TIME MINIMIZING TRAJECTORIES IN LORENTZIAN GEOMETRY

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Abstract

This is survey article about recent results concerning some unidimensional variational problems in Lorentzian geometry (Refs. [5, 6, 7, 8, 9, 12, 18, 19]). We give a short historical introduction to the classical brachistochrone problem and we give two formulations of a time minimizing variational problem in the context of General Relativity. The solutions to this variational problem may be interpreted as worldline of massive objects moving under the action of a gravitational field and subject to suitable constraint forces, whose trajectory is a (local) time minimizer among all trajectories that have a fixed energy in a given reference system. In agreement with the terminology adopted in previous references, we will call such solutions *relativistic brachistochrones*. We distinguish between the *travel time* and the *arrival time* brachistochrones, which are curves extremizing the time measured respectively by a watch which is traveling together with the massive object and by a watch fixed at the arrival point in space. Two variational principles are discussed and some existence results for brachistochrones of both types are presented. Finally, we announce some results concerning the second variation of the travel time, aimed to develop a Morse theory for the travel time brachistochrones.

Resumo

Daremos uma breve introdução histórica ao clássico problema da braquistócrona e duas formulações no contexto da Relatividade Geral. Uma braquistócrona relativística é interpretada como a linha de universo de um ponto material movendo-se sob a ação de um campo gravitacional cuja trajetória minimiza (localmente) o tempo dentre todas as possiveis trajetórias tendo uma energia pre-fixada num dado sistema de referência (observadores). Distiguiremos entre o "tempo de viagem" e "tempo de chegada" que são dados , respectivamente, por um relógio que viaja com o ponto material e por um relógio fixado no ponto de chegada. Dois princípios variacionais serão discutidos e apresentaremos alguns resultados de existência para cada um dos casos.

1. A Historical Introduction and the General Relativistic Formulation

The classical brachistochrone[1] problem dates back to the end of the seventeenth century, when Johann Bernoulli challenged his contemporaries to solve the following problem.

> If in a vertical plane two points A and B are given, then it is required to specify the orbit AMB of the movable point M, along which it, starting from A, and under the influence of its own weight, arrives at B in the shortest possible time. *Acta Eruditorum, June 1696* (Fig. 1)

This problem attracted the attention of many important mathematicians of the time, including Newton, Leibniz, L'Hôpital, and Johann's brother, Jakob Bernoulli. The papers written on the subject may be considered the fundaments of a new field in mathematics, the *Calculus of Variations.* A beautiful historical exposition of the brachistochrone problem may be found in Reference [22], where the authors' thesis is that the brachistochrone problem also *marks the birth of Optimal Control.*

Still now the classical brachistochrone problem is very popular, and its importance is witnessed by the fact that there is hardly any book on Calculus of Variations that does not use this problem as a takeoff point. The well known solution to the brachistochrone problem is a cycloid, which is the curve described by a point P on a circle that rolls without slipping (see Fig. 2).

The cycloid curve was introduced by Galileo, who was actually the first scientist to formulate the brachistochrone problem several decades before Bernoulli, in his *Discorsi e dimostrazioni matematiche intorno a due nuove scienze*, of 1638. Curiously enough, Galileo did not find the correct answer to the problem; apparently, he simply noticed that an arc of a circle joining A and B would give a faster travel time than the straight segment.

[1] from the greek: $\beta\rho\alpha\chi\iota\sigma\tau o\varsigma$=shortest, $\chi\rho o\nu o\varsigma$=time.

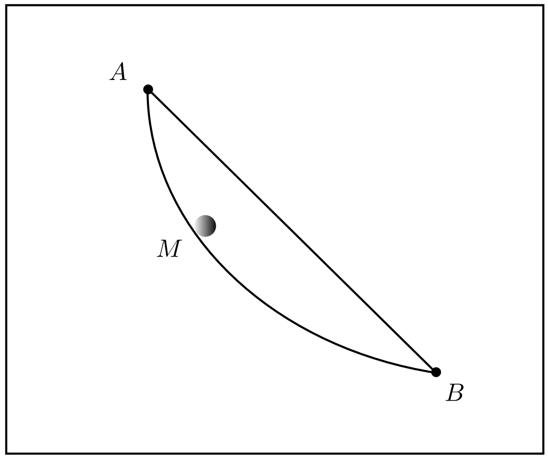

Figure 1: the brachistochrone problem

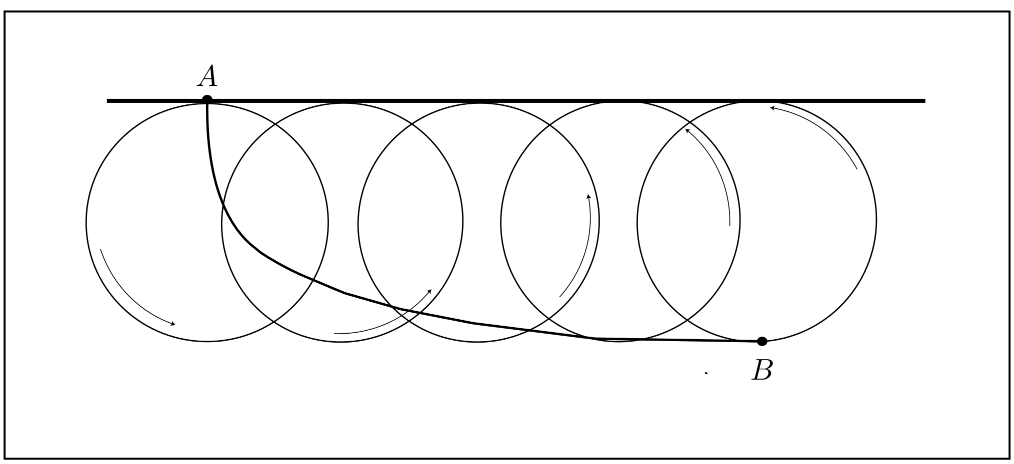

Figure 2: the cycloid

Huygens had discovered another remarkable property of the cycloid: it is the only curve such that a body, falling under its own weight, is guided by this curve so as to oscillate with a period that is independent of the initial point where the body is released. For this reason, Huygens called this curve the *tautochrone.*[2]

The classical brachistochrone problem has several generalizations, e.g., the homogeneous gravitational field could be replaced with an arbitrary Newtonian potential, and instead of releasing the particle from rest one could prescribe an arbitrary value for the initial speed, leaving the initial direction of the velocity undetermined.

In modern terminology, the Newtonian brachistochrone problem can be stated as follows. Given a manifold $\mathcal{M}_0$ endowed with a Riemannian metric g_0, to be interpreted as the configuration space, and a smooth function $V : \mathcal{M}_0 \longmapsto I\!\!R$, representing the gravitational potential, a brachistochrone of energy $E > 0$ between two points x_0 and x_1 of $\mathcal{M}_0$ is a curve $x : [0, T_x] \longmapsto \mathcal{M}$ joining x_0 and x_1 that extremizes the travel time T_x in the space of all unit speed curves y joining x_0 and x_1 and satisfying the conservation of energy law:

$$\frac{1}{2}\, g(\dot{x}, \dot{x}) + V(x) \equiv E. \tag{1.1}$$

(throughout this paper we will consider the motion of particles with unit mass) A well known variational principle states that a curve x joining x_0 and x_1 is a brachistochrone of fixed energy E if and only if x is a geodesic with respect to the conformal Riemannian metric $\phi_E \cdot g_0$, with conformal factor $\phi_E = (E - V)^{-1}$.

Figure 3 shows a picture of the brachistochrone curves in the Kepler potential $V(x, y, z) = -M(x^2 + y^2 + z^2)^{\frac{1}{2}}$ in $\mathcal{M}_0 = I\!\!R^3 \setminus \{0\}$ (see Ref. [18]).

The brachistochrone problem can also be formulated in the context of general relativity. We want to emphasize here that the original solution to the brachistochrone problem offered by Johann Bernoulli, can be made absolutely rigorous also in a general relativistic context. Namely, Johann's ingenious solution was based on a *plausibility argument* that the moving body could be imagined as a (massless) light ray moving in a plane medium where the speed

[2]from the greek, $\tau\alpha\upsilon\tau o\varsigma$=equal or same, and $\chi\rho o\nu o\varsigma$=time.

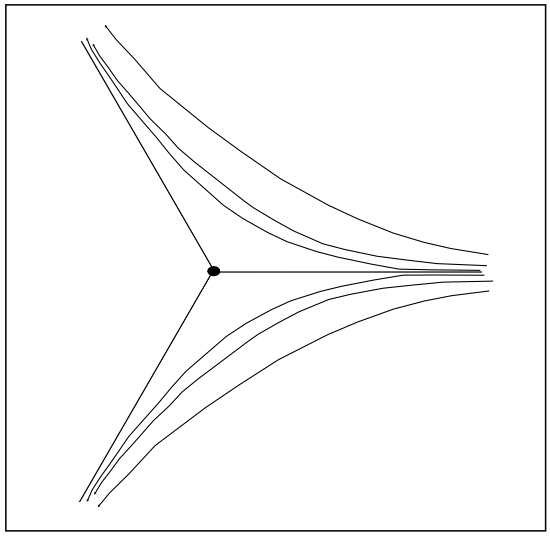

Figure 3: Brachistochrones of energy $k = 1$ issuing from infinity in the Kepler potential.

of light varies continuously. Under these circumstances, the problem can be studied using the laws of Optics developed by Snellius, Fermat and Huygens. As it was proven by Fermat, these laws imply that the trajectory of a light ray is a path of extremal travel time, and the classical cycloid solution of the brachistochrone problem can be derived using the Fermat variational principle.

In General Relativity it is possible to prove that the trajectory of a freely falling massive object, which is represented by a timelike geodesic in a Lorentzian manifold, is characterized by extremizing its arrival time measured by means of a smooth parameterization of the receiving observer. This is the so called general relativistic timelike Fermat Principle, suggested in [13] and rigorously proven in [4]. Hence, the general relativity offers a natural environment for the extension of the classical brachistochrone problem.

The first relativistic versions of the brachistochrone problem appear in [9] and [12]. V. Perlick (see [18]) has determined the brachistochrone equation in a *regular* stationary Lorentzian manifold, i.e., in a time-independent split gravitational field according to general relativity, and the other three authors in [8] have generalized Perlick's result to the case of a possibly non regular stationary Lorentzian manifold by reformulating the brachistochrone problem in the context of sub-Riemannian geometry. The variational principle proven in [8] was then used in [5] to prove some results concerning the existence and the multiplicity of relativistic brachistochrones with fixed energy between a fixed event and a fixed observer of a stationary spacetime.

More generally, the brachistochrone problem can be formulated on possibly non stationary Lorentzian manifolds in the following way.

Let $(\mathcal{M}, g)$ be a 4-dimensional Lorentzian manifold, i.e., an arbitrary spacetime in the sense of general relativity and fix a timelike smooth vector field Y on $\mathcal{M}$. For simplicity, we assume that Y is complete, i.e., its integral lines are defined over the entire real line. The integral curves of Y can be interpreted as the worldlines of *observers*. Please note that we do not require Y to be normalized, i.e., in general the worldlines of our observers are not parameterized by proper time. The reason is that in the stationary case, i.e., if $(\mathcal{M}, g)$ admits a timelike Killing vector field, it is convenient to choose this Killing vector field for Y and not a renormalized version of it.

To formulate the brachistochrone problem with respect to our arbitrarily chosen observer field Y, we fix a point p in $\mathcal{M}$, a (maximal) integral curve $\gamma : I\!R \longmapsto \mathcal{M}$ of Y and a real number $k > 0$. The *trial paths* for our variational problem are all timelike smooth curves $\sigma : [0,1] \longmapsto \mathcal{M}$ which are nowhere tangent to Y and satisfy the following conditions.

$$\sigma(0) = p; \tag{1.2}$$

$$\sigma(1) \in \gamma(I\!R); \tag{1.3}$$

$$g(\dot\sigma(0), Y(\sigma(0))) = -k\,\big(-g(\dot\sigma(0), \dot\sigma(0))\big)^{1/2}; \tag{1.4}$$

$$g(\nabla_{\dot\sigma}\dot\sigma, \dot\sigma) = 0; \tag{1.5}$$

$$g(\nabla_{\dot\sigma}\dot\sigma, Y) = 0. \tag{1.6}$$

Here ∇ denotes the Levi-Civita connection of the Lorentzian metric g. The space of trial path for our variational problem is represented in Figure 4.

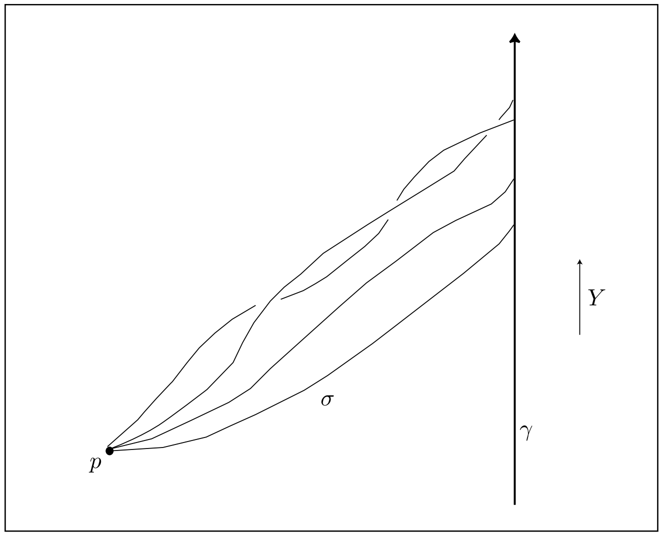

Figure 4: the space of trial paths for the general relativistic brachistochrone problem

If we interpret each integral curve of Y as a "point in space", (1.2) and (1.3) mean that all trial paths connect the same two points in space, where the starting time is fixed whereas the arrival time is not. Condition (1.5) says that all trial paths start with the same speed with respect to the observer field Y. By condition (1.5), the quantity T_σ defined by $-T_\sigma^2 = g(\dot\sigma, \dot\sigma)$ is a constant for each trial path σ (but takes different values for different trial paths). This implies that the curve parameter s along σ is related to proper time τ by an affine transformation, $\tau = T_\sigma s + const.$ As a consequence, the 4-velocity along each trial path is given by $T_\sigma^{-1}\dot\sigma$, whereas the 4-acceleration is given by $T_\sigma^{-2}\nabla_{\dot\sigma}\dot\sigma$. Hence, conditions (1.5) and (1.6) require the 4-acceleration to be perpendicular to the plane spanned by $\dot\sigma$ and Y. In other words, with respect to the observer field Y there are only forces perpendicular to the direction of motion. Such forces can be interpreted as constraint forces supplied by a frictionless slide which is at rest with respect to the observer field Y.

The brachistochrone problem can now be formulated in the following way.

Among all trial paths that satisfy the above-mentioned conditions, we want to find those curves for which the travel time is minimal or, more generally, stationary.

A different general relativistic brachistochrone problem can be formulated, by requiring that the solutions be stationary points for the *arrival time* functional, given by $AT(\sigma) = \gamma^{-1}(\sigma(1))$. In other words, $AT(\sigma)$ is the value of the time of the receiver at the arrival event; this is proper time if and only if Y is normalized along γ. In physical terms, the two brachistochrone problems differ by the way of measuring time: in the first case the time is measured by a watch traveling along the trajectory of the mass, in the second case the time is measured by the observer that receives the mass at the end of its trajectory. The two variational problems are essentially different; in this paper we present some recent results on the *travel time* brachistochrone (see [8, 5, 7]) and we announce some new results on the *arrival time* brachistochrone, that are the subject of a followup paper (Ref. [6]).

For a physical interpretation of our brachistochrone problem, the timelike vector field Y should be related to some observable quantities, i.e., Y should be co-moving with some bodies. For instance, if we are in the solar system and Y is comoving with the planets, the solutions to our brachistochrone problem will give worldlines of particles that minimize the travel time among all curves that have a fixed specific energy in the rest system of the planets. If Y is at rest with respect to the sun and to the distant stars, then the brachistochrones will be the worldlines of massive objects that minimize the travel time among all curves that have fixed energy in a reference system oriented at distant stars.

It is also possible to return to the original interpretation of of the brachistochrone problem and think of the body guided by a frictionless slide, in which case Y is determined by being the rest system of the slide.

We emphasize that in the formulation of our variational problem we consider trajectories subject to the *constraint* given by equation (1.6). The corresponding solutions are in general *not* given by trajectories of freely falling bodies, and thus the brachistochrones are not geodesics in the spacetime metric.

If $(\mathcal{M}, g)$ is a stationary spacetime and Y is a *Killing* vector field, i.e., the flow of Y preserves the metric g, then the condition (1.4) means that the product $g(\dot{\sigma}, Y)$ is constant along σ. The value of this constant can be easily computed using condition (1.4), that gives $g(\dot{\sigma}, Y) \equiv -kT_\sigma$. Hence, in the stationary case, the conditions (1.4) and (1.6) can be resumed in the condition:

$$g(\dot{\sigma}, Y) = -kT_\sigma. \tag{1.7}$$

The condition (1.7) is the relativistic counterpart of the energy conservation law (1.1) in the Newtonian case. Although physically meaningful, the mathematical approach to the general relativistic brachistochrone problem in the non stationary case presents difficulties of higher order than in the stationary case. For instance, it is not even clear whether the non stationary brachistochrones are solutions to a second order differential equation; in Reference [19], the authors used a Lagrange multiplier technique to derive a system of differential equations for the travel time brachistochrones and for the Lagrangian multipliers. Unfortunately, it does not seem to be possible to eliminate the Lagrangian multipliers from the system without introducing integrals, unless in the stationary case. Thus, it looks as if the brachistochrones in the non-stationary case are not determined by a second-order differential equation, but rather by an integro-differential equation.

For these technical reasons, we will stick to the case of a manifold $\mathcal{M}$ with metric g which is stationary with respect to the observer field Y.

In this paper we want to review some recent results obtained by the authors. For the proofs and the details of the arguments presented, the reader is referred to the original articles, especially references [4, 5, 6, 7, 8, 17, 18, 19]. For the basic notions of Lorentzian geometry and their physical interpretation, we refer to classical textbooks, like [2, 10, 15, 20].

We organize the paper as follows. In Section 2 we will present the functional framework of our variational problems. In Section 3 we will discuss the travel time brachistochrone problem, and we present a variational principle for the arrival time brachistochrones relating such curves to geodesics in a suitable Riemannian structure defined on $\mathcal{M}$. As a consequence of the principle we obtain the differential equation that characterizes the travel time brachistochrones. The results presented in Sections 2 and 3 are fully detailed in the articles [5, 8, 18, 19]. Section 4 is devoted to the arrival time brachistochrones; in this case, the solution curves to our variational problem are influenced by a sort of *Coriolis force*, which causes the arrival time brachistochrone to trace out a path different from the travel time brachistochrones with the same energy. The results presented in Section 4 are contained in [6, 18]. In Section 5 we present some existence and multiplicity results for both types of brachistochrones, obtained using the variational principles and techniques of Critical Point Theory; these results are from the references [5, 6]. Finally, in Section 6 we will study the second variation of the travel time functional and we will discuss a few applications of the second variational formula for the travel time brachistochrones. We announce partial results about the Morse theory for this kind of brachistochrones; the details are contained in a forthcoming paper [7].

2. The Functional Setup: The Space of Trial Paths

Throughout this paper we will denote by $(\mathcal{M}, g)$ a stationary Lorentzian manifold, with g a Lorentzian metric tensor on $\mathcal{M}$, and Y is a smooth timelike Killing vector field on $\mathcal{M}$, which is assumed to be complete.

The symbol $\langle\cdot,\cdot\rangle$ will denote the bilinear form induced by g on the tangent spaces of $\mathcal{M}$; the usual nabla symbol ∇ will denote the covariant derivative relative to the Levi–Civita connection of g. Given a smooth function ϕ on $\mathcal{M}$, for $q \in \mathcal{M}$ we denote by $\nabla\phi(q)$ the gradient of ϕ at q with respect to g, which is the vector in $T_q\mathcal{M}$ defined by $\langle \nabla\phi(q), \cdot\rangle = \mathrm{d}\phi(q)[\,\cdot\,]$; the Hessian $H^\phi(q)$ of ϕ at q is the symmetric bilinear form on $T_q\mathcal{M}$ given by $H^\phi(q)[v_1, v_2] = \langle \nabla_{v_1}\nabla\phi, v_2\rangle$, for $v_1, v_2 \in T_q\mathcal{M}$. We introduce for convenience the auxiliary Riemannian metric g_{R} on $\mathcal{M}$, given by:

$$g_{\mathrm{R}}(q)(v_1, v_2) = \langle v_1, v_2\rangle_{(\mathrm{R})} = \langle v_1, v_2\rangle - \frac{\langle v_1, Y(q)\rangle \cdot \langle v_2, Y(q)\rangle}{\langle Y(q), Y(q)\rangle}, \tag{2.1}$$

for $q \in \mathcal{M}$ and $v_1, v_2 \in T_q\mathcal{M}$. It is easy to see that Y is Killing also for the metric g_{R}; moreover, the restrictions of g and g_{R} to the orthocomplement of Y coincide.

For $A \subseteq \mathcal{M}$, the symbol $C^1([0,1],A)$ will denote the set of C^1-curves defined on $[0,1]$ and with image in A; we also define the space $H^2([0,1],A)$ of curves in A satisfying the H^2-Sobolev regularity condition:

$$H^2([0,1],A) = \Big\{\sigma \in C^1([0,1],A) \ : \ \dot\sigma \text{ is absolutely continuous, and } \nabla_{\dot\sigma}\dot\sigma \in L^2([0,1],T\mathcal{M})\Big\}. \tag{2.2}$$

It is not too difficult to prove that the definition of the space $H^2([0,1],A)$ does *not* indeed depend on the choice of the Riemannian metric g_{R}, nor on the choice of the linear connection ∇ that appears in (2.2). As a matter of fact, the space $H^2([0,1],A)$ can be defined intrinsically for any differentiable manifold A using local charts (see [16]) or, equivalently, using auxiliary structures on A, like for instance a Riemannian metric. In the sequel, we will assume the definition of the space $H^2([0,1],T\mathcal{M})$.

If A is a smooth submanifold of $\mathcal{M}$, in particular if A is an open subset, then $H^2([0,1],A)$ has the structure of an infinite dimensional Hilbertian manifold, modeled on the Sobolev space $H^2([0,1],\mathbb{R}^m)$; for $\sigma \in H^2([0,1],A)$, the tangent space $T_\sigma H^2([0,1],A)$ can be identified with the Hilbert space:

$$T_\sigma H^2([0,1],A) = \Big\{\zeta \in H^2([0,1],T\mathcal{M}) : \zeta \ \text{ vector field along } \sigma\Big\}. \tag{2.3}$$

The inner product in $T_\sigma H^2([0,1],A)$ is given by:

$$\langle\zeta,\zeta\rangle_* = \int_0^1 \Big(\langle\zeta,\zeta\rangle_{(\mathrm{R})} + \langle\nabla_{\dot\sigma}\zeta,\nabla_{\dot\sigma}\zeta\rangle_{(\mathrm{R})} + \langle\nabla^2_{\dot\sigma}\zeta,\nabla^2_{\dot\sigma}\zeta\rangle_{(\mathrm{R})}\Big)\,\mathrm{d}t, \tag{2.4}$$

where $\nabla^2_{\dot\sigma}\zeta = \nabla_{\dot\sigma}(\nabla_{\dot\sigma}\zeta)$.

Let k be a fixed positive constant, with $-k^2 < \sup\limits_{\mathcal{M}} \langle Y(q),Y(q)\rangle$, and U_k be the open set:

$$U_k = \Big\{q \in \mathcal{M} : \langle Y(q),Y(q)\rangle + k^2 > 0\Big\}. \tag{2.5}$$

Since Y is Killing, the quantity $\langle Y,Y\rangle$ is constant along the integral lines of Y, hence U_k is invariant with respect to the flow of Y.

We will denote by p a fixed event of U_k and by $\gamma : I\!R \longmapsto U_k$ a given integral line of Y which does not pass through p. We introduce the space

$$\Omega^{(2)}_{p,\gamma} = \Omega^{(2)}_{p,\gamma}(U_k) = \Big\{\sigma \in H^2([0,1], U_k) : \sigma(0) = p,\ \sigma(1) \in \gamma(I\!R)\Big\}. \tag{2.6}$$

It is well known that $\Omega^{(2)}_{p,\gamma}$ is a smooth submanifold of $H^2([0,1], U_k)$; for $\sigma \in \Omega^{(2)}_{p,\gamma}$, the tangent space $T_\sigma\Omega^{(2)}_{p,\gamma}$ is given by:

$$T_\sigma\Omega^{(2)}_{p,\gamma} = \Big\{\zeta \in T_\sigma H^2([0,1], U_k) : \zeta(0) = 0,\ \zeta(1) \in I\!R \cdot Y(\sigma(1))\Big\}, \tag{2.7}$$

which is a Hilbert space with respect to the inner product:

$$\langle\zeta, \zeta\rangle_2 = \int_0^1 \Big(\langle\nabla_{\dot\sigma}\zeta, \nabla_{\dot\sigma}\zeta\rangle_{(\mathrm{R})} + \langle\nabla^2_{\dot\sigma}\zeta, \nabla^2_{\dot\sigma}\zeta\rangle_{(\mathrm{R})}\Big)\, \mathrm{d}t. \tag{2.8}$$

Observe that the inner products $\langle\cdot,\cdot\rangle_*$ and $\langle\cdot,\cdot\rangle_2$ of formulas (2.4) and (2.8) are equivalent in $T_\sigma\Omega^{(2)}_{p,\gamma}$.

Finally, for all positive constant $k \in I\!R^+$, we introduce the space $\mathcal{B}^{(2)}_{p,\gamma}(k)$ by:

$$\begin{aligned}\mathcal{B}^{(2)}_{p,\gamma}(k) \ = \ & \Big\{\sigma \in \Omega^{(2)}_{p,\gamma} : \exists\, T_\sigma \in I\!R^+ \text{ such that} \\ & \langle\dot\sigma, Y\rangle \equiv -k\, T_\sigma \text{ and } \langle\dot\sigma,\dot\sigma\rangle \equiv -T_\sigma^2\Big\}.\end{aligned} \tag{2.9}$$

The space $\mathcal{B}^{(2)}_{p,\gamma}(k)$ has the structure of an infinite dimensional smooth Hilbert manifold:

Proposition 2.1. *$\mathcal{B}^{(2)}_{p,\gamma}(k)$ is a smooth submanifold of $\Omega^{(2)}_{p,\gamma}$. For $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$, the tangent space $T_\sigma\mathcal{B}^{(2)}_{p,\gamma}(k)$ can be identified with the Hilbert space:*

$$\begin{aligned}T_\sigma\mathcal{B}^{(2)}_{p,\gamma}(k) = & \Big\{\zeta \in T_\sigma\Omega^{(2)}_{p,\gamma} : \exists\, C_\zeta \in I\!R \textit{ such that} \\ & \langle\nabla_{\dot\sigma}\zeta, Y\rangle - \langle\zeta, \nabla_{\dot\sigma}Y\rangle \equiv C_\zeta \textit{ and } \langle\nabla_{\dot\sigma}\zeta, \dot\sigma\rangle \equiv \frac{T_\sigma C_\zeta}{k}\Big\},\end{aligned} \tag{2.10}$$

endowed with the inner product $\langle\cdot,\cdot\rangle_2$ of formula (2.8).

3. The Travel Time Brachistochrones

We consider the *action* functional F on $\Omega^{(2)}_{p,\gamma}$, given by:

$$F(\sigma) = \frac{1}{2}\int_0^1 \langle\dot\sigma, \dot\sigma\rangle\, \mathrm{d}t. \tag{3.1}$$

It is well known that F is smooth; for $\sigma \in \Omega^{(2)}_{p,\gamma}$ and $V \in T_\sigma \Omega^{(2)}_{p,\gamma}$, the Gateaux derivative $\mathrm{d}F(\sigma)[V]$ is given by:

$$\mathrm{d}F(\sigma)[V] = \int_0^1 \langle \nabla_{\dot\sigma} V, \dot\sigma \rangle \, \mathrm{d}t. \tag{3.2}$$

If $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$, then from (3.1) we obtain immediately:

$$F(\sigma) = -\frac{1}{2}\, T^2_\sigma; \tag{3.3}$$

hence, in $\mathcal{B}^{(2)}_{p,\gamma}(k)$ the *travel time* functional T, given by $T(\sigma) = T_\sigma$, has the following form:

$$T(\sigma) = \sqrt{-2\, F(\sigma)}. \tag{3.4}$$

In particular, from (3.3) and (3.4), one can write the travel time in an integral form, so it is possible to study its critical points using the Euler–Lagrange formalism (see Ref. [7]).

We have the following:

Proposition 3.1. *The travel time functional T on $\mathcal{B}^{(2)}_{p,\gamma}(k)$ is smooth on $\mathcal{B}^{(2)}_{p,\gamma}(k)$. For $\zeta \in T_\sigma \mathcal{B}^{(2)}_{p,\gamma}(k)$, the Gateaux derivative $\mathrm{d}T(\sigma)[\zeta]$ is given by:*

$$\mathrm{d}T(\sigma)[\zeta] = -\frac{C_\zeta}{k}. \tag{3.5}$$

After setting up our variational framework, we are ready to give the following definition:

Definition 3.2. *A* travel time brachistochrone *of energy k between p and γ is a stationary point for the travel time functional T on $\mathcal{B}^{(2)}_{p,\gamma}(k)$. A travel time brachistochrone curve σ is said to be* minimal *if σ is a minimum point for T on $\mathcal{B}^{(2)}_{p,\gamma}(k)$. From Proposition 3.1 it follows easily that a curve $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$ is a travel time brachistochrone if and only if, for every $\zeta \in T_\sigma \mathcal{B}^{(2)}_{p,\gamma}(k)$ it is $C_\zeta = 0$.*

Since T is strictly positive on $\mathcal{B}^{(2)}_{p,\gamma}(k)$, then its critical points coincide with the critical points in $\mathcal{B}^{(2)}_{p,\gamma}(k)$ of the restriction of the action functional $F = -\frac{1}{2}T^2$.

The minimal brachistochrones of energy k are *maximum* points of F on $\mathcal{B}^{(2)}_{p,\gamma}(k)$.

Figure 5 shows a picture of the spatial part of the travel time brachistochrones in the exterior Schwarzschild's spacetime, with metric given in spherical coordinates by $g = (1 - \frac{2M}{r})^{-1}\,\mathrm{d}r^2 + r^2(\sin^2\theta\,\mathrm{d}\phi^2 + \mathrm{d}\theta^2) - (1 - \frac{2M}{r})\,\mathrm{d}t^2$.

In the equatorial plane $\theta = \frac{\pi}{2}$ they are determined by two constants of motion:

$$\frac{(1 - \frac{2M}{r})^{-1}\,\dot r^2 + r^2\dot\phi^2}{(1 - \frac{2M}{r})^{-1} - 1} = 1$$

and

$$\ell = \frac{r^2\dot\phi}{(1 - \frac{2M}{r})^{-1} - 1}.$$

Substituting $u = \frac{2M}{r}$, using the constants of motion we see that the travel time brachistochrones in the exterior Schwarzschild spacetime are solutions to the hyperelliptic integral:

$$\mathrm{d}\phi = \frac{\pm\sqrt{u}\,\mathrm{d}u}{\sqrt{1-u}\,\sqrt{4M^2\ell^{-2}(1-u) - u^3}}.$$

We now give a different description of the travel time brachistochrone curves, as curves that minimize *locally* their travel time.

If q is any point in U_k, we denote by γ_q the maximal integral line of Y through q. Moreover, if $I = [a,b] \subseteq [0,1]$ is any interval, and if q_1, q_2 are any two points in U_k, we define $\mathcal{B}^{(2)}_{q_1,\gamma_{q_2}}(k, I)$ as the space of curves $\tau \in H^2(I, U_k)$ such that $\tau(a) = q_1$, $\tau(b) \in \gamma_{q_2}(\mathbb{R})$, and satisfying $\langle\dot\tau, Y\rangle \equiv -k\,T_\tau$, $\langle\dot\tau,\dot\tau\rangle \equiv -T_\tau^2$ for some $T_\tau \in \mathbb{R}^+$.

Observe that if $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$, then, for every $I = [a,b] \subseteq [0,1]$, the restriction of σ to I is a curve in $\mathcal{B}^{(2)}_{\sigma(a),\gamma_{\sigma(b)}}(k, I)$.

Definition 3.3. *A curve* $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$ *is said to be a* local minimizer *for the travel time if, for all* $0 \le a < b \le 1$ *such that* $b - a$ *is sufficiently small, the restriction of* σ *to the interval* $I = [a,b]$ *is a minimum point for the travel time*

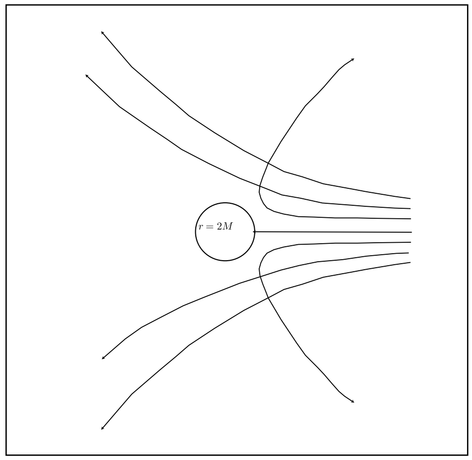

Figure 5: Travel time brachistochrones of energy $k = 1$ in Schwarzschild's spacetime issuing from a point at infinity (see Ref. [18]). They all return to infinity after passing through a point of minimal distance from the center; for large values of the constant of motion ℓ^2 the travel time brachistochrones remain in the region $r \gg 2M$ and differ little from the classical brachistochrones in Kepler's potential (Fig. **??**).

functional in the space $\mathcal{B}^{(2)}_{\sigma(a),\gamma_{\sigma(b)}}(k, I)$

As in the classical case, the travel time brachistochrone problem can be reduced to a geodesic problem with respect to a suitable Riemannian structure.

We denote by Δ the smooth distribution on $\mathcal{M}$ given by the orthocomplement of the vector field Y. Observe that, since Y is timelike, Δ is *spacelike*, i.e., the restriction of the Lorentzian metric g on Δ is positive definite.

Let $\psi : \mathcal{M} \times I\!R \longmapsto \mathcal{M}$ be the flow of Y, i.e., for $q \in \mathcal{M}$ and $t \in I\!R$, $\psi(q, t)$ is the value $\gamma_q(t)$, where γ_q is the maximal integral line of Y satisfying $\gamma_q(0) = q$. Since Y is Killing, $\psi(\cdot, t)$ is a local isometry for all $t \in I\!R$; moreover, it is easy to see that the distribution Δ is ψ-invariant, which means that $\psi_x(q, t_0)(\Delta_q) = \Delta_{\psi(q,t_0)}$, where $\psi_x(q, t_0)$ denotes the differential of the map $\psi(\cdot, t_0)$ at the point q. A function $\phi : \mathcal{M} \longmapsto I\!R$ is said to be Y-invariant if it is constant along each flow line of Y; if ψ is C^1, this amounts to saying that $\langle Y, \nabla\phi\rangle \equiv 0$.

We define $\Omega^{(2)}_{p,\gamma}(\Delta)$ to be the subset of $\Omega^{(2)}_{p,\gamma}$ consisting of curves whose tangent vector at each point lies in Δ:

$$\Omega^{(2)}_{p,\gamma}(\Delta) = \Big\{ w \in \Omega^{(2)}_{p,\gamma} : \dot w(t) \in \Delta_{w(t)},\ \forall\, t \in [0,1] \Big\}. \tag{3.6}$$

Using the language of sub-Riemannian geometry, we will call *horizontal* the curves in $\Omega^{(2)}_{p,\gamma}$. By the same arguments of Proposition 2.1, one checks immediately that $\Omega^{(2)}_{p,\gamma}(\Delta)$ is a smooth submanifold of $\Omega^{(2)}_{p,\gamma}$, and that, for $w \in \Omega^{(2)}_{p,\gamma}(\Delta)$, the tangent space $T_w\Omega^{(2)}_{p,\gamma}(\Delta)$ is given by:

$$T_w\Omega^{(2)}_{p,\gamma}(\Delta) = \Big\{ V \in T_w\Omega^{(2)}_{p,\gamma} : \langle \nabla_{\dot w} V, Y\rangle - \langle V, \nabla_{\dot w} Y\rangle = 0 \Big\}. \tag{3.7}$$

We single out the following simple fact:

Lemma 3.4. *Let ϕ be a smooth Y-invariant positive function. Then, the functional*

$$E_\phi(w) = \frac{1}{2}\int_0^1 \phi(w)\, \langle \dot w, \dot w\rangle_{(R)}\, \mathrm{d}t \tag{3.8}$$

on $\Omega^{(2)}_{p,\gamma}$ and its restriction to $\Omega^{(2)}_{p,\gamma}(\Delta)$ have the same critical points. These critical points are geodesics in $\mathcal{M}$ with respect to the Riemannian metric $\phi \cdot g_{\mathrm{R}}$ that join p and γ and that are orthogonal to γ.

The functional E_ϕ of (3.8) is called the *energy* functional relative to the metric $\phi \cdot g_{\mathrm{R}}$. The critical points of E_ϕ in $\Omega^{(2)}_{p,\gamma}$ (or equivalently in $\Omega^{(2)}_{p,\gamma}(\Delta)$) will be called *horizontal geodesics* between p and γ with respect to the Riemannian metric $\phi \cdot g_{\mathrm{R}}$.

In order to state properly our variational principle, we introduce an operator $\mathcal{D}$ that *deforms* curves in $\Omega^{(2)}_{p,\gamma}$ into horizontal curves using the flow of Y.

Let $\mathcal{D}$ be the map:

$$\mathcal{D} : \Omega^{(2)}_{p,\gamma} \longmapsto \Omega^{(2)}_{p,\gamma}(\Delta)$$

defined by $\mathcal{D}(\sigma) = w$, where

$$w(t) = \psi(\sigma(t), \tau_\sigma(t)), \tag{3.9}$$

and τ_σ is the unique solution on $[0,1]$ of the Cauchy problem:

$$\tau_\sigma' = -\frac{\langle \dot\sigma, Y\rangle}{\langle Y, Y\rangle}, \qquad \tau_\sigma(0) = 0. \tag{3.10}$$

Using the Killing property of Y it is easily checked that $\mathcal{D}$ is well defined, i.e., the maximal solution of (3.10) is defined on the entire interval $[0,1]$ and the corresponding curve w given by (3.9) is horizontal. Moreover, the smooth dependence on σ of the solution τ_σ of (3.10) proves that $\mathcal{D}$ is a smooth map.

We have the following:

Proposition 3.5. *The map $\mathcal{D}$ is smooth. For $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$ and $\zeta \in T_\sigma\mathcal{B}^{(2)}_{p,\gamma}(k)$, the Gateaux derivative* $\mathrm{d}\mathcal{D}(\sigma)[\zeta]$ *is given by:*

$$\mathrm{d}\mathcal{D}(\sigma)[\zeta] = \mathrm{d}_x\psi(\sigma, t_\sigma)\left[\zeta + \tau_\zeta \cdot Y(\sigma)\right], \tag{3.11}$$

where $\mathrm{d}_x\psi$ denotes the partial derivative of ψ with respect to the first variable and $\tau_\zeta : [0,1] \longmapsto I\!\!R$ is the function:

$$\tau_\zeta(t) = -\int_0^t \frac{C_\zeta \langle Y, Y\rangle + 2k\, T_\sigma \langle \nabla_\zeta Y, Y\rangle}{\langle Y, Y\rangle^2}\,\mathrm{d}r. \tag{3.12}$$

In particular, if σ is a brachistochrone, then τ_ζ takes the following form:

$$\tau_\zeta(t) = -2k\, T_\sigma \int_0^t \frac{\langle \nabla_\zeta Y, Y\rangle}{\langle Y, Y\rangle^2}\,\mathrm{d}r. \tag{3.13}$$

Moreover, for all $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$, the differential $\mathrm{d}\mathcal{D}(\sigma) : T_\sigma\mathcal{B}^{(2)}_{p,\gamma}(k) \longmapsto T_{\mathcal{D}}(\sigma)\Omega^{(2)}_{p,\gamma}$ is injective.

Now everything is ready to state the following:

Proposition 3.6 (Variational Principle for Travel Time Brachistochrones). *Let σ be a curve in $\mathcal{B}^{(2)}_{p,\gamma}(k)$. The following are equivalent:*

1. *σ is a brachistochrone of energy k between p and γ;*
2. *σ is a local minimizer for the travel time;*
3. *$w = \mathcal{D}(\sigma)$ is a horizontal geodesic between p and γ with respect to the Riemannian metric $\phi_k \cdot g_{\mathrm{R}}$, where g_{R} is the Riemannian metric defined in (**??**), and ϕ_k is given by:*

$$\phi_k = -\frac{\langle Y, Y\rangle}{k^2 + \langle Y, Y\rangle}. \tag{3.14}$$

Moreover, if one of the conditions above is satisfied, then $E_{\phi_k}(w) = \frac{1}{2}T_\sigma^2$*, where* E_{ϕ_k} *is the energy functional relative to the metric* $\phi_k \cdot g_{\mathrm{R}}$.

Observe that the result of Lemma 3.4 applies to the function ϕ_k; Proposition 3.6 is easily proven using the following equality:

$$F = -E_{\phi_k} \circ \mathcal{D} \quad \text{in } \mathcal{B}^{(2)}_{p,\gamma}(k). \tag{3.15}$$

As a Corollary of the variational principle, we obtain the following characterization of the travel time brachistochrones:

Proposition 3.7. *A curve* $\sigma \in \Omega^{(2)}_{p,\gamma}$ *is a travel time brachistochrone of energy* k *between* p *and* γ *if and only if* σ *is smooth and there exists a* $T_\sigma > 0$ *such that* σ *satisfies the following second order differential equation:*

$$\begin{aligned} \nabla_{\dot\sigma}\dot\sigma \quad + \quad & 2k^2 \frac{\langle \nabla_{\dot\sigma} Y, Y\rangle}{\langle Y, Y\rangle\,(k^2 + \langle Y, Y\rangle)}\,\dot\sigma + \frac{2k\,T_\sigma}{\langle Y, Y\rangle}\,\nabla_{\dot\sigma} Y + \\ & -2k\,T_\sigma \frac{\langle \nabla_{\dot\sigma} Y, Y\rangle}{\langle Y, Y\rangle\,(k^2 + \langle Y, Y\rangle)}\,Y = 0. \end{aligned} \tag{3.16}$$

with initial velocity $\dot\sigma(0)$ *satisfying:*

$$\langle \dot\sigma(0), \dot\sigma(0)\rangle = -T_\sigma^2, \quad \text{and} \quad \langle \dot\sigma(0), Y(p)\rangle = -k\,T_\sigma. \tag{3.17}$$

4. The Arrival Time Brachistochrones

We now discuss a different variational problem in $\mathcal{B}^{(2)}_{p,\gamma}(k)$, whose solution curves are the arrival time brachistochrones. To this aim, we will now assume that $(\mathcal{M}, g)$ satisfies the strong causality axiom:

- there exists no closed, or *almost* closed, causal curve in $\mathcal{M}$.

The axiom above is very natural in General Relativity; its physical interpretation is that objects (massive or massless) traveling at a speed less than or equal to the speed of light will never come close to themselves at an earlier stage of their life.

Then, it is not difficult to prove that γ is an embedded curve in $\mathcal{M}$. We define the following functional $\tau : \Omega^{(2)}_{p,\gamma} \longmapsto \mathbb{R}$:

$$\tau(\sigma) = \gamma^{-1}(\sigma(1)). \tag{4.1}$$

The functional τ is interpreted as the time measured by the observer γ at the arrival event of σ. The following result is proven:

Proposition 4.1. *The functional τ is smooth on $\Omega^{(2)}_{p,\gamma}$, hence its restriction AT to $\mathcal{B}^{(2)}_{p,\gamma}(k)$ is also smooth. For all $\sigma \in \Omega^{(2)}_{p,\gamma}$ and all $\zeta \in T_\sigma\Omega^{(2)}_{p,\gamma}$, the Gateaux derivative* $\mathrm{d}\tau(\sigma)[\zeta]$ *is given by:*

$$\mathrm{d}\tau(\sigma)[\zeta] = \frac{\langle \zeta(1), Y(\sigma(1))\rangle}{\langle Y(\sigma(1)), Y(\sigma(1))\rangle}. \tag{4.2}$$

Observe that the value of the functional $\tau(\sigma)$ is independent of the parameterization of the curve σ; also the spaces $\Omega^{(2)}_{p,\gamma}$ and $\Omega^{(2)}_{p,\gamma}(\Delta)$ are invariant under reparameterization, while the space $\mathcal{B}^{(2)}_{p,\gamma}(k)$ is *not*. In particular, it follows immediately from Proposition 4.1 that if w is a critical point of τ in the space $\Omega^{(2)}_{p,\gamma}$ or in $\Omega^{(2)}_{p,\gamma}(\Delta)$, then any reparameterization $\tilde{w}$ of w is still a critical point of τ in the same space as w.

We are ready to give the following definition:

Definition 4.2. *An* arrival time brachistochrone *of energy k between p and γ is a stationary point for the arrival time functional AT on $\mathcal{B}^{(2)}_{p,\gamma}(k)$. An arrival time brachistochrone curve σ is said to be* minimal *if σ is a minimum point for AT on $\mathcal{B}^{(2)}_{p,\gamma}(k)$. Again, from Proposition 4.1 it follows immediately that a curve $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$ is an arrival time brachistochrone if and only if, for all $\zeta \in T_\sigma\mathcal{B}^{(2)}_{p,\gamma}(k)$, it is $\zeta(1) = 0$.*

Figure 6 shows a picture of the spatial part of the arrival time brachistochrones issuing from a point at infinity in Schwarzschild's spacetime. For $r \gg 2M$ sufficiently large, both the travel time and the arrival time brachistochrones differ little from the Newtonian brachistochrones in Kepler's potential.

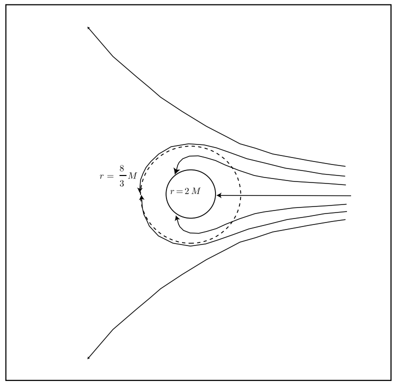

Figure 6: Arrival time brachistochrones of energy $k = 1$ issuing from a point at infinity in Schwarzschild's spacetime (see Ref. [18]). They cover the region $r > 2M$ completely; for $\ell^2 > \ell_0^2 = \frac{4^5 M^2}{27}$ they return to $r = \infty$, for $\ell^2 = \ell_0^2$ they approach the sphere at $r = \frac{8M}{3}$ asymptotically spiraling on and on forever, for $\ell^2 < \ell_0^2$ they terminate at $r = 2M$.

From the viewpoint of Calculus of Variations, the arrival time brachistochrone problem is much more delicate than the travel time time brachistochrone problem, mainly due to the following reasons:

- the functional AT is not given by an integral, i.e., it cannot be expressed in terms of a Lagrangian function $\mathcal{L}$ as in the case of the travel time functional (see formulas (3.1), (3.3) and (3.4));
- there exists no *natural* notion of local minimization for AT (although one could use the arrival time with respect to other integral lines of Y for a localization of the problem);
- the functional AT is invariant under reparameterization, yielding a lack of good compactness properties of AT, as well as the regularity and discreteness of its critical points and the finiteness of their Morse index.

In order to state our variational principle for the arrival time geodesics with fixed energy value k, we define the following functional $\tilde{\tau}_k$ on $\Omega^{(2)}_{p,\gamma}(\Delta)$:

$$\tilde{\tau}_k(w) = \tau(w) - k \int_0^1 \frac{\sqrt{\phi_k(w)\langle \dot{w}, \dot{w}\rangle}}{\langle Y, Y\rangle}\, \mathrm{d}t, \tag{4.3}$$

where ϕ_k is given by (**??**). It is easily seen that $\tilde\tau_k$ is smooth on $\Omega^{(2)}_{p,\gamma}(\Delta)$; moreover, it is invariant under reparameterization, as well as the space $\Omega^{(2)}_{p,\gamma}(\Delta)$, and a quick computation shows that:

$$\tilde\tau_k \circ \mathcal{D} = AT \tag{4.4}$$

on $\mathcal{B}^{(2)}_{p,\gamma}(k)$, where $\mathcal{D}$ is the deformation map defined by (3.9).

We prove the following variational principle:

Proposition 4.3 (Variational Principle for Arrival Time Brachistochrones). *If $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$ is an arrival time brachistochrone of energy k between p and γ, then there exists a unique reparameterization $\tilde w$ of $w = \mathcal{D}(\sigma)$ which is a critical point for the functional $\tilde\tau_k$ in $\Omega^{(2)}_{p,\gamma}(\Delta)$.*

Conversely, if w is a critical point of $\tilde\tau_k$ in $\Omega^{(2)}_{p,\gamma}(\Delta)$ and $\tilde w$ is the unique reparameterization of w such that the quantity $\phi_k(\tilde w)\langle \tilde w', \tilde w'\rangle$ is constant, then there exists a unique arrival time brachistochrone $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$ such that $\mathcal{D}(\sigma) = \tilde w$. Moreover, $\tilde w$ is a minimal point for $\tilde\tau_k$ if and only if σ is minimal for AT.

Proposition 4.3 allows to prove the regularity and to give the following differential characterizations of the arrival time brachistochrones:

Proposition 4.4. *Let $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$ be fixed and z be the unique affine reparameterization of σ defined on the interval $[0, T_\sigma]$. Then, σ is an arrival time brachistochrone of energy k between p and γ if and only if z is a smooth curve satisfying the second order differential equation:*

$$\nabla_{\dot z}\dot z - \frac{2}{k}\nabla_{\dot z}Y - \frac{2}{k}\,\frac{\langle \nabla_{\dot z}Y, Y\rangle}{k^2 + \langle Y, Y\rangle}\,(k\,\dot z - Y) = 0, \tag{4.5}$$

with initial tangent vector $\dot z(0)$ satisfying:

$$\langle \dot z(0), \dot z(0)\rangle = -1, \quad \text{and} \quad \langle \dot z(0), Y(z(0)\rangle = -k. \tag{4.6}$$

5. Some Existence Results

As it is natural to expect, in order to obtain existence results for brachistochrones, one needs to assume the completeness of $\mathcal{M}$ with respect to some Riemannian structure.

The following two theorems are proven in [5] in the case of travel time brachistochrones, using techniques of Critical Point Theory and Global Analysis on Manifolds. The case of arrival time brachistochrones is proven in [6], under the assumption of causality for $(\mathcal{M}, g)$.

Theorem 5.1. *Let $\mathcal{M}$ be a stationary Lorentzian manifold, Y a timelike Killing vector field on $\mathcal{M}$ and $k \in \mathbb{R}^+$ a fixed real constant.*

Suppose that Y is complete, k^2 is a regular value for the function $-\langle Y, Y\rangle$, and assume that Y is bounded from below in U_k, i.e., there exists a positive constant $\nu > 0$ such that:

$$-\langle Y(q), Y(q)\rangle \geq \nu > 0, \quad \forall\, q \in U_k. \tag{5.1}$$

Let p be a point in U_k and γ a maximal integral line of Y inside U_k.

Then, if $\overline{U}_k = U_k \bigcup \partial U_k$ is complete with respect to the Riemannian metric g_{R}, there exists at least one travel time brachistochrone $\sigma^{(t)}$ and at least one arrival time brachistochrone $\sigma^{(a)}$ of energy k joining p and γ in U_k.

Under the extra assumption that the closed set $\overline{U}_k$ be non contractible, we also have the following multiplicity result for brachistochrones of fixed energy:

Theorem 5.2. *Under the hypotheses of Theorem 5.1, if $\overline{U}_k$ is not contractible, then there exists a sequence $\{\sigma_n^{(t)}\}_{n\in\mathbb{N}}$ of travel time brachistochrones and a sequence $\{\sigma_n^{(a)}\}_{n\in\mathbb{N}}$ of arrival time brachistochrones of energy k between p and γ such that:*

$$\lim_{n\to\infty} T_{\sigma_n^{(t)}} = \lim_{n\to\infty} AT(\sigma_n^{(a)}) = +\infty. \tag{5.2}$$

Theorem 5.2 is the analogue of Serre's Theorem (see [21]) concerning the multiplicity of geodesics joining two fixed points in a non contractible Riemannian manifold.

6. The Second Variation of the Travel Time. Morse Theory for the Travel Time Brachistochrones

In order to investigate whether a given stationary point σ in $\mathcal{B}^{(2)}_{p,\gamma}(k)$ for the travel time functional is a local minimum, maximum, or a saddle point, one needs a second order variational formula. In this section we present briefly some results from [7] concerning the second variation of the travel time functional at a given arrival time brachistochrone.

Let M be a Banach manifold and $f : M \longmapsto I\!R$ be a smooth map. If $x_0 \in M$ is a critical point for f, i.e., $\mathrm{d}f(x_0) = 0$, then it makes sense to define the *Hessian* of f at x_0, denoted by $H^f(x_0)$, which is a continuous symmetric bilinear form on $T_{x_0}M$, in the following way.

Choose a coordinate system around x_0, $\phi : U \subset M \longmapsto U_0 \subset E$, where E is some Banach space. Define:

$$H^f(x_0)[v,w] = \mathrm{d}^2(f \circ \phi^{-1})(\phi(x_0))[\mathrm{d}\phi(x_0)[v], \mathrm{d}\phi(x_0)[w]], \tag{6.1}$$

for $v, w \in T_{x_0}M$. Using the fact that x_0 is critical for f, it is easy to see that this definition will not depend on the chart (U, ϕ). Indeed, is is easily seen that for every smooth curve $s \longmapsto y_s \in M$ such that $y_0 = x_0$ and $y_0' = v \in T_{x_0}M$, it is:

$$\frac{\mathrm{d}^2(f(y_s))}{\mathrm{d}s^2}\,|_{s=0} = H^f(x_0)[v,v]. \tag{6.2}$$

Formula (6.2) provides a simple way of computing $H^f(x_0)[v,v]$; the general formula for $H^f(x_0)[v,w]$ is easily obtained by polarization. If M is finite dimensional and it is endowed with a semi–Riemannian metric g, then one defines the Hessian $H^f(m)$ of a C^2-function $f : M \longmapsto I\!R$ at any point $m \in M$ as the symmetric bilinear form on T_mM given by $H^f(m)[v_1, v_2] = g(\nabla_{v_1}\nabla f, v_2)$, where $v_1, v_2 \in T_mM$, ∇ is the Levi–Civita connection of g and ∇f is the gradient of f with respect to g.

It is easy to prove the following:

Proposition 6.1. *Let M and N be Banach manifolds and $\mathcal{D} : M \longmapsto N$ be a smooth map; let $f : N \longmapsto I\!R$ be a smooth function. If $x_0 \in M$ is such that $\mathcal{D}(x_0)$ a critical point for f, then x_0 is a critical point for $f \circ \mathcal{D}$, and the Hessians $H^f(\mathcal{D}(x_0))$ and $H^{f\circ\mathcal{D}}(x_0)$ are related by:*

$$H^f(\mathcal{D}(x_0))[\mathrm{d}\mathcal{D}(x_0)[v], \mathrm{d}\mathcal{D}(x_0)[w]] = H^{f\circ\mathcal{D}}(x_0)[v, w], \tag{6.3}$$

for all $v, w \in T_{x_0}M$.

Observe that, from (3.3) and (6.2) we obtain easily:

$$H^F(\sigma) = -T_\sigma \cdot H^T(\sigma) \tag{6.4}$$

for all brachistochrone $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$, where T is the travel time and F is the Lorentzian action functional defined in (3.1). In particular, using Proposition 3.6, we obtain immediately the following *second order variational principle* for the travel time brachistochrones:

Proposition 6.2. *Let $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$ be a brachistochrone and $w = \mathcal{D}(\sigma)$. Then, for all $\zeta_1, \zeta_2 \in T_\sigma\mathcal{B}^{(2)}_{p,\gamma}(k)$:*

$$H^F(\sigma)[\zeta_1, \zeta_2] = -H^{E_{\phi_k}}(w)[\mathrm{d}\mathcal{D}(w)[\zeta_1], \mathrm{d}\mathcal{D}(w)[\zeta_2]]. \tag{6.5}$$

On the other hand, the Hessian $H^{E_{\phi_k}}$ of the energy functional of the Riemannian metric $\phi_k \cdot g_{\mathrm{R}}$ in the space $\Omega^{(2)}_{p,\gamma}(\Delta)$ is computed directly in the next Proposition.

We recall that, given a *non degenerate* submanifold Σ of $\mathcal{M}$, i.e., a submanifold of $\mathcal{M}$ such that for each $m \in \Sigma$ the restriction of g to $T_m\Sigma$ is non degenerate, and given any orthogonal vector $n \in T_m\Sigma^\perp$, the *second fundamental form* S^Σ_n of Σ at m in the direction of n is the symmetric bilinear form on $T_m\Sigma$ given by:

$$S^\Sigma_n(v_1, v_2) = \langle \nabla_{v_1} V_2, n \rangle,$$

where V_2 is any vector field on Σ that extends v_2.

Let's assume that the timelike curve γ has no self intersection, which in particular implies that $\gamma(I\!R)$ is a non degenerate submanifold of $\mathcal{M}$.

Proposition 6.3. *Let $w \in \Omega^{(2)}_{p,\gamma}(\Delta)$ be a horizontal geodesic between p and γ with respect to the Riemannian metric $\phi_k \cdot g_{\rm R}$. Then, the Hessian $H^{E_{\phi_k}}(w)$ is given by the following symmetric bilinear map on $T_w\Omega^{(2)}_{p,\gamma}(\Delta)$:*

$$
\begin{aligned}
H^{E_{\phi_k}}(w)[V,V] = \int_0^1 \phi_k(w)\Big[\langle\nabla_{\dot w}V,\nabla_{\dot w}V\rangle + \langle R(V,\dot w)\,V,\dot w\rangle\Big]\,{\rm d}t\, + \\
+\int_0^1 \Big[2\langle\nabla\phi_k(w),V\rangle\,\langle\nabla_{\dot w}V,\dot w\rangle + \frac{1}{2}\langle H^{\phi_k}(w)V,V\rangle\langle\dot w,\dot w\rangle\Big]\,{\rm d}t\, + \\
+\phi(w(1))\cdot S^{\gamma}_{\dot w(1)}(V(1),V(1)),
\end{aligned}
\tag{6.6}
$$

where $\nabla\phi_k$ and H^{ϕ_k} are the gradient and the Hessian of ϕ_k with respect to the Lorentzian metric g of $\mathcal{M}$ and $S^{\gamma}_{\dot w(1)}$ is the second fundamental form of γ in the direction of the normal vector $\dot w(1)$, with respect to g.

We recall the definition of the Morse index at a critical point of a C^2-functional:

Proposition 6.3. *Let M be a Hilbert manifold, $f : M \longmapsto I\!R$ be a map of class C^2 and x_0 a critical point for f in M. The Morse index $\mu^f(x_0)$ is the dimension of a maximal subspace of $T_{x_0}M$ on which the Hessian $H^f(x_0)$ is* negative *definite.*

Roughly speaking, the Morse Index $\mu^T(\sigma)$ of the travel time T at a brachistochrone σ gives the number of *essentially different* directions in which the curve σ can be deformed to obtain a curve of shorter travel time.

A travel time brachistochrone σ is a local minimum for T if and only if $\mu^T(\sigma) = 0$; moreover, if σ is a local maximum, it is necessarily $\mu^T(\sigma) = +\infty$.

Observe that, from (3.15) and (6.4), since the differential ${\rm d}\mathcal{D}(\sigma)$ is injective (see Proposition 3.5), we obtain immediately:

$$
\mu^T(\sigma) \le \mu^{E_{\phi_k}}(\mathcal{D}(\sigma)),
\tag{6.7}
$$

for all brachistochrone $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$.

It is well known that the Morse index of a Riemannian geodesic is finite (see [14]), hence formula (6.7) gives us immediately the following:

Corollary 6.5. *Let $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$ be a brachistochrone. Then, the Morse index $\mu^T(\sigma)$ is* finite. *In particular, σ is never a local maximum for the travel time functional T.*

Studying the Morse index of the bilinear form (6.6) by direct computation can be quite a challenging task. However, in the papers [1, 3, 11] it is presented a Morse Index Theorem for geodesics between submanifolds of a Riemannian manifold that relates the Morse index of a Riemannian action functional at a given geodesic with some geometrical properties of the geodesic. We recall here the main facts of the Theory.

Given a horizontal geodesic w in between p and γ with respect to the Riemannian metric $\phi_k \cdot g_{\mathrm{R}}$, let $\nabla^{\{k\}}$ and $R^{\{k\}}$ denote respectively the Levi–Civita connection and the curvature tensor of the metric $\phi_k \cdot g_{\mathrm{R}}$, and let$\mathcal{J}^{\{k\}}$ be the finite dimensional vector space of all the Jacobi fields J along w with respect to $\phi_k \cdot g_{\mathrm{R}}$, i.e., all smooth vector fields satisfying the second order differential equation:

$$\nabla^{\{k\}}_{\dot w}\nabla^{\{k\}}_{\dot w} J + R^{\{k\}}(\dot w, J)\,\dot w = 0. \tag{6.8}$$

Moreover, let $\mathcal{J}^{\{k\}}(\gamma, t_0)$ be the subspace of $\mathcal{J}^{\{k\}}$ consisting of all Jacobi fields J along w satisfying:

1. $J(1) \parallel Y(w(1))$;
2. $J(t_0) = 0$;
3. $\langle \nabla_{\dot w(1)} J, Y\rangle + S^{\gamma}_{\dot w(1)}(J(1), Y) = 0$.

A point $w(t_0)$ along w is said to be a *γ-focal point* if $\dim(\mathcal{J}^{\{k\}}(t_0)) > 0$; the *multiplicity* of the a γ-focal point $w(t_0)$ is the dimension of $\mathcal{J}^{\{k\}}(t_0)$.

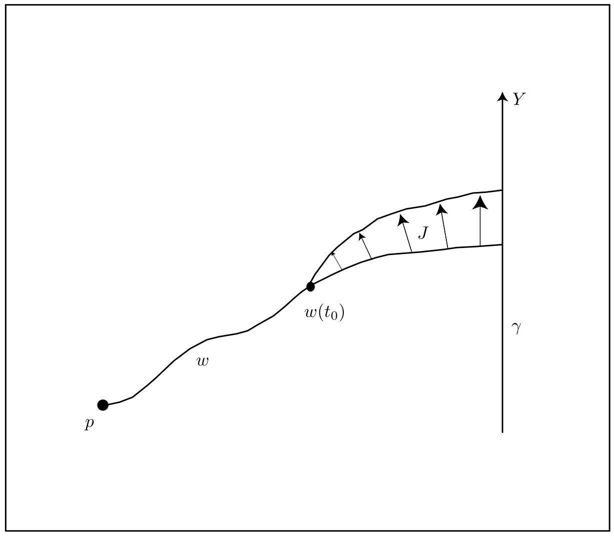

Figure 7: a γ-focal point along w

Equation (**??**) is obtained by linearizing the geodesic equation in the metric $\phi_k \cdot g_{\rm R}$; hence, it is satisfied by vector fields along w that correspond to variations of w consisting of geodesics. Loosely speaking, the arrow-head of J traces out infinitesimally close neighboring geodesics to w (see Figure 7).

The condition 1 means that, in a first order approximation, these geodesics arrive on γ; condition 2 means that they pass through $w(t_0)$. Condition 3 means that these geodesics arrive orthogonally at γ; observe that orthogonality to the vector field Y is equivalent in the three metrics g, $g_{\rm R}$ and $\phi_k \cdot g_{\rm R}$, and for this reason it is possible to write this condition using the Lorentzian Levi–Civita connection ∇ and the Lorentzian second fundamental form S^γ of γ.

The Morse Index Theorem says that, if p is not a γ-focal point along w, the Morse index $\mu^{E_{\phi_k}}(w)$ of E_{ϕ_k} in the space $T_w\Omega^{(2)}_{p,\gamma}$ (or, equivalently, in $T_w\Omega^{(2)}_{p,\gamma}(\Delta)$) is given by the number of γ-focal points along w, counted with multiplicity.

Thus, we have the following:

Proposition 6.6. *Let $\sigma \in \mathcal{B}^{(2)}_{p,\gamma}(k)$ be a brachistochrone and $w = \mathcal{D}(\sigma)$. Suppose that there are no γ-focal points along w. Then, σ is a local minimum for the* *arrival time functional T.*

Details and further results concerning the Morse theory for the travel time brachistochrones are found in [7].

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