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Combinatorial Applications of the Compactness Theorem

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Abstract This work discusses a formalization in Isabelle/HOL of the compactness theorem for propositional logic. The formalization is based on the model existence theorem approach. Further, the paper presents applications of this theorem to formalize combinatorial theorems over countable structures: the De Bruijn-Erdös Graph coloring theorem for countable graphs, König Lemma, and set- and graph-theoretical versions of Hall's Theorem for countable families of sets and graphs.

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1 Introduction

The propositional compactness theorem is of principal importance for any meta-logical development because of the myriad of applications in logic and combinatorics. Typically, this theorem is presented as a simple consequence of the completeness theorem. However, constructive proofs based on Henkin's-style model existence theorem provide the mathematical machinery to design proofs of combinatorial properties in areas such as set theory and graph theory through the construction of logical interpretations and models.

This paper discusses a formalization in Isabelle/HOL of the compactness theorem for propositional logic according to Smullyan's approach given in the third chapter of his influential textbook on mathematical logic [48], and based on Henkin's model existence theorem. The formalization follows the impeccable presentation in Fitting's textbook [12]. In addition, we present three applications of this formalization of the compactness theorem, detailing how models and interpretations are built for proving landmark theorems such as the de Bruijn-Erdös k-colouring theorem, and Hall's theorem, both them for the countable infinite case and König's lemma.

The proofs described in this paper add to the meta-logic available in Isabelle/HOL, a proof of the compactness theorem for propositional logic for the countable case. The formalization is adapted from Serrano's thesis [41]. The formalization of Hall's theorem for countable sets is only briefly discussed since it was reported in detail in [42]. Also, a graph-theoretical version of Hall's theorem for countable graphs was presented in [43]. The formalizations of the de Bruijn-Erdös k-coloring theorem for countable graphs and of König Lemma by the model construction technique and application of the compactness theorem are unpublished.

1.1 Organization

Initially, Section 2 discusses the formalization of the compactness theorem; afterward, Section 3 details the three applications mentioned above; finally, after a brief discussion on related work in Section 4, Section 5 concludes. The paper includes links to all discussed crucial points of the development Compactness Theory ↗.

2 Compactness Theorem

For the preamble of this section, we present a selection of comments extracted from the excellent discussion on the compactness theorem by Paseau and Leek [33] and on Gödel's mathematical work in [11].

2.1 Proofs of the Compactness Theorem

The compactness theorem is a fundamental property for the model theory of (classical) propositional and first-order logic. Besides algebra and combinatorics, the compactness theorem also has implications in topology and foundations of mathematics. In general, it implies that any compact logic extending first-order logic cannot express the notions of finitude or infinitude (of a model). Also, it implies that any first-order theory of arithmetic satisfied by the standard model has a non-standard model. It also can be used to prove the Order-Extension Principle: any partial order may be extended to a linear order.

Also, according to Paseau and Leek [33], the first proof of a compactness theorem for countable versions of propositional and first-order logic was published by Gödel: (Satz X in [19]), who also proved the general version for arbitrary languages applying transfinite recursion in [20]. Mal'cev [31] also proved the compactness theorem for propositional logic, again using the full strength of the Axiom of Choice. His proof relies on transfinite induction.

The first explicit, published proofs of a compactness theorem from completeness, which is the one presented in several contemporary textbooks in logic, were given independently by Henkin and Robinson for the first-order functional calculus [25] and [37], and for the simple theory of types [26]. Indeed, Paseau and Leek [33] adequately remark that "proofs of compactness via completeness are not satisfactory because they are based on properties incidental to the semantic property of interest. Such proofs conclude compactness, a semantic property, from a property of the logic relating its syntax to its semantics." The authors also cited Keisler's connections between ultraproducts technique and compactness and essential and unessential applications of such method. In particular, Keisler's viewpoint about such proofs of compactness [29]: "Unlike the completeness theorem, the compactness theorem does not involve the notion of a formal deduction, and so it is desirable to prove it directly without using that notion." They finish with the following commentary: "From the perspective of a model theorist who sees talk of syntax as a heuristic for the study of certain relations between structures that happen to have syntactic correlates, proving compactness via completeness is tantamount to heresy ([34], page 53)."

Our formalization uses Smullyan's approach in Fitting's textbook [12], which is a direct proof of the Compactness Theorem for propositional logic without using the Completeness Theorem.

2.2 Formalization of the Compactness Theorem

The formalization was first given in [41] and follows Smullyan's proof as presented in Fitting's famous textbook [12]. König's Lemma can be used to prove the compactness theorem for propositional logic in the countable case. Consider a set of formulas S. It is enough to order the countable set of sentences in S, say as $F_1, F_2, \ldots$, and to build a countable tree with successful evaluations of the propositional letters validating the subsets of formulas $\{F_1\}, \{F_1, F_2\}$, and so on. The infinite branch gives an interpretation of S. There are other proofs of this theorem also given as part of a collection of classical propositional formalizations aiming at applications and teaching logic. For instance, Michaelis and Nipkow developed a formalization, part of IsaFOL, based on an enumeration of all formulas and saturation accordingly to Enderton's textbook proof ([10]) [32]. The formalization in this paper is based on the so-called "model existence theorem". It shows first Hintikka's Lemma: Hintikka sets of propositional formulas are satisfiable. Such a set is defined as a set of propositional formulas that does neither include both A and $\neg A$ for a propositional letter nor $\bot$, or $\neg\top$. Additionally, if it includes $\neg\neg F$, F is included; if it includes a conjunctive formula, which is an α formula, then the two components of the conjunction are included; and finally, if it includes a disjunction, which is a β formula, at least one of the components of the disjunction is included (specified as hintikkaP).

The satisfiability of any Hintikka set is proved by assuming a valuation that maps all propositional letters in the set to true and all other propositional letters to false (map IH in the specification). The second step consists in proving that families of sets of propositional formulas, which hold the so-called "propositional consistency property" (definition consistenceP), consist of satisfiable sets. The last is indeed the model existence theorem (Theo_ExistenceModels). The model existence theorem compiles the essence of completeness: a family of sets of propositional formulas that holds the propositional consistency property can be extended, preserving this property to a set collection that is closed for subsets and satisfies the finite character property. The finite character property states that a set belongs to the family if and only if each of its finite subsets belongs to the family. With the model existence theorem in hands, the compactness theorem (Compacteness_Theorem) is obtained easily: given a set of propositional formulas S such that all its finite subsets are satisfiable, one considers the family $\mathcal{C}$ of subsets in S such that all their finite subsets are satisfiable. S belongs to the family $\mathcal{C}$ and the latter holds the propositional consistence property.

The main theorems are given below.

Theorem 1 (Model Existence (Theorem 3.6.2 in [12])) *If $\mathcal{C}$ is a propositional consistency property, and $S \in \mathcal{C}$, then S is satisfiable.*

Theorem 2 (Compactness Theorem (Theorem 3.6.3 in [12])) *Let S be a set of propositional formulas. If every finite subset of S is satisfiable, so is S.*

We present the most important definitions and proofs used in the formalization.

The language of propositional formulas is specified through the following datatype.

```
Datatype 'b formula =
    ⊥
  | ⊤
  | atom 'b
  | negation     'b formula                    (¬.(-) [110] 110)
  | conjuntion   'b formula    'b formula      (infixl ∧.  109)
  | disjunction  'b formula    'b formula      (infixl ∨.  108)
  | implication  'b formula    'b formula      (infixl →.  100)
```

To evaluate the *truth-value* of propositional formulas over an interpretation we specify the operator *t-v-evaluation*.

```
Primrec t-v-evaluation :: ('b ⇒ truth-value) ⇒ 'b formula ⇒ truth-value
where
   t-v-evaluation I ⊥ = Ffalse
|  t-v-evaluation I ⊤ = Ttrue
|  t-v-evaluation I (Atom P) = I P
|  t-v-evaluation I (¬. F) = (v-negation (t-v-evaluation I F))
|  t-v-evaluation I (F ∧. G) = (v-conjunction (t-v-evaluation I F) (t-v-evaluation I G))
|  t-v-evaluation I (F ∨. G) = (v-disjunction (t-v-evaluation I F) (t-v-evaluation I G))
|  t-v-evaluation I (F →. G) = (v-implication (t-v-evaluation I F) (t-v-evaluation I G))
```

The operator *t-v-evaluation* uses the definitions below.

Definition *v-negation* :: *truth-value* ⇒ *truth-value* **where**
v-negation x ≡ (*if x* = *Ttrue then Ffalse else Ttrue*)

Definition *v-conjunction* :: *truth-value* ⇒ *truth-value* ⇒ *truth-value* **where**
v-conjunction x y ≡ (*if x* = *Ffalse then Ffalse else y*)

Definition *v-disjunction* :: *truth-value* ⇒ *truth-value* ⇒ *truth-value* **where**
v-disjunction x y ≡ (*if x* = *Ttrue then Ttrue else y*)

Definition *v-implication* :: *truth-value* ⇒ *truth-value* ⇒ *truth-value* **where**
v-implication x y ≡ (*if x* = *Ffalse then Ttrue else y*)

The notion of satisfiability is specified through the existence of *models*.

Definition *model* :: (′*b* ⇒ *truth-value*) ⇒ ′*b formula set* ⇒ *bool* (- *model* - [*80*,*80*] *80*)
where *I model S* ≡ (∀ *F* ∈ *S*. *t-v-evaluation I F* = *Ttrue*)

Definition *satisfiable* :: ′*b formula set* ⇒ *bool* **where** *satisfiable S* ≡ (∃ *v*. *v model S*)

The notion of compactness is specified using the Isabelle specification for finite sets and a specification for countable sets.
The next lemma, from Isabelle, formalized the fact that a *finite* set A is finite if and only if there exists a surjective function f from I_n onto A, where $I_n = \{m \in \mathbb{N} \mid m < n\}$, for some $n \in \mathbb{N}$.

Lemma *finite A* ⟷ (∃ *n f*. *A* = *f* ‘ {*i*::*nat*. *i* < *n*})

We specify countable sets using the notion of *enumeration*, i.e., the existence of a surjective function with domain ℕ, given below.

Definition *enumeration* :: (*nat* ⇒′*b*) ⇒ *bool* **where** *enumeration f* = (∀ *y*.∃ *n*. *y* = (*f n*))

Hintikka's lemma is formalized as the following corollary.

Corollary Hintikka-satisfiable :
assumes hintikkaP H
shows satisfiable H

The formalization of Hintikka's lemma is by induction on the structure of the formulas in a Hintikka set H by applying the technical theorem hintikkaP_model_aux . This theorem applies a series of lemmas to address the evaluation of all possible cases of formulas in H. Indeed, considering the Boolean evaluation IH that maps all propositional letters in H to true and all other letters to false, the most interesting cases of the inductive proof are those related to implicational formulas in H and the negation of arbitrary formulas in H. These cases are not straightforward since implicational and negation formulas are not considered in the definition of Hintikka sets. For an implicational formula, say $F_1 \longrightarrow F_2$, it is necessary to prove that if it belongs to H, its evaluation by IH is true. Also, whenever $\neg(F_1 \longrightarrow F_2)$ belongs to H its evaluation is false. The proof is obtained by relating such formulas, respectively, with β and α formulas (case P6). The second interesting case is the one related to arbitrary negations. In this case, it is proved that if $\neg F$ belongs to H, its evaluation by IH is true, and in the case that $\neg\neg F$ belongs to H, its evaluation by IH is also true (Case P7) .

As previously mentioned, both these theorems require the definition of propositional consistency. Let $\mathcal{C}$ be a collection of sets of propositional formulas. We call $\mathcal{C}$ a propositional consistency property if it meets the conditions for each $S \in \mathcal{C}$, given in the definition *consistenceP*, as specified below. In this definition, *FormulaAlpha* and *FormulaBeta* correspond respectively to conjunctive (α) and disjunctive (β) propositional formulas as defined in [12].

Definition *consistenceP* :: *'b formula set set* ⇒ *bool* **where**
consistenceP $\mathcal{C}$ =
$(\forall S.\ S \in \mathcal{C} \longrightarrow (\forall P.\ \neg\ (atom\ P \in S \land (\neg.atom\ P\ ) \in S)) \land$
$\bot\ \notin S \land (\neg.\top) \notin S \land$
$(\forall F.\ (\neg.\neg.F) \in S \longrightarrow S \cup \{F\} \in\ \mathcal{C}) \land$
$(\forall F.\ ((FormulaAlpha\ F) \land F{\in}S) \longrightarrow (\ S \cup \{Comp1\ F,\ Comp2\ F\}) \in \mathcal{C}) \land$
$(\forall F.\ ((FormulaBeta\ F) \land F{\in}S) \longrightarrow (\ S \cup \{Comp1\ F\} \in \mathcal{C}) \lor$
$(\ S \cup \{Comp2\ F\} \in \mathcal{C})))$

The specifications of the model existence and the compactness theorems are given below.

Theorem *TheoremExistenceModels*:
assumes *h1*: $\exists\, g.\ enumeration\ (g{::}\ nat \Rightarrow\ 'b\ formula)$
and *h2*: *consistenceP* $\mathcal{C}$
and *h3*: $(S{::}\ 'b\ formula\ set) \in \mathcal{C}$
shows *satisfiable* S

The formalization of the existence model theorem requires a series of properties. In the theory T5Closedness, closedness properties of the propositional consistency property are proved. Such properties leave to conclude that if $\mathcal{C}$ holds the property, then $(\mathcal{C}^{+})$, which is the closure o $\mathcal{C}$ under subsets, does too (Closed_ConsistenceP).

The finite character property previously mentioned is specified below.

Definition *finite-character* :: *'a set set* ⇒ *bool* **where**
finite-character $\mathcal{C} = (\forall S.\ S \in \mathcal{C} = (\forall S'.\ finite\ S' \longrightarrow S' \subseteq S \longrightarrow S' \in \mathcal{C}))$

In order to formalize the finite character property for subset closed families of sets of propositional formulas that satisfy the propositional consistency property, it is necessary to show a series of properties for extensions of the families of sets. It is proved that a finite character property of families of sets of propositional formulas implies subset closedness (finite_character_closed).

Finally, the theorem that states that subset closed propositional consistency properties can be extended to satisfy the finite character property is specified below.

Theorem *cfinite-consistenceP*:
assumes *hip1*: *consistenceP* $\mathcal{C}$ **and** *hip2*: *subset-closed* $\mathcal{C}$
shows *consistenceP* $(\mathcal{C})$

The proof is by induction on the structure of propositional formulas based on the analysis of cases for the possible different types of formula in the sets of the collection of sets that hold the propositional consistency property (lemmas cond_characterP1 to cond_characterP5 in the theory T6Finiteness).

An interesting corollary of the model existence theorem is that each subset of a set of formulas $\mathcal{C}$ that satisfies the propositional consistency property, which is built over a countable set of propositional letters, is satisfiable. This corollary requires proving that the set of formulas built over a countable set of propositional letters is countable. The last result is formalized in the theory FormulaEnumeration.

The following auxiliary lemma, *ConsistenceCompactness*, is required to apply *TheoremExistenceModels* to obtain the compactness theorem. This lemma states the general fact that the collection $\mathcal{C}$ of all sets of propositional formulas such that all their subsets are satisfiable is a propositional consistency property *ConsistenceP*. The collection of such sets is defined below.

$$\mathcal{C} = \{W \mid \forall A\, (A \subseteq W \wedge A \text{ } \mathit{finite} \rightarrow A \text{ } \mathit{satisfiable})\}$$

Lemma *ConsistenceCompactness*:
shows $\mathit{consistenceP}\{W{::}'b\ \mathit{formula\ set}.\ \forall\, A.\ (A \subseteq\ W\ \wedge\ \mathit{finite}\ A) \longrightarrow\ \ \mathit{satisfiable}\ A\}$

With this lemma in hand, since any countable set of formulas that belongs to $\mathcal{C}$ is satisfiable as a consequence of the theorem of the existence of models, one obtains the formalization of the Compactness Theorem. Indeed, given a set S of formulas, all whose finite subsets of formulas are satisfiable, it is only necessary to prove it belongs to $\mathcal{C}$.

Theorem *Compactness-Theorem*:
assumes $\exists\, g.\ \mathit{enumeration}\ (g{::}\ \mathit{nat} \Rightarrow\ 'b\ \mathit{formula})$
and $\forall\, A.\ (A \subseteq (S{::}\ 'b\ \mathit{formula\ set}) \wedge \mathit{finite}\ A) \longrightarrow \mathit{satisfiable}\ A$
shows $\mathit{satisfiable}\ S$

So, the key technical part of the formalization of the Compactness Theorem from the theorem of existence of models is the above lemma *ConsistenceCompactness*. This lemma is formalized unfolding the definition *consistenceP* through a series of auxiliary lemmas *consistenceP_Prop1* to *consistenceP_Prop6* for each of the conditions of the definition. For instance, the auxiliary lemma consistenceP_Prop5 states the required satisfiability property for the case of formulas α:

$$\begin{gathered}\forall F.(F \in W \wedge\ \ \mathit{FormulaAlpha}\ F)\\ \forall A.(A \subseteq W \wedge A\ \mathit{finite}) \longrightarrow A\ \mathit{satisfiable}\\ \longrightarrow\\ \forall A.(A \subseteq W\{\mathit{Comp1}\ F, \mathit{Comp2}\ F\} \wedge A\ \mathit{finite}) \longrightarrow A\ \mathit{satisfiable}\end{gathered}$$

This lemma is formalized by applying another auxiliary lemma (such as for the case of the property of formulas β in the definition of *consistenceP*) satisfiableUnion2 that states the more simple property below.

$$\begin{gathered}\mathit{FormulaAlfa}\ F \wedge\ \ \mathit{satisfiable}\ (A \cup \{F\}) \longrightarrow\\ \mathit{satisfiable}\ (A \cup \{\mathit{Comp1}\ F,\ \ \mathit{Comp2}\ F\})\end{gathered}$$

Another application of the model existence theorem for propositional logic, formalized in Serrano's Thesis ([41]), is Craig's interpolation theorem. In addition, and always following Fitting's textbook elegant presentation ([12]), the Thesis includes formalizations of a variety of results for first-order logic as the model existence theorem, the Löwenheim-Skolem theorem, obtained as an application of such theorem, and the completeness of natural deduction.

3 Applications of the Compactness Theorem

This section discusses the formalization of three important applications of the Compactness Theorem; namely, the de Bruijn-Erdös k-coloring theorem, König Lemma, and Hall's theorem.

3.1 De Bruijn-Erdös Graph Couloring Theorem

The theory k_coloring formalizes the de Bruijn-Erdös k-coloring theorem for countable graphs. The proof follows as a consequence of the compactness theorem for propositional logic. We start with the definition of digraphs.

Definition 1 (Digraph) A digraph $G = (V, E)$ is a sorted pair, where V is a set of vertices and $E \subseteq V \times V$ is a binary irreflexive relation called edges.
A pair of vertices $u, v \in V$ are called to be adjacent if $(u, v) \in E$ or $(v, u) \in E$

The essential components of a digraph are specified below.

synonym *'v digraph* = *('v set)* × *(('v* × *'v) set)*

abbreviation *vert* :: *'v digraph* ⇒ *'v set* (*V*[-] [*80*] *80*) **where**
V[*G*] ≡ *fst G*

abbreviation *edge* :: *'v digraph* ⇒ (*'v* × *'v*) *set* (*E*[-] [*80*] *80*) **where**
E[*G*] ≡ *snd G*

definition *is-graph* :: *'v digraph* ⇒ *bool* **where**
is-graph G ≡ ∀ *u v*. (*u*,*v*) ∈ *E*[*G*] ⟶ *u* ∈ *V*[*G*] ∧ *v* ∈ *V*[*G*] ∧ *u* ≠ *v*

Notice how the irreflexibility of the edge relation is obtained from the definition above, excluding self-loops.

Definition 2 (Induced digraph) Let $G = (V, E)$ be a digraph, and $S \subseteq V$. The digraph $G_S = (S, E \cap (S \times S))$ is the subgraph of G induced by S.

The above definition of the subgraph induced by a subset of vertices is specified below, followed by a lemma on its well-definedness.

definition *is-induced-subgraph* :: *'v digraph* ⇒ *'v digraph* ⇒ *bool* **where**
is-induced-subgraph H G ≡
(*V*[*H*] ⊆ *V*[*G*]) ∧ *E*[*H*] = *E*[*G*] ∩ ((*V*[*H*]) × (*V*[*H*]))

lemma
assumes *is-graph G* **and** *is-induced-subgraph H G*
shows *is-graph H*

A digraph is k-colorable, for $k \in \mathbb{N}$, if its vertices can be mapped to the set $\{1, \ldots, k\}$ avoiding mapping adjunct vertices to the same natural.

Definition 3 (k-Coloring) Let k be a positive integer. A k-coloring of a digraph $G = (V, E)$ is a function $c : V \to [k] = \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for all $(u, v) \in E$.
A graph G is said to be k-colorable if there is a k-coloring of G.

The above definition is specified in Isabell/HOL below.

definition *coloring* :: $('v \Rightarrow nat) \Rightarrow nat \Rightarrow 'v\ digraph \Rightarrow bool$ **where**
coloring c k G $\equiv$
$(\forall u.\ u \in V[G] \longrightarrow c(u) \leq k) \wedge (\forall u\ v.(u,v) \in E[G] \longrightarrow c(u) \neq c(v))$

definition *colorable* :: $'v\ digraph \Rightarrow nat \Rightarrow bool$ **where**
colorable G k $\equiv \exists c.$ *coloring c k G*

3.1.1 Informal proof of the de Bruijn-Erdös Theorem

The de Bruijn-Erdös theorem is stated below. The "pen-and-paper" proof applies the compactness theorem.

Theorem 3 (de Bruijn-Erdös) *Let $G = (V, E)$ be a countable graph and k be a positive integer. If for all finite $S \subseteq V$, G_S is k-colorable, then G is k-colorable.*

Proof Let us fix a set of propositional symbols,

$$\mathcal{P} = \{C_{u,i} \mid u \in V, 1 \leq i \leq k\}$$

where $C_{u,i}$ is interpreted as "the vertex u has color i". We define three propositional formula sets:

1. $\mathcal{F} = \{C_{u,1} \vee C_{u,2} \vee \cdots \vee C_{u,k} \mid u \in V\}$;
2. $\mathcal{G} = \{\neg(C_{u,i} \wedge C_{u,j}) \mid u \in V, 1 \leq i, j \leq k, i \neq j\}$;
3. $\mathcal{H} = \{\neg(C_{u,i} \wedge C_{v,i}) \mid u, v \in V, (u, v) \in E, 1 \leq i \leq k\}$.

The previous sets express the following properties regarding G and k, respectively:

1. each vertex corresponds to at least a color;
2. no vertex is associated with more than one color; and,
3. adjacent vertices are associated with different colors.

Let $\mathcal{T} = \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$. The compactness theorem is applied to prove that $\mathcal{T}$ is satisfiable.
Let S be a finite subset of $\mathcal{T}$ and $V_0 = \{u_1, \ldots, u_n\}$ be the set of all vertices u such that $C_{u,i}$ for some i, occurs in some formula in S.
Let $G_{V_0} = (V_0, E_0)$ be the subgraph of G induced by V_0.
Let $c : V_0 \to [k]$ be a k-coloring of G_{V_0}.
We define the interpretation $v : \mathcal{P} \to \{\mathsf{T}, \mathsf{F}\}$ as

$$v(C_{u,i}) = \begin{cases} \mathsf{T} \text{ if } u \in V_0 \text{ and } c(u) = i, \\ \mathsf{F} \text{ otherwise.} \end{cases}$$

We have $v(F) = \mathsf{T}$ for all $F \in S$ since c is a k-coloring and $F \in \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$. Thus, $\mathcal{T}$ is finitely satisfiable; hence, by the compactness theorem, it is satisfiable.
Let $I : \mathcal{P} \to \{\mathsf{T}, \mathsf{F}\}$ be an interpretation that satisfies $\mathcal{T}$. We establish a correspondence $c : V \to [k]$ defined as $c(u) = i$ if and only if $I(C_{u,i}) = \mathsf{T}$.
Therefore, by the definition of $\mathcal{T}$ and since $I(F) = \mathsf{T}$ for all $F \in \mathcal{T}$, one has that c is a k-coloring of $G = (V, E)$. Indeed, since $\mathcal{F}$ and $\mathcal{G}$ are satisfiable, to each vertex $v \in V$ corresponds exactly a color in $[k]$, thus, c is a function. Finally, since $\mathcal{H}$ is satisfiable, adjacent vertices have different colors. □

3.1.2 Formalization of the Bruijn-Erdös k-coloring Theorem

This subsection discusses details of the formalization of the k-coloring theorem following the proof of the previous Theorem 3.

The following Isabelle definitions specify the set related to the set of propositional symbols $\mathcal{P}$, and the sets of formulas $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and $\mathcal{T}$.

```
primrec atomic-disjunctions :: 'v ⇒ nat ⇒ ('v × nat)formula where
 atomic-disjunctions v 0 = atom (v, 0)
| atomic-disjunctions v (Suc k) =
  (atom (v, Suc k)) ∨. (atomic-disjunctions v k)

definition 𝓕 :: 'v digraph ⇒ nat ⇒ (('v × nat)formula) set where
  𝓕 G k ≡ (⋃v∈V[G]. {atomic-disjunctions v k})

definition 𝓖 :: 'v digraph ⇒ nat ⇒ ('v × nat)formula set where
  𝓖 G k ≡ {¬.(atom (v, i) ∧. atom(v,j))
                    | v i j. (v∈V[G]) ∧ (0≤i ∧ 0≤j ∧ i≤k ∧ j≤k ∧ i≠j)}

definition 𝓗 :: 'v digraph ⇒ nat ⇒ ('v × nat)formula set where
  𝓗 G k ≡ {¬.(atom (u, i) ∧. atom(v,i))
                    |u v i . (u∈V[G] ∧ v∈V[G] ∧ (u,v)∈E[G]) ∧ (0≤i ∧ i≤k)}

definition 𝓣 :: 'v digraph ⇒ nat ⇒ ('v × nat)formula set where
  𝓣 G k ≡ (𝓕 G k) ∪ (𝓖 G k) ∪ (𝓗 G k)
```

The set of vertices occurring in a formula and a set S of formulas denoted as V_0 in the proof of Theorem 3 are defined below.

```
primrec vertices-formula :: ('v × nat)formula ⇒ 'v set where
  vertices-formula FF = {}
| vertices-formula TT = {}
| vertices-formula (atom P) = {fst P}
| vertices-formula (¬. F) = vertices-formula F
| vertices-formula (F ∧. G) = vertices-formula F ∪ vertices-formula G
| vertices-formula (F ∨. G) = vertices-formula F ∪ vertices-formula G
| vertices-formula (F →.G) = vertices-formula F ∪ vertices-formula G

definition vertices-set-formulas :: ('v × nat)formula set ⇒ 'v set where
vertices-set-formulas S = (⋃F∈ S. vertices-formula F)
```

Several auxiliary lemmas are formalized that relate a subset of formulas S, the sets of propositional symbols in $\mathcal{P}$, representing vertices and their possible colors, and the set $\mathcal{T}$. For instance, the next lemma specifies that the subset of vertices occurring in any subset of formulas S of $\mathcal{T}$ is a subset of the set of vertices of G.

```
lemma vertices-subset-formulas:
  assumes S ⊆ (𝓣 G k)
  shows vertices-set-formulas S ⊆ V[G]
```

The next definition specifies the subgraph given by a set of vertices V of a graph. Let S be a finite subset of $\mathcal{T}$, and $V_0 = \{u_1, \ldots, u_n\}$ be the set of vertices u such that $C_{u,i}$, for some i, occurs in some formula in S. The lemma *finite-subraph* below, formalizes the fact that the subgraph of G induced by V_0, $G_{V_0} = (V_0, E_0)$, also is a finite graph.

definition *subgraph-aux* :: *'v digraph* ⇒ *'v set* ⇒*'v digraph* **where**
subgraph-aux G V ≡ (*V*, *E*[*G*] ∩ (*V* × *V*))

lemma *finite-subgraph*:
assumes *is-graph G* **and** *S* ⊆ (𝒯 *G k*) **and** *finite S*
shows *finite-graph* (*subgraph-aux G* (*vertices-set-formulas S*))

The theorem *coloring-satisfiable* states that a coloring of G_{V_0} enables the construction of a model of S. The formalization uses the function *graph-interpretation*, showing that it gives a k-coloring of the subgraph induced by the vertices in the set of formulas S.

fun *graph-interpretation* :: *'v digraph* ⇒ (*'v* ⇒ *nat*) ⇒ ((*'v* × *nat*) ⇒ *v-truth*) **where**
graph-interpretation G f = (λ(*v*,*i*).(*if v* ∈ *V*[*G*] ∧ *f*(*v*) = *i then Ttrue else Ffalse*))

theorem *coloring-satisfiable*:
assumes *is-graph G* **and** *S* ⊆ (𝒯 *G k*) **and**
coloring f k (*subgraph-aux G* (*vertices-set-formulas S*))
shows *satisfiable S*

An interpretation $I : \mathcal{P} \to \{\mathsf{T}, \mathsf{F}\}$ that holds $\mathcal{T}$ establishes a coloring $c : V \to [k]$ given by $c(u) = i$ if and only if $I(C_{u,i}) = \mathsf{T}$.

fun *graph-coloring* :: ((*'v* × *nat*) ⇒ *v-truth*) ⇒ *nat* ⇒ (*'v* ⇒ *nat*)
where
graph-coloring I k = (λ*v*.(*THE i*. (*t-v-evaluation I* (*atom* (*v*,*i*)) = *Ttrue*) ∧ *0*≤*i* ∧ *i*≤*k*))

The following lemma establishes the existence of the previous coloring function when I is a model of $\mathcal{T}$. It is formalized using a series of auxiliary lemmas that state the existence and unicity of the color associated with each vertex regarding any interpretation I model of $\mathcal{T}$.

lemma *coloring-function*:
assumes *u* ∈ *V*[*G*] **and** *I model* (𝒯 *G k*)
shows ∃!*i*. (*t-v-evaluation I* (*atom* (*u*,*i*)) = *Ttrue* ∧ *0*≤*i* ∧ *i*≤*k*) ∧
graph-coloring I k u = *i*

The following lemma establishes that if the set of formulas $\mathcal{T}$ for a graph G and a natural k is satisfied, then G is k-colourable. The proof assumes a model I for $\mathcal{T}$ by the satisfiability hypothesis. Applying the previous lemma `coloring-function`, the function *graph-coloring* will give a unique color i, $0 \leq i \leq k$ for each vertex u in the graph. This happens since the evaluation of the formulas F and G for the model I will guarantee the existence of a unique atom $c_{u,i}$ that is true. Finally, by applying another auxiliary lemma called distinct-color (that states *graph-coloring* gives different colors for adjacent vertices), since I is also a model for $\mathcal{H}$, one guarantees that the evaluation of I for adjacent vertices u and v is such that the unique atoms $c_{u,i}$ and $c_{v,j}$ evaluated as true are such that $i \neq j$.

theorem *satisfiable-coloring*:
  assumes *is-graph G* **and** *satisfiable* $(\mathcal{T}\ G\ k)$
  shows *colorable G k*

Finally, the de Bruijn-Erdös theorem (Theorem 3) is specified below. Its formalization applies theorem *coloring_satisfiable* to prove that any finite subgraph H of G induces a finite subset S of formulas of $\mathcal{T}$ that is satisfiable; therefore, by the compactness theorem one has that $\mathcal{T}$ is satisfiable, concluding by application of theorem *satisfiable_coloring* that G is colorable.

theorem *deBruijn-Erdos-coloring*:
  assumes *is-graph* $(G{::}('vertices{::}\ countable)\ set \times ('vertices \times 'vertices)\ set)$
  and $\forall H.$ (*is-induced-subgraph H G* $\wedge$ *finite-graph H* $\longrightarrow$ *colorable H k*)
shows *colorable G k*

3.2 Formalization of König's Lemma

Using the Compactness Theorem for propositional logic, we formalize König's Lemma for countable trees:

Any infinite countable finitely branching tree has an infinite path.

The steps of the formal proof follow the approach sketched in [5].
In the following, we provide the definitions and properties regarding trees needed to formalize König's Lemma. Also, the specification of each one is presented.

Definition 4 (Basic Relations) Let R be a binary relation on a set A.

i) R is irreflexive if and only if

$$\forall x \in A, (x, x) \notin R.$$

ii) R is transitive if and only if

$$\forall x \in A\, \forall y \in A\, \forall z \in A\, ((x, y) \in R \wedge (y, z) \in R \longrightarrow (x, z) \in R).$$

iii) R is total if and only if

$$\forall x \in A\, \forall y \in A\, (x \neq y \longrightarrow (x, y) \in R \vee (y, x) \in R).$$

iv) An element $a \in A$ is a minimum element of A if and only if

$$\forall x \in A\, (x \neq a \longrightarrow (a, x) \in R).$$

v) Consider $a \in A$. We define the set of predecessors, $Pr(a)$, of a as

$$Pr(a) = \{x \in A \mid (x, a) \in R\}.$$

Specifications of these relations in Isabelle are presented below.

type-synonym *'a rel* = (*'a* × *'a*) *set*

definition *irreflexive-on* :: *'a set* ⇒ *'a rel* ⇒ *bool*
where *irreflexive-on A r* ≡ (∀ *x*∈*A*. (*x*, *x*) ∉ *r*)

definition *transitive-on* :: *'a set* ⇒ *'a rel* ⇒ *bool*
where *transitive-on A r* ≡
(∀ *x*∈*A*. ∀ *y*∈*A*. ∀ *z*∈*A*. (*x*, *y*) ∈ *r* ∧ (*y*, *z*) ∈ *r* ⟶ (*x*, *z*) ∈ *r*)

definition *total-on* :: *'a set* ⇒ *'a rel* ⇒ *bool*
where *total-on A r* ≡ (∀ *x*∈*A*. ∀ *y*∈*A*. *x* ≠ *y* ⟶ (*x*, *y*) ∈ *r* ∨ (*y*, *x*) ∈ *r*)

definition *minimum* :: *'a set* ⇒ *'a* ⇒*'a rel* ⇒ *bool*
where *minimum A a r* ≡ (*a*∈*A* ∧ (∀ *x*∈*A*. *x* ≠ *a* ⟶ (*a*,*x*) ∈ *r*))

definition *predecessors* :: *'a set* ⇒*'a* ⇒*'a rel* ⇒ *'a set*
where *predecessors A a r* ≡ {*x*∈*A*.(*x*, *a*) ∈ *r*}

Definition 5 (Height, Level, and Immediate successors) Let R be a binary relation on A such that for all $a \in A$, $Pr(a)$ is finite.

1. For all $a \in A$, we define the height of a, $Hg(a)$, as the number of its predecessors:

$$Hg(a) = |Pr(a)|.$$

2. For each integer number $n \geq 0$, the n-th level of R is the set of elements of A, whose height is n; that is,

$$Lv(n) = \{a \in A \mid Hg(a) = n\}.$$

3. For each $a \in A$, the set of immediate successors of a, $Suc(a)$, is defined as

$$Suc(a) = \{y \in A \mid (a, y) \in R \wedge Hg(y) = Hg(a) + 1\}.$$

The above definitions are specified in Isabelle below.

definition *height* :: *'a set* ⇒*'a* ⇒ *'a rel* ⇒ *nat*
where *height A a r* ≡ *card* (*predecessors A a r*)

definition *level* :: *'a set* ⇒ *'a rel* ⇒ *nat* ⇒*'a set*
where *level A r n* ≡ {*x*∈*A*. *height A x r* = *n*}

definition *imm-successors* :: *'a set* ⇒ *'a* ⇒ *'a rel* ⇒ *'a set*
where *imm-successors A a r* ≡
{*x*∈*A*. (*a*,*x*)∈ *r* ∧ *height A x r* = (*height A a r*)+*1*}

The uniqueness of the minimum in an SPO is given by the following lemma and formalized in Isabelle as the subsequent specification.

Definition 6 (Strict Partial Order (SPO) and Linear Order) Let R be a binary relation on A.

1. The pair (A, R) is called a strict partial order if and only if R is irreflexive and transitive.
2. (A, R) is a linear order if and only if it is a strict partial order and R is a total relation.

The class of strict partial orders is specified below.

definition *strict-part-order* :: *'a set ⇒ 'a rel ⇒ bool*
where *strict-part-order A r ≡ irreflexive-on A r ∧ transitive-on A r*

Lemma 1 (Uniqueness of Minimum in SPO) *Let (A, R) be a strict partial order. If A has a minimum element, then such an element is unique.*

lemma *spo-uniqueness-min*:
assumes *strict-part-order A r* **and** *minimum A a r* **and** *minimum A b r*
shows *a=b*

Next, the definitions and the specifications of (finite and infinite) trees are given.

Definition 7 (Tree) Let $R \neq \emptyset$ be a binary relation on a set A. A pair $T = (A, R)$ is a tree if and only if:

1. (A, R) is a strict partial order.
2. A has a minimum element, which we call the root of T.
3. For all $a \in A$, the set $Pr(a)$ is finite and the restriction of R to $Pr(a)$ is total.

The elements of A are called nodes of T.

definition *tree* :: *'a set ⇒ 'a rel ⇒ bool*
where *tree A r ≡*
r ⊆ A × A ∧ r≠{} ∧ (strict-part-order A r) ∧ (∃ a. minimum A a r) ∧
(∀ a∈A. finite (predecessors A a r) ∧ (total-on (predecessors A a r) r))

Definition 8 (Finite Tree) A tree $T = (A, R)$ is finite if and only if the set of nodes A is finite; otherwise, T is infinite.

definition *finite-tree*:: *'a set ⇒ 'a rel ⇒ bool*
where
finite-tree A r ≡ tree A r ∧ finite A

abbreviation *infinite-tree*:: *'a set ⇒ 'a rel ⇒ bool*
where
infinite-tree A r ≡ tree A r ∧ ¬ finite A

The definition and specification of a finitely branching tree are given below.

Definition 9 (Finitely Branching Tree) Let $T = (A, R)$ be a tree. T is finitely branching if and only if for each $a \in A$, the set $Suc(a)$ is finite.

definition *finitely-branching* :: *'a set ⇒ 'a rel ⇒ bool*
where *finitely-branching A r ≡ (∀ x∈A. finite (imm-successors A x r))*

Next, finite and infinite paths in trees are defined.

Definition 10 (Path) Let $T = (A, R)$ be a tree. A set of nodes $B \subseteq A$ is a path of T if and only if (B, R) is a linear order and B is maximal (regarding the subset relation). If B is finite, it is called a finite path; otherwise, B is an infinite path.

Notice that a finitely branching tree having an infinite path has an infinite branch. Specifications of sub-linear orders and trees are given below.

definition *sub-linear-order* :: $'a\ set \Rightarrow\ 'a\ set \Rightarrow\ 'a\ rel \Rightarrow bool$
where *sub-linear-order* $B\ A\ r\ \equiv\ B \subseteq A \wedge (strict\text{-}part\text{-}order\ A\ r) \wedge (total\text{-}on\ B\ r)$

definition *path* :: $'a\ set \Rightarrow\ 'a\ set \Rightarrow\ 'a\ rel \Rightarrow bool$
where *path* $B\ A\ r\ \equiv$
$(sub\text{-}linear\text{-}order\ B\ A\ r) \wedge$
$(\forall\, C.\ B \subseteq C \wedge sub\text{-}linear\text{-}order\ C\ A\ r \longrightarrow B = C)$

definition *finite-path* :: $'a\ set \Rightarrow\ 'a\ set \Rightarrow\ 'a\ rel \Rightarrow bool$
where *finite-path* $B\ A\ r \equiv\ path\ B\ A\ r \wedge finite\ B$

definition *infinite-path* :: $'a\ set \Rightarrow\ 'a\ set \Rightarrow\ 'a\ rel \Rightarrow bool$
where *infinite-path* $B\ A\ r \equiv\ path\ B\ A\ r \wedge\ \neg\ finite\ B$

The following lemmas (Lemmas 2, 3, 4 and 5) are crucial and form the basis to prove König's lemma.

Lemma 2 (Finiteness of levels in Finitely Branching Trees) *Let $T = (A, R)$ be a tree. The following statements are equivalent:*

1. *T is finitely branching.*
2. *For all $n \geq 0$, the set $Lv(n)$ is finite.*

Although we formalized the equivalence between the propositions in Lemma 2, only the essential condition to verify König's lemma (namely, (1) $\Rightarrow$ (2)) is shown below as the lemma *finite-level*.

lemma *finite-level*:
assumes *tree A r* **and** *finitely-branching A r*
shows *finite (level A r n)*

Lemmas 3 and 4 guarantee the existence of a path from any node to the root of a tree and the non-emptiness of each level in a finitely branching infinite tree, respectively. They are formalized as lemmas *path-to-node* and *all-levels-non-empty*.

Lemma 3 (Root Reachability in Trees) *Let $T = (A, R)$ be a tree. If $n \geq 0$ and $x \in Lv(n+1)$ then for all k, $0 \leq k \leq n$, there is y_k such that $(y_k, x) \in R$ and $y_k \in Lv(k)$.*

lemma *path-to-node*:
assumes *tree A r* **and** $x \in (level\ A\ r\ (n{+}1))$
shows $\forall\, k.(0 \leq k \wedge k \leq n) \longrightarrow (\exists\, y.\ (y,x) \in r \wedge y \in (level\ A\ r\ k))$

Lemma 4 (Non-emptiness of Levels) *Consider $T = (A, R)$ a finitely branching infinite tree. Thus, for all $n \geq 0, Lv(n) \neq \emptyset$.*

lemma *all-levels-non-empty*:
  assumes *infinite-tree A r* **and** *finitely-branching A r*
  shows $\forall n.$ *level A r n* $\neq \{\}$

Lemma 5 states that the elements in the same set of predecessors are at distinct levels.

Lemma 5 (Emptyness of Level Intersection) *Let $T = (A, R)$ be a tree. Suppose that $(x, z) \in R$, $(y, z) \in R$, and $x \neq y$. If $x \in Lv(n)$ and $y \in Lv(m)$ then $Lv(n) \cap Lv(m) = \emptyset$.*

lemma *emptyness-inter-diff-levels*:
  assumes *tree A r* **and** $(x,z) \in r$ **and** $(y,z) \in r$
  and $x \neq y$ **and** $x \in$ (*level A r n*) **and** $y \in$ (*level A r m*)
shows *level A r n* $\cap$ *level A r m* $= \{\}$

3.2.1 Informal proof of König's Lemma

In this section, we discuss the "pen-and-paper" proof of König's Lemma (Theorem 4) obtained as a consequence of the Compactness Theorem.

Theorem 4 (König Lemma) *Every finitely branching infinite (countable) tree has an infinite branch.*

Proof Let $T = (A, R)$ be a finitely branching infinite countable tree. Consider the following set of propositional symbols indexed by the vertices of T:

$$\mathcal{P} = \{B_u \mid u \in A\}.$$

From the set $\mathcal{P}$, one can define a set of formulas $\mathcal{T}$, such that if $\mathcal{T}$ is satisfiable then for any interpretation I, which is model of $\mathcal{T}$, the set of vertices $\mathcal{B}$ is an infinite path of T:

$$\mathcal{B} = \{u \in A \mid I(B_u) = \mathsf{T}\}$$

$\mathcal{T}$ is given by the union of the following three sets of propositional formulas.

1. For each $n \in \mathbb{N}$,
$$\mathcal{F} = \{ \bigvee_{u \in Lv(n)} B_u \mid n \in \mathbb{N}\},$$
where $\bigvee_{u \in Lv(n)} B_u$ is the disjunction of the atomic formulas corresponding to the elements of the level $Lv(n)$, which is a finite set by the Lemma 2.
2. $\mathcal{G} = \{B_u \longrightarrow B_v \mid u, v \in A, (v, u) \in R\}$,
3. $\mathcal{H} = \{\neg(B_u \wedge B_v) \mid u, v \in Lv(n), u \neq v, n \in \mathbb{N}\}$.

The previous sets allow the characterization of an infinite path in a tree. Indeed, if a set B of vertices of T satisfies such sets, then for any $n \in \mathbb{N}$, there is at least one vertex of T in the level n which belongs to B; every predecessor of any element of B belongs to B, and B has only a vertex in the level n.

Now, we show that the set $\mathcal{T} = \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$ is satisfiable by applying the Compactness Theorem.

Let S be a finite subset of $\mathcal{T}$. Since S is finite, the set

$$N = \{u \in A \mid B_u \text{ occurs in some formula of } S\}$$

is also finite; consequently, the set of the heights of vertices from N has a maximum element h. Additionally, one has that $Lv(h+1) \neq \emptyset$ since T is infinite and finitely branching (Lemma 4).

Consider $t \in Lv(h+1)$ and define the interpretation $I : \mathcal{P} \to \{\mathsf{T}, \mathsf{F}\}$ as,

$$I(B_u) = \begin{cases} \mathsf{T} \text{ , if } (u,t) \in R \\ \mathsf{F} \text{ , otherwise.} \end{cases}$$

Notice that, $I(J) = \mathsf{T}$ for every formula $J \in S$. In fact:

1. If $J \in \mathcal{F}$ then $J = \bigvee_{u \in Lv(n)} B_u$, which corresponds to the disjunction of the atomic formulas associated with the vertices of the level n, for some $n \leq h$. Since the vertices that occur in J have height $n < h+1$, there exists $u \in Lv(n)$ such that $(u,t) \in R$ (Lemmas 4, 3). Consequently, $I(B_u) = \mathsf{T}$ and $I(J) = \mathsf{T}$.
2. If $J \in \mathcal{G}$ then there exist $u, w \in A$ such that $J = B_u \longrightarrow B_w$ and $(w,u) \in R$. If $I(J) = \mathsf{F}$ then $I(B_u) = \mathsf{T}$ and $I(B_w) = \mathsf{F}$. Consequently, $(u,t) \in R$ and $(w,t) \notin R$ which is impossible considering that $(w,u) \in R$ and R is transitive relation. Thus, $I(J) = \mathsf{T}$.
3. If $J \in \mathcal{H}$ then there exist $u, w \in Lv(n)$, for some $n \geq 0$, such that $u \neq w$ and $J = \neg(B_u \wedge B_w)$. Since u and w belong to the same level, one has that $(u,t) \notin R$ or $(w,t) \notin R$ (Lemma 5). Consequently, $I(B_u) = \mathsf{F}$ or $I(B_w) = \mathsf{F}$, and $I(J) = \mathsf{T}$.

Therefore, $\mathcal{T}$ is finitely satisfiable and, as a consequence of the Compactness Theorem, $\mathcal{T}$ is satisfiable.

Let $I : \mathcal{P} \to \{\mathsf{T}, \mathsf{F}\}$ be a model for $\mathcal{T}$. Then,

$$\mathcal{B} = \{u \in A \mid I(B_u) = \mathsf{T}\}$$

is an infinite path of T:

Since I satisfies $\mathcal{F}$ and $\mathcal{H}$, one has that, for each level n, the intersection $\mathcal{B} \cap Lv(n)$ is a singleton vertex. In the following, we show that $(\mathcal{B}, R)$ is a total and maximal relation and $\mathcal{B}$ is infinite.

(a) $(\mathcal{B}, R)$ is a total relation: consider $u, w \in \mathcal{B}$ such that $u \neq w$. Assume that $Hg(u) < Hg(w)$. Let $n = Hg(u)$ and x be the predecessor of w at level n. Then, $B_w \longrightarrow B_x \in \mathcal{G}$, hence $I(B_w \longrightarrow B_x) = \mathsf{T}$. Since $I(B_w) = \mathsf{T}$, $I(B_x) = \mathsf{T}$. Therefore, $x \in \mathcal{B}$ and, since $u, x \in Lv(n)$, one concludes that $u = x$. Thus, $(u,w) \in R$. The case $Hg(w) < Hg(u)$ is proved analogously. Therefore, one concludes that $(\mathcal{B}, R)$ is total.

(b) $(\mathcal{B}, R)$ is maximal: we prove that if $\mathcal{B} \subseteq \mathcal{B}'$ and $(\mathcal{B}', R)$ is total then $\mathcal{B}' \subseteq \mathcal{B}$. Let $x \in \mathcal{B}'$, $n = Hg(x)$ and u be a vertex that belongs to the intersection of $\mathcal{B}$ and the vertices at level $Lv(n)$. Since $u \in \mathcal{B}'$ and $(\mathcal{B}', R)$ is total, if $u \neq x$, then either $(u,x) \in R$ or $(x,u) \in R$, which is impossible since in a strict order, comparable elements with a finite number of predecessors are at different levels. Therefore, $x = u$, which implies $\mathcal{B}' \subseteq \mathcal{B}$.

(c) $\mathcal{B}$ is infinite: since I satisfies $\mathcal{F}$, it is enough to prove that for all $n \geq 0$, $Lv(n) \neq \emptyset$. This implies that there exists u such that $u \in \mathcal{B} \cap Lv(n)$, therefore, $\mathcal{B}$ is infinite. Suppose there exists n such that $Lv(n) = \emptyset$. This implies that for all $m > n$, $Lv(m) = \emptyset$ too. Consequently, since T is finitely branching, it would be finite. To conclude, one also needs to consider that $Lv(n) \cap Lv(m) = \emptyset$ for all $n \neq m$, and therefore $\bigcup_{n\in\mathbb{N}} \mathcal{B} \cap Lv(n)$ is infinite. □

3.2.2 Formalization of König's Lemma

In this subsection, we explain the crucial steps in the formalization of this proof.

The Isabelle formalizations of the sets $\mathcal{F}, \mathcal{G}, \mathcal{H}$, and $\mathcal{T}$ use the recursive constructor *disjuction-nodes* of disjuction of atoms below.

primrec *disjunction-nodes* :: *'a list* ⇒ *'a formula* **where**
disjunction-nodes [] = *FF*
| *disjunction-nodes* (*v*#*D*) = (*atom v*) ∨. (*disjunction-nodes D*)

The specification of $\mathcal{F}, \mathcal{G}, \mathcal{H}$, and $\mathcal{T}$ are given below. Notice that $\mathcal{H}$ is built as the union of all the sets $\mathcal{H}n$ of negations of formulas of the form $(B_u \wedge B_v)$ for nodes at the same level (n).

definition $\mathcal{F}$ :: *'a set* ⇒ *'a rel* ⇒ (*'a formula*) *set* **where**
$\mathcal{F}$ *A r* ≡ (⋃*n*. {*disjunction-nodes*(*set-to-list* (*level A r n*))})

definition $\mathcal{G}$ :: *'a set* ⇒ *'a rel* ⇒ (*'a formula*) *set* **where**
$\mathcal{G}$ *A r* ≡ {(*atom u*) →. (*atom v*) |*u v*. *u*∈*A* ∧ *v*∈*A* ∧ (*v*,*u*)∈ *r*}

definition $\mathcal{H}n$ :: *'a set* ⇒ *'a rel* ⇒ *nat* ⇒ (*'a formula*) *set* **where**
$\mathcal{H}n$ *A r n* ≡ {¬.((*atom u*) ∧. (*atom v*))
|*u v* . *u*∈(*level A r n*) ∧ *v*∈(*level A r n*) ∧ *u*≠*v* }
definition $\mathcal{H}$ :: *'a set* ⇒ *'a rel* ⇒ (*'a formula*) *set* **where**
$\mathcal{H}$ *A r* ≡ ⋃*n*. $\mathcal{H}n$ *A r n*

definition $\mathcal{T}$ :: *'a set* ⇒ *'a rel* ⇒ (*'a formula*) *set* **where**
$\mathcal{T}$ *A r* ≡ ($\mathcal{F}$ *A r*) ∪ ($\mathcal{G}$ *A r*) ∪ ($\mathcal{H}$ *A r*)

The definition *maximum-height* specifies the maximum height of nodes in a set of formulas. The specification uses *nodes-set-formulas*, which specify the union of the nodes in a finite set of formulas.

definition *maximum-height* :: *'v set* ⇒*'v rel* ⇒ *'v formula set* ⇒ *nat* **where**
maximum-height A r S = *Max* (⋃*x*∈*nodes-set-formulas S*. {*height A x r*})

Let S be a set of formulas, and h be the maximum height of the set of nodes occurring in the formulas of S. The next function returns some node at level $Lv(h+1)$.

fun *node-sig-level-max* :: *'v set* ⇒ *'v rel* ⇒ *'v formula set* ⇒ *'v*
where *node-sig-level-max A r S* =
(*SOME u*. *u* ∈ (*level A r* ((*maximum-height A r S*)+*1*)))

The next lemma shows that any finite subset S of $\mathcal{T}$ is satisfiable:

lemma *satisfiable-path*:
assumes *infinite-tree A r*

and *finitely-branching A r* **and** $S \subseteq (\mathcal{T}\ A\ r)$
and *finite S*
shows *satisfiable S*

The formalization of previous lemma builds a very simple model in the following manner: we select a node, say u, in the tree at level $h+1$, where h is the maximum level of the set of nodes occurring in the formulas of S. The truth value of all nodes (atomic formulas) except the predecessors of u, which have truth value true, is false. This is built through a simple interpretation *path-interpretation*:

fun *path-interpretation* :: $'v$ *set* $\Rightarrow 'v$ *rel* $\Rightarrow\ 'v \Rightarrow ('v\ \Rightarrow$ *v-truth*)
where *path-interpretation A r u* =
(λv. (*if* $(v,u) \in r$ *then Ttrue else Ffalse*))

In this way, using lemmas (4, 3 and 5, resp.) *all-levels-non-empty*, *path-to-node* and *emptyness-inter-diff-levels* one concludes that such interpretation holds in S.

So, $\mathcal{T}$ is finitely satisfiable, and so is satisfiable by the compactness theorem.

The next definition of the set of nodes $\mathcal{B}$, which are true in an interpretation I, gives the construction of the infinite path used in the proof of König's lemma (Theorem 4).

definition $\mathcal{B}$ :: $'a$ *set* $\Rightarrow ('a\ \Rightarrow$ *v-truth*) $\Rightarrow\ 'a$ *set* **where**
$\mathcal{B}\ A\ I \equiv \{u|u.\ u \in A \wedge$ *t-v-evaluation I* (*atom u*) = *Ttrue*$\}$

The properties of $\mathcal{B}$ are described by the following lemmas.

The next lemma states that if $\mathcal{B}$ is built from an infinite finitely branching tree and I is an interpretation that satisfies $\mathcal{F}$, then $\mathcal{B}$ has at least a node in each level of the tree. The proof is by induction on the number of nodes at any level of the tree.

lemma *intersection-branch-set-nodes-at-level*:
assumes *infinite-tree A r* **and** *finitely-branching A r*
and I: $\forall F \in (\mathcal{F}\ A\ r)$. *t-v-evaluation I F* = *Ttrue*
shows $\forall n.\ \exists x.\ x \in$ *level A r n* $\wedge\ x \in (\mathcal{B}\ A\ I)$

The following lemma states that for each tree and interpretation I that satisfies $\mathcal{H}$, the set $\mathcal{B}$ has at most one node with a truth value true at each level of the tree. The formalization follows by contradiction.

lemma *intersection-branch-emptyness-below-height*:
assumes I: $\forall F \in (\mathcal{H}\ A\ r)$. *t-v-evaluation I F* = *Ttrue*
and $x \in (\mathcal{B}\ A\ I)$ **and** $y \in (\mathcal{B}\ A\ I)$ **and** $x \neq y$ **and** n: $x \in$ *level A r n*
and m: $y \in$ *level A r m*
shows $n \neq m$

In addition, by applying the previous two lemmas, one formalizes that if the tree is an infinite finitely branching tree, and the interpretation I is a model of $\mathcal{F}$ and $\mathcal{H}$, the set $\mathcal{B}$ has only a node at each level of the tree:

lemma *intersection-branch-level*:
assumes *infinite-tree A r* **and** *finitely-branching A r*
and I: $\forall F \in (\mathcal{F}\ A\ r) \cup (\mathcal{H}\ A\ r)$. *t-v-evaluation I F* = *Ttrue*
shows $\forall n.\ \exists u.\ (\mathcal{B}\ A\ I) \cap$ *level A r n* $= \{u\}$

The next lemma states that for any tree and interpretation I that satisfies $\mathcal{G}$, all predecessors of a node in the set $\mathcal{B}$ belong to $\mathcal{B}$.

lemma *predecessor-in-branch*:
  assumes I: $\forall F \in (\mathcal{G}\ A\ r)$. *t-v-evaluation* $I\ F = Ttrue$
  and $y \in (\mathcal{B}\ A\ I)$ **and** $(x,y) \in r$ **and** $x \in A$ **and** $y \in A$
shows $x \in (\mathcal{B}\ A\ I)$

By applying all previous lemmas, one formalizes that for an infinite finitely branching three and an interpretation I of $\mathcal{T}$, the set $\mathcal{B}$ is a path:

lemma *is-path*:
  assumes *infinite-tree* $A\ r$ **and** *finitely-branching* $A\ r$
  and I: $\forall F \in (\mathcal{T}\ A\ r)$. *t-v-evaluation* $I\ F = Ttrue$
shows *path* $(\mathcal{B}\ A\ I)\ A\ r$

To close the sequence of auxiliary lemmas on the set $\mathcal{B}$, the lemma below shows that this set is infinite whenever it is built from a model of $\mathcal{F}$, for an infinite finitely branching tree.

lemma *infinite-path*:
  assumes *infinite-tree* $A\ r$ **and** *finitely-branching* $A\ r$
  and I: $\forall F \in (\mathcal{F}\ A\ r)$. *t-v-evaluation* $I\ F = Ttrue$
shows *infinite* $(\mathcal{B}\ A\ I)$

Finally, the formalization of König's lemma (Theorem 4) is obtained by applying the lemma *satisfiable-path* that proves that any finite subset S of an infinite finitely branching tree satisfies $\mathcal{T}$, then, applying the compactness theorem to conclude that the tree also satisfies $\mathcal{T}$. In the sequence, assuming that I is a model of $\mathcal{T}$ for the tree, and building the set $\mathcal{B}$ and by applying the auxiliary lemmas one obtains that the tree has an infinite path.

theorem *Koenig-Lemma*:
  assumes *infinite-tree* $(A::'nodes\ set)\ r$
  and *enumeration* $(g::\ nat \Rightarrow 'nodes)$
  and *finitely-branching* $A\ r$
shows $\exists B$. *infinite-path* $B\ A\ r$

3.3 Formalizations of Hall's Theorem

This subsection briefly discusses the application of the compactness theorem in the Isabelle/HOL formalizations of Hall's theorem for countable sets and graphs described in detail in [42] and [43].

The Hall's Theorem, also called "marriage theorem," proved primarily by Philip Hall [21], provides necessary and sufficient conditions to choose a distinct representative for each set in a finite family of finite sets $\mathcal{A}$ over elements in a set S.

Given S, an arbitrary set, and $\{S_i\}_{i \in I}$ a collection of not necessarily distinct subsets of S with indices in the set I, a function $f : I \rightarrow \bigcup_{i \in I} S_i$ is a *system of distinct representatives* (SDR) for $\{S_i\}_{i \in I}$ if:

1. for all $i \in I$, $f(i) \in S_i$, and;
2. f is an injective function.

From the definition of an SDR, one can state Hall's Theorem for sets as follows.

Theorem 5 (Hall's Theorem | finite case) *Consider an arbitrary set S and a positive integer n. A finite collection $\{S_1, S_2, \ldots, S_n\}$ of finite subsets of S has an SDR if and only if the so-called* marriage condition *(M) below is satisfied.*

For every $1 \leq k \leq n$ and an arbitrary set of k distinct indices $1 \leq i_1, \ldots, i_k \leq n$, one has that $|S_{i_1} \cup \ldots \cup S_{i_k}| \geq k$. *(M)*

Hall's theorem is a landmark result that is equivalent to several other significant theorems in combinatory and graph theory (cf. [4], [5], [36]), namely: Menger's theorem (1929), König's minimax theorem (1931), König–Egerváry theorem (1931), Dilworth's theorem (1950), Max Flow-Min Cut theorem (Ford-Fulkerson algorithm), among others. Consequently, a complete formalization of Hall's Theorem gives rise to formally proving those equivalent results. Considering Isabelle/HOL theorem prover, Jiang and Nipkow [27] formalized Hall's theorem by implementing both Halmos and Vaughan's [22] and Rado's [35] techniques.

More general versions of Hall's Theorem were established [35]. In particular, Hall's Theorem, as enunciated in Theorem 6, holds for a countable collection of finite subsets $\{S_i\}_{i \in I}$ of a set S.

Theorem 6 (Hall's Theorem | countable case) *Let S be an arbitrary set and I an enumerable set of indices of finite subsets of S. The family $\{S_i\}_{i \in I}$ has an SDR if and only if the condition (M^*) below holds.*

For every finite subset of indices $J \subseteq I$, one has that $|\bigcup_{j \in J} S_j| \geq |J|$. *($M^*$)*

This theorem is formalized in the theory Hall_Theorem.

As another application of the Compactness Theorem for propositional logic, Serrano et al. formalized Theorem 6 in Isabelle/HOL. Such a development combines the formalization of the Compactness Theorem as in [41], described in Section 2, and of Jiang and Nipkow's for the finite case of Hall's Theorem. The formal proof of the countable case of Hall's Theorem in Isabelle/HOL was recently published in [42] and gives rise to provide mechanisms to formally establish general versions of results that are equivalent to Theorem 6.

For instance, besides the set-theoretical version of Hall's Theorem for countable families of sets 6, another well-known version is Hall's Theorem for graphs was also formalized.

Theorem 7 (Hall's Theorem graph version | countable case) *Let $G = \langle X, Y, E\rangle$ be a digraph such that the set of vertices $X \cup Y$ is countable, the set of edges holds $E \subseteq X \times Y$, and for each vertex $x \in X$, the set of neighborhoods of x $N(x) = \{y \mid (x, y) \in E\}$ is finite. Then G contains a perfect matching covering the set of vertices X if and only if ($M^\dagger$) below holds.*

For every finite subset of vertices $J \subseteq X$, one has that $|\bigcup_{j \in J} N(j)| \geq |J|$. *($M^\dagger$)*

This theorem is formalized in the theory Hall_Th_Graph_Theo.

Previously, we cited some combinatorial theorems equivalent to Hall's theorem. Depending on the result, the proof of such an equivalence can be adapted to either the set-theoretical or graph-theoretical versions. For example, König–Egerváry theorem states that the minimum cover in a finite bipartite graph has the same cardinality as a maximum matching. Thus, if we assume Hall's theorem for finite graphs, one possible way to infer König–Egerváry theorem will consist of building a reduction from the latter to the former. Considering the nature of König–Egerváry theorem, it is clear that the graph-theoretical version of Hall's theorem is more appropriate than the set version to establish the equivalence between these theorems.

In the preprint [43], by applying authors' development in [42], the infinite graph-theoretical version of Hall's theorem was formalized in Isabelle/HOL. The mechanization focuses on maintaining specifications and proofs as closely as possible to textbooks since our primary objective was to increase mathematicians' interest in using interactive proof assistants. Although this, the specification also includes a concise and more automatized proof using locales, which can be seen at the end of the theory Hall_Th_Graph_Theo.

Interestingly, other combinatorial well-known results equivalent to Hall's theorem in the finite case are not straightforwardly equivalent in the infinite case; for instance, the infinite version of König-Egerváry theorem that as reported in [2] cannot be inferred from the compactness theorem. Thus, another of the aspects we are interested in is to explore if possible restricted variations of infinite versions of König-Egerváry theorem can be obtained as a consequence of the Compactness Theorem.

4 Related Work

4.1 Formalizations of the compactness theorem

As mentioned in Subsection 2.2, another proof in Isabelle/HOL of the compactness theorem is given by Michaelis and Nipkow as part of IsaFOL [32]. In general, formalizations of the compactness theorem belong to collections of developments for propositional and first-order logic, as is the case of IsaFOL (e.g., [14], [40], [13]). In particular, Michaelis and Nipkow formalized proof systems for propositional logic, such as sequent calculus, natural deduction, and Hilbert systems; they added to ISAFOL proofs of soundness, completeness, cut-elimination, interpolation, and the model existence theorem. However, the formalization of compactness follows a different approach, as the one of this paper, which is based on an enumeration of all formulas and saturation [32].

Among a variety of solid formal developments in classical logic, which provide elements for formalizations of theorems as those treated in this paper, one can include Shankar's pioneering formalizations of the Church-Rosser and the first Gödel incompleteness theorem in the Boyer-Moore theorem prover [44]. Also, it deserves to mention Harrison's formalization in HOL Light of important results such as the compactness and the Löwenheim-Skolem theorems [23]. Harrison's formalization of the propositional compactness theorem is also for the countable case and applies Zorn's lemma to extend satisfiable sets to maximal satisfiable sets of propositional formulas (as in the proof given in Enderton's textbook [10]).

4.2 Formalizations of König's Lemma, and de Bruijn-Erdös and Hall's Theorems

Nowadays, proof assistants include robust proof engines and elaborated mathematical libraries that make the formalization of König's lemma an easy routine exercise. An earlier proof of König's lemma in the Boyer-Moore theorem prover is reported by Kaufmann in [28]. The formalization uses the NQTHM extension of this prover to deal with quantification by applying the technique of (event) Skolemization. The existence of an infinite path in a finitely branching infinite tree is obtained using the predicate "for any node with infinite descendants there exists a successor with infinite descendants." Bancerek developed another earlier formalization of this theorem in Mizar [3]. The formalization states the lemma proving the existence of an infinite branch whenever the tree has arbitrary long finite chains.

Despite the fact of the existence of excellent libraries on graph theory for different interactive theorem provers (e.g., those related to Gonthier's formalization of the four-color theorem for planar graphs in Coq [15, 17, 16]), to the best of our knowledge there are no formalizations of the de Bruijn-Erdös k-colouring theorem, neither for the finite nor for the countable case.

Considering the finite version of Hall's Theorem, Romanowicz and Grabowski [38] reported the first formalization of this result in Mizar. Jiang and Nipkow [27] presented two formalizations in Isabelle/HOL: in addition to a formalization of Rado's proof ([35]), also used in Mizar, the Isabelle/HOL development formalizes Vaughan's proof ([22]). Also, a formalization in Coq applies Dilworth's decomposition theorem and bi-partitions in graphs [45]. Dilworth's theorem is formalized in Mizar in [39]. Recently, Gusakov, Mehta, and Miller [18] reported different formalizations in Lean of the finite version of Hall's theorem; the first, in terms of indexed families of finite subsets, the second, in terms of the existence of injections that saturate binary relations over finite sets and, the third, in terms of matchings in bipartite graphs. Related combinatorial results are reported in recent works by Doczkal et al. in their graph theory Coq library (e.g., [7], [9], and [8]). Additionally, Singh and Natarajan formalized in Coq other combinatorial results as the perfect graph theorem and a weak version of this theorem (e.g., [46], [47]).

Adaptations to the infinite case from theorems equivalent to the finite case of Hall's marriage theorem may be elaborated. Moreover, such adaptations would not necessarily be derivable from the compactness theorem. An example is König's duality theorem that states that in every bipartite graph $G = \langle X, Y, E\rangle$, there exists a matching $M \subseteq E$ such that selecting one vertex from each arc in M one has a *cover* of the graph [1, 2]. This theorem is a strong form of the König-Egerváry theorem, stating that in a finite bipartite graph, the size of a maximal matching is equal to that of a minimal *cover* [30]. The key difference of the duality theorem is that such a cover of the graph cannot be extracted from any matching; namely, given any matching of the graph, it is possible to build a cover of the same cardinality as the cardinality of the matching, but not that covers the graph entirely. So, the notion of *König cover* came to arise, which is defined as a cover of the graph that consists of a selection of one vertex from each arc of a matching.

Lifting results from the finite to the infinite through the application of compactness (of König's lemma) corresponds to a recursive construction of a procedure that produces the target solution in the degree of unsolvability of the halting problem [2]. Such a recursive construction is possible for Dilworth's theorem (restricting the maximal anti-chains in infinite partial ordered sets to be finite - [6], see also Sec. 2.5 in [24]) but not for König's duality theorem. Indeed, Aharoni et al. [2] proved that the complexity of constructing covers exceeds the complexity of the halting problem; it is even a problem of higher complexity than answering all first-order questions about arithmetic. Also, they proved that the compactness theorem and König's lemma do not suffice to prove the duality theorem and other related results in matching theory.

There are two formalizations of the countable set-theoretical version of Hall's theorem: one by the authors detailed in [42], and another by Gusakov, Mehta, and Miller presented in [18]. Also, we formalized a countable graph-theoretical version derived from the set-theoretical formalization presented in [43]. The distinguishing feature of our formalization in Isabelle/HOL is the application of the compactness theorem. In the Lean formalization, the authors use an *inverse limit* version of the König's lemma. This lemma states that if $\{X_i\}_{i\in\mathbb{N}}$ is an indexed family of nonempty finite sets with functions $f_i : X_{i+1} \to X_i$, for each $i \in \mathbb{N}$, then there exists a family of elements $x \in \prod_i X_i$ such that $x_i = f_i(x_{i+1})$, for all $i \in \mathbb{N}$. König's lemma follows from this infinite limit version by choosing as set X_i the paths of length i from the root vertex v_0 in a tree. So, the function f_i maps paths in X_{i+1} into the paths without their last arc, which are paths that belong to X_i. The inverse limit consists of the infinite chain of functions $f_1, f_2, \ldots$. König's lemma is applied to prove the countable version of Hall's theorem by taking M_n as the set of all matchings on the first n indices of I (i.e., the set of all possible SDRs for the sets $S_1, \ldots, S_n$), and $f_n : M_{n+1} \to M_n$ as the restriction of a match to a smaller set of indices. Since the marriage condition holds for the finite indexed families, each M_n is nonempty, and by König's lemma, an element of the inverse limit gives a matching on I.

5 Conclusions and Future Work

We presented a complete formalization of the propositional compactness theorem based on the construction of models. The compactness theorem was applied to build complete and constructive proofs of three relevant applications: Hall's theorem for countable sets and graphs, de Bruijn-Erdös theorem for countable graphs, and König's lemma.

The whole Isabelle/HOL development discussed in this paper, available through the link Compactness Theory, consists of a directory called *ModelExistence* with all required elements to prove the model existence theorem. The total number of lines in the theories related to the logical notions and properties required on the proof of the model existence theorem is 3218, in which proofs of seventeen theorems are included (see the "subtotal" row in the Table 1). The theory *Compactness* uses the formalization of the model existence theorem and adds 15 lemmas to formalize the compactness theorem. Table 1 also contains information about the theories related to the discussed applications. It is remarkable to notice that the elements required to apply the compactness theorem to prove König's lemma are almost twice the size of the other applications. Also, notice that the formalization of Hall's theorem for countable graphs is smaller since this uses directly the set-theoretical version of Hall's theorem without building any model.

As mentioned in the section on related work (Subsection 4.2), potential applications would lift combinatorial results from the infinite to the countable cases. Exploring such extensions is of remarkable interest since it is well-known that the finite cases of Hall's and de Bruijn-Erdös theorems are equivalent to other relevant combinatorial theorems.

Table 1 Theories of the development

Theory Name	**Line Num.**	**Number of Proved Formulas** Lemmas	Corollaries	Theorems
SyntaxAndSemantics.thy	691	17		3
UniformNotation.thy	694	29		
Closedness.thy	180	7		1
Finiteness.thy	337	7		2
MaximalSet.thy	235	5	1	4
Hintikka_theory.thy	429	8	3	1
MaximalHintikka.thy	158	6		1
BinaryTreeEnumeration.thy	172	11		
FormulaEnumeration.thy	129	4	3	1
ModelExistence.thy	147	1	2	4
Subtotal	3172	95	9	17
Compactness.thy	374	15		1
Total	3546	110	9	18
Applications				
k_coloring.thy	881	30		3
KoenigLemma.thy	1966	66		1
Hall_Theorem.thy	997	44		4
Hall_Th_Graph_Theo.thy	461	7		3

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