The Abel's Theorem via monodromy groups

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January 19, 2021

Abstract

We present a topological proof of Abel's Theorem that establishes the impossibility of quintic equations through of radicals. Usually the demonstration this theorem is done through of Galois' classic theory via field extensions and Galois groups. However, this work we follow the proof effectued by the Professor V.I. Arnold [1], that show topological character of Abel's Theorem. We study the connection between groups theory and complex functions theory, mainly exploring Riemann surfaces, algebraic functions and monodromy.
Monodromy are a topological concept. Given a polynomial

$$p_z(w) = w^n + a_{n-1}w^{n-1} + \cdots + a_1 w + z$$

in $\mathbb{C}[w]$, we consider the algebraic function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \{w \in \mathbb{C};\ p_z(w) = 0\}$. This function is multivalued and determines a Riemann surface. Closed paths around their branch points induce permutations in their sheets, generating what we call of monodromy groups.

Theorem 0.1 *If a complex functions $h(z)$ is representable by radicals, its monodromy group is soluble.*

The solubility this groups determines when the function $f(z)$ and, consequently the roots of $p_z(w)$, are expressed by radicals.

Theorem 0.2 (Abel's Theorem) *For $n \geq 5$ the general algebraic equation of degree n*

$$w^n + a_{n-1}w^{n-1} + \cdots + a_1 w + z = 0$$

is not solvable by radicals.

The work is the result of a scientific initiation project carried out at PIVIC-UFV.

*Partially supported by XXX, e-mail: `ellen.peixoto@ufv.br`

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