An Infinitary System for the Least Fixed-Point Logic restricted to Finite Models

Alexandre M. Arruda - alexandre@lia.ufc.br

Ana Teresa Martins - ana@lia.ufc.br

Universidade Federal do Ceará

Motivations

- Fixed-Point in CS: Recursive Definitions
- Least Fixed-Point Logic: $LFP+ < \equiv PTIME$
- Finite Model Theory: Descritive Complexity
- Trahtenbrot's Theorem
  - The set of FOL sentences valid in all finite structures is not recursively enumerable.
- Infinitary Systems
- Proof Theory for Arithmetic: the ω-rule

General Goals

- A Proof Theory for the
  Least Fixed-Point Logic (LFP)
- Finite Model Theory X Proof Theory
- A Theorem Prover for
  Database Query Languages

Specific Goals

- An Infinitary System for FOL restricted to finite models: FOL_{fin}
- An Infinitary System for LFP restricted to finite models: LFP_{fin}
- Normalization
  - FOL + equality axioms
  - LFP_{fin} (with restrictions)

The First-Order Logic in Finite Models (FOL_{fin})

Definition 1 *A set Γ of FOL formulas is finitely satisfiable if there is a finite model that satisfies it.*

Definition 2 $\Gamma \models_{FOL_{fin}} \varphi$ *is all finite interpretation which satisfy Γ also satisfies φ.*

The First-Order Logic in Finite Models (FOL_{fin})

$$\lambda_{\geq n} := \exists v_0 \ldots \exists v_{n-1}(\neg v_0 \equiv v_1 \wedge \ldots \wedge \neg v_0 \equiv v_{n-1} \wedge \ldots \wedge \neg v_{n-2} \equiv v_{n-1})$$

$\lambda_{\geq n}$:= “There is at least n elements in A”.

$$\Phi_\infty := \{\lambda_{\geq n} \mid n \geq 2\}$$

$\mathfrak{I} \models \Phi_\infty$ iff A contains infinitely many elements.

The First-Order Logic in Finite Models (FOL_{fin})

FOL_{fin} := FOL + $(FIN\!-\perp)$

$$\frac{\begin{array}{c}\Pi_1\\ \exists v_1 \exists v_2 (\neg v_1 \equiv v_2)\end{array} \quad \begin{array}{c}\Pi_2\\ \exists z_1 \exists z_2 \exists z_3 (\neg z_1 \equiv z_2 \wedge \neg z_1 \equiv z_3 \wedge \neg z_2 \equiv z_3)\end{array} \quad \ldots}{\perp}\ (FIN\!-\perp)$$

or

$$\frac{\begin{array}{c}\Pi_1\\ \lambda_{\geq 2}\end{array} \quad \begin{array}{c}\Pi_2\\ \lambda_{\geq 3}\end{array} \quad \ldots \quad \begin{array}{c}\Pi_{i-1}\\ \lambda_{\geq i}\end{array} \quad \ldots}{\perp}\ (FIN\!-\perp).$$

.

We will denote this system by $\vdash_{FOL_{fin}}$.

The First-Order Logic in Finite Models (FOL_{fin})

Theorem 1 (Soundness) *For* $\Gamma \in L^S$ *and* $\varphi \in L^S$ *we have:*

$$\text{If } \Gamma \vdash_{FOL_{fin}} \varphi \text{ then } \Gamma \models_{FOL_{fin}} \varphi.$$

and

Theorem 2 (Completeness) *For* $\Gamma \in L^S$ *and* $\varphi \in L^S$ *we have:*

$$\text{If } \Gamma \models_{FOL_{fin}} \varphi \text{ then } \Gamma \vdash_{FOL_{fin}} \varphi.$$

The First-Order Logic in Finite Models : An Example

$$(\forall x \forall y(fx \equiv fy \to x \equiv y) \to \forall x \exists y(x \equiv fy)).$$

This sentence holds in every finite interpretation. Thus, we can prove in FOL_{fin}

$$\vdash_{FOL_{fin}} (\forall x \forall y(fx \equiv fy \to x \equiv y) \to \forall x \exists y(x \equiv fy)).$$

The scheme of this proof is:

$$\dfrac{\dfrac{\begin{array}{ccc}\Pi_1 & \Pi_2 & \\ \lambda_{\geq 2} & \lambda_{\geq 3} & \cdots\end{array}}{\bot}\ (FIN- \bot)}{(\forall x \forall y(fx \equiv fy \to x \equiv y) \to \forall x \exists y(x \equiv fy))}\ (\bot_c), 0$$

The First-Order Logic in Finite Models : An Example

where $\Pi_1 =$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\forall x \forall y(fx \equiv fy \to x \equiv y) \to \forall x \exists y(x \equiv fy))^0}{\forall x \forall y(fx \equiv fy \to x \equiv y) \wedge \neg \forall x \exists y(x \equiv fy)}}{\neg \forall x \exists y(x \equiv fy)}}{\exists x \forall y \neg(x \equiv fy)} \qquad \dfrac{\dfrac{\dfrac{\forall y \neg(z \equiv fy)^1}{\neg(z \equiv fz)}}{\exists v_1 \neg(z \equiv v_1)}}{\exists v_0 \exists v_1 \neg(v_0 \equiv v_1)}}{\exists v_0 \exists v_1 \neg(v_0 \equiv v_1)}$$

The First-Order Logic in Finite Models : An Example

and $\Pi_2 =$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\neg(\forall x \forall y (fx \equiv fy \to x \equiv y) \to \forall x \exists y (x \equiv fy))^0}{\forall x \forall y (fx \equiv fy \to x \equiv y) \wedge \neg \forall x \exists y (x \equiv fy)}
}{\neg \forall x \exists y (x \equiv fy)}
}{\exists x \forall y \neg (x \equiv fy)}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\forall y \neg (z \equiv fy)}{\neg (z \equiv fz)} \quad \Sigma
}{\neg (z \equiv fz) \wedge \neg (z \equiv ffz) \wedge \neg (fz \equiv ffz)}
}{\exists v_2 (\neg (z \equiv fz) \wedge \neg (z \equiv v_2) \wedge \neg (fz \equiv v_2))}
}{\exists v_1 \exists v_2 (\neg (z \equiv v_1) \wedge \neg (z \equiv v_2) \wedge \neg (v_1 \equiv v_2))}
}{\exists v_0 \exists v_1 \exists v_2 (\neg v_0 \equiv v_1 \wedge \neg v_0 \equiv v_2 \wedge \neg v_1 \equiv v_2)}
}{\exists v_0 \exists v_1 \exists v_2 (\neg v_0 \equiv v_1 \wedge \neg v_0 \equiv v_2 \wedge \neg v_1 \equiv v_2)}
$$

The First-Order Logic in Finite Models : An Example

where $\Sigma =$

$$\dfrac{\dfrac{\forall y \neg (z \equiv fy)}{\neg (z \equiv ffz)} \qquad \dfrac{\dfrac{\dfrac{\forall y \neg (z \equiv fy)}{\neg (z \equiv fz)} \qquad \dfrac{fz \equiv ffz \qquad \dfrac{\dfrac{\dfrac{\dfrac{\neg (\forall x \forall y (fx \equiv fy \to x \equiv y) \to \forall x \exists y (x \equiv fy))^0}{\forall x \forall y (fx \equiv fy \to x \equiv y) \wedge \neg \forall x \exists y (x \equiv fy)}}{\forall x \forall y (fx \equiv fy \to x \equiv y)}}{\forall y (fz \equiv fy \to z \equiv y)}}{fz \equiv ffz \to z \equiv fz}}{(z \equiv fz)}}{\dfrac{\bot}{\neg (fz \equiv ffz)}}}{\neg (z \equiv ffz) \wedge \neg (fz \equiv ffz)}$$

and $\Pi_i = \ldots.$

Fixed-Point Logics

Definition 3 (Operator) *Let U be a set. An operator in U is a map $F : \wp(U) \to \wp(U)$.*

Definition 4 (Monotone Operator) *An operator in U is monotone if:*

$$X \subseteq Y \text{ implies } F(X) \subseteq F(Y).$$

Fixed-Point Logics

Definition 5 (The Fixed-Point of an Operator) *A set $X \subseteq U$ is a fixed-point of F if $F(X) = X$.*

Definition 6 (An Inductive Operator) *Consider the sequence:*

$$X^0 = \emptyset,\ X^{i+1} = F(X^i).$$

We call F inductive if this sequence is increasing, that is, $X^i \subseteq X^{i+1}$, for all $i \in N$.

Definition 7 (The Least Fixed-Point of an Operator) *A set $X \subseteq A$ is the least fixed-point of F if it is a fixed-point, and, for every other fixed-point Y of F, we have $X \subseteq Y$.*

Fixed-Point Logics

Theorem 3 (Tarski-Knapster) *Every monotone operator F has a least fixed-point $\mathbf{lpf}(F)$ which can be defined as:*

$$\mathbf{lpf}(F) = \bigcap \{Y \mid Y = F(Y)\}.$$

Futhermore, $\mathbf{lpf}(F) = X^\infty = \bigcup_i X^i$

Fixed-Point Logics

In order to add the concept of fixed-point to FOL, let $\varphi(R, x_1, ..., x_k)$ be a formula of $S \cup \{R\}$.

Then, for each S-interpretation $\mathfrak{I}$, the formula $\varphi(R, \vec{x})$ gives rise to an operator $F_\varphi : \wp(A^k) \rightarrow \wp(A^k)$ defined as:

$$F_\varphi(X) = \{(a_1, \ldots a_k) \mid \mathfrak{I}\frac{a_1 \ldots a_k}{x_1 \ldots x_k} \models \varphi(X/R, x_1, \ldots, x_k)\}.$$

The Least Fixed-Point Logic (LFP)

The Formation Rule:

- If $\varphi(R, \vec{x})$ is a positive formula in R, then

$$[\mathbf{lfp}_{R,\vec{x}}\varphi(R, \vec{x})](\vec{t})$$

is a formula.

The semantics is defined as:

$$\Im \models [\mathbf{lfp}_{R,\vec{x}}\varphi(R, \vec{x})](t_1, \ldots, t_k) \text{ iff}$$
$$(\Im(t_1), \ldots, \Im(t_k)) \in \mathbf{lfp}(F_\varphi).$$

The Least Fixed-Point Logic (LFP)

Each computing stage X^i of the fixed-point can be defined by a formula φ^i of LFP:

$$\varphi^0(\vec{x_0}) \equiv \neg(x = x) \quad \text{, where } x \text{ is a variable in } \vec{x_0}$$
$$\varphi^{i+1}(\vec{x_{i+1}}) \equiv \varphi(\varphi^i/R, \vec{x_{i+1}}) \quad .$$

(1)

where the notation $\varphi(\varphi^i/R, \vec{x_{i+1}})$ means that each ocorrence of $R(\vec{y})$ in φ is replaced by $\varphi^i(\vec{y})$.

The Least Fixed-Point Logic (LFP): an Example

Let $\varphi(R, x, y)$ be $E(x, y) \vee \exists z(E(x, z) \wedge R(z, y))$. Then:

$$
\begin{array}{rcl}
\varphi^0(x_0, y_0) & \equiv & \neg(x_0 = x_0) \\
\varphi^1(x_1, y_1) & \equiv & E(x_1, y_1) \ \vee \ \exists z_1(E(x_1, z_1) \ \wedge \ \varphi^0(z_1, y_1)) \quad \leftrightarrow \\
& & E(x_1, y_1) \\
\varphi^2(x_2, y_2) & \equiv & E(x_2, y_2) \ \vee \ \exists z_2(E(x_2, z_2) \ \wedge \ \varphi^1(z_2, y_2))) \quad \leftrightarrow \\
& & E(x_2, y_2) \vee \exists z_2(E(x_2, z_2) \wedge E(z_2, y_2)) \\
\ldots & & \ldots
\end{array}
$$

Computing the stages of the transitive closure.

The Least Fixed-Point Logic in Finite Models (LFP_{fin})

We add to FOL_{fin} the following rules:

$$\dfrac{\begin{matrix} \Pi \\ [\mathbf{lfp}_{R,\vec{x}}\ \varphi(R,\vec{x})](\vec{t}) \end{matrix} \qquad \begin{matrix} \varphi^0(\vec{t})^0 \\ \Pi_0 \\ \sigma \end{matrix} \qquad \begin{matrix} \varphi^1(\vec{t})^1 \\ \Pi_1 \\ \sigma \end{matrix} \qquad \ldots \qquad \begin{matrix} \varphi^i(\vec{t})^i \\ \Pi_i \\ \sigma \end{matrix} \qquad \ldots}{\sigma} \ (LFP_{fin} - E), 0, 1, 2, \ldots,$$

discharging hypotheses $0, 1, 2 \ldots$

$$\dfrac{\varphi^i(\vec{t})}{[\mathbf{lfp}_{R,\vec{x}}\ \varphi(R,\vec{x})](\vec{t})} \ (LFP_{fin} - I), \text{ where } i \geq 0 \text{ for } i \in N,$$

$$\dfrac{[\mathbf{lfp}_{R,\vec{x}}\ \varphi(R,\vec{x})](\vec{t}) \qquad \neg\lambda_{\geq i+1}}{\varphi^{i^{|\vec{x}|}}(\vec{t})} \ (LFP_{fin} - FP), \text{ where } i \geq 1 \text{ for } i \in N.$$

The Least Fixed-Point Logic in Finite Models (LFP_{fin})

Definition 8 $\Gamma \vdash_{LFP_{fin}} \alpha$ *if there is a derivation* Π *in* LFP_{fin} *with hypotheses not discharged in* Γ *and conclusion* α*.*

Theorem 4 (Soundness)

If $\Gamma \vdash_{LFP_{fin}} \varphi$ *then* $\Gamma \models_{LFP_{fin}} \varphi$*.*

The Fixed-Point Logic in Finite Models (LFP_{fin}) - Completeness

Theorem 5 (Completeness)

If $\Gamma \models_{LFP_{fin}} \varphi$ *then* $\Gamma \vdash_{LFP_{fin}} \varphi$.

If a set Φ is consistent, then Φ is also satifiable

The Fixed-Point Logic in Finite Models (LFP_{fin}) - Completeness

Given a set Φ consider:

$$t_1 \backsim t_2 : iff\ \Phi \vdash_{LFP_{fin}} t_1 \equiv t_2.$$

and

$$\bar{t} := \{t' \in T^S \mid t \backsim t'\},$$

We define an interpretation $\mathfrak{I}^\Phi = (\mathfrak{T}^\Phi, \beta^\Phi)$ such that:

$$T^\Phi := \{\bar{t} \mid t \in T^S\} \text{ is the domain.}$$

and also:

$$R^{\mathfrak{T}^\Phi} \bar{t}_1 \ldots \bar{t}_n \ : iff\ \Phi \vdash_{LFP_{fin}} Rt_1 \ldots t_n,$$

$$c^{\mathfrak{T}^\Phi} := \bar{c},$$

$$\beta^\Phi(x) := \bar{x}.$$

The Fixed-Point Logic in Finite Models (LFP_{fin}) - Completeness

Theorem 6 (Henkin) *Let Φ be a consistent set which is negation complete and contains witnesses, then for all γ in LFP,*

$$\mathfrak{I}^{\Phi} \models_{LFP_{fin}} \gamma \text{ iff } \Phi \vdash_{LFP_{fin}} \gamma.$$

Definition 9

(a) Φ is negation complete iff for every formula γ,

$$\Phi \vdash_{LFP_{fin}} \gamma \text{ or } \Phi \vdash_{LFP_{fin}} \neg\gamma.$$

(b) Φ contains witnesses iff for every formula $\exists x\gamma$ there is a term t such that $\Phi \vdash_{LFP_{fin}} (\exists x\gamma \rightarrow \gamma\frac{t}{x})$.

The Fixed-Point Logic in Finite Models (LFP_{fin}) - Completeness

Definition 10 *A set Φ contains f-witness if for some $i \geq 2$, $\neg\lambda_{\geq i} \in \Phi$.*

Theorem 7 *If a set Φ contains f-witness, then for every formula of form $[\textbf{\textit{lfp}}_{R,\vec{x}}\, \varphi(R,\vec{x})](\vec{t})$ we have*

$$\Phi \vdash_{LFP_{fin}} \forall x_1 \ldots \forall x_n ([\textbf{\textit{lfp}}_{R,\vec{x}}\, \varphi(R,\vec{x})](x_1,\ldots,x_n) \rightarrow \varphi^{i^{|\vec{t}|}}(x_1,\ldots,x_n)),$$

for some i such that $\neg\lambda_{\geq i+1} \in \Phi$.

Theorem 8 *For every set Φ which contains f-witness, if Π is a derivation of φ from Φ, then there is a derivation Π' of φ from Φ such that $|\Pi'| < \omega$.*

The Fixed-Point Logic in Finite Models (LFP_{fin}) - Completeness

Corollary 1 (Compactness for LFP_{fin}) *For every consistent set Φ which contains f-witness, the following holds: Φ is consistent iff every finite $\Phi_0 \subset \Phi$ is consistent.*

Theorem 9 *For every consistent set Φ_0 there is a consistent set Φ such that $\Phi_0 \subseteq \Phi$ contains f-witness.*

Theorem 10 *Consider a consistent set Φ. Then there is a consistent set Ψ such that $\Phi \subseteq \Psi$ and Ψ contains witness.*

The Fixed-Point Logic in Finite Models (LFP_{fin}) - Completeness

Theorem 11 *Given Ψ consistent, then there is a consistent set Θ such that $\Psi \subseteq \Theta$ and Θ is negation complete.*

Theorem 12 (Completeness: second version) *Given a consistent set Φ, then Φ is satisfiable.*

The Fixed-Point Logic in Finite Models (LFP_{fin}) - Example

Consider graphs with the edge relation represented by E and the formula $\alpha(S, x)$ given by:

$$\forall y(E(y, x) \rightarrow S(y)).$$

we have:

- The formula is positive in S.
- The operator F, associated to the i-th stage, returns all vertices which is a path of size at most i.
- The fixed-point is the set of all vertices such that all paths that end in them are finite.

The Fixed-Point Logic in Finite Models (LFP_{fin}) - Example

- The formula $\forall u[\mathbf{lfp}_{S,\vec{x}}\ \alpha(S,\vec{x})](u)$ tests if a graph is acyclic.
- An acyclic graph with more than two vertices is not complete.
- A graph is complete if, for every two distincts vertices, there is an edge. We can express this with the formula $\forall x \forall y(\neg x \equiv y \leftrightarrow E(x,y))$.
- We prove in LFP_{fin} that an acyclic graph with more than two vertices (and a finite number of them) cannot be complete.

The Fixed-Point Logic in Finite Models (LFP_{fin}) - Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\exists x_1 \exists x_2 \neg x_1 \equiv x_2 \wedge \forall z[\mathbf{lfp}_{S,\vec{x}}\ \alpha(S,\vec{x})](z)^0}{\exists x_1 \exists x_2 \neg x_1 \equiv x_2} \quad \cfrac{\exists x_2 \neg v \equiv x_2 \quad \Sigma}{\perp}}{\perp}}{\neg \forall x \forall y(\neg x \equiv y \leftrightarrow E(x,y))}\ ,1}{(\exists x_1 \exists x_2 \neg x_1 \equiv x_2 \wedge \forall z[\mathbf{lfp}_{S,\vec{x}}\ \alpha(S,\vec{x})](z)) \rightarrow \neg \forall x \forall y(\neg x \equiv y \leftrightarrow E(x,y))}\ ,0$$

where $\Sigma =$

$$\cfrac{\cfrac{\cfrac{\exists x_1 \exists x_2 \neg x_1 \equiv x_2 \wedge \forall z[\mathbf{lfp}_{S,\vec{x}}\ \alpha(S,\vec{x})](z)}{\forall z[\mathbf{lfp}_{S,\vec{x}}\ \alpha(S,\vec{x})](z)}}{[\mathbf{lfp}_{S,\vec{x}}\ \alpha(S,\vec{x})](y)} \quad \begin{matrix}\Pi_1\\ \perp\end{matrix} \quad \begin{matrix}\Pi_2\\ \perp\end{matrix} \quad \begin{matrix}\Pi_3\\ \perp\end{matrix} \quad \ldots}{\perp}$$

The Fixed-Point Logic in Finite Models (LFP_{fin}) - Example

where $\Pi_1 =$

$$\dfrac{\neg y \equiv y \qquad \overline{y \equiv y}}{\bot}$$

and $\Pi_2 =$

$$\dfrac{\dfrac{\dfrac{\forall z_1(E(z_1, y) \to \neg y \equiv y)}{E(v, y) \to \neg y \equiv y} \qquad \dfrac{\neg v \equiv y \qquad \dfrac{\forall x \forall y(\neg x \equiv y \leftrightarrow E(x, y))^1}{\neg v \equiv y \to E(v, y)}}{E(v, y)}}{\neg y \equiv y} \qquad \overline{y \equiv y}}{\bot}$$

The Fixed-Point Logic in Finite Models (LFP_{fin}) - Example

and $\Pi_3 =$

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall z_2(E(z_2,y)\to\forall z_1(E(z_1,z_2)\to\neg z_2\equiv z_2))}{E(v,y)\to\forall z_1(E(z_1,v)\to\neg v\equiv v)} \qquad \dfrac{\neg v\equiv y \qquad \dfrac{\forall x\forall y(\neg x\equiv y\leftrightarrow E(x,y))^1}{\neg v\equiv y\to E(v,y)}}{E(v,y)}}{\dfrac{\forall z_1(E(z_1,v)\to\neg v\equiv v)}{E(y,v)\to\neg v\equiv v}} \qquad \Sigma_1}{\neg v\equiv v} \qquad \dfrac{}{v\equiv v}}{\bot}$$

where $\Sigma_1 =$

$$\dfrac{\dfrac{\neg v\equiv y}{\neg y\equiv v} \qquad \dfrac{\forall x\forall y(\neg x\equiv y\leftrightarrow E(x,y))^1}{\neg y\equiv v\to E(y,v)}}{E(y,v)}$$

and $\Pi_i = \ldots$

Normalization for FOL + Equality Axioms

- Normal Proof
- Subformula Property
- Maximum Segment
- Normalization Procedure: Reduction Steps
- Classical Normalization Procedure for FOL
- Normalization for FOL + equality axioms
- Equality maximum segment

Normalization for FOL + Equality Axioms - Equality Maximum Segment

Consider the derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Sigma_1 \\ P(x)\end{matrix} \quad \begin{matrix}\Sigma_2 \\ Q(x)\end{matrix}}{P(x) \wedge Q(x)} \quad \begin{matrix}\Sigma_3 \\ x \equiv y\end{matrix}}{P(y) \wedge Q(y)}}{P(y)}$$

We note this type of redundance is not captured by the concept of classical maximum segment.

We be able to eliminate this redudance putting up an application of the equality rule, as follows:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Sigma_1 \\ P(x)\end{matrix} \quad \begin{matrix}\Sigma_3 \\ x \equiv y\end{matrix}}{P(y)} \quad \dfrac{\begin{matrix}\Sigma_2 \\ Q(x)\end{matrix} \quad \begin{matrix}\Sigma_3 \\ x \equiv y\end{matrix}}{Q(y)}}{P(y) \wedge Q(y)}}{P(y)}$$

Normalization for FOL + Equality Axioms - Equality Maximum Segment

A segment $\sigma = \varphi_1, \varphi_2, \ldots, \varphi_n$ is an equality maximum segment when:

(α) 1. φ_1 is the conclusion of an introduction rule, or an intutionistic absurdum rule, and it is also the first premisse of the rule $EQ2$.

Normalization for FOL + Equality Axioms - Degree of a Derivation

Definition 11 *The degree of a derivation Π is defined as:*

$$d(\Pi) = max\{d_c(\Pi), d_e(\Pi)\}.$$

Normalization for FOL + Equality Axioms

- Operational Reduction Steps

- $\wedge I$:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ \varphi\frac{t}{z}\end{matrix} \quad \begin{matrix}\Pi_2 \\ \psi\frac{t}{z}\end{matrix}}{(\varphi \wedge \psi)\frac{t}{z}} \quad \begin{matrix}\Pi_3 \\ t \equiv t'\end{matrix}}{(\varphi \wedge \psi)\frac{t'}{z}} \Rightarrow \dfrac{\chi\left(\dfrac{\begin{matrix}\Pi_1 \\ \varphi\frac{t}{z}\end{matrix} \quad \begin{matrix}\Pi_3 \\ t \equiv t'\end{matrix}}{\varphi\frac{t'}{z}}\right) \quad \chi\left(\dfrac{\begin{matrix}\Pi_2 \\ \psi\frac{t}{z}\end{matrix} \quad \begin{matrix}\Pi_3 \\ t \equiv t'\end{matrix}}{\psi\frac{t'}{z}}\right)}{(\varphi \wedge \psi)\frac{t'}{z}}$$

Normalization for FOL + Equality Axioms

- Permutational Reduction Steps

- $\vee E$:

$$
\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \varphi_1 \vee \varphi_2\end{matrix} \quad \begin{matrix}[\varphi_1]\\ \Pi_2\\ \psi\frac{t}{z}\end{matrix} \quad \begin{matrix}[\varphi_2]\\ \Pi_3\\ \psi\frac{t}{z}\end{matrix}}{\psi\frac{t}{z}} \qquad \begin{matrix}\Pi_4\\ t \equiv t'\end{matrix}}{\psi\frac{t'}{z}}
\quad \Rightarrow \quad
\dfrac{\begin{matrix}\Pi_1\\ \varphi_1 \vee \varphi_2\end{matrix} \quad \dfrac{\begin{matrix}[\varphi_1]\\ \Pi_2\\ \psi\frac{t}{z}\end{matrix} \quad \begin{matrix}\Pi_4\\ t \equiv t'\end{matrix}}{\psi\frac{t'}{z}} \quad \dfrac{\begin{matrix}[\varphi_2]\\ \Pi_3\\ \psi\frac{t}{z}\end{matrix} \quad \begin{matrix}\Pi_4\\ t \equiv t'\end{matrix}}{\psi\frac{t'}{z}}}{\psi\frac{t'}{z}}
$$

Normalization for FOL + Equality Axioms
- Absurdum Reduction Steps

- $\perp_C$:

$$\dfrac{\dfrac{\begin{array}{c}[\neg\varphi]^i\\ \Pi\\ \perp\end{array}}{\varphi}\,,i \qquad \dfrac{\begin{array}{c}\dfrac{[\neg\xi_1]\quad \begin{array}{c}\Pi_1\\ t_1\equiv t_1^{'}\end{array}}{\neg\xi_2}\\ \vdots\\ \neg\xi_k\end{array} \qquad \begin{array}{c}\Pi_k\\ t_k\equiv t_k^{'}\end{array}}{\neg\varphi}}{\perp} \quad \Rightarrow \quad \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{[\neg\xi_1]\quad \begin{array}{c}\Pi_1\\ t_1\equiv t_1^{'}\end{array}}{\neg\xi_2}\\ \vdots\\ \neg\xi_k\end{array} \qquad \begin{array}{c}\Pi_k\\ t_k\equiv t_k^{'}\end{array}}{[\neg\varphi]^i}\\ \Pi\\ \perp\end{array}$$

Normalization for FOL + Equality Axioms

The proof is based as follows:

- Given a derivation Π with degree $d(\Pi) = n$, we prove by induction we be able to reduce it to Π' such that $d(\Pi') < d(\Pi)$.
- The proof is based in the fact that we can reduce both $d_e(\Pi)$ and $d_c(\Pi)$ such that $d(\Pi)$ do not increase.

Theorem 13 (Weak Normalization for FOL + Equality Axioms) *Let Π be a derivation of $\Gamma \vdash \varphi$, then Π can be reduced to a normal derivation Π' of $\Gamma \vdash \varphi$.*

Normalization for LFP_{fin}

- $FIN - \perp$ and Subformula Property
- Normalization restricted to sets that contains f-witnesses
- Finite version of the rules
- $LFP_{fin} - FP$ is derivable

Elimination of rule $LFP_{fin}\ FP$

With this two last theorems, we be able to eliminate the rule $(LFP_{fin}\ FP)$ from the system, because given an derivation:

$$\dfrac{\begin{array}{c}\Pi_1\\ [\mathbf{lfp}_{R,\vec{x}}\ \varphi(R,\vec{x})](\vec{t})\end{array} \quad \begin{array}{c}\Pi_2\\ \neg\lambda_{\geq i+1}\end{array}}{\begin{array}{c}\varphi^{i^{|\vec{t}|}}(\vec{t})\\ \Sigma\end{array}}\ (LFP_{fin}\ FP)$$

We simply replace the above for

$$\dfrac{\begin{array}{c}\Pi_1\\ [\mathbf{lfp}_{R,\vec{x}}\ \varphi(R,\vec{x})](\vec{t})\end{array} \quad \begin{array}{c}\varphi^0(\vec{t})\\ \Sigma_0\\ \varphi^{i^{|\vec{t}|}}(\vec{t})\end{array} \quad \ldots \begin{array}{c}\varphi^{i^{|\vec{t}|}}(\vec{t})\\ \Sigma_i\\ \varphi^{i^{|\vec{t}|}}(\vec{t})\end{array} \quad \begin{array}{c}\varphi^{i^{|\vec{t}|}+1}(\vec{t})\\ \Sigma_{i^{|\vec{t}|}+1}\\ \varphi^{i^{|\vec{t}|}}(\vec{t})\end{array} \quad \ldots}{\begin{array}{c}\varphi^{i^{|\vec{t}|}}(\vec{t})\\ \Sigma\end{array}}\ (LFP_{fin}\ E)$$

,

where the derivations Σ_j with $j < i^{|\vec{t}|}$ use only FOL rules.

Finite Version of $LFP_{fin}E$

For a set containing f-witness, we can restrict the rule $(LFP_{fin}\ E)$.

$$\cfrac{\begin{matrix}\Pi_1 \\ [\mathbf{lfp}_{R,\vec{x}}\ \varphi(R,\vec{x})](\vec{t})\end{matrix} \qquad \begin{matrix}\varphi^0(\vec{t}) \\ \Sigma_0 \\ \psi\end{matrix} \qquad \ldots \qquad \begin{matrix}\varphi^{i^{|\vec{t}|}}(\vec{t}) \\ \Sigma_{i^{|\vec{t}|}} \\ \psi\end{matrix}}{\begin{matrix}\psi \\ \Sigma\end{matrix}}\ (LFP\ E)_{res}.$$

Normalization for LFP_{fin}

Theorem 14 (Weak Normalization for restricted LFP_{fin})
Let Π be a derivation in restricted LFP_{fin}, then Π can be reduced to a normal derivation.

Conclusions: Results

- An infinitary system for FOL$_{fin}$.
- An infinitary system for LFP$_{fin}$.
- Soundness and Completeness
- Normalization for FOL + equality
- Normalization for LFP$_{fin}$ with restrictions

Conclusions: Comparisons

Compton's work [Compton 1993]:

- Existential Fragment of LFP
- Sequent Calculus
- Similar left/right rules for **lfp**
- Models of any cardinality.
- No rules to deal with cardinality

Future Works

- Normalization for LFP_{fin} without restrictions.
- Investigation of new (proofs of) theorems in Finite Model Theory using Proof Theory, and vice-versa.
- A theorem prover for LFP_{fin}: a database query language processor

References

[1] K.J. Compton. A Deductive System for Existential Least Fixpoint Logic. Journal of Logic and Computation, 3(2): 197-213, 1993.

[2] H.D. Ebbinghaus & J.Flum. *Finite Model Theory*. Springer, 1995.

[3] H.D. Ebbinghaus, J.Flum & W.Thomas. *Mathematical Logic*. 2nd Edition, Undergraduate Texts in Mathematics, Springer, New York, 1994.

[4] R. Fagin. Generalized First-order Spectra and Polynomial-time Recognizable Sets. In Complexity of Computation, R. Karp, ed., SIAM-MAS Proceedings, 7, 1974, 43-73.

[5] R. Fraïssé. Sur Quelques Classifications des Systèmes de Relations. *Université d'Alger, Publications Scientifiques, Série A*, 1, 1954, 35-182.

References

[6] G.Gentzen. Die Widerspruchsfreiheit der reinen Zahlentheorie, *Mathematische Annalen*, 112, 493-565. Translation in M.E.Szabo, editor, *The Collected Papers of Gerhard Gentzen,* pages 132-170, North-Holland, Amsterdam, 1969.

[7] G.Gentzen. Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der Transfiniten Induktion in der reinen Zahlentheorie, *Mathematische Annalen*, 119, 140-161. Translation in M.E.Szabo, editor, *The Collected Papers of Gerhard Gentzen,* pages 287-311, North-Holland, Amsterdam, 1969.

[8] L.Henkin. The Completeness of the First-Order Functional Calculus. Journal of Symbolic Logic, 14, 1949.

[9] N. Immerman. Upper and Lower Bounds for First Order Expressability. Journal of Computer and System Sciences, 25, 1982, 76-98.

[10] N. Immerman. Relational Queries Computable in Polynomial Time (extended abstract). In ACM Symposium on Theory of Computation. ACM Press, 1982, 147-152.

References

[11] L. Libkin. *Elements of Finite Model Theory*. Springer, Berlin, 2004.

[12] Pereira, L.C., and Massi, C.D.B., *Normalização para a Lógica Clássica*, O que nos faz pensar, Cadernos de Filosofia da PUC-RJ **2** (1990), 49-53.

[13] D. Prawitz. Natural Deduction: A Proof-Theoretical Study. In *Stockholm Studies in Philosophy 3*, Almqvist and Wiksell, Stockholm, 1965. Acta Universitatis Stockholmiensis.

[14] K. Schütte. Beweistheoretische Erfassung der Unendliche Induktion in der Zahlentheorie. Mathemastiche Annalen, 122:369Ű389, 1950.

[15] D. A. Schmidt. *Denotational Semantics: a Methodology for Language Development.* Allyn and Bacon, INC, Boston, 1986.

References

[16] D. Scott. Domains for Denotational Semantics. LNCS 140: Proc. 9th ICALP, pp.577–613, Springer, Berlin, 1982.

[17] B.A.Trahtenbrot. The Impossibility of an Algorithm for the Decision Problem for Finite Models. Doklady Academii Nauk SSSR, 70, 1950. 569-572.

[18] A.S.Troelstra & H.Schwichtenberg. *Proof Theory*. Cambridge University Press, Cambridge, 1996.

[19] D. Van Dalen. *Logic and Structure*. 2nd Edition. Springer Verlag. Berlin. 1989.

[20] M.Y.Vardi. The complexity of relational query languages. In Proc. ACM Symp. on Theory of Computing, 1982, 137-146.

Questions?