Numerical Semigroups and Music

Maria Bras-Amorós*
Department of Computer Science and Mathematics
Universitat Rovira i Virgili
Tarragona, Catalonia, Spain

Abstract

We will elaborate on the algebraic structure of the sequence of harmonics when combined with equal temperaments. Fractals and the golden ratio appear surprisingly on the way. The sequence of physical harmonics is an increasingly enumerable submonoid of $(\mathbb{R}^+, +)$ whose pairs of consecutive terms get arbitrarily close as they grow. These properties suggest the definition of a new mathematical object which we denote a tempered monoid. Mapping the elements of the tempered monoid of physical harmonics from $\mathbb{R}$ to $\mathbb{N}$ may be considered tantamount to defining equal temperaments. The number of equal parts of the octave in an equal temperament corresponds to the multiplicity of the related numerical semigroup. Analyzing the sequence of musical harmonics we will derive two important properties that tempered monoids may have: that of being product-compatible and that of being fractal. We will demonstrate that, up to normalization, there is only one product-compatible tempered monoid, which is the logarithmic monoid, and there is only one nonbisectional fractal monoid which is generated by the golden ratio. The example of half-closed cylindrical pipes imposes a third property to the sequence of musical harmonics, the so-called odd-filterability property. We will prove that the maximum number of equal divisions of the octave such that the discretizations of the golden fractal monoid and the logarithmic monoid coincide, and such that the discretization is odd-filterable is 12. This is nothing else but the number of equal divisions of the octave in classical Western music.

References

[1] M. Bras-Amorós, *Increasingly Enumerable Submonoids of R: Music Theory as a Unifying Theme*, The American Mathematical Monthly, Mathematical Association of America, 127(1), (2020), 33-44.

[2] M. Bras-Amorós, *Tempered Monoids of Real Numbers, the Golden Fractal Monoid, and the Well-Tempered Harmonic Semigroup*, Semigroup Forum, Springer, 99(2), (2019), 496-516.

*Partially supported by the Spanish government under grant TIN2016-80250-R and by the Catalan government under grant 2014 SGR 537, e-mail: maria.bras@urv.cat