On the convergence of reduction-based and model-based methods in proof theory

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Cut elimination

Many results in proof theory corollaries of cut elimination theorems:

- unprovability results
- completeness of proof search methods
- disjunction and witness properties
- possibility to extract programs from proofs

Cut elimination

Two kinds of methods: reduction-based and model-based

Recently: a convergence of these two methods: two sides of the same coin

The (5) steps of this convergence

I. The problem of the axioms

A preliminary step: give a sufficiently general notion of cut

This required: modify the notion of theory

An example: the disjunction property for constructive logic

If $A \vee B$ is provable in the empty theory then A or B is provable

If $A \vee B$ is provable then it has a cut free proof

A cut free proof in the empty theory ends with an intro. rule

Hence a cut free proof of $A \vee B$ in the empty theory ends with an $\vee$-intro: either A or B provable

When we add axioms axioms

"Last rule is an introduction" does not generalize

Harrop's formulae, etc.

Disjunction property for arithmetic ?

The cut elimination theorems

Extend the cut elimination theorem first

A notion of cut specific to arithmetic (induction cuts)

A cut elimination theorem for each theory of interest

An obstacle to the development of a general theory of cut elimination

Deduction modulo

A partial solution to this problem

Axioms of the form $t = u$ or $P \Leftrightarrow A$

Replace them by rewrite rules $t \longrightarrow u$ and $P \longrightarrow A$

Deduction modulo the congruence defined by these rules

Deduction modulo

Many theories expressed with rewrite rules only (no axioms)

Not all theories have the cut elimination property

But, a notion of cut independent of the theory of interest

Well-formed question: "Does the theory $\mathcal{T}$ have the cut elimination property ?"

II. Models

Soundness and completeness

Soundness:

For every A, if $\vdash A$, then for every $\mathcal{M}$, $\mathcal{M} \models A$

Completeneness:

For every A, if (for every $\mathcal{M}$, $\mathcal{M} \models A$), then $\vdash A$

Soundness and completeness

Soundness:

For every A, if $\vdash A$, then for every $\mathcal{M}$, $\mathcal{M} \models A$

Completeneness:

For every A, there exists $\mathcal{M}$ such that if $\mathcal{M} \models A$, then $\vdash A$

Can we prove

Uniform completeneness:

there exists $\mathcal{M}$ such that for every A, if $\mathcal{M} \models A$, then $\vdash A$

?

$\{0,1\}$-models: No

replace $\{0,1\}$ by an arbitrary boolean algebra: Yes (Lindenbaum)

For constructive logic: Heyting algebras

Model-based cut elimination proofs

Sharpen the completeness theorem

For every A, there exists $\mathcal{M}$ such that if $\mathcal{M} \models A$, then $\vdash^{cf} A$

$\vdash A \rightarrow A$ valid in all $\mathcal{M}$ (sound.) $\rightarrow \vdash^{cf} A$ (sharpened comp.)

Sharpened uniform completeness

there exists $\mathcal{M}$ such that for every A, if $\mathcal{M} \models A$, then $\vdash^{cf} A$

$\vdash A \rightarrow \mathcal{M} \models A$ (sound.) $\rightarrow \vdash^{cf} A$ (sharpened uniform comp.)

III. Reductions

Reductions

An algorithm that transforms every non cut free proof into another closer to a cut free proof

Iterate this algorithm

If termination: cut free proof

Tait's proof

Prove termination by induction over proof structure

Needs to strengthen the IH from termination to reducibility

Reducibility

Parametrized by the proposition the proof is a proof of

1. define "being a reducible proof of A" (induction on A)
2. prove that all proofs of A are reducible proofs of A

(induction over proof structure)

Reducibility

"being a reducible proof of A" = set of reducible proofs of A: R_A

induction: $R_{A \Rightarrow B} = (R_A \tilde{\Rightarrow} R_B)$, etc.

IV. Bridging the gap

1. Reducibility candidates

Girard (70) reduction-based proof of cut elimination for STT

Define R_A: quantify over all R_B

Circular

Avoid circularity: define $\mathcal{C}$ *a priori* quantify over $\mathcal{C}$ instead, prove *a posteriori* that $R_B \in \mathcal{C}$

$\mathcal{C}$: the place where R_A live

Later (Parigot): $\mathcal{C}$ is just the smallest set closed by $\tilde{\Rightarrow}$, $\tilde{\forall}$, ...

2. Reducibility candidates as truth values

Type theory

Werner (and others)

$$R_A \longrightarrow [\![A]\!]$$

3. Introducing domains

Together with Werner: a reduction-based proof of cut elimination for a large class of theories in Deduction modulo

In DM (unlike type theory): terms distinct from proofs-terms

Interpret not only propositions but also terms

Introduction of a domain of interpretation for terms

$[\![P(t_1, ..., t_n)]\!] = \hat{P}([\![t_1]\!], ..., [\![t_n]\!])$

Pre-models: $\mathcal{C}$-valued models

4. Truth values algebras

In a models, propositions can be valued in $\{0, 1\}$, a boolean algebra or a Heyting algebra

Can $\mathcal{C}$ be seen as a Heyting algebra ?

Can we find an order relation $\leq$ on $\mathcal{C}$ such that $\tilde{\wedge}$ is the greatest lower bound, $\tilde{\vee}$ the least upper bound, ... ?

No: in all Heyting algebras $(\tilde{\top} \tilde{\Rightarrow} \tilde{\top}) = \tilde{\top}$

but not in $\mathcal{C}$

4. Truth values algebras

What does it take to be a set of truth values ?

An order ? Why ?

Operations $\tilde{\Rightarrow}$, $\tilde{\forall}$, $\tilde{\wedge}$, to interpret propositions

and a notion of positive truth values to characterize valid propositions

For soundness the set of positive truth values must be closed by deduction rules : if $a \tilde{\Rightarrow} b$ and a are positive, then b must be positive

Truth values algebras

4. Truth values algebras

Coquand: any Truth value algebra can be equipped with $\leq$

$a \leq b$ if $(a \tilde{\Rightarrow} b)$ positive

verifies all prop. of Heyting algebras except: antisymmetry of $\leq$

An old remark: the set of propositions can be equipped with a relation $\leq$

$A \leq B$ if $A \Rightarrow B$ provable

reflexive and transitive

but to make it antisymmetric: need to take a quotient: Lindenbaum

an alternative "solution" drop the antisymmetry requirement: TVA

4. Truth values algebras

A $\mathcal{B}$-valued models generalizes models valued in a HA and pre-models

The constructions of $\mathcal{C}$-valued models (with Werner) used no properties of $\mathcal{C}$

Generalize to all Truth values algebras

Super-consistent theory: a $\mathcal{B}$-model, not only for one, but for all $\mathcal{B}$

Super-consistency $\longrightarrow$ termination of proof reduction

5. From Super-consistency to model-based cut elimination proofs

Together with Hermant

Simplify the proof that Super-consistency implies cut elimination

More modest goal: existence of a cut free proof and not termination of proof reduction

Each proof in a candidate collapses to its conclusion (sequent)

$\mathcal{C}'$-valued model transformed into a HA-valued model

Validity in this model implies cut elimination: sharpened uniform completeness

Reduction based and model bases cut elimination proofs

Two sides of the same coin

Both can of proof in two steps

1. the theory of interest is super-consistent
2. super-consistency implies cut elimination (independent of the theory)

Only difference: choice of the TVA for 2: $\mathcal{C}$ to get termination of proof reduction, $\mathcal{C}'$ to get just existence of cut free proofs

What's next ?

Cross fertilization of the two methods

Proof theoretical results directly from super-consistency

From Deduction modulo back to type theory