 3.4.5. *There exist $A,\ B \in \mathbb{R}$ which do not depend on y, such that $0 < A \leq t_y \leq B$ for $|y|$ big enough.*

Proof. Since $w_y = v_y^- + t_y u_0^+(\cdot - y) \in M$ and the number R given by Lemma 3.4.1 is positive and does not depend on y, one has

$$\begin{aligned} R^2 &\geq \|w_y\|^2 = \|v_y^-\|^2 + t_y^2 \|u_0^+(\cdot - y)\|^2 \\ &\geq t_y^2 \Big(\|u_0(\cdot - y)\|^2 - \|u_0^-(\cdot - y)\|^2 \Big). \end{aligned}$$

As proven previously, in Claim 3.4.2, we can take $|y|$ large enough to ensure that

$$\|u_0^-(\cdot - u)\|^2 \leq \frac{C}{2} \|u_0\|_\infty^2$$

where $C > 0$ does not depend on y and satisfies $\|u_0(\cdot - y)\|^2 \geq C \|u_0\|_\infty^2$. Thus,

$$R^2 \geq t_y^2 \Big(\|u_0(\cdot - y)\|^2 - \|u_0^-(\cdot - y)\|^2 \Big) \geq t_y^2 \Big(C\|u_0\|_\infty^2 - \frac{C}{2}\|u_0\|_\infty^2 \Big) = \frac{t_y^2}{2} \|u_0\|_\infty^2,$$

that is, $t_y^2 \leq 2R/(\|u_0\|_\infty^2) := B^2$.

On the other hand, from estimates given by Lemma 3.3.1 with $2 < p < 2^*$, for $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, if $u \in E^+$ with $\|u\| = \rho > 0$ then

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \geq \frac{1}{2}\rho^2 - \varepsilon \|u\|_{L^2}^2 - C_\varepsilon \|u\|_{L^p}^p. \tag{3.4.18}$$

By Sobolev embedding and the equivalence of the norms there exist constants $C_5,\ C_6 > 0$ which make (3.4.18) in

$$I(u) \geq \frac{1}{2}\|u\|^2 - \varepsilon C_5 \|u\|^2 - C_6 \|u\|^p \geq \frac{1}{2}\rho^2 - \varepsilon C_5 \rho^2 - C_6 \rho^p = \left(\frac{1}{2} - \varepsilon C_5 \right) \rho^2 - C_6 \rho^p.$$

Let $\varepsilon > 0$ be such that $D_\varepsilon := \dfrac{1}{2} - \varepsilon C_5 > 0$. Choosing $\rho > 0$ sufficiently small so that $D_\varepsilon \rho^2 - C_6 \rho^p > 0$, that is, $0 < \rho < (D_\varepsilon / C_6)^{1/(p-2)}$, we obtain that

$$I(u) \geq D_\varepsilon \rho^2 - C_6 \rho^p := \rho_0 > 0, \tag{3.4.19}$$

for all $u \in E^+$ with $\|u\| = \rho$ where ρ_0 does not depend on y. Thus, we take $t_0 > 0$, which does not depend on y so that $\|t_0 u_0^+(\cdot - y)\| \leq \rho < R$ to conclude that $I(t_0 u_0^+(\cdot - y)) \geq \rho_0 > 0$. Consequently,

$$I(v_y^- + t_y u_0^+(\cdot - y)) = \max_{w \in M} I(w) \geq I(t_0 u_0^+(\cdot - y)) \geq \rho_0,$$

that is,

$$\frac{t_y^2}{2}\|u_0^+(\cdot - y)\|^2 - \frac{1}{2}\|v_y^-\|^2 - \int_{\mathbb{R}^N} F(x - y, v_y^- + t_y u_0^+(x - y))dx = I(v_y^- + t_y u_0^+(\cdot - y)) \geq \rho_0.$$

Therefore, since F is nonnegative,

$$\frac{t_y^2}{2}\|u_0^+(\cdot - y)\|^2 \geq \rho_0.$$

This shows that

$$t_y^2 \geq \frac{2\rho_0}{C\|u_0\|_\infty^2} =: A^2$$

where $C > 0$ does not depend on $|y|$ and satisfies $\|u_0^+(\cdot - y)\|^2 \leq C\|u_0\|_\infty^2$. The lemma is proved. □

Now let us present the proof of the Lemma 3.4.2.

Proof of Lemma 3.4.2. To simplicity, we will denote $u_{0,y}(x) = u_0(x - y)$ and C will denote a positive constant, not necessarily the same one. By the definition of the functional I and I_∞ and of the norms $\|\cdot\|$ and $\|\cdot\|_\infty$, we have

$$\begin{aligned}
I(v_y^- + t_y u_{0,y}^+) &= \frac{t_y^2}{2}\|u_{0,y}^+\|^2 - \frac{1}{2}\|v_y^-\|^2 - \int_{\mathbb{R}^N} F(x - y, v_y^- + t_y u_{0,y}^+)dx \\
&\leq \frac{t_y^2}{2}\|u_{0,y}^+\|^2 - \int_{\mathbb{R}^N} F(x - y, v_y^- + t_y u_{0,y}^+)dx + \int_{\mathbb{R}^N} F(x - y, t_y u_{0,y})dx \\
&\quad - \int_{\mathbb{R}^N} H(t_y u_{0,y})dx + \int_{\mathbb{R}^N} \Big[H(t_y u_{0,y}) - F(x - y, t_y u_{0,y})\Big]dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{t_y^2}{2}\int_{\mathbb{R}^N}(\xi(x)|\nabla u_{0,y}^+|^2+V(x)(u_{0,y}^+)^2)dx+\frac{t_y^2}{2}\int_{\mathbb{R}^N}(\xi_\infty|\nabla u_{0,y}^+|^2+V_\infty(u_{0,y}^+)^2)dx\\
&\quad -\frac{t_y^2}{2}\int_{\mathbb{R}^N}(\xi_\infty|\nabla u_{0,y}^+|^2+V_\infty(u_{0,y}^+)^2)dx-\int_{\mathbb{R}^N}H(t_yu_{0,y})dx+\int_{\mathbb{R}^N}(H(t_yu_{0,y})\\
&\quad -F(x-y,t_yu_{0,y}))dx+\int_{\mathbb{R}^N}(F(x-y,t_yu_{0,y})-F(x-y,v_y^-+t_yu_{0,y}^+))dx
\end{aligned}
$$

since F is nonnegative. By hypotheses (f_2) and (f_4) we have that the term satisfies $\int_{\mathbb{R}^N}[H(t_yu_{0,y})-F(x-y,t_yu_{0,y})]dx\leq 0$ and thus

$$
\begin{aligned}
I(v_y^-+t_yu_{0,y}^+) &\leq \frac{t_y^2}{2}\|u_{0,y}^+\|_\infty^2-\int_{\mathbb{R}^N}H(t_yu_{0,y})dx+\frac{t_y^2}{2}\int_{\mathbb{R}^N}(\xi(x)-\xi_\infty)|\nabla u_{0,y}^+|^2dx\\
&\quad +\frac{t_y^2}{2}\int_{\mathbb{R}^N}(V(x)-V_\infty)(u_{0,y}^+)^2dx+\int_{\mathbb{R}^N}(F(x-y,t_yu_{0,y})-F(x-y,v_y^-+t_yu_{0,y}^+))dx\\
&\leq I_\infty(t_yu_{0,y})+\frac{t_y^2}{2}\int_{\mathbb{R}^N}(\xi(x)-\xi_\infty)|\nabla u_{0,y}^+|^2dx+\frac{t_y^2}{2}\int_{\mathbb{R}^N}(V(x)-V_\infty)(u_{0,y}^+)^2dx\\
&\quad +\int_{\mathbb{R}^N}(F(x-y,t_yu_{0,y})+F(x-y,v_y^--t_yu_{0,y}^-)-F(x-y,v_y^-+t_yu_{0,y}^+))dx. \quad (3.4.20)
\end{aligned}
$$

Now, let us estimate the last integral in the above inequality. Taking $w_y^-=v_y^--t_yu_{0,y}^-$, we want to estimate $\mathcal{I}_y$ defined by

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\Big[F(x-y,v_y^--t_yu_{0,y}^-)+F(x-y,t_yu_{0,y})-F(x-y,v_y^-+t_yu_{0,y}^+)\Big]dx\\
&=\int_{\mathbb{R}^N}\Big[F(x-y,v_y^--t_yu_{0,y}^-)+F(x-y,t_yu_{0,y})-F(x-y,v_y^--t_yu_{0,y}^-+t_yu_{0,y}^-+t_yu_{0,y}^-)\Big]dx\\
&=\int_{\mathbb{R}^N}\Big[F(x-y,w_y^-)+F(x-y,t_yu_{0,y})-F(x-y,w_y^-+t_yu_{0,y})\Big]dx=:\mathcal{I}_y.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\mathcal{I}_y &= \int_{\mathbb{R}^N}-\Big[F(x-y,w_y^-+t_yu_{0,y})-F(x-y,w_y^-)-F(x-y,t_yu_{0,y})\Big]dx\\
&= \int_{\mathbb{R}^N}\Big[F(x-y,w_y^-+t_yu_{0,y})-F(x-y,w_y^-)-F(x-y,t_yu_{0,y})\\
&\quad -f(x-y,w_y^-)t_yu_{0,y}-f(x-y,t_yu_{0,y})w_y^-\Big]dx-\int_{\mathbb{R}^N}f(x-y,w_y^-)(t_yu_{0,y})dx\\
&\quad -\int_{\mathbb{R}^N}f(x-y,t_yu_{0,y})(w_y^-)dx\\
&\leq C_\rho|w_y^-(t_yu_{0,y})|^\mu+\int_{\mathbb{R}^N}|f(x-y,w_y^-)||t_yu_{0,y}|dx+\int_{\mathbb{R}^N}|f(x-y,t_yu_{0,y}||w_y^-|dx.
\end{aligned}
$$

Since $w_y^- = v_y^- - t_y u_{0,y}^- \in M$ and, hence $\|w_y^-\|^2 \leq R^2$, we can rewrite w_y^- as a linear combination of the eigenfuctions $\phi_1, \cdots, \phi_k$ because v_y^-, $u_{0,y}^- \in E^-$. Due to $\dim E^- < \infty$, we may repeat the estimates in (3.4.14) with w_y^- in the place of $u_{0,y}^-$ and using the Lemma 3.4.5 to show that there exists a constant $C > 0$ which does not depend on y, such that

$$\begin{aligned} |w_y^-(x)| &= |v_y^-(x) - t_y u_{0,y}^-(x)| = \left| \sum_{i=1}^k \eta_i(y)\phi_i(x) - t_y \sum_{i=1}^k \zeta_i(y)\phi_i(x) \right| \\ &= \left| \sum_{i=1}^k (\eta_i(y) - C\zeta_i(y))\phi_i(x) \right| \leq \sum_{i=1}^k \left|\eta_i(y) - C\zeta_i(y)\right| \left|\phi_i(x)\right| \\ &\leq \sum_{i=1}^k (|\eta_i(y)| + C|\zeta_i(y)|)|\phi_i(x)| \\ &\leq \sum_{i=1}^k \Big(\max\{\eta_i(y)\} + C\max\{\zeta_i(y)\} \Big) |\phi_i(x)| \\ &\leq C\sum_{i=1}^k |\phi_i(x)| \leq C\sum_{i=1}^k \sup_{\mathbb{R}^N} |\phi_i(x)| =: D, \end{aligned} \tag{3.4.21}$$

for all $x \in \mathbb{R}^N$. Without loss of generality, we may suppose that $D > 1$, also satisfies $|u_{0,y}(x)| \leq D$ for all $x \in \mathbb{R}^N$ since $u_{0,y} \in L^\infty(\mathbb{R}^N)$. Now, we can apply Lemma 3.4.3 and the hypothesis (f_2) to obtain a constant $C > 0$, such that

$$\begin{aligned} \mathcal{I}_y &\leq \int_{\mathbb{R}^N} C|w_y^-|^\mu |t_y u_{0,y}|^\mu dx + t_y \int_{\mathbb{R}^N} a(x)|w_y^-||u_{0,y}| dx + \int_{\mathbb{R}^N} a(x)|t_y u_{0,y}||w_y^-| dx \\ &\leq C t_y^\mu \int_{\mathbb{R}^N} |w_y^-|^\mu |u_{0,y}|^\mu dx + 2t_y a_\infty \int_{\mathbb{R}^N} |w_y^-||u_{0,y}| dx \end{aligned} \tag{3.4.22}$$

where $\mu > 1$ is given by Lemma 3.4.3. Now, taking $\eta = \lambda_1 < 0 < V_\infty$ in the Theorem 3.1.3, it holds that any eigenfunctions ϕ_i, $i = 1, \cdots, k$ satisfies

$$|\phi_i(x)| \leq Ce^{-\delta|x|},$$

for all $x \in \mathbb{R}^N$ and some $\sqrt{V_\infty/\xi_\infty} < \delta < \sqrt{V_\infty - \eta}$. Therefore, from the first inequality of (3.4.21), for $|y|$ sufficiently large, we have

$$|w_y^-(x)| \leq Ce^{-\delta|x|}, \quad \text{for all } x \in \mathbb{R}^N.$$

Since u_0 is a solution of equation (3.2.4) given by Berestick and Lions in [6] we have that $|u_0(x)|\leq Ce^{-\sqrt{V_\infty/\xi_\infty}|x|}$, for all $x\in\mathbb{R}^N$. It follows from Lemma 3.4.4 that

$$\int_{\mathbb{R}^N}|w_y^-||u_{0,y}|dx\leq\int_{\mathbb{R}^N}Ce^{-\delta|x|}Ce^{-\sqrt{V_\infty/\xi_\infty}|x-y|}dx\leq Ce^{-\sqrt{V_\infty/\xi_\infty}|y|}. \tag{3.4.23}$$

Analogously, by Lemma 3.4.4 we have

$$\int_{\mathbb{R}^N}|w_y^-|^\mu|u_{0,y}|^\mu dx\leq\int_{\mathbb{R}^N}Ce^{-\delta\mu|x|}Ce^{-\mu\sqrt{V_\infty/\xi_\infty}|x-y|}dx\leq Ce^{-\mu\sqrt{V_\infty/\xi_\infty}|y|}, \tag{3.4.24}$$

because $\mu>1$. Estimates (3.4.23) and (3.4.24) applied in (3.4.22) yield

$$\mathcal{I}_y\leq Ct_y^\mu e^{-\mu\sqrt{V_\infty/\xi_\infty}|y|}+2t_yCa_\infty e^{-\sqrt{V_\infty/\xi_\infty}|y|}\leq Ce^{-\sqrt{V_\infty/\xi_\infty}|y|}, \tag{3.4.25}$$

where the constant $C>0$ does not depend on y since t_y is uniformly bounded by Lemma 3.4.5.

By (3.2.1) and a change of variable, there exists a positive constant C_1 such that

$$\begin{aligned}\frac{t_y^2}{2}\int_{\mathbb{R}^N}(\xi(x)-\xi_\infty)|\nabla u_{0,y}|^2dx &\leq -C_1\int_{\mathbb{R}^N}e^{-\gamma_1|x|}|\nabla u_{0,y}|^2dx\\ &= -C_1\int_{\mathbb{R}^N}e^{-\gamma_1|z+y|}|\nabla u_0(z)|^2dz\\ &\leq -C_1e^{-\gamma_1|y|}\int_{\mathbb{R}^N}e^{-\gamma_1|z|}|\nabla u_0(z)|^2dz\\ &\leq -C_1e^{-\gamma_1|y|}.\end{aligned} \tag{3.4.26}$$

Or else by (3.2.2) and a change of variables, there exists a positive constant C_2 such that

$$\begin{aligned}\frac{t_y^2}{2}\int_{\mathbb{R}^N}(V(x)-V_\infty)(u_{0,y})^2dx &\leq -C_2\int_{\mathbb{R}^N}e^{-\gamma_2|x|}u_{0,y}^2dx\\ &= -C_2\int_{\mathbb{R}^N}e^{-\gamma_2|z+y|}u_0^2(z)dz\\ &\leq -C_2e^{-\gamma_2|y|}\int_{\mathbb{R}^N}e^{-\gamma_2|z|}u_0^2(z)dz\\ &\leq -C_2e^{-\gamma_2|y|},\end{aligned} \tag{3.4.27}$$

for $|y|$ sufficiently large. Thus, it follows from (3.4.25), (3.4.26) and (3.4.27) that (3.4.20) can be rewrite

$$I(v_y^- + t_y u_{0,y}^+) \leq I_\infty(t_y u_{0,y}) - C_1 e^{-\gamma_1|y|} - C_2 e^{-\gamma_2|y|} + Ce^{-\sqrt{V_\infty/\xi_\infty}|y|}.$$

Since $0 < \gamma_1 < \sqrt{V_\infty/\xi_\infty}$ by (3.2.1) or $0 < \gamma_2 < \sqrt{V_\infty/\xi_\infty}$ by (3.2.2), we get

$$-C_1 e^{-\gamma_1|y|} - C_2 e^{-\gamma_2|y|} + Ce^{-\sqrt{V_\infty/\xi_\infty}|y|} < 0.$$

And thus,

$$I(v_y^- + t_y u_{0,y}^+) < \max_{t\geq 0} I_\infty(tu_0)$$

for $|y|$ sufficiently large.

Claim 3.4.4. *The maximum* $\max_{t\geq 0} I_\infty(tu_0)$ *is attained at* $t = 1$.

Indeed, since u_0 is a positive, radial and symmetric solution given by Berestick and Lions in [6], then

$$\begin{aligned}
\frac{d}{dt} I_\infty(tu) &= \frac{d}{dt}\left[\frac{\|tu\|_\infty^2}{2} - \int_{\mathbb{R}^N} H(tu)dx\right] \\
&= t\|u\|_\infty^2 - \int_{\mathbb{R}^N} h(tu)(tu)u dx \\
&= t\int_{\mathbb{R}^N} h(u)u^2 dx - t\int_{\mathbb{R}^N} h(tu)u^2 dx.
\end{aligned}$$

By hypotheses $(f_2), (f_4)$ and (f_6) if, $t > 1$ we have that $\frac{d}{dt} I_\infty(tu) < 0$ and if, $0 < t < 1$, then $\frac{d}{dt} I_\infty(tu) > 0$, which give us that the maximum may be attained exactly at $t = 1$.

It follows this claim that $I_\infty(u) = \max_{t>0} I_\infty(tu)$. And from the definition of the value $c > 0$ we get

$$c \leq \max_{w\in M} I(w) = I(v_y^- + t_y u_{0,y}^+) < \max_{t>0} I_\infty(tu) = I_\infty(u) \leq c_\infty,$$

and the lemma is proved. □

The next lemma has the same proof of Lemma 2.1.4 and we will be stated for completeness.

Lemma 3.4.6. *Let (u_n) be a bounded sequence in $H^1(\mathbb{R}^N)$ such that*

$$I(u_n) \to c \quad \text{and} \quad \|I'(u_n)\|(1+\|u_n\|) \to 0.$$

Then, up to a subsequence, there exists a solution of (P_3), a number $m \in \mathbb{N}$, m functions $u_1, \ldots, u_m$ and m sequences $(y_k^j) \subset \mathbb{R}^N$ $1 \leq j \leq m$, satisfying one of the following alternatives:

(1) $u_k \to u_0$ in $H^1(\mathbb{R}^N)$; or

(2) u^j are nontrivial solutions of problem (3.2.4), such that:

(a) $|y_k^j| \to \infty$ and $|y_k^i - y_k^j| \to \infty$, $i \neq j$;

(b) $u_k - \sum_{j=1}^{k} u^j(\cdot - y_k^j) \to u_0$ in $H^1(\mathbb{R}^N)$;

(c) $c = I(u_0) + \sum_{j=1}^{k} I_\infty(u^j)$.

Lemma 3.4.7. *Let v_n be a solution of the following problem*

$$\begin{cases} -div(\xi(x)\nabla v_n) + V(x)v_n = f(x, v_n), & \text{in} \quad \mathbb{R}^N, \\ v_n \in H^1(\mathbb{R}^N), \quad \text{with} \quad N \geq 3, \\ v_n(x) \geq 0, \quad \text{for all } x \in \mathbb{R}^N. \end{cases}$$

Assuming that $(\xi_1)-(\xi_3)$, $(V_1)-(V_4)$, $(f_1)-(f_5)$ holds and that $v_n \to v$ in $H^1(\mathbb{R}^N)$ with $v \not\equiv 0$, then $v_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $\|v_n\|_{L^\infty} \leq C$ for all $n \in \mathbb{N}$. Furthermore,

$$\lim_{|x|\to\infty} v_n(x) = 0, \text{ uniformly in } n.$$

Proof. For any $R > 0$, $0 < r \leq R/2$, let $\eta \in C^\infty(\mathbb{R}^N)$, $0 \leq \eta \leq 1$ with $\eta(x) = 1$ if $|x| \geq R$ and $\eta(x) = 0$ if $|x| \leq R - r$ and $|\nabla\eta| \leq 2/r$. Note that, by Lemma 3.3.1 and by Sobolev's embedding for $2 \leq p \leq 2^*$, we obtain the following growth condition for f:

$$f(x, s) \leq \varepsilon|s| + C_\varepsilon|s|^{p-1} \leq \varepsilon|s| + C_\varepsilon|s|^{2^*-1}. \tag{3.4.28}$$

For each $n \in \mathbb{N}$ and for $L > 0$, let

$$v_{L,n}(x) = \begin{cases} v_n(x), & v_n(x) \leq L, \\ L, & v_n(x) \geq L, \end{cases}$$

$z_{L,n} = \eta^2 v_{L,n}^{2(\beta-1)} v_n$ and $w_{L,n} = \eta v_n v_{L,n}^{\beta-1}$ with $\beta > 1$ to be determinated later. Taking $z_{L,n}$ as a test function, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)}|\nabla v_n|^2 dx \;=\; & -2(\beta-1)\int_{\mathbb{R}^N} \xi(x) v_{L,n}^{2\beta-3}\eta^2 v_n \nabla v_n \nabla v_{L,n} dx \\ & + \int_{\mathbb{R}^N} f(x,v_n)\eta^2 v_n v_{L,n}^{2(\beta-1)} dx - \int_{\mathbb{R}^N} V(x) v_n^2 \eta^2 v_{L,n}^{2(\beta-1)} dx \\ & -2\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla \eta dx.\end{aligned}$$

Note that, $-2(\beta-1)\int_{\mathbb{R}^N} \xi(x) v_{L,n}^{2\beta-3}\eta^2 v_n \nabla v_n \nabla v_{L,n} dx \leq 0$, then

$$\begin{aligned}\int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)}|\nabla v_n|^2 dx \leq & -2\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla \eta dx - \int_{\mathbb{R}^N} V(x)\eta^2 v_{L,n}^{2(\beta-1)} v_n^2 dx \\ & + \int_{\mathbb{R}^N} f(x,v_n)\eta^2 v_n v_{L,n}^{2(\beta-1)} dx.\end{aligned}$$

Using the estimate in (3.4.28) we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)}|\nabla v_n|^2 dx \;\leq\; & -2\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla \eta dx - V_0\int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^2 dx \\ & +\varepsilon\int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^2 dx + C_\varepsilon \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^{2^*} dx.\end{aligned}$$

Now, by hypothesis (V_1) we have

$$\begin{aligned}\int_{\mathbb{R}^N} \xi(x)\eta^2 v_{L,n}^{2(\beta-1)}|\nabla v_n|^2 dx \leq & -2\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla \eta dx \\ & + (V_0+\varepsilon)\int_{\mathbb{R}^N} v_n^2\eta^2 v_{L,n}^{2(\beta-1)} dx + C_\varepsilon\int_{\mathbb{R}^N} \eta^2 v_n^{2^*} v_{L,n}^{2(\beta-1)} dx \\ \leq & C_\varepsilon \int_{\mathbb{R}^N} \eta^2 v_n^{2^*} v_{L,n}^{2(\beta-1)} dx + (V_0+\varepsilon)\int_{\mathbb{R}^N} v_n^2\eta^2 v_{L,n}^{2(\beta-1)} dx \\ & + 2\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla \eta dx.\end{aligned}$$

For each $\varepsilon > 0$, using the Young's inequality we get

$$\int_{\mathbb{R}^N} \xi(x)\eta v_{L,n}^{2(\beta-1)}|\nabla v_n|^2 dx \leq C_\varepsilon \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} v_n^{2^*} dx + (V_0+\varepsilon)\int_{\mathbb{R}^N} v_n^2\eta^2 v_{L,n}^{2(\beta-1)} dx$$

$$+2\varepsilon\int_{\mathbb{R}^N}\xi(x)\eta^2v_{L,n}^{2(\beta-1)}|\nabla v_n|^2dx+2C_\varepsilon\int_{\mathbb{R}^N}\xi(x)v_n^2v_{L,n}^{2(\beta-1)}|\nabla\eta|^2dx.$$

Using the immersion of $L^{2^*}(\mathbb{R}^N)\hookrightarrow L^2(\mathbb{R}^N)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\xi(x)\eta v_{L,n}^{2(\beta-1)}|\nabla v_n|^2dx &\leq C_\varepsilon\int_{\mathbb{R}^N}\eta^2v_{L,n}^{2(\beta-1)}v_n^{2^*}dx+C(V_0+\varepsilon)\int_{\mathbb{R}^N}v_n^{2^*}\eta^2v_{L,n}^{2(\beta-1)}dx\\ &+2\varepsilon\int_{\mathbb{R}^N}\xi(x)\eta^2v_{L,n}^{2(\beta-1)}|\nabla v_n|^2dx+2C_\varepsilon\int_{\mathbb{R}^N}\xi(x)v_n^2v_{L,n}^{2(\beta-1)}|\nabla\eta|^2dx\\ &=\leq\tilde{C}_\varepsilon\int_{\mathbb{R}^N}\eta^2v_{L,n}^{2(\beta-1)}v_n^{2^*}dx+2\varepsilon\int_{\mathbb{R}^N}\xi(x)\eta^2v_{L,n}^{2(\beta-1)}|\nabla v_n|^2dx\\ &+2C_\varepsilon\int_{\mathbb{R}^N}\xi(x)v_n^2v_{L,n}^{2(\beta-1)}|\nabla\eta|^2dx.\end{aligned}$$

Choosing $\varepsilon>0$ sufficiently small,

$$\int_{\mathbb{R}^N}\xi(x)\eta^2v_{L,n}^{2(\beta-1)}|\nabla v_n|^2dx\leq C\int_{\mathbb{R}^N}\eta^2v_{L,n}^{2(\beta-1)}v_n^{2^*}dx+C\int_{\mathbb{R}^N}\xi(x)v_n^2v_{L,n}^{2(\beta-1)}|\nabla\eta|^2dx. \tag{3.4.29}$$

Now, from Sobolev's embedding, by (3.4.29) and by (ξ_1) we have

$$\begin{aligned}\xi_0\|w_{L,n}\|_{L^{2^*}}^2 &\leq\int_{\mathbb{R}^N}\xi(x)\eta^2v_n^2v_{L,n}^{2(\beta-1)}dx\leq\int_{\mathbb{R}^N}\xi(x)\eta^2v_{L,n}^{2(\beta-1)}|\nabla v_n|^2dx\\ &\leq C\Big[\int_{\mathbb{R}^N}\eta^2v_{L,n}^{2(\beta-1)}v_n^{2^*}dx+\int_{\mathbb{R}^N}\xi(x)v_n^2v_{L,n}^{2(\beta-1)}|\nabla\eta|^2dx\Big].\end{aligned} \tag{3.4.30}$$

To complete the proof, follow the same steps from (1.2.7) to (1.2.8) as in the proof of Lemma 1.2.4 in Chapter 1. □

Proof of Theorem 3.2.1. As previously mentioned, for $R>0$ and $y\in\mathbb{R}^N$ the following sets were considered:

$$\begin{aligned}M &=\Big\{w=t_yu_0^+(\cdot-y)+v^-:\ \|w\|\leq R,\ t\geq 0,\ v^-\in E^-\Big\},\\ M_0 &=\Big\{w=t_yu_0^+(\cdot-y)+v^-:\ v^-\in E^-,\ \|w\|=R,\ t\geq 0 \text{ or } \|w\|\leq R,\ t=0\Big\}.\end{aligned}$$

Moreover, consider the set

$$N_\rho=\Big\{w\in E^+:\ \|w\|=\rho>0\Big\}.$$

Let us show that $\inf\limits_{N_\rho} I > \max\limits_{M_0} I$. By Lemma 3.4.1, we have $I|_{M_0} \leq 0$ and so $\max\limits_{M_0} I \leq 0$. Therefore, it is enough to verify that $\inf\limits_{N_\rho} I$.

From (3.4.19) we have $I(w) > 0$, since $w \in E^+$ with $\|w\| = \rho > 0$. It follows that $\inf\limits_{w \in N_\rho} I(w) > 0$ and thus $\inf\limits_{N_\rho} I > \max\limits_{M_0} I$.

By Linking Theorem 3.4.1 there exists a Cerami sequence (u_n) to the functional I at level $c > 0$. By Lemma 3.3.2, up to a subsequence, (u_n) is bounded. Therefore, $u_n \rightharpoonup u$ for some $u \in H^1(\mathbb{R}^N)$. By Lemma 3.4.2 $c < c_\infty$, and by item (i) of Lemma 3.4.6, up to a subsequence, $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$. Indeed, we have that $I(u) > 0$, from hypothesis (f_4) and due to the fact that u is a solution of (P_3), we have that

$$I(u) = I(u) - \frac{1}{2} I'(u)u = \int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u)u - F(x,u) \right) dx > 0.$$

Therefore, if item (2) is valid for item (c) we would have

$$c = I(u) + \sum_{j=1}^{m} I_\infty(u^j) \geq c_\infty$$

which is a contradiction by Lemma 3.4.2.

Thus, $u_n \to u$ and $I(u) = c > 0$ with $I'(u) = 0$ since I is a functional C^1. Hence, $u \in H^1(\mathbb{R}^N)$ is a weak solution of problem (P_3).

To show that u is nonnegative we can assume in the beginning $f(x,s) = 0$ for all $s \leq 0$, then $I'(u)u^- = 0$ and with the same calculations done in (1.2.9) we obtain $u^- \equiv 0$. Hence $u \geq 0$ in $\mathbb{R}^N$. By Lemma 3.4.7 we have that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{loc}(\mathbb{R}^N)$ for some $0 < \alpha < 1$. Then, Harnarck's inequality [2], as in (1.2.10), guarantees that $u > 0$ for all $u(x) > 0$ for all $x \in \mathbb{R}^N$. Therefore, u is a nontrivial and positive solution of (P_3). $\square$

Appendix A

Auxiliary Results

The following lemma, as seen in Stuart [32], deals with the behavior of any solution of problem (1.1.4).

Lemma A.1. *Consider $q \in C(\mathbb{R}^N)$ such that $\lim_{|x|\to\infty} q(x) = 0$. If $u \in C^2(\mathbb{R}^N)$ is a solution of the problem*

$$\begin{cases} -\Delta u - \lambda u = q(x)u, \quad \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{A.0.1}$$

with $\lambda < 0$, then

$$\lim_{|x|\to\infty} u(x)e^{\alpha|x|} = 0, \tag{A.0.2}$$

for all $\alpha \in (0, \sqrt{|\lambda|})$.

Proof. Consider $\alpha \in (0, \sqrt{|\lambda|})$ fixed and $\delta = |\delta| - \alpha^2$. Since $\lim_{|x|\to\infty} q(x) = 0$, then there exists $R > 0$ such that $|q(x)| \leq \delta$ for all $|x| \geq R$. Now, for $x \neq 0$, consider the function

$$w(x) = Me^{\alpha(|x|-R)},$$

where $M = \max\{|u(x)|;\ |x| = R\}$ and for $L > R$, let

$$\Omega(L) = \{x \in \mathbb{R}^N :\ R < |x| < L \text{ and } u(x) > w(x)\}.$$

Then, $\Omega(L)$ is open. Coupled with the fact that $u(x) > 0$ in $\Omega(L)$ and $x \in \Omega(L)$, we have that

$$\begin{aligned}\Delta(w-u)(x) &= \left(\alpha^2 - \frac{\alpha(N-1)}{|x|}\right) w(x) + (\lambda + q(x))u(x) \\ &\leq \alpha^2 w(x) + (-|\lambda| + \delta)u(x) \\ &= \alpha^2(w(x) - u(x)) < 0.\end{aligned}$$

By maximum principle, for all $x \in \Omega$, we have

$$w(x) - u(x) \geq \min\{(w-u)(x) : \ x \in \partial\Omega(L)\} = \min\{0, \min_{|x|=L}(w-u)(x)\}.$$

Since $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} w(x) = 0$, as $L \to \infty$, we obtain that

$$w(x) - u(x) \geq 0, \tag{A.0.3}$$

for all $|x| \geq R$. In the same way, taking for $-u$, we obtain

$$u(x) - w(x) \geq 0, \tag{A.0.4}$$

thus, from (A.0.3) and (A.0.4), we have that $|u(x)| \leq w(x)$, for all $|x| \geq R$ and the result follows. □

Remark A.1. *For our case, in Chapter 1, we consider* $\lambda = -\sqrt{1/\xi_\infty}$*, and in Chapter 2* $\lambda = -\sqrt{V_\infty/\xi_\infty}$*. And in both chaters we have* $q(x) = \dfrac{f(x,s)}{s}$.

The following definition and theorem are due to Ghoussoub-Preiss. It can be found in [14], Chapter IV, Definition 5, and Theorem 6.

Definition A.1. *A closed subspace* F *separates two points* z_0 *and* z_1 *in* X *if* z_0 *and* z_1 *belong to disjoint connected components in* X/F.

Theorem A.1 (Ghoussoub-Preiss)**.** *Let* X *be a Banach space and* $\Phi : X \to \mathbb{R}$ *a continuous, Gâteaux-differentiable function, such that* $\Phi' : X \to X$ *is continuous from the norm topology of* X *to weak* topology of* X^**. Take we two points* (z_0, z_1) *in* X *and consider the set* Γ *for all continuous paths from* z_0 *to* z_1*:*

$$\Gamma := \left\{c \in C^0([0,1], X) : c(0) = z_0, \ c(1) = z_1\right\}.$$

Define a number γ by:

$$\gamma := \inf_{c\in\Gamma} \max_{0\le t\le 1} \Phi(c(t)).$$

Assume there is a closed subset F of X such that:

$$F \cap \Phi_\gamma \quad \textit{separates } z_0 \textit{ and } z_1$$

with $\Phi_\gamma := \{x \in X : \Phi(x) \ge \gamma\}$. Then, there is a sequence x_n in X such that

$$\delta(x_n, F) \to 0, \quad \Phi(x_n) \to \gamma \quad \textit{and} \quad (1+\|x_n\|)\|F'(x_n)\|_* \longrightarrow 0.$$

Remark A.2. *In Chapters 1 and 2, we consider $X = E^\tau$, $\Phi = I_\infty\Big|_{E^\tau}$, $\gamma = c^\tau$ and $F = \mathcal{P}$.*

The next lemma presents an important inequality given by Alves, Carriõ and Medeiros in [3].

Lemma A.2. *Let $F \in C^2(\mathbb{R}, \mathbb{R}^+)$ be a convex function and even such that $F(0) = 0$ and $f(s) = F'(s) \ge 0$ for all $s \in [0,\infty)$. Then, for all $u, v \ge 0$,*

$$\Big|F(u-v) - F(u) - F(v)\Big| \le 2\Big(f(u)v + f(v)u\Big). \tag{A.0.5}$$

Let ∂B_1 be the boundary of B_1, where B_1 is the open ball of radius 1 in a finite dimensional space spanned by the functions $u_0^+(\cdot - y)$, $\phi_1, \cdots, \phi_k$.

The lemma to be proved next contributes to the proof of Lemma 3.4.1, which guarantees us the first geometry of the Linking Theorem.

Lemma A.3. *The limit*

$$\lim_{R\to\infty} \int_{\mathbb{R}^N} \left(\frac{a(x)}{2} - \frac{F(x,Ru)}{(Ru)^2} \right) u^2 dx = 0,$$

is uniformly for $u \in \partial B_1$.

Proof. For each $R = n \in \mathbb{R}$, consider $J_n : \partial B_1 \to \mathbb{R}$ the function given by $J_n(u) = \int_{\mathbb{R}^N} \left(\frac{a(x)}{2} - \frac{F(x,nu)}{(nu)^2} \right) u^2 dx$. The continuity of the function F shows that J_n is a continuous functional for each fixed n. Hypothesis (f_4) and equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_E$ show that there exists a constant $C > 0$ such that

$$0 \le J_n(u) = \int_{\mathbb{R}^N} \left(\frac{a(x)}{2} - \frac{F(x,nu)}{(nu)^2} \right) u^2 dx \le a_0 \|u\|_E^2 \le C$$

for all $u \in \partial B_1$, where $a_0 = \sup_{\mathbb{R}^N} a(x)$. Hence the continuity of the functional J_n in the compact set ∂B_1 ensures that, for each fixed n, the functional J_n assumes its maximum at $u_n \in \partial B_1$. Consider (u_n) the sequence of these maxima. Since $\|u_n\| = 1$ for each n and the space spanned by the functions $u_0^+(\cdot - y)$, $\phi_1, \cdots, \phi_k$ is finite dimensional, there exists $\overline{u} \in \partial B_1$ such that, up to a subsequence,

$$u_n \to \overline{u} \text{ as } n \to \infty \tag{A.0.6}$$

strongly in the norm $\|\cdot\|$. For all $u \in \partial B_1$ and for each n $0 \leq J_n(u) \leq J)n(u_n)$, that is,

$$0 \leq \int_{\mathbb{R}^N} \left(\frac{a(x)}{2} \frac{F(x, nu_n)}{(nu_n)^2} \right) u_n^2 dx \tag{A.0.7}$$

for all u and for each n. Taking the limit $n \to \infty$, firstly, note that

$$u_n(x) \to \overline{u}(x) \text{ a. e. in } \mathbb{R}^N.$$

Thus, if, $\overline{u}(x) \neq 0$, it follows that $|n\overline{u}(x)| \to \infty$ if $n \to \infty$. Hence hypothesis (f_4) yields

$$\left(\frac{a(x)}{2} - \frac{F(x, nu_n(x))}{(nu_n(x))^2} \right) u_n^2(x) \to 0 \tag{A.0.8}$$

if $n \to \infty$. If $\overline{u}(x) = 0$, que also have (A.0.8). By the strongly convergence in (A.0.6), there exist a function $\overline{h} \in L^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$0 \leq \left(\frac{a(x)}{2} - \frac{F(x, nu_n(x))}{(nu_n(x))^2} \right) u_n^2(x) \leq a_0 |u_n^2(x)| \leq a_0 \overline{h}(x) \in L^1(\mathbb{R}^N). \tag{A.0.9}$$

Finally, by (A.0.8) and (A.0.9), Lebesgue dominated convergence theorem ensures that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \left(\frac{a(x)}{2} - \frac{F(x, nu_n(x))}{(nu_n(x))^2} \right) u_n^2(x) dx = 0.$$

Therefore, taking $n \to \infty$ in (A.0.7), we have

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \left(\frac{a(x)}{2} - \frac{F(x, nu)}{(nu)^2} \right) u^2(x) dx = 0.$$

uniformly for $u \in \partial B_1$. □

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